Similarity and incomplete similarity

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Outline

Similarity: Rayleigh-Benard Convection

Incomplete similarity: fractals

The renormalization group

Inclusive and semi-inclusive cross sections

Keywords and References

Keywords

Convection, Fourier's law, Boussinesq's approximation, shear viscosity, Joule's constant, Boltzmann's constant, Prandtl number, Rayleigh number, Rayleigh-Benard convection, scaling, incomplete scaling, anomalous dimension, fractal, fractal dimension, broken scale invariance, scaling and the renormalization group, inclusive cross sections, semi-inclusive cross sections, scaling, broken scaling, strong interactions.

Books

Scaling, by G. I. Barenblatt, Ch 1, 7.

The experimental setup

The simplest setup for studying convection would be fluid contained within two walls— the lower one at temperature Θ_0 and the upper one at $\Theta_0 + d\Theta$ (we need not assume that $d\Theta$ is positive). The distance between these walls is H.

Heat is transferred in two ways. One is by conduction. Fourier's law for heat conduction states that the flux of heat, q, is given by

$$q = -\lambda \frac{d\Theta}{H},$$

where λ is called the heat conductivity. The heat flux is the heat that passes through a unit area in unit time. The second method is by convection. Materials become lighter on being heated, and rise against gravity, carrying the heat stored in the material. This latter depends on the specific heat, c, defined as the increase in the heat content of a material per unit mass per degree rise in temperature.

Other relevant variables

Heated fluids rise against gravity since the density decreases with increase in temperature. If $d\Theta$ is small, then one can apply Boussinesq's approximation. This states that the buoyancy will be αg, where α is the fractional increase in volume per degree rise in temperature and g is the acceleration due to gravity. One also needs the density of the fluid, ρ, measured at temperature Θ_0. Since the fluid is moving, its viscosity, η, at the temperature T_0, may play a role in the problem. Newton's experiment utilizes the same setup to measure viscosity, by measuring the strain (force per unit area), τ required to move the upper plate at a constant velocity, U, against the fluid. Then, one has

$$\tau = \eta \frac{U}{H},$$

We will use units with Joule's constant, $J = 1$, and Boltzmann's constant, $k_B = 1$.

Dimensions

The dimensions needed are the usual mechanical quantities M, L, T. Since $J = k_B = 1$, the units of heat and temperature can be taken to be those of energy. From the definitions

$$[H] = L, \quad [\rho] = ML^{-3}, \quad [\eta] = ML^{-1}T^{-1}, \quad [d\Theta] = ML^2T^{-2}.$$

The new thermal quantities are

$$[\alpha g] = M^{-1}L^{-1}, \quad [c] = M^{-1}, \quad [\lambda] = L^{-1}T^{-1}.$$

With 7 quantities and 3 different dimensions, one will have 4 independent dimensionless quantities. Fluid properties encode microscopic properties of materials, and the experimentally determined parameters are the mesoscopic scales of the experiment. There are two of them: H is the length scale of the experiment, and $d\Theta$ is the energy scale. We would like to find the relation between them when convection sets in.

Intrinsic length scales

Note that ρ and αg involve only length and mass. So one can construct two quantities of dimension length using these two and c. One is $r = 1/(c\rho)^{1/3}$; for water we find it is about 0.15 nm. The other quantity is $\ell = c/(\alpha g)$. For water we find that $\ell = 2000$ Km. Note that both of these length scales are intrinsic to the fluid, and must be due to its molecular properties. Clearly, H is a mesoscopic scale which lies between the two.

The dimensionless quantities we build out of these two length are

$$\Pi_4 = c\rho H^3 = \left(\frac{H}{r}\right)^3, \qquad \Pi_3 = \frac{c}{\alpha g H} = \frac{\ell}{H}.$$

Since the experiment will typically involve $H \simeq 0.1$ m. Then, of course, $\Pi_4 \gg 1$ and $\Pi_3 \gg 1$. Also, since $\ell/r \simeq 10^{15}$, there is a large range of H for which $\Pi_{3,4} \gg 1$.

Two intrinsic time scales

There are two time scales in the problem: one is a frictional time scale $t = 1/(\eta H c)$, the other is due to heat conduction $\tau = 1/(\lambda H)$. Both are due to the interplay between intrinsic fluid parameters and the external experimental setup.
The dimensionless Prandtl number is a combination which depends only on properties of the fluid—

$$\Pi_2 = \mathrm{Pr} = \frac{\eta c}{\lambda} = \frac{\tau}{t}.$$

This determines the relative importance of frictional losses. For water at about 300 K, Pr is of the order of 10^6.
For $H = 0.1$ m, the larger of the two times scales, $\tau \simeq 10^{-22}$ s. A mesoscopic time scale is set by $\sqrt{H^2/d\Theta c}$. If we set $d\Theta \simeq 10$ K, then this is about 0.5 ms. Since there is a tremendous mismatch between microscopic and experimental time scales, the former may not be easy guides to the physics of interest.

Comparison of forces

Instead, we should focus on the actual physics of convection. Heating causes expansion, and the resulting buoyancy forces the fluid to move. This is opposed by dissipative forces, due entirely to intrinsic quantities.

The dissipative force has to be proportional to η, and must involve the microscopic volume $r^3 = 1/c\rho$. The simplest quantity which makes a force using these is $f = \eta\lambda/(c\rho)$. Since the driving force is buoyancy, it must involve $\alpha g \rho d\Theta$. Since the fluid volume is H^3, the driving force $F = \alpha g \rho d\Theta H^3$. The ratio is called the Raleigh number,

$$\Pi_1 = \mathrm{Ra} = \frac{F}{f} = \frac{\alpha g c \rho^2 d\Theta H^3}{\eta\lambda}.$$

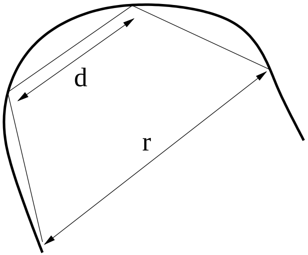
The solution of the problem is a relation $\mathrm{Ra} = f(\Pi_2, \Pi_3, \Pi_4)$. Clearly, this problem would have been simplest to solve in terms of units of force, length and time.

Similarity flow

$\mathrm{Ra} = f(\Pi_2, \Pi_3, \Pi_4)$ could either tend to a constant or increase without bound as $\Pi_{3,4} \to \infty$. If it is constant, then we can take $f(\Pi_2, \infty, \infty) = \overline{f}(\Pi_2)$. This gives a similarity flow. The nomenclature comes from the fact Π_2 is fixed for a given liquid, and so, flows for different H and $d\Theta$ must have the same value of Ra. This gives the scaling law $d\Theta \propto 1/H^3$. If this works, then it gives evidence that our hypothesis is correct.

For $\mathrm{Ra} < R_c$ there is no convection; viscous friction wins. For $\mathrm{Ra} > R_c$, buoyancy wins, convection sets in and cells of oppositely circulating fluid are created. $R_c \simeq 650$ when the lower surface of the fluid is in contact with a rigid surface, and the upper surface is free. When both bounding surfaces are rigid, one finds $R_c \simeq 1700$. Our analysis has not touched on the size of cells.

Perimeters and polygons



The arc length of a curve, L, whose ends are a distance r apart, can be approximated by a segment of a polygon with sides of length, d. A dimensional argument tells us that

$$L(r) = r \lim_{d \to 0} \Phi\left(\frac{r}{d}\right),$$

since one gets successively better approximations to the curve by decreasing the size of the edge of the approximating polygon.

Similarity and incomplete similarity

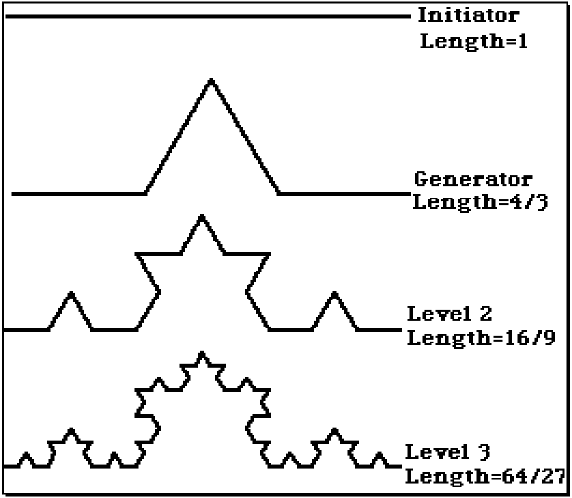
In the limit $\Pi = r/d$ goes to infinity. A similarity hypothesis is that $\Phi(\infty)$ has a good limit. For example, for a semicircle, $\Phi(\infty) = \pi/2$. These are simple curves described by functions which are smooth, with continuous first derivatives.

However, for most curves the limit does not exist. For a larger class of these curves a fairly simple hypothesis called incomplete scaling may hold. This is the assumption that $\Phi(\Pi) = \Pi^\alpha$, where α is a positive number. In this case $L(r) \propto r^{1+\alpha}$, although the limit does not exist. The curve is called a fractal; α is called an anomalous dimension and $D = 1 + \alpha$ is called a fractal dimension.

Fractals have broken scale invariance

For fractals the unit of the measuring scale, d, leaves a trace. Invariance with respect to d is broken, and this quantity does not disappear from the formula. This is the origin of the anomalous dimension.

The Koch curve



The set of fractals is not empty!

Dimensional analysis and homogenous functions

Recall how dimensional analysis involves homogenous functions. In a physical problem list all the n variables of interest, choose the basis set of m of them, U_1, U_2, $\cdots$, U_m, and write the others as A_1, A_2, $\cdots$, A_k, where $k + m = n$. Then a physical relation will be

$$A_1 = F(U_1, U_2, \cdots, U_m, A_2, A_3, \cdots, A_k).$$

One can construct k dimensionless quantities, $\Pi_i = A_i / \prod U_j^{a_{ij}}$. In terms of these one can write the same relation as

$$A_1 = U_1^{a_{11}} U_2^{a_{12}} \cdots U_m^{a_{1m}} \; f\left(\frac{A_2}{U_1^{a_{21}} U_2^{a_{22}} \cdots U_m^{a_{2m}}}, \cdots \frac{A_k}{U_1^{a_{k1}} U_2^{a_{k2}} \cdots U_m^{a_{km}}} \right),$$

or, more compactly as,

$$\Pi_1 = f(\Pi_2, \Pi_3, \cdots, \Pi_k).$$

Similarity and incomplete similarity

The function f is said to possess similarity if it has a finite and non-zero limit as any one of these Π_i go to zero or infinity. Then, in that limit, the functional dependence can be dropped to give

$$\Pi_1 = f(\Pi_2, \Pi_3, \cdots, \Pi_{k-1}) \qquad \Pi_k \to \infty.$$

Since the labelling of the variables is immaterial, in the case when $k - \ell$ of the dependences can be dropped we write

$$\Pi_1 = f(\Pi_2, \Pi_3, \cdots, \Pi_\ell) \qquad \Pi_i \to \infty \quad \forall i > \ell.$$

Clearly this is a very special class of functions. In general no simplification can occur. However, there is a wider class which has the property of incomplete similarity

$$\Pi_1 = \left[\prod_{i>\ell} \Pi_i^{\alpha_{1i}}\right] g\left(\frac{\Pi_2}{\prod_{i>\ell} \Pi_i^{\alpha_{2i}}}, \cdots, \frac{\Pi_\ell}{\prod_{i>\ell} \Pi_i^{\alpha_{\ell i}}}\right).$$

The exponents α_{ji} are called anomalous dimensions.

Incomplete similarity

Incomplete symmetry also involves homogeneous functions and can be written in terms of renormalized variables, $\Pi_j^* = \Pi_j / \prod_{i>\ell} \Pi_i^{\alpha_{ji}}$, as

$$\Pi_1^* = g(\Pi_2^*, \Pi_3^*, \cdots, \Pi_\ell^*).$$

Note that similarity is included within incomplete similarity as the special case when all $\alpha_{ij} = 0$.

We saw earlier that the tool for developing dimensional analysis is the invariance of physics under scaling of units of measurement. The tool for investigating incomplete symmetry is also such scaling, restricted to the domain where some of the dimensionless parameters become large or small. This analysis is called a renormalization group analysis.

Finally, note that there are many functions f which do not allow compression of variables beyond ordinary dimensional analysis.

Renormalization group transformations

The renormalization group arises if there is a scale invariance of the form

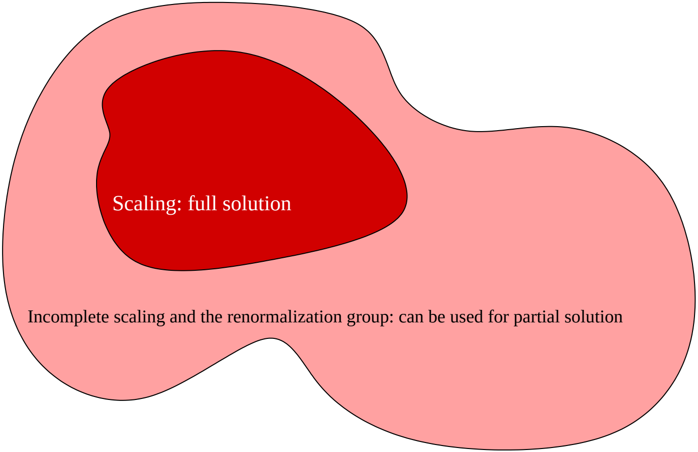
$$\Pi_i = \xi_i \Pi_i \quad \text{for} \quad \ell + 1 \le i \le k$$
$$\Pi_j = \left[\prod_{i>\ell} \xi_i^{\alpha_{ji}}\right] \Pi_i \quad \text{for} \quad 1 \le j \le \ell.$$

Since this is supposed to hold only for very large or very small values of Π_i (with $i > \ell$), the range of ξ_i is restricted to be not very different from 1. The proof that this leads to incomplete similarity follows the same lines as dimensional analysis.

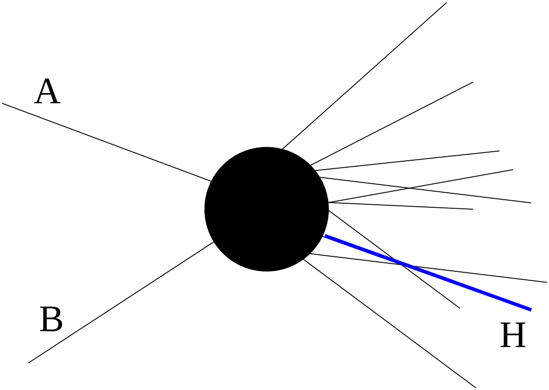
In the most general case, even this kind of restricted scaling does not hold. Then dimensional analysis gives much weaker results.

What dimensional analysis leads to

Only dimensional analysis: can be used to discover new phenomena



Inclusive scattering



Inclusive cross section for the reaction $A + B \to$anything: σ.
Semi-inclusive cross section for $A + B \to H+$anything: σ_H^{AB}. The only observation about the final state is whether H is produced or not.

Identify the variables

We want to study the inclusive scattering cross section, σ. The initial state momenta must be important variables. In the CM frame of the initial particles

$$P_A = (E, 0, 0, p), \qquad P_B = (E', 0, 0, -p).$$

The cross section is Lorentz invariant, so it cannot depend on the components of the momenta, only on dot products of vectors. Define

$$S = (P_A + P_B)^2.$$

Since $P_A + P_B = (E + E', 0, 0, 0)$, therefore $\sqrt{S}$ is the CM energy. M_A and M_B could also be important variables. So we have

$$\sigma = f(S, M_A, M_B).$$

Dimensional analysis

Since we have a relativistic quantum problem, we set $\hbar = c = 1$. Then

$$[\sigma] = M^{-2}, \quad [S] = M^2, \quad [M_A] = [M_B] = M.$$

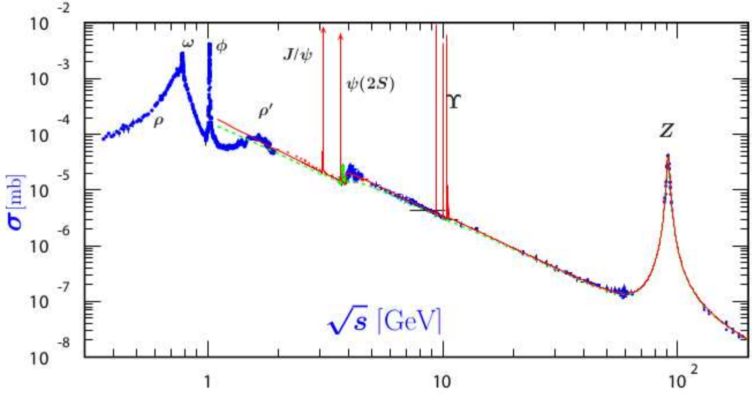
There are 4 variables and one dimension, so there are 3 dimensionless combinations. Write them as

$$\Pi_1 = \sigma S, \quad \Pi_2 = \frac{M_A}{\sqrt{S}}, \quad \Pi_3 = \frac{M_B}{\sqrt{S}}.$$

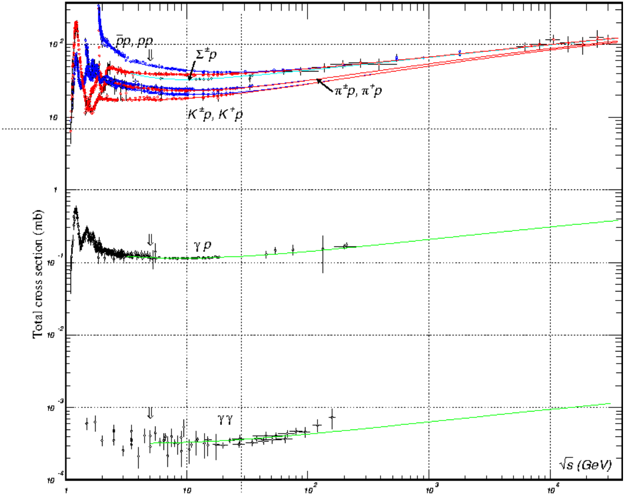
Then we have $\sigma = f(\Pi_2, \Pi_3)/S$.

The behaviour at very high energy could scale. Since $\Pi_{2,3} \to 0$, we can assume that f goes to a constant. Then $\sigma = f/S$.

Electron cross sections: scaling



Hadron cross sections: weird



Broken scaling

Dimensional analysis of cross sections gives $\sigma = f(\Pi_2, \Pi_3)/S$. For asymptotically high energy, as $\Pi_{2,3} \to 0$, if there is broken scaling, then one may be able to write

$$\sigma \propto S^{\alpha/2-1}.$$

The anomalous dimension α is compensated by an appropriate power of one of the masses.

This result comes from the behaviour $f(\Pi_2, \Pi_3) \simeq \Pi_2^\alpha g(\Pi_3/\Pi_2^\beta)$. In that case there may be subleading corrections of the form of $g(S^{(\beta-1)/2})$. Cross sections which rise asymptotically are possible examples of such broken scaling. Computing the values of the anomalous dimensions α and β from the theory of strong interactions is an open problem.