Quanta, waves, and vector spaces

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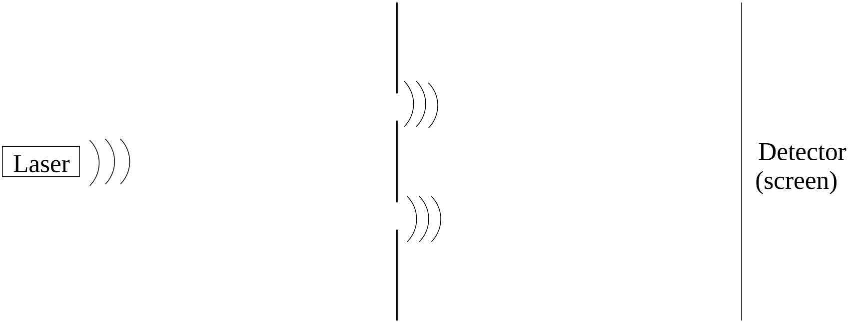
Wave-particle unity

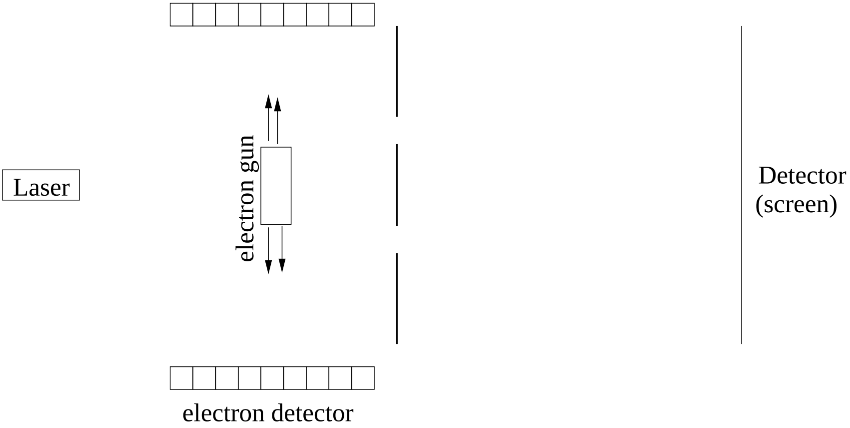


Use a laser which emits one photon at a time. Use the screen as a detector. Interference is due to multiple open paths.

Wave-particle unity

- We need to think of a quantum particle as both a classical particle and a classical wave.
- Wavelength corresponding to a given particle momentum: $\lambda = \mathbf{k}/(2\pi) = \mathbf{p}/(2\pi\hbar)$. Distance between maxima of interference patterns?
- Classical waves describe state of any quantum particle: wave function can be expressed as a function of x or of k.
- Detectors observe a quantum particle as a classical particle. This phenomenon is called collapse of the wave-function.
- Recall: waves do not require a medium, just harmonic dependence on space or time. The mathematics of waves is a generalization of vectors.

Which-path experiment



Electrons scatter from photons. Back-to back electrons, if no photon. Scattering locates the electron but destroys the fringe.

Polarization of light waves

Recall electromagnetic waves are vector waves, where $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ can be obtained from each other using Maxwell's equations. Consider light of a single wavelength, so that $\mathbf{k}$ is fixed. Suppose the z-direction to be the direction of $\mathbf{k}$, *i.e.*, $\mathbf{k} = (0, 0, k)$. Then $\mathbf{E}$ lies in the xy-plane since $\mathbf{k} \cdot \mathbf{E} = 0$.

Plane polarized light has $\mathbf{k}$ and $\mathbf{E}$ in a single plane. Then we can write

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}^* \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}.$$

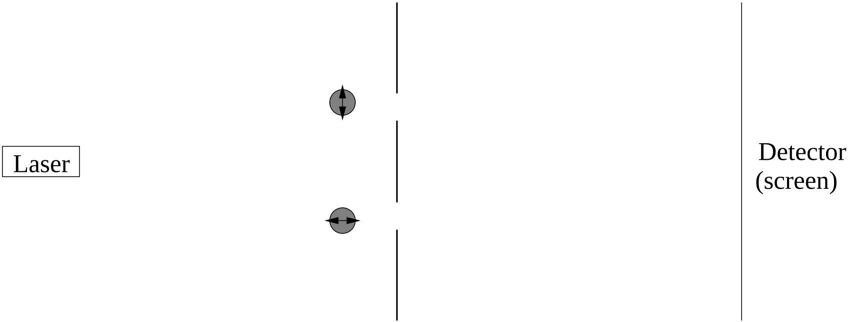
Two orthogonal choices are $\mathbf{E}^*$ defines the x-direction, and $\mathbf{E}^*$ defines the y-direction. Any linear combination is also allowed.

Circularly polarized light has no fixed plane of $\mathbf{E}$. One can write

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}^*(t)\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}, \quad \mathbf{E}^*_\pm(t) = |E^*|(\hat{\mathbf{x}} \sin \omega t \pm i\hat{\mathbf{y}} \cos \omega t).$$

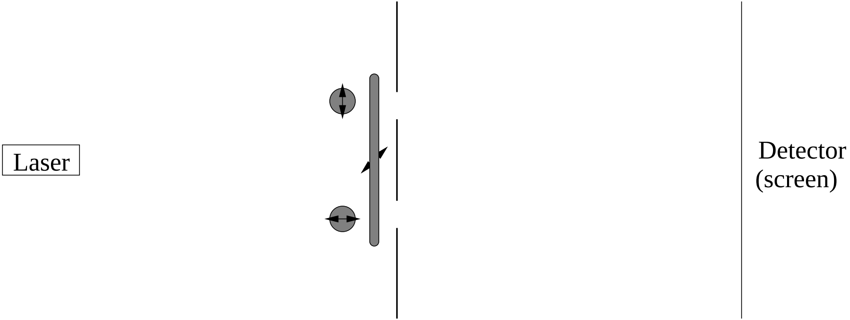
So, complex linear combinations of x and y polarizations are physical.

Simpler which-path experiment



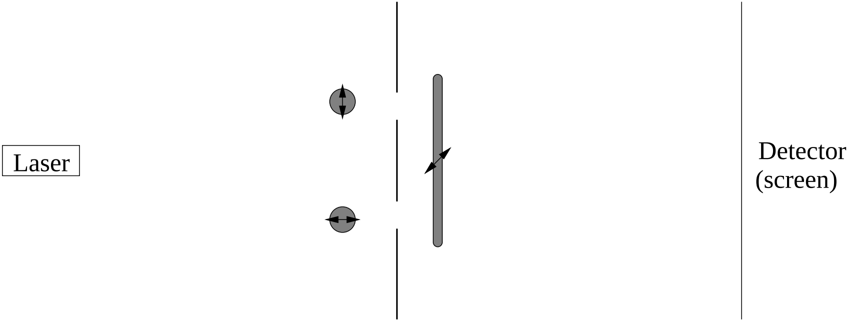
Arrange each slit to have a fixed polarization of photons. But photons of different polarizations do not interfere.

Quantum eraser



If you erase the which-path information, then interference fringes reappear

Delayed choice quantum eraser



Interference fringes reappear whether the which-path information is erased before interference or after interference.

Vectors

Take vectors such as $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. Let a, b, c, *etc.*, be scalars. Then, the following are true—

1. A vector multiplied by a scalar is a vector: $a\mathbf{x} = (ax_1, ax_2, ax_3)$.
2. Two vectors can be added to give a new vector: $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$. Vector addition is commutative: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$, and associative: $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$. Multiplication by scalars is distributive over vector addition: $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$. Addition of scalars is compatible with multiplication by a vector. Multiplication of scalars is compatible with multiplication by a vector.
3. There exists a zero vector: $\mathbf{0}$ such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for any vector $\mathbf{x}$. There exists a negative of every vector.
4. There is an inner product (dot product): $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$.

Square integrable functions are vectors!

Take functions $f(x)$, $g(x)$, *etc.*, such that

$$\int dx|f(x)|^2 \qquad \text{and} \qquad \int dx|g(x)|^2$$

exist. Let a, b, c, *etc.*, be constrants. Then, the following are true—

1. $af(x)$ is square integrable.
2. $f(x) + g(x)$ is square integrable. The sum is clearly commutative and associative. Multiplication by a constant is distributive over the addition of the two functions; scalar addition is distributive over multiplication by functions; scalar multiplication is compatible with multiplication by a function.
3. There exists a zero function. Every function has a negative.
4. An inner product exists: $\int dxf(x)g^*(x)$. (Prove this)

Coordinates: linear independence and bases

Take a collection of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots, \mathbf{x}_N\}$, such that it is impossible to find any set of N non-zero numbers which satisfy the vector equation:

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_N\mathbf{x}_N = 0.$$

Then, the collection is called linearly independent.
The dimension of the vector space is the largest size of a linearly indpendent collection of vectors. Start from any vector $\mathbf{x}_1$. Find a second vector which is linearly independent of this one. If you cannot find one, then the vector space is one-dimensional. If you find one then you have the set $\{\mathbf{x}_1, \mathbf{x}_2\}$. Find a third vector which is linearly indpendent of these two. If you cannot find one, then the space is 2 dimensional. Otherwise you have the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, *etc.*. Any set of D linearly independent vectors in a vector space of dimension D is called a basis.

Components

- Given a basis, every vector not in this basis is linearly dependent on the basis vectors. Hence one has an unique decomposition $\mathbf{v} = v_1\mathbf{x}_1 + v_2\mathbf{x}_2 + \cdots v_D\mathbf{x}_D$. The numbers v_i are the components of the vector in this basis.
- The basis is orthonormal if $\mathbf{x}_i \cdot \mathbf{x}_j = \delta_{ij}$. (The Kronecker delta, δ_{ij}, is 1 if $i = j$ and zero otherwise).
- In an orthonormal basis the components of a vector are obtained by taking a dot product with the basis vectors: $v_i = \mathbf{v} \cdot \mathbf{x}_i$.
- A change from one orthonormal basis to another is performed by taking linear combinations of the components. This linear combination is specified by an orthogonal matrix: $\mathbf{v}' = M\mathbf{v}$, where $\mathbf{v}'$ is the same as $\mathbf{v}$ but in the new basis. Since $\mathbf{v} \cdot \mathbf{v} = 1 = \mathbf{v}' \cdot \mathbf{v}'$, one must have $M^T M = 1$ (M^T is the transpose of the matrix M).

The Fourier basis

- The vector space of square integrable functions is infinite dimensional. The recursive procedure for constructing a linearly independent set does not end in any number of steps. (Prove this).
- The plane waves, $\phi_k(x) = \exp(ikx)$, provide a basis (almost) on these functions. They are orthogonal under the inner product defined before. However, they are not square integrable. We will assume that this problem can be cured (the cure will come later) and continue. (Cure this)
- Each square integrable function can be completely specified by its Fourier coefficients: $\widetilde{f}(k) = f(x) \cdot \phi_k(x)$, under the inner product defined earlier. The standard representation as an infinite component is the Fourier transform: $f(x) \equiv \widetilde{f}(k)$.

Keywords and References

Keywords

Interference, waves, which-path information, polarizers, quantum eraser, delayed-choice quantum eraser, vector space, inner product, square-integrable function, linear independence, basis, orthonormal basis, Kronecker delta, coordinates, rotations.

References

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