Dimensional analysis

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Outline

Copyright statement

Keywords and References

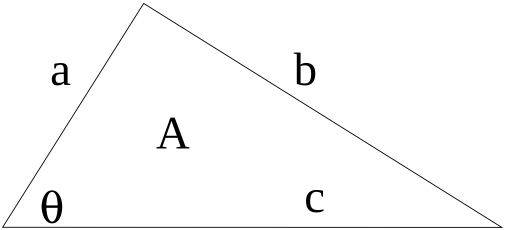
Keywords

Dimensional analysis, similarity, scaling, Pythagoras' theorem, radius of curvature, choice of units, dimensional analysis, vector spaces, homogeneous functions, slenderness ratio, fracture, buckling, elasticity, diastolic pressure, cardiac output, giraffe, Brachiosaurus.

Books

Scaling, by G. I. Barenblatt, Ch 2.
Anatomy and Physiology, by Tortora and Gradowski, Ch 20.

The area of a right triangle

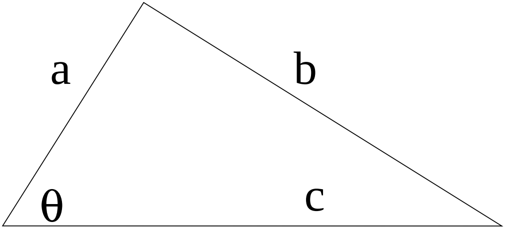


The area of a right angled triangle must depend on one of the angles and one of the sides.

$$A = \mathcal{A}(c, \theta).$$

Here $\mathcal{A}$ is an unknown function.

The dimensional argument

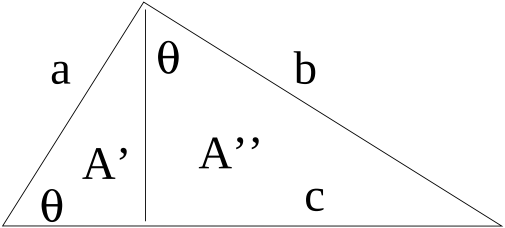


A has dimensions L^2, c has dimensions L and θ has dimensions L^0. So we must have

$$A = c^2 \mathbf{A}(\theta).$$

Since the area is non vanishing for any non-zero value of θ, the function $\mathbf{A}(\theta)$ is non-vanishing.

Similarity



If we cut up the triangle into two similar triangles, then

$$A' = a^2 \mathbf{A}(\theta), \quad \text{and} \quad A'' = b^2 \mathbf{A}(\theta).$$

Since $A = A' + A''$, we must have

$$c^2 \mathbf{A}(\theta) = a^2 \mathbf{A}(\theta) + b^2 \mathbf{A}(\theta).$$

Since $\mathbf{A}(\theta)$ is non-vanishing, we have $c^2 = a^2 + b^2$.

Questions

1. Why does the argument fail for $\theta = 0$? Does the theorem of Pythagoras fail for $\theta = 0$?
2. Solve the problem by computing the function $\mathbf{A}(\theta)$.
3. If the triangle were drawn on a surface with radius of curvature R, how would the argument change? What would replace the formula of Pythagoras?
4. The dimensional argument is true of all planar curves. Can the similarity argument be used for other shapes?

A principle of relativity

Relativity

An equation of physics cannot depend on the units in which the quantity is measured. If it does, then the relationship it describes is accidental.

$$\mathbf{F} = m\mathbf{a}, \quad E = mc^2, \quad \frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2},$$

remain true whether distances are measured in hands or light years, times in ghati or seconds, *etc.*.

The numerical value of m in the first equation depends on the choice of units. If we change units so that the numerical value is changed by a factor of λ, then the right hand side must also change by λ. As a result, the unit of force must also be revised to take this into account. Proceeding so with length and time gives us the dimensions of force.

Dimensionless variables

The dimensions of a physical quantity X will be denoted as

$$[X] = M^a L^b T^c.$$

The principle of relativity then tells us that all terms in an equation must have the same dimensions.

Faced with a new problem we can begin to analyze it *without a theory* by just listing all the possible relevant variables, X_α, in the problem and combining them into a number of dimensionless constants, Π_i. Then any physical relation between these are necessarily of the form $f(\Pi_1, \Pi_2, \cdots) = \text{constant}$.

Example: For a simple pendulum, the important quantities are possibly its length, ℓ, the acceleration due to gravity, g, the mass of the bob, m, and its period, t. Now $[\ell] = L$, $[g] = LT^{-2}$, $[m] = M$, $[t] = T$. The only dimensionless variable one can construct from these is $\Pi = t^2 g/\ell$. Since Π is constant, $t \propto \sqrt{\ell/g}$.

Questions

1. Is it necessary to always consider all three of L, T and M? Are there domains of problems where we could deal with a smaller number of dimensions?
2. Is it sufficient to consider only dimensions of L, T and M? Could we possibly require more?
3. We know that for a pendulum $t \propto \sqrt{\ell/g}$ only at small amplitudes. If we add the angular amplitude, Θ, what becomes of the dimensional argument? If we take into account the damping due to frictional losses, can we still perform a dimensional analysis? What is the result?
4. Your Body Mass Index (BMI) is defined as your mass in Kg divided by the square of your height in m. Normal people are supposed to have BMI between 18.5 and 25; there could be increased risk of lifestyle related diseases outside this range. What physics could lie behind such a measure?

A vector space

Dimensional analysis as manipulations in a vector space

Use the notation U_1, U_2, $\cdots$, U_m for the basic dimensional quantities, if there are m dimensions in a problem. Then the dimension of any other quantity A can be written as an ordered m-tuplet of numbers a_1, a_2 *etc.*,

$$[A] = U_1^{a_1} U_2^{a_2} \cdots U_m^{a_m} \equiv (a_1, a_2, \cdots a_m)$$

The addition of two vectors is the operation $[AB] = [A][B]$. Multiplication by a scalar λ is the operation $[A^\lambda] = [A]^\lambda$.

Dimensional analysis is the problem of finding sets of vectors which sum to zero! Any set of linearly independent vectors in the space of dimensional quantities can be used for this.

Example: Instead of setting dimensions by M, L and T, one may set them by angular momentum, J, force, F, and energy, E.

Homogeneous functions

In a physical problem list all the n variables of interest, choose the basis set of m of them, U_1, U_2, $\cdots$, U_m, and write the others as A_1, A_2, $\cdots$, A_k, where $k + m = n$. Then a physical relation will be

$$A_1 = F(U_1, U_2, \cdots, U_m, A_2, A_3, \cdots, A_k).$$

One can construct k dimensionless quantities, $\Pi_i = A_i / \prod U_j^{a_{ij}}$. In terms of these one can write the same relation as

$$\Pi_1 = f(U_1, U_2, \cdots, U_m, \Pi_2, \Pi_3, \cdots, \Pi_k).$$

Now by a change in the units, one has $U_j \to \lambda_j U_j$, but the Π_i do not change. So the function f cannot depend on the U_j, and one must have

$$\Pi_1 = f(\Pi_2, \Pi_3, \cdots, \Pi_k).$$

This is the content of dimensional analysis. It expresses the fact that the measuring scale does not matter.

Physical quantities

Pillars which have to take a certain load have an optimum cross sectional area, A. Too thin, and the structure can fall. Too thick, and the cost goes up.

Important physical quantities to consider are the mass load the pillar has to support, M, the height of the pillar, h, and the density of the material of the pillar, ρ. We seek a relationship of the form

$$A = f(M, h, \rho).$$

The dimensions of various quantities are

$$[M] = M, \quad [h] = L, \quad [A] = L^2, \quad [\rho] = ML^{-3}.$$

There are four quantities with only two dimensions, so one should be able to write two dimensionless quantities. We can build these as

$$\Pi_1 = \frac{A}{h^2} \qquad \text{and} \qquad \Pi_2 = \frac{\rho h^3}{M}$$

The shape of pillars

The quantity $1/\sqrt{\Pi_1}$ is essentially what engineers call the slenderness ratio, and denote as λ or SR. Since there are only two quantities, one should be able to write $\Pi_1 = F(\Pi_2)$, *i.e.*,

$$A = h^2 F\left(\frac{\rho h^3}{M}\right).$$

With increasing M, if h is kept fixed, then Π_2 decreases. Clearly then Π_1 must increase, since A must increase to support the load. This solution would lead to short and fat pillars.
One could try a solution in which Π_1 is fixed as M increases, then this implies that Π_2 is also fixed, since ρh^3 is then proportional to the mass of the pillar, m. This provides an easy rule of the thumb for a builder: to double the load, build twice as high, also doubling the radius of the pillar. From eye level the pillars would still look more massive as the building grew taller.

The failure of pillars

The previous theory of the shape of pillars can be applied when the pillar is stable. Pillars can fail in various ways. Short and squat pillars mainly fail by cracking. Slender pillars buckle before they crack. The use of slender pillars is historically recent: from around the 18th century. Euler's theory of buckling dates from around that time.

Discovery of Elasticity

The failure of pillars can depend on the shape, Π_1 and the stress applied to it. The stress is defined to be the loading force per unit area, $\sigma = F/A$. Clearly $[\sigma] = ML^{-1}T^{-2}$. We have not encountered anything in the theory developed till now which can be used to construct a dimensionless quantity, Π_3, from σ. Therefore, there must be a property of materials which has dimensions of σ. Our previous theory of pillars must be applicable for fixed Π_3.

Strength of materials

Material	Short column	Long column
Structural Steel	< 40	> 150
Aluminium Alloy 6061	< 9.5	> 66
Aluminium Alloy 2014	< 12	> 55
Wood	< 11	> 18–30
Stone	< 18	

Table: The slenderness ratio of a material determines the failure mode. The data is taken from [1] except for stone, which is from [2]. Short columns fail by fracture, long columns by buckling. In the intermediate range, the notion of an inelastic limit is important.

1) http://www.efunda.com/formulae/solid_mechanics/columns/intro.cfm

2) http://www.rwgrayprojects.com/synergetics/s06/p4000.html

Questions

1. Can the scaling function $F(\Pi_2)$ be determined using only experiments with paper?
2. Can this theory be applied to the legs of animals? Can you say something about the shapes of the legs of horses versus hippopotami? Can you say something about bipedal, quadrupedal and millipedal animals?
3. Can the theory of the shape of pillars be applied to the shape of cables suspended from a point? Can you apply it equally to ropes or to the kinds of steel cables which hold elevators? Can you apply it to a space elevator?
4. What else can the theory of pillars be applied to?
5. Does the same theory hold on the moon? Does one have to incorporate the acceleration due to gravity, g, into the theory of stable pillars?

High neck, stout heart

The heart must pump blood to the head against gravity. Some quantities which may be important for the problem are the the height of its head above the heart, h, the volume of blood pumped per unit beat (stroke volume), V_s, the density of blood, ρ, and the acceleration due to gravity, g, as well as a measure of the animal's blood pressure, P. Then

$$[h] = L, \quad [\rho] = ML^{-3}, \quad [g] = LT^{-2},$$
$$[P] = ML^{-1}T^{-2}, \quad [V_s] = L^3.$$

Taking as control parameters, h, ρ and g, we have

$$\Pi_1 = \frac{P}{\rho g h}, \qquad \text{and} \qquad \Pi_2 = V_s h^3.$$

The medical unit of pressure is mm of Hg $= 10^{-3} g\rho_0$ m (ρ_0 is the density of Hg), so in these units, $10^3\Pi_1 = p\rho_0/(\rho h)$, if the h is in meters, and p is the pressure in medical units.

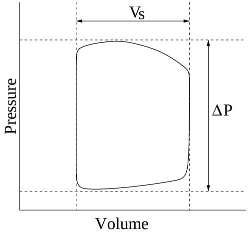
The heart as a pump

If Π_2 is held constant between two species, then we must have the scaling law—

$$\frac{p_1}{p_2} = \frac{h_1}{h_2},$$

if the density of blood is the same for two species. If the subscript 1 refers to giraffes and 2 to humans, then $h_2 = 0.3$ m, $h_1 = 2$ m, so a giraffe's blood pressure is about 7 times that of a human.

But which blood pressure is this? We need a simple model of the heart. It is a pump, whose PV characteristic is shown in the figure. The difference in the systolic and diastolic pressures is the p in the previous analysis. Π_1 is the fraction of the area of the dashed rectangle inside the closed curve.



The model heart and new physics

If we set $\Pi_1 = 1$, then we have a model relating animal hearts and heights:

$$\Delta P = 10^3 \left(\frac{\rho}{\rho_0}\right) \frac{h}{m}.$$

Since blood is only mildly thicker than water, we can take $\rho_0/\rho = 13.6$. Since the distance between a human head and heart is 0.3 m, we find that human $\Delta P = 22$ mm of Hg. For a normal heart, the diastolic pressure is 80 mm and the systolic is 120 mm, so we get only about half the pressure.

We could improve the model by measuring Π_1 more accurately, and putting in an improved value for ρ. However, this makes only about 30% change to the result. The remaining about 10 mm of pressure corresponds to the work that the heart has to do in pushing blood against viscous losses. The physics of viscosity is a fairly new ingredient in medicine.

The blood pressure of a giraffe

Our scaling law, neglecting viscosity, now predicts for a 3 m tall giraffe $\Delta P = 150$ mm of Hg. If its diastolic pressure were the same as a human's then the systolic pressure predicted would be about 300 mm. This is about 40% of the atmospheric pressure. If the blood pressure had to increase by about 50% to accomodate viscous losses, then the actual pressure would have to be about 60% of atmospheric pressure.

A measurement at the level of the heart on three immobilized but non-anesthesized giraffes gave ΔP between 85 and 30 mm of Hg [1]. The difference in ranges came from different individuals, of presumably different heights. The model underpredicts human ΔP, but over predicts a giraffe's, so there must be differences in the circulatory system of these two species.

1) R. H. Goetz *et al.*, Circ. Res. 8, 1049 (1960);
2) R. L. van Citters *et al.*, Science, 152, 384 (1966)

The shape of a giraffe

The scaling variable Π_2 is a little too sketchy. In order to find the scaling of cardiac size, one should consider other dimensions of a giraffe. Let us lump this into a variable A, with dimensions $[A] = L^2$. Assume that the mass of the neck and head, $M_n \propto Ah$, and the mass of the remainder is $M_t \propto A^{3/2}$.

The ratio, $\Pi_3 = M_t/M_n = \sqrt{A}/h$. This variable captures the odd shape of a giraffe. We have no measurements of M_n and M_t separately, so approximate Π_3 by the ratio of the heart to head length, h, and the toe to heart height, H, *i.e.*, $\Pi_3 \simeq H/h$. Then for humans, $\Pi_3 \simeq 5.5$, whereas for a giraffe $\Pi_3 \simeq 1$. Finally, we define a corrected variable $\tilde{\Pi}_2 = V_s/(H^2 h) = \Pi_2/\Pi_3^2$.

Since $V_s \simeq 70$ ml in humans, one might expect that for a giraffe $V_s \simeq 1.3$ l. The measured values are between 0.87 and 1.1 l [4].

A giraffe's heart

The work done by the heart is $\Delta P V_s \propto H^2 h^2$. This work is done by muscle tissue. So the amount of muscle, *i.e.*, the mass of the heart, $M \propto H^2 h^2$. We can write a scaling law between species,

$$\frac{M_1}{M_2} = \left(\frac{H_1 h_1}{H_2 h_2}\right)^2.$$

Since the human heart weighs about 0.25 kg, the giraffe's heart is about 30 kg. Published data includes a range of values: $M \simeq 11$ kg [3] or $M < 8$ kg [4].
The scaling law is not very accurate. Either we have missed shape variables or cardiac muscles can deliver more power per fiber than a human cardiac muscle.

3) R. H. Goetz and O. Budtz-Olsen, S. A. Tyd. Gen. 773 (1955)

4) G. Mitchell and J. D. Skinner, Comp. Bioch. Physio. A 154, 523 (2009)

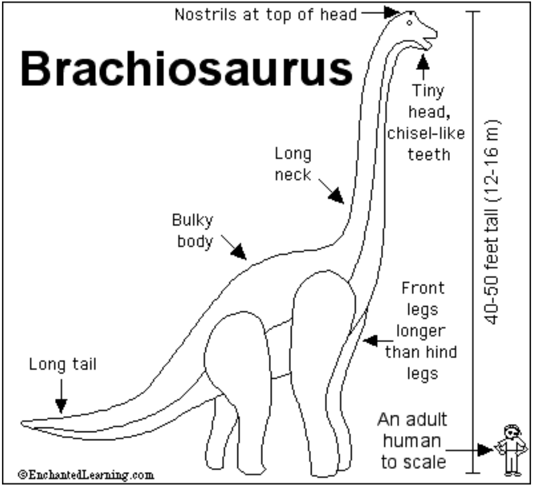
Is the model essentially correct?

We may have made some overly simple assumptions in developing a model of the mass of the giraffe's heart, but do we at least have the right physics?

If the mass of the heart is determined by the work it must do, then $\Phi = M/(\Delta P V_s)$ must be roughly equal for equally efficient hearts. For humans $\Phi = 89$ kg/(m^3 mm of Hg), whereas for giraffes $\Phi = 62$ kg/(m^3 mm of Hg). Since they are roughly similar, the heart probably works in similar ways for the two.

For elephants, it turns out that $M = 12$–21 kg, $\Delta P = 60$ mm of Hg, and $V_s \simeq 2.5$ l. So $\Phi > 80$ kg/(m^3 mm of Hg). For a dog $M \simeq 0.1$ kg, $\Delta P \simeq 60$ mm of Hg, and $V_s = 0.02$ l. This gives $\Phi = 83$ kg/(m^3 mm of Hg). Neither of these is very different from the human value. So it would seem that the giraffe heart is about 30% more efficient than most mammalian hearts in work output per unit mass of the heart.

Brachiosaurus



Brachiosaurus

At 12 meters to 16 meters tall, Brachiosaurus was among the tallest dinosaurs. It was about 26 meters long from head to tail. At the time of its discovery in 1903, Brachiosaurus was declared the largest dinosaur ever.

Paleontologists also debate whether Brachiosaurus kept its head raised continually or just when the herbivore dinosaur was foraging for food. A very large muscular heart would have been needed to raise its head to its full vertical height off the ground (12 meters), so some scientists believe that it held its head in a horizontal position much of the time when not in search of food in higher places.

Unlike its portrayal in the movie "Jurassic Park," paleontologists do not believe that Brachiosaurus could rear up on its slender, columnar hind legs.

http://www.livescience.com/25024-brachiosaurus.html

Questions

1. Brachiosaurus had forelegs which were about 4 m long. It seems that unlike most dinosaurs, this had forelegs longer than its hind legs; what sense would this make? Assuming that its blood pressure at the head (and other physiology) was similar to humans, what would its cardiac output have to be? How does a avian heart differ from a mammalian heart? Would this change the scaling law?
2. In discussing the height of animals, we considered only the cardiac system. However, blood takes up oxygen in the lungs. Could it be important to include the pulmonary system in our discussion?
3. Tall animals would extend significant hydrostatic pressure at the legs. What possible adaptation could this lead to? Would there be limits on the height of animals coming from this?