Some problems

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Using symmetry to solve matrix problems

Does the following matrix have unique eigenvectors?

$$\begin{pmatrix} 1 & 7 & 7 & 7 \\ 7 & 1 & 7 & 7 \\ 7 & 7 & 1 & 7 \\ 7 & 7 & 7 & 1 \end{pmatrix}.$$

Solution 1 Solve the quartic. It can be easily factorized. There are degenerate eigenvalues, so the eigenvectors are not unique.
Solution 2 $(1, 1, 1, 1)^T$ is an eigenvector. This has the eigenvalue 22. Other eigenvectors can be constructed by requiring them to be orthogonal to this and each other. The remaining eigenvalues are degenerate.
Solution 3 $M = 1 + 7C + 7C^2 + 7C^3$, where C represents cyclic permutations of 4 objects. Since $C^4 = 1$, it has eigenvalues ± 1 and $\pm i$. Hence, the eigenvalues of M are 22 and a triply degenerate eigenvalue -6.

Functions of operators

Compute the operator A such that

$$\exp A = \exp(ix\hat{\mathbf{r}})\exp(iq\hat{\mathbf{p}})$$

where $\hat{\mathbf{r}}$ is the position operator and $\hat{\mathbf{p}}$ is the momentum operator.
Solution: This is a straight application of the Baker Campbell Hausdorf formula.

Functional bases

A set of polynomials is generated by the recurrence relation $P_n(x) = xP_{n-1}(x) - P'_{n-1}(x)$, where $P_0(x) = 1$ and prime denotes differentiation. Are these polynomials orthogonal under the inner product rule given below?

$$\langle f|g\rangle = \int_{-\infty}^{\infty} dx \mathrm{e}^{-x^2} f(x)g(x).$$

Solution: For even (odd) n the polynomials are of even (odd) order, and hence even (odd) under the reflection $x \to -x$. Hence, all polynomials of odd order are orthogonal to those of even order (since the integrand is odd). We need to check this for two even order polynomials or two odd orders. Now $P_1(x) = x$ and $P_2(x) = x^2 - 1$. We have

$$\begin{aligned} \langle P_0|P_2\rangle &= 2\int_0^{-\infty} dx \mathrm{e}^{-x^2}(x^2-1) = \int_0^{-\infty} dy \mathrm{e}^{-y}(y^{1/2} - y^{-1/2}) \\ &= \Gamma\left(\frac{3}{2}\right) - \Gamma\left(\frac{1}{2}\right) \neq 0. \end{aligned}$$

Spin precession

A stream of Ag^+ ions with fixed momentum are polarized in a uniform magnetic field transverse to the direction of the momentum. They are then allowed to pass through a tube of fixed length containing a constant magnetic field pointing along the axis of the tube. What fraction of the atoms emerging from the tube retain their original polarization?

Solution: The initial field polarizes the ions such that their spins point perpendicular to the direction of momentum. When they enter the tube, their momentum is parallel to the magnetic field, so there is no acceleration on the ions. However, their spins are orthogonal to the direction of the field, and hence the spin will precess. The solution is an application of Rabi's formula.

Canonical commutation relations.

Find the commutator of the Hamiltonian for a one-dimensional harmonic oscillator with a generic translation operator in one dimension.
Solution: The Hamiltonian is $H = p^2/2m + m\omega^2 x^2/2$. The translation operator is $T = \exp(i\xi p/\hbar)$. Hence the commutator is

$$[H, T] = \frac{m\omega^2}{2}[x^2, T] = \frac{m\omega^2}{2}\{x^2 - Tx^2 T^{-1}\}T = -\frac{m\omega^2\xi}{2}(2x + \xi)T.$$

su(3)

How many real numbers are required to specify the most general 3×3 Hermitean matrix? Write down a set of traceless Hermitean matrices Σ_j, analogous to the Pauli matrices, such that any 3×3 Hermitean matrix, $H = a_0 + \sum_j a_j \Sigma_j$ for real a_0. How many of the Σ_j commute among themselves?

Solution: A 3×3 Hermitean matrix needs 3 real diagonal elements to be specified, along with 3 complex elements above the diagonal; hence a total of 9 complex numbers. If the trace is zero, then only two of the diagonal elements are independent. Hence there are exactly eight matrices Σ_j. Six of these are those which have values 1 or i in the 12, 13 and 23 places (everything else being zero). Two others are diagonal and traceless. Since only two are simultaneously diagonal, exactly two commute.

Counting

Write down the most general density matrix for a system consisting of two spin $1/2$ particles. Construct the density matrix (in the same basis) in the special case when the two-particle wavefunctions are projected into total spin 1.

Solution: The most general density matrix for two spin $1/2$ particles needs 3 real parameters for the three independent diagonal elements, and $2 \times 6 = 12$ real parameters for the off-diagonal elements (since the density matrix is the most general 4×4 Hermitean matrix with trace unity). When the two particle state is projected on to total spin unity, then one has the most general 3×3 Hermitean matrix with unit trace, *i.e.*, 8 reals. 6 real parameters.

The energy eigenvalues and eigenvectors

Find the 2×2 scattering matrix for waves of a fixed magnitude of the momentum, q, for the one-dimensional single step potential: $V(x) = V_0 > 0$ when $|x| \leq a$ and 0 otherwise. From this find the resonant values of q.
Solution: The answer can be read off from a textbook.

Benzene

A molecule consists of six atoms arranged in a hexagonal ring. Assume that the atom contains a single valence electron and the energy splitting between the filled and valence shells is large enough that the coupling between them can be neglected, and that one can also neglect the energy of interaction of the valence electrons with each other. The energy of the valence electron in each of the isolated atoms is E_0. In the molecule each valence electron lowers its energy by E_1 by hopping to either of its neighbours. What are the energies in the eigenstates of the Hamiltonian of the molecule for each valence electron?

Solution: The conditions on the problem ask us to investigate a single electron problem. One basis is that in which the electron is localized in the valence orbital of any one of the atoms. There are six such basis vectors, $|i\rangle$, where $i = 1, 2 \cdots 6$. The Hamiltonian has a diagonal part $\langle i|H|i\rangle = E_0$ and an off-diagonal part $\langle i|H|j\rangle = -E_1\delta_{i,j\pm1}$. In other words, $H = E_0 - E_1(C + C^5)$, where $C^6 = 1$. This can be solved in close analogy with the first problem.

The isotropic harmonic oscillator in three dimensions

Consider the energy level $E = 5\hbar\omega/2$ of an isotropic three dimensional harmonic operator. What is the degeneracy of this level? A basis for the eigenstates of the three dimensional oscillator is that of three independent harmonic oscillators along each of the three orthogonal directions, x, y and z. Another basis is that in which the angular momentum operator is diagonalized simultaneously with the Hamiltonian. For the energy level given above, find the unitary matrix which performs the transformation between the two bases.