Representations of angular momentum

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Kinds of representations

For any set of Hermitean operators, H_i, consider the algebra $[H_i, H_j] = f_{ijk} H_k$. Using these H_i, we construct a maximum set of commuting operators. Given any matrix representation of the H_i, we find the basis which diagonalizes the maximum commuting set. In this basis the remaining operators are block-diagonal. As a result, exponentials such as $U = \exp(iu_j H_j)$ are also block diagonal.

A matrix representation of all the H_i which cannot all be reduced to smaller blocks is called an **irreducible representation**. All other representations can be reduced to smaller blocks by unitary transformations and are therefore called **reducible representations**.

Example: In the **scalar** (or trivial) representation, we can set all $H_i = 0$. The group generated by exponentiating, $U = \exp(iu_j H_j)$, is then represented by $U = 1$ for all U. The trivial representation of any group is an irreducible representation of any group.

Example: The representation of **J** by the Pauli matrices gives rise to an irreducible representation of the group of rotations.

Building things up and breaking them down

We will build representations of larger j through direct products (also called tensor products) of lower representations. A direct product of two matrices N and M is the matrix

$$N \otimes M = \begin{pmatrix} n_{11}M & n_{12}M & n_{13}M & \cdots \\ n_{21}M & n_{22}M & n_{23}M & \cdots \\ n_{31}M & n_{32}M & n_{33}M & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{pmatrix}.$$

The dimension of $N \otimes M$ is the product of the dimensions of each matrix. In general, $N \otimes M \neq M \otimes N$. Direct products of vectors follow from this definition. A direct sum of two matrices $N \oplus M$ is the block diagonal form

$$N \oplus M = \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix}.$$

The dimension of the direct sum is the sum of the dimensions of each matrix. We will now try to reduce direct products into direct sums.

Outline

Summing two momenta

If $|\mathbf{k}_1\rangle$ is the basis state of one particle and $|\mathbf{k}_2\rangle$ of another, then the direct product state $|\mathbf{k}_1;\mathbf{k}_2\rangle = |\mathbf{k}_1\rangle \otimes |\mathbf{k}_2\rangle$. The operator $\mathbf{p}_1 = \mathbf{p} \otimes 1$ acts only on the Hilbert state of the first particle, and the operator $\mathbf{p}_2 = 1 \times \mathbf{p}$ on the second. These operators commute since they act on different Hilbert spaces. The total momentum $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$.

Since all representations which we have built are one-dimensional, the direct product state is also one dimensional. One has

$$\mathbf{P}|\mathbf{k}_1;\mathbf{k}_2\rangle = (\mathbf{p}_1 + \mathbf{p}_2)\,|\mathbf{k}_1;\mathbf{k}_2\rangle = (\mathbf{k}_1 + \mathbf{k}_2)\,|\mathbf{k}_1;\mathbf{k}_2\rangle.$$

Therefore, the direct product state is the representation with momentum equal to the sum of the two momenta:

$$|\mathbf{k}_1\rangle \otimes |\mathbf{k}_2\rangle = |\mathbf{k}_1 + \mathbf{k}_2\rangle.$$

This is a fairly trivial example of direct product spaces. The case of direct products of angular momentum states is significantly different.

Outline

Summing two spins: counting dimensions

If $|j_1, m_1\rangle$ is the basis states of one particle, and $|j_2, m_2\rangle$ of another, then the direct product $|j_1, m_1\rangle \otimes |j_2, m_2\rangle = |j_1, m_1; j_2, m_2\rangle$. The operator $\mathbf{J}^{(1)} = \mathbf{j} \otimes 1$, *i.e.*, the operator for the first particle acts only on the Hilbert space of the first particle. Similarly, $\mathbf{J}^{(2)} = 1 \otimes \mathbf{j}$. All components of these two operators commute, since they act on different spaces. The total angular momentum of the system is $\mathbf{J} = \mathbf{J}^{(1)} + \mathbf{J}^{(2)}$. But

$$J^2|j_1, m_1; j_2, m_2\rangle \neq (j_1 + j_2)(j_1 + j_2 + 1)\hbar^2|j_1, m_1; j_2, m_2\rangle.$$

This is because the dimension of the direct product is $(2j_1 + 1)(2j_2 + 1)$ and this is not equal to $(2j_1 + 2j_2 + 1)$ unless either j_1 or j_2 (or both) is zero.

Example: The direct product of two $j = 1/2$ particles has dimension 4. This is either a $j = 3/2$ representation (which has dimension 4) or a direct sum of a $j = 0$ (dimension 1) and a $j = 1$ (dimension 3) representation. If the direct product can be reduced to a direct sum, then **all** components of $\mathbf{J}$ can be block diagonalized in this fashion.

Summing two spins: the spectrum of J_z

By the definition of the direct product, one has

$$J_z^1 = \begin{pmatrix} \hbar m_1 I & 0 & 0 \cdots & \\ 0 & \hbar m_2 I & 0 & \cdots \\ 0 & 0 & \hbar m_3 I & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{pmatrix}, \qquad J_z^2 = \begin{pmatrix} j_z & 0 & 0 \cdots & \\ 0 & j_z & 0 & \cdots \\ 0 & 0 & j_z & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{pmatrix}.$$

This remains diagonal. As a result, the quantum number M corresponding to the total J_z is the sum $m_1 + m_2$. In other words

$$J_z|j_1, m_1; j_2, m_2\rangle = (m_1 + m_2)\hbar|j_1, m_1; j_2, m_2\rangle.$$

Example: For the direct product of two $j = 1/2$ particles, the possible values of M are 1, -1, and 0 (twice). As a result, this direct product cannot be the representation $j = 3/2$. Therefore

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1.$$

A problem: summing two spins

Using the definition of the direct product for $|1/2, m_1; 1/2, m_2\rangle$, one has

$$J_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}, \; J_+ = \frac{\hbar}{2}\begin{pmatrix}0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}, \; J_- = \frac{\hbar}{2}\begin{pmatrix}0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0\end{pmatrix}.$$

Use this to find J^2 and check how to diagonalize it while keeping J_z diagonal. Using these results, show that

$$\begin{aligned}|1,0\rangle &= \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2}\right\rangle + \left|\frac{1}{2},-\frac{1}{2};\frac{1}{2},\frac{1}{2}\right\rangle\right),\\ |0,0\rangle &= \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2}\right\rangle - \left|\frac{1}{2},-\frac{1}{2};\frac{1}{2},\frac{1}{2}\right\rangle\right).\end{aligned}$$

Find the matrices corresponding to $\mathbf{J}$ in the $j = 1$ representation.

Summing two spins: the Clebsch-Gordan series

By the argument just presented, the states $|j, m; 1/2, m'\rangle$ can have total M ranging from $(j+1/2)$ to $-(j+1/2)$. The extreme eigenvalues are single, every other eigenvalue occurs twice. As a result,

$$j \otimes \frac{1}{2} = \left(j + \frac{1}{2}\right) \oplus \left(j - \frac{1}{2}\right).$$

By an inductive argument one can prove that the direct product states $|j_1, m_1; j_2, m_2\rangle$ can be decomposed as

$$j_1 \otimes j_2 = (j_1 + j_2) \oplus (j_1 + j_2 - 1) \oplus \cdots \oplus (j_1 - j_2),$$

where each J occurs only once. The reduction of a direct product to direct sums of terms is called the **Clebsch-Gordan series**. In the CG series any value of $M = m_1 + m_2$, except the extremes, are degenerate. An unitary transformation among the various $|j_1, m_1; j_2, M - m_1\rangle$ is required to produce the angular momentum eigenstates $|J, M\rangle$. The unitary matrix is $|J, M\rangle\langle j_1, m_1; j_2, m_2|$. The matrix elements are called **Clebsch-Gordan coefficients**.

Examples of Clebsch-Gordan coefficients

1. The trivial CG coefficients are

$$\langle j_1 + j_2, j_1 + j_2 | j_1, j_1; j_2, j_2 \rangle = 1.$$

One can in general write this as $\exp(i\psi)$ for some real ψ. The choice of ψ has to be compatible with the phase choices for the angular momentum eigenstates.

2. In the problem of the coupling of two spin 1/2 particles, the unitary transformation that rotates from the eigenbasis of the two uncoupled spins to the eigenbasis of the coupled spins is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

The CG coefficients $\langle 1, 1 | 1/2, 1/2; 1/2, 1/2 \rangle$, $\langle 1, -1 | 1/2, -1/2; 1/2, -1/2 \rangle$, $\langle 1, 0 | 1/2, m; 1/2, -m \rangle$, $\langle 0, 0 | 1/2, m; 1/2, -m \rangle$ can be read off this matrix.

Outline

References

1. Quantum Mechanics (Non-relativistic theory), by L. D. Landau and E. M. Lifschitz. The material in this lecture are scattered through chapters 4, 8 and 14 of this book.
2. Quantum Mechanics (Vol 1), C. Cohen-Tannoudji, B. Diu and F. Laloë. Chapter 6 of this book discusses angular momentum. The presentation in these lectures follow this chapter sometimes.
3. Quantum Mechanics (Vol 2), C. Cohen-Tannoudji, B. Diu and F. Laloë. Chapters 9 and 10 of this book discuss angular momentum.
4. Classical groups for Physicists, by B. G. Wybourne. This book is highly recommended for a good exposition on Lie groups.
5. A Handbook of Mathematical Functions, by M. Abramowicz and I. A. Stegun. This is a handy place to look up useful things about various classes of functions.