Regularization and renormalization

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Outline

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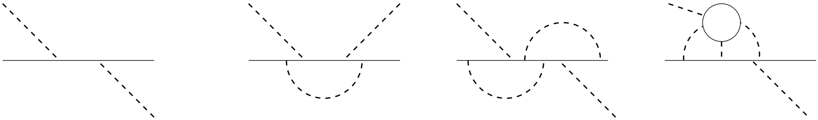
Perturbation theory: expansion of amplitudes in loops



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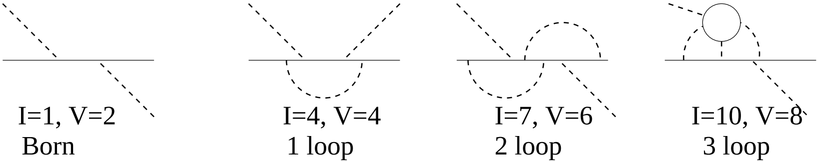
Perturbation theory: expansion of amplitudes in loops



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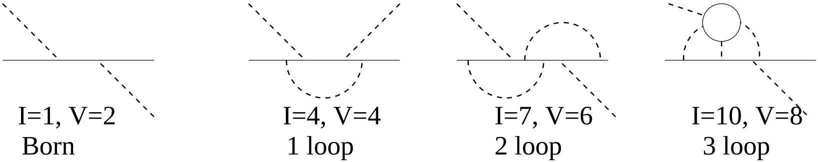

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$$L = 1 + I - V$$

Problem 2.1

Prove the equation. Prove that the expansion in loops is an expansion in $\hbar$, so is a semi-classical expansion. The number of unconstrained momenta is equal to the number of loops, giving an integral over each loop momenta. (Hint: See section 6.2 of *Quantum Field Theory*, by Itzykson and Zuber.)

Ultraviolet divergences

Typical loop diagrams give rise to integrals of the form

$$I_n^m = \int \frac{d^4k}{(2\pi)^4} \, \frac{k^{2m}}{(k^2+\ell^2)^n}$$

where k is the loop momentum and ℓ may be some function of the other momenta and the masses. When $2m+4 \geq 2n$, then the integral diverges.

This can be regularized by putting an UV cutoff, Λ.

$$I_n^m = \frac{\Omega_4}{(2\pi)^4} \int_0^\Lambda \frac{k^{2m+3}dk}{(k^2+\ell^2)^n} = \frac{\Omega_4}{(2\pi)^4} \, \ell^{2(m-n)+4} F\left(\frac{\Lambda}{\ell}\right),$$

where Ω_4 is the result of doing the angular integration. The cutoff makes this a completely regular integral. As a result, the last part of the answer can be obtained entirely by dimensional analysis. What can we say about the limit $\Lambda \to \infty$?

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What can we say about the limit $\Lambda \to \infty$? ???

The old renormalization

We start with a Lagrangian, for example, the 4-Fermi theory:

$$\mathcal{L} = \frac{1}{2}\overline{\psi}\partial\!\!\!/\psi - \frac{1}{2}m\overline{\psi}\psi + \lambda(\overline{\psi}\psi)^2 + \cdots$$

Here all the parameters are finite, but the perturbative expansion diverges, as we saw. We add counter-terms

$$\mathcal{L}_c = \frac{1}{2}A\overline{\psi}\partial\!\!\!/\psi - \frac{1}{2}Bm\overline{\psi}\psi + \lambda C(\overline{\psi}\psi)^2 + \cdots$$

where A, B, C, *etc.*, are chosen to cancel all divergences in amplitudes. This gives the renormalized Lagrangian

$$\mathcal{L}_r = \frac{1}{2}\overline{\psi}_r\partial\!\!\!/\psi_r - \frac{1}{2}m_r\overline{\psi}_r\psi_r + \lambda_r(\overline{\psi}_r\psi_r)^2 + \cdots$$

Clearly, $\psi_r = Z_\psi\psi$ where $\psi_r = \sqrt{1+A}$, $m_r = m(1+B)/(1+A)$, $\lambda_r = \lambda(1+C)/(1+A)^2$, *etc.*. The 4-Fermi theory is an unrenormalizable theory since an infinite number of counter-terms are needed to cancel all the divergences arising from $\mathcal{L}$.

Review problems: understanding the old renormalization

Problem 2.2: Self-study

Study the proof of renormalizability of QED to see how one identifies all the divergences which appear at fixed-loop orders, and how it is shown that taking care of a fixed number of divergences (through counter-terms) is sufficient to render the perturbation theory finite. The curing of the divergence requires fitting a small set of parameters in the theory to experimental data (a choice of which data is to be fitted is called a renormalization scheme). As a result, the content of a QFT is to use some experimental data to predict others.

Problem 2.3

Follow the above steps in a 4-Fermi theory and find a 4-loop diagram which cannot be regularized using the counter-terms shown in $\mathcal{L}_c$. Would your arguments also go through for a scalar ϕ^4 theory? Unrenormalizable theories require infinite amount of input data.

Dimensional regularization

The UV divergences we are worried about can be cured if $D < 4$. So, instead of the four-dimensional integral, try to perform an integral in $4+\delta$ dimensions, and then take the limit $\delta \to 0^-$. Since everything is to be defined by analytic continuation, we will not worry about the sign of δ until the end.

The integrals of interest are

$$I_n^m = \int \frac{d^4k}{(2\pi)^4}\,\frac{k^{2m}}{(k^2+\ell^2)^n} \to \int \frac{d^{4+\delta}k}{\mu^\delta(2\pi)^{4+\delta}}\,\frac{(k_\delta^2+k^2)^m}{(k_\delta^2+k^2+\ell^2)^n},$$

where we have introduced an arbitrary mass scale, μ, in the second form of the integral in order to keep the dimension of I_n unchanged. Also, the square of the $4+\delta$ dimensional momentum, k, has been decomposed into its four dimensional part, k^2, and the remainder, k_δ^2.

Doing the integral in one step

Usually one does the integral in $4+\delta$ dimensions in one step:

$$
\begin{aligned}
I_n^m &= \mu^{4-D}\int \frac{d^Dk}{(2\pi)^D}\,\frac{k^{2m}}{(k^2+\ell^2)^n} \\
&= \ell^{2m+4-2n}\left(\frac{\ell}{\mu}\right)^{D-4}\,\frac{\Omega_D}{(2\pi)^D}\,\frac{\Gamma(m+D/2)\Gamma(n-m-D/2)}{2\Gamma(n)},
\end{aligned}
$$

where $\Omega_D=\Gamma(D/2)/(2\pi)^{D/2}$ is the volume of an unit sphere in D dimensions.

For $m=0$ and $n=1$, setting $D=4-2\epsilon$, the ϵ-dependent terms become

$$
\left(\frac{\ell^2}{4\pi\mu^2}\right)^{-\epsilon}\Gamma(-1+\epsilon) = -\frac{1}{\epsilon}+\gamma-1+\log\left[\frac{\ell^2}{4\pi\mu^2}\right]+\mathcal{O}(\epsilon),
$$

where γ is the Euler-Mascheroni constant.

Doing the integral in two steps

One can do this integral in two steps, as indicated by the decomposition given below

$$I_n^0 = \int \frac{d^4k}{(2\pi)^4} \, \frac{1}{(2\pi\mu)^\delta} \int \frac{d^\delta k}{(k_\delta^2 + k^2 + \ell^2)^n},$$

Simply by power counting, one knows that the internal integral should be a k-independent multiple of $(k^2+\ell^2)^{-n+\delta/2}$. In fact, this is most easily taken care of by the transformation of variables $k_\delta^2 = (k^2+\ell^2)x^2$. This gives

$$\int \frac{d^\delta k/(2\pi\mu)^\delta}{(k_\delta^2 + k^2 + \ell^2)^n} = \frac{1}{(k^2+\ell^2)^n} \left(\frac{k^2+\ell^2}{2\pi\mu}\right)^\delta \Omega_\delta \int \frac{x^{\delta-1}dx}{(1+x^2)^n}$$

where Ω_δ is the angular integral in δ dimensions. The last two factors depend only on δ and n, the first factor reproduces I_n, so the regularization is due to the second factor.

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Recognizing the regularization

The regulation becomes transparent by writing

$$\left(\frac{k^2+\ell^2}{2\pi\mu}\right)^{\delta} = \exp\left[\delta \log\left(\frac{k^2+\ell^2}{2\pi\mu}\right)\right].$$

For fixed μ, the logarithm goes to a constant when $k \to 0$. Also, the logarithm goes to $-\infty$ when $k \to \infty$. As a result, the regulating factor goes to zero provided $\delta < 0$. This is exactly the intuition we started from.

In the context of dimensional regularization, the quantity μ is called the renormalization scale. We have seen that it gives an ultraviolet cutoff. The important thing is that the scale μ is completely arbitrary, and has nothing to do with the range of applicability of the QFT.

Outline

The old renormalization

In 1929, Heisenberg and Pauli wrote down a general formulation for QFT and noted the problem of infinities in using perturbation theory. After 1947 the problem was considered solved. The general outline of the method is the following:

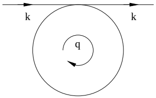
- Analyze perturbation theory for the loop integrals which have ultraviolet divergences.
- Regulate these divergences by putting an ultraviolet cutoff in some consistent way.
- Identify the independent sources of divergences, and add to the Lagrangian counter-terms which precisely cancel these divergences.
- QFTs are called renormalizable if there are a finite number of counter-terms needed to render perturbation theory useful.
- Use only renormalizable Lagrangians as models for physical phenomena.

Unrenormalizable terms

In this view, the unrenormalizable Lagrangian

$$\mathcal{L}_{\rm int} = -\lambda(\overline{\psi}\psi)^2,$$

was deemed impossible as a model for physical phenomena, since it needs an infinite number of counter-terms.



Examine its contribution to the fermion mass:

$$im\lambda \int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2-m^2} \propto \lambda m\Lambda^2,$$

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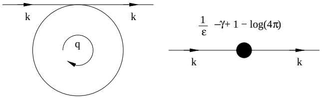

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Irrelevant terms

Today the same Lagrangian is written as

$$\mathcal{L}_{int} = -\frac{\lambda}{\Lambda^2}(\overline{\psi}\psi)^2,$$

where Λ is interpreted as a scale below which one should apply the theory.
The contribution to the mass is

$$\frac{im\lambda}{\Lambda^2}\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2-m^2} = \frac{m^3}{16\pi^2\Lambda^2}\left(-\frac{1}{\epsilon}+\gamma-1+\log\left[\frac{m^2}{4\pi\mu^2}\right]\right),$$

where the integral regulated by doing it in $4-2\epsilon$ dimensions. In the $\overline{\mathrm{MS}}$ renormalization scheme the counter-term subtracts the pole and the finite parts $\gamma-1-\log 4\pi$, leaving

$$\frac{\delta m}{m} = \frac{\lambda}{16\pi^2}\left(\frac{m}{\Lambda}\right)^2\log\left[\frac{m^2}{\mu^2}\right].$$

Separation of scales

The cutoff scale in the problem, Λ, is dissociated from the renormalization scale, μ, in dimensional regularization. This is not true in cutoff regularization. This separation of scales allows us to recognize two things:

- There is no divergence in the limit $\Lambda \to \infty$; instead the coupling becomes irrelevant. The theory remains predictive, because the effect of these terms is bounded.
- There are no large logarithms such as $\log(m/\Lambda)$. The amplitudes, computed to all orders are independent of μ, although fixed loop orders are not. In practical fixed loop-order computations, it is possible to choose $\mu \simeq m$, and reduce the dependence on this spurious scale.

Regularization schemes which do this are called mass-independent regularization. They are a crucial technical step in the new Wilsonian way of thinking about renormalization.

Is cutoff regularization wrong?

All regularizations must give the same results when the perturbation theory is done to all orders. Cutoff regularization is just more cumbersome.

Cutoff regularization retains all the problems of the old view: since the cutoff and renormalization scales are not separated, higher dimensional counter-terms are needed to cancel the worsening divergences at higher loop orders. When all is computed and cancelled, the m^2/Λ^2 and $\log(m/\Lambda)$ emerge.

In mass-independent regularization schemes, higher dimensional terms give smaller corrections because of larger powers of m/Λ.

In a renormalizable theory, since the number of counter-terms is finite and small, the equivalence of different regularizations is easier to see.

Chiral symmetry: an important secondary issue

In this example we find that $\delta m \propto m$; if the bare mass were zero, then the renormalized mass remains zero. There is a symmetry reason behind this.

A Dirac spinor can be resolved into left and right handed components using the projection operators $1 \pm \gamma_5$. The two components are decoupled in the kinetic term, but coupled by the mass term. In the absence of the mass term at the tree level, the theory has chiral symmetry: $\psi \to \gamma_5 \psi$.

If chiral symmetry is not broken by $\mathcal{L}_{\rm int}$, then the mass renormalization must vanish as $m \to 0$. Chiral Ward identities can say which terms in $\mathcal{L}_{\rm int}$ are allowed by chiral symmetry.

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Keywords and References

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Loop integrals; ultraviolet cutoff scale; cutoff regularization; large logarithms; dimensional regularization; mass-independent regularization; counter-terms; renormalization scheme; renormalization scale; *msbar* renormalization scheme; un-renormalizable theory; renormalizable Lagrangians; super-renormalizable couplings; Chiral Ward identities;

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