Wilsonian renormalization

Sourendu Gupta

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Outline

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Using the tool called QFT

Every QFT contains a certain number of input parmeters (couplings and masses). These are computed by matching some experimental results to a computation in the QFT. In order to do this we choose an UV cutoff Λ on the QFT, and a renormalization scale μ, and do computations of the input experiments to a certain accuracy (retaining a certain order of irrelevant terms, using loops up to some order). By matching these computations to data, we extract the parameters.

The QFT is now ready to make predictions for all experiments with energy scales up to $E \ll \Lambda$. However, if we want to describe experiments at much larger energies, then we may need to increase Λ. In that case we need to set up the tool again using the changed cutoff. Examining this process reveals the real meaning of renormalization.

The physical content of renormalization

Wilson fixed his attention on the quantum field theory which emerges as the cutoff Λ is pushed to infinity while the low-energy physics is held fixed. According to him, one should define a renormalization group (RG) transformation as the following—

1. Include field with momenta in $[\Lambda, \zeta\Lambda]$, and perform a wave-function renormalization by scaling the field to give the same kinetic term as before.
2. Find the Lagrangian of the new fields which reproduces the IR dynamics (matching experiments) of the original system. The couplings in the Hamiltonians "flow" $g(\Lambda) \to g(\zeta\Lambda)$. This flow defines the Callan-Symanzik beta-function

$$\beta(g) = \frac{\partial g}{\partial \log \zeta}.$$

A fixed point of the RG has $\beta(g) = 0$.

Linearized Renormalization Group transformation

Assume that there are multiple couplings G_i with beta-functions B_i. At the fixed point the values are G_i^*. Define $g_i = G_i - G_i^*$. Then,

$$\beta_i(G_1, G_2, \cdots) = \sum_j B_{ij} g_j + \mathcal{O}(g^2).$$

Diagonalize the matrix B whose elements are B_{ij}. In cases of interest the eigenvalues, y, turn out to be real. Under an RG transformation by a scaling factor ζ an eigenvector of B scales as $v \to \zeta^y v$

Eigenvectors corresponding to negative eigenvalues scale away to zero under RG, and so correspond to super-renormalizable couplings. We have already set up the correspondence of these with relevant couplings. For positive eigenvalues, we find un-renormalizable couplings, *i.e.*, irrelevant couplings. Those with zero eigenvalues are the marginal operators.

Understanding the beta-function

We examine the β-function in a model field theory with a single coupling g. If the β-function is computed in perturbation theory then we know its behaviour only near $g = 0$. But imagine that we know it at all g.

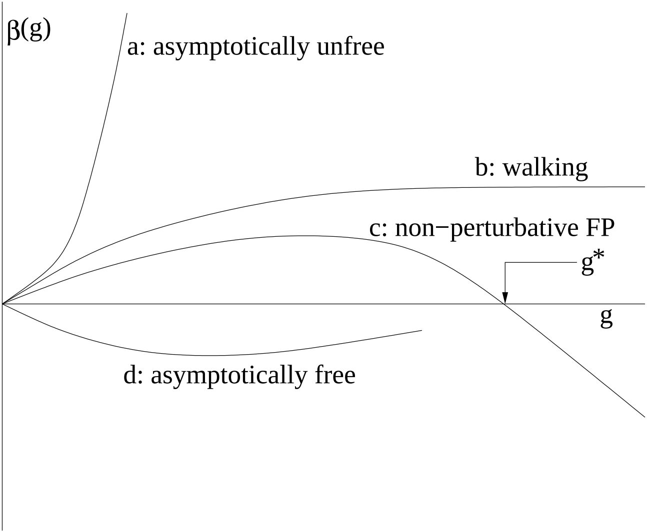
The solution of the Callan-Symanzik equation gives us a running coupling, obtained by inverting the equation

$$\zeta = \exp\left[\int_0^{g(\zeta)} \frac{dg}{\beta(g)}\right].$$

This happens since the coupling which gives a fixed physics can change as we change the cutoff scale.

Since larger ζ means that we can examine larger momenta, the behaviour of $g(\zeta)$ at large ζ tells us about high-energy scattering.

The behaviour of model field theories



Enumerating the cases

1. Asymptotically unfree: if $\beta(g)$ grows sufficiently fast, then the integral converges. This means that the upper limit of the integral can be pushed to infinity with ζ finite. This happens with the one-loop expression for QED and scalar theory.
2. Walking theories: if $\beta(g)$ grows slowly enough, then the integral does not converge. As a result, $g(\zeta)$ grows very slowly as $\zeta \to \infty$.
3. Non-perturbative fixed point: there is a new fixed point at g^*. The scaling dimensions of the fields may be very different here.
4. Asymptotic freedom: if $\beta(g) < 0$ near $g = 0$, then, the coupling comes closer to $g = 0$ as $\zeta \to \infty$. There is no special significance to $\beta(g)$ changing sign at some g^*, except that it means that for all couplings below g^*, the renormalized theory is asymptotically free.

Wilson's change of perspective

In a QFT we want to compute amplitudes with bounded errors, and to systematically improve the error bounds, if required. With just a small change in the point of view, Wilsonian renormalization gives a new non-perturbative computing technique.

If we need amplitudes at a low momentum scale, then we can use the RG to systematically lower the cutoff scale, by integrating over the range $[\Lambda/\zeta, \Lambda]$. This corresponds to coarse graining the fields and examining the long-distance behaviour of the theory. Now the couplings follow the changed equation

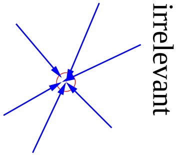
$$\frac{\partial g}{\partial \log \zeta} = -\beta(g).$$

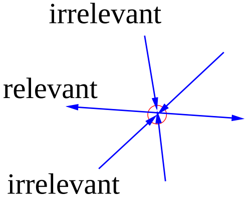
Asymptotically unfree theories may be perturbative at long distances; while asymptotically free theories may become highly non-perturbative if the corresponding beta-function crosses zero at some $g^*\equiv 0$.

Renormalization Group trajectories

$\xi = \Lambda/m$. Fixed points: $\xi = 0$ (stable) or $\xi = \infty$ (unstable).

Renormalization Group trajectories





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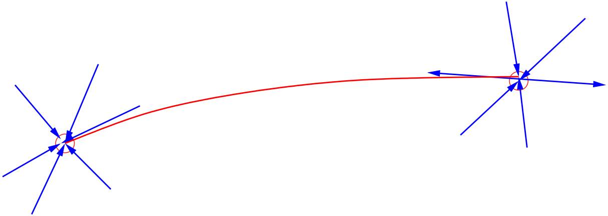

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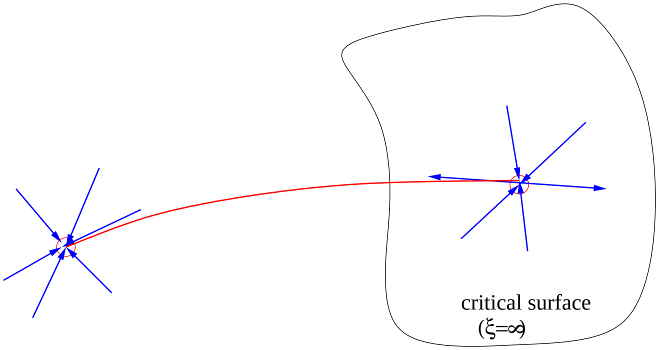

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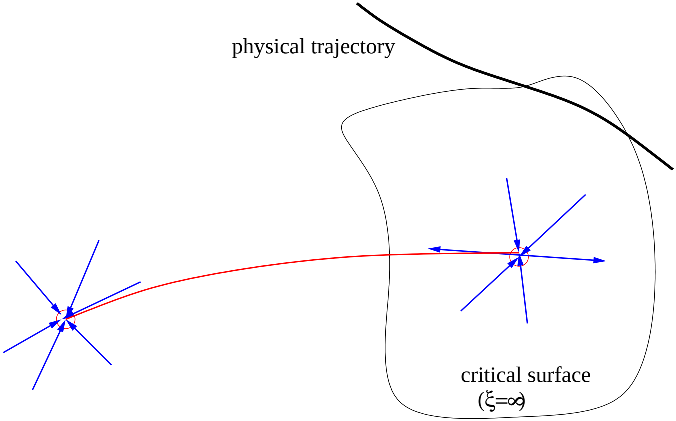

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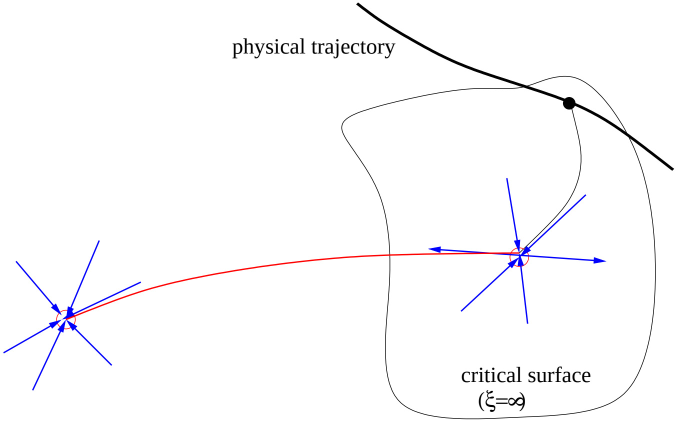

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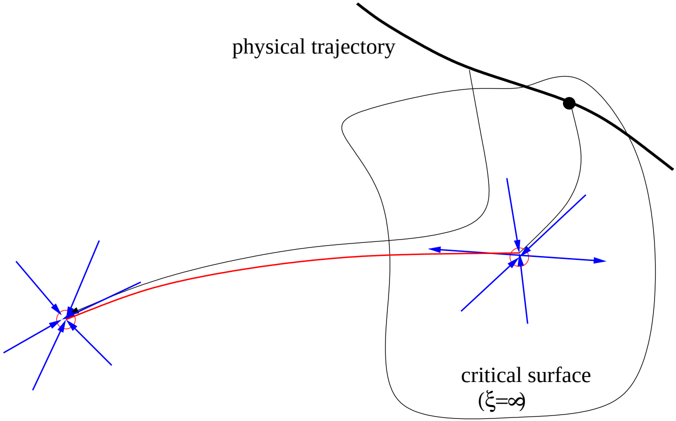

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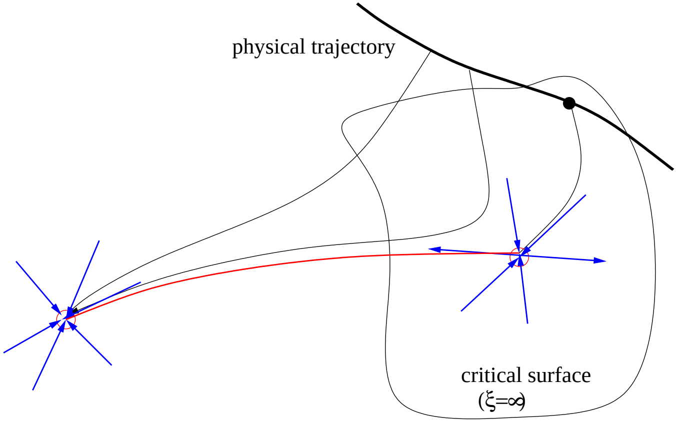

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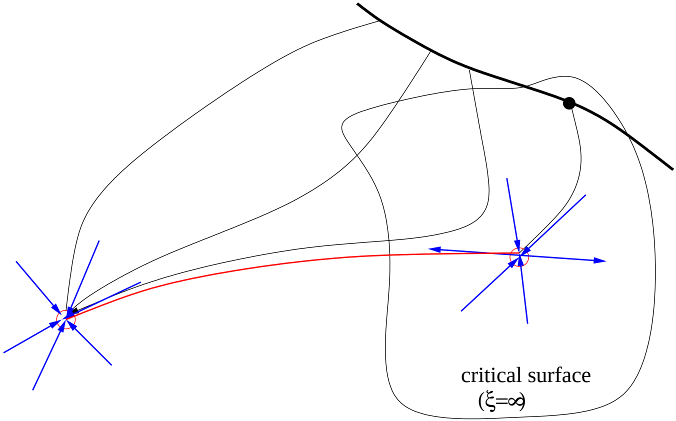

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Probability theory as a trivial field theory

Consider a random variate x with a probability density $P(x)$. In the commonest applications x is real. One needs to compute

$$\langle f \rangle = \int_{-\infty}^{\infty} dx\, f(x)P(x), \quad \text{where} \quad \langle 1 \rangle = 1.$$

Since $P(x) \geq 0$, one finds $S(x) = -\ln P(x)$ is real.
Define the characteristic function, $Z[j]$ and cumulants

$$Z(j) = \int_{-\infty}^{\infty} dx \mathrm{e}^{-S(x)-jx}, \quad \text{and} \quad [x^n] = \left.\frac{\partial^n F(j)}{\partial j^n}\right|_{j=0},$$

where the generating function: $F[j] = -\log Z[j]$. Note the analogy of $S(x)$ with the action of a zero dimensional field theory, of $Z(j)$ with the path integral and $F(j)$ with the generating function for the correlators. The cumulants, $[x^n]$, and are just connected parts of n-point functions of the field x. The connection between the cumulants, $[x^n]$ and the moments, $\langle x^n \rangle$, is left as an exercise in `Mathematica`.

Setting up the RG

Now suppose we take m of the random variates and average them, then what are the cumulants of the distribution of

$$y_m = \frac{1}{m}\sum_{i=1}^{m} x_i?$$

This is an RG. The sum over many random variates corresponds to taking low-frequency modes of quantum fields, and m corresponds to Λ.

Clearly,

$$Z_m(j) = \int dy \left[\int \prod_{i=1}^{m} dx_i P(x_i)\, \delta\left(y - \frac{1}{m}\sum_{i=1}^{m} x_i\right)\right] \mathrm{e}^{-jy} = \left[Z\left(\frac{j}{m}\right)\right]^m.$$

So the RG gives us $F_m(j) = mF(j/m)$.

The central limit theorem

Since the cumulants are Taylor coefficients of the generating function, one has

$$F(j) = \sum_{n=1}^{\infty} [x^n] \frac{j^n}{n!},$$

and similarly for $F_m(j)$. Then comparing the coefficients of j^n gives the RG flow

$$[y^n] = \frac{1}{m^{n-1}} [x^n].$$

This procedure corresponds to matching the "low-momentum" correlation function.

The mean is unchanged by the RG, and the variance scales as $1/m$. All the higher cumulants scale by successively higher powers of m, and can be neglected if m is large enough. The RG flow proves the central limit theorem: the fixed point of probability distributions under RG is the Gaussian distribution.

Perturbation theory is insufficient

- The β-function of QED, obtained at 1-loop order, is positive and grows so fast that the running coupling becomes infinite at finite energy: this is called the Landau pole. As a result, QED does not work at high energy.
- The 1-loop effective action for non-Abelian gauge fields is minimized at a finite constant field strength [Savvidy: 1977]. In such a background, the gauge fields have an instability [Nielsen, Olesen: 1978]. So a perturbative expansion around this does not work.
- There are arguments which lead us to believe that the Euclidean path integral of a non-Abelian gauge theory is not dominated by a minimum of the classical action [Pagels, Tomboulis: 1978]. As a result a perturbative expansion around the quantum ground state cannot work.

What is quantum field theory?

The quantum theory of fixed number of particles can be solved in many different ways. Perturbation theory is only one of these.

The older view of renormalization tied the definition of a quantum field theory completely to the perturbation expansion. But since perturbation theory is insufficient, it became necessary to develop a definition, *i.e.*, a computational method, for quantum field theory independent of the perturbation expansion.

The Wilsonian view of renormalization yields a new way of defining computational techniques for quantum field theory: the method of effective field theory. These can be treated in perturbation theory (as in this course). Or one can treat it exactly by creating a Wilson flow in the space of Lagrangians, as in lattice field theory.

A space-time lattice

If a Green's function has an UV divergence, then that means that the product of field operators separated by short distances diverges. An UV cutoff means that the shortest distances are not allowed.

A simple way to implement this is to put fields on a space-time lattice. If the lattice spacing is a, then this corresponds to an UV cutoff, $\Lambda \simeq 1/a$. Derivative operators are simple:

$$\partial_\mu \phi(x) = \frac{1}{a}\left[\phi(x + a\hat{\mu}) - \phi(x)\right].$$

The discretization of the derivative operator is not unique; there are others which differ by higher powers of a. This means that the difference between different definitions of the derivative are irrelevant operators.

The reciprocal lattice: momenta

Making a lattice in space-time means putting an upper bound to the momenta. It is also possible to make an infrared (IR) cutoff by putting the field theory in a finite box. If the box size is $L = Na$, and one puts periodic boundary conditions, then only the momenta $2\pi n/(Na)$ are allowed. The spacing between allowed momenta is $2\pi/(Na)$, the lowest momentum possible is 0, and the highest possible momentum is $2\pi(N-1)/(Na)$. This range is called the Brillouin zone.

Fourier transforms of fields become discrete Fourier series, and momentum integrals become computable sums.

Problem 2.4

Explicitly construct the Fourier transforms of scalar and Dirac fields with periodic and anti-periodic boundary conditions on a hypercubic lattice in 4-dimensions of size N^4.

Pure Higgs theory

Take the scalar field theory in Euclidean space-time:

$$S = \int d^4x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + \lambda \phi^4 \right], \qquad (m, \lambda > 0),$$

and put it on a space-time lattice. The discretisation of the derivative in the kinetic term gives products of fields at neighbouring lattice sites. Everything else becomes an on-site interaction of the fields. If we take $\lambda \to \infty$, then the fields are pinned to the minimum of the potential. We can render the fields dimensionless using the lattice spacing a, and scale the field value at the minimum of the potential to ± 1. Then the scalar field theory reduces to

$$S = \sum_{x,\mu} s_x s_{x+\hat{\mu}}, \qquad (s_j = \pm 1),$$

which is the Ising model. (Problem 2.5: Complete this construction.)

Outline

Keywords and References

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Wave-function renormalization; Callan-Symanzik beta-function; fixed point; running coupling; coarse-graining; central limit theorem; Landau pole; lattice field theory; Ising model.

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