Canonical Transformations

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Symplectic structure

Hamilton's equations treat q_k and p_k asymmetrically. It is possible to hide the asymmetry inside clever notation. Define a column vector $\mathbf{x} = (q_1, q_2, \cdots, q_D, p_1, p_2, \cdots, p_D)^T$. Then, in terms of this, one can write

$$\dot{\mathbf{x}} = J\frac{\partial H}{\partial \mathbf{x}}, \quad \text{where} \quad J = \begin{pmatrix} 0 & 0 & \cdots & 1 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 1 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ -1 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & -1 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

The antisymmetric matrix J is called the symplectic form. Note that $J^{-1} = -J = J^T$.

Restricted canonical transformations

Transformation of coordinates on phase space, $\boldsymbol{\xi}(\mathbf{x})$ are called restricted canonical transformations if they do not change the form of Hamilton's equations. They are called restricted because they do not depend explicitly on time.

The Jacobian of the transformation, M, is given by the derivatives $M_{ij} = \partial\xi_i/\partial x_j$. Clearly, $\dot{\boldsymbol{\xi}} = M\dot{\mathbf{x}}$. Also, one has the relation

$$\frac{\partial H}{\partial x_i} = \frac{\partial H}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_i}, \qquad \text{so} \qquad \frac{\partial H}{\partial \mathbf{x}} = M^T\frac{\partial H}{\partial \boldsymbol{\xi}}.$$

As a result, one may write the transformed Hamilton's equations as

$$\dot{\boldsymbol{\xi}} = MJ\frac{\partial H}{\partial \mathbf{x}} = MJM^T\frac{\partial H}{\partial \boldsymbol{\xi}}.$$

The new coordinates can be interpreted as generalized coordinates and momenta provided $\boxed{MJM^T = J}$. Such transformations are called symplectic transformations or canonical transformations.

Some problems

Problem 21: A canonical transformation

It is common to call the top half of the components of ξ the new coordinates, Q_i, and the bottom half the new momenta, P_i. Take the transformation $P_i = q_i$ and $Q_i = -p_i$. Is this a canonical transformation?

Problem 22: Elementary Poisson brackets

With the notation $\mathbf{x} = (q_1, q_2, \cdots, p_1, p_2, \cdots)^T$, write down the matrix of Poisson brackets $P_{ij} = [x_i, x_j]$. How do these Poisson brackets change under restricted canonical transformations?

Problem 23: Poisson's theorem

If f and g are two conserved quantities, show that $[f, g]$ is also a conserved quantity.

Infinitesimal transformations

Consider the infinitesimal transformations

$$\xi_i = x_i + \delta\xi_i = x_i + \epsilon f_i(\mathbf{x}), \quad \text{with Jacobian} \quad M_{ij} = \frac{\partial \xi_i}{\partial x_j} = \delta_{ij} + \epsilon \frac{\partial f_i}{\partial x_j},$$

which we will write as $\mathbf{x} \xrightarrow{M} \xi$. In other words, we can write $M = I + \epsilon A$. For the transformation to be canonical, one must have

$$J = MJM^T = (I + \epsilon A)J(I + \epsilon A^T) = J + \epsilon(AJ + JA^T) + \mathcal{O}(\epsilon^2).$$

If one wants to satisfy the condition that the coefficient of ϵ vanishes, it is sufficient to arrange that $A = JG$, where G is a symmetric matrix, so clearly, $A^T = -G^T J$. The symmetry of G is guaranteed if we start with a generating function $\mathcal{G}(\mathbf{x})$ and write

$$\xi = \mathbf{x} + \epsilon J \nabla \mathcal{G}(\mathbf{x}), \quad i.e. \quad f_i = \frac{\partial \mathcal{G}}{\partial x_i} \quad \text{and} \quad G_{ij} = \frac{\partial^2 \mathcal{G}}{\partial x_i \partial x_j}.$$

The group of canonical transformations

Infinitesimal canonical transformations satisfy

1. The **identity** transformation, $\xi = \mathbf{x}$, is canonical.
2. If $\mathbf{x} \xrightarrow{L} \xi$ is a canonical transformation and $\xi \xrightarrow{M} \zeta$ is also canonical, then the **product** transformation $\mathbf{x} \xrightarrow{P=LM} \zeta$ is also canonical, because $PJP^T = LMJM^TL^T = LJL^T = J$.
3. If the transformation $\mathbf{x} \xrightarrow{M} \xi$ is canonical, so is the **inverse** transformation $\xi \xrightarrow{M^{-1}} \mathbf{x}$.
4. Canonical transformations are **associative**, *i.e.*, when making three successive transformations, $\mathbf{x} \xrightarrow{L} \xi$, $\xi \xrightarrow{M} \zeta$ and $\zeta \xrightarrow{N} \chi$, we get the same transformation $\mathbf{x} \xrightarrow{P=LMN} \chi$ whether we compose them as $\mathbf{x} \xrightarrow{LM} \zeta$ and $\zeta \xrightarrow{N} \chi$ or $\mathbf{x} \xrightarrow{L} \xi$ and $\xi \xrightarrow{MN} \chi$, since matrix multiplication is associative: $(LM)N = L(MN)$.

These properties define groups of transformations. Because there is a continuous infinity of them, they form a Lie group.

Active and passive views of transformations

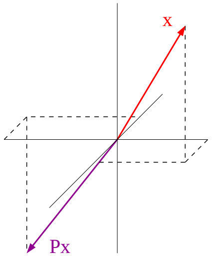
When we start discussing transformations of geometrical objects such as vectors, we must be aware of a choice of convention.

One view of transformations is the so-called passive view, in which we take the vector as being unchanged and merely describing it in some new coordinate system: the components of the vectors change but the vector is geometrically the same. Consider a rotation in two dimensional space. Take a vector with components $\mathbf{x} = (r\cos\theta, r\sin\theta)$. If the coordinate system is rotated in the anti-clockwise direction by an angle ϕ, then the new components of the same vector are $\mathbf{x} = (r\cos(\theta-\phi), r\sin(\theta-\phi))$.

The other view of transformations is the active view, *i.e.*, one in which the coordinate system is unchanged but the vector changes. If the same vector $\mathbf{x} = (r\cos\theta, r\sin\theta)$ is rotated in the anti-clockwise direction by an angle ϕ then the components of the new vector $\mathbf{x}' = (r\cos(\theta+\phi), r\sin(\theta+\phi))$.

In these lectures we adopt the active view.

An example: parity in three dimensions

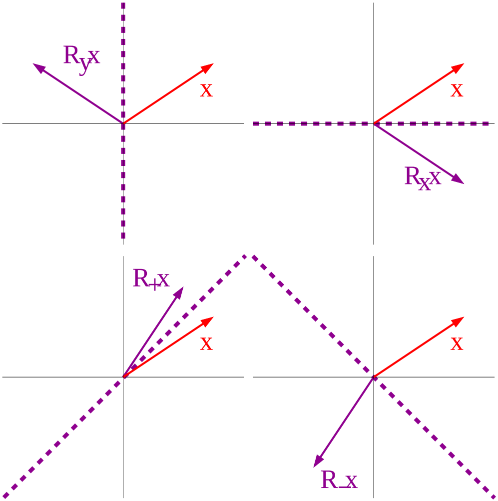


Matrix representations: $I = I_3$, $P = -I_3$.

Consider the two transformations acting on a vector $\mathbf{x}$: I (for identity) which does nothing to $\mathbf{x}$, and P (for parity) which changes $\mathbf{x} \to -\mathbf{x}$, *i.e.*, it flips the sign of each component. Clearly $P^2 = I$. This set of transformations forms a group since:

1. There is an **identity**.
2. The set is **closed** under products of transformations.
3. P is its own **inverse**.
4. Any product of three successive transformations (III, IPI, IPP, *etc.*) is **associative**.

An example: reflections in two dimensions



An example: reflections in two dimensions

In two dimensional space with vectors $\mathbf{x} = (x, y)$ place a mirror along the line $x = 0$. Reflections, R_y, in this mirror transform vectors as $\mathbf{x} \to R_y\mathbf{x} = (-x, y)$. In terms of matrices acting on vectors, we have

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \text{and} \qquad R_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Clearly, $R_y^2 = 1$, and the set of transformations $\{I, R_y\}$ form a group since the set has the identity, is closed, every element has an inverse, and transformations are associative.
Take the reflections in a mirror placed along any line passing through the origin; they form a similar group. We find

$$R_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad R_+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad R_- = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Two element groups

The group Z_2

Take a group with two elements. One has to be the identity. The other element has to be its own inverse (otherwise the group is not closed). Hence all groups with two elements are equivalent. They are all called Z_2.

The product operation can be shown as a table:

	I	P
I	I	P
P	P	I

The set of integers $\{1, -1\}$ under the operation of the product of integers also forms the group Z_2. The set of integers $\{0, 1\}$ under the operation of addition modulo 2 (add the integers and then take the remainder after division by 2) also form the group Z_2.

Three element groups

Consider a group with three elements $\{I, A, B\}$, with identity I. The products A^2, B^2 and AB must be elements of the group. We get one consistent solution if we set $B = A^2$ and $AB = I$ (for then $A^3 = I$ and hence $B^2 = A^4 = A$). This gives a group which is called Z_3 (addition of integers modulo 3). No other possibilities exist. The group multiplication table for Z_3 is

	I	A	B
I	I	A	B
A	A	B	I
B	B	I	A

Problem 24: Four-element groups

How many groups exist with four elements: $\{I, A, B, C\}$?

Continuous groups: a rotation group

Consider rotations around a point (the origin) in two dimensions. Under such transformations we have $\mathbf{x} \to R\mathbf{x}$ with

$$R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Clearly, $R(0) = I$, so identity is one of the rotations. Now, using the addition theorems for cos and ϕ, we can verify that $R(\phi)R(\theta) = R(\phi + \theta)$, so the set is closed. Since $R(\phi)R(-\phi) = I$, each element has an inverse. The group operation is associative because matrix multiplication is associative. So rotations in two dimensions form a group.

The group is called continuous because the components of $R(\phi)\mathbf{x}$ are continuous functions of ϕ. Since are also differentiable functions of ϕ, this is called a Lie group. This is given the name SO(2), *i.e.*, the group of orthogonal 2×2 matrices with determinant unity.

Infinitesimal elements of SO(2)

Note that the determinant of $R(\epsilon)$ is unity. This is a characteristic of rotations. However the determinants of P, R_x, R_y, R_+ and R_- are all negative. This is a characteristic of reflections.

Rotations by an infinitesimal angle ϵ are clearly given by

$$R(\epsilon) = \begin{pmatrix} 1 & \epsilon \\ -\epsilon & 1 \end{pmatrix} = I + i\epsilon\sigma, \quad \text{where} \quad \sigma = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Problem 25: Exponentiation

A definition of $\exp x$ is that it is the limit of $(1 + x/N)^N$ when $N \to \infty$. So, by analogy one can write

$$R(\phi) = \lim_{N\to\infty} \left(I + \frac{i\phi\sigma}{N} \right)^N = \exp(i\phi\sigma).$$

How would you actually compute the exponential of a matrix?

One-dimensional representation of the rotation group

Suppose we write the vector $\mathbf{x}$ as the complex number $z = x + iy$. Then the rotated vector, $R(\phi)\mathbf{x}$ can be represented as the complex number

$$\begin{aligned} & \cos\phi x + \sin\phi y + i(-\sin\phi x + \cos\phi y) \\ = \quad & \cos\phi(x + iy) + \sin\phi(y - ix) = (\cos\phi - i\sin\phi)z = \mathrm{e}^{-i\phi}z. \end{aligned}$$

The unimodular complex numbers, $\exp(i\phi)$ form a group which is given the name U(1) (the group of 1×1 unitary matrices). Clearly SO(2)=U(1). Both are Abelian groups, *i.e.*, all their elements commute with each other.
The eigenvalues of the matrix $R(\phi)$ are $\exp(\pm i\phi)$. The action of the rotation on z is $z \to \exp(-i\phi)z$. All 2d vectors can also be mapped on the complex conjugate $\overline{z} = x - iy$. Rotations act on this by $\overline{z} \to \exp(i\phi)\overline{z}$.

Rotations in three dimensions

Rotations in three dimensions can be specified by giving an axis of rotation, $\hat{\mathbf{n}}$, and the angle of rotation, ψ around $\hat{\mathbf{n}}$, which we write as $R(\hat{\mathbf{n}}, \psi)$. The specification of an unit vector in three dimensions requires two angles

$$\hat{\mathbf{n}}^T = (\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\phi).$$

So a rotation in three dimensions can be specified by three angles. Rotations around a fixed axis $\hat{\mathbf{n}}$ are isomorphic to rotations in two dimensions,

$$R(\hat{\mathbf{n}}, \psi')R(\hat{\mathbf{n}}, \psi) = R(\hat{\mathbf{n}}, \psi + \psi').$$

However, rotations around different axes have quite different properties

$$R(\hat{\mathbf{n}}, \psi)R(\hat{\mathbf{n}}', \psi') \neq R(\hat{\mathbf{n}}', \psi')R(\hat{\mathbf{n}}, \psi).$$

Lorentz transformations

In relativity, the norm of a 4-vector (the first component of $\mathbf{x}$ is the time and the others are space) is $\mathbf{x}\cdot\mathbf{x} = \mathbf{x}^T G\mathbf{x}$ where $G = \mathrm{diag}(-1,1,1,1)$. A boost acts linearly and homogenously on the coordinates, so one has $\mathbf{x} \to M\mathbf{x}$ under boosts (also called Lorentz transformations). A boost leaves relativistic norms unchanged. Hence, $M^T GM = G$.

Problem 26: Relativistic boosts

1. Prove that Lorentz transformations form a group.
2. If space were one-dimensional, then $G = \mathrm{diag}(-1,1)$, and Lorentz transformations would be represented by 2×2 matrices. Since a boost is specified by one parameter, each M must be specified by a single parameter. What can you say about such matrices. Can you use the group property to find the relativistic law of addition of velocities?

Keywords and References

Keywords

antisymmetric matrix, symplectic form, restricted canonical transformations, Jacobian, symplectic transformations, canonical transformations, generating function, groups, passive view, active view, mirror, rotation in two dimensions, Lie group, exponential of a matrix, Abelian groups, rotation in three dimensions, isomorphic, 4-vectors, relativistic boosts, Lorentz transformations

References

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