Units and dimensions

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The meaning of dimensionless constants

Dimensionless constants relate the scales of different phenomena. The discovery of a dimensionless constant in the analysis of a physical system is usually a step towards the discovery of new phenomena.

The most important is Avogadro's number: $N_A \simeq 6.022 \times 10^{23}$. It relates the fundamental scale of molecular phenomena to "accidental" human scales. For example, the proton mass is essentially

$$m_p = \frac{1 \text{ gm}}{N_A} = 1.67 \times 10^{-27} \text{ Kg}.$$

Many of the dimensionless constants discovered in the 18th and 19th centuries are of this kind. All of them can be simply rewritten in terms of N_A.

The insight of Gauss

Coulomb's law of attraction between charges e_1 and e_2 placed a distance r apart is

$$\mathbf{F} = \kappa_C \frac{e_1 e_2}{r^3} \mathbf{r}.$$

An interesting question is whether one needs to introduce two quantities, κ_C and the charge, or just one quantity. Gaussian units prevent a proliferation of dimensions by making κ_C into a dimensionless quantity: unity. Then $[e] = M^{1/2} L^{3/2} T^{-1}$. Gauss' insight is that any non-trivial κ_C can be absorbed into the definition of charge.

Solution by committee

The solution by committee is called SI (Systeme Internationale). This introduces a new unit, the Ampere, with $[e] = TA$. It also forces us to write $\kappa_C = (4\pi\epsilon_0)^{-1}$, with $[\epsilon_0] = M^{-1} L^{-3} T^4 A^2$.

Gaussian theory of dimensional constants

If universal dimensional constants involve distinct physical quantities, then they correct our knowledge of basic physics.

1. The Joule's constant relates units of energy and heat. It tells us that these two notions are the same.
2. The speed of light in vacuum relates time and space. It says that these two quantities are not different. It also relates energy and mass.
3. Planck's constant relates energy and frequency (time). It tells us that energy is the same as frequency.

There are other fundamental dimensional constants, for example, the charge of the electron and its mass, which defy current understanding in terms of more basic phenomena

$$m_e = 9.109 \times 10^{-31}\ \mathrm{Kg}, \qquad e = 1.602 \times 10^{-19}\ \mathrm{C}.$$

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Appropriate units

Natural units are those in which the dimensional constants are unity. Take Planck's relation in natural units,

$$E = \hbar\omega, \qquad \hbar = 1.054589 \times 10^{-34}\ \mathrm{J/Hz} = 1.$$

Units which are more appropriate to molecular physics are $1\ \mathrm{eV} = 1.602 \times 10^{-19}$ J, and $1\ \mathrm{nm} = 10^{-9}$ m. As a result, in these units

$$1 = \hbar = 6.58 \times 10^{-16}\ \mathrm{eV/Hz}.$$

In applications to the thermodynamics of collections of molecules, it is also natural to set

$$k_B = 11 \times 10^3 \mathrm{K/ev} = 1.$$

Quantum mechanics and natural units

When $\hbar = 1$, then $[E] = T^{-1}$. As a result, angular momentum no longer has dimensions $[J] = ML^2T^{-1}$, but becomes dimensionless. Interestingly, this connects to electrodynamics, by converting the electric charge into a velocity $[e^2] = LT^{-1}$. Also, $[\mathbf{E}^2] = [\mathbf{B}^2] = L^{-3}T^{-1}$.

These last relations become simpler when relativity and quantum mechanics are combined. Then, we can set

$$c = 2.998 \times 10^8 \text{ m Hz} = 1.$$

The choice $\hbar = c = 1$ defines natural units. In this, mass and energy have the same dimensions and length and time have the inverse dimensions: $[\ell] = M^{-1}$. As a result, e^2 is dimensionless. This is an important simplification for the quantum mechanics of photons.

The Gaussian route to Black-body radiation

Black body radiation is the thermodynamics of photons. In a volume filled with electromagnetic radiation fields at a fixed external temperature Θ, the appropriate scale of length, time, and energy are set by Θ.

The variables of interest are the frequency of photons, ω, and the spectrum, U, *i.e.*, the energy density per unit range of ω. Clearly $[U] = M^3$, so the dimensionless variables are U/ω^3 and Θ/ω, and

$$U/\omega^3 = f(\omega/\Theta).$$

This is subject to the Stefan-Boltzmann law for the energy density;

$$\mathcal{U} = \int_0^\infty d\omega U(\omega,\Theta) = \sigma\Theta^4, \qquad \text{so} \qquad \int_0^\infty x^3 f(x) dx = \sigma.$$

where the dimensionless constant, σ, is obtained from experiment. Planck, and, later, Bose could derive $f(x)$, and hence σ, by a simple argument.

Classical black body radiation: UV catastrophe

In classical physics, where $\hbar = 0$, natural units cannot be used. However, we can still set $c = 1$ and do the dimension counting $[\Theta] = M$, $[\omega] = L^{-1}$ and $[U] = ML^{-2}$. As a result, one can make only a single dimensionless variable out of these: $U/(\Theta\omega^2)$. The only possible physical relation between the three quantities is therefore the Rayleigh-Jeans law

$$U(\omega, \Theta) = \kappa\Theta\omega^2,$$

where κ is a dimensionless constant.

However, then one has

$$\mathcal{U} = \int_0^\infty d\omega U(\omega, \Theta) = \kappa\Theta \int_0^\infty \omega^2 d\omega,$$

which diverges. This ultraviolet catastrophe can be arrested if there is a shortest length scale a. Then $\omega < c/a$, so we would obtain $\mathcal{U} \propto \Theta/a^3$.

Contrafactual: the history which might have been

No fundamental shortest length scale *a* has been observed with light or charge particles. Instead one has the SB law.

If the SB law had been recognized as a fundamental new relation, then the correct step, in the spirit of Gauss, would have been to recognize that σ is a fundamental constant of nature. It relates temperature (energy) to the frequency; we would get $[\Theta] = [\omega]$. This would solve the problem of the UV catastrophe. Planck's law would have emerged as a consequence.

Taking $\hbar = 1$ is physically equivalent to taking $\sigma = 1$. The differences are in the units of energy.

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Multiple scales in a hydrogen atom: Bohr's analysis

The hydrogen atom involves an electron and a proton, so the speed of light is not an appropriate quantity to use. However, since charges are involved, we can use the dimension counting $[e^2] = LT^{-1}$ to identify the velocity scale of the problem: $v = e^2$.

Also, since $H = mv^2/2 - e^2/r$, where m is the electron mass, there is a natural distance scale in the problem: $1/(me^2)$. This length is called the Bohr radius.

Using the velocity and the distance scales, we can create a scale of energy: $v/a_0 = me^4$. This scale is called the Rydberg energy scale, R.

Since $v \ll c$, let us write $\alpha = v/c \ll 1$. Then the Compton wavelength is $\lambda = 1/m$. Clearly, $a_0 = \lambda/\alpha^2$. Also, $1/R = a_0/\alpha^2$. In natural units there are three length scales in the problem, $\lambda \ll a_0 \ll 1/R$, or three energy scales: $m \gg 1/a_0 \gg R$.

Some problems

Problem 11.1: The hydrogen atom

Recall that the electron's mass is 0.511 MeV. Also recall that $e^2 = c/137$. Compute the values of the Bohr radius and the Rydberg. Is the wavelength of the photons radiated by the electron of the order of the Bohr radius or the inverse Rydberg?

Problem 11.2: Muonic atoms

Muonic atoms are rare atoms created in the lab where the electron in orbit around the proton is replaced by a negatively charged muon. The mass of the muon is 105.7 MeV. What is the corresponding Bohr radius and the Rydberg? The charge radius of a proton is 0.88 fm. Could the muonic atom serve as a probe of the charge distribution inside a proton?

Problem 11.3: the many scales of molecular spectra

	ω (Hz)	λ (nm)	E (eV)	T (K)
Electronic transitions	10^{15}–10^{18}	100–0.1	1–10^3	$10^4 - 10^7$
Vibrational transitions		10^3–10^6	0.1–10^{-4}	1–10^3
Rotational transitions		10^6–10^9		
Ammonia maser	10^{10}			

Energy scale of electronic transitions of the order of $Z\times$ Rydberg. So understood in terms of the quantum Coulomb problem.

Moment of inertia of O_2 molecule: $16M_p(8a_0)^2$. Therefore, the typical rotational energy is $RM_e/(256M_p) \simeq 10^{-6}R$. So understood in terms of rotations of molecules.

What is the physics of the other energy scales? Make semi-quantitative estimates of the above kind to explain them.

Problem 11.4: Angular momenta

Using the definition $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$, note that the scaling $\hat{\mathbf{r}} \to \xi \hat{\mathbf{r}}$ and $\hat{\mathbf{p}} \to \hat{\mathbf{p}}/\xi$ leaves $\hat{\mathbf{L}}$ unchanged. So it is enough to use unit vectors for $\hat{\mathbf{r}}$. Define the spherical harmonics $Y_{lm}(\mathbf{r}) = \langle \mathbf{r} | lm \rangle$.

Introduce the operators $\hat{r}_\pm = \hat{r}_x \pm i\hat{r}_y = \exp(\pm i\hat{\phi}) \sin\hat{\theta}$ and $\hat{r}_z = \cos\hat{\theta}$. Find the commutators $[\hat{\mathbf{L}}, \hat{r}_\pm]$.

Use the definition $L_+ |00\rangle = 0$ to set up a differential equation for $Y_{00}(\mathbf{r})$ and solve it to find the normalized eigenfunction.

Use that fact that $\hat{\mathbf{r}}$ is a vector operator, and hence acting on $|00\rangle$, must generate the $|1m\rangle$. Use this repeatedly to obtain the normalized eigenfunctions $Y_{lm}(\mathbf{r})$. (You may have to use the ladder operators as well.)

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Gaussian units, Systeme Internationale, natural units, black body radiation, Stefan-Boltzmann law, Rayleigh-Jeans law, ultraviolet catastrophe, Bohr radius, Rydberg energy, Compton wavelength, spherical harmonics.

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