The use of symmetries

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Quantum Mechanics 1
Thirteenth lecture

Outline

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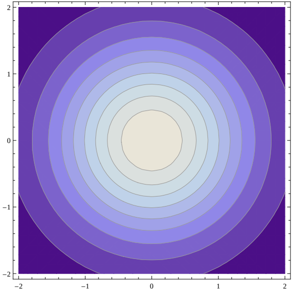
Problem 13.1

Find the eigenvectors of the following matrix using only its algebraic properties:

$$M = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

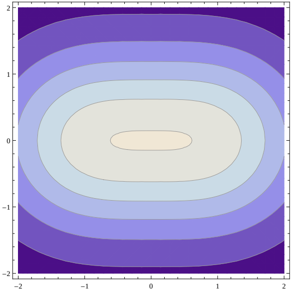
In order to do this, first find a set of simpler matrices which commute with M. Find the commutators of these matrices with each other. Do these simpler matrices form a group? Construct the group multiplication table, and understand the eigenvectors in terms of the group property.

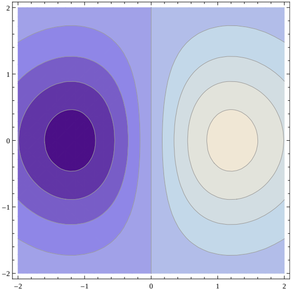
Problem 13.2



Suppose a transformation of coordinates $\mathbf{x} \to R\mathbf{x}$ does not change a potential, $V(\mathbf{x})$. Then is there a transformation on the Hilbert space of states which does not change any property of the system? Does this imply that the eigenstates of the Hamiltonian are degenerate? Does it imply any other properties?

Problem 13.3





Suppose $F(\mathbf{x}) = Exp[-(x_1^2 + x_2^2)/2]$. Let $V(\mathbf{x}) = F(\mathbf{x} - \mathbf{x}_0) + F(\mathbf{x} + \mathbf{x}_0)$, $W(\mathbf{x}) = F(\mathbf{x} - \mathbf{x}_0) - F(\mathbf{x} + \mathbf{x}_0)$, be two potentials. What are the transformations which leave the potentials invariant? What are the symmetries of the Hilbert space of states?

Problem 13.4

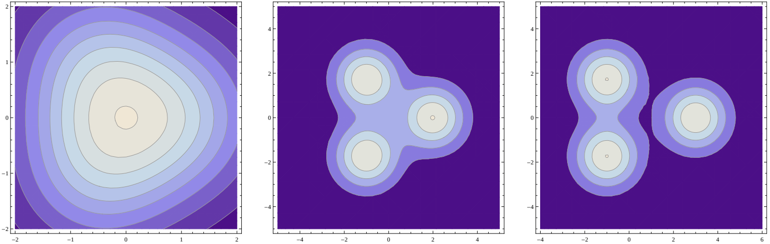
Take a two state system. What are the possible symmetries of Hamiltonians of such a system?

Take a three state system. What are the possible symmetries of Hamiltonians of such a system?

Take a four state system. What are the possible symmetries of Hamiltonians of such a system?

Take an N-state system. What are the possible symmetries of Hamiltonians of such a system?

Problem 13.5a



What are the symmetry groups of these potentials? How many elements do they have? Write down the group multiplication table.

Problem 13.5b

Given the symmetry groups of the potentials in Problem 13.5a, what can one say about the degeneracies of energy levels? Can one relate this to the irreducible representations of the group?

Now construct the Hilbert space of 2, 3, *etc.*particle states with the same Hamiltonian. What can one say about the degeneracies of the energies of these states? What does this have to do with the Clebsch-Gordan series for these groups?

How does one find the Clebsch-Gordan coefficients systematically for these groups? Does this decomposition force symmetries under the interchange of the particles?

Problem 13.6

1. Suppose R_1, R_2, *etc.*, are transformations of the coordinates which leave the potential unchanged. Do these form a group?
2. Next suppose that Π_1, Π_2, *etc.*, are transformations of the Hilbert space which correspond to the action of R_1, R_2, *etc.*, on space. Then, do these form the same group? Does the group multiplication table determine the commutators of Π_1, Π_2, *etc.*?
3. Does H commute with all of Π_1, Π_2, *etc.*?
4. What determines the degeneracies of the eigenvalues of H?

Problem 13.7a

An isotropic harmonic oscillator in 3 dimensions has the potential

$$V \propto \frac{1}{2}\left(x^2 + y^2 + z^2\right).$$

This is invariant under 3-dimensional rotations, *i.e.*, under the Lie group SO(3). However, the Hamiltonian

$$H = \frac{\omega}{2}\left[p_x^2 + p_y^2 + p_z^2 + x^2 + y^2 + z^2\right],$$

lives in a 6-dimensional phase space, which transforms under the symplectic group Sp(6,R). A subgroup of this, called U(3), leaves H invariant.

In the quantum theory, one finds the algebra of this Lie group is generated by Hermitean operators constructed from the bilinear forms $a_i a_j^\dagger$. How many such Hermitean operators are there? What is the algebra of these operators? Can you show that they commute with the Hamiltonian?

Problem 13.7b

In the previous part you checked that the symmetries of the Hamiltonian of the 3D isotropic harmonic oscillator is invariant under the action of the group U(3). What does this imply for the degeneracies of H? How would you relate these to the irreducible representations of SU(3)?

What will happen to 2-particle systems with this Hamiltonian? How will you construct the Clebsch-Gordan series for SU(3)? How will you construct the Clebsch-Gordan coefficients? Does any of these properties have anything to do with symmetries of the interchange of the two particles?

How does one generalize this to three or more particles?

Problem 13.8

Previously we have only considered the symmetries of the potential. The example of the isotropic harmonic oscillator shows that the symmetry group of the (classical) Hamiltonian can be larger than the symmetry group of the potential.

In Problems 13.3 and 13.4, does this give you more information about the degeneracies of the Hamiltonian?

Outline

Keywords and References

Keywords

Hilbert space, group, transformation of coordinates, group multiplication table, degenerate eigenvalues, invariants, symmetry groups, symmetries of Hamiltonians, symmetries under interchange, permutation groups, degeneracies of energy levels, irreducible representations, Clebsch-Gordan series, Clebsch-Gordan coefficients, isotropic harmonic oscillator, Lie group, orthogonal group, symplectic group, unitary group, SO(3), Sp(6,R), U(3), bilinear forms.

References

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Lie Groups, Physics, and Geometry: an Introduction for Physicists