Gauss' views on measurement

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Outline

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Natural units

Dimensional analysis as a tool of discovery

Keywords and References

Keywords

Principle of equivalence, Coulomb's law, Gaussian units, SI units, Lorentz force, electric field, magnetic field, Biot-Savart law, Maxwell's equations, speed of light, Joule's constant, Boltzmann's constant, Planck's constant, black body, Planck spectrum, Stefan-Boltzmann constant, ultraviolet catastrophe, Compton wavelength, Bohr radius, Rydberg's constant, Thompson scattering, protein folding, vacuum decay, Planck mass

Books

Measuring the World, by Daniel Kehlmann.
Electrodynamics, by J. D. Jackson.
Qualitative methods in QM, by A. B. Migdal.

The principle of equivalence

Newton's law for the gravitational force between bodies with masses m_1 and m_2 placed a distance r away from each other is—

$$\mathbf{F} = \kappa_N \frac{m_1 m_2}{r^2} \hat{\mathbf{r}}$$

where $[\kappa_N] = M^{-1} L^3 T^{-2}$. We have assumed the principle of equivalence: the masses which appear in the force law for gravity are the same as those which appear in the relation between force and acceleration.

Let us study the simple pendulum without it. The quantities of relevance are possibly the length, ℓ, the mass, m, the time period, t, and the force of gravity due to the earth, f. We can build only one dimensionless constant $\Pi = ft^2/(m\ell)$, so that $t \propto \sqrt{m\ell/f}$. The period of the pendulum could then depend on the material of the bob, or even the mass of the bob. Galileo's negative observations perhaps led to the demonstration at the tower of Pisa.

An aside on pendulums in physics education

In elementary physics courses pendulums are used in the classroom as an example of an easily solvable system, and an introduction to an oscillator. In the laboratory it is used as an instrument for a simple but precise measurement of $\mathbf{g}$.

Over the years its use as a simple but precise probe of the principle of equivalence (PoE) has been forgotten. By changing the mass of the bob, the size of the bob, the shape of the bob, the material of which the bob is made, *etc.*, one can introduce the PoE as a practical tool.

Galileo's purported experiment in Pisa (EiP) is counter-intuitive for students. Introducing the PoE using pendulums opens the way to using variants of the EiP to introduce the notion of viscosity and drag.

Electrodynamics: the insight of Gauss

Another classic force law is Coulomb's law of attraction between charges e_1 and e_2 placed a distance r apart

$$\mathbf{F} = \kappa_C \frac{e_1 e_2}{r^2} \hat{\mathbf{r}}.$$

Experiments show that e is completely independent of the mass of a body, since the same mass can have many different charge states. An interesting question is whether one needs to introduce two quantities, κ_C and the charge, or just one quantity.
Gaussian units prevent a proliferation of dimensions by making κ_C into a dimensionless quantity: unity. Then $[e] = M^{1/2} L^{3/2} T^{-1}$.
Gauss' insight is that any non-trivial κ_C can be absorbed into the definition of charge.
The solution by committee is called SI (Systeme International). This introduces a new unit, the Ampere, with $[e] = TA$. It also forces us to write $\kappa = (4\pi\epsilon_0)^{-1}$, with $[\epsilon_0] = M^{-1} L^{-3} T^4 A^2$.

Consequences of inappropriate choice of units

Another fundamental dynamical relation is the Lorentz force on a particle of charge e moving with speed $\mathbf{v}$ in external electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$—

$$\mathbf{F} = e\left(\kappa_E \mathbf{E} + \kappa_B \mathbf{v} \times \mathbf{B}\right).$$

The coupling $\kappa_E = 1$ is a dimensionless unity in both Gaussian and SI units. In Gaussian units $\kappa_B = 1/c$, the speed of light, whereas $\kappa_B = 1$ in SI. As a result, in Gaussian units, $[\mathbf{E}] = [\mathbf{B}] = M^{1/2}L^{-1/2}T^{-1}$. Then $[\mathbf{E}^2]$ and $[\mathbf{B}^2]$ are energy densities.

These clear physical relations are lost in SI, where $[\mathbf{E}] = MLT^{-3}A^{-1}$ and $[\mathbf{B}] = MT^{-2}A^{-1}$.

Completing consistency checks

The Biot-Savart law describes the magnetic field, $\mathbf{B}$, generated at a distance r from a wire carrying current J. If a length element of the wire is denoted by $d\mathbf{x}$ and $\mathbf{r}$ points from the wire to the point where the field is measured, then

$$\mathbf{B} = \kappa_S \oint \frac{J}{r^3} d\mathbf{x} \times \mathbf{r}.$$

In Gaussian units the integral on the right has dimensions $M^{1/2}L^{1/2}T^{-2}$, and as a result, $[\kappa_S] = L^{-1}T$. Measurements show that $\kappa_S = 1/c$.

SI becomes really ugly at this point. The integral has dimensions $L^{-1}A$, whereas $[\mathbf{B}] = MT^{-2}A^{-1}$. The constant is written as $\kappa_s = \mu_0/(4\pi)$, where $[\mu_0] = MLT^{-2}A^{-2}$. This ugly formulation remains consistent because $\epsilon_0\mu_0 = 1/c^2$.

Irrationality

Different field definitions

Another elegant consequence of Gaussian units is the clear recognition that there is no physical difference between the displacement field, **D**, and the electric field **E**. In fact, $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$.

However, in SI the relations are $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$. The appearance of $\epsilon_0 = 1/(4\pi\kappa_c)$ makes it clear that this difference has no physical basis, but is merely a consequence of choice of units.

Gauss' law

The removal of irrational numbers like 4π from equations is sometimes listed among the positive features of SI. Gauss' law becomes $\nabla \cdot \mathbf{E} = \rho$ in SI. This can be counted as a gain if the definition of solid angles is changed.

Maxwell's equations

Faraday's law of induction in Gaussian units is

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}.$$

This clearly presages the physics of relativity, where **E** and **B** are combined into a single tensor field, and ct and x are identified. In SI the $1/c$ on the right is absorbed into units of fields.
The Ampere-Maxwell equation also shows physics very clearly in Gaussian units

$$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}.$$

The relation between t and x is clear, and the equivalence of the pairs **D** and **E** on one hand and **H** and **B** on the other, makes the transition between fields in material and vacuum very easy to understand.

Units: Electrodynamics in Systeme Internationale

Since SI contains a redundant unit, we will not use symbols such as [**E**] to talk about dimensions in SI, but write instead $\{\mathbf{E}\}$. For reference, we collect together these definitions.

$$\begin{aligned}
\{\mathbf{J}\} &= \mathrm{A} \qquad \text{ampere} \\
\{\mathbf{e}\} &= \mathrm{A\,s} = \mathrm{C} \qquad \text{coulomb} \\
\{\mathbf{E}\} &= \mathrm{kg\,m}/(\mathrm{A\,s}^2) \\
\{\Phi\} &= \mathrm{kg\,m}^2/(\mathrm{A\,s}^3) = \mathrm{V} \qquad \text{volt} \\
\{\mathbf{H}\} &= \mathrm{kg}/(\mathrm{A\,s}^{-2}) = \mathrm{T} \qquad \text{tesla} \\
\{\mathbf{B}\} &= \mathrm{kg\,m}^2/(\mathrm{A\,s}^2) = \mathrm{Wb} \qquad \text{weber}
\end{aligned}$$

Gaussian ideas in relativity

Experiments tell us that the beautiful consequences of Gauss' ideas about measurements can be taken further.

The speed of light in vacuum, c, is a limiting speed, and frame invariant. By setting $c = 1$, one will not obscure any physics. Instead one can actually improve understanding in some ways. The first consequence is $[T] = L$; kinematics becomes geometry. Acceleration has unit $[g] = L^{-1}$, which is the same as curvature. The geometry is that of world lines in spacetime.

Dynamics simplifies because $[F] = ML^{-1}$ and $[E] = M$. This last relation is the famous equation $E = m$. There are immediate consequences for Gaussian electrodynamics: $[e] = M^{1/2}L^{1/2}$, $[J] = M^{1/2}L^{1/2}$ and $[E] = [B] = ML^{-3/2}$. There are also consequences for gravity: $[\kappa_N] = M^{-1}L$: mass induces curvature of worldlines.

Gaussian ideas in thermodynamics

Joule's constant expresses the fact that energy and heat are interconvertible. Setting $J = 1$ means that heat has the units of energy.

A fundamental relation in thermodynamics is the combination of various classical laws concerning ideal gases which are combined into the equation relating the pressure, volume, and temperature: $PV = Nk_B\Theta$, where N is the number of molecules under consideration and k_B is called the Boltzmann constant. This is measured to be an universal constant and therefore we are allowed to set $k_B = 1$ in our choice of units. Then $[\Theta] = [PV]$, so temperature can be measured in units of energy.

With this choice, $P/\Theta = N/V$, where the right hand side is clearly the number of molecules per unit volume. In other words, the mean interparticle spacing is $\sqrt[3]{\Theta/P}$.

More relativity

We need to put together relativity and thermodynamics when we study materials whose constituents travel at or near the speed of light: photons, neutrinos, quarks and gluons. Then one has $[Q] = [\Theta] = M$ where Q denotes heat. Since transport of energy is not dimensionally distinguishable from transport of material, the distinction between heat conduction and convection has to be rethought.

Another line of development is to set $\kappa_N = 1$ in the context of relativity. This would reduce the number of dimensions further, because then $[M] = L$. Dynamics becomes geometry: forces become dimensionless, energy becomes length. The principle of equivalence can be extended to say that an accelerated frame of reference cannot be distinguished from the effects of gravity by any local measurement.

Quantum mechanics

The Planck's constant, $\hbar$ relates the energy E of a quantum of light with its frequency ω: $E = \hbar\omega$. Since $\hbar$ is an universal constant, we can treat it as part of our definition of units and set $\hbar = 1$. Then $[E] = T^{-1}$. As a result, angular momentum no longer has dimensions $[J] = ML^2T^{-1}$, but becomes dimensionless. This also connects to electrodynamics, by converting the electric charge into a velocity $[e^2] = LT^{-1}$. Also, $[\mathbf{E}^2] = [\mathbf{B}^2] = L^{-3}T^{-1}$.

These last relations become clearer when relativity and quantum mechanics are combined. Then, since $c = \hbar = 1$, we find that in Gaussian units mass and energy have the same dimensions and length and time have the inverse dimensions: $[\ell] = M^{-1}$. As a result, e^2 is dimensionless, and $[\mathbf{E}^2] = [\mathbf{B}^2] = M^4$. Gravity also simplifies, with $[\kappa_N] = M^{-2}$. If we also set $\kappa_N = 1$, there are no dimensions left in physics.

Black-body radiation

Since quantum mechanics was first recognized in the context of black body radiation, we would expect this physics to become simple in natural units.

Black body radiation is a volume filled with electromagnetic radiation fields at a fixed external temperature Θ. Since Θ is the only external parameter, it must provide the common scale of length, time, and energy. The failure of classical theory was that it had $\hbar = 0$, and therefore required another scale, which was not available in the problem.

We are interested in the spectrum of the field, $U(\omega, \Theta)$, which is the energy density per unit range of ω. Clearly $[U] = M^3$. As a result, U/ω^3 is a dimensionless variable, and the spectrum must be of the form

$$U(\omega, \Theta) = \omega^3 f(\omega/\Theta).$$

A Gaussian route from black bodies to quantum mechanics

This correct quantum solution, $U(\omega) = \omega^3 f(\omega/\Theta)$, is subject to the Stefan-Boltzmann law for the energy density of a black body;

$$\mathcal{U} = \int_0^\infty d\omega U(\omega, \Theta) = \sigma\Theta^4, \qquad \text{so} \qquad \int_0^\infty x^3 f(x) dx = \sigma.$$

where the constant, σ, is obtained from experiment. Planck, and, later, Bose could derive $f(x)$ by a simple argument.
In classical physics, where $\hbar = 0$, the dimension counting becomes $[\Theta] = M$, $[\omega] = L^{-1}$ and $[U] = ML^{-2}$. This leads to the Rayleigh-Jeans law $U(\omega) \propto \Theta\omega^2$. As a result, $\mathcal{U}$ diverges. This ultraviolet catastrophe can be arrested by a mathematical artefact: cut off the range of integration with a short length scale a. Then we would obtain $\mathcal{U} \propto \Theta/a^3$, but not the SB law. The crucial physics to understand is that if σ is made dimensionless, then $[\Theta] = [\omega]$, and the problem is solved.

Physics is an experimental science

A critique of pure reason

The universe is not only queerer than we suppose, it is queerer than we *can* suppose.
J. B. S. Haldane in *Possible Worlds and Other Papers*, 1927.

No amount of reasoning without experiment could have convinced an 18th century physicist that heat and energy are the same: it required painstaking experiments to establish that. More than a century of experiments was needed to establish Maxwell's equations, and even a couple of decades more to establish that the speed of light was frame independent, and that the universe really had a limiting speed. Several decades of experiments led to the black body laws. Theory is used to draw new conclusions: such that there are no dimensional quantities in the universe. But without experiments, all such reasoning is vacuous.

Appropriate units for atomic phenomena

The unit of mass appropriate for computations at the level of atomic phenomena is the atomic mass unit (AMU). Define $\overline{\mathrm{AMU}}$ by the equation $m_p = 1\ \overline{\mathrm{AMU}}$. The charge of a proton is $\alpha = e^2 \simeq 1/137$.

Since $m_e = 1/2000\ \overline{\mathrm{AMU}}$, the reduced mass of the electron-proton system is almost the same as the electron's. As a result, atomic properties depend only on m_e and α. Here are the interesting scales:

1. The electron's Compton wavelength: $\lambda_e = 1/m_e$.
2. The Bohr radius: $a_0 = 1/(\alpha m_e)$
3. The electron's classical radius: $r_e = \alpha/m_e$
4. The Rydberg's constant: $R = 2\pi^2\alpha^2 m_e$
5. The Thompson scattering cross section: $\sigma = 8\pi\alpha^2/(2m_e^2)$.

The powers of m_e come purely from dimensional analysis. The interesting dynamical question is where the powers of α come from.

Units: The atomic scale and SI

The atomic unit of energy, eV, is the energy gained by an electron when it falls through a potential difference of a Volt, defined to be Joule per Coulomb. The Coulomb is the SI derived unit of charge, which is one Ampere second. From the electron's charge in SI

$$e = 1.60217646 \times 10^{-19}\ \mathrm{C} \quad \text{we obtain} \quad \mathrm{eV} = 1.60217646 \times 10^{-19}\ \mathrm{J}.$$

The electron and proton masses in these units are

$$m_e = 0.510999\ \mathrm{MeV}, \qquad \text{and} \qquad m_p = 938.272\ \mathrm{MeV}.$$

Derived units of length are sometimes used for convenience in atomic and high energy physics. The commonly used units are nm $= 10^{-9}$ m and fm (femtometer or fermi) $= 10^{-15}$ m. The conversion to MeV units is most conveniently done using the relation $\hbar c = 1$, so

$$1 = 197.326972\ \mathrm{MeV\,fm}.$$

Obtaining the scales of atomic physics

The speed of an electron in a Hydrogen atom bound state is a dimensionless number. As a result, $v \simeq \alpha$.

The momentum of an electron in a bound state must therefore be $p \simeq \alpha m_e$. The uncertainty principle therefore implies that the typical size of such a bound state is $a_0 \simeq 1/(\alpha m_e)$.

The typical kinetic energy of such a bound state is $p^2/(2m) \simeq \alpha^2 m_e$. If the virial theorem holds, then this is also the scale of the binding energy, R. Alternatively, the frequency of an electron in its orbit is $v/a_0 = \alpha/a_0 = \alpha^2 m_e$, so this is R.

The above three arguments depend on there being no other dimensionless scale in the problem. In quantum states of large angular momentum, J is such a scale, and the arguments may not hold. However, inner shells have $J \simeq 1$, so the arguments above can be applied to inner shells.

Units: Migdal's molecular measuring-scale

Since electronic velocities are of order α, the expansion of scales in powers of α is a non-relativistic expansion. All it means is that c is not an appropriate unit in the molecular domain.

Migdal suggests that it is more appropriate to define scales using $\hbar = 1$ and $\alpha = 1$. We have seen earlier that setting $\hbar = 1$ gives α the units of velocity, and setting this to unity gives $[\ell] = T^{-1}$. In these units

$$c = \frac{1}{\alpha} = 137.035999.$$

The appropriate mass unit is not $\overline{\mathrm{AMU}}$, but m_e. In Migdal's units λ_e, a_0 and $1/R$ all have the same magnitude: 1.

How strong are the strong forces?

A nucleus contains uncharged and positively charged particles only. The fact that many nuclei are stable means that electrical forces are not the cause of the binding. There has to be a new attractive force which binds the nucleus; this is called the strong force.

A simple estimate of its strength can be made from the binding energy of the simplest nucleus: deuteron. It contains one proton and one neutron. The reduced mass of the system, $M = m_p/2 \simeq 469$ MeV. The binding energy, $B \simeq 2$ MeV.

If there is a dimensionless coupling, f_N, which is responsible for the binding, and the proton and neutron are non-relativistic, then $v = f_N$. Also, $B \simeq f_N^2 M$, giving $f_N \simeq 0.065$. This is about 10 times larger than α. More sophisticated analyses give similar results.

V. Stoks, R. Timmermans and J. J. de Swart, *Phys. Rev.* **C 47 (1993) 512.**

Biomolecular energy scales

The shape of proteins is important for their function, but what causes it? A protein can be denatured by heating it. Raising the temperature of an egg by 40 K is enough to denature the proteins in the egg white.

$$k_B = 1.3806503 \times 10^{-23} \mathrm{kg\,m^2/(s^2\,K)}, \quad c = 2.99792458 \times 10^8 \mathrm{m/s}.$$

Also,

$$\hbar = 1.05457148 \times 10^{-34} \mathrm{kg\,m^2/s}, \quad \frac{1}{m_e} = 2.4263102 \times 10^{-12} \mathrm{m}.$$

So writing the dimensionless quantity $k_B/(\hbar c m_e)$ and setting $\hbar = c = 1$, we find $k_B = 1.06 \times 10^{-9} m_e/\mathrm{K}$. The typical energy of a chemical bond is $\alpha^2 m_e \approx 5 \times 10^{-5} m_e$. The thermal energy of 40 K is about 1000 times smaller. So the shape of proteins is not due to covalent chemical bonds.

Nonlinear electrodynamics: instability of the vacuum

Electrodynamics couples particles and fields. Currents, $\mathbf{J}$, due to the motion of particles are sources of the fields; $\mathbf{E}$ and $\mathbf{B}$ act back on moving charges through the Lorentz force. Now $[\mathbf{E}] = [\mathbf{B}] = M^2$ but $[\mathbf{J}] = M^3$. So the theory of coupled electrons and photons (electrodynamics: ED) is non-linear. We do not notice this because very often fields are small.

ED certainly becomes non-linear at the scale where the vacuum breaks down. The energy density in a field, $\mathcal{U}$, is $\mathcal{U} = |\mathbf{E}|^2$. The only other quantity in ED with this dimension is m_e^4. Hence the scale at with ED becomes non-linear is $|\mathbf{E}| \simeq m_e^2$. The breakdown is due to the fact that when $\mathcal{U}$ is larger, it is energetically more favourable for electron-positron pairs to be created everywhere in the vacuum to dynamically generate an opposite field to cancel the external field.

Tesla to mass conversions

The unit of **B** in SI is Tesla: $\mathrm{T} = \mathrm{kg A^{-1} s^{-2}}$. The Ampere can be related to the Coulomb by $\mathrm{A} = \mathrm{C\,s^{-1}}$, and

$$1\ \mathrm{C} = 96485.3399 \times (6.022 \times 10^{23}) e = 4.9634 \times 10^{27}.$$

Also,

$$\hbar = 1.05457148 \times 10^{-34} \mathrm{kg\,m^2/s}, \quad \frac{1}{m_e} = 2.4263102 \times 10^{-12} \mathrm{m}.$$

Putting these together, we find that

$$\mathrm{T} = 1.125 \times 10^{-16} m_e^2.$$

The strongest magnetic field generated in the laboratory is less than 100 T. The strogest magnetic fields seen in the universe occur around a kind of neutron stars called magnetars. These have fields of about 10 GT, about 10 million times smaller than the critical field for the vacuum.

How weak is the weak interaction?

Free neutrons decay into protons with the emission of an electron and an antineutrino in a process called β-decay. The half-life, $\tau = 887$ s, is controlled by the energy difference between the initial and final states: $\delta m \simeq m_n - m_p - m_e = 0.782$ MeV. The dimensionless constant

$$\tau \delta m = 5 \times 10^{-4}.$$

If the decay were due to electromagnetic forces, then the constant would have been of the order of magnitude of $\alpha^2 = 5 \times 10^{-5}$. If it was due to strong interactions, then it would have been of the order of $f_N^2 = 0.004$.

In fact the dimensionless number is almost the geometric mean of the two! So it would seem that there is a fourth force which is responsible for β-decays. This is called the weak force.

How strong is gravity?

A comparison of two kinds of force laws would obviouly have to involve dimensionless numbers: for example comparing the forces under standardized conditions. Since both the Coulomb's and Newton's laws involve the same power of the separation, we can compare them at any distance. So the relative strength is

$$\phi = \frac{\kappa_N m^2}{e^2} = \left(\frac{\sqrt{\kappa_N} m}{e}\right)^2, \qquad \text{and} \qquad \kappa_N = 1/(1.30 \times 10^{19}\ \overline{AMU})^2.$$

For an electron $\phi \simeq 1.5 \times 10^{-45}$. Most particles have charges which are small integer multiples of the electron's charge. However, their masses can be very different: $m_p \simeq 2000 m_e$. However, even for the proton, $\phi \simeq 6 \times 10^{-39}$.

The inverse of the square root of Newton's constant is called the Planck mass. The statement of the weakness of gravity is the same as the statement that m_{Pl} is large on the atomic scale. In fact, $m_{Pl} \simeq 22\ \mu$g.