# FUN WITH DIRAC EIGENVALUES

Michael Creutz

Brookhaven National Laboratory

Eigenvalues popular for discussion

- chiral condensate and density of small eigenvalues
  - Banks-Casher formula
- approximations to the Ginsparg Wilson relation
  - eigenvalues near ‘‘circles’’
- projection issues for the overlap/domain wall operators
  - undefined when $D_W^\dagger D_W$ not invertible
  - need a gap in the Wilson operator spectrum

Dangers

- eigenvalues depend on gauge fields
- gauge fields depend on eigenvalues
- Highly non-linear system!

Generic path integral

$$Z = \int (dA)(d\psi)(d\overline{\psi})\ e^{-S_G(A)+\overline{\psi}D(A)\psi}$$

Integrate out fermions

$$Z = \int (dA)\ |D(A)|\ e^{-S_G(A)}$$

Determinant is product of eigenvalues

$$D(A)\psi_i = \lambda_i \psi_i$$

$$Z = \int (dA)\ e^{-S_G(A)}\ \prod_i \lambda_i$$

## Eigenvalue density

$$\rho(x+iy) = \frac{1}{VZ}\int (dA)\, |D(A)|\, e^{-S_G(A)} \sum_i \delta(x-\mathrm{Re}\lambda_i(A))\delta(y-\mathrm{Im}\lambda_i(A))$$

- $V$ dimension of $D$; proportional to system volume
- Hermiticity condition

$$\gamma_5 D \gamma_5 = D^\dagger$$
$$\rho(z) = \rho(z^*)$$

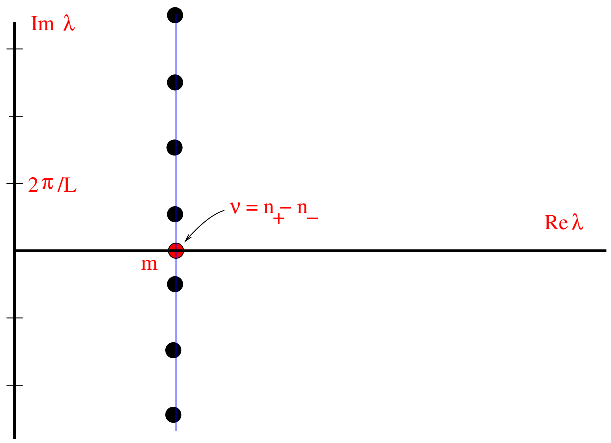
Repeat warning:

- $\lambda$ depends on $A$ which is weighted by $\lambda$ which depends on $A$ ...

Continuum

$$D = \gamma_\mu(\partial_\mu + igA_\mu) + m$$

$$\rho(x+iy) = \delta(x-m)\tilde\rho(y)$$




Banks and Casher, multiple flavors, vanishing mass

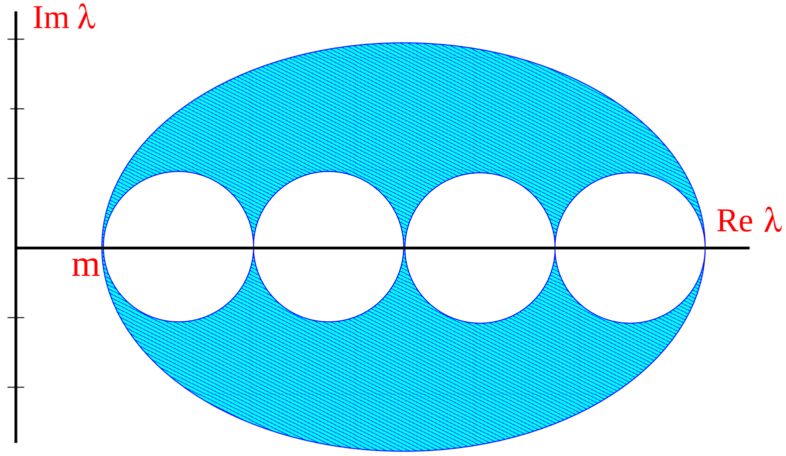
- $\langle\overline{\psi}\psi\rangle \neq 0$ correlates with $\tilde\rho(0) \neq 0$

Index theorem: consider eigenmodes with real eigenvalues

- $\gamma_5$ commutes with $D$ when restricted to this set
- chirality $\pm 1$
- winding number $\nu = n_+ - n_-$
- matches winding from smooth gauge field topology

Lattice

- free Wilson fermions
- doublers given large real part

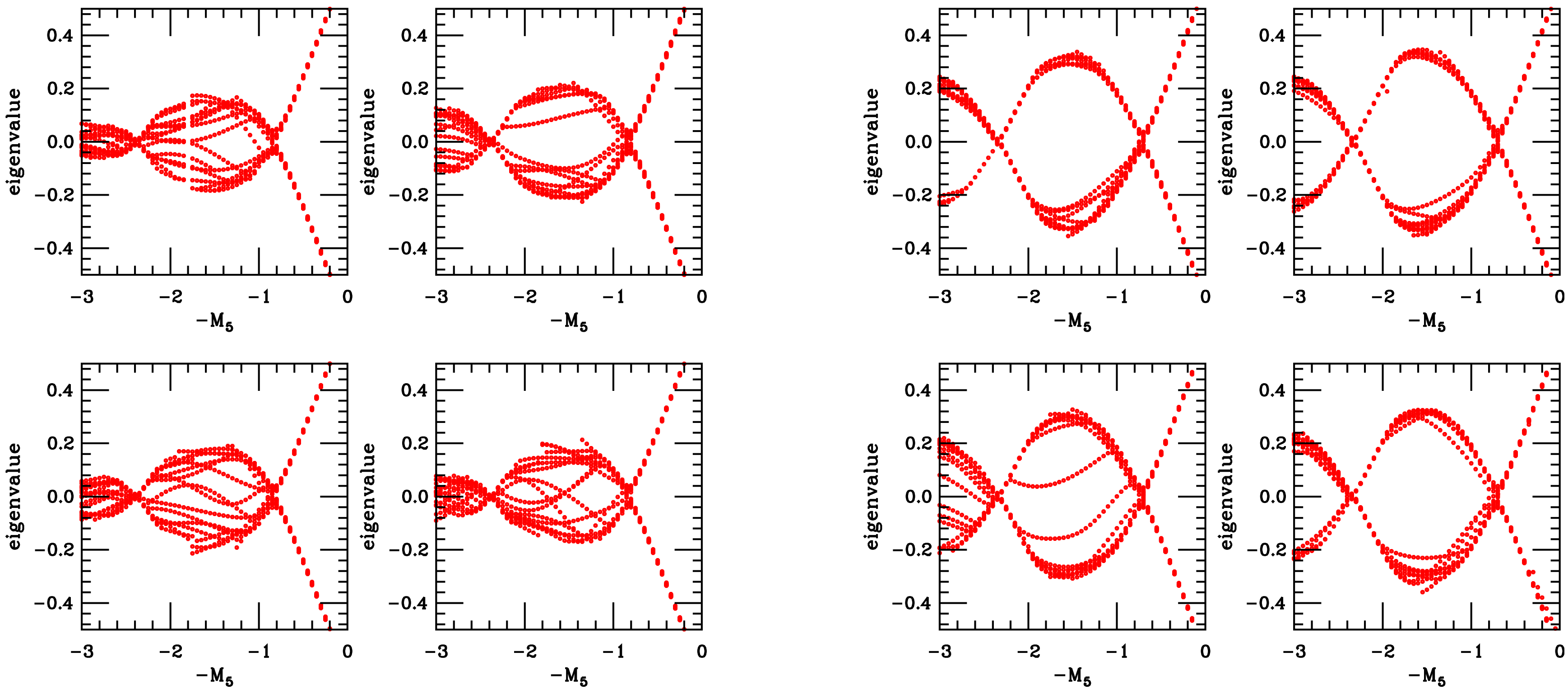



Turn on gauge fields

- $D$ no longer normal, i.e. $[D, D^{\dagger}] \neq 0$
- eigenvalues spread out, remain in complex conjugate pairs
- some eigenvalue pairs collide and become real
    - continuous spectrum of eigenvalues along real axis

From hep-lat/0211023, Aoki et al.

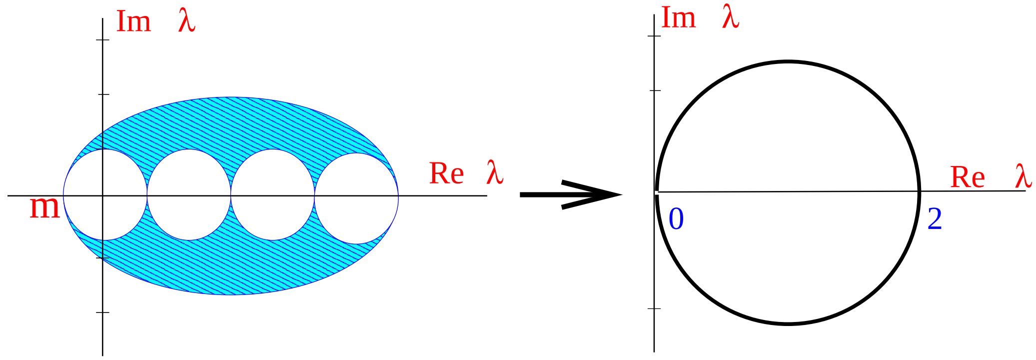
- lowest few eigenvalues of $\gamma_5 D_W$
- typical quenched configurations




Wilson gauge fields

Iwasaki gauge fields

Overlap: project Wilson eigenvalues onto circle




- $D = 1 + V$
- $V = (D_W D_W^\dagger)^{-1/2} D_W$
- $V^\dagger V = 1$ Ginsparg-Wilson condition
- normality restored
- $m < 0$: Wilson hopping parameter "supercritical"

Exact chiral symmetry

$$\psi \rightarrow e^{i\theta\gamma_5}\psi$$

$$\overline{\psi} \rightarrow \overline{\psi} e^{i\theta\hat{\gamma}_5}$$

$$\hat{\gamma}_5 = V\gamma_5$$

$$\nu = \frac{1}{2}\mathrm{Tr}\hat{\gamma}_5$$

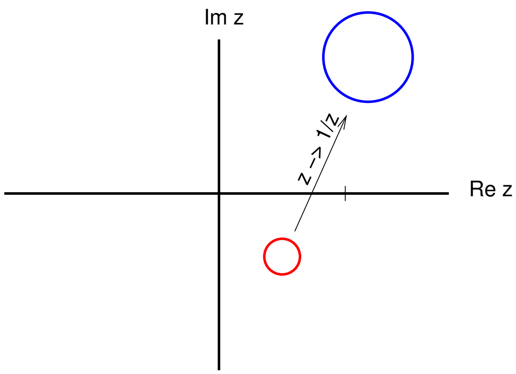
# A Cheshire Chiral Condensate

Consider the overlap

- Eigenvalues in complex conjugate pairs on a circle
  - $D = 1 + V$
  - $V^\dagger V = 1$
- Calculate the condensate

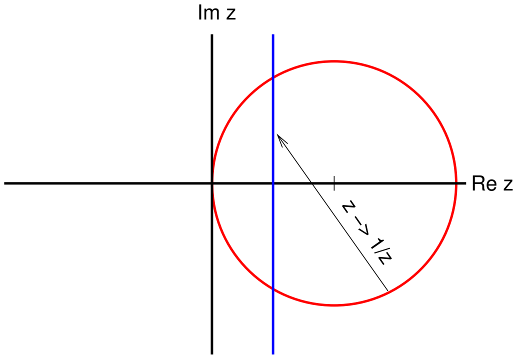
$$\langle \overline{\psi}\psi \rangle = \langle \mathrm{Tr} D^{-1} \rangle = \left\langle \sum \frac{1}{\lambda_i} \right\rangle = \left\langle \sum \mathrm{Re} \frac{1}{\lambda_i} \right\rangle$$

Inverting a complex circle gives another circle




Circle for $D$ touches the origin

- inverses collapse onto line $\mathrm{Re}\frac{1}{\lambda} = \frac{1}{2}$
- For all eigenvalues!




For the condensate

$$\langle \overline{\psi}\psi \rangle = \sum \mathrm{Re}\frac{1}{\lambda_i} = \sum \frac{1}{2} = \frac{N}{2} \neq 0$$

- $N$ is the dimension of $D$
- Independent of any dynamics!?

Do we have the wrong operator?

- $\overline{\psi}\psi$ nontrivial under generalized chiral symmetry
- is $\langle\overline{\psi}(1-D/2)\psi\rangle$ better?
  - goes to its negative on chiral rotation

The second term is also easy to calculate

$$\langle\overline{\psi}D\psi\rangle = \mathrm{Tr}D^{-1}D = \mathrm{Tr}1 = N$$

Combining:

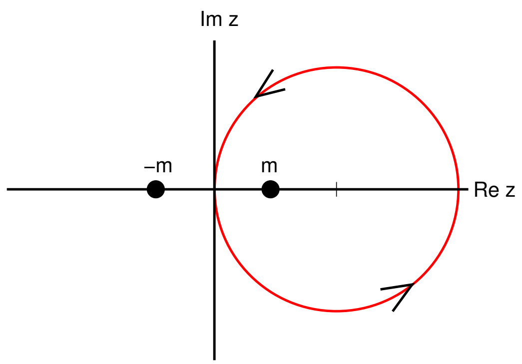
$$\langle\overline{\psi}(1-D/2)\psi\rangle = N/2 - N/2 = 0$$

Oops, the condensate is gone?

Resolution: $V \to \infty$ and $m \to 0$ limits don't commute

- add a small mass
- $\langle \overline{\psi}\psi \rangle = \sum \frac{1}{\lambda + m}$
- look for a jump as $m$ passes through zero

Contour integral around the GW circle $\lambda = 1 + e^{i\theta}$




$$i \int_0^{2\pi} d\theta \frac{\rho(\theta)}{1 + e^{i\theta} + m}$$

- pole at $-m$ moves from inside to outside the circle
- residue $\rho(0) \equiv \lim_{\theta \to 0} \rho(\theta)$
- integral jumps by $2\pi\rho(0)$

This is the Banks-Casher relation for the overlap

# Another Puzzle

Two flavors

- expect spontaneous chiral symmetry breaking
- explains light pions
- should have $\rho(0) \neq 0$

One flavor

- anomaly breaks all chiral symmetry
- $\langle \overline{\psi}\psi \rangle$ behaves smoothly at $m \sim 0$
- should have $\rho(0) = 0$
  - note: zero modes give smooth contribution to $\langle \overline{\psi}\psi \rangle$ (see later)

But

- one flavor has one power of $|D|$ — $Z = \int (dA)\ |D|^1\ e^{-S_G}$
- two flavors have two powers — $Z = \int (dA)\ |D|^2\ e^{-S_G}$
- Two flavors should naively supress small eigenvalues more!

How can two flavors have the bigger $\rho(0)$???

# $\rho$ depends on distribution of $A$ depends on $\rho$

Not just low eigenvalues are relevant

- fermions tend to smooth out gauge fields
- more fermions smooth things more
- involves all scales
- smoother fields give more low eigenvalues
- overcomes suppression from more powers of the determinant

$$\int dA\, |D|^{N_f}\, e^{-S_g(A)}$$

Increasing $N_f$ can increase density of small eigenvalues!

# Zero modes?

Again insert a small mass

$$Z = \int dA\, e^{-S_g} \prod (\lambda_i + m)$$

As $m$ goes to zero any configurations involving a $\lambda = 0$ drop out

- are "instantons" irrelevant in the chiral limit?

No: add sources $\eta, \overline{\eta}$

$$Z(\eta, \overline{\eta}) = \int dA\, d\psi\, d\overline{\psi}\; e^{-S_g + \overline{\psi}(D+m)\psi + \overline{\psi}\eta + \overline{\eta}\psi}$$

- integrate out fermions

$$Z = \int dA\; e^{-S_g + \overline{\eta}(D+m)^{-1}\eta} \prod(\lambda_i + m)$$

If source overlaps with the zero mode eigenvector $(\psi_0, \eta) \neq 0$

- $1/m$ in source term cancels $m$ from determinant
- With multiple flavors
  - need a factor from each flavor: "t'Hooft vertex"

Instantons drop out of $Z$

- but survive in correlation functions
  - small mass extrapolations are numerically difficult

# Masses and topology

One massless flavor

- ’t Hooft vertex quadratic in fermion fields
- generates smooth contribution to $\langle\overline{\psi}\psi\rangle$
- an additive mass shift        “renormalon”
  - non-perturbative
  - depends on scale and regulator

Overlap operator is not unique

- depends on
  - particular input $D$ chosen
  - Wilson mass (domain wall height)

Scheme dependent additive mass shift

- $m = 0$ is not a physical concept for a single flavor
- $m_u = 0$ cannot solve strong $CP$ problem

## $m = 0 \longleftrightarrow$ vanishing topological susceptibility?

Winding number ambiguous when $D_W D_W^\dagger$ not invertible

- occurs with eigenvalues near domain wall height

Admissibility condition

- strong constraint on allowed plaquettes
- disallows rough configurations, making winding unique
- Violates reflection positivity!

Is the topological susceptibility a well defined observable?

- do we care?

# Final remarks

Eigenvalues can give some insight

- Banks-Casher

But can be misleading

- adding flavors enhances low eigenvalues

Unresolved issues

- do we understand non-perturbative ambiguities?
  - is topological susceptibility an observable?
  - are rough gauge fields essential?
- how do these issues interplay with quark masses?
  - is $m_u = 0$ a definable concept?