Angular Momentum

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Quantum Mechanics 1
Tenth lecture

Outline

1. The algebra of angular momentum operators
2. Rotations
3. Properties of operators under rotations
4. Keywords and References

Angular momentum

- In SI $\hbar$ has dimensions of angular momentum. So the expectation value of an angular momentum in natural units is a dimensionless number.
- Classically the angular momentum of a particle is defined to be $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. The quantum operator is obtained by taking the operator expressions for $\mathbf{r}$ and $\mathbf{p}$ in this definition.
- The quantum operator is unchanged if one takes the equivalent classical expression $\mathbf{L} = -\mathbf{p} \times \mathbf{r}$. Check.
- Classically $\mathbf{r} \cdot \mathbf{p} = \mathbf{p} \cdot \mathbf{r}$. But the quantum operators corresponding to the two expressions are not equal. Why?
- The commutation relations of the components of $\mathbf{L}$ follow from the definition above: $[L_j, L_k] = i\epsilon_{jkl} L_l$. The operator L^2 commutes with all L_j. Hence, L^2 and one of the L_j (usually L_z) can be simultaneously diagonalized. (check)

Raising and lowering operators

- Any set of three operators, **J**, satisfying the algebra

$$[J_j, J_k] = i\epsilon_{jkl} J_l, \quad \text{and} \quad [J^2, J_j] = 0 \quad \text{where } J^2 = J_x^2 + J_y^2 + J_z^2,$$

can be identified with angular momentum operators.

- Introduce the raising and lowering operators

$$J_+ = J_x + iJ_y \quad \text{and} \quad J_- = J_x - iJ_y.$$

- They have the properties

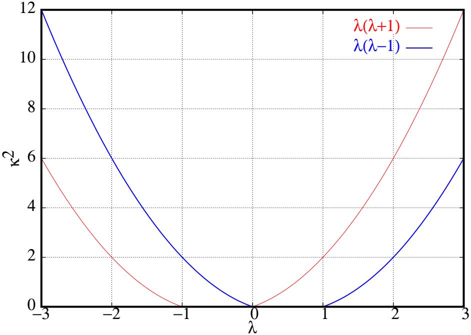
$$J_+J_- = J^2 - J_z^2 + J_z \quad \text{and} \quad J_-J_+ = J^2 - J_z^2 - J_z.$$

- The new operators have the commutation relations

$$[J_+, J_-] = 2J_z, \qquad [J^2, J_\pm] = 0, \qquad [J_z, J_\pm] = \pm J_\pm.$$

Limits on eigenvalues

- Choose a $|\psi\rangle$ such that $J_z|\psi\rangle = m|\psi\rangle$ and $J^2|\psi\rangle = \kappa^2|\psi\rangle$ with $\kappa^2 > 0$. Why?
- If $|\psi\pm\rangle = J_\pm|\psi\rangle$, then using the fact that the square of the norm of a vector is positive, one can show that $\kappa^2 - m(m \pm 1) \geq 0$.



The representations are finite dimensional

- For a fixed κ, when m satisfies the relation $m(m+1)=\kappa^2$, then the norm of $|\psi-\rangle$ vanishes. As a result, $|\psi-\rangle = 0$.
- Similarly, for a fixed κ, when m satisfies the relation $m(m-1)=\kappa^2$, then $|\psi+\rangle = 0$.
- Since $J_\pm$ commute with J^2, the vectors $|\psi\pm\rangle$ are also eigenvectors of J^2 with eigenvalues κ^2. However,

  $$J_z J_\pm |\psi\rangle = (\pm J_\pm + J_\pm J_z) |\psi\rangle = (m \pm 1) J_\pm |\psi\rangle ,$$

  so $|\psi\pm\rangle$ are eigenvectors of J_z with eigenvalues $(m\pm 1)$.
- It follows that, for every κ, the range of possible values of m goes between the lowest and highest allowed values in steps of unity. Therefore the number of states is finite for each κ,

The eigenvalues

- For a given κ the lowest m is
$$m = \frac{1}{2} - \sqrt{\kappa^2 + \frac{1}{4}},$$
and the highest is obtained by flipping the signs. The difference must be an integer, since one goes from one to the other by the action of raising and lowering operators.
- As a result,
$$\kappa^2 = \frac{1}{4}(n^2 - 1) = \frac{1}{4}(n+1)(n-1) = j(j+1),$$
where $n \geq 1$ is an integer, and $j = n/2 - 1/2$ is integer for n odd and half integer for n even. Hence the eigenvalues of J^2 are $j(j+1)$ for $j = 0,\ 1/2,\ 1,\ 3/2,$ *etc.*.
- The values of m therefore lie in the range $-j$ to j and are seperated by 1. The size of the representation space is $2j+1$. The eigenstates can be labelled $|j, m\rangle$.

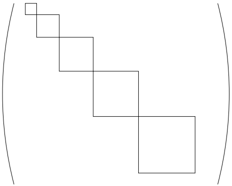
The representations

- $|j, m\rangle$ is an eigenstate of J^2 and J_z; *i.e.*,

  $$J^2 |j, m\rangle = j(j+1) |j, m\rangle , \qquad \text{and} \qquad J_z |j, m\rangle = m |j, m\rangle .$$

  An overall phase factor still remains to be specified.
- The action of the ladder operators is

  $$\begin{aligned} J_+ |j, m\rangle &= \sqrt{j(j+1) - m(m+1)}\, |j, m+1\rangle , \\ J_- |j, m\rangle &= \sqrt{j(j+1) - m(m-1)}\, |j, m-1\rangle . \end{aligned}$$
- If A commutes with J^2, then it is block diagonalized in the basis $|j, m\rangle$, *i.e.*, it can be put in the form



Two representations

- The representation with $j = 0$ is called the scalar representation. This has dimension 1, *i.e.*, all the angular momentum operators are represented by scalars. J_x, J_y and J_z act on the eigenstates by multiplying them by zero. The eigenstate is $|0,0\rangle$.
- The representation with $j = 1/2$ is called the spinor representation. All components of angular momentum are represented by 2×2 Hermitean matrices, and hence must be proportional to the Pauli matrices. Since J_z must be diagonal and the eigenvalues have to be $\pm 1/2$, one can write $J_k = \sigma_k/2$. Then $J^2 = 3/4$, which is exactly what is expected. With this choice one has

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \text{and} \qquad \left|\frac{1}{2},-\frac{1}{2}\right\rangle = \begin{pmatrix}0\\1\end{pmatrix}.$$

Problem 9.1: Angular momenta

Using the definition $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$, note that the scaling $\hat{\mathbf{r}} \to \xi\hat{\mathbf{r}}$ and $\hat{\mathbf{p}} \to \hat{\mathbf{p}}/\xi$ leaves $\hat{\mathbf{L}}$ unchanged. So it is enough to use unit vectors for $\hat{\mathbf{r}}$. Define the spherical harmonics $Y_{lm}(\mathbf{r}) = \langle \mathbf{r} | lm \rangle$.

Introduce the operators $\hat{r}_\pm = \hat{r}_x \pm i\hat{r}_y = \exp(\pm i\hat{\phi}) \sin\hat{\theta}$ and $\hat{r}_z = \cos\hat{\theta}$. Find the commutators $[\hat{\mathbf{L}}, \hat{r}_\pm]$.

Use the definition $L_+ |00\rangle = 0$ to set up a differential equation for $Y_{00}(\mathbf{r})$ and solve it to find the normalized eigenfunction.

Use that fact that $\hat{\mathbf{r}}$ is a vector operator, and hence acting on $|00\rangle$, must generate the $|1m\rangle$. Use this repeatedly to obtain the normalized eigenfunctions $Y_{lm}(\mathbf{r})$. (You may have to use the ladder operators as well.)

Outline

Translations and unitary transformations

- Since the components of the spatial momenta, $\mathbf{p}$, commute with each other, all the components can be simultaneously diagonalized. The eigenvectors, $|\mathbf{k}\rangle \equiv |k_x, k_y, k_z\rangle$, are labelled by the eigenvalues, $\mathbf{k}$, of these operators. Each such eigenvector gives a one dimensional representation of the momentum operators.
- Translations in space act on the eigenvectors of $\mathbf{p}$ through unitary operators, $T(\mathbf{x}) = \exp(i\mathbf{x} \cdot \mathbf{p})$. Since the components of $\mathbf{p}$ commute, so do the translation operators (check). In other words, translations form an Abelian group.
- Translations act on the space of each $|\mathbf{k}\rangle$ through a multiplication by a single complex number (a 1×1 unitary matrix), $\exp(i\mathbf{x} \cdot \mathbf{k})$. That is, $T(\mathbf{x})\,|\mathbf{k}\rangle = \exp(i\mathbf{x} \cdot \mathbf{k})\,|\mathbf{k}\rangle$.

Rotations and unitary transformations

- Any rotation is specified by an axis of rotation, $\hat{\mathbf{n}}$, and the angle of rotation, ψ, about this axis.
- The result of a rotation in space is an unitary transformation on the eigenvectors of the angular momentum operator. The rotation operator is given by $R(\hat{\mathbf{n}}, \psi) = \exp(-i\psi \mathbf{J} \cdot \hat{\mathbf{n}})$.
- When $\hat{\mathbf{n}}$ is the z axis, then, $R(\hat{\mathbf{z}}, \psi) = \exp(-i\psi J_z)$. In this case, $R(\hat{\mathbf{z}}, \psi) |j, m\rangle = \exp(-im\psi) |j, m\rangle$.
- In general, however, acting on the $(2j + 1)$-dimensional eigenbasis $|j, m\rangle$ (for $-j \le m \le j$), the rotation operators mix the basis. This is a consequence of the fact that the z axis is not (in general) fixed under rotations, and hence the axis used to define m changes.
- In the $j = 0$ representation, since all $J_k = 0$, the rotation operators $R(\hat{\mathbf{n}}, \psi) = 1$. Thus, the eigenfunction is completely unchanged under rotations, *i.e.*, it is a scalar.

Rotation matrices in the spinor representation

In the $j = 1/2$ representation (also called the spinor) one has

$$R(\hat{\mathbf{n}}, \psi) = \mathrm{e}^{-i\sigma\cdot\hat{\mathbf{n}}\psi/2}.$$

In terms of the components of $\hat{\mathbf{n}}$ in spherical coordinates in the original frame, before the rotation, one has

$$\sigma \cdot \hat{\mathbf{n}} = \begin{pmatrix} \cos\theta & -i\sin\theta\mathrm{e}^{i\phi} \\ i\sin\theta\mathrm{e}^{-i\phi} & -\cos\theta \end{pmatrix}.$$

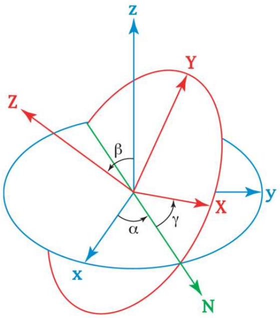
If this is diagonalized by an unitary transformation V, then (check)

$$R(\hat{\mathbf{n}}, \psi) = V \begin{pmatrix} \mathrm{e}^{-i\psi/2} & 0 \\ 0 & \mathrm{e}^{i\psi/2} \end{pmatrix} V^\dagger$$

Note that when $\psi = 4\pi$, then $R = 1$, and the rotation has no effect. However, when $\psi = 2\pi$, the rotated wavefunctions are not the same as the original wavefunctions. This is non-classical behaviour.

Euler angles

The specification of rotations used three angles— two to define the axis of rotation and one to specify the angle of rotation. Other conventions for defining the rotation are due to Euler.



The fixed system is xyz; the rotated system is XYZ.

1. Rotate about the z axis by angle α.
2. Rotate about the new x axis by angle β.
3. Rotate about the new z axis by angle γ.

Other definitions of the Euler angles are possible. This definition (called the zxz convention) is used by Goldstein and Landau and Lifschitz.

Wigner's D matrices

Using Euler's angles in the zyz convention, a rotation operator can be written as

$$R(\alpha,\beta,\gamma) = \mathrm{e}^{-i\alpha J_z}\mathrm{e}^{-i\beta J_y}\mathrm{e}^{-i\gamma J_z}.$$

Clearly, these rotation matrices commute with J^2 and hence can be evaluated for each j separately. Wigner's D matrices are

$$D^j_{mm'}(\alpha,\beta,\gamma) = \langle jm'|\, R(\alpha,\beta,\gamma)\, |jm\rangle = \mathrm{e}^{-i(m'\alpha+m\gamma)} d^j_{mm'}(\beta).$$

We introduced the matrix elements of the rotation about the x-axis through Wigner's d matrices

$$d^j_{mm'}(\beta) = \langle jm'|\, \exp(-i\beta J_y)\, |jm\rangle\,.$$

In this convention the d matrices are purely real.

Outline

Scalar operators

A scalar operator remains unchanged under rotations, *i.e.*,

$$S = R(\hat{\mathbf{n}}, \psi) S R^\dagger(\hat{\mathbf{n}}, \psi).$$

Taking the rotation angle to be infinitesimal, and expanding the rotation operator to leading order in the Taylor expansion in the angle, one finds that $[S, J_k] = 0$. This means that scalar operators can be diagonalized together with J^2 and J_z, and the matrix elements are independent of the eigenvalue of J_z. In other words,

$$\langle j, m | \, S \, \left| j', m' \right\rangle = \delta_{jj'} \delta_{mm'} (j|S|j),$$

where the reduced matrix element $(j|S|j)$ is a function only of j. Thus, for each j, all scalar operators are multiples of identity. This is a special case of the Wigner-Eckart theorem.
Examples of scalar operators are J^2, r^2, p^2 and $\mathbf{r} \cdot \mathbf{p}$. Hamiltonians with central potentials, $V(|r|)$, are scalar.

Vector operators

J is a typical (axial) vector. Other vector operators must have the same algebraic property. Each vector operator must have 3 components, all of which have the same properties under rotations as **J**. So this is a matter of arranging the right commutation relations. As a result, any vector operator, **V**, has the property that $[V_j, J_k] = i\epsilon_{jkl}V_l$. So under infinitesimal rotations vector operators transform to

$$V_j' = V_j - \psi\epsilon_{jkl}\hat{\mathbf{n}}_k V_l.$$

Examples of vector operators are **J**, **r** and **p**.

Vector operators neither commute with J_z nor with J^2 (check). So they cannot be diagonalized or block diagonalized in the $|j, m\rangle$ basis. So $\langle j', m'| \mathbf{V} |jm\rangle \neq 0$. Purely geometric considerations (called the Wigner Eckart theorem) allows us to reduce such matrix elements to a simpler form.

Raising and lowering in j

For a vector operator $\mathbf{V}$, introduce $V_\pm = V_1 \pm iV_2$. Then,

$$[J_z, V_\pm] = \pm V_\pm, \quad \text{so} \quad J_z V_\pm |j, \pm j\rangle = \pm(j+1) V_\pm |j, \pm j\rangle .$$

One can check that $[J^2, \mathbf{V}] = 2\mathbf{V} + 2i\mathbf{V} \times \mathbf{J}$. As a result, one finds

$$[J^2, V_\pm] = 2V_\pm + 2(V_\pm J_z - V_z J_\pm).$$

This implies that

$$J^2 V_\pm |j, \pm j\rangle = [j(j+1)+2+2j] V_\pm |j, \pm j\rangle = (j+1)(j+2) V_\pm |j, \pm j\rangle .$$

Together, these two conditions imply that $V_\pm |j, \pm j\rangle \propto |j+1, \pm(j+1)\rangle$. The Wigner-Eckart theorem concerns the constant of proportionality.
All irreps of $SU(2)$ can be constructed from the scalar and spinor irreps by the action of $V_\pm$ and $J_\pm$.

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Angular momentum operators, raising and lowering operators, scalar representation, spinor representation, translation operator, rotation operator, Euler angles, Wigner's D matrices, Wigner's d matrices, scalar operator, reduced matrix element, Wigner-Eckart theorem, vector operators.

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