Symmetries for fun and profit

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Outline

The energy eigenvalues and eigenvectors

The isotropic harmonic oscillator in two dimensions is specified by the two position variables x_1 and x_2 and the two conjugate momenta p_1 and p_2. Isotropy implies that the angular frequency, ω is the same in all directions. Then introducing the scaled quantities $X_i = x_i\sqrt{m\omega/\hbar}$ and $P_i = p_i/\sqrt{m\omega\hbar}$, one has the Hamiltonian

$$H = \frac{\hbar\omega}{2}(P_1^2 + P_2^2 + X_1^2 + X_2^2).$$

Introducing the shift operators $a_j = (X_j + iP_j)/\sqrt{2}$ and their Hermitean conjugates, $a_j^\dagger$, as before, one can show that H can be written in terms of two number operators $N_j = a_j^\dagger a_j$ in the form

$$H = \hbar\omega[(N_1 + 1/2) + (N_2 + 1/2)].$$

The energy eigenstates can be specified in the form $|n_1, n_2\rangle$ where n_i are the eigenfunctions of N_i. The energies of these states are $E = \hbar\omega(n_1 + n_2 + 1)$. Each eigenvalue is $(N + 1)$-fold degenerate. Therefore there is a larger symmetry in the problem.

Extended symmetry

If one changes n_1 and n_2 simultaneously, while keeping the sum then the energy does not change. An operator of the form $a_1^\dagger a_2$ does exactly this. The two Hermitean operators

$$s_1 = a_1^\dagger a_2 + a_2^\dagger a_1 \qquad \text{and} \qquad s_2 = i a_1^\dagger a_2 - i a_2^\dagger a_1$$

act on $|N, n_1\rangle$ to produce linear combinations of $|N, n_1 - 1\rangle$ and $|N, n_1 + 1\rangle$. So one must have $[H, s_1] = 0 = [H, s_2]$. (**check**). The combinations $s_0 = N_1 + N_2$, s_1, s_2 and $s_3 = N_1 - N_2$ have the commutation relations of the Pauli matrices. (**Check that in the subspace of $E = 2\hbar\omega$ these operators are exactly the Pauli matrices**). When acting in the eigenspace of larger values of E, the operators are represented by larger matrices. (**Compute the operator $S^2 = s_1^2 + s_2^2 + s_3^2$ in the degenerate subspace of any E. Also compute the matrix representation of s_3 in this subspace.**)

The symmetry group SU(2)

An arbitrary (new) linear combination of the degenerate eigenstates of the isotropic two-dimensional harmonic oscillator is generated by the unitary matrix $U = \exp\left(i\sum_j \theta_j s_j\right)$. Note that $\mathrm{Det}\, U = 1$ (because the trace of its logarithm is zero). Since these linear combinations all have the same energy, all these U must commute with the Hamiltonian.
In particular, this is true of the two-dimensional subspace with $N = 1$. All 2×2 unitary matrices with unit determinant form a group. This is called **the group SU(2)**. Since all these matrices commute with H, the symmetry group of this problem is SU(2). The higher dimensional matrices generated by the above prescription do not exhaust all possible unitary matrices of that size, but a subgroup which is isomorphic to SU(2). These matrices of different sizes are called different **representations** of SU(2). The trace of S^2 in each representation is characteristic of that representation. The Hermitean operators s_1, s_2 and s_3 are called the **generators** of SU(2), or elements of **the algebra su(2)**.

A problem

Consider the isotropic harmonic oscillator in three dimensions. In analogy with the construction we have presented here, find the complete group of symmetries of this problem: it is called SU(3).

1. Construct the complete algebra of operators from Hermitean combinations of the bilinears of the shift operators which leave the energy unchanged.
2. Find the commutators of these operators, and construct the completion of this algebra. How many operators are there in the algebra?
3. Find a complete set of commuting operators among these.
4. In the degenerate space of eigenstates corresponding to the energy eigenvalue $E = 5\hbar\omega/2$, construct the representations of the elements of the algebra.
5. Construct the representation of the algebra in the space of energy eigenstates with eigenvalue $E = 7\hbar\omega/2$.

Outline

Extended raising and lowering operators

Extend the notion of raising and lowering operators to

$$A_\pm = \pm\frac{ip}{\sqrt{2m}} + W(x), \qquad A_-^\dagger = A_+ \quad \text{if } W(x) \text{ is real.}$$

$$H_\pm = A_\pm A_\mp = \frac{p^2}{2m} + V_\pm(x) \quad \text{where} \quad V_\pm(x) = W^2 \pm \frac{\hbar}{\sqrt{2m}}\frac{dW}{dx}.$$

The last equation is called the Ricatti equation. If one finds a solvable Hamiltonian, then one can check whether this is one of a pair of solutions of the Ricatti equation.
If H_- has a discrete spectrum, $H_-|\psi_N^-\rangle = E_N^-|\psi_N^-\rangle$, for $N = 0, 1, 2\cdots$, then one can prove (as for the harmonic oscillator) that $E_N^- \geq 0$. Also, if $A_+|\psi_0^-\rangle = 0$, then $E_0^- = 0$ and

$$\left[W(x) + \frac{\hbar}{\sqrt{2m}}\frac{d}{dx}\right]\psi_0^-(x) = 0, \; ie, \; \psi_0^-(x) \propto \exp\left(-\frac{\sqrt{2m}}{\hbar}\int W(x)dx\right).$$

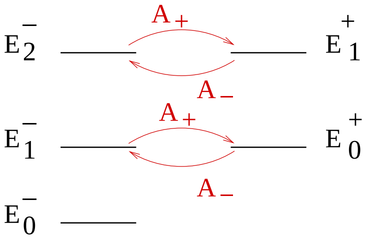
Solving for H_+ given a solved H_-

The eigenfunctions $H_+|\psi_N^+\rangle = E_N^+|\psi_N^+\rangle$ can be related to those of H_- using the shift operators. Note that

$$H_+A_+|\psi_N^-\rangle = A_+H_-|\psi_N^-\rangle = E_N^-A_+|\psi_N^-\rangle, \quad \text{so} \quad A_+|\psi_N^-\rangle = \sqrt{E_N^-}|\psi_{N-1}^+\rangle.$$

The proportionality constant can be found by taking the norm of both sides in the last equation. Similarly, one finds that

$$A_-|\psi_N^+\rangle = \sqrt{E_N^+}|\psi_{N+1}^-\rangle, \qquad \text{hence} \qquad E_N^+ = E_{N+1}^-.$$



Supersymmetry

Given the degeneracy of all except one state of the Hamiltonian

$$H_s = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} = \frac{p^2}{2m} + W^2(x) - \frac{\hbar}{\sqrt{2m}}\frac{dW}{dx}\sigma_3,$$

one expects some new symmetry of this extended Hamiltonian. Introduce the operators and states

$$Q = \begin{pmatrix} 0 & 0 \\ A_+ & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & A_- \\ 0 & 0 \end{pmatrix}, \quad |N\rangle_- = \begin{pmatrix} |\psi_N^-\rangle \\ 0 \end{pmatrix}, \quad |N\rangle + = \begin{pmatrix} 0 \\ |\psi_N^+\rangle \end{pmatrix},$$

where $|N\rangle\pm$ are the eigenstates of H_s and Q and $Q^\dagger$ are exhibited in this basis. Clearly $H_s = \{Q^\dagger, Q\}$, and

$$Q^2 = 0 = (Q^\dagger)^2, \qquad [Q, H_s] = 0 = [Q^\dagger, H_s].$$

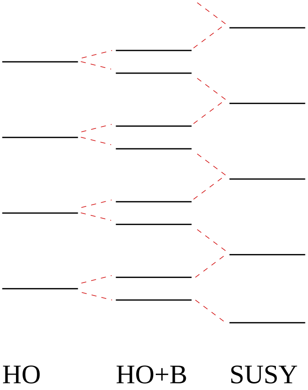
H_s is called the **supersymmetric Hamiltonian**, Q and $Q^\dagger$ are called **supersymmetry charges** and W is called the **superpotential**. Note that $Q = A_+\sigma_-/2$ where $\sigma_- = \sigma_1 - i\sigma_2$.

An example

Take $W(x) = \omega x\sqrt{m/2}$. Then

$$H_s = \begin{pmatrix} H_{ho} - \hbar\omega/2 & 0 \\ 0 & H_{ho} + \hbar\omega/2 \end{pmatrix}.$$

The state at $E = 0$ is a singlet. Every other state is a doublet. This Hamiltonian is obtained by taking a harmonic oscillator potential H_{ho}, and filling each of its levels with an electron. Each state is doubly degenerate because of the two spin states of the electron. Then if one switches on a small magnetic field, the degeneracy is broken. By tuning the field appropriately, one can send the lowest energy level to zero, and make the remaining states doubly degenerate again.



Other solvable potentials

1. The Morse potential is
$$V(x) = D\left\{\mathrm{e}^{-2x/a} - 2\mathrm{e}^{-x/a}\right\}.$$
2. The Rosen-Morse potential
$$V(x) = -U_0 \mathrm{sech}^2\frac{x}{a} + U_1 \tanh\frac{x}{a}.$$
3. The first Pöschl-Teller potential is
$$V(x) = \frac{\hbar^2}{2ma^2}\left\{\frac{\mu(\mu-1)}{\sin^2(x/a)} + \frac{\lambda(\lambda-1)}{\cos^2(x/a)}\right\}.$$
4. The second Pöschl-Teller potential
$$V(x) = \frac{\hbar^2}{2ma^2}\left\{\frac{\mu(\mu-1)}{\sinh^2(x/a)} + \frac{\lambda(\lambda-1)}{\cosh^2(x/a)}\right\}.$$
5. The supersymmetric partner of a given H_- is not unique; hence corresponding to each solvable potential, many others can be found.

Outline

A permutation symmetric matrix

Consider the matrix

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 4 \end{pmatrix}.$$

It has the property that permuting the first two rows and columns simultaneously does not change the matrix. Hence, $[M, P_{12}] = 0$ where P_{12} is the matrix that permutes the first two elements of a vector and leaves the third unchanged. Now P_{12} generates a Z_2 group. However, this is a subgroup of the group of permutations of 3 objects, a group called S_3. What are the other matrices in S_3 which commute with P_{12}?

Once all the real symmetric 3×3 matrices, S_i, which commute with P_{12}, are found, one could write $M = \sum_i m_i S_i$. Then diagonalizing the S_i and P_{12} simultaneously, one can diagonalize all matrices such as M.

Alternately, one takes the vector $(1, 1, 0)^T$. Since this is an eigenvector of P_{12}, it must also be of M. Since M is real symmetric, its eigenvectors are real and orthogonal. Hence $(1, -1, 0)^T$ and $(0, 0, 1)^T$ are two others.

A non-invertible matrix

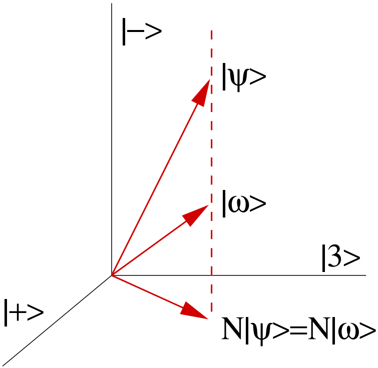
Consider the two matrices

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 4 \end{pmatrix}, \qquad N = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 4 \end{pmatrix}.$$

It seems that the matrix N has higher symmetry than M. However, the obvious symmetries of both are exactly the Z_2 group generated by P_{12}. Using the method used for M, we find that the three eigenvectors can be chosen as $|+\rangle = (1,1,0)^T$, $|-\rangle = (1,-1,0)^T$ and $|3\rangle = (0,0,1)^T$. Since $|+\rangle$ and $|3\rangle$ have equal eigenvalues, we have missed a symmetry.

$|-\rangle$ has eigenvalue zero. The vanishing of an eigenvalue is also clear from the fact that two rows of N are equal, so that $\mathrm{Det}\, N = 0$. Since any $|\psi\rangle = \psi_+|+\rangle + \psi_-|-\rangle + \psi_3|3\rangle$, clearly $|\phi\rangle = N|\psi\rangle = 4(\psi_+|+\rangle + \psi_3|3\rangle)$. Any component of $|-\rangle$ vanishes. Thus, any trial N^{-1} cannot decide how much of ψ_- to add back to $N^{-1}|\phi\rangle$. So there is no N^{-1}. For vectors with $\psi_- = 0$ there is a perfectly sensible inverse.

The geometry of non-invertible matrices



When the matrix is not invertible, many vectors project down to the same result. Given the projection, it is impossible to decide which vector it cam from. This is the same as saying that the set of simultaneous equations $N|\psi\rangle = |\phi\rangle$ cannot be solved. However, if one knows that the component $\psi_- = 0$, then the solution is straightforward.

Degenerate eigenvalues

If an $N \times N$ matrix, M, has degenerate eigenvalues corresponding to two choices of eigenvectors $|1\rangle$ and $|2\rangle$, then there is a permutation symmetry P_{12} between these. We have treated this as the discrete group Z_2. However, in this 2×2 space, $P_{12} = \sigma_1$, which is a Hermitean matrix. Hence $\exp(i\theta P_{12})$ is unitary. In fact, all matrices of this form constitute an Abelian group. This is the unitary group U(1).
Actually the symmetry group is even larger, since any unitary transformation of a vector in the space spanned by $|1\rangle$ and $|2\rangle$ commutes with M. This is the group $U(2)$. If the applications are to quantum mechanics, where overall phases are unimportant, then one needs only the subgroup $SU(2)$.
The cases of 3 or more degenerate eigenvalues are less common in general, but their analysis parallels this discussion.

Outline

References

1. Practical Quantum Mechanics, by S. Flugge. This book solves many of the older known solvable potential problems before supersymmetry was discovered. Take a look at the solutions of the Morse and Rosen-Morse potentials for later use.
2. Classical groups for Physicists, by B. G. Wybourne. This book is highly recommended for a good exposition on Lie groups.
3. The article by M. Harvey in the book Advances in Nuclear Physics, vol 1 (Plenum Press, New York) discussed the Elliott Model, which is an application of the SU(3) symmetry of the three dimensional harmonic oscillator to problems in nuclear physics.
4. A nice introduction to supersymmetric quantum mechanics is in the paper by R. Dutt, A. Khare and U. P. Sukhatme, "Supersymmetry, shape invariance and exactly solvable potentials", *American Journal of Physics*, **56** (1988), 163–168.