Some results on Su Doku

Sourendu Gupta

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1 Proofs of widely known facts

Definition **1.** A Su Doku **grid** contains $M \times M$ **cells** laid out in a square with M cells to each side.

Definition **2.** For every non-prime M, the Su Doku grid is divided into rectangular **blocks** of cells, R in a row and C in a column ($R \neq 1$m $C \neq 1$ and $M = RC$). This is called $R \times C$ Su Doku.

3×3 Su Doku is also called by the generic name Su Doku. For $M = 12$ one can define either a 4×3 Su Doku or a 6×2 Su Doku. Due to this non-uniqueness, Su Dokus should be classified by the pair R and C rather than their product M.

Definition **3.** The **One Rule** of Su Doku is that a consistent filling of a grid is to place one of M symbols (such as numbers) in each cell so that each row, column and block has each of the M symbols appearing exactly once.

Su Dokus can be defined for prime M, but the group theory and counting of these objects is different from that for composite M.

Theorem 1. *The group of symmetries of a Su Doku grid of order $M = R \times C$, $SD(R, C)$, is generated by the following—*

1. *All permutations of the M symbols, $M!$ in number.*
2. *All interchanges of two rows of blocks, $C!$ in number.*
3. *All interchanges of two columns of blocks, $R!$ in number.*
4. *All interchanges of two rows inside a block, $R!$ in number.*
5. *All interchanges of two columns inside a block, $C!$ in number.*
6. *All rotations and reflections of the square.*

This group has been investigated in [1]. Writing $SD(3, 3)/S_9$ (where S_9 is the group of permutations of the 9 symbols) it has been shown that G is of order 3359232, has a trivial center, has exponent 72, is solvable but not nilpotent and has derived length 5. For Su Doku, G has 275 conjugacy classes.

Definition **4.** A **solution** is a filling of the Su Doku grid consistent with the One Rule.

Solutions of Su Doku are subsets of Latin Squares.

Starting from a solution, S, apply all elements of $SD(R, C)$. The set of solutions $g \circ S$ where $g \in SD(R, C)$ generates the orbit of S. Solutions which cannot be generated from each other by the action of a symmetry are essentially different. They must come from different orbits.

Each orbit can be represented by any one member of the orbit. The subgroup of $SD(R, C)$ which leaves the representative invariant is called the little group of the orbit, $L \in SD(R, C)$. If $g \in L$ then $g \circ S = S$, where S is a representative of the orbit. Each solution in the orbit labels the coset $SD(R, C)/L$. If $L = \{I\}$ for an orbit, then one says that the orbit has no symmetry. The total number of solutions of $R \times C$ Su Doku will equal the product of the number of orbits and the order of $SD(R, C)$ if and only if all orbits have no symmetry.

The counting of the total number of solutions of 3×3 Su Doku was completed by Felgenhauer and Jarvis [2] and subsequently generalized to other Su Dokus by many other people [3]. The counting of the orbits of the solutions of 3×3 Su Doku (often called the number of essentially different solutions) was completed by Felgenhauer and Jarvis [1] and subsequently generalized to other Su Dokus by many other people [3].

Theorem 2. *The little group of any complete Su Doku solution cannot be any of the following subgroups of* $SD(R, C)$*—*

1. *Permutations of the numbers*
2. *Permutations of columns within a column of blocks*
3. *Permutations of rows within a row of blocks*
4. *Permutations of columns of blocks*
5. *Permutations of rows of blocks*
6. *The group of symmetries of the square*
7. *Any subgroup of the above*

Proof. Introduce the notation $S[r, c]$ for the number that appears in the row r and column c of the solution S.

1. If a Su Doku solution is invariant under the exchange of i and j, then if we find an r and c such that $S[r, c] = i$, then one must also have $S[r, c] = j$ for $i \neq j$. This is impossible since each cell must have an unique entry.

2. If a Su Doku solution is invariant under the exchange of two columns within a block, then $S[r, c] = S[r, c']$, where c and c' are two columns within the same block. Thus the same number appears twice in a block, thus violating the One Rule.

3. Since $S[r, c] = S[r', c]$ for two rows r and r' in a block, the same numbers appear at least twice within a block, hence violating the One Rule.

4. If a Su Doku solution is invariant under the exchange of two columns of blocks, then the same numbers appear at least twice within a row, hence violating the One Rule.

5. If a Su Doku solution is invariant under the exchange of two rows of blocks, then the same numbers appear at least twice within a column, hence violating the One Rule.

6. Let $M = R \times C$. Under a reflection about the vertical axis, $S[1, 1] = S[1, M]$. Under reflections about a diagonal one has $S[1, 2] = S[2, 1]$. Neither is allowed by the One Rule. Under a rotation of $\pi/2$, $S[1, 1] = S[1, M]$, which is ruled out by the One Rule. Under a rotation of π one has to consider the cases of even and odd M separately. For odd M, let $a = (M - 1)/2$ and $b = (M + 1)/2$. Then a rotation of π gives $S[a, a] = S[b, b]$ which is forbidden by the One Rule since these two squares are both in the same block.

7. The steps in the proof above constitute the proof of the last part of the theorem.

□

Definition **5.** A **puzzle**, P, is any incomplete Su Doku.

We denote the solution of a puzzle by $\mathfrak{S}(P)$.

Definition **6.** A **proper puzzle** is one that has an unique solution, $\mathfrak{S}(P)$.

A proper puzzle has the property that for any $g \in SD(M)$, $\mathfrak{S}(g \circ P) = g \circ \mathfrak{S}(P)$.

Definition **7.** A **reducible proper puzzle** is one which remains proper even when a clue is removed.

Definition **8.** The **puzzle tree** of a solution S, denoted by $T(S)$, is a directed graph generated from the root S whose nodes are all proper puzzles, P, such that $\mathfrak{S}(P) = S$. The edges are directed from P to all proper puzzles P' obtained by removing exactly one of the clues.

Clearly, the terminal nodes (leaves) of $T(S)$ and irreducible puzzles, P with $\mathfrak{S}(P) = S$ are in one-to-one correspondence.

Definition **9.** The **puzzle forest** of $R \times C$ Su Doku, $\mathfrak{F}(R, C)$, is the collection of puzzle trees of all orbits of solutions. The **leaves** of $\mathfrak{F}(R, C)$ is the set of all proper puzzles in $\mathfrak{F}(R, C)$.

Counting of the nodes and leaves in the puzzle forest is currently the biggest open problem in the mathematics of Su Doku. Since all proper puzzles (and irreducible puzzles) can be generated from $\mathfrak{F}(R,C)$ (respectively, leaves of $\mathfrak{F}(R,C)$) by the action of $SD(R,C)$, the results of [1] would allow us to count all puzzles as soon as the counting of the puzzle forest is done.

$N_c(P)$ is the number of clues in P. The minimum problem of Su Doku is to find the minimum number of clues, $N_c(min)$, in the leaves of $\mathfrak{F}(R,C)$. The maximum problem of Su Doku is to find the maximum number of clues, N_c^{max}, in the leaves of $\mathfrak{F}(R,C)$. The minimum problem need not be restricted to the leaves because any reducible puzzle will have larger number of clues. The unrestricted maximum problem has solution at the trivial problem at the root of the tree, *i.e.*, the maximum number of clues is clearly M^2. It is more fruitful to restrict the search for the maximum number of clues to the leaves, *i.e.*, to the irreducible puzzles.

Theorem 3. *Let a puzzle P have one solution $S = \mathfrak{S}(P)$. If P is invariant under the action of any element $g \in SD(R,C)$ which is not an element of the little group of S, then P is not a proper puzzle.*

Proof. Let $g \in SD(R,C)$ but not in the little group of S. Then $S' = gS \neq S$. However, $S' = gS = g\mathfrak{S}(P) = \mathfrak{S}(gP) = \mathfrak{S}(P)$. Thus, the puzzle has more than one solution, and hence is not a proper puzzle. □

Some of the corollaries of the above theorem are—

1. If two numbers are missing from the set of clues, then the puzzle is not proper.
2. If two rows in the same block, or two columns in the same block, have no clues, then the puzzle is not proper.
3. If the i-th column and i-th row both contain no clues then the puzzle is not proper.

Theorem 4. $N_c^{min} \geq M - 1$

Proof. With $M - 2$ clues the puzzle is improper, since there are at least two symbols which do not appear. □

This bound is saturated for $M = 1$, 2 and 3 [4].

Theorem 5. $N_c^{max} = 0$ *for* $M = 1$ *and* $N_c^{max} \leq M^2 - 3$ *for* $M > 1$.

Proof. For $M = 1$ no clue is needed to complete the puzzle, and the estimate is obvious. Otherwise, if in a solution S two symbols i and j appear in the four corners of a rectangle such that each corner lies in a different block, then the deletion of these four results in an improper puzzle. Hence at least one of these must appear to make the puzzle proper. □

The bound above is saturated for $M = 2$ [4]. No lower bound on N_c^{max} or upper bound on N_c^{min} is known, except the one that follows from the the theorems above because $N_c^{min} \leq N_c^{max}$.

2 Shi Doku

Definition **10.** **Shi Doku** is 2×2 Su Doku.

Theorem 6. *A proper Shi Doku puzzle cannot be made with 3 clues.*

Proof.

1. Using the spatial symmetries of the grid and the permutation symmetry of the symbols, one can set $S(1,1) = 1$. The little group of the resulting (as yet improper) puzzle contains
   (a) Permutations on the remaining 3 symbols (S_3)
   (b) Permutations of columns 3 and 4 (S_2)
   (c) Permutations of rows 3 and 4 (S_2)
   (d) Transposition of the grid (S_2)
2. If the second clue is placed in either the first row or column, then there would be two rows and columns left blank after placing the third clue, hence definitely resulting in an improper puzzle. Hence the second clue is in the remaining 3×3 square.
3. The value of the 2nd clue can always be assumed to be 2. This reduces the permutation symmetry of the resulting little group to S_2. This clue may be placed in the 2nd row or column or this row and column may be blank.
4. If the 2nd clue is placed in the second column below the diagonal, then one can use a transposition to bring it in the second row above the diagonal. Thus, one assumes that the second clue is on or above the diagonal, and removes the transposition symmetry from the little group.
   (a) If $S(2,2) = 2$ then if the 3rd clue is in the second row or column then there are two columns or rows blank and the puzzle is improper. However, of the 3rd clue is elsewhere, it can always be brought to the third row by permutations of rows 3 and 4. The clue can be assigned value 3 by using the remaining permutation symmetry. The resulting grid has trivial little group. Nevertheless there is an ambiguity about which cell in the first block has value 3— $S(1,2)$ or $S(2,1)$. Thus this puzzle is improper.
   (b) Otherwise we can set $S(2,3) = 2$ by using column permutations. Now row permutations and the symmetry of the remaining symbols allow us to place a 3 in either $S(3,2)$ or $S(3,4)$. In either case, the little group is trivial, but there is a two-fold ambiguity about where to place a 3 in the first row. Hence the puzzle is improper.
5. The last remaining case is that one sets $S(3,3) = 2$ using the row and column permutations of the last block. Then the third clue can be placed above the diagonal in one of the remaining rows by using the transposition

symmetry, and make this clue equal to 3 using the remaining permutation symmetry. This fixes the little group to be the trivial group. However, there still remains a two-fold ambiguity about where to place a 3 in the first row. Hence the puzzle is improper.

□

Theorem 7. *A proper Shi Doku puzzle can be made with 4 clues.*

Proof. The simplest proof is to explicitly construct a proper puzzle. □

References

[1] E. Russell and F. Jarvis, http://www.afjarvis.staff.shef.ac.uk/sudoku/sudgroup.html

[2] B. Felgenhauer and F. Jarvis, "Enumerating possible Su Doku grids", http://www.afjarvis.staff.shef.ac.uk/sudoku/sudoku.pdf

[3] Results are often announced in the voluminous Sudoku Forum (http://www.sudoku.com/forums/). Nevertheless it remains a very fragmentary record of discoveries about Su Doku.

[4] See http://theory.tifr.res.in/ sgupta/sudoku/expert.html