Simple potentials in one dimension

Assignment 2
Quantum Mechanics 1, 2008

These assignments will be graded, and the grades will count towards the final consolidated performance in the course. The assignment is due on September 11. Every day's delay after that will cause a 10% loss in marks.

1. All the problems below need Mathematica.
2. The assignments in this sheet should be solved within a single notebook.
3. The notebook should be submitted to Jyotirmoy Bhattacharya by email.
4. Two problems should not be in the same cell of the notebook.
5. Each cell should have a comment on top saying which problem you are solving in that cell.
6. An example notebook can be found on the web page.

1. Write a Mathematica function for the transfer matrix at a step in a potential. It should take as input the momenta on the two sides of the step, and the position of the step. It should output the 2×2 matrix which is the transfer matrix. Perform the following tests on it—
   (a) If the two momenta are equal (*i.e.*, the step height becomes zero) then do you get the result you expect?
   (b) Does your function give the correct reflection coefficient at the step barrier?
   (c) Does reversing the order of the momenta give the inverse matrix?
   (d) For the square barrier of range $2a$ and height V_0, the transfer matrix can be found as the product of two step transfer matrices. Does this give the right result?
2. Use the Mathematica function that you have written to find the transfer matrices in a square well ($V_0 < 0$) and square barrier ($V_0 > 0$). Rewrite the results you obtain from your Mathematica code in terms of the dimensionless quantities $r^2 = 2m|V_0|a^2/\hbar^2$ $z^2 = E/|V_0|$ using Mathematica's ability to make substitutions.
   (a) How are the two transfer matrices related?
   (b) Compare the transmission coefficients, $T(z)$, of the square well and the square barrier. Find the differences (if any) in the expressions for the two quantities. Plot the two as functions of z on the same graph. (If numerical values are required, use a couple of different values of r).
   (c) At what values of z does one find resonances in each case?
   (d) Expand the expressions for $T(z)-1$ near the n-th resonance, z_n^*, to fourth order in δ (defined to be $z-z_n^*$). Use the Mathematica function which creates the series expansion of an expression.
      i. What is the leading power of δ in each case?
      ii. What are the coefficients of each of the terms, in both cases?
      iii. Some quantity is universal if it is independent of either the form of the potential or the value of n, or both. List the universal quantities and say in which sense they are universal.
   (e) If the incoming wave is the right mover on the left with amplitude unity, then find the amplitude of the outgoing wave (right mover) on the right using the transfer matrix in both cases. For each z the outgoing amplitude is a complex number. The locus of this complex number as z changes is called the Argand diagram. Draw the Argand diagram in the two cases. What happens at each of the resonances, z_n^*?
   (f) The phase shift of a transmitted waves is the phase of the amplitude of the transmitted wave when the incoming wave amplitude is taken to be real. How are the phase shifts of the barrier and well related? Plot the phase shifts as functions of z on the same graph. Are there any universal features in the phase shifts as resonances are approached.
3. Using your Mathematica function, construct the transmission coefficient for a potential which is a square barrier of height V_0 and width $2a$ followed at distance $2a$ by a square well of depth V_0 and width $2a$.

(a) Check whether the transmission coefficient depends on the direction from which the incoming wave comes.

(b) Of the quantities which you found to be universal in some sense in the previous problem, which remain universal in this new potential?

(c) Construct the scattering matrix for this potential.

4. Take a potential which is a series of square barriers (each of height V_0 and range $2a$). The barriers are separated from each other by a distance of $2a$. Find the amplitude of the transmitted wave as a function of k and plot it (use scaled variables and choose appropriate numerical values for $r^2 = 2mV_0a^2/\hbar^2$). Do this when there is only a single barrier, two barriers, three barriers, *etc.*.

5. (Extra credit: not mandatory, can be submitted any time before the end of the course) Write a Mathematica program to construct the propagation of a Gaussian wave packet through potentials constructed from square wells and square barriers. It could be convenient to consider the propagation in the Fourier basis, and to make the Fourier transform only when plotting the wave packet.