Coordinates, phase space, constraints

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Mechanics of a single particle

For the motion of a particle (of constant mass m and position specified by the vector $\mathbf{x}$) viewed from an inertial frame, one has the equations of motion:

$$m\ddot{\mathbf{x}} = \mathbf{f}, \quad ie, \quad \dot{\mathbf{x}} = \frac{\mathbf{p}}{m}, \ \dot{\mathbf{p}} = \mathbf{f}.$$

Here $\mathbf{f}$ is the force acting on the particle and $\mathbf{p} = m\dot{\mathbf{x}}$ is its momentum. The function $\mathbf{x}(t)$ is the trajectory of the particle. This is found with enough initial conditions: $\mathbf{x}(0)$ and $\mathbf{p}(0)$ are both needed.

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Phase space

The 6-dimensional space with coordinates $\{\mathbf{x}, \mathbf{p}\}$ is called the phase space of the particle. The solutions of the EoM give $\mathbf{x}(t)$ and $\mathbf{p}(t)$ which together define a phase space trajectory.

The free particle

Problem 1

Assume that a particle is constrained to travel along a straight line.

1. How many dimensions is the phase space?
2. What is the dimension of phase space volume?
3. Solve the equations of motion when $f = 0$. What are the trajectories? The phase space trajectories?
4. If different particles (with $f = 0$) have initial conditions which are within a small square of phase space at the initial time $t = 0$, then what is the shape of the area covered by these particles after some time?
5. Does this phase space volume change with time?

The square well potential

Problem 2

Assume that a particle is constrained to travel along a straight line subject to the potential

$$V(x) = \begin{cases} -V_0 & (|x| \leq a) \\ 0 & (|x| > a) \end{cases}$$

1. Solve the equations of motion. What are the trajectories? The phase space trajectories?
2. If different particles (with the same equations of motion) have initial conditions all of which are within a small square of phase space at the initial time $t = 0$, what happens to this under time evolution?

A little mathematical generalization

An ordinary differential equation of order N for any vector $\mathbf{x}_1$

$$\frac{d^N \mathbf{x}_1}{dt^N} = \mathbf{f}\left(t, \mathbf{x}_1, \frac{d\mathbf{x}_1}{dt}, \cdots, \frac{d^{N-1}\mathbf{x}_1}{dt^{N-1}}\right),$$

is equivalent to the system of equations

$$\dot{\mathbf{x}}_1 = \mathbf{x}_2, \quad \dot{\mathbf{x}}_2 = \mathbf{x}_3, \quad \dot{\mathbf{x}}_N = \mathbf{f}(t, \mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{N-1}).$$

One can define a "phase space" with coordinates $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N\}$. When these are given as functions of time one has a "phase space trajectory". Clearly this requires complete specification of N initial conditions, one for each of the $\mathbf{x}_i$.

If the force is not an explicit function of time, then the equation is autonomous. In an autonomous equation (or system of equations) the phase space trajectories do not intersect. Prove this

The principle of relativity

Galileo's principle of relativity

There are frames where the laws of physics are terribly simple: if no forces act on a particle then it remains at rest. This is one inertial frame. Any frame which moves at a constant velocity with respect to this is also an inertial frame.

Here is the transformation law from one inertial frame to another:

$$\mathbf{p}' = \mathbf{p} + m\mathbf{V}, \qquad \mathbf{x}' = \mathbf{x} + \mathbf{V}t.$$

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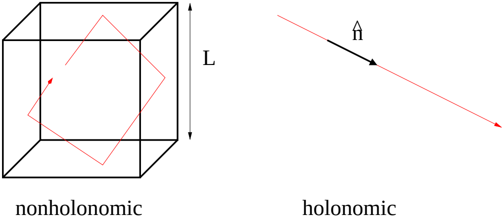
Problem 3

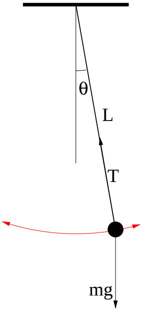
Is a frame fixed with respect to the earth an inertial frame? If not then how badly non-inertial is it?

Kinds of constraints

- Non-holonomic constraint. Example: particle confined in a box. The coordinates are constrained to be $|\mathbf{x}| \leq L$ where L is the size of the box. Typically non-holonomic constraints are expressed as inequalities on the coordinates.
- Holonomic constraint. Example: particle moving in a line. The coordinates are constrained to be $\mathbf{x} = s\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is an unit vector in some direction. Another way to write this is $\mathbf{x} \times \hat{\mathbf{n}} = 0$, *i.e.*, the components of $\mathbf{x}$ orthogonal to $\hat{\mathbf{n}}$ vanish. Typically holonomic constraints are expressed as equalities.



Forces of contraint



- Consider the motion of a simple pendulum: a bob suspended from a fixed support by means of an inelastic thread. If the length of the thread is L, then (if the initial condition is $\mathbf{p} = 0$) the CM of the bob moves along the arc of a circle of radius L. The thread is in tension. This force of constraint makes the motion one dimensional. There is no work done by the force of constraint. Prove this

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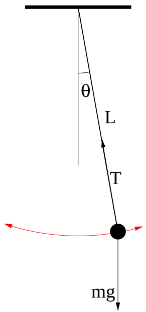

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- In actual fact any thread is elastic, so the length of the pendulum can vary. Similarly, the support is never exactly fixed. In both cases the motion is no longer one-dimensional, and the constraint can do work. A holonomic constraint is a mathematical idealization. When is it a good idealization?

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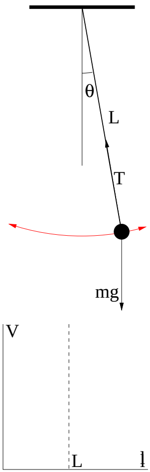

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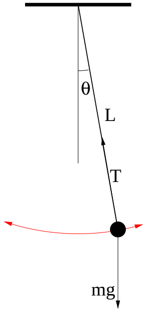

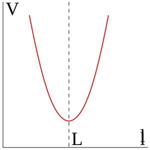

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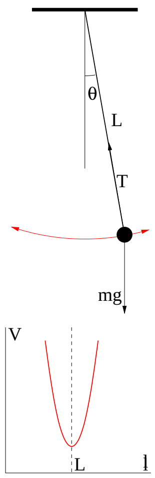

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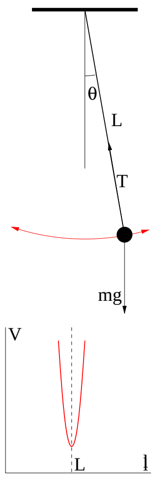

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Effective theories

1. The energy scale of the pendulum is $E = mgh$, where h is the maximum height of the bob. In terms of the generalized coordinate, $h = L(1 - \cos\theta)$, so a typical energy scale of the pendulum is $E_0 = mgL$.
2. Changes in the length of the thread cost some energy, $E_{\rm el}$. Similarly, a motion of the point of support is due to the elastic energy of the bending of the supporting body: $E'_{\rm el}$.
3. Elastic deformations can be ignored as long as $E_0 \ll E_{\rm el}$ or $E_0 \ll E'_{\rm el}$.

Effective theory

The dynamics of any system is often an effective low-energy theory, obtained by neglecting physical effects which become important at much higher energy. The approximation of rigidity of bodies is one such effective theory.

When can a body be considered rigid?

Problem 4: How rigid is a rigid body?

1. A simple pendulum is hung from a spring with spring constant K. The motion of the pendulum is a sum of two harmonic motions (one is the swinging of the bob, the other the extension of the spring) assumed not to be in resonance. How does energy flow between these two oscillators? Does the energy flow modify the frequency of the pendulum?
2. A particle tries to bounce off an inflated balloon. Are there any conditions under which there is no bounce? When there is a bounce can the kinetic energy of the particle remain unchanged?
3. Under what conditions does a stone bounce off the surface of water?
4. Is it possible to run on loose sand without slowing down?

Forces of constraint do no work

A particle is restricted to a surface by a constraint force **R**.
Why not a curve? The equation of motion gives

$$0 = \mathbf{v}\cdot(m\dot{\mathbf{v}} - \mathbf{R}) = \dot{T} - \mathbf{v}\cdot\mathbf{R},$$

where the kinetic energy $T = mv^2/2$. If the force of the constraint does no work then T is conserved and $\mathbf{v}\cdot\mathbf{R} = 0$. If the normal to the surface at the point $\mathbf{x}$ is in the direction $\hat{\mathbf{n}}$, then $\hat{\mathbf{n}}\cdot\mathbf{v} = 0$, which means that $\hat{\mathbf{n}}\times\mathbf{R} = 0$.
If we try to write the constraint force in terms of a potential

$$V' = \sum_{i=1}^{3} \alpha_i x_i^2,$$

since the gradient of the potential is zero along the surface, the only non-vanishing component is $\hat{\mathbf{n}}\cdot\nabla V'$. The constraint is recovered by taking $\alpha_i \to \infty$ subject to this condition.

Keywords and References

Keywords

equations of motion, force, inertial frame, kinetic energy, momentum, trajectory, phase space, phase space trajectory, phase space volume, ordinary differential equation, autonomous system of equations, initial conditions, non-holonomic constraint, holonomic constraint, constraint force, rigid bodies, effective low-energy theory

References

Goldstein, chapter 1.
Landau, section 1 and 3.