Scale Anomalies and Dimensional Analysis

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Ultraviolet divergences in QFT

Typical loop diagrams give rise to integrals of the form

$$I_n^m = \int \frac{d^4k}{(2\pi)^4} \frac{k^{2m}}{(k^2+\ell^2)^n}$$

where k is the loop momentum and ℓ may be some function of the other momenta and the masses. When $2m+4 \geq 2n$, then the integral diverges.

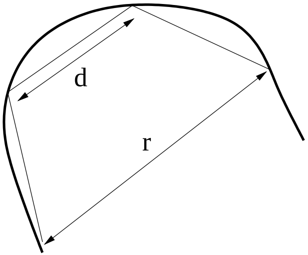
This can be regularized by putting an UV cutoff, Λ.

$$I_n^m = \frac{\Omega_4}{(2\pi)^4} \int_0^\Lambda \frac{k^{2m+3}dk}{(k^2+\ell^2)^n} = \frac{\Omega_4}{(2\pi)^4} \ell^{2(m-n)+4} F\left(\frac{\Lambda}{\ell}\right),$$

where Ω_4 is the result of doing the angular integration. The cutoff makes this a completely regular integral. As a result, the last part of the answer can be obtained entirely by dimensional analysis.

What can we say about the limit $\Lambda \to \infty$?

Perimeters of polygons



The arc length of a curve, L, whose ends are a distance r apart, can be approximated by a segment of a polygon with sides of length, d. A dimensional argument tells us that

$$L(r) = r \lim_{d \to 0} \Phi\left(\frac{r}{d}\right),$$

since one gets successively better approximations to the curve by decreasing the size of the edge of the approximating polygon.

Scale invariance and breaking

Nice curves are scale invariant. By this we mean that $\Phi(r/d \to \infty)$ has a finite limit. For example, for a semicircle, $\Phi(\infty) = \pi/2$. Scale invariance holds for a very small set of curves.

For a larger set of curves a scaling hypothesis may hold: $\Phi(\Pi) = \Pi^\alpha$, where α is a positive number. Then $L(r) \propto r^{1+\alpha}$. Such curves are called fractals; $D = 1 + \alpha$ is called a fractal dimension. In analogy with QFT, α can be called an anomalous dimension.

Broken scale invariance

For fractals the unit of the measuring scale, d, leaves a trace. Invariance with respect to d is broken, and this quantity does not disappear from the formula. This is the origin of the anomalous dimension.

A fractal

Fractals model quantum field theories!

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Dimensional analysis and homogenous functions

Dimensional analysis involves homogenous functions. In a physical problem list all the n variables of interest, choose the basis set of m of them, U_1, U_2, $\cdots$, U_m, and write the others as A_1, A_2, $\cdots$, A_k, where $k+m=n$. Then a physical relation will be

$$A_1 = F(U_1, U_2, \cdots, U_m, A_2, A_3, \cdots, A_k).$$

One can construct k dimensionless quantities, $\Pi_i = A_i / \prod U_j^{a_{ij}}$. In terms of these one can write the same relation as

$$A_1 = U_1^{a_{11}} U_2^{a_{12}} \cdots U_m^{a_{1m}}\; f\left(\frac{A_2}{U_1^{a_{21}} U_2^{a_{22}} \cdots U_m^{a_{2m}}}, \cdots \frac{A_k}{U_1^{a_{k1}} U_2^{a_{k2}} \cdots U_m^{a_{km}}}\right),$$

A more compact notation is

$$\Pi_1 = f(\Pi_2, \Pi_3, \cdots, \Pi_k).$$

Scaling symmetry and scale anomaly

The function f is said to possess a scaling symmetry if it has a finite and non-zero limit as any one of these Π_i go to zero or infinity. Then, in that limit, the functional dependence can be dropped to give

$$\Pi_1 = f(\Pi_2, \Pi_3, \cdots, \Pi_{k-1}) \qquad \Pi_k \to \infty.$$

Since the labelling of the variables is immaterial, in the case when $k - \ell$ of the dependences can be dropped we write

$$\Pi_1 = f(\Pi_2, \Pi_3, \cdots, \Pi_\ell) \qquad \Pi_i \to \infty \quad \forall i > \ell.$$

Clearly this is a very special class of functions. We saw a wider class which has the property of having a scale anomaly

$$\Pi_1 = \left[\prod_{i>\ell} \Pi_i^{\alpha_{1i}}\right] g\left(\frac{\Pi_2}{\prod_{i>\ell} \Pi_i^{\alpha_{2i}}}, \cdots, \frac{\Pi_\ell}{\prod_{i>\ell} \Pi_i^{\alpha_{\ell i}}}\right).$$

The exponents α_{ji} are called anomalous dimensions.

Scale transformations

The function under study

$$A_1 = F(U_1, U_2, \cdots, U_m, A_2, A_3, \cdots, A_k),$$

has a necessary invariance under scale transformations of units

$$U_i = \lambda_i U_i, \text{ for } 1 \le i \le m, \qquad A_j = \left[\prod_{i=1}^{m} \lambda_i^{a_{ji}}\right] A_j.$$

These transformations form a group because

1. all non-negative values of λ_i are allowed (closure),
2. $\lambda_i = 1$ is an allowed transformation (existence of unity),
3. for any λ_i the inverse scaling $1/\lambda_i$ is allowed (unique inverse),
4. and a sequence of scalings can be composed in any way (associativity).

The Π_i are invariants of this group. Dimensional analysis expresses the covariance of physical equations under this group.

Scale anomaly and the renormalization group

Scale anomalies also involve homogeneous functions and can be written in terms of renormalized variables, $\Pi_j^* = \Pi_j / \prod_{i>\ell} \Pi_i^{\alpha_{ji}}$. This means that there is an anomalous scale invariance of the form

$$\Pi_i = \xi_i \Pi_i \quad \text{for} \quad \ell+1 \le i \le k$$
$$\Pi_j = \left[\prod_{i>\ell} \xi_i^{\alpha_{ji}}\right] \Pi_i \quad \text{for} \quad 1 \le j \le \ell.$$

Since this is supposed to hold only for very large or very small values of Π_i (with $i > \ell$), the range of ξ_i is restricted to be not very different from 1. The proof that this leads to incomplete similarity follows the same lines as dimensional analysis.

These transformations satisfy the existence of unity, unique inverse and associativity. However, the restriction that ξ_i be not very large or small tells us that the property of closure is not satisfied. So the renormalization group is, almost but not quite, a group.

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Power counting of scattering amplitudes

Scattering processes may be classified by the number of external states they contain. If there are F fermions and B bosons in a process then the amplitude for the process contains F Dirac fields and B boson fields. As a result, the dimension of the process is $3F/2 + B$.

Compton scattering has 1 electron and 1 photon in both the initial and final states, so $F = 2$ and $B = 2$, and the amplitude has dimension 5. Bhabha scattering has $F = 4$ and $B = 0$, so the amplitude has dimension 6.

Problem 3.1

In jet production, the process $gg \to gg$ has dimension 4, the process $\overline{q}q \to \overline{q}q$ has dimension 6. The cross sections must be the same dimension, but the 2-body phase spaces also have the same dimensions. What is the resolution?

The physical origin of anomalous scaling

If we write the amplitude for a process as $\Gamma(\alpha, m_e, Q, \Lambda)$, where α is the fine structure constant, m_e the electron's mass, Q the typical momentum scale in the process, and Λ is the UV cutoff, then dimensional analysis tells us that

$$\Gamma(\alpha, m_e, Q, \Lambda) = Q^{3F/2+B} F\left(\alpha, \frac{m_e}{\Lambda}, \frac{Q}{\Lambda}\right).$$

Quantum field theories do not exhibit scaling in the limit $Q/\Lambda \to 0$.

Probe an electron with photons of momentum Q; what happens when Q changes? In an interacting field theory "elementary" particles are not point-like. With changing Q one sees more or less structure: virtual photons, virtual e^+e^- pairs. So, the essential physics behind anomalous scaling is the uncertainty principle.

Stückelberg-Petermann-Gell-Mann-Low-Callan-Symanzik

Quantum field theories show anomalous scaling. In terms of our dimensional analysis, we can define renormalized quantities,

$$\alpha^* = \alpha(Q/\Lambda)^\beta \quad \text{and} \quad m_e^* = m_e(Q/\Lambda)^\gamma,$$

in terms of which one can write

$$\Gamma(\alpha, m_e, Q, \Lambda) = Q^{3F/2+B+\eta} G\left(\alpha^*, \frac{m_e^*}{\Lambda}\right).$$

We have the renormalization group equations

$$\frac{d\alpha^*}{d\log Q} = \beta\alpha^*, \qquad \text{and} \qquad \frac{dm_e^*}{d\log Q} = \gamma m_e^*.$$

Which parameters of the theory do β and γ depend on? Since they are independent of Q and Λ, they cannot depend on m_e, by a dimensional argument. Therefore, β and γ can depend only on α. One can compute them from the usual perturbation series.

Modified power counting

Anomalous dimensions change the power counting encountered in tree-level effective theories. In perturbative expansions the anomalous dimensions are small, since they start at order g. As a result, the nature of relevant or irrelevant operators is not changed. However, marginal operators are usually changed to either irrelevant or relevant. For example, the anomalous dimension of the scalar field in a $\lambda\phi^4$ theory is increased. As a result, the interaction term becomes irrelevant.

Non-trivial IR fixed points, $g^*{\equiv}0$, can change the counting drastically. We believe that QCD has such a fixed point where the effective theory no longer involves quarks and gluons but contains hadrons. The pion field, which is nominally a scalar field made of a quark and an anti-quark does not have dimension close to 3.

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Keywords and References

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Regularization; scale invariance; scaling hypothesis; fractals; anomalous dimension; fractal dimension; scale anomaly; renormalized variables; renormalization group; homogenous functions; Compton scattering; Bhabha scattering; renormalization group equations.

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