Generalized Coordinates, Lagrangians

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Generalized coordinates

Consider again the motion of a simple pendulum. Since it is one dimensional, use arc length as a coordinate. Since radius is fixed, use the angular displacement, θ, as a generalized coordinate. The equation of motion involves $\ddot{\theta}$, as it should, although the coordinate is dimensionless.

Problem 5: Simple pendulum

Choose θ as the generalized coordinate for a simple pendulum. What is an appropriate generalized momentum, so that its time derivative is equal to the force? What is the engineering dimension of the generalized momentum. Draw phase space trajectories for the pendulum: periodic motion corresponds to closed trajectories. What is the dimension of the area enclosed by such a trajectory? What is the physical interpretation of this area?

Many particles

Following the motion of N particles requires keeping track of N vectors, $\mathbf{x}_1$, $\mathbf{x}_2$, $\cdots$, $\mathbf{x}_N$. The configuration space has $3N$ dimensions; the phase space has $6N$ dimensions. We say that there are $3N$ degrees of freedom. Phase space volume has engineering dimension of $(\mathrm{energy} \times \mathrm{time})^{3N}$.

The equations of motion are

$$\dot{\mathbf{x}}_i = \frac{\mathbf{p}_i}{m_i}, \qquad \dot{\mathbf{p}}_i = \mathbf{f}_i(\{\mathbf{x}, \mathbf{p}\}).$$

If these are subject to some non-holonomic constraints, then there is no reduction in the number of degrees of freedom. If there are M scalar equations expressing holonomic constraints, then the number of degrees of freedom reduces to $D = 3N - M$. There is a consequent change in the dimension of phase space and the engineering dimension of phase space volume.

Generalized coordinates

If there are M constraints of the form $f_\alpha(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N) = 0$ with $1 \le \alpha \le M$, then all the coordinates of the N particles are given in terms of generalized coordinates q_i where $1 \le i \le D = 3N - M$. In other words, one has N vector-valued functions $\mathbf{x}_j = \mathbf{x}_j(q_1, q_2, \cdots q_D, t)$. If the generalized coordinates are to provide a complete description of the dynamics then knowledge of all the q_k should be equivalent to specifying all the $\mathbf{x}_j$. A counting of the number of scalar equations shows that this is possible. Clearly, the velocities are

$$\mathbf{v}_j = \dot{\mathbf{x}}_j = \sum_k \frac{\partial \mathbf{x}_j}{\partial q_k} \dot{q}_k + \frac{\partial \mathbf{x}_j}{\partial t}.$$

As a result one has the important identity

$$\frac{\partial \mathbf{v}_j}{\partial \dot{q}_k} = \frac{\partial \dot{\mathbf{x}}_j}{\partial \dot{q}_k} = \frac{\partial \mathbf{x}_j}{\partial q_k}.$$

Generalized forces

The equations of motion are equivalent to the principle that if one makes an instantaneous virtual displacement of a mechanical system, then the work done by the forces goes into a change of the total kinetic energy. In other words

$$\sum_j \delta \mathbf{x}_j \cdot \left(\mathbf{F}_j - \dot{\mathbf{p}}_j\right) = 0.$$

Now one can use the generalized coordinates to rewrite the work done by the forces

$$\delta W = \sum_j \delta \mathbf{x}_j \cdot \mathbf{F}_j = \sum_{jk} \mathbf{F}_j \cdot \frac{\partial \mathbf{x}_j}{\partial q_k} \delta q_k = \sum_k Q_k \delta q_k,$$

where one has defined the generalized forces

$$Q_k = \sum_j \mathbf{F}_j \cdot \frac{\partial \mathbf{x}_j}{\partial q_k}.$$

The change in kinetic energy

One can write

$$\begin{aligned}
\sum_j m_j \ddot{\mathbf{x}}_j \cdot \delta \mathbf{x}_j &= \sum_k \delta q_k \left[\sum_j m_j \boxed{\dot{\mathbf{v}}_j \frac{\partial \mathbf{x}_j}{\partial q_k}} \right] \\
&= \sum_k \delta q_k \left[\sum_j m_j \left\{ \boxed{\frac{d}{dt}\left(\mathbf{v}_j \frac{\partial \mathbf{x}_j}{\partial q_k} \right) - \mathbf{v}_j \frac{\partial \mathbf{v}_j}{\partial q_k}} \right\} \right] \\
&= \sum_k \delta q_k \left[\sum_j m_j \left\{ \frac{d}{dt}\left(\mathbf{v}_j \frac{\partial \mathbf{v}_j}{\partial \dot{q}_k} \right) - \mathbf{v}_j \frac{\partial \mathbf{v}_j}{\partial q_k} \right\} \right] \\
&= \sum_k \delta q_k \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} \right]
\end{aligned}$$

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&= \sum_k \delta q_k \left[\sum_j m_j \left\{ \frac{d}{dt} \left(\mathbf{v}_j \frac{\partial \mathbf{v}_j}{\partial \dot{q}_k} \right) - \mathbf{v}_j \frac{\partial \mathbf{v}_j}{\partial q_k} \right\} \right] \\
&= \sum_k \delta q_k \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} \right]
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&= \sum_k \delta q_k \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} \right]
\end{aligned}$$

Equation of motion

Since the virtual displacements of the generalized coordinates are all independent, one can set each coefficient independently to zero. Then we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k.$$

If the particles move in a field of conservative forces then

$$Q_k = \frac{\partial V}{\partial q_k} \qquad \text{and} \qquad \frac{\partial V}{\partial \dot{q}_k} = 0.$$

Then the equations of motion can be written in terms of the Lagrangian function $L = T - V$,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0.$$

The Euler-Lagrange equations of motion

The generalized momenta are defined as

$$p_k = \frac{\partial L}{\partial \dot{q}_k}.$$

One recovers the usual definition for systems where the velocities appear only in the kinetic part of the energy. Similarly, if one considers the kinds of systems where the coordinates only appear in the potential, then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} \qquad \text{implies} \qquad \dot{p}_k = Q_k.$$

The Euler-Lagrange equations reduce to the usual form of Newton's equations of motion in these cases. Interesting generalizations arise in other cases.

Particle in an electromagnetic field

The Lorentz force on a particle in an electromagnetic field is

$$\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right) = q\left(-\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \frac{1}{c}\mathbf{v}\times\nabla\times\mathbf{A}\right),$$

where $q =$ charge, $c =$ speed of light, $\mathbf{v}$ the velocity, $\mathbf{E}$ and $\mathbf{B}$ the electric and magnetic fields, and ϕ and $\mathbf{A}$ the scalar and vector potentials.

The Lagrangian formalism continues to be useful if one can write down a velocity dependent potential $V(q,\dot{q})$ such that

$$Q = \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}}\right) - \frac{\partial V}{\partial q}.$$

Now using the identity

$$\mathbf{v}\times\nabla\times\mathbf{A} = \nabla(\mathbf{v}\cdot\mathbf{A}) + \frac{\partial \mathbf{A}}{\partial t} - \frac{d\mathbf{A}}{dt},$$

one finds that $V = q(\phi - \mathbf{v}\cdot\mathbf{A}/c)$ gives the Lorentz force.

Dissipation

The problem of dissipative forces lies a little away from the developments made till now. However, models of frictional forces show that they are proportional to the velocity. Hence, for the dissipative forces on a body one may write the relation

$$Q = -\frac{\partial \mathcal{F}}{\partial \dot{q}}.$$

This introduces the Rayleigh term, $\mathcal{F}(\dot{q})$, which is usually chosen to be quadratic in $\dot{q}$. The equations of motion are then written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial \mathcal{F}}{\partial \dot{q}} = 0.$$

In order to describe the motion of a body in a dissipative environment both the Lagrangian L and the Rayleigh term $\mathcal{F}$ need to be specified.

Keywords and References

Keywords

engineering dimensions, conservative forces, configuration space, degrees of freedom, virtual displacement, generalized coordinates, generalized forces, generalized momenta, Lagrangian function, Lorentz force, Rayleigh term

References

Goldstein, chapter 1.