Physics of Continuous media

Sourendu Gupta

TIFR, Mumbai, India

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Deformations of continuous media

If a body is deformed, we say that the point which originally had coordinates $\mathbf{x}$ has coordinates $\mathbf{x}'$ are the deformation. The deformation itself is given by $\mathbf{u} = \mathbf{x}' - \mathbf{x}$. The strain is a measure of the deformation of the body. In order to make a quantitative definition, we begin by measuring lengths of line elements in the body. The length of a piece of the material may change:

$$dl'^2 = |d\mathbf{x}'|^2 = dx_i dx_i + 2\frac{\partial u_i}{\partial x_k} dx_i dx_k + \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_k} dx_j dx_k$$

The factor 2 in the term in the middle comes after interchanging dummy indices.

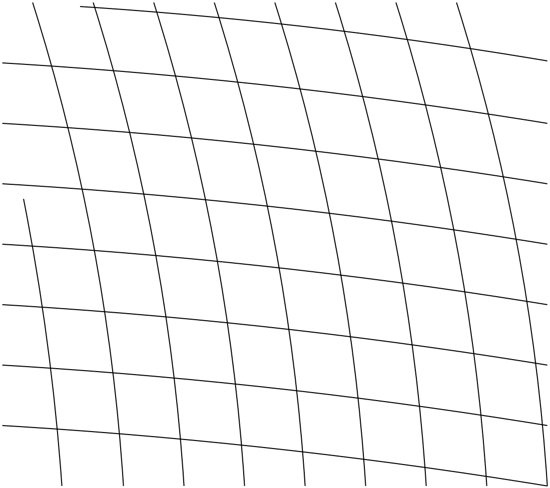
The strain tensor measures changes of lengths in various direction:

$$\Delta dl^2 = dl'^2 - dl^2 = 2u_{ij} dx_i dx_j, \qquad u_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_k}\right).$$

Strain

The instantaneous deformation of a body defines a tensor field u_{ij}.

Strain



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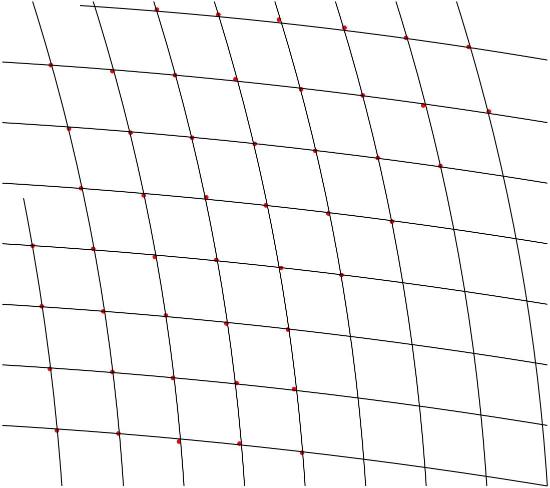

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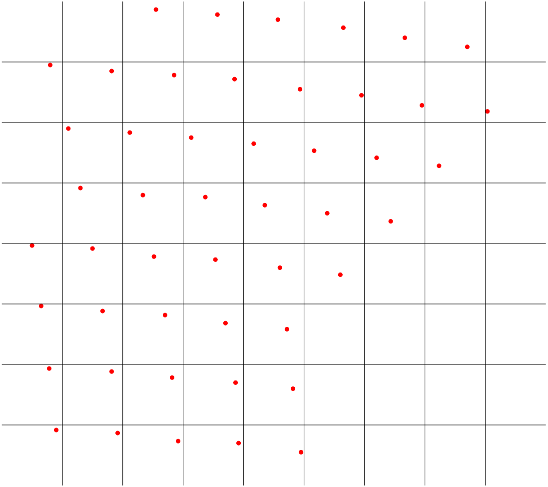

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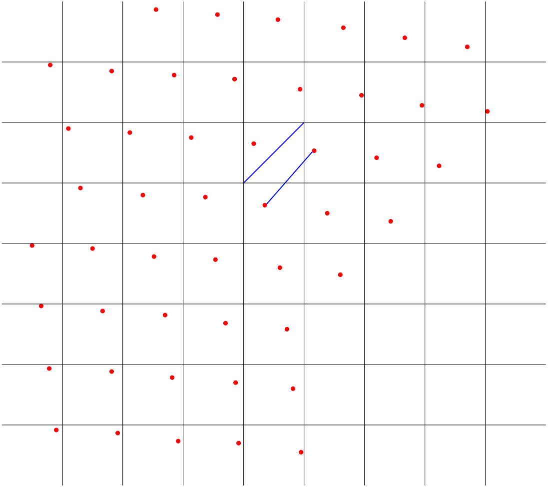

The instantaneous deformation of a body defines a tensor field u_{ij}.

Properties of the strain tensor

Since strain is a symmetric second rank tensor, it is possible to choose axes so that it is diagonal. With these principal axes one has

$$dl'2 = (\delta_{ij}+2u_{ij})dx_i dx_j = (1+2u_1)dx_1^2+(1+2u_2)dx_2^2+(1+2u_3)dx_3^2.$$

In other words, the strain corresponds to stretching or compression along certain principal directions.
Clearly, the change in the volume due to a strain is given by

$$\Delta dV = dV' - dV = \det u dV = u_1 u_2 u_3 dV.$$

If the dimensions of the body are large compared with the strain at any point, then one can linearize the expression for the strain:

$$u_{ij} \simeq \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

The linear expression may break down for thin rods or plates.

Analogy to a metric

A metric tensor is a symmetric tensor of rank two which allows us to define lengths, through the formula

$$dl^2 = g_{ij} dx_i dx_j .$$

In Euclidean space with Cartesian coordinates the metric tensor is $g_{ij} = \delta_{ij}$. A body under strain can be thought of as a curved space with a changed metric tensor,

$$dl'^2 = g'_{ij} dx_i dx_j, \qquad \text{where} \qquad g'_{ij} = g_{ij} + 2u_{ij} .$$

Problem 65: Exploring the analogy

In a curved space defining a derivative operator involves introducing parallel transporters called Christoffel symbols. Explore the definition of derivatives in the curved space of the strained body. What is the physical interpretation of the Christoffel symbols?

The stress tensor

When a body is strained there are clearly internal forces which try to return it to its original shape: if the external forces causing the strain are removed, then the body returns to its original shape. These forces are short ranged (except when electric fields are produced by the strain).
Since the net force on every small volume vanishes, we can write this as an integral formula by introducing the force on a small volume element, **f**, through

$$0 = F_i = \int dV \; f_i = \oint \sigma_{ij} ds_i.$$

Here we used Stoke's theorem to convert the volume integral into an integral over the surface bounding that volume. In order to do this, we need the divergence formula $f_i = \partial_j \sigma_{ij}$. This new tensor of rank two is called the stress tensor.

Internal moments

The equilibrium of every element of the body also requires that the net moment at each point vanishes. In other words

$$
\begin{aligned}
0 = M_{ij} &= \int dV(x_i f_j - x_j f_i) = \int dV \left(x_i \frac{\partial \sigma_{jk}}{\partial x_k} - x_j \frac{\partial \sigma_{ik}}{\partial x_k} \right) \\
&= \int dV \frac{\partial}{\partial dx_k} (x_i \sigma_{jk} - x_j \sigma_{ik}) - \int dV \left(\frac{\partial x_i}{\partial x_k} \sigma_{jk} - \frac{\partial x_j}{\partial x_k} \sigma_{ik} \right) \\
&= \oint ds_k (x_i \sigma_{jk} - x_j \sigma_{ik}) - \int dV (\sigma_{ij} - \sigma_{ik})
\end{aligned}
$$

This is a surface integral if the tensor is symmetric.
More generally, the second integral vanishes if the difference is a divergence of a rank-3 tensor with suitable symmetries. However, since the only physical quantity that we construct from σ_{ij} is the net force, these symmetries can always be imposed to make the stress tensor symmetric.

Equilibrium of bodies

If external forces **F** act on every unit volume of a body, then the condition for the equilibrium of the body is

$$\frac{\partial \sigma_{ij}}{\partial x_j} = F_i.$$

The diagonal elements of σ_{ij} correspond to hydrostatic pressure. Assume that the force acting along the normal to a surface element ds_i is pds_i, then the stress tensor must be given by $\sigma_{ij} = -p\delta_{ij}$. If one works in a frame where there are off-diagonal elements of the stress tensor, then these correspond to shear forces acting along the surface of the body. In the most general case one has pressure as well shear acting on deformable bodies.

The energetics of a deformed body

If the displacements u_i in a deformed body change by an amount δu_i, then the work done by the internal forces is

$$\delta W = \int dV \delta u_i \partial_j \sigma_{ij} = \oint ds_j \delta u_i \sigma_{ij} - \int dV \sigma_{ij} \partial_j \delta u_i.$$

If there are no surface deformations, then

$$\delta W = -\frac{1}{2} \int dV \sigma_{ij} (\partial_j \delta u_i + \partial_i \delta u_j) = - \int dV \sigma_{ij} u_{ij}.$$

As a result, one may write the internal energy $dU = TdS + \sigma_{ij} du_{ij}$, the Helmholtz free energy, $dF = -SdT + \sigma_{ij} du_{ij}$, and the Gibbs free energy $dG = -SdT - u_{ij} d\sigma_{ij}$. As a result, in a state of thermodynamic equilibrium one may write

$$\sigma_{ij} = \left. \frac{\partial F}{\partial u_{ij}} \right|_T, \qquad \text{and} \qquad u_{ij} = \left. \frac{\partial G}{\partial \sigma_{ij}} \right|_T.$$

Thermodynamics of a deformed body

Thermodynamics is applicable if the stresses and strains are variables of state, *i.e.*, on reversing the strain the body returns to an earlier state. This regime of deformations is called elastic. If the strains are large, then the body may permanently deform. This regime of deformations is called plastic. In the elastic regime, $\sigma_{ij} = 0$ when $u_{ij} = 0$, so the expression for the free energy cannot contain a linear term. The quadratic term is expected to be a rotational scalar, so one may write

$$F = F_0 + \frac{\lambda}{2} u_{ii}^2 + \mu u_{ij}^2,$$

where the constants λ and μ are called Lamé coefficients.
The state of thermodynamic equilibrium is obtained by minimizing the free energy. When the strains vanish, this corresponds to $F = F_0$. For finite strains, the free energy must be higher. This implies that $\lambda > 0$ and $\mu > 0$ for thermodynamic stability.

Hooke's law

Alternatively, one may decompose the strain into a pure shear and a pure compression through the formula

$$u_{ij} = \left(u_{ij} - \frac{1}{3}\delta_{ij}u_{kk}\right) + \frac{1}{3}\delta_{ij}u_{kk}.$$

Then the free energy may be written as

$$F = F_0 + \frac{K}{2}u_{ii}^2 + \mu\left(u_{ij} - \frac{1}{3}\delta_{ij}u_{kk}\right)^2.$$

K is called the bulk modulus or the modulus of compressibility and μ is called the shear modulus.

We can now write $dF = [Ku_{kk}\delta_{ij} + 2\mu(u_{ij} - u_{kk}\delta_{ij}/3)]du_{ij}$. Using this we obtain Hooke's Law

$$\sigma_{ij} = Ku_{kk}\delta_{ij} + 2\mu\left(u_{ij} - \frac{1}{3}u_{kk}\delta_{ij}\right).$$

This is valid only within the quadratic approximation to F.

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A simple case

Solving a problem of elastic deformations means one should be able to write down all the components of the stress and strain tensors using information on external forces and the elastic moduli.

Hydrostatic equilibrium

Assume that a body is subjected to an uniform external pressure P, so that the tensor fields u_{ij} and σ_{ij} are constant through the body. Since the shear on the body vanishes, $u_{ij} = U\delta_{ij}$. The strain tensor is given by $\sigma_{ij} = 3KU\delta_{ij}$. The condition of equilibrium of forces is $\sigma_{ij} = -P\delta_{ij}$, since the force on every surface within the solid is given by the inward directed force equal to the pressure times the area of the surface. The solution of the problem is $U = -P/(3K)$. The strain is known once P and K are given.

A rod under tension (1)

Take a rod with cross sectional area A, and choose the z-axis to be aligned in the long direction. Assume that the rod is under constant tension due to applied outward forces PA on each end of the rod. Assume also that the stresses and strains are uniform through the body. Since there are no sideward forces, one has $\sigma_{ij} n_j = 0$ for every sideward normal. Also, equilibrium at the ends implies $\sigma_{zz} = P$. So the strain tensor has the form

$$\sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & P \end{pmatrix}.$$

From Hooke's law we find $u_{ii} = \sigma_{ii}/(3K)$. Substituting this back gives

$$\sigma_{ij} = \frac{1}{3}\sigma_{kk}\delta_{ij} + 2\mu\left(u_{ij} - \frac{1}{9K}\sigma_{kk}\delta_{ij}\right).$$

This allows us to solve for u_{ij}.

A rod under tension (2)

Solving for u_{ij} gives

$$u_{ij} = \left(\frac{\sigma_{kk}}{9K}\right)\delta_{ij} + \frac{1}{2\mu}\left(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}\right).$$

This formula is more general than the specific case we will apply it to. Sometimes this form is called Hooke's law.

In the case at hand we find

$$u_{zz} = \frac{P}{3}\left(\frac{1}{3K} + \frac{1}{\mu}\right), \quad u_{xx} = u_{yy} = \frac{P}{3}\left(\frac{1}{3K} - \frac{1}{2\mu}\right).$$

The Young's modulus is defined to be

$$Y = \frac{P}{u_{zz}}, \qquad \text{hence} \qquad Y = \frac{9K\mu}{3K+\mu}.$$

Poisson's ratio is defined as

$$\sigma = -\frac{u_{xx}}{u_{zz}} = \frac{1}{2}\,\frac{3K-2\mu}{3K+\mu}.$$

Some problems

Problem 66: Elevator cables

Take an elevator cable of transverse area A and a mass per unit length ρ. Assume that it is of length L, being hung from a support at the top and hanging vertically due to gravity. Neglect transverse forces on the cable. The vertical downward force at any point along the cable is due to the mass of all the material below it. At the point of suspension the upward force is just balanced by the mass of the full cable. Solve for the stresses and strains given that these vary along the length of the cable.

Problem 67: A space elevator

A suggestion by Arthur Clarke for a space elevator is essentially that one end of an elevator cable is in circular geostationary orbit and the other end reaches down to the surface of the earth. Compute the stresses and strains on this elevator cable.

Keywords and References

Keywords

strain tensor, principal axes, thin rods, plates, parallel transporters, Stoke's theorem, stress tensor, hydrostatic forces, shear forces, elastic deformation, plastic deformation, Lamé coefficients, thermodynamic stability, pure shear, pure compression, bulk modulus, modulus of compressibility, shear modulus, Hooke's Law, Young's modulus, Poisson's ratio, elevator cable, space elevator

References

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