# Canonical transformations II

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# Symplectic invariants

In terms of the $2D$-dimensional vector in phase space, $\mathbf{x} = (q_1, q_2, \cdots, q_D, p_1, p_2, \cdots, p_D)^T$, the Poisson bracket is

$$[f, g]_x = \nabla f \cdot J \cdot \nabla g.$$

Clearly, under a canonical transformation, $\xi(\mathbf{x})$, with Jacobian $M$, one has

$$[f, g]_\xi = \nabla f \cdot M^T J M \cdot \nabla g = \nabla f \cdot J \cdot \nabla g = [f, g]_x.$$

Therefore, the Poisson bracket is a symplectic invariant.
The phase space volume element transforms through the determinant of the Jacobian,

$$d^{2D}\xi_k = |M| d^{2D}\mathbf{x}_k.$$

But since $|M| = |M^T|$, the invariance relation $MJM^T = J$ implies that $|M|^2 = 1$. As a result the phase space volume is also a symplectic invariant.

# Infinitesimal Canonical Transformations

Earlier we wrote an ICT in terms of a generating function $\mathcal{G}(\mathbf{x})$ in the form $\xi = \mathbf{x} + \epsilon J \nabla \mathcal{G}(\mathbf{x})$. Now we note that for any phase space function $u(\mathbf{x})$,

$$[\mathbf{x}, u] = \nabla \mathbf{x} \cdot J \cdot \nabla u = J \nabla u, \quad \text{since} \quad \nabla \mathbf{x} = I.$$

So one can write the ICT as $\xi = \mathbf{x} + \epsilon[\mathbf{x}, \mathcal{G}]$.

## Time evolution is a canonical transformation

Since Hamilton's equations can be written as $\dot{\mathbf{x}} = [\mathbf{x}, H]$, one can write an infinitesimal time evolution in the form

$$\mathbf{x}(t + dt) = \mathbf{x}(t) + dt[\mathbf{x}, H].$$

This is in the form of ICT with generating function $H(\mathbf{x})$.

From this it follows that phase space volume is conserved under time evolution. This is called Liouville's theorem.

# Transformations of phase space functions

If $u(\mathbf{x})$ is a phase space function, then its change under an ICT generated by $\mathcal{G}$ is clearly given by

$$\delta u = \nabla u \cdot \delta \mathbf{x} = \epsilon \nabla u \cdot J \cdot \nabla \mathcal{G} = \epsilon [u, \mathcal{G}].$$

## Momenta generate translations

Select $\mathcal{G}(\mathbf{x}) = p_i = x_{D+i}$. Then clearly $\delta q_j = \epsilon \delta_{ij}$ and $\delta p_j = 0$.

## Angular momenta generate rotations

Select $\mathcal{G}(\mathbf{x}) = J_i = \epsilon_{imn} q_m p_n$. Then $\delta q_j = -\epsilon \epsilon_{ijk} q_k$ and $\delta p_j = -\epsilon \epsilon_{ijk} p_k$. Angular momenta are rotation generators.

Factoid: Define a product of two phase space functions $u$ and $v$ through its Poisson bracket, *i.e.*, $u \otimes v = [u, v]$. This product is non-associative since $[u[v, w]] \neq [[u, v], w]$! The Jacobi identity is a particular replacement of the law of associativity.

# Finite Canonical Transformations

A Taylor expansion gives

$$\begin{aligned} \mathbf{x}(t+\delta t) &= \mathbf{x}(t) + \Delta t \dot{\mathbf{x}} + \frac{(\Delta t)^2}{2!} \ddot{\mathbf{x}} + \cdots \\ &= \mathbf{x}(t) + \Delta t [\mathbf{x}(t), H] + \frac{(\Delta t)^2}{2!} [[\mathbf{x}(t), H], H] + \cdots \\ &= \mathbf{x}(t) \mathrm{e}^{\Delta t H}. \end{aligned}$$

where the exponential stands for the series expansion shown in the previous line. Similarly, the other generating functions can also be written formally as exponentials with the same meaning

$$T(\mathbf{y}) = \exp(\mathbf{y} \cdot \mathbf{p}), \qquad R(\hat{\mathbf{n}}, \psi) = \exp(\psi \mathbf{J} \cdot \hat{\mathbf{n}}),$$

where $T(\mathbf{y})$ is the operator which generates translations in space by an amount $\mathbf{y}$ and $R(\hat{\mathbf{n}}, \psi)$ generates rotations around the axis $\hat{\mathbf{n}}$ by the amount $\psi$.

# Keywords and References

## Keywords

Poisson bracket, symplectic invariant, phase space volume, generating function, time evolution, Liouville's theorem, translation generators, rotation generators, non-associative algebra, Jacobi identity

## References

Landau, Sections 3,8,9,13