No alternatives to entanglement

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Outline

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Commuting measurements

The dynamics of a quantum system involves some Hermitean operators, O_i, which generally do not all commute. We take the maximum commuting set, $\mathcal{C}$, and label states by eigenvalues, λ_α, of $O_\alpha \in \mathcal{C}$. Such states are a basis in the Hilbert space of the system. We will assume, for the moment, that the Hamiltonian, $H \in \mathcal{C}$.

Given a state, Abed can make measurements of the $O_\alpha \in \mathcal{C}$ one by one. After each measurement, he knows that the state has an eigenvalue λ_α. So, after all the measurements are done, he knows that the state is $|\lambda_1, \lambda_2, \cdots\rangle$. This is usually called preparing a state. By our assumption, this is a stationary state.

Now Bela makes many measurements of any number of these $O_\alpha \in \mathcal{C}$ in any order. Since they all commute, her measurements will give the same values λ_α's over and over again.

Non-commuting measurements

Now Abed makes a measurement of one of the $O_i \notin \mathcal{C}$ on the state. The result will be one of the eigenvalues of O_i. However, the corresponding eigenstate is not one of the basis states above. As a result, the new state is some superposition of these basis states.

Clearly, if Bela now makes a measurement of one of the $O_\alpha \in \mathcal{C}$ she does not necessarily get the old value. This happens because the measurements do not commute,

$$O_i O_\alpha \left|\lambda_1, \lambda_2, \cdots\right\rangle \neq O_\alpha O_i \left|\lambda_1, \lambda_2, \cdots\right\rangle.$$

This will happen whether Abed and Bela observe a single-particle quantum state or a multi-particle quantum state. This logic applied to a single atom with its many electrons and nucleons (or quarks) seems quite reasonable. But it begins to seem a little odd when they apply it to a quantum state which is not localized.

Two-particle states

We will construct generic two-particle states. We will assume that they have equal mass, spin, and that there are no forces acting on them. Let $\mathbf{r}_1$ and $\mathbf{p}_1$ be the position and momentum of the first particle, and $\mathbf{r}_2$ and $\mathbf{p}_2$ of the second. Basis states of the two particle system could be $|\mathbf{r}_1\rangle \otimes |\mathbf{r}_2\rangle$. But it will be useful to construct states labelled by the CM momentum, $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$ and $\mathbf{r} = \mathbf{r}_1 - \mathbf{p}_2$. There is a unitary transformation to the basis $|\mathbf{P}, \mathbf{r}\rangle$.

Elsewhere it will be useful to discuss the spin states. Since the particles have equal spin, s, we can drop this label from the states, and use only the projection in the z-direction: m_1 and m_2. The basis states are $|m_1\rangle \otimes |m_2\rangle$. We can also couple the spins into a total spin S state, with z-projection of M. There is an unitary transformation into the basis $|S, M\rangle$.

The Einstein-Podolsky-Rosen construction

Abed and Bela set up the two-particle system $|0, \mathbf{r}_0\rangle$, where initially $\mathbf{P} = 0$. Now Abed makes a very accurate measurement of the momentum of particle 1 and finds it to be $\mathbf{p}_1$. Clearly, $\mathbf{p}_2 = -\mathbf{p}_1$. The positions of the two particles have large uncertainties, but $\mathbf{r} = \mathbf{r}_0 + 2t\mathbf{p}_1/m$.

When the particles have travelled very far from each other, Bela makes a very accurate measurement of the position of the particle 2 and finds it to be $\mathbf{r}_2$. Now do we know all of $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{r}_1$ and $\mathbf{r}_2$ with arbitrary accuracy?

We know that we do not. As soon as Bela measures $\mathbf{r}_2$, the momentum $\mathbf{p}_2$ becomes completely uncertain, so instantaneously making $\mathbf{p}_1$ also very uncertain. This happens no matter how large $\mathbf{r}$ is.

This "action at a distance" is called quantum entanglement.

Bohm's version of the EPR construction

Suppose we choose to work with spins, and build an initial state with $S = 0$ where each $s = 1/2$. Then

$$|00\rangle = \frac{1}{\sqrt{2}}\Big(|1/2\rangle \otimes |-1/2\rangle - |-1/2\rangle \otimes |1/2\rangle\Big).$$

Abed and Bela make this state. Then Abed measures m_1. If this is $1/2$ then he deduces that $m_2 = -1/2$.

Bela makes a measurement of s_x, on particle 2, and finds it to be $1/2$. Since the initial state was $|00\rangle$, she deduces immediately that Abed's particle should yield $-1/2$ if he were to measure s_x. Now does Abed know both the eigenvalues of s_z and s_x for particle 1?

No. Instead as soon as Bela makes her measurement, the state of the entangled pair changes. Now if Abed measures s_z again on the particle, there is a 50% chance that he will not get the same value as before.

Outline

Hidden variables

Charu believes that there must be a dynamical variable μ, hidden to Abed and Bela, which completely describes the spins of the particles. She sets up a function $S(\hat{\mathbf{r}}, \mu) = \pm 1$ which gives the sign of the projection of the spin in the direction $\hat{\mathbf{r}}$. Since the total spin is zero, the conservation of angular moment forces S to give opposite values for the spins of particles 1 and 2.

In Charu's mechanics, the variable μ may take any value, but once the initial state is set up, its value is fixed. There is a probability distribution $p(\mu)$ from which the value of μ is drawn, with $\int d\mu p(\mu) = 1$. Abed can measure the spin projection in direction $\hat{\mathbf{a}}$, and Bela in direction $\hat{\mathbf{b}}$. By repeated measurements they can set up a correlation funtion

$$\left\langle (\mathbf{s}_1 \cdot \hat{\mathbf{a}})(\mathbf{s}_2 \cdot \hat{\mathbf{b}}) \right\rangle = -\left\langle (s_1 \cdot \hat{\mathbf{a}})(\mathbf{s}_1 \cdot \hat{\mathbf{b}}) \right\rangle = -\frac{1}{4} \int d\mu p(\mu) S(\hat{\mathbf{a}}, \mu) S(\hat{\mathbf{b}}, \mu).$$

Quantum correlation function

In quantum mechanics, one must use the operator $\sigma/2$ for the spin. As a result, one can write the operator

$$\left\langle (\mathbf{s}_1\cdot\hat{\mathbf{a}})(\mathbf{s}_2\cdot\hat{\mathbf{b}})\right\rangle = -\left\langle (\mathbf{s}_1\cdot\hat{\mathbf{a}})(\mathbf{s}_1\cdot\hat{\mathbf{b}})\right\rangle = -\frac{1}{4}\hat{\mathbf{a}}\cdot\hat{\mathbf{b}} + \frac{i}{2}(\hat{\mathbf{a}}\times\hat{\mathbf{b}})\cdot\langle\mathbf{s}_1\rangle,$$

where we have used the fact that $\mathbf{s}_2 = -\mathbf{s}_1$. The expectation value of $\mathbf{s}_1$ is zero, so in QM,

$$\left\langle (\mathbf{s}_1\cdot\hat{\mathbf{a}})(\mathbf{s}_2\cdot\hat{\mathbf{b}})\right\rangle = -\frac{1}{4}\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}.$$

Problem 15.1: Hidden variable theory

It is possible to choose $S(\hat{\mathbf{a}},\mu)$ and $p(\mu)$ in such a way that the results of quantum mechanics is obtained. Find such functions S and p.

Charu wonders whether any experiment can distinguish between quantum mechanics and hidden variable theories.

Bell's inequality

$$\left\langle(\mathbf{s}_1\cdot\hat{\mathbf{a}})(\mathbf{s}_2\cdot\hat{\mathbf{b}})\right\rangle - \left\langle(\mathbf{s}_1\cdot\hat{\mathbf{b}})(\mathbf{s}_2\cdot\hat{\mathbf{c}})\right\rangle =$$
$$\frac{1}{4}\int d\mu p(\mu) S(\hat{\mathbf{b}},\mu)\left[S(\hat{\mathbf{c}},\mu) - S(\hat{\mathbf{a}},\mu)\right]$$
$$\frac{1}{4}\int d\mu p(\mu) S(\hat{\mathbf{b}},\mu) S(\hat{\mathbf{c}},\mu)\left[1 - S(\hat{\mathbf{a}},\mu) S(\hat{\mathbf{c}},\mu)\right].$$

Where we used the fact $S^2(\hat{\mathbf{c}},\mu) = 1$. Taking the absolute value

$$\left|\left\langle(\mathbf{s}_1\cdot\hat{\mathbf{a}})(\mathbf{s}_2\cdot\hat{\mathbf{b}})\right\rangle - \left\langle(\mathbf{s}_1\cdot\hat{\mathbf{b}})(\mathbf{s}_2\cdot\hat{\mathbf{c}})\right\rangle\right| \le \frac{1}{4}\int d\mu p(\mu)\left[1 - S(\hat{\mathbf{a}},\mu) S(\hat{\mathbf{c}},\mu)\right]$$

So hidden variable theories obey Bell's inequality

$$\left|\left\langle(\mathbf{s}_1\cdot\hat{\mathbf{a}})(\mathbf{s}_2\cdot\hat{\mathbf{b}})\right\rangle - \left\langle(\mathbf{s}_1\cdot\hat{\mathbf{b}})(\mathbf{s}_2\cdot\hat{\mathbf{c}})\right\rangle\right| \le \frac{1}{4} + \langle(\mathbf{s}_1\cdot\hat{\mathbf{a}})(\mathbf{s}_2\cdot\hat{\mathbf{c}})\rangle.$$

$\cdots$ in quantum mechanics

Now Abel and Bela can choose $\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}=0$, so that the corresponding correlation vanishes. Next, on choosing $\hat{\mathbf{c}}=\cos\theta\hat{\mathbf{a}}+\sin\theta\hat{\mathbf{b}}$, they have

$$\left\langle(\mathbf{s}_1\cdot\hat{\mathbf{b}})(\mathbf{s}_2\cdot\hat{\mathbf{c}})\right\rangle=-\frac{1}{4}\sin\theta,\quad \langle(\mathbf{s}_1\cdot\hat{\mathbf{a}})(\mathbf{s}_2\cdot\hat{\mathbf{c}})\rangle=-\frac{1}{4}\cos\theta.$$

As a result, one finds

$$\left|\left\langle(\mathbf{s}_1\cdot\hat{\mathbf{a}})(\mathbf{s}_2\cdot\hat{\mathbf{b}})\right\rangle-\left\langle(\mathbf{s}_1\cdot\hat{\mathbf{b}})(\mathbf{s}_2\cdot\hat{\mathbf{c}})\right\rangle\right|=\frac{1}{4}|\sin\theta|,$$
$$\frac{1}{4}+\langle(\mathbf{s}_1\cdot\hat{\mathbf{a}})(\mathbf{s}_2\cdot\hat{\mathbf{c}})\rangle=\frac{1}{4}(1-\cos\theta)\,.$$

Bell's inequality requires the difference

$$\mathcal{B}(\theta)=\frac{1}{4}\left[|\sin\theta|+\cos\theta-1\right]$$

to be negative. However, it turns out that for $0<|\theta|<\pi/2$ this function is positive.

Experiments

Bell's analysis was published in 1964. It was the first example of a correlation function which made it possible to distinguish exerimentally between hidden variable theories and quantum mechanics.

Further inequalities can be developed with correlations of four different measurements. Such an experiment was performed in 1982 by Alain Aspect and collaborators. The results support quantum mechanics.

More subtle correlations have also been measured since then. All measurements until now support quantum mechanics.

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Keywords and References

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preparing a state, quantum entanglement, entangled pair, correlation funtion, hidden variable theories, Bell's inequality.

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