Similarity and incomplete similarity

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Methods in Science 2013
19 and 21 February, 2013

Outline

Similarity: Rayleigh-Benard Convection

Incomplete similarity: fractals

The renormalization group

Relativistic quantum fields

Keywords and References

Keywords

Convection, Fourier's law, Boussinesq's approximation, shear viscosity, Joule's constant, Boltzmann's constant, Prandtl number, Rayleigh number, Rayleigh-Benard convection, scaling, incomplete scaling, anomalous dimension, fractal, fractal dimension, broken scale invariance, scaling and the renormalization group, quantum field theory, ultraviolet divergence, broken scale invariance, renormalized coupling, renormalized mass, renomalization group equations.

Books

Scaling, by G. I. Barenblatt, Ch 1, 7.

Copyright statement

The experimental setup

The simplest setup for studying convection would be fluid contained within two walls— the lower one at temperature T_0 and the upper one at $T_0 + dT$ (we need not assume that dT is positive). The distance between these walls is H.
Heat is transferred in two ways. One is by conduction. Fourier's law for heat conduction states that the flux of heat, q, is given by

$$q = -\lambda \frac{dT}{H},$$

where λ is called the heat conductivity. The heat flux is the heat that passes through a unit area in unit time. The second method is by convection. Materials become lighter on being heated, and rise against gravity, carrying the heat stored in the material. This latter depends on the specific heat, c, defined as the increase in the heat content of a material per unit mass per degree rise in temperature.

Other relevant variables

Heated fluids rise against gravity since the density decreases with increase in temperature. If dT is small, then one can apply Boussinesq's approximation. This states that the buoyancy will be αg, where α is the fractional increase in volume per degree rise in temperature and g is the acceleration due to gravity. One also needs the density of the fluid, ρ, measured at temperature T_0. Since the fluid is moving, its viscosity, η, at the temperature T_0, may play a role in the problem. Newton's experiment utilizes the same setup to measure viscosity, by measuring the strain (force per unit area), τ required to move the upper plate at a constant velocity, U, against the fluid. Then, one has

$$\tau = \eta \frac{U}{H},$$

We will use units with Joule's constant, $J = 1$, and Boltzmann's constant, $k_B = 1$.

Dimensions

The dimensions needed are the usual mechanical quantities M, L, T. The units of heat and temperature are taken to be those of energy. From the definitions

$$[H] = L, \quad [\rho] = ML^{-3}, \quad [\eta] = ML^{-1}T^{-1}, \quad [dT] = ML^2T^{-2}.$$

The new thermal quantities are

$$[\alpha g] = M^{-1}L^{-1}, \quad [c] = M^{-1}, \quad [\lambda] = LT^{-1}.$$

With 7 quantities and 3 different dimensions, one will have 4 independent dimensionless quantities. One is a trivial comparison between energy scales of temperature and heat:

$$\Pi_4 = c\rho H^3.$$

It turns out that $r = 1/(c\rho)^{1/3}$ is about 0.15 nm for water. So $\Pi_4 \ll 1$ in all experimental situations.

Dimensionless quantities

The dimensionless Prandtl number can be built purely from the material properties of the fluid—

$$\Pi_3 = \mathrm{Pr} = \frac{\eta c}{\lambda}.$$

Another dimensionless number is the ratio of two lengths

$$\Pi_2 = \frac{c}{\alpha g H} = \frac{\ell}{H} \quad \text{where} \quad \ell = \frac{c}{\alpha g}.$$

The length scale ℓ characterizes the fluid. For water, we find $\ell = 2000$ Km. The third dimensionless quantity is the Raleigh number

$$\Pi_1 = \mathrm{Ra} = \frac{\alpha g c \rho^2 dTH^3}{\eta\lambda}.$$

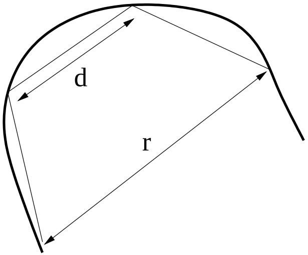
The solution of the problem is a relation $\Pi_1 = f(\Pi_2, \Pi_3, \Pi_4)$.

Similarity flow

Since $\ell/r \simeq 10^{15}$, one has a large range of H for which $\Pi_2 \gg 1$ and $\Pi_4 \ll 1$. Now $f(\infty, \Pi_3, 0)$ could either increase without bound or tend to a constant. If we take $f(\infty, \Pi_3, 0) = \overline{f}(\Pi_3)$, then we have postulated a similarity flow. The nomenclature comes from the fact Π_1 is then fixed for a given liquid, and a result, flows for different H are geometrically similar with $dT \propto 1/H^3$. This is a very strong assumption, and may not be justified in many cases (experimental evidence tells us that it works here).

Ra is the ratio of two forces: one is the buoyant force $\rho\alpha g dTH^3$, and the other a frictional force $\eta\lambda/(c\rho) = \eta^2/(\rho\,\mathrm{Pr})$. For $\mathrm{Ra} < R_c$ there is no convection; viscous friction wins. For $\mathrm{Ra} > R_c$, buoyancy wins, convection sets in and cells of oppositely circulating fluid are created. $R_c \simeq 650$ when the lower surface of the fluid is in contact with a rigid surface, and the upper surface is free. Our analysis has not touched on the size of cells.

Perimeters and polygons



The arc length of a curve, L, whose ends are a distance r apart, can be approximated by a segment of a polygon with sides of length, d. A dimensional argument tells us that

$$L(r) = r \lim_{d \to 0} \Phi\left(\frac{r}{d}\right),$$

since one gets successively better approximations to the curve by decreasing the size of the edge of the approximating polygon.

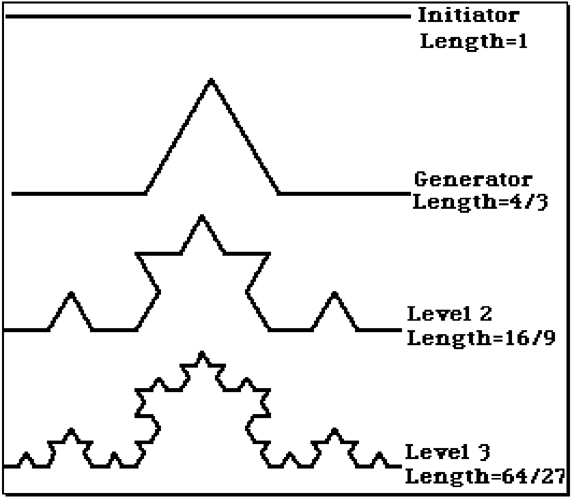
Similarity and incomplete similarity

In the limit $\Pi = r/d$ goes to infinity. A similarity hypothesis is that $\Phi(\infty)$ has a good limit. For example, for a semicircle, $\Phi(\infty) = \pi/2$. These are simple curves described by functions which are smooth, with continuous first derivatives.
However, for most curves the limit does not exist. For a larger class of these curves a fairly simple hypothesis called incomplete scaling may hold. This is the assumption that $\Phi(\Pi) = \Pi^\alpha$, where α is a positive number. In this case $L(r) \propto r^{1+\alpha}$, although the limit does not exist. The curve is called a fractal; α is called an anomalous dimension and $D = 1 + \alpha$ is called a fractal dimension.

Fractals have broken scale invariance

For fractals the unit of the measuring scale, d, leaves a trace. Invariance with respect to d is broken, and this quantity does not disappear from the formula. This is the origin of the anomalous dimension.

The Koch curve



The set of fractals is not empty!

Dimensional analysis and homogenous functions

Recall how dimensional analysis involves homogenous functions. In a physical problem list all the n variables of interest, choose the basis set of m of them, U_1, U_2, $\cdots$, U_m, and write the others as A_1, A_2, $\cdots$, A_k, where $k+m=n$. Then a physical relation will be

$$A_1 = F(U_1, U_2, \cdots, U_m, A_2, A_3, \cdots, A_k).$$

One can construct k dimensionless quantities, $\Pi_i = A_i / \prod U_j^{a_{ij}}$. In terms of these one can write the same relation as

$$A_1 = U_1^{a_{11}} U_2^{a_{12}} \cdots U_m^{a_{1m}} \; f\left(\frac{A_2}{U_1^{a_{21}} U_2^{a_{22}} \cdots U_m^{a_{2m}}}, \cdots \frac{A_k}{U_1^{a_{k1}} U_2^{a_{k2}} \cdots U_m^{a_{km}}}\right),$$

or, more compactly as,

$$\Pi_1 = f(\Pi_2, \Pi_3, \cdots, \Pi_k).$$

Scale transformations: the relativity of measurements

The function under study

$$A_1 = F(U_1, U_2, \cdots, U_m, A_2, A_3, \cdots, A_k),$$

has a necessary invariance under scale transformations of units

$$U_i = \lambda_i U_i, \text{ for } 1 \leq i \leq m, \qquad A_j = \left[\prod_{i=1}^{m} \lambda_i^{a_{ji}}\right] A_j.$$

These transformations form a group because

1. all non-negative values of λ_i are allowed (closure),
2. $\lambda_i = 1$ is an allowed transformation (existence of unity),
3. for any λ_i the inverse scaling $1/\lambda_i$ is allowed (unique inverse),
4. and a sequence of scalings can be composed in any way (associativity).

The Π_i are invariants of this group. Dimensional analysis expresses the covariance of physical equations under this group.

Similarity and incomplete similarity

The function f is said to possess similarity if it has a finite and non-zero limit as any one of these Π_i go to zero or infinity. Then, in that limit, the functional dependence can be dropped to give

$$\Pi_1 = f(\Pi_2, \Pi_3, \cdots, \Pi_{k-1}) \qquad \Pi_k \to \infty.$$

Since the labelling of the variables is immaterial, in the case when $k - \ell$ of the dependences can be dropped we write

$$\Pi_1 = f(\Pi_2, \Pi_3, \cdots, \Pi_\ell) \qquad \Pi_i \to \infty \quad \forall i > \ell.$$

Clearly this is a very special class of functions. In general no simplification can occur. However, there is a wider class which has the property of incomplete similarity

$$\Pi_1 = \left[\prod_{i>\ell} \Pi_i^{\alpha_{1i}}\right] g\left(\frac{\Pi_2}{\prod_{i>\ell} \Pi_i^{\alpha_{2i}}}, \cdots, \frac{\Pi_\ell}{\prod_{i>\ell} \Pi_i^{\alpha_{\ell i}}}\right).$$

The exponents α_{ji} are called anomalous dimensions.

Incomplete similarity

Incomplete symmetry also involves homogeneous functions and can be written in terms of renormalized variables, $\Pi_j^* = \Pi_j / \prod_{i>\ell} \Pi_i^{\alpha_{ji}}$, as

$$\Pi_1^* = g(\Pi_2^*, \Pi_3^*, \cdots, \Pi_\ell^*).$$

Note that similarity is included within incomplete similarity as the special case when all $\alpha_{ij} = 0$.

We saw earlier that the tool for developing dimensional analysis is the invariance of physics under scaling of units of measurement. The tool for investigating incomplete symmetry is also such scaling, restricted to the domain where some of the dimensionless parameters become large or small. This analysis is called a renormalization group analysis.

Finally, note that there are many functions f which do not allow compression of variables beyond ordinary dimensional analysis.

Renormalization group transformations

The renormalization group arises if there is a scale invariance of the form

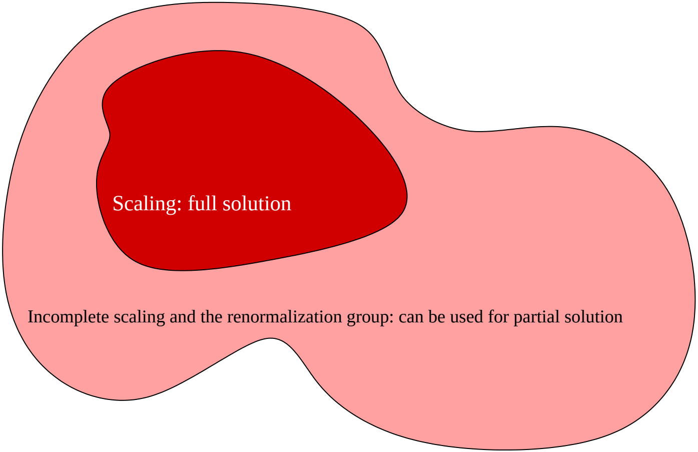
$$\Pi_i = \xi_i \Pi_i \quad \text{for} \quad \ell + 1 \le i \le k$$
$$\Pi_j = \left[\prod_{i>\ell} \xi_i^{\alpha_{ji}} \right] \Pi_i \quad \text{for} \quad 1 \le j \le \ell.$$

Since this is supposed to hold only for very large or very small values of Π_i (with $i > \ell$), the range of ξ_i is restricted to be not very different from 1. The proof that this leads to incomplete similarity follows the same lines as dimensional analysis.

These transformations satisfy the existence of unity, unique inverse and associativity. However, the restriction that ξ_i be not very large or small tells us that the property of closure is not satisfied. So the renormalization group is, almost but not quite, a group.

What dimensional analysis leads to

Only dimensional analysis: can be used to discover new phenomena



Counting dimensions in dynamics

In relativistic quantum theory there is only the dimension of mass, so it is possible to associate with every physical quantity, ϕ, a single number called its dimension, d, such that $[\phi] = M^d$.

Spacetime coordinates have dimension -1. Momenta have dimension 1. As a result, phase space volume has dimension 0, so action also has dimension 0. Langrangian densities, $\mathcal{L}$, therefore have dimension 4.

A theory of a relativistic scalar field, ϕ, must have a term in $\mathcal{L}$ of the form $m^2\phi^2$. So clearly, ϕ has dimension 1. The kinetic term has to be $\partial_\mu\phi\partial^\mu\phi$, and this also gives the same dimension to the field. A relativistic Dirac field, ψ, has a mass term $m\overline{\psi}\psi$, so ψ has dimension 3/2. For a gauge field A_μ one has the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_n u A_\mu$, and the action $F_{\mu\nu}F^{\mu\nu}$. So A_μ has dimension 1.

Power counting of processes

Processes are classified by the number of external states they contain. This also happens to count the dimensions of the quantum amplitude. If there are F fermions and B bosons in a process (say $F/2$ fermions in the initial and final states, 0 bosons in the initial state and B in the final state), then the amplitude for the process contains F Dirac fields and B boson fields. As a result, the dimension of the process is $3F/2 + B$.

In Compton scattering, one has initially 1 electron and 1 photon, and finally also 1 electron and 1 photon. So $F = 2$ and $B = 2$. The dimension of the amplitude is 5. Electron-positron annihilation also has $F = 2$ and $B = 2$, so its amplitude also has dimension 5. In Bhabha scattering, one has $F = 4$ and $B = 0$, so the amplitude has dimension 6.

Scale invariance

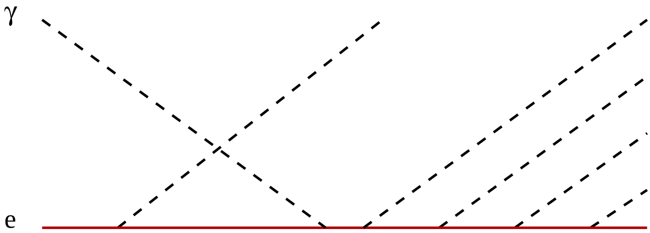
If we write the amplitude for a process as $\Gamma(\alpha, m_e, Q)$, where α is the fine structure constant, m_e the electron's mass, and Q the typical momentum scale in the process, then dimensional analysis tells us that

$$\Gamma(\alpha, m_e, Q) = Q^{3F/2+B}\Gamma\left(\alpha, \frac{m_e}{Q}\right).$$

If $m_e = 0$, then there is no intrinsic scale to the problem and the theory could be scale invariant.

Recall the usual arguments behind dimensional analysis. Since there is only one dimension, scale all quantities by powers of a single parameter λ. Scale $Q \to \lambda Q$. Distances and times scale as $1/\lambda$. Scalar and gauge fields scale as λ and fermion fields as $\lambda^{3/2}$, whereas α does not scale. This scaling is taken care of in the above formula.

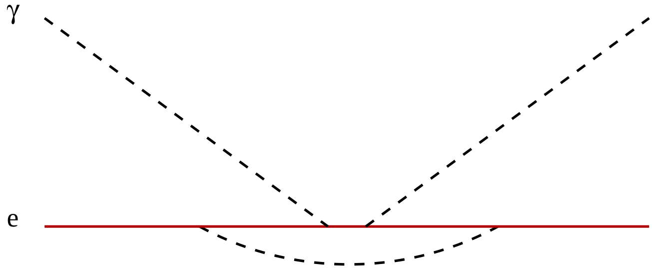
Example of Compton scattering



Many momenta: Q_1, Q_2, Q_3, *etc.*. Completely computable. Set all momenta $Q_i \simeq Q$; take limit $m_e \to 0$ perfectly fine. Scale invariant theory.

Breaking of scale invariance

However, there are two kinds of scale breaking in these theories: one is through divergences which arise when $m_e = 0$ and $Q \to 0$. These are called infrared divergences. The other kind occurs for any m_e when $Q \to \infty$. These are called ultraviolet divergences. We met one of these in the ultraviolet catastrophe of black body radiation. In either case $\Gamma(\alpha, 0)$ does not have a good limit.



A divergence arising from the sum over all possible states.

Incomplete similarity

The ultraviolet catastrophe of the black body spectrum went away as soon as we introduced a new constant relating two different dimensions ($\hbar$). We not longer have this freedom. The only way to take care of this is to introduce an ultraviolet cutoff Λ.

However, if there is a new scale in the problem, Λ, then one must replace the scaling formula by

$$\Gamma(\alpha, m_e, Q, \Lambda) = Q^{3F/2+B}\Gamma\left(\alpha, \frac{m_e}{\Lambda}, \frac{Q}{\Lambda}\right).$$

Could we expect quantum field theories to have incomplete similarity in the limit $Q/\Lambda \to \infty$? Why incomplete similarity and not similarity?

Probe an electron with photons of momentum Q; what phenomena do we see as we increase Q? Increase of Q decreases wavelength and therefore reveals more structure. Is an electron structureless?

Stückelberg-Petermann-Gell-Mann-Low-Callan-Symanzik

Renormalizable field theories are those which have incomplete scaling. For these theories we can define renormalized quantities, $\alpha^* = \alpha(Q/\Lambda)^\beta$ and $m_e^* = m_e(Q/\Lambda)^\gamma$ in terms of which one can write

$$\Gamma(\alpha, m_e, Q, \Lambda) = Q^{3F/2+B+\eta}\Gamma\left(\alpha^*, \frac{m_e^*}{\Lambda}\right).$$

We have the renormalization group equations

$$\frac{d\alpha^*}{d\log Q} = \beta\alpha^*, \qquad \text{and} \qquad \frac{dm_e^*}{d\log Q} = \gamma m_e^*.$$

Which parameters of the theory do β and γ depend on? Since they are independent of Q and Λ, they cannot depend on m_e, by a dimensional argument. Therefore, β and γ can depend only on α. One can compute them from the usual perturbation series.