Adding angular momenta

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Quantum Mechanics 1
Eleventh Lecture

Outline

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Kinds of representations

For any set of Hermitean operators, H_i, consider the algebra $[H_i, H_j] = f_{ijk} H_k$. Using these H_i, we construct a maximum set of commuting operators. Given any matrix representation of the H_i, we find the basis which diagonalizes the maximum commuting set. In this basis the remaining operators are block-diagonal. As a result, exponentials such as $U = \exp(iu_j H_j)$ are also block diagonal. A matrix representation of all the H_i which cannot all be reduced to smaller blocks is called an irreducible representation. All other representations can be reduced to smaller blocks by unitary transformations and are therefore called reducible representations.

Example: In the scalar (or trivial) representation, we can set all $H_i = 0$. The group generated by exponentiating, $U = \exp(iu_j H_j)$, is then represented by $U = 1$ for all U. The trivial representation of any group is an irreducible representation of any group.

Example: The representation of $\mathbf{J}$ by the Pauli matrices gives rise to an irreducible representation of the group of rotations.

Building things up and breaking them down

A direct product (also called tensor product) of two matrices N and M is

$$N \otimes M = \begin{pmatrix} n_{11}M & n_{12}M & n_{13}M & \cdots \\ n_{21}M & n_{22}M & n_{23}M & \cdots \\ n_{31}M & n_{32}M & n_{33}M & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{pmatrix}.$$

The dimension of $N \otimes M$ is the product of the dimensions of each matrix. Direct products of vectors follow from this definition.

A direct sum of two matrices $N \oplus M$ is the block diagonal form

$$N \oplus M = \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix}.$$

The dimension of the direct sum is the sum of the dimensions of each matrix. We will now try to reduce direct products into direct sums.

States of two particle systems

We examine two particle systems, where each particle admits the same maximal commuting set of measurement: $\{H_i\}$. Then we can choose the basis in which all the $\{H_i\}$ are diagonal, and the basis states are labelled by the eigenvalues of these H_i: $|h_1, h_2, \cdots\rangle$. Use the notation h_i^j to mean the eigenvalue of H_i for the j-th particle. Two particle states are completely specified when the quantum numbers of each state is completely specified, *i.e.*, the basis states can be chosen to be

$$\left|h_1^1, h_2^1, \cdots ; h_1^2, h_2^2, \cdots\right\rangle = \left|h_1^1, h_2^1, \cdots\right\rangle \otimes \left|h_1^2, h_2^2, \cdots\right\rangle .$$

Then the measurement of H_i on particle 1 can be represented by $\hat{H}_i^1 = \hat{H}_i \otimes 1$, and on particle 2 can be represented by $\hat{H}_i^2 = 1 \otimes \hat{H}_i$. The measurement on the two particle system is specified by $\hat{H}_i = \hat{H}_i^1 + \hat{H}_i^2$. Prove that $\hat{H}_i^1$ and $\hat{H}_i^2$ commute, and therefore commute with $\hat{H}_i$.

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Summing two momenta

If $|\mathbf{k}_1\rangle$ is the basis state of one particle and $|\mathbf{k}_2\rangle$ of another, then the direct product state $|\mathbf{k}_1;\mathbf{k}_2\rangle = |\mathbf{k}_1\rangle \otimes |\mathbf{k}_2\rangle$. The operator $\hat{\mathbf{p}}^1 = \hat{\mathbf{p}} \otimes 1$ acts on the state vector of the first particle, and the operator $\hat{\mathbf{p}}^2 = 1 \otimes \mathbf{p}$ on the second. The total momentum $\hat{\mathbf{P}} = \hat{\mathbf{p}}^1 + \hat{\mathbf{p}}^2$.

Since all representations which we have built are one-dimensional, the direct product state is also one dimensional. One has

$$\hat{\mathbf{P}}\,|\mathbf{k}_1;\mathbf{k}_2\rangle = \left(\hat{\mathbf{p}}^1 + \hat{\mathbf{p}}^2\right)|\mathbf{k}_1;\mathbf{k}_2\rangle = (\mathbf{k}_1 + \mathbf{k}_2)\,|\mathbf{k}_1;\mathbf{k}_2\rangle\,.$$

Therefore, the direct product state is the representation with momentum equal to the sum of the two momenta:

$$|\mathbf{k}_1\rangle \otimes |\mathbf{k}_2\rangle = |\mathbf{k}_1 + \mathbf{k}_2\rangle\,.$$

Since every representation is one-dimensional, this is a trivial example of direct product spaces.

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Summing two spins: counting dimensions

If $|j_1, m_1\rangle$ is the basis states of one particle, and $|j_2, m_2\rangle$ of another, then $|j_1, m_1; j_2, m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle$. The operator $\hat{\mathbf{J}}^1 = \hat{\mathbf{j}} \otimes 1$, and $\hat{\mathbf{J}}^2 = 1 \otimes \hat{\mathbf{j}}$. All components of these two operators commute, since they act on different spaces. The total angular momentum of the system is $\hat{\mathbf{J}} = \hat{\mathbf{J}}^1 + \hat{\mathbf{J}}^2$. But

$$J^2 |j_1, m_1; j_2, m_2\rangle \neq (j_1 + j_2)(j_1 + j_2 + 1) |j_1, m_1; j_2, m_2\rangle .$$

This is because the dimension of the direct product is $(2j_1 + 1)(2j_2 + 1)$ and this is not equal to $(2j_1 + 2j_2 + 1)$ unless either j_1 or j_2 (or both) is zero.

Example: The direct product of two $j = 1/2$ particles has dimension 4. This is either a $j = 3/2$ representation (which has dimension 4) or a direct sum of a $j = 0$ (dimension 1) and a $j = 1$ (dimension 3) representation. If the direct product can be reduced to a direct sum, then all components of $\mathbf{J}$ can be block diagonalized in this fashion.

Summing two spins: the spectrum of J_z

By the definition of the direct product, one has

$$J_z^1 = \begin{pmatrix} \mu_1 I & 0 & 0 \cdots & \\ 0 & \mu_2 I & 0 & \cdots \\ 0 & 0 & \mu_3 I & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{pmatrix}, \qquad J_z^2 = \begin{pmatrix} J_z & 0 & 0 \cdots & \\ 0 & J_z & 0 & \cdots \\ 0 & 0 & J_z & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{pmatrix}.$$

This remains diagonal. As a result, the quantum number M corresponding to the total $J_z = J_z^1 + J_z^2$ is the sum $m_1 + m_2$. In other words

$$J_z \left| j_1, m_1; j_2, m_2 \right\rangle = (m_1 + m_2) \left| j_1, m_1; j_2, m_2 \right\rangle .$$

Example: For the direct product of two $j = 1/2$ particles, the possible values of M are 1, -1, and 0 (twice). As a result, this direct product cannot be the representation $j = 3/2$. Therefore

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1.$$

Problem 10.1: summing two spins

Using the definition of the direct product for $|1/2, m_1; 1/2, m_2\rangle$, one has

$$J_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad J_+ = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Use this to find J^2 and check how to diagonalize it while keeping J_z diagonal. Using these results, show that

$$\begin{aligned} |1,0\rangle &= \frac{1}{\sqrt{2}} \left(\left| \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, -\frac{1}{2} \right\rangle + \left| \frac{1}{2}, -\frac{1}{2}; \frac{1}{2}, \frac{1}{2} \right\rangle \right), \\ |0,0\rangle &= \frac{1}{\sqrt{2}} \left(\left| \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, -\frac{1}{2} \right\rangle - \left| \frac{1}{2}, -\frac{1}{2}; \frac{1}{2}, \frac{1}{2} \right\rangle \right). \end{aligned}$$

Find the matrices corresponding to $\mathbf{J}$ in the $j = 1$ representation.

Summing two spins: the Clebsch-Gordan series

Clearly the states $|j, m; 1/2, m'\rangle$ have $j + 1/2 \leq M \leq -j - 1/2$. The extreme eigenvalues are single, every other eigenvalue occurs twice. So

$$j \otimes \frac{1}{2} = \left(j + \frac{1}{2}\right) \oplus \left(j - \frac{1}{2}\right).$$

By an inductive argument one can prove that the direct product states $|j_1, m_1; j_2, m_2\rangle$ can be decomposed as

$$j_1 \otimes j_2 = (j_1 + j_2) \oplus (j_1 + j_2 - 1) \oplus \cdots \oplus (j_1 - j_2),$$

where each J occurs only once. The reduction of a direct product to direct sums of terms is called the Clebsch-Gordan series. In the CG series any value of $M = m_1 + m_2$, except the extremes, are degenerate. An unitary transformation connects the states $|j_1, m_1; j_2, M - m_1\rangle$ to the angular momentum eigenstates $|J, M\rangle$. The unitary matrix can be written as $|J, M\rangle \langle j_1, m_1; j_2, m_2|$. The matrix elements are called Clebsch-Gordan coefficients.

Examples of Clebsch-Gordan coefficients

1. The trivial CG coefficients are
$$\langle j_1+j_2, j_1+j_2 | j_1, j_1; j_2, j_2\rangle = 1.$$
One can in general write this as $\exp(i\psi)$ for some real ψ. The choice of ψ has to be compatible with the phase choices for the angular momentum eigenstates.

2. In the problem of the coupling of two spin 1/2 particles, the unitary transformation that rotates from the eigenbasis of the two uncoupled spins to the eigenbasis of the coupled spins is
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$
The CG coefficients $\langle 1,1|1/2,1/2;1/2,1/2\rangle$, $\langle 1,-1|1/2,-1/2;1/2,-1/2\rangle$, $\langle 1,0|1/2,m;1/2,-m\rangle$, $\langle 0,0|1/2,m;1/2,-m\rangle$ can be read off this matrix.

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Keywords and References

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Irreducible representation, reducible representations, direct products, direct sums, tensor products, block diagonal matrices, two particle systems, Clebsch Gordan series, and coefficients.

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