Field Theory at Finite Temperature

Sourendu Gupta

SERC Main School 2014, BITS Pilani Goa, India

Effective Field Theories
December, 2014

Outline

Outline

Outline

A neutral conductor

Definition

A plasma is any material which is overall charge neutral but contains mobile charge.

Any charge neutral medium which conducts electricity is likely to be plasma. Tap water is the most common plasma: it conducts well enough that a wet hand is dangerous near an electrical mains. The cellular material of animals, plants and bacteria are plasmas. Most gas in galaxies are plasmas. All plasmas need not be fluid; metals are plasmas.

Sometimes the notion of a charged plasma is also introduced. These are of technological importance: for example beams of protons used in the LHC. In this lecture we will confine ourselves to plasmas in the older sense: those which are electrically neutral.

The mythical one-component plasma

We assume that the mobile charges are in thermal equilibrium when no external field is present. The temperature, T, then sets the scale of kinetic energy of the charges. Assume that the charge of each particle is e, and their number density is n. The average distance between charges is $r_0 = 1/\sqrt[3]{n}$.
The scattering cross section between charges must involve α. Dimensional considerations dictate that the Rutherford scattering cross section is

$$\sigma \simeq \frac{\alpha^2}{T^2}.$$

The Rutherford scattering formula has another factor for the angular dependence, which is divergent for small angle scattering: $1/\psi^4$. Small angle scattering happens when the charges approach each other with large impact parameter. Then should other charges in the plasma be taken into account?

The plasma parameter

In fact, the microscopic variables in the plasma can be used to create a dimensionless variable, called the plasma parameter,

$$\mathcal{P} = \frac{\alpha}{\sqrt[3]{n}T} = \frac{\alpha}{r_0 T}.$$

This is a comparison between the average electrostatic potential energy between two charges in the plasma, α/r_0, and the kinetic energy, T.

If $\mathcal{P} \ll 1$, then one says that the plasma is weakly coupled. On the other hand, if $\mathcal{P} \simeq 1$ or greater, then the plasma is said to be strongly coupled. $1/\sqrt{\mathcal{P}}$ is the smallest angle that can be taken into account through two-body scattering in a plasma.

The length scale, $r_D = r_0/\mathcal{P}$ is called the Debye screening length. At distances of approach below this, the Coulomb force between two particles in a plasma is not strongly modified.

Space charge formation and screening

The number of charge carriers in a volume of radius r_D is $nr_D^3 \simeq 1/\mathcal{P}^3$. For a weakly coupled plasma this is large. The modification of the Coulomb potential is due to a many-body effect.

This many body effect is the formation of a space charge. If an external charge is introduced into a plasma, then it attracts opposite charges towards itself, and repels similar charges. For $\mathcal{P} \ll 1$, the thermal energy is much larger than the Coulomb energy, so the charges can knock each other around. So a larger volume is needed to shield (screen) the charge, and appreciably decrease the Coulomb force between it and a distant charge.

Another way to state this is to say that the charge e depends on the distance at which it is measured, $\alpha(r)$. This can happen because of the generation of a new length scale: $\alpha(r) = \alpha f(r/r_D)$.

Excitation by external fields

If an electromagnetic wave impinges on the plasma, the response of the medium depends on the frequency. Write $k_D = 1/r_D = \alpha/(Tr_0^2)$. For waves with $k \gg k_D$, the wave will drive charged particles individually. The energy of the wave will be lost in accelerating each particle separately.

In the opposite limit $k \ll k_D$, there will be collective effects. In the limit $k \to 0$, *i.e.*, for static fields, there will be a space charge separation. If the field begins to oscillate slowly, so will the space charge. So, for very slow and long waves, the different charges in the plasma begin to separate out, and then oscillate against each other. These are called plasma oscillations.

Plasma oscillations are also called Langmuir waves. Quantized plasma oscillations are important in metals; they are called plasmons.

The plasma frequency

The quantities of interest are α, n and the mass of the charges, m. Dimensionally, it is clear that the only frequency one can construct is the plasma frequency

$$\omega_p \simeq \sqrt{\frac{\alpha n}{m}} = \sqrt{\frac{\alpha}{m r_0^3}}.$$

An incident wave with $\omega = \omega_p$ can destabilize the plasma by pumping in energy and separating out the charges.

This formula is true for strongly interacting plasmas. When $\mathcal{P} \ll 1$ thermal motion can randomize the coherent effect of the wave. As a result, T will control the magnitude of the dissipative terms.

Since, the charges have speed $v \simeq \sqrt{T/m}$, one might expect the plasma wave dispersion relation to be modified to

$$\omega^2 = \omega_p^2 + k^2 v^2.$$

The plasma parameter again

It is also interesting to construct the submicroscopic length scale

$$r_c = \mathcal{P} r_0 = \frac{\alpha}{T}.$$

Clearly, this is the distance at which the Coulomb and thermal energies become equal, *i.e.*, the distance of closest approach of two charges. For a weakly coupled plasma we have $r_c \ll r_0 \ll r_D$, and the plasma parameter, $\mathcal{P}$, controls the separation of these scales.

A typical microscopic time scale in the plasma is the frequency of low angle scattering of charge carriers

$$\tau = \frac{r_D}{v} \simeq \frac{r_0^2 T}{\alpha} \sqrt{\frac{m}{T}}. \quad \text{So} \quad (\tau\omega_p)^2 = \frac{r_0 T}{\alpha} = \frac{1}{\mathcal{P}}.$$

Another time scale in the plasma is the frequency of large angle scattering, $\tau_\ell = r_0/v = \tau/\mathcal{P}$. This means that there is a similar hierarchy of time scales, $\tau_\ell \ll \tau \ll 1/\omega_p$.

Landau damping

Take a generic electromagnetic wave travelling in a plasma with wave-number $\mathbf{k}$ and frequency ω. If there is a charged particle with velocity $\mathbf{v}$ which satisfies $\mathbf{k}\cdot\mathbf{v}=\omega$, then it is resonantly coupled to the wave because it sees a static field: the particle position is $\mathbf{v}t$, and the wave has phase $(\mathbf{k}\cdot\mathbf{v}-\omega)t=0$ at the position of the particle.

Particles travelling slightly faster than resonance will be decelerated and lose energy to the wave. Particles travelling slower will be accelerated, and will gain energy from the wave.

If there are more particles slightly slower than the wave, then they will damp out the wave. This process is called Landau damping. This is common because momentum distributions of particles typically drop with increasing momentum.

Some plasmas

System	n (m^{-3})	T (K)	$\mathcal{P}$	r_D (m)
Interstellar gas	10^6	10^4	2321	2.3×10^1
Gaseous nebulae	10^8	10^4	1077	2.3×10^0
Ionosphere	10^{12}	10^3	73	7.3×10^{-3}
Solar				$\times$
corona	10^{12}	10^6	2321	2.3×10^{-1}
atmosphere	10^{20}	10^4	11	2.3×10^{-6}
interior	10^{33}	10^7	2	2.3×10^{-11}
Lab plasma				$\times$
tenuous	10^{17}	10^4	34	7.3×10^{-5}
dense	10^{22}	10^5	16	7.3×10^{-7}
thermonuclear	10^{22}	10^8	500	2.3×10^{-5}
Metal	10^{29}	10^2	0.03	7.3×10^{-12}

Outline

Relativity

In relativistic plasmas, there is a new energy scale, m. So there is a new dimensionless number,

$$\gamma = \frac{T}{m}.$$

This is the usual relativistic Lorentz factor. When $\gamma \ll 1$ the previous classical theory is applicable.

When $\gamma \gg 1$, the system is ultra-relativistic, and the mass of the particle is much less than the kinetic energy. Under these circumstances, it is easy to create and destroy particles. It is therefore unproductive to examine relativistic classical plasmas. It is more useful and realistic to examine plasmas within quantum field theory. This is the case in the early universe. Also, since the universe is flat, there is no scale from the curvature.

Quantum field theory

In a gauge theory, T, m and $\alpha = g^2/(4\pi)$ are the parameters. Since m/T and α are the dimensionless parameters in the problem, one should be able to write $n = T^3 f(\alpha, m/T)$. It turns out that n does not have a good limit for $m \ll T$, since low-energy pairs can be produced in arbitrarily high numbers. One may replace n by the entropy density, $S \simeq T^3$, but then $r_0 \simeq 1/T$, so this is not an independent scale. The energy density, $\epsilon = T^4 g(\alpha, m/T)$ has a good limit $\epsilon = T^4 g(\alpha)$.

Debye screening occurs in such a plasma, and one should be able to write $r_D^{-1} = T\rho(\alpha, m/T)$. This has interesting, curable, infra-red problems in the limit $m \ll T$. There is a plasmon in the problem, with dispersion relations shown before. The plasma frequency, ω_p, acts like a mass in a relativistic dispersion relation.

The partition function

In order to set up the QFT, we need to evaluate the partition function

$$Z = \mathrm{Tr}\, \exp(-H/T),$$

which needs QFT in a fixed frame. Lorentz covariance is recovered by introducing a 4-vector u which gives the velocity of the heat-bath with respect to a inertial observer: $u = \gamma(1, \beta_x, \beta_y, \beta_z)$, so that $u^2 = 1$. Then $E/T = u \cdot P/T$.

The path integral for Z defines QFT at finite T. This is exactly the same as the unitary evolution operator of a QFT taken over a finite Euclidean time $t = 1/T$. Hence exactly the same Euclidean path integral, but over a finite range.

Thermal quantum field theory

Need to know boundary conditions in order to perform the trace. Poles of the Bose and Fermi distributions given by

$$\exp(p_0/T) = \pm 1, \qquad i.e., \qquad p_0 = \begin{cases} 2\pi n T & \textit{bosons} \\ 2\pi\left(n+\frac{1}{2}\right) T & \textit{fermions} \end{cases}$$

So bosons are periodic, and fermions are anti-periodic in Euclidean time. Spatial volume must be infinite for thermodynamics, so spatial boundary conditions irrelevant.

Problem 7.1

Do the boundary conditions break supersymmetry at finite T? What happens to the SuSy algebra? Which states dominate the partition function?

Dimensional reduction

With the identification in a QFT of $r_0 \simeq 1/T$, the plasma parameter,

$$\mathcal{P} = \frac{\alpha}{r_0 T} \simeq \alpha.$$

As a result $m_D = 1/r_D = \alpha T$. If T is large enough, then the running coupling $\alpha \ll 1$, the plasma is weakly coupled, and $m_D \ll T$. Due to this separation of scales it should be possible to write an EFT for physics at the Debye scale.

The UV cutoff $\Lambda \simeq T$. This implies that all Euclidean fields with $p_0 > 0$ are integrated out, leaving a $D = 3$ dimensionally reduced theory. This EFT does not contain any fermions. If the $D = 4$ theory had scalars then the EFT also contains scalars. If the $D = 4$ theory had gauge fields, A_μ, then the $D = 3$ theory contains gauge fields, A_i, made of the spatial components of the original, and a scalar field, A_0, which transforms adjointly under the gauge group.

Dimensionally reduced scalar theory

The EFT describing Debye scale physics of a scalar theory has a Lagrangian

$$\mathcal{L}_{3D} = \frac{1}{2}(\partial_i \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!}\lambda\phi^4 + \cdots .$$

Since $[\mathcal{L}] = 3$, the kinetic term shows $[\phi] = 1/2$. The second term has dimension 1 so one can write $m^2 = c_2\Lambda^2$. The third term has dimension 2, so $\lambda = c_4\Lambda$. All other terms are irrelevant. Since the ϕ^4 term arises from the $g^2\phi^4$ term in $D = 4$ at tree level, one has $c_4 = \mathcal{O}(g^2)$. Also, c_2 is generated at 1-loop from the tadpole, so $c_2 = \mathcal{O}(g^2)$.

If we use dimensional regularization, then, since $\Lambda \simeq T$, we have

$$\lambda \simeq g^2 T, \qquad \text{and} \qquad m^2 \simeq g^2 T \log(m_0/\mu).$$

Terms in the EFT of order $\partial^{2m}\phi^{2n}$ has dimension $2m + n$. As a result its coupling will be of order g^{2n}/T^{2m+n-3} for $n > 1$.

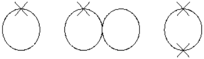
Another relevant term

The partition function is

$$Z = \mathrm{e}^{-fV} \int \mathcal{D}\phi \, \mathrm{e}^{-\int d^3x, \mathcal{L}_{3D}}.$$

The term f (cosmological constant!) is needed to match the free energies in the EFT and the 4D theory. This can be done via a perturbation theory in g^2 in both the theories. The expansion in g^2 in the 4D theory involves the ϕ^4 term, and hence diagrams such as:

The expansion in the 3D theory involves the $\lambda\phi^4$ term as well as the $m^2\phi^2$ term. The former gives the above diagrams, the latter adds on:



The Debye screening mass

The Debye screening mass can be computed from the 2-point function at finite temperature. The leading term $m_D = m^2 \simeq g^2 T^2$. The next terms come from an expansion in $\lambda \simeq g^2 T$, so one might think that m_D has an expansion in g^2.

However, the one loop contribution is

$$\lambda \int \frac{d^3 k}{k^2 + m^2} \simeq m\lambda \simeq g^3 T^2.$$

The power counting for the two loop contributions gives

$$\lambda^2 \int \frac{(d^3 k)^2}{(k^2 + m^2)^3} \simeq g^4 T^2.$$

The one loop result shows that the weak coupling expansion in FTQFT is not analytic in the coupling. This is also true for the pressure. The effective theory implements Braaten-Pisraski resummation.

Non-Abelian gauge theories

In QED at $T > 0$, the electric polarization of the photon gets a mass, and therefore electric fields die out in a distance of about $1/m_D$. However, the magnetic polarization does not get a mass. Magnetic fields are unscreened in a plasma, as a result of which the long-distance EFT of a QED plasma is magneto hydro-dynamics.

In a non-abelian theory it turns out that there $m_D \simeq gT$. However, the magnetic polarizations are also screened at order g^2T. One then has an yet softer EFT at scale g^2T which can be used to compute effects which arise due to magnetic screening.

Outline

Keywords and References

Keywords

plasma; mobile charges; plasma parameter; Debye screening; space charge; plasma oscillations; Langmuir waves; plasmons; plasma frequency; Landau damping; dimensional reduction; free energy; pressure; Braaten-Pisarski resummation; magneto hydro-dynamics; non-Abelian magnetic screening.

References

E. Braaten and A. Nieto, *Phys. Rev.* D 51 (1995) 6990 [hep-ph/9501375];
K. Farakos etal, *Nucl. Phys.* B 425 (1994) 67 [hep-ph/9404201].

Copyright statement