Lattice QCD with chemical potential

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1. Why chemical potential? Why sweat?
2. Reweighting = Taylor series expansion.
3. Quark number susceptibilities: fluctuations and strangeness
4. Determining the equation of state: the pressure at finite μ
5. Condensates and masses
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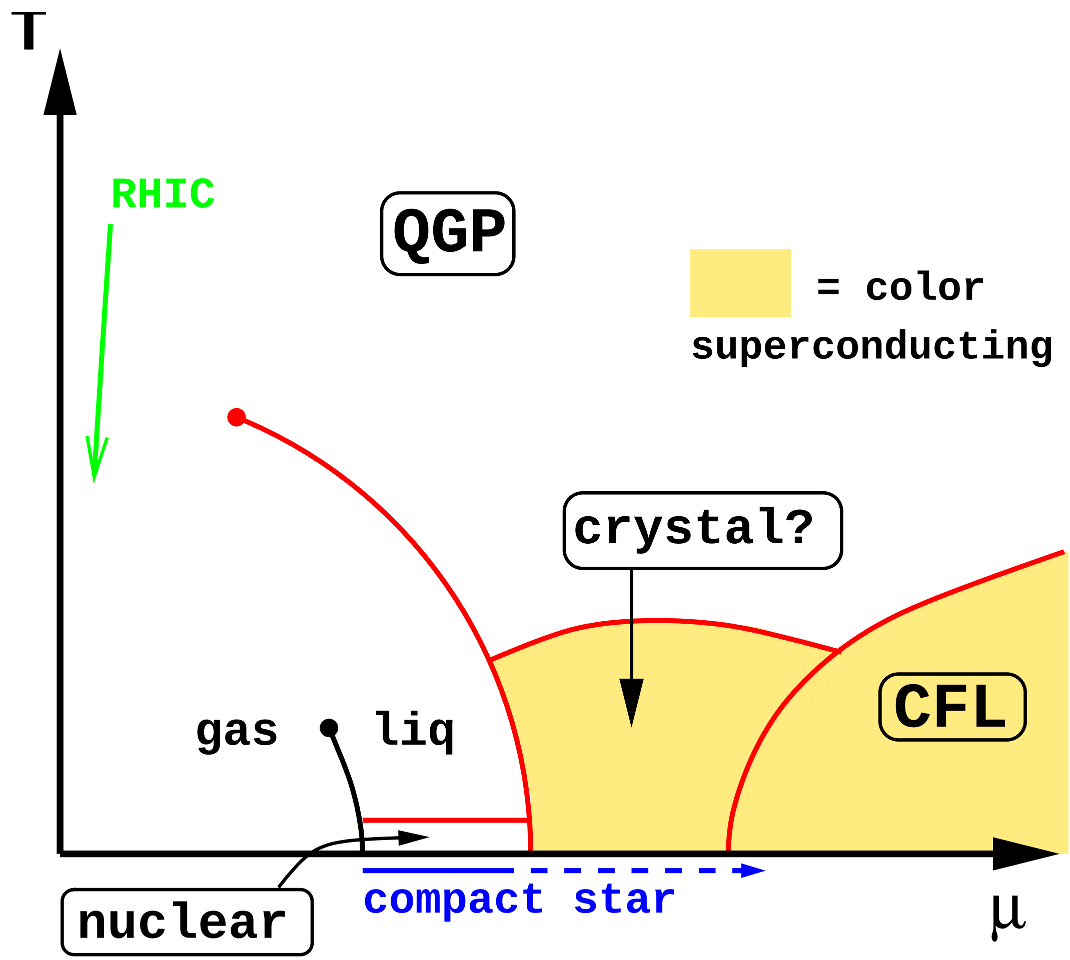
Why chemical potential?



Why sweat?

$$Z = \mathrm{e}^{-F/T} = \int DU \mathrm{e}^{-S} \prod_f \det M(U, m_f, \mu_f)$$

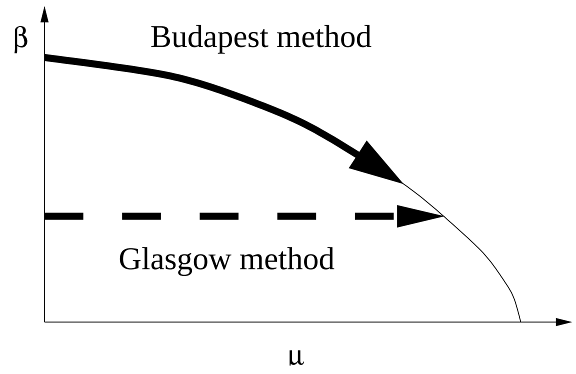
- If $M^\dagger = Q^\dagger M Q$ for some Q, then clearly $\det M$ is real.
- For $\mu = 0$ $Q = \gamma_5$. For $\mu \neq 0$ no Q exists.
- Monte Carlo simulations of Z fail.
- However, Z remains real and non-negative: thermodynamics is safe.

All lattice computations done with $m_u = m_d$ ($N_f = 2$). Some also with $m_s/T_c \approx 1$ ($N_f = 2 + 1$). Many with $\det M = 1$ ($N_f = 0$).

Reweighting

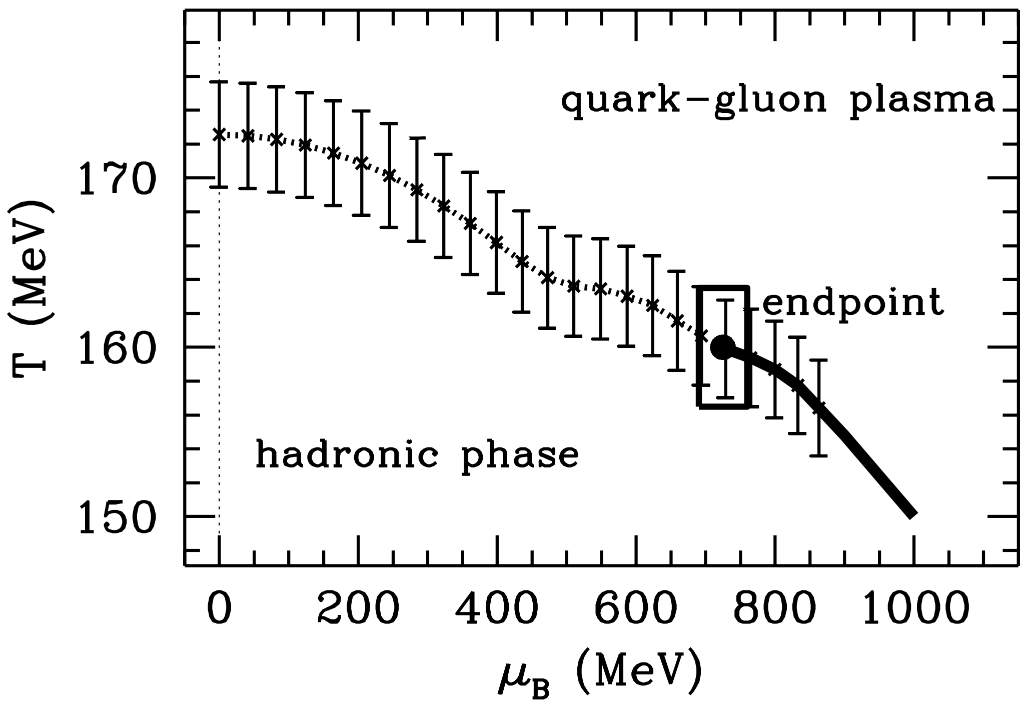
Do simulations at $\mu_f = 0$, re-express expectation values in terms of these—

$$\langle O \rangle_\mu = \frac{\langle O \exp(-\Delta\mathcal{S}) \rangle}{\langle \exp(-\Delta\mathcal{S}) \rangle} \quad \text{where} \quad \mathcal{S} = S - \sum_f \mathrm{Tr}\ \log M_f,$$



Reweighting: results

Reweighting done for coarse lattices ($N_t = 4$) and $N_f = 4$, 2 and 2+1.



Z. Fodor and S. D. Katz, *J. H. E. P.*, 03 (2002) 014.

Reweighting: variants

- Express the reweighting in terms of derivatives of Z with respect to chemical potential.
  C. R. Allton *et al.*, *Phys. Rev.*, D 66 (2002) 074507

- Simulate imaginary chemical potential (positive $\det M$) and do analytic continuation. This actually the same as above.
  M. D'Elia and M.-P. Lombardo, hep-lat/0209146
  P. De Forcrand and O. Philipsen, *Nucl. Phys.*, B642 (2002) 290

Reweighting suffers from finite lattice effects.

Systematic Taylor series expansion cures this.

Taylor Expansion

Since $PV = -F = T \log Z$, the Taylor expansion of P is the same as of F!

$$\frac{1}{V} P(T, \mu_u, \mu_d) = \frac{1}{V} P(T, 0, 0) + \sum_f n_f \mu_f + \frac{1}{2!} \sum_{fg} \chi_{fg} \mu_f \mu_g + \cdots$$

where the quark number densities and susceptibilities are—

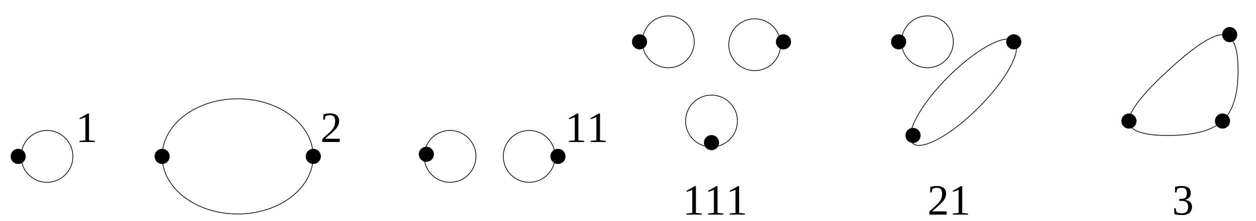
$$\begin{aligned} n_f &= \frac{T}{V} \left. \frac{\partial \log Z}{\partial \mu_f} \right|_{\mu_f=0} \\ \chi_{fg} &= \frac{T}{V} \left. \frac{\partial^2 \log Z}{\partial \mu_f \partial \mu_g} \right|_{\mu_f=\mu_g=0} \\ \chi_{fgh\cdots} &= \frac{T}{V} \left. \frac{\partial^n \log Z}{\partial \mu_f \partial \mu_g \partial \mu_h \cdots} \right|_{\mu_f=\mu_g=\cdots=0} \end{aligned}$$

Derivatives

Derivatives of $\log Z$ can be expressed in terms of derivatives of Z. The latter can be constructed by the chain rule.

$$Z_f = \frac{\partial Z}{\partial \mu_f} = \int DU \mathrm{e}^{-\mathcal{S}} \mathrm{Tr}\; M_f^{-1} M_f'.$$

Note: $M' = \gamma_0$ and $M^{-1} = \psi\overline{\psi}$, so $\mathrm{Tr}\; M^{-1}M' = \psi^\dagger\psi$. Odd derivatives vanish for $\mu_f = 0$ by CP symmetry. S. Gottlieb *et al.*, *Phys. Rev. Lett.*, 59 (1987) 2247



S. Gupta, *Acta Phys. Pol.*, B 33 (2002) 4259

Quark number susceptibilities

- Fluctuations of conserved quantities in heavy-ion collisions are related to χ_{uu}. Isospin fluctuations are related to $\chi_3 = \chi_{uu} - \chi_{ud}$, charge fluctuations can also be constructed out of these. M. Asakawa *et al.*, *Phys. Rev. Lett.*, 85 (2000) 2072; S. Jeon and V. Koch, *ibid.*, 85 (2000) 2076

- Under certain conditions strangeness production rate can be related to the strange susceptibility, χ_{ss}. R. V. Gavai *et al.*, *Phys. Rev.*, D 65 (2002) 054506

- The pressure at finite chemical potential is essentially determined by the susceptibility. R. V. Gavai and S. Gupta, hep-lat/0303013

- χ_3 is the zero momentum Euclidean finite temperature vector propagator and hence closely related to a transport coefficient— the DC electrical conductivity of quark matter.

Some notation

With two degenerate flavours of quarks, in flavour space the linear susceptibilities form the matrix

$$\begin{pmatrix} \chi_u & \chi_{ud} \\ \chi_{ud} & \chi_u \end{pmatrix}$$

Redefining $\mu_0 = \mu_u + \mu_d$ and $\mu_3 = \mu_u - \mu_d$, this matrix becomes

$$\begin{pmatrix} \chi_u + \chi_{ud} & 0 \\ 0 & \chi_u - \chi_{ud} \end{pmatrix}$$

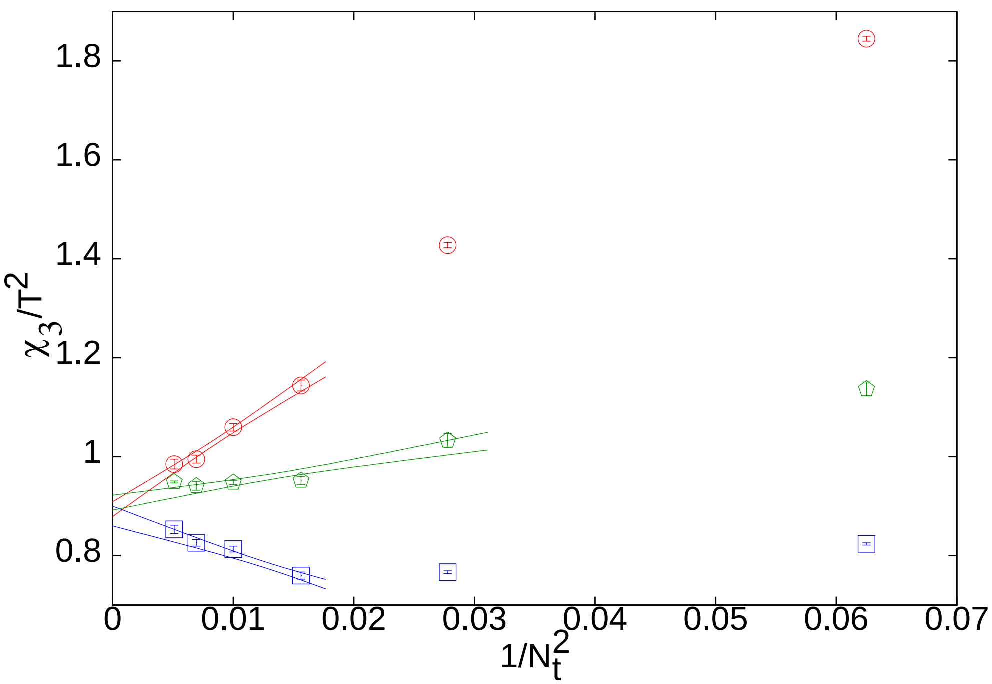
We define

$$\begin{aligned} \chi_3 &= \chi_u - \chi_{ud} = \left\langle \mathrm{Tr}\ M^{-1}M'M^{-1}M' - \mathrm{Tr}\ M^{-1}M'' \right\rangle \\ \chi_{ud} &= \left\langle \left(\mathrm{Tr}\ M^{-1}M'\right)^2 \right\rangle \quad \text{and} \quad \chi_0 = \chi_3 + 2\chi_{ud} \end{aligned}$$

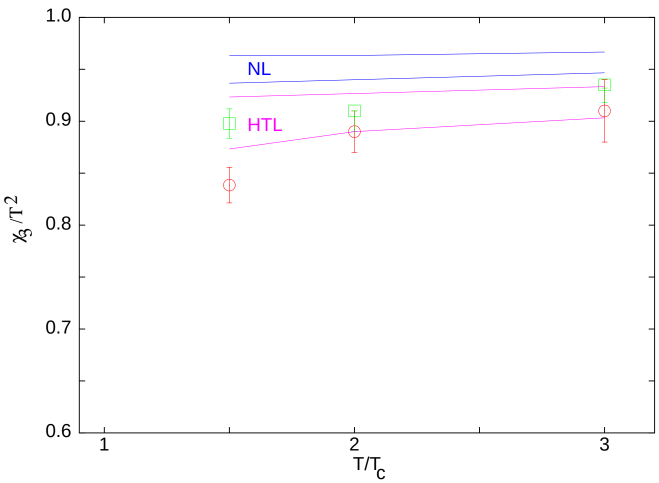
Finding the continuum limit

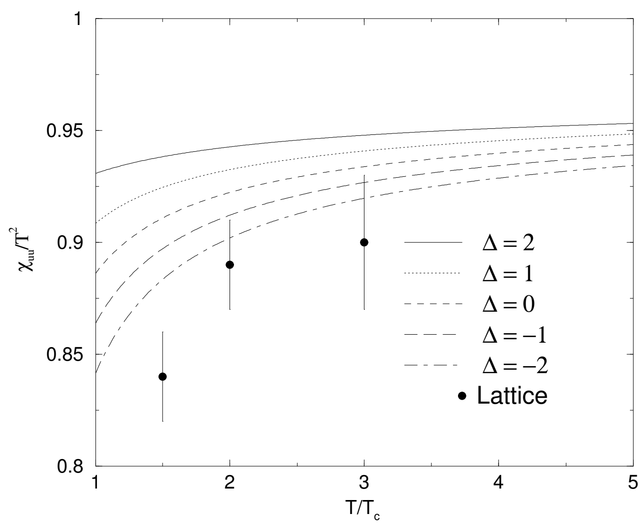
Main technical problem is to control the extrapolation to zero lattice spacing. For this we use two different kinds of Fermions (staggered and Naik) and perform simultaneous extrapolation with both: in the quenched theory.

R. V. Gavai and S. Gupta, *Phys. Rev.* D 67 (2003) 034501



Perturbation theory





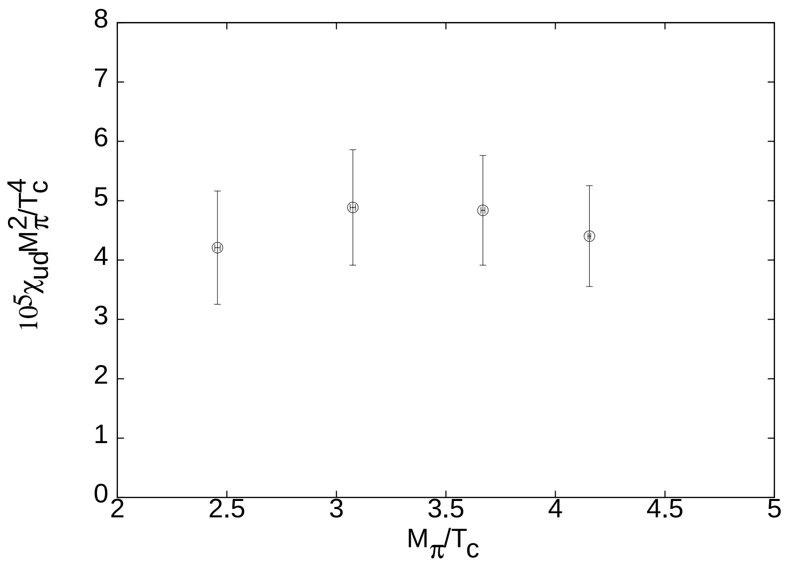
J.P. Blaizot, E. Iancu and A. Rebhan, *Phys. Lett.*, B 523 (2001) 143

A. Vuorinen, hep-ph/0212283

χ_{ud}

$\chi_{ud} = 0$ for all $T > T_c$, but not $T < T_c$.

Perturbation theory is unable to explain this.



R. V. Gavai *et al.*, *Phys. Rev.*, D 65 (2002) 054506

Event to event fluctuations

Each heavy-ion collision event, followed by the hadronisation, is one realisation of the whole ensemble of possible thermodynamic systems. Within a given rapidity region, the total amount of any conserved charge fluctuates from one event to another. The variance is determined by the response function of QCD matter in equilibrium.

M. Asakawa *et al.*, *Phys. Rev. Lett.*, 85 (2000) 2072

S. Jeon *et al.*, *Phys. Rev. Lett.*, 85 (2000) 2076

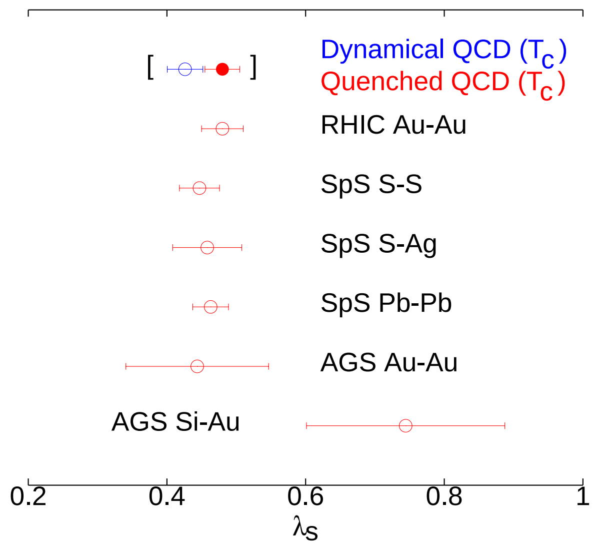
D. Bower and S. Gavin, *Phys. Rev.*, C 64 (2001) 051902

From lattice computations it is seen that

$$\chi_B < \chi_Q < \chi_s \qquad (T > T_c)$$
$$\chi_B > \chi_Q > \chi_s \qquad (T < T_c)$$

R. V. Gavai, S. Gupta, P. Majumdar, *Phys. Rev.*¡ D 65 (2002) 054506

Strangeness production



$$\lambda_s = \frac{\langle n_s \rangle}{\langle n_u + n_d \rangle}$$

J. Cleymans, *J. Phys.*, G 28 (2002) 1575,

R. V. Gavai and S. Gupta, *Phys. Rev.*, D 65 (2002) 094515.

Prescription

Chemical potential on the lattice is prescription dependent. Why? The continuum Dirac operator specifies effects of an infinitesimal time translation. On the lattice we deals with finite translations (by lattice spacing a). There are many ways of doing this which lead to the same infinitesimal transformation.

This is the origin of problems with reweighting: it gives no indication of how large the lattice artifacts are.

Taylor series expansion is prescription dependent beyond 2nd order at every finite lattice spacing a, but prescription independent for $a \to 0$. With explicit Taylor expansion we can take the continuum limit.

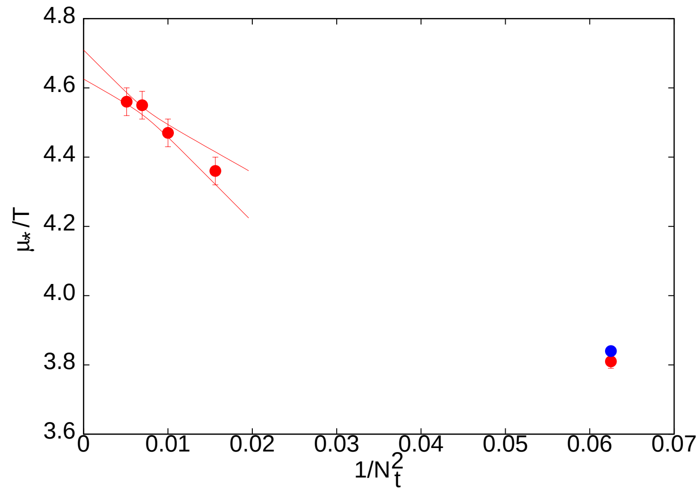
R. V. Gavai and S. Gupta, hep-lat/0303013

Obtaining pressure

$$
\begin{aligned}
P(T,\mu) = -F/V &= P(T,0) + \chi_3(T)\mu^2 + \frac{1}{12}\chi_{uuuu}(T)\mu^4 + \mathcal{O}\left(\mu^6\right) \\
&= P(T,0) + \chi_3(T)\mu^2 \left[1 + \left(\frac{\mu}{\mu_*}\right)^2 + \mathcal{O}\left(\frac{\mu^4}{\mu_*^4}\right)\right].
\end{aligned}
$$

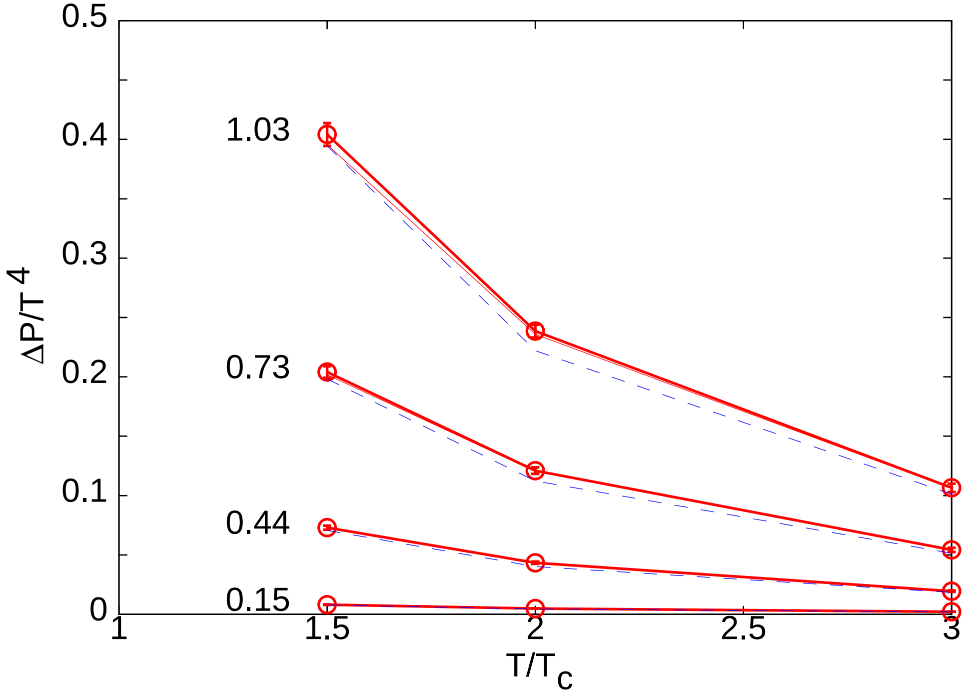
where $\mu_* = \sqrt{12\chi_3/\chi_{uuuu}}$ and other 2nd and 4th order terms have been neglected. Well-behaved for $\mu \ll \mu_*$ if all the higher order terms are small enough. All results can be obtained in the continuum. Term by term improvement of the series is possible. Series should fail to converge near a critical point. Series extrapolation methods should then be used to locate the critical point nearest to $\mu = 0$.

Radius of convergence



4-th order estimate of μ_* at $T = 1.5T_c$. At finite N_t, the series is insensitive to prescription when $\mu \ll \mu_*$. In the continuum μ_* is the first estimate of the radius of convergence of the series.

The pressure



$$\Delta P(T) = P(T, \mu) - P(T, 0)$$

(Reweighting) Z. Fodor, S. D. Katz and K. K. Szabo, hep-lat/0208078,
(Taylor expn) R. V. Gavai and S. Gupta, hep-lat/0303013

Condensates and masses

Talor expansions can also be made for expectation values of any operator. We are investigating this for

1. Condensates: $\langle\overline{\psi}\psi\rangle$ changes quadratically with μ, and the quadratic coefficient is the same for isovector and baryon chemical potential. This number is also related to λ_s in strangeness production through a Maxwell relation.

2. Masses: The mass splitting of charged pions at finite isovector chemical potential is linear in μ, but that of the neutral pion is quadratic. This quadratic coefficient is the same as shift in pion mass at finite baryon chemical potential.

O. Miyamura *et al.*, *Phys. Rev.*, D 66 (2002) 077502,

S. Gupta, hep-lat/0202005, S. Gupta and Rajarshi Ray, in progress

Summary of Results

- Susceptibilities provide a systematic and easy way of computing quantities non-perturbatively at finite chemical potential in the continuum.
- Computation of several high order susceptibilities may allow estimation of the critical end point by series extrapolation methods.
- Fluctuations and strangeness production rate in heavy-ion collisions are related to susceptibilities.
- Susceptibilities allow extension of the equation of state to finite chemical potential.
- Taylor expansions yield identities between behaviour of various quantities at finite isovector and baryon chemical potential.