Experimental tests of QCD at finite density

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ILGTI: TIFR

Non-perturbative problems in Field Theory
IACS Kolkata, India
December 20, 2010

Introduction
- Phase structure
- A conjectured phase diagram
- The sign problem

Avoiding the sign problem
- A Madhava-Maclaurin series expansion
- Radius of convergence
- Extrapolating measurements to $\mu > 0$

Connecting to experiments
- The questions
- Theoretical developments
- Observational tests

Summary

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Summary

Phase diagrams

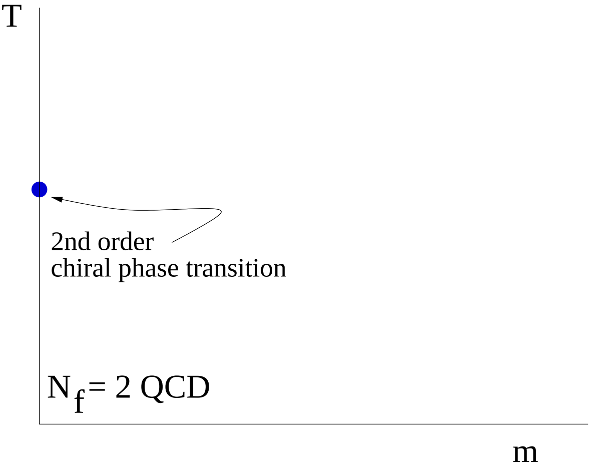
Dimension of Gibbs space

The free energy of a system is a function of intensive thermodynamic variables: one for each conserved quantum number. In QCD: T, μ_B, μ_Q and μ_S. Other intensive variables correspond to other parameters in the action; quark masses: $m_{ud} \ll \Lambda_{QCD}$ and $m_s \simeq \Lambda_{QCD}$.

A phase diagram

Every phase diagram is a plot of the location of the singularities of the free energy.
An order parameter is an extensive thermodynamic quantity which is exactly constant in one phase and changes in others: hence locates a singularity. However, singularities more general: so there can be phase transitions without order parameters.

The phase diagram of $N_f = 2$ QCD



Pisarski and Wilczek, PR D 29, 338 (1984)

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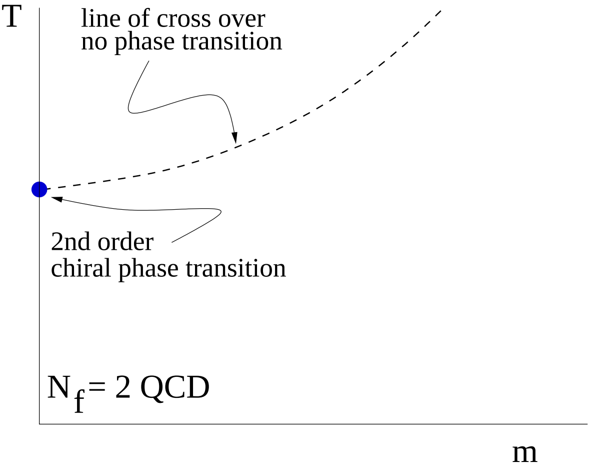

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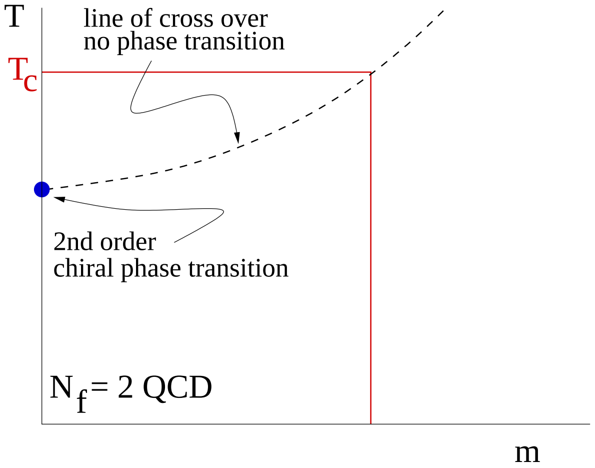

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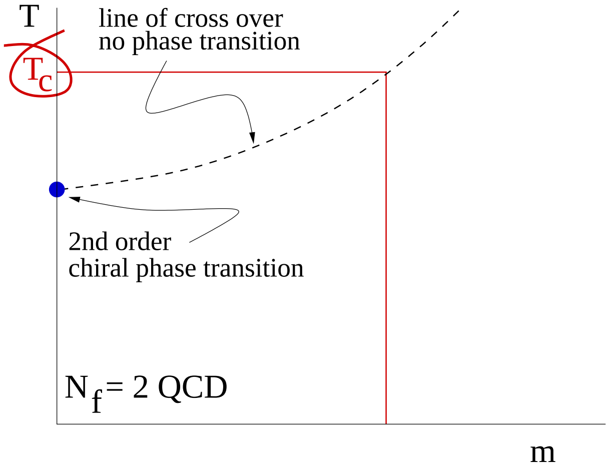
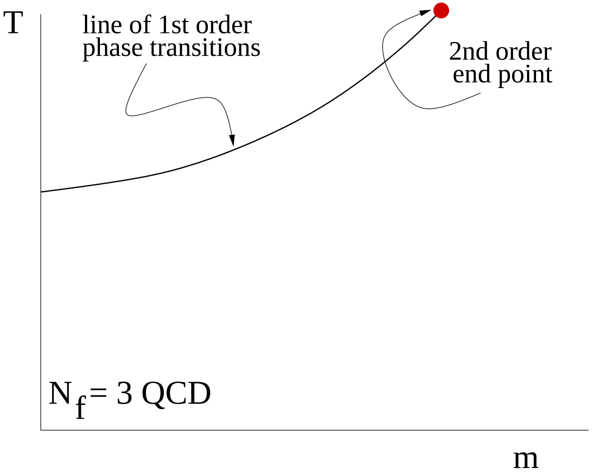


Pisarski and Wilczek, PR D 29, 338 (1984)

The phase diagram of $N_f = 3$ QCD



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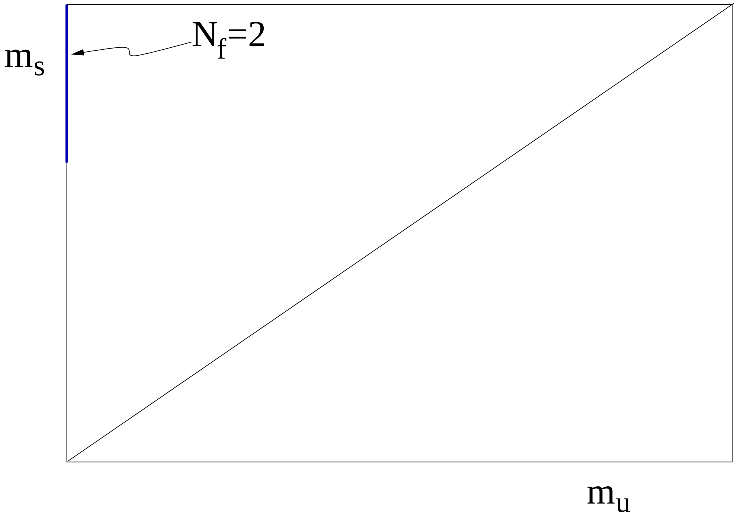
The Columbia plot



Brown et al, PRL 65, 2491 (1990) Not a phase diagram: flag diagram

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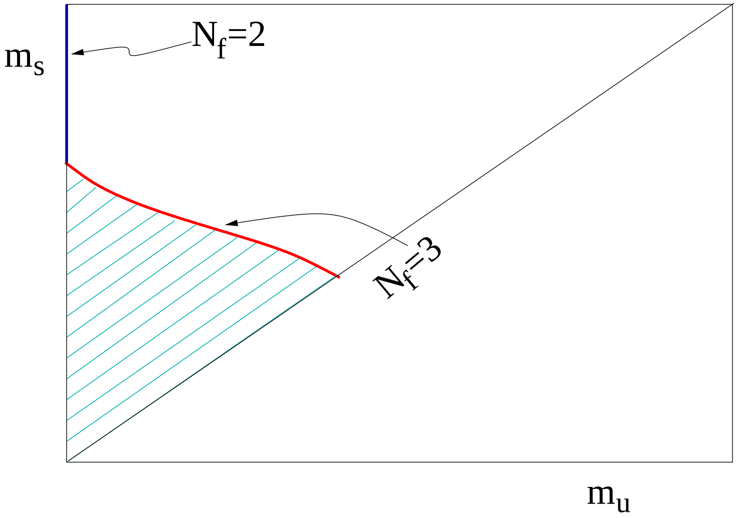

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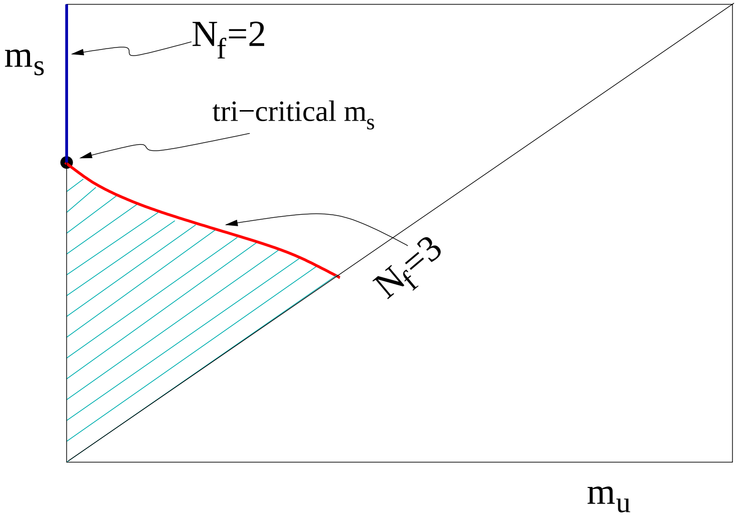
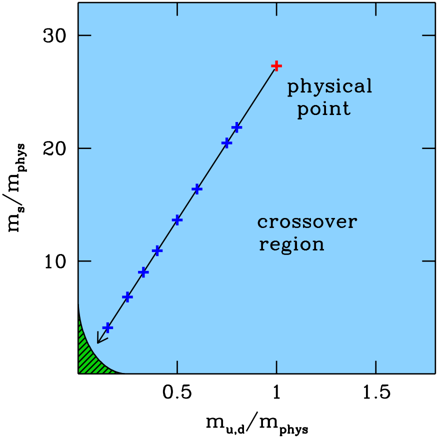


Brown et al, PRL 65, 2491 (1990) Not a phase diagram: flag diagram

Lattice results for the Columbia Plot



In $N_f = 2 + 1$:

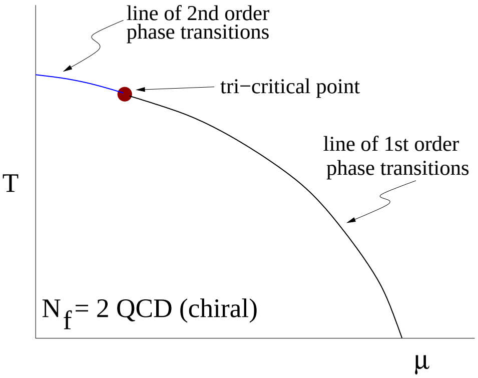
$$m_\pi^{crit} \begin{cases} = 0.07 m_\pi & (N_t = 4) \\ < 0.12 m_\pi & (N_t = 6) \end{cases}$$

Endrodi etal, 0710.0988 (2007)

Similarly for $N_f = 3$.

Karsch etal, hep-lat/0309121 (2004)

A conjectured phase diagram



Rajagopal, Stephanov, Shuryak 1998 and 1999

Other effects: anomaly, large-N counting, condensed phases?

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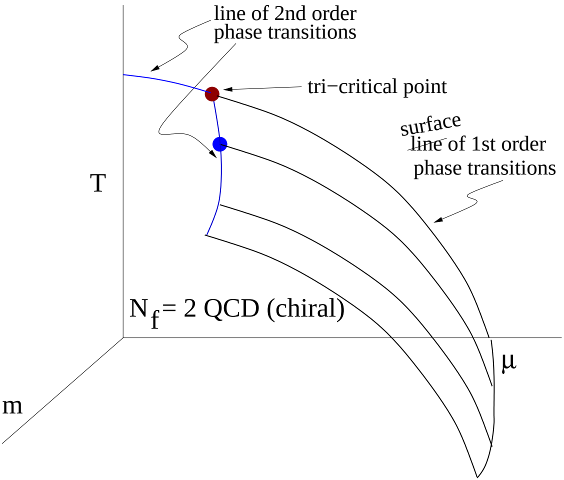

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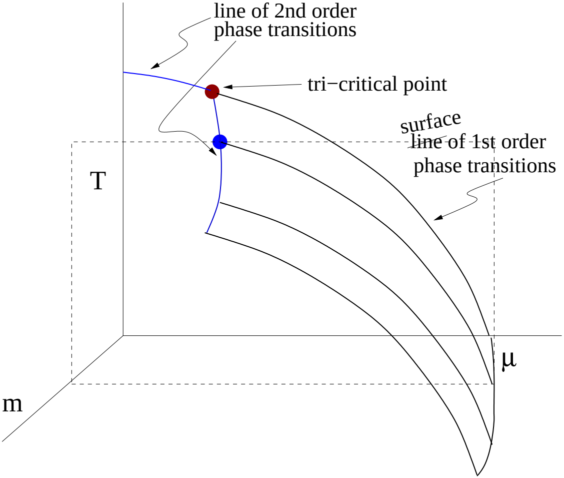

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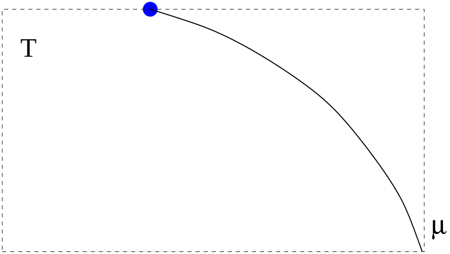

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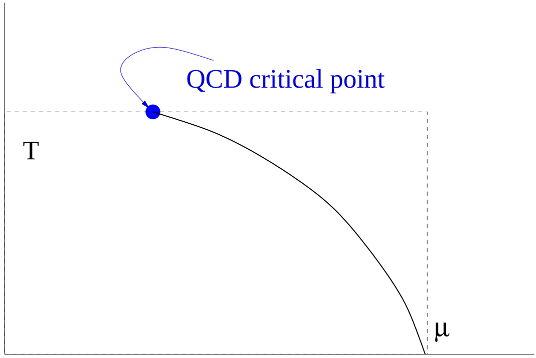


Rajagopal, Stephanov, Shuryak 1998 and 1999

Other effects: anomaly, large-N counting, condensed phases?

The axial anomaly

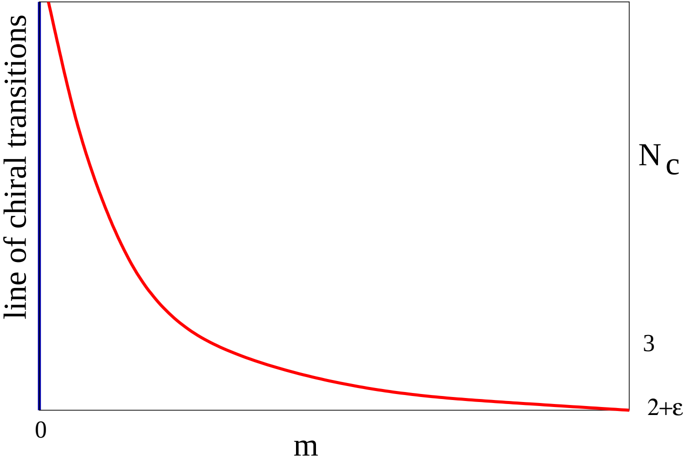
First results indicate that $U_A(1)$ may not be restored in the high temperature phase of QCD.

In the absence of instantons one would find with chiral fermions;

$$\chi_{PS} = -\chi_S, \qquad \chi_{PS} = \int d^4x C_{PS}(x).$$

Quenched overlap computation does not find this up to $2T_c$. Verified with a computation of non-vanishing topological susceptibility.
Gavai, SG, Lacaze: 2001, 2009
Similar result now obtained with dynamical domain wall quarks. However, no index theorem, so exact correspondence not yet available in this case.
RBRC: BYOPD Mumbai, December 2010

Flag diagram of large-N QCD



Finite radius of convergence of the $1/N$ expansion.

Datta and SG: 2009

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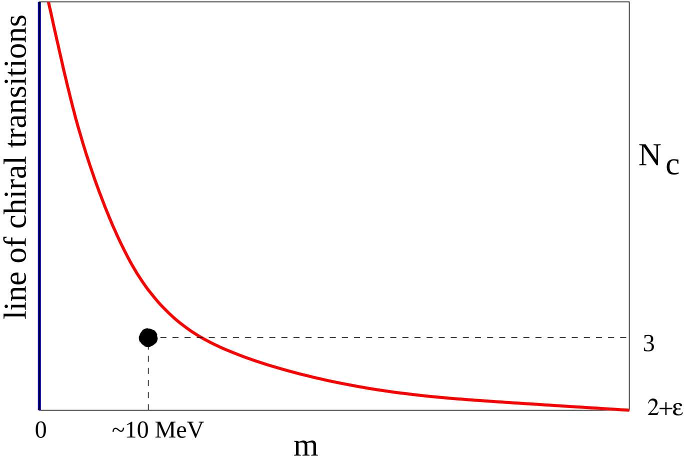

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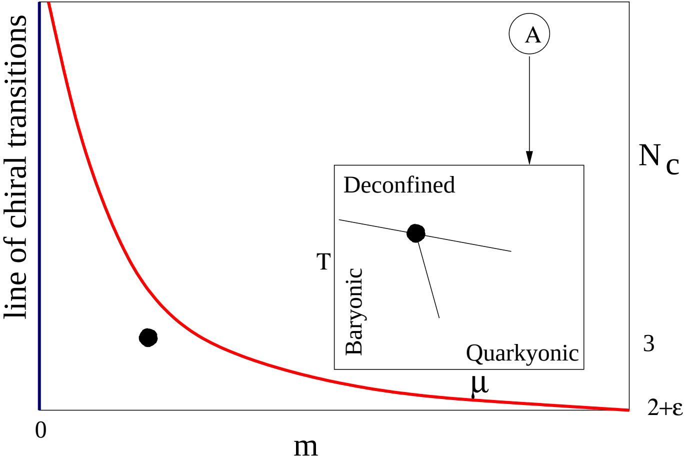

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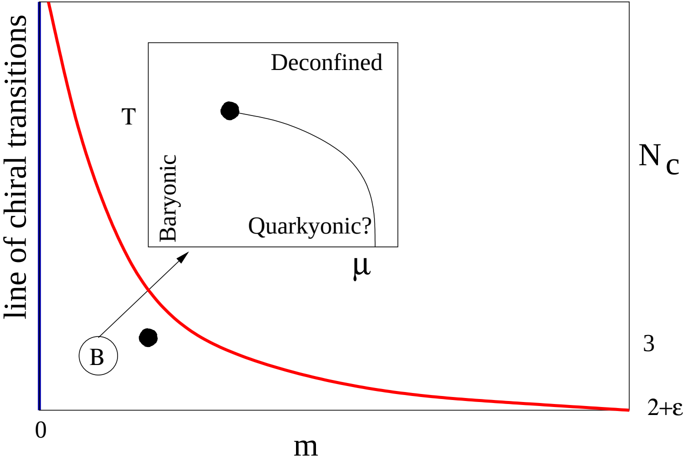

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The sign problem

Gauge action positive; not changed by introduction of flavour chemical potentials.
Fermion determinant contains sign problem:

$$\det(D + m + \mu\gamma_0)^* = \det(D + m - \mu^*\gamma_0)$$

Cannot be free of sign problems when μ is real non-zero.

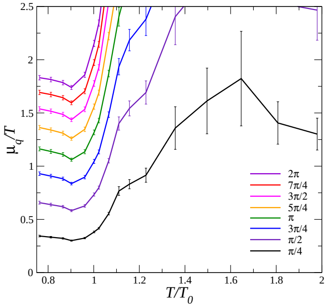
Importance sampling fails: no Monte Carlo procedure.

Problem could be representation dependent; clever reformulation may resolve the problem: for example, by changing to new variables.

How bad is the sign problem?

For $\mu < m_\pi/2$ distribution of signs is Gaussian. At larger μ it becomes Lorentzian. (Analysis in baryonless random matrix theory). Hard in both cases.

Lombardo, Splittorff and Verbaarschot, 0910.5842



Effect of baryons? Effect of finite temperature?

Splittorff *et al.*, Lattice 2010

Contour lines of the variance of the phase of the determinant: problem easier at high temperature.

Bielefeld-Swansea, PR D 71 2005

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Madhava-Maclaurin (Taylor) series expansion

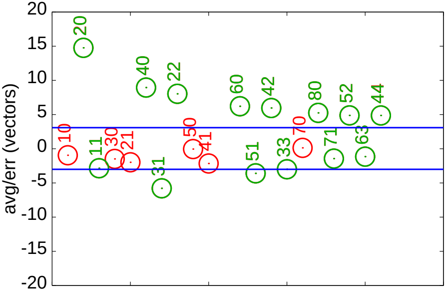
The pressure in a grand canonical ensemble allows a Maclaurin series expansion:

$$P(T,\mu) = P(T) + \frac{\mu^2}{2!}\chi^{(2)}(\mathbf{T}) + \frac{\mu^4}{4!}\chi^{(4)}(\mathbf{T}) + \cdots$$

The coefficients are evaluated at $\mu = 0$ where there is no sign problem.

Evaluate the non-linear susceptibilities (NLS) $\chi^{(n)}$ directly as expectation values of operators.
Gavai, SG, 2003
Evaluate the susceptibilities by constructing the pressure (or its derivatives) at series of imaginary chemical potentials and then fitting extrapolating functions to the data.
Cosmai *et al.*, Falcone *et al.*: 2009, 2010

Statistical significance of measurements



Covariance over configurations: $\sigma_{O_4,O_6} \simeq \sigma_{O_6,O_8} \simeq 0.7$

1. Staggered: 4.24^3 lattice, $m_\pi = 230$ MeV, $T = 0.75\,T_c$, 400 vectors. (Red symbols: supposed to vanish) SG, 2004

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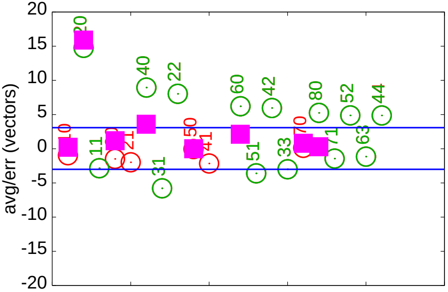

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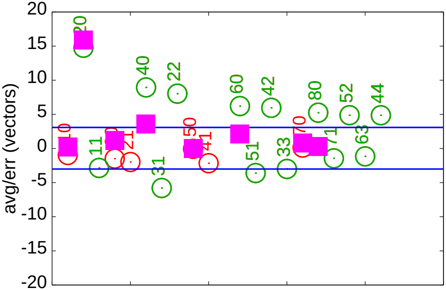

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3. Asqtad quarks: up to 50% of the noise due to stochastic estimators with 400–800 vectors. MILC, 1003.5682

CPU effort

$LT = 4$ lattices

At T_c autocorrelations: 200–250 trajectories
Number of CG inversions per trajectory: 200
One measurement every decorrelated configuration: 500×18 CG inversions
Measurement/configuration: $500 \times 18/(200 \times 200) = 0.24$

At $2T_c$ autocorrelations: 4 trajectories
Number of CG inversions per trajectory: 100
One measurement every decorrelated configuration: 100×18 CG inversions
Measurement/configuration: $100 \times 18/(100 \times 4) = 4.5$

Series Analysis: radius of convergence

Series analysis for spin models

Analysis of series for critical behaviour since 1960s. Well-developed when series coefficients are exactly known. First step: evaluate radius of convergence. Then check whether singularity is due to physical parameter values.

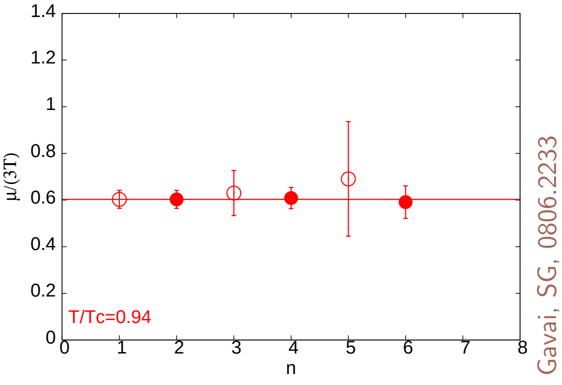
Domb and Green, vol 2

Series analysis for $\mu \neq 0$ QCD

Similar idea, but needs to be adapted to specific problem. Series coefficients have statistical errors; coefficients are volume dependent. Some subtleties.

Gavai, SG, 2004, 2008

Finite volume effects

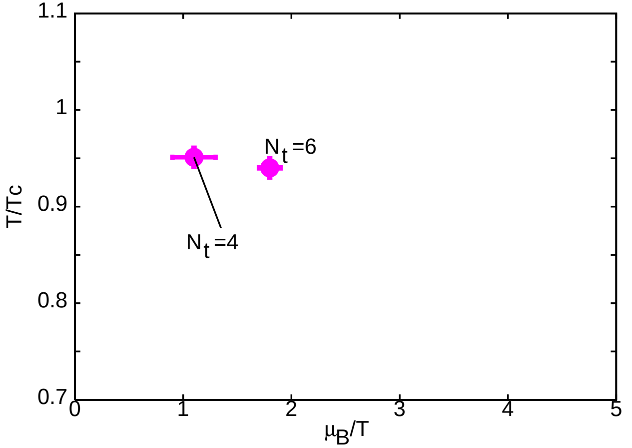


Filled symbols: $r_n = \sqrt{(n+3)!\chi^{(n+1)}/(n+1)!\chi^{(n+3)}}$
Unfilled symbols: $r_n = ((n+2)!\chi^{(2)}/2!\chi^{(n+2)})^{1/n}$
$LT \geq 4$ and $Lm_\pi \geq 5$; plateau develops.

Finite volume effects and order of expansion

1. Increasing order of series expansion and finite volume scaling closely tied together.
2. Susceptibility never diverges on finite volume, but grows higher and sharper with increasing volume. Major effect: growth of peak; minor effect: shift of peak.
3. Series expansion of such a sequence of functions should show lack of divergence for each volume if pushed to large enough order.
4. At finite order, signal of eventual divergence should build up.
5. With increasing volume, there should be a plateau of stability for radius of convergence before radius diverges.

Cutoff dependence and the effect of strange quarks



Staggered: $N_f = 2$, $m_\pi = 230$ MeV, $LT \geq 4$ Gavai, SG, 0806.2233
P4: $N_f = 2 + 1$, $m_\pi = 220$ MeV, $LT = 4$ Schmidt, 2010

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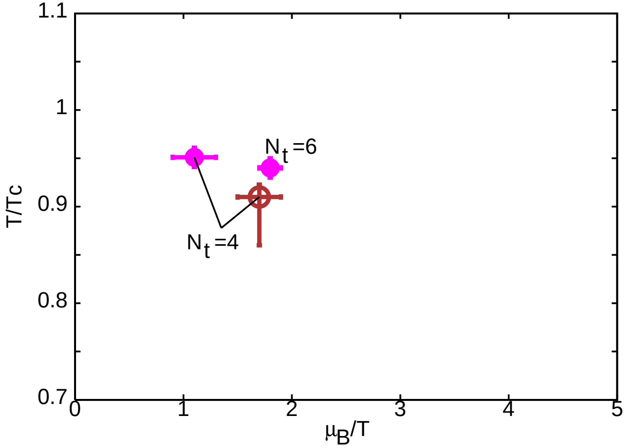

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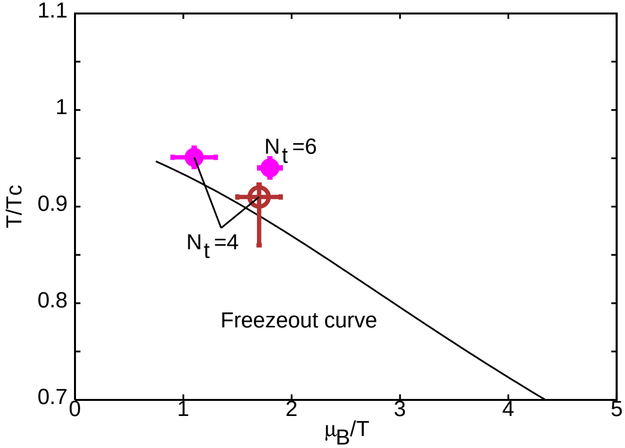

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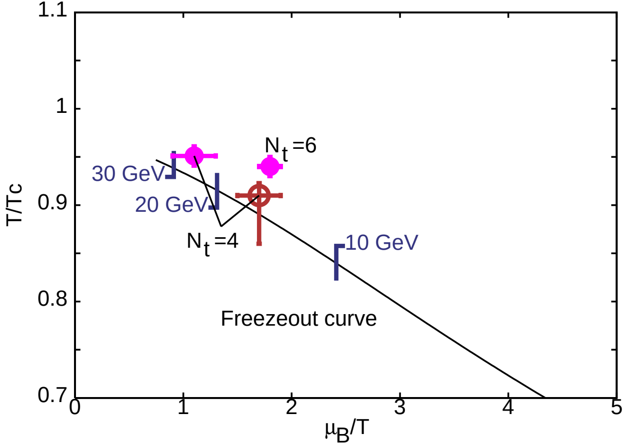

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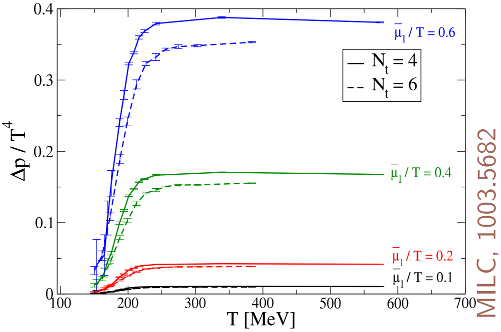
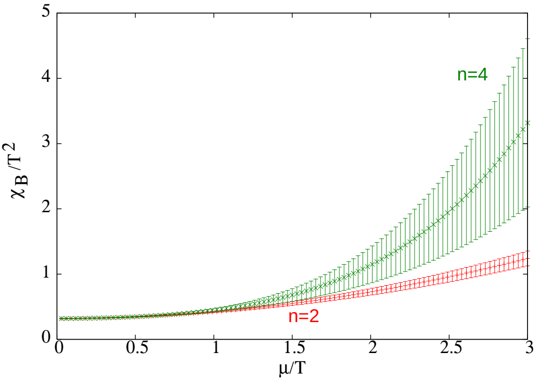
The pressure



$\Delta p = p(T, \mu) - p(T, 0)$. May be interesting to try a resummation.

Extrapolating measurements



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Infinite series diverges, but truncated series finite and smooth: sum is bad. Resummations needed to reproduce critical divergence. Padé resummation useful.

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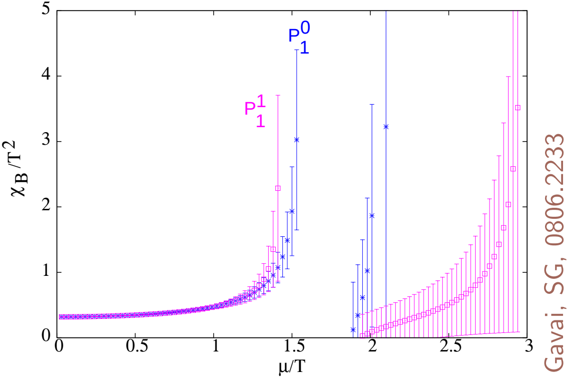

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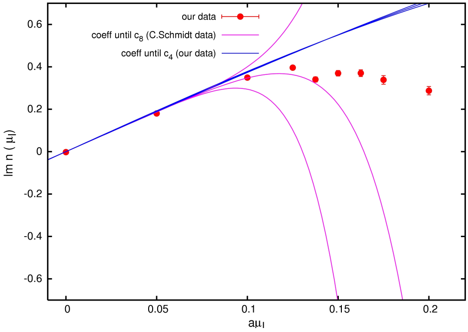
Series at imaginary μ



Falcone *et al.*, Lattice 2010

More terms in the series needed. Does a resummation help?

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The experimental reflex

" We didn't have flint when when I was a kid, we had to rub two sticks together. "

The set of questions

Can experiment test any non-perturbative predictions of QCD?

In heavy-ion collisions QCD often enters indirectly: as the result of a long secondary computation such as hydro. Instead, can one get directly at QCD?

Can experiment test the existence of a critical point of QCD?

Do heavy-ion experiments have anything to say about the phase diagram? Or are they just dirtier versions of proton-proton collisions?

Non-linear susceptibilities

Taylor expansion of the pressure in μ_B

$$P(T, \mu_B + \Delta\mu_B)/T^4 = \sum_n \frac{1}{n!} \left[\chi^{(n)}(T, \mu_B) T^{n-4}\right] \left(\frac{\Delta\mu_B}{T}\right)^n$$

has Taylor coefficients called non-linear susceptibilities (NLS). When $\mu_B = 0$ they can be computed directly on the lattice, otherwise reconstructed from such computations.
(Gavai, SG: 2003, 2010)
Cumulants of the event-to-event distribution of baryon number are directly related to the NLS:

$$[B^2] = T^3 V \left(\frac{\chi^{(2)}}{T^2}\right), \quad [B^3] = T^3 V \left(\frac{\chi^{(3)}}{T}\right), \quad [B^4] = T^3 V \chi^{(4)}.$$

V unknown, can be removed by taking ratios.
(SG: 2009)

Tests and assumptions

$$m_1: \quad \frac{[B^3]}{[B^2]} = \frac{\chi^{(3)}(T,\mu_B)/T}{\chi^{(2)}(T,\mu_B)/T^2}$$

$$m_2: \quad \frac{[B^4]}{[B^2]} = \frac{\chi^{(4)}(T,\mu_B)}{\chi^{(2)}(T,\mu_B)/T^2}$$

$$m_3: \quad \frac{[B^4]}{[B^3]} = \frac{\chi^{(4)}(T,\mu_B)}{\chi^{(3)}(T,\mu_B)/T}$$

Also for cumulants of electric charge, Q, and strangeness, S.

1. Two sides of the equation equal if there is thermal equilibrium and no other sources of fluctuations.
2. Right hand side computed in the grand canonical ensemble (GCE). Can observations simulate a grand canonical ensemble? What T and μ_B?
3. Why should hydrodynamics and diffusion be neglected?

Why thermodynamics and not dynamics?

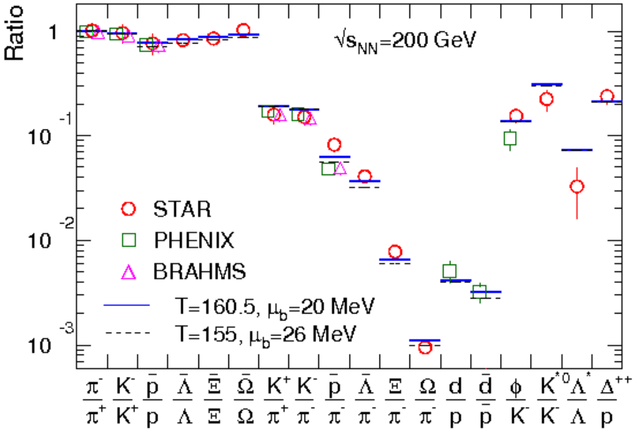
Chemical species may diffuse on the expanding background of the fireball, so why should we neglect diffusion and expansion?

First check whether the system size, ℓ, is large enough compared to the correlation length ξ: **Knudsen's number** $K = \xi/\ell$. If $K \ll 1$, ie, $\ell \gg \xi$ then central limit theorem will apply.
Next, compare the relative importance of diffusion and advection through a dimensionless number (**Peclet's number**):

$$\mathcal{W} = \frac{\ell^2}{t\mathcal{D}} = \frac{\ell v_{flow}}{\mathcal{D}} = \frac{\xi v_{flow}}{K\mathcal{D}} = \frac{v_{flow}}{Kc_s} = \frac{M}{K}.$$

When $\mathcal{W} \ll 1$ diffusion dominates. After chemical freeze-out K is small but **Mach's number** $M \simeq 1$, so flow dominates: fluctuations are frozen in. So detector observes thermodynamic fluctuations <u>at chemical freeze out</u>.
(Bhalerao, SG: 2009)

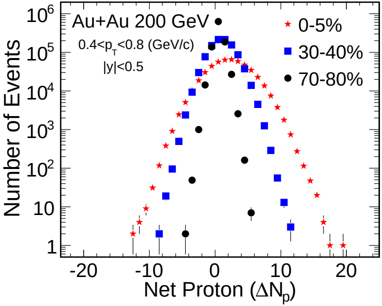
The fireball thermalizes



Chemical freeze out: $T = 160.5$ MeV, $\mu = 20$ MeV.

Andronic et al, nucl-th/0511071

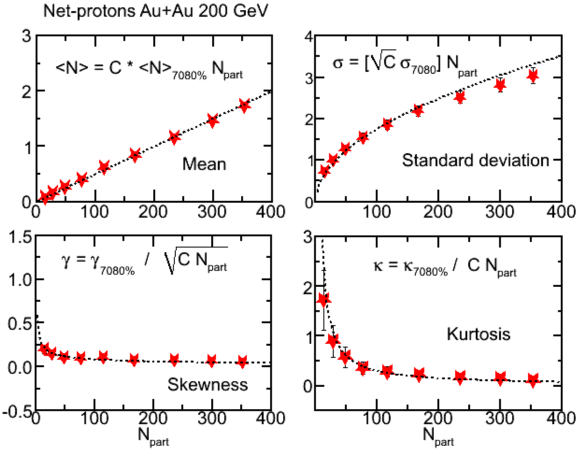
Event distributions of conserved charges



STAR, 1004.4959

- Fluctuations of conserved quantities are Gaussian: provided large volume and equilibrium
- Proton number a substitute for baryon number: how good?
- Is this Gaussian due (entirely or largely) to thermal fluctuations?

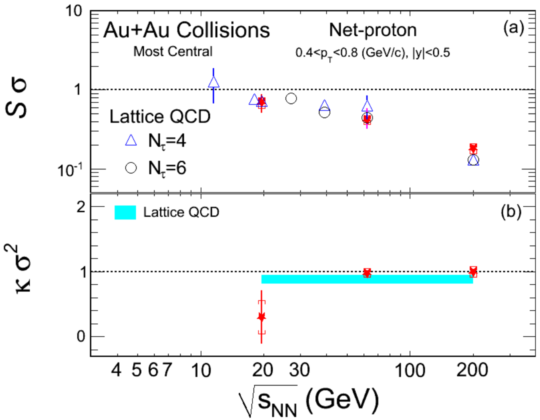
STAR measurements: 2009



$\ell \gg \xi$ ($K \ll 1$) tested and found true.

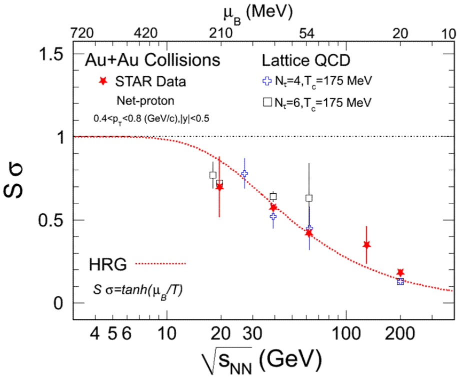
STAR Collaboration: QM 2009, Knoxville

STAR measurements: beginning 2010



First ever agreement between lattice and experiment for bulk matter! STAR Collaboration: 2010

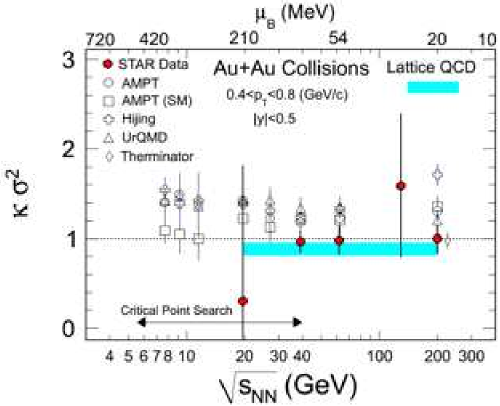
STAR measurements: end 2010



Continuing agreement between bulk matter lattice and experiment!

STAR Collaboration (preliminary): ICPAQGP, Goa, December 2010

New STAR data



Intriguing structure in m_2: not predicted by models which have no critical point.

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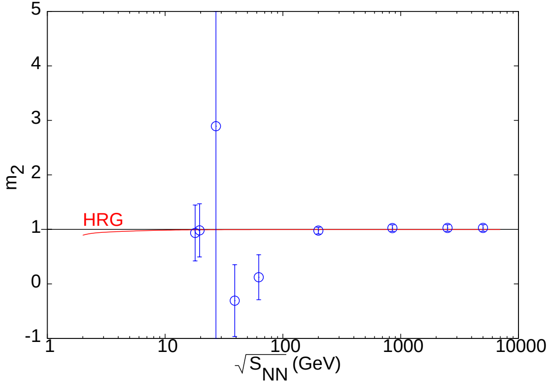

Intriguing structure in m_2: not predicted by models which have no critical point.

Outline

Introduction
- Phase structure
- A conjectured phase diagram
- The sign problem

Avoiding the sign problem
- A Madhava-Maclaurin series expansion
- Radius of convergence
- Extrapolating measurements to $\mu > 0$

Connecting to experiments
- The questions
- Theoretical developments
- Observational tests

Summary

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1. The strange quark is heavy; light quarks determine the shape of the phase diagram. The cross over temperature now under control: $T_c \simeq 170$ MeV. SU(2) flavour symmetry breaking unlikely to change T_c.
2. Lattice determines series expansion of pressure; indicates a critical point in QCD. Lattice spacing effects under reasonable control. Physical quantities can be found be resumming the series expansion (*e.g.*, Padé approximants).
3. Imaginary μ is an alternative method for analytic continuation. Many studies of systematics can be tested. Consistency with Taylor expansion now being established.
4. First direct comparison of lattice results with experimental data done; good agreement. A landmark in the field: good evidence for thermalization of fireball.