Search for the critical point of QCD

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Basics

Experiment

Statistics

Summary

Thermodynamics and statistical mechanics

Experiment sees thermodynamics

Observing nearly Gaussian fluctuations

Detailed test of thermalization

What could happen near the critical point

Error analysis for ratios

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Outline

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Can experiments see the critical point?

\[ T \]

\[ \mu \]

freezeout

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QGP@SERC-EHEP: Lecture 4
Can experiments see the critical point?
Can experiments see the critical point?
Thermodynamic fluctuations

Thermodynamics deals with matter in bulk: Avogadro number of particles \( N_A = 0.602 \times 10^{24} \). Short range interactions, so energy is additive. Particle number, charge, etc are all additive. All give rise to thermodynamic extensive quantities.

Equilibrium system: all extensive quantities have a precise and measurable value. Disturb the system a little and let it settle to equilibrium: new precise and measurable values.

Thermal fluctuations deduced by Boltzmann, Maxwell. Observation (1827) by Robert Brown of position fluctuation of tracer particles in water explained in 1905 by Einstein as due to thermal fluctuations of molecules.
Grand canonical ensemble (GCE)

To study fluctuations, divide material into small part which is observed (system) and large unobserved part (heat bath). Extensive thermodynamic variables of system fixed; but microscopic variables can still fluctuate. Need a collection (ensemble) of different realizations of the microscopic states in thermodynamic equilibrium.

Grand canonical ensemble: all conserved quantities allowed to fluctuate by exchanging energy and material with the heat bath. Possible to realize by observing a small portion of the fireball by putting angular cuts.

Which ensemble is created in an experiment? Different realizations of a system may be same system at different times: normal in tabletop experiments. More natural in colliders to use measurements in different collisions to have many realizations of the same system.
Fluctuations and correlations
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For almost ideal gases $\xi \sim \lambda$. 
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For almost ideal gases $\xi \simeq \lambda$. 
Gaussian fluctuations

When $V_{obs} \gg \xi^3$ then distribution of any extensive quantity is Gaussian: central limit theorem. Mean value of Gaussian is the extensive thermodynamic equilibrium value. Thermodynamic quantities alone have little connection to microscopic properties.

Study of microscopic properties therefore requires study of fluctuations. Width of Gaussian related to specific heat. Width proportional to $\sqrt{N_A}$, therefore usually negligible, with respect to mean.

When system is small enough then deviations from Gaussian are visible. These are reflections of microscopic physics.

Central limit theorem fails when $\xi \simeq L$, i.e., $\xi^3 \simeq V_{obs}$. Then no difference between microscopic and macroscopic.
Non-linear susceptibilities

Taylor expansion of the pressure in $\mu_B$

$$P(T, \mu_B + \Delta\mu_B)/T^4 = \sum_n \frac{1}{n!} \left[ \chi^{(n)}(T, \mu_B) T^{n-4} \right] \left( \frac{\Delta\mu_B}{T} \right)^n$$

has Taylor coefficients called non-linear susceptibilities (NLS). When $\mu_B = 0$ they can be computed directly on the lattice, otherwise reconstructed from such computations.

Gavai, SG: 2003, 2010

Cumulants of the event-to-event distribution of baryon number are directly related to the NLS:

$$[B^2] = T^3 V \left( \frac{\chi^{(2)}}{T^2} \right), \quad [B^3] = T^3 V \left( \frac{\chi^{(3)}}{T} \right), \quad [B^4] = T^3 V \chi^{(4)}.$$

$V$ unknown, can be removed by taking ratios.

(SG: 2009)
What are the cumulants?

\[
[B] = \langle B \rangle \\
[B^2] = \langle (B - \langle B \rangle)^2 \rangle \\
[B^3] = \langle (B - \langle B \rangle)^3 \rangle \\
[B^4] = \langle (B - \langle B \rangle)^4 \rangle - 3[B^2]^2 \\
[B^5] = \langle (B - \langle B \rangle)^5 \rangle - 10[B^2][B^3] \\
\]
Tests and assumptions

\[ m_1 : \frac{[B^3]}{[B^2]} = \frac{\chi^3(T, \mu_B)}{\chi^2(T, \mu_B)/T^2} \]
\[ m_2 : \frac{[B^4]}{[B^2]} = \frac{\chi^4(T, \mu_B)}{\chi^2(T, \mu_B)/T^2} \]
\[ m_3 : \frac{[B^4]}{[B^3]} = \frac{\chi^4(T, \mu_B)}{\chi^3(T, \mu_B)/T} \]

Also for cumulants of electric charge, \( Q \), and strangeness, \( S \).

1. Two sides of the equation equal if there is thermal equilibrium and no other sources of fluctuations.
2. Right hand side computed in the grand canonical ensemble (GCE). Can observations simulate a grand canonical ensemble? What \( T \) and \( \mu_B \)?
3. Why should hydrodynamics and diffusion be neglected?
Thermodynamics and fluctuations

Observations

In a single heavy-ion collision, each conserved quantity \((B, Q, S)\) is exactly constant when the full fireball is observed. In a small part of the fireball they fluctuate: from part to part and event to event.
Thermodynamics and fluctuations

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Thermodynamics
If \( \xi^3 \ll V_{\text{obs}} \ll V_{\text{fireball}} \), then fluctuations can be explained in the grand canonical ensemble: energy and B, Q, S allowed to fluctuate in one part by exchange with rest of fireball (diffusion: transport).
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Thermodynamics
If $\xi^3 \ll V_{obs} \ll V_{fireball}$, then fluctuations can be explained in the grand canonical ensemble: energy and $B$, $Q$, $S$ allowed to fluctuate in one part by exchange with rest of fireball (diffusion: transport).

Comparison
When $V_{obs} \ll V_{fireball}$, Gaussian as $V_{obs}/\xi^3 \to \infty$. Finite size effects are mainly controlled by NLS. Otherwise system is in the critical regime.
Ensuring control

Ensembles used in tabletop experiments keep intensive quantities fixed by controlling the heatbath: fixed temperature, fixed composition etc.

Collider experiments find this harder. Events may be classified by impact parameter, but there may still be fluctuations in the total energy content of the fireball. Similarly, the chemical composition of the fireball may need to be controlled. Compositional control may be harder to achieve at low densities.

LHC will play an important role in ensuring that these control mechanisms are implemented properly.
Why thermodynamics and not dynamics?

Chemical species may diffuse on the expanding background of the fireball, so why should we neglect diffusion and expansion?

First check whether the system size, $L$, is large enough compared to the correlation length $\xi$: Knudsen’s number $K = L/\xi$. If $K \gg 1$, ie, $L \gg \xi$ then central limit theorem will apply. Next, compare the relative importance of diffusion and flow through a dimensionless number (Peclet’s number):

$$\mathcal{W} = \frac{L^2}{tD} = \frac{Lv_{\text{flow}}}{D} = \frac{K\xi v_{\text{flow}}}{\xi c_s} = \frac{Kv_{\text{flow}}}{c_s} = MK,$$

where $M$ is the Mach number. When $\mathcal{W} \ll 1$ diffusion dominates. Near chemical freeze-out $K \simeq 1$ and $M \simeq 1$, so flow dominates: fluctuations are frozen in. Detector observes thermodynamic fluctuations at chemical freeze out.

(Bhalerao, SG: 2009)
Define Peclet length $\lambda_p \simeq \xi / M$ where $\mathcal{W} = 1$. $\lambda_p$ remains fairly constant until time $R_0/c_s$ then falls rapidly as flow becomes fully 3d. So freezeout time is not very strongly dependent on rapidity window.
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- Observing nearly Gaussian fluctuations
- Detailed test of thermalization
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Event by event fluctuations

Open questions:

1. Is the system in equilibrium?
2. Is the observed volume much smaller than full fireball?
3. Do the wide bins of impact parameter sufficiently control intensive parameters?
4. Is proton number a good substitute for baryon number?

STAR, 1004.4959
Grand canonical thermodynamics?

When \( V_{\text{obs}}/\xi^3 \to \infty \) and \( V_{\text{fireball}}/V_{\text{obs}} \to \infty \), then thermodynamics in the grand canonical ensemble works; all distributions of conserved quantities are Gaussian. For a Gaussian the only non-vanishing cumulants are the mean, \([B]\), and the variance \([B^2]\).

Observation of any other non-vanishing cumulant \([B^n]\) means the fluctuations are non-Gaussian. Is it because the system is not large enough for thermodynamics to be applied? If it is a finite size effect, we are still ok.

If finite size effect, then trivial volume dependence of cumulants, i.e., all cumulants scale as \(V\).
Is the fluctuation Gaussian?

- Higher cumulants scale down with larger powers of $V$.
- $N_{part}$ is a proxy for $V$.
- Cumulants observed to scale correctly as $N_{part}$.
- Can one connect to QCD?

STAR: QM 2009, Knoxville
Same data, different plot

from STAR data [naive error propagation, no systematic errors]
200 GeV Au Au

Cumulants

Central limit theorem follows. Scaling implies $\xi^3 \ll V_{obs}$ at some $\sqrt{S}$. 
Same data, different plot

from STAR data [naive error propagation, no systematic errors]
62.4 GeV Au Au

Central limit theorem follows. Scaling implies $\xi^3 \ll V_{obs}$ at some $\sqrt{S}$. 
Central limit theorem follows. Scaling implies $\xi^3 \ll V_{obs}$ at some $\sqrt{S}$. 

from STAR data [naive error propagation, no systematic errors] 
19.6 GeV Au Au 

Cumulants 

$N_{part}$ 

$[B^2]$ 

$[B^1]$ 

$[B^3]$ 

$[B^4]$
Very important deduction

The cumulants are linear in the volume, so the higher order fluctuations will scale away in the limit when the volume becomes infinite. The fluctuations will then become Gaussian.

This happens at almost all $\sqrt{S}$. This is consistent with thermodynamics at an ordinary point.

This means that departure from Gaussian at one $\sqrt{S}$ could be significant, and should be investigated as a signal of a critical point.
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We are so lucky

Look at $m_1$, $m_2$ and $m_3$ along the freeze-out curve.
$m_1 = S / \sigma$

Gavai, SG: 2010
$m_1 = S/\sigma$

Gavai, SG: 2010
\[ m_2 = K \sigma^2 \]
\[ m_2 = \mathcal{K} \sigma^2 \]
$m_3 = S\sigma / \mathcal{K}$

Filled circles: negative values

Gavai, SG: 2010
\[ m_3 = S\sigma/K \]

![Graph showing the relationship between \( m_3 \) and \( \sqrt{S\text{_{NN}}} \) (GeV)](image)

**Filled circles:** negative values

**HRG**

**Power law**

Gavai, SG: 2010
Comparison with lattice predictions

Surprising agreement with lattice QCD:

- implies non-thermal sources of fluctuations are very small
- $T$ does not vary across the freezeout surface.
- tests QCD in non-perturbative thermal region

Gavai, SG, 1001.3796
Experiment vs lattice QCD

![Graph showing Au+Au Collisions and Net-proton data compared to lattice QCD predictions. The graph plots $S_\sigma$ and $\kappa \sigma^2$ as functions of $\sqrt{s_{NN}}$ (GeV). The lattice QCD predictions are shown for $N_t=4$ and $N_t=6$.](image-url)
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Finite size effects damp divergences

1. System size limits correlation lengths near the critical point: \( \ell \sim \xi \). The Knudsen number is never small near the CEP, so central limit theorem will stop working. Check the scaling of \( \sigma^2, S \) and \( K \) and see whether there are violations of the central limit theorem. (SG: 2009)

2. As a result, the Peclet number need not be large, and diffusion may play an important role even close to kinetic freeze-out. Then fluctuations of conserved quantities may not be comparable to thermal equilibrium values at chemical freeze-out!

3. Another way of saying this is: critical divergences are limited due to finite size effects: no singularities, hence no direct measurement of the critical exponents. System drops out of equilibrium due to finite lifetime. (Stephanov: 2008; Berdnikov, Rajagopal: 1998)
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The meaning of errors and error propagation

One usually uses the central limit theorem to say that the error in a measurement, $\Delta x$, is related to the variance of the measurement, $\sigma^2(x)$, and the number of measurements, $N$, by the formula

$$(\Delta x)^2 = \frac{\sigma^2(x)}{N}.$$ 

This works if the distribution of the estimates of the measurements is Gaussian. The error is the 65% confidence limit.

It is very tempting to use standard error analysis for products and ratios of numbers. If $t = x/y$ and then one usually writes

$$\left(\frac{\Delta t}{t}\right)^2 = \left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2.$$ 

What does this mean?
The error formula for ratios is meaningless

One can prove that the ratio (or product) of two Gaussian distributed numbers is not Gaussian. In fact the ratio has infinite variance! So the formula for error propagation gives some $\Delta t$ which has no meaning as a confidence limit. In fact, since the variance is infinite, such a $\Delta t$ is perfectly meaningless. (Geary: 1930)
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You do not believe this?
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You do not believe this?

Do a Monte Carlo experiment. Draw two numbers $x$ and $y$ each from a Gaussian distribution. Take the ratio $t$. Repeat this process $N$ times so you have $N$ values of $x$, $y$ and $t$. Now compute the variance of each quantity. As you increase $N$ you find $\sigma^2(x)$ and $\sigma^2(y)$ are finite numbers but $\sigma^2(t)$ grows without bound.
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Do not use the error propagation formula for ratios. Other methods are possible.
The variance of a ratio grows with statistics

Result of a simple Monte Carlo test.
Alternatives to variance

Measures of error exist even when variance is infinite.
Phase transitions and statistics are very closely linked!
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1. The ensemble of collider events can be used to simulate the grand canonical ensemble provided
   - the fireball is thermalized
   - one observes only a small part of the fireball
   - this part is much larger than the correlation volume

GCE fluctuations can then be measured through event by event fluctuations.

2. Crucial check: cumulants of $B$, $Q$, and $S$, must scale linearly with system volume (proxy: $N_{part}$).

3. If the ratios of cumulants equal the results of the lattice, then the fireball is in thermal equilibrium.

4. Near a critical point the fireball cannot come to thermal equilibrium.