Interference, vector spaces

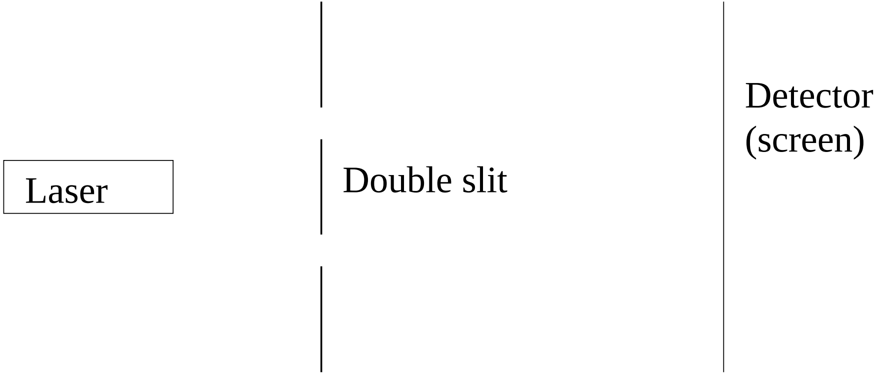
Sourendu Gupta

TIFR Graduate School

Quantum Mechanics 1
August 6, 2008

Outline

Wave-particle unity

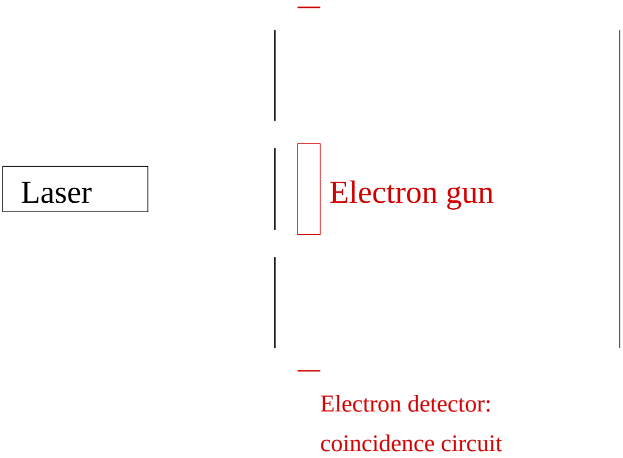


Use a laser which emits one photon at a time. Use the screen as a detector.

Wave-particle unity

- The laser emits one photon at a time. Each photon appears as a single pointlike flash when it arrives at the screen. This is exactly what you expect of a particle.
- When you plot the positions of many such flashes, then the density of points is governed by the interference fringes. The variation of the intensity through interference is exactly what you expect of a wave.
- A quantum particle is both a wave and a particle: a wave when it travels through the slit, a particle when seen by the detector.
- The state of a quantum particle must have a wave description. This is one complex number at each point of space (**Schrödinger**). The square of the modulus of the wave is proportional to the probability of finding the particle at a point (**Born**).

Which-slit experiment



Electrons scatter from photons. Back-to back electrons, if no photon. One of the electrons missing if photon passes through one of the slits.

Phase coherence

- Wave function of photon passing through top slit: $\Psi_1(x)$.
- Wave function of photon passing through bottom slit: $\Psi_2(x)$.
- Full wave function: $\Psi(x) = \Psi_1(x) + \Psi_2(x)$.
- Fixed phase relationship between the two needed for interference, *i.e.*, if $\Psi_2(x) = \Psi_1(x)A(x)\mathrm{e}^{i\phi(x)}$, then

$$|\Psi(x)|^2 = |\Psi_1(x)|^2 \times \left\{1 + A(x)^2 + 2A(x)\cos\phi(x)\right\}.$$

  Usual case of interference: $\phi(x) = k \times \mathrm{path\ difference}(x)$.
- If phase relationship randomized, then interference disappears.

Quantum cool

(Scully and Drühl, 1982) Quantum eraser.

1. What happens if we collect information about "which path" but never use it?
2. What happens if we collect information about "which path" but erase the information before looking at it?
3. What happens if we collect information about "which path" but look at it long after the photons have reached the screen?

(Schrödinger, 1935) Schrödinger's cat.
Macroscopic quantum states, entanglement.
(Bell, 1964) Locality vs reality.
Bell's inequalities: quantum entanglement is non-local but cannot be used to send faster-than-light messages.

Outline

Vectors

Take vectors such as $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. Let a, b, c, *etc.*, be scalars. Then, the following are true—

1. A vector multiplied by a scalar is a vector: $a\mathbf{x} = (ax_1, ax_2, ax_3)$.
2. Two vectors can be added to give a new vector: $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$. Vector addition is commutative: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$, and associative: $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$. Multiplication by scalars is distributive over vector addition: $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$. Scalar addition is distributive over multiplication by a vector. Scalar multiplication is compatible with multiplication with a vector.
3. There exists a zero vector: $\mathbf{0}$ such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for any vector $\mathbf{x}$. There exists a negative of every vector.
4. There is an inner product (dot product): $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + x_3y_3$.

The number of linearly independent vectors is the dimension of the space, D. An basis is a choice of D linearly independent vectors, in terms of which one can expand any vector.

Square integrable functions are vectors!

Take functions $f(x)$, $g(x)$, *etc.*, such that

$$\int dx|f(x)|^2 \qquad \text{and} \qquad \int dx|g(x)|^2$$

exist. Let a, b, c, *etc.*, be constrants. Then, the following are true—

1. There is an inner product: $\int dx f(x) g^*(x)$. (**Prove this**)
2. $af(x)$ is square integrable.
3. $f(x) + g(x)$ is square integrable. The sum is clearly commutative and associative. Multiplication by a constant is distributive over the addition of the two functions; scalar addition is distributive over multiplication by a functions; scalar multiplication is compatible with multiplication by a function.
4. There exists a zero function. Every function has a negative.

Coordinates: linear independence and bases

Take a collection of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots, \mathbf{x}_N\}$. such that it is impossible to find any set of N numbers which satisfy the vector equation:

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_N\mathbf{x}_N = 0.$$

Then, the collection is called **linearly independent**.
The dimension of the vector space is the largest size of a linearly indpendent collection of vectors. Start from any vector $\mathbf{x}_1$. Find a second vector which is linearly independent of this one. If you cannot find one, then the vector space is one-dimensional. If you find one then you have the set $\{\mathbf{x}_1, \mathbf{x}_2\}$. Find a third vector which is linearly indpendent of these two. If you cannot find one, then the space is 2 dimensional. Otherwise you have the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, *etc.*. Any set of D linearly independent vectors in a vector space of dimension D is called a **basis**.

Components

- Given a basis, every vector not in this basis is linearly dependent on the basis vectors. Hence one has an unique decomposition
$$\mathbf{v} = v_1\mathbf{x}_1 + v_2\mathbf{x}_2 + \cdots v_D\mathbf{x}_D.$$
The numbers v_i are the components of the vector in this basis.
- The basis is **orthonormal** if $\mathbf{x}_i \cdot \mathbf{x}_j = \delta_{ij}$. (The Kronecker delta, δ_{ij}, is 1 if $i = j$ and zero otherwise).
- In an orthonormal basis the components of a vector are obtained by taking a dot product with the basis vectors: $v_i = \mathbf{v} \cdot \mathbf{x}_i$.
- A change from one orthonormal basis to another is performed by taking linear combinations of the components. This linear combination is specified by an orthogonal matrix. In other words: $\mathbf{v}' = M\mathbf{v}$, where $\mathbf{v}'$ is the same as $\mathbf{v}$ but in the new basis. Since $\mathbf{v} \cdot \mathbf{v} = 1$, and the length of the vector is not changed by the choice of basis, one must have $M^T M = 1$ (M^T is the transpose of the matrix M).

The Fourier basis

- The vector space of square integrable functions is infinite dimensional. The recursive procedure for constructing a linearly independent set does not end in any number of steps. (**Prove this**).
- The plane waves, $\phi_k(x) = \exp(ikx)$, provide a basis (almost) on these functions. They are orthogonal under the inner product defined before. However, they are not square integrable. We will assume that this problem can be cured (the cure will come later) and continue. (**Cure this**)
- Each square integrable function can be completely specified by its Fourier coefficients: $\widetilde{f}(k) = f(x) \cdot \phi_k(x)$, under the inner product defined earlier. The standard representation as an infinite component is the Fourier transform: $f(x) \equiv \widetilde{f}(k)$.

Outline

References

1. The Feynman Lectures in Physics (Vol 3), by R. P. Feynman *et al*. The material in this (and the previous) lecture correspond roughly to the first chapter of this book.
2. Quantum Mechanics (Non-relativistic theory), by L. D. Landau and E. M. Lifschitz. The material in this (and the previous) lecture roughly correspond to a few sections of the first chapter of this book. first chapter of this book.
3. Mathematical Methods for Physicists, by G. Arfken. This book contains chapters on matrices and Fourier transforms which will be useful throughout this course.
4. The quantum eraser is discussed in an article in the April 2007 issue of Scientific American.