Hamilton-Jacobi Method and perturbation theory

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Classical Mechanics 2012
October 29, 2012

A clever canonical transformation

The motion of a mechanical system with D degrees of freedom can be easily integrated if there are D conserved quantities. Examples are: one dimensional systems with time-independent Hamiltonians, the Kepler problem, motion of rigid bodies, *etc.*.
One can exhibit this through a canonical transformation to a new set of coordinates and momenta which are constant. If they are constant, then

$$\dot{Q}_i = \frac{\partial \mathcal{H}}{\partial P_i} = 0, \qquad \text{and} \qquad \dot{P}_i = -\frac{\partial \mathcal{H}}{\partial Q_i} = 0.$$

It is sufficient to choose the transformed Hamiltonian, $\mathcal{H}$, to be zero. This implies that

$$\mathcal{H} = H(q_i, p_i, t) + \frac{\partial S(q_i, P_i, t)}{\partial t} = 0,$$

where S is a generating function of the transformation.

The Hamilton-Jacobi equation

Using the fact that S generates p_i, this equation becomes

$$H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) + \frac{\partial S}{\partial t} = 0, \qquad \text{since} \qquad p_i = \frac{\partial S}{\partial q_i}.$$

This is the Hamilton-Jacobi equation.

This first order differential equation has $D+1$ variables, and therefore, can be integrated with $D+1$ constants of integration to give the solution $S(q_i, \alpha_i, t)$. Since S does not appear in the equations, but only its derivatives do, one of the constants of integration is an additive constant. This is immaterial to the dynamics, and hence can be dropped. Retaining D constants of integration, α_i, we can choose

$$P_i = \alpha_i, \quad Q_i = \frac{\partial S}{\partial \alpha_i} = \beta_i, \quad p_i = \frac{\partial S}{\partial q_i}.$$

The constants α_i and β_i are related to the initial conditions. This solves the dynamics of the system.

The generating function

Since

$$\frac{dS}{dt} = \frac{\partial S}{\partial q_i}\dot{q}_i + \frac{\partial S}{\partial t} = p_i\dot{q}_i - H = L.$$

In other words, the generating function is the action,

$$S = \int L dt.$$

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When H is independent of time, then the only time dependence in

$$\frac{\partial S}{\partial t} + H\left(q_i, \frac{\partial S}{\partial q_i}\right) = 0,$$

is in the time derivative. Hence the solution must be of the form $S = W - \alpha_1 t$. So the equation becomes

$$H\left(q_i, \frac{\partial W}{\partial q_i}\right) = \alpha_1.$$

The new and old Hamiltonians are the same, and α_1 is its value.

More on time-independent Hamiltonians

For a time-independent Hamiltonian, the Hamilton-Jacobi equation can be written in terms of W, which generates a quite different canonical transformation:

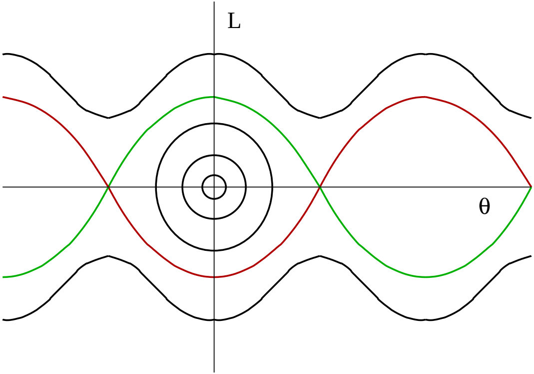
$$p_i = \frac{\partial W}{\partial q_i} \quad \text{and} \quad Q_i = \frac{\partial W}{\partial \alpha_i}, \quad \text{with} \quad H(p_i, q_i) = \alpha_1.$$

The canonical equations after transformation are

$$P_i = \alpha_i, \quad \dot{Q}_i = \delta_{i1}, \qquad Q_i = t\delta_{i1} + \beta_i.$$

One is at liberty to choose P_i to be some functions of α_i, which would also lead to conserved values of P_i. However, then the canonical equation would not lead to $\dot{Q}_i = 0$. Instead one would find the Q_i to be cyclic coordinates; these are called action-angle variables. Phase space trajectories would then lie on tori. Such systems are said to be integrable.

Tori in phase space



Two problems

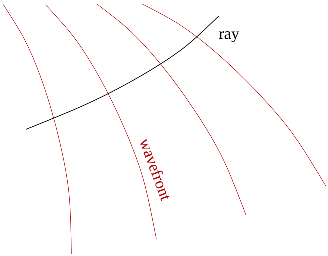
Problem 68: Small oscillations

Investigate the harmonic oscillator in D dimensions and show that it has D constants of motion. Apply the Hamilton-Jacobi equations to solve this problem and hence show that small oscillations of non-rigid systems is an integrable problem.

Problem 69: The Kepler problem

Solve the Kepler problem using the Hamilton Jacobi method. Is motion in a $1/r$ potential integrable in all dimensions of space? Investigate one, two, three and four dimensions and try to find the number of conserved quantities in each in order to answer this question.

Analogy with optics



For time-independent Hamiltonians, if $S = \sigma$, then $W = \sigma - Et$, *i.e.*, the change in surfaces of constant S lead to travelling wavefronts of constant W, with $dW = Edt$, since motion is periodic.

However, along the ray $dW = \nabla W ds$ so the phase velocity of the wave:

$$u = \frac{ds}{dt} = \frac{W}{\nabla W} = \frac{E}{\sqrt{2m(E-V)}} = \frac{E}{\sqrt{2mT}},$$

provided that the kinetic energy is quadratic in momenta.

Exact extension of the Hamilton-Jacobi method

Write the Hamiltonian of the problem at hand, H, as the sum of the Hamiltonian of an integrable system, H_0, and the remaining terms,

$$H(q_i, p_i, t) = H_0(q_i, p_i, t) + \Delta H(q_i, p_i, t).$$

All three Hamiltonians, H, H_0 and ΔH act on the same phase space. We know a canonical transformation $\{q_i, p_i\} \to \{\beta_i, \alpha_i\}$ generated by $S(q_i, \alpha_i, t)$ which solves the dynamics of H_0.
The transformation generated by S is canonical irrespective of the Hamiltonian, so the Hamiltonian of the perturbed problem is

$$K(\beta_i, \alpha_i, t) = H_0 + \frac{\partial S}{\partial t} + \Delta H = \Delta H(\beta_i, \alpha_i, t).$$

As a result, the equations of motion of the new coordinates and momenta are

$$\dot{\alpha}_i = -\frac{\partial \Delta H}{\partial \beta_i}, \qquad \text{and} \qquad \dot{\beta}_i = \frac{\partial \Delta H}{\partial \alpha_i}.$$

A simple example: first step

For the harmonic oscillator problem, use the free particle Hamiltonian as the integrable system, *i.e.*, $H_0 = p^2/2m$. So the Hamilton-Jacobi equation for this problem is

$$\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{\partial S}{\partial t} = 0,$$

which, we know, has the solution $S = \alpha x - \alpha^2 t/(2m)$. The transformed momentum is α. So the new coordinate is

$$\beta = \frac{\partial S}{\partial \alpha} = x - \frac{\alpha t}{m}.$$

As we already know, the new coordinates are precisely the initial conditions on p and x. We can exhibit this by inverting the equation for β to get

$$x = \beta + \frac{\alpha t}{m}.$$

A simple example: second step

The perturbation Hamiltonian is $V = m\omega^2 x^2/2$. In terms of the new coordinates, we have

$$\Delta H(\beta, \alpha, t) = \frac{m\omega^2}{2}\left(\beta + \frac{\alpha t}{m}\right)^2.$$

As a result, the equations of motion now become

$$\dot{\alpha} = -m\omega^2\left(\beta + \frac{\alpha t}{m}\right), \qquad \text{and} \qquad \dot{\beta} = \omega^2 t\left(\beta + \frac{\alpha t}{m}\right).$$

These equations are exact, and can be solved exactly. In order to do that note first that

$$\dot{\beta} + \frac{\dot{\alpha} t}{m} = 0.$$

As a result, one obtains

$$\ddot{\alpha} = -\omega^2 \alpha.$$

So the solution for α is exactly the simple harmonic motion that one expects. Prove that the solution for x is also harmonic.

Perturbation theory

It often turns out to be hard to solve the exact equations

$$\dot{\alpha}_i = -\frac{\partial \Delta H}{\partial \beta_i}, \qquad \text{and} \qquad \dot{\beta}_i = \frac{\partial \Delta H}{\partial \alpha_i}.$$

Instead we want to develop an approximation for the case when ΔH is small. This seems to be straightforward when we can write $\Delta H = \lambda h$, where $\lambda \ll 1$, then we can try the series expansions

$$\alpha_i = \sum_{j=0}^{\infty} \lambda^j \alpha_i^{(j)}, \qquad \text{and} \qquad \beta_i = \sum_{i=0}^{\infty} \lambda^j \beta_i^{(j)}.$$

The terms $\alpha_i^{(0)}$ and $\beta_i^{(0)}$ are solution of the dynamics of H_0, and the series is called a perturbation series. We need to insert the series expansion into the expression for ΔH and equate equal powers of λ in the canonical equations.

The same simple example by perturbation

Defining $\lambda = m\omega^2$, we get $\Delta H = \lambda(\beta + \alpha t/m)^2/2$. Now inserting the series expansions for β and α into this, we get

$$\Delta H = \frac{\lambda}{2}\left(\beta^{(0)} + \frac{\alpha^{(0)}t}{m}\right)^2 + \mathcal{O}\left(\lambda^2\right).$$

So the leading order perturbative equations are $\dot{\alpha}^{(0)} = \dot{\beta}^{(0)} = 0$, as expected. The next order equations are

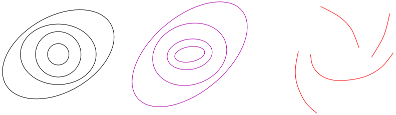
$$\dot{\alpha}^{(1)} = -\left(\beta^{(0)} + \frac{\alpha^{(0)}t}{m}\right), \qquad \text{and} \qquad \dot{\beta}^{(1)} = \frac{t}{m}\left(\beta^{(0)} + \frac{\alpha^{(0)}t}{m}\right).$$

Choosing, for simplicity, the initial condition $\beta^{(0)} = 0$, these integrate to the leading terms in the series expansions of circular functions, *i.e.*,

$$\alpha^{(1)} = -\frac{\alpha^{(0)}t^2}{2m}, \qquad \text{and} \qquad \beta^{(1)} = \frac{\alpha^{(0)}t^3}{3m^2}.$$

Secular terms

The example of the harmonic oscillator is technically simple, and we know from the exact solution that the problem is well behaved. However, the perturbative solution illustrates one of the main technical difficulties with perturbation theory. Note that both $\alpha^{(1)}$ and $\beta^{(1)}$ increase unboundedly with t. Terms in the perturbative solution which grow with t are called secular terms.



The appearance of secular terms put into question our initial assumption that $\alpha^{(0)}$ and $\beta^{(0)}$ provide good initial approximations to the solution. KAM theory, to be discussed later, tells us whether the series expansion in λ merely distorts the tori or destroys them.

Perturbation of orbits

One may be more interested in the perturbation of the elements of an orbit than in the orbit itself: for example, how the eccentricity of a Keplerian orbit changes due to a perturbation. The elements of the orbit are functions on phase space, $\zeta_k(\beta_i, \alpha_i)$. So their time dependence is given exactly by

$$\dot{\zeta}_k = [\zeta_k, \Delta H] = \nabla_\xi \zeta_k J \nabla_\xi \Delta H = \nabla_\xi \zeta_k J \nabla_\xi \zeta_l \frac{\partial \Delta H}{\partial \zeta_l} = [\zeta_k, \zeta_l] \frac{\partial \Delta H}{\partial \zeta_l}.$$

A perturbation theory can be set up, as before, by making series expansions in λ and matching terms in equal powers of λ.

Problem 70: Precession of perihelia

The perihelia of Keplerian orbits are given by the Runge-Lenz vector. For perturbations of Newton's law of gravity by a term in $1/r^n$ (fixed n) set up the equation for the motion of perihelia. Find the first order perturbative solution for $n = 2$, 3 and 4.

Variation of parameters

Consider a system in which some parameter, a, is slowly varying: such as the length of a pendulum. For constant a, we can find action-angle variables $\{\beta, \alpha\}$ such that the Hamiltonian is $H(\alpha, a)$. Assume that this is obtained by a canonical transformation $\mathcal{S}(q, \beta, a)$ instead of the usual $S(q, \alpha, a)$. When a varies, then the appropriate Hamiltonian is

$$K(\beta, \alpha, a) = H(\alpha, a) + \frac{\partial \mathcal{S}}{\partial t} = H(\alpha, a) + \dot{a}\frac{\partial \mathcal{S}}{\partial a}.$$

Clearly, then the variation of α is governed by

$$\dot{\alpha} = -\frac{\partial K}{\partial \beta} = -\dot{a}\frac{\partial}{\partial \beta}\,\frac{\partial \mathcal{S}}{\partial a}.$$

If a varies slowly, one can take it to be constant over the original period of motion, T, and average this over that interval to get

$$\overline{\dot{\alpha}} = -\frac{\dot{a}}{T}\int_0^T dt \frac{\partial}{\partial \beta}\,\frac{\partial \mathcal{S}}{\partial a} + \mathcal{O}(\dot{a}^2, \ddot{a}).$$

Adiabatic invariants

Since S is the action, it increases by α in time T. Because S and $\mathcal{S}$ are related by Legendre transforms, so does $\mathcal{S}$. Hence $\mathcal{S}$ and its derivatives can be expanded in a Fourier series over multiples of the frequency $2\pi/T$. They have no constant coefficient, and the integral vanishes. This is therefore an adiabatic invariant.

Problem 71: Plasma confinement

Assume that a charged particle moves in a constant and nearly uniform magnetic field, $\mathbf{H}(\mathbf{x})$. Also assume that it has initially a very small velocity in the direction of the field. Show that when it enters a region where the field strength is larger than it encountered initially, the particle can be reflected. Use this phenomenon to design a plasma bottle.

Matching of Tori

Assume that there is an invariant torus T in phase space which contains the motion of the system. If the angle variable is x, then in time t under the action of the integrable Hamiltonian, H_0, the system moves to a new point $P_0(x) = x + 2\pi t$. It is useful to work in an extended phase space labelled by $\{x, t\}$ and define $P_0(x, t) = \{x + 2\pi t, t\}$. Also assume that under the perturbed Hamiltonian the new point is $P_\lambda(x, t) = \{x + 2\pi t + \epsilon f(x, t), t\}$, where λ is small.

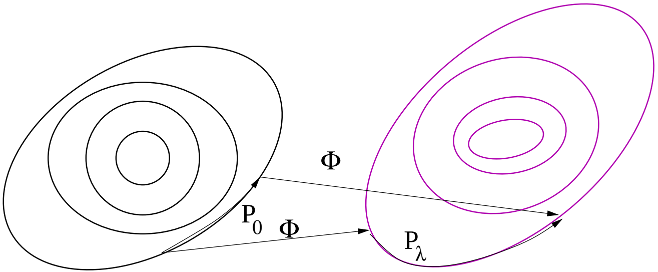
If we try to match the trajectories of the original and unperturbed system then we may have to change the coordinates as well as the time. This involves a function

$$\Phi_\lambda(x, t) = \{x + \lambda\xi(x, t), t + \lambda\tau(t)\}.$$

If such a mapping preserves the tori, then one must find that

$$P_\lambda(\Phi_\lambda(x, t)) = \Phi_\lambda(P_0(x, t)).$$

The commutative diagram



The diagram above illustrates the meaning of the desired equality

$$P_\lambda(\Phi_\lambda(x,t)) = \Phi_\lambda(P_0(x,t)).$$

Matching conditions

The left hand side of the desired equality is

$$P_\lambda(\Phi_\lambda(x,t)) = \{x+\lambda\xi(x,t)+2\pi t+2\pi\lambda\tau(t) \\ +\lambda f(x+\lambda\xi(x,t), t+\lambda\tau(t)), t+\lambda\tau(t)\}.$$

The right hand side gives

$$P_0(\Phi_\lambda(x,t)) = \{x+2\pi t+\lambda\xi(x+2\pi t, t), +\lambda f(x,t), t+\lambda\tau(t)\}.$$

Equating them reduces the desired equation into

$$\xi(x+2\pi t, t)-\xi(x,t) = 2\pi\tau(t)+f(x+\lambda\xi(t), t+\lambda\tau(t)).$$

This can be simplified by expanding each function in a series in λ and equating equal powers, as before. The result is

$$\xi_0(x+2\pi t, t)-\xi_0(x,t) = 2\pi\tau_0(t)+f_0(x,t).$$

Large denominators

This can be solved by Fourier transforming in the angle variable—

$$\xi_0(x,t) = \sum_k \xi_0(k,t)\mathrm{e}^{ikt}, \qquad \text{and} \qquad f_0(x,t) = \sum_k f_0(k,t)\mathrm{e}^{ikt},$$

if the perturbation has commensurate frequencies. Since $\tau_0(t)$ is independent of x, it is a term in $k=0$. The equation

$$\xi_0(x+2\pi t,t) - \xi_0(x,t) = 2\pi\tau_0(t) + f_0(x,t)$$

then gives the solution

$$\xi_0(k,t) = \frac{f_0(k,t) + 2\pi\tau_0(t)\delta_{k0}}{\exp(2\pi ikt) - 1}.$$

Clearly, when the perturbing function f_0 is small the solution is small except when the denominator is large. Large denominators occur when kt is integer. It seems possible that these can destroy invariant tori.

The Kolmogorov-Arnold-Moser Theorem

In general each invariant torus corresponds to motion with a certain frequency (and its harmonics). On the other hand, the perturbation Hamiltonian will, in general, have only a few fixed frequencies. The KAM theorem states that only those tori which correspond to motion at rational harmonics of the perturbation will be destroyed. The intuition gained in the above analysis is correct.

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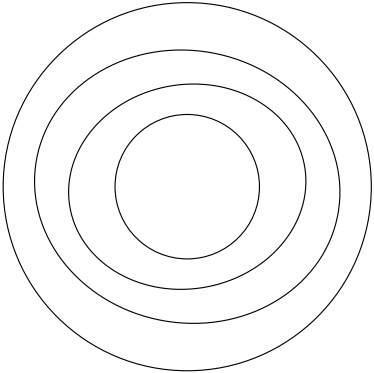
1. Unless the earth-moon-sun system is unstable, the length of the day and the year must be incommensurate. Similarly, the lengths of the lunar month and the year have to be incommensurate.
2. No moons of any planet will have rational periods with respect to the orbital period of the parent planet.
3. No asteroids will be observed with periods which are rational multiples of the orbital period of Jupiter.

Cantori



Some of the initial tori are distorted by perturbation, others are destroyed.

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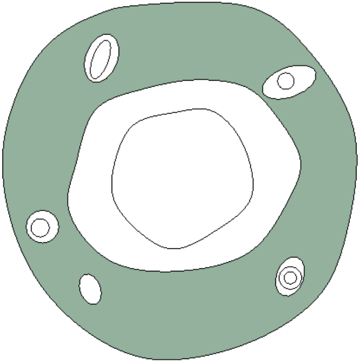

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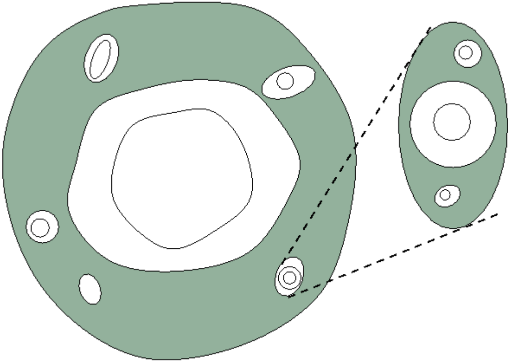
Cantori



Some of the initial tori are distorted by perturbation, others are destroyed. The width of the unstable region increases with λ. The structure in the unstable region is fractal, like a Cantor set, hence sometimes called cantori.

Keywords and References

Keywords

Hamilton-Jacobi equation, action, action-angle variables, tori, integrable systems perturbation series, secular terms, precession of perihelia, adiabatic invariant, plasma bottle, commensurate frequencies, large denominators, KAM theorem, Cantor set, cantori

References

Appropriate chapters of Goldstein and Landau