

# Angular momentum

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Quantum Mechanics 1

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- 1 The algebra of angular momentum operators
- 2 Rotations
- 3 Properties of operators under rotations
- 4 References

# Outline

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- 2 Rotations
- 3 Properties of operators under rotations
- 4 References

# Angular momentum

- Classically the angular momentum of a particle is defined to be  $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ . The quantum operator is obtained by taking the operator expressions for  $\mathbf{r}$  and  $\mathbf{p}$  in this definition.
- $\hbar$  has dimensions of angular momentum, hence any expectation value of an angular momentum is a dimensionless number times  $\hbar$ .
- The quantum operator is unchanged if one takes the equivalent classical expression  $\mathbf{L} = -\mathbf{p} \times \mathbf{r}$ . **Check.**
- Classically  $\mathbf{r} \cdot \mathbf{p} = \mathbf{p} \cdot \mathbf{r}$ . **Check that the quantum operators corresponding to the two expressions are not equal.**
- The commutation relations of the components of  $\mathbf{L}$  follow from the definition above:  $[L_j, L_k] = i\hbar\epsilon_{jkl}L_l$ . The operator  $L^2$  commutes with all  $L_j$ . Hence,  $L^2$  and one of the  $L_j$  (usually  $L_z$ ) can be simultaneously diagonalized. (**check**)

# Raising and lowering operators

- Any set of three operators,  $\mathbf{J}$ , satisfying the algebra

$$[J_j, J_k] = i\hbar\epsilon_{jkl}J_l, \quad \text{and} \quad [J^2, J_j] = 0 \quad \text{where} \quad J^2 = J_x^2 + J_y^2 + J_z^2,$$

can be identified with angular momentum operators.

- Introduce the raising and lowering operators

$$J_+ = J_x + iJ_y \quad \text{and} \quad J_- = J_x - iJ_y.$$

- They have the properties

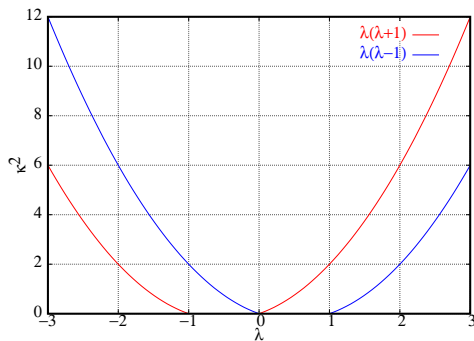
$$J_+J_- = J^2 - J_z^2 + \hbar J_z \quad \text{and} \quad J_-J_+ = J^2 - J_z^2 - \hbar J_z.$$

- The new operators have the commutation relations

$$[J_+, J_-] = 2\hbar J_z, \quad [J^2, J_{\pm}] = 0, \quad [J_z, J_{\pm}] = \pm\hbar J_{\pm}.$$

# Limits on eigenvalues

- The eigenvalues of  $J^2$  must be positive. Assume that  $|\psi\rangle$  is a simultaneous eigenvector of  $J^2$  and  $J_z$  with eigenvalues  $\kappa^2\hbar^2$  and  $m\hbar$  respectively.
- If  $|\psi\pm\rangle = J_{\pm}|\psi\rangle$ , then using the fact that the square of the norm of a vector is positive, one can show that  $\kappa^2 - m(m\pm 1) \geq 0$ .



# The representations are finite dimensional

- For a fixed  $\kappa$ , when  $m$  satisfies the relation  $m(m+1) = \kappa^2$ , then the norm of  $|\psi-\rangle$  vanishes. As a result,  $|\psi-\rangle = 0$ .
- Similarly, for a fixed  $\kappa$ , when  $m$  satisfies the relation  $m(m-1) = \kappa^2$ , then  $|\psi+\rangle = 0$ .
- Since  $J_{\pm}$  commute with  $J^2$ , the vectors  $|\psi\pm\rangle$  are also eigenvectors of  $J^2$  with eigenvalues  $\kappa^2\hbar^2$ . However,

$$J_z J_{\pm} |\psi\rangle = (\pm\hbar J_{\pm} + J_{\pm} J_z) |\psi\rangle = \hbar(m \pm 1) J_{\pm} |\psi\rangle,$$

so  $|\psi\pm\rangle$  are eigenvectors of  $J_z$  with eigenvalues  $(m \pm 1)\hbar$ .

- It follows that, for every  $\kappa$ , the range of possible values of  $m$  goes between the lowest and highest allowed values in steps of unity. Therefore the number of states is finite for each  $\kappa$ ,

# The eigenvalues

- For a given  $\kappa$  the lowest  $m$  is

$$m = \frac{1}{2} - \sqrt{\kappa^2 + \frac{1}{4}},$$

and the highest is obtained by flipping the signs. The difference must be an integer, since one goes from one to the other by the action of raising and lowering operators.

- As a result,

$$\kappa^2 = \frac{1}{4}(n^2 - 1) = \frac{1}{4}(n+1)(n-1) = j(j+1),$$

where  $n \geq 1$  is an integer, and  $j = n/2 - 1/2$  is integer for  $n$  odd and half integer for  $n$  even. Hence the eigenvalues of  $J^2$  are  $j(j+1)\hbar^2$  for  $j = 0, 1/2, 1, 3/2, \text{etc.}$

- The values of  $m$  therefore lie in the range  $-j$  to  $j$  and are separated by 1. The size of the representation space is  $2j+1$ . The eigenstates can be labelled  $|j, m\rangle$ .



# The representations

- The state  $|j, m\rangle$  is defined as that which diagonalizes  $J^2$  and  $J_z$ ; i.e.,  

$$J^2|j, m\rangle = j(j+1)\hbar^2|j, m\rangle, \quad \text{and} \quad J_z|j, m\rangle = m\hbar|j, m\rangle.$$

An overall phase factor still remains to be specified.

- The action of the ladder operators is

$$J_+|j, m\rangle = \hbar\sqrt{j(j+1) - m(m+1)}|j, m+1\rangle,$$

$$J_-|j, m\rangle = \hbar\sqrt{j(j+1) - m(m-1)}|j, m-1\rangle.$$

- If an operator,  $A$ , commutes with  $J^2$ , then it is block diagonalized in the basis  $|j, m\rangle$ , i.e., it can be put into the form

$$\begin{pmatrix} \square & & & & \\ & \square & & & \\ & & \square & & \\ & & & \square & \\ & & & & \square \end{pmatrix}$$

## Two representations

- The representation with  $j = 0$  is called the scalar representation. This has dimension 1, *i.e.*, all the angular momentum operators are represented by scalars.  $J_x$ ,  $J_y$  and  $J_z$  act on the eigenstates by multiplying them by zero. The eigenstate is  $|0, 0\rangle$ .
- The representation with  $j = 1/2$  is called the spinor representation. All components of angular momentum are represented by  $2 \times 2$  Hermitean matrices, and hence must be proportional to the Pauli matrices. Since  $J_z$  must be diagonal and the eigenvalues have to be  $\pm\hbar/2$ , one can write  $J_k = \hbar\sigma_k/2$ . Then  $J^2 = 3\hbar^2/4$ , which is exactly what is expected. With this choice one has

$$\left| \frac{1}{2}, \frac{1}{2} \right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \left| \frac{1}{2}, -\frac{1}{2} \right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

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- 1 The algebra of angular momentum operators
- 2 Rotations**
- 3 Properties of operators under rotations
- 4 References

# Translations and unitary transformations

- Since the components of the spatial momenta,  $\mathbf{p}$ , commute with each other, all the components can be simultaneously diagonalized. The eigenvectors,  $|\mathbf{k}\rangle \equiv |k_x, k_y, k_z\rangle$ , are labelled by the eigenvalues,  $\mathbf{k}$ , of these operators. Each such eigenvector gives a one dimensional representation of the momentum operators.
- Translations in space act on the eigenvectors of  $\mathbf{p}$  through unitary operators,  $T(\mathbf{x}) = \exp(i\mathbf{x} \cdot \mathbf{p}/\hbar)$ . Since the components of  $\mathbf{p}$  commute, so do the translation operators (**check**). In other words, translations form an Abelian group.
- Translations act on the space of each  $|\mathbf{k}\rangle$  through a multiplication by a single complex number (a  $1 \times 1$  unitary matrix),  $\exp(i\mathbf{x} \cdot \mathbf{k}/\hbar)$ . That is,  $T(\mathbf{x})|\mathbf{k}\rangle = \exp(i\mathbf{x} \cdot \mathbf{k}/\hbar)|\mathbf{k}\rangle$ .

# Rotations and unitary transformations

- Any rotation is specified by an axis of rotation,  $\hat{\mathbf{n}}$ , and the angle of rotation,  $\psi$ , about this axis.
- The result of a rotation in space is an unitary transformation on the eigenvectors of the angular momentum operator. The rotation operator is given by  $R(\hat{\mathbf{n}}, \psi) = \exp(-i\psi \mathbf{J} \cdot \hat{\mathbf{n}}/\hbar)$ .
- When  $\hat{\mathbf{n}}$  is the  $z$  axis, then,  $R(\hat{\mathbf{z}}, \psi) = \exp(-i\psi J_z/\hbar)$ . In this case,  $R(\hat{\mathbf{z}}, \psi)|j, m\rangle = \exp(-im\psi)|j, m\rangle$ .
- In general, however, acting on the  $(2j + 1)$ -dimensional eigenbasis  $|j, m\rangle$  (for  $-j \leq m \leq j$ ), the rotation operators mix the basis. This is a consequence of the fact that the  $z$  axis is not (in general) fixed under rotations, and hence the axis used to define  $m$  changes.
- In the  $j = 0$  representation, since all  $J_k = 0$ , the rotation operators  $R(\hat{\mathbf{n}}, \psi) = 1$ . Thus, the eigenfunction is completely unchanged under rotations. This is the reason why this is called the scalar representation.

# Rotation matrices in the spinor representation

In the  $j = 1/2$  representation (also called the spinor) one has

$$R(\hat{\mathbf{n}}, \psi) = e^{-i\sigma \cdot \hat{\mathbf{n}}\psi/2}.$$

In terms of the components of  $\hat{\mathbf{n}}$  in spherical coordinates in the original frame, before the rotation, one has

$$\sigma \cdot \hat{\mathbf{n}} = \begin{pmatrix} \cos \theta & \sin \theta e^{i\phi} \\ \sin \theta e^{-i\phi} & \cos \theta \end{pmatrix}.$$

If this is diagonalized by an unitary transformation  $V$ , then ([check](#))

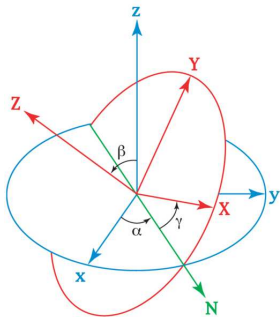
$$R(\hat{\mathbf{n}}, \psi) = V \begin{pmatrix} e^{-i\psi/2} & 0 \\ 0 & e^{i\psi/2} \end{pmatrix} V^\dagger = \begin{pmatrix} \cos \frac{\psi}{2} & i \sin \frac{\psi}{2} \\ i \sin \frac{\psi}{2} & \cos \frac{\psi}{2} \end{pmatrix}.$$

Note that when  $\psi = 4\pi$ , then  $R = 1$ , and the rotation has no effect. However, when  $\psi = 2\pi$ , the rotated wavefunctions are not the same as the original wavefunctions. This is non-classical behaviour.

# Euler angles

The specification of rotations used three angles— two to define the axis of rotation and one to specify the angle of rotation. Other conventions for defining the rotation are due to Euler.

The fixed system is  $xyz$ ; the rotated system is  $XYZ$ .



- ❶ Rotate about the  $z$  axis by angle  $\alpha$ .
- ❷ Rotate about the new  $x$  axis by angle  $\beta$ .
- ❸ Rotate about the new  $z$  axis by angle  $\gamma$ .

Other definitions of the Euler angles are possible. This definition (called the  $zxz$  convention) is used by Goldstein and Landau and Lifschitz.

# Wigner's D matrices

Using Euler's angles in the zyz convention, a rotation operator can be written as

$$R(\alpha, \beta, \gamma) = e^{-i\alpha J_z/\hbar} e^{-i\beta J_y/\hbar} e^{-i\gamma J_z/\hbar}.$$

Clearly, these rotation matrices commute with  $J^2$  and hence can be evaluated for each  $j$  separately. Wigner's D matrices are

$$D_{mm'}^j(\alpha, \beta, \gamma) = \langle jm' | R(\alpha, \beta, \gamma) | jm \rangle = e^{-i(m'\alpha + m\gamma)} d_{mm'}^j(\beta).$$

We introduced the matrix elements of the rotation about the x-axis through Wigner's d matrices

$$d_{mm'}^j(\beta) = \langle jm' | \exp(-i\beta J_y/\hbar) | jm \rangle.$$

In this convention the  $d$  matrices are purely real.



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# Scalar operators

A **scalar operator** under rotations is one which remains unchanged under rotations, *i.e.*,

$$S = R(\hat{n}, \psi) S R^\dagger(\hat{n}, \psi).$$

Taking the rotation angle to be infinitesimal, and expanding the rotation operator to leading order in the Taylor expansion in the angle, one finds that  $[S, J_k] = 0$ . This means that scalar operators can be diagonalized together with  $J^2$  and  $J_z$ , and the matrix elements are independent of the eigenvalue of  $J_z$ . In other words,

$$\langle j, m | S | j', m' \rangle = \delta_{jj'} \delta_{mm'} (j | S | j),$$

where the **reduced matrix element**  $(j | S | j)$  is a function only of  $j$ . Thus, for each  $j$ , all scalar operators are multiples of identity. This is a special case of the **Wigner-Eckart theorem**.

Examples of scalar operators are  $J^2$ ,  $r^2$ ,  $p^2$  and  $\mathbf{r} \cdot \mathbf{p}$ . Hamiltonians with central potentials,  $V(|r|)$ , are scalar.

# Vector operators

One recognizes  $\mathbf{J}$  as a typical (axial) vector. Hence one expects a **vector operator** to have the same algebraic property. Each vector operator must have three components, and the three must have the same properties under rotations as  $\mathbf{J}$ . However, as we discovered already, this is a matter of arranging the right commutation relations.

By definition then, vector operators have the property that  $[V_j, J_k] = i\hbar\epsilon_{jkl}V_l$ . This implies that under infinitesimal rotations vector operators transform to

$$V'_j = V_j - \frac{\psi}{\hbar}\epsilon_{jkl}\hat{n}_k V_l.$$

Examples of vector operators are  $\mathbf{J}$ ,  $\mathbf{r}$  and  $\mathbf{p}$ .

Vector operators neither commute with  $J_z$  (by definition) nor with  $J^2$  (**check**). As a result they can neither be diagonalized, nor block diagonalized, in the  $|j, m\rangle$  basis. In other words, one has  $\langle j', m' | \mathbf{V} | j, m \rangle \neq 0$ . Purely geometric considerations (called the **Wigner Eckart theorem**) allows us to reduce such matrix elements to a simpler form.

# Raising and lowering in $j$

For a vector operator  $\mathbf{V}$ , introduce  $V_{\pm} = V_1 \pm iV_2$ . Then,

$$[J_z, V_{\pm}] = \pm \hbar V_{\pm}, \quad \text{so} \quad J_z V_{\pm} |j, \pm j\rangle = \pm \hbar(j+1) V_{\pm} |j, \pm j\rangle.$$

One can check that  $[J^2, \mathbf{V}] = 2\hbar^2 \mathbf{V} + 2i\hbar \mathbf{V} \times \mathbf{J}$ . As a result, one finds

$$[J^2, V_{\pm}] = 2\hbar^2 V_{\pm} + 2\hbar(V_{\pm} J_z - V_z J_{\pm}).$$

This implies that

$$J^2 V_{\pm} |j, \pm j\rangle = \hbar^2 [j(j+1) + 2 + 2j] V_{\pm} |j, \pm j\rangle = \hbar^2 (j+1)(j+2) V_{\pm} |j, \pm j\rangle.$$

Together, these two conditions imply that  $V_{\pm} |j, \pm j\rangle \propto |j+1, \pm(j+1)\rangle$ .

The Wigner-Eckart theorem concerns the constant of proportionality.

All irreps of  $SU(2)$  can be constructed from the scalar and spinor irreps by the action of  $V_{\pm}$  and  $J_{\pm}$ .

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# References

- ❶ Quantum Mechanics (Non-relativistic theory), by L. D. Landau and E. M. Lifschitz. The material in this lecture are scattered through chapters 3 and 15 of this book. The later chapter deals with the topic of Landau levels.
- ❷ Quantum Mechanics (Vol 1), C. Cohen-Tannoudji, B. Diu and F. Laloë. Chapter 6 of this book discusses angular momentum. The presentation in these lectures follow this chapter sometimes.
- ❸ Quantum Mechanics (Vol 2), C. Cohen-Tannoudji, B. Diu and F. Laloë. Chapters 9 and 10 of this book discuss angular momentum.
- ❹ Classical groups for Physicists, by B. G. Wybourne. This book is highly recommended for a good exposition on Lie groups.