# The early stages of Heavy Ion Collisions in the Color Glass Condensate framework

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# Outline

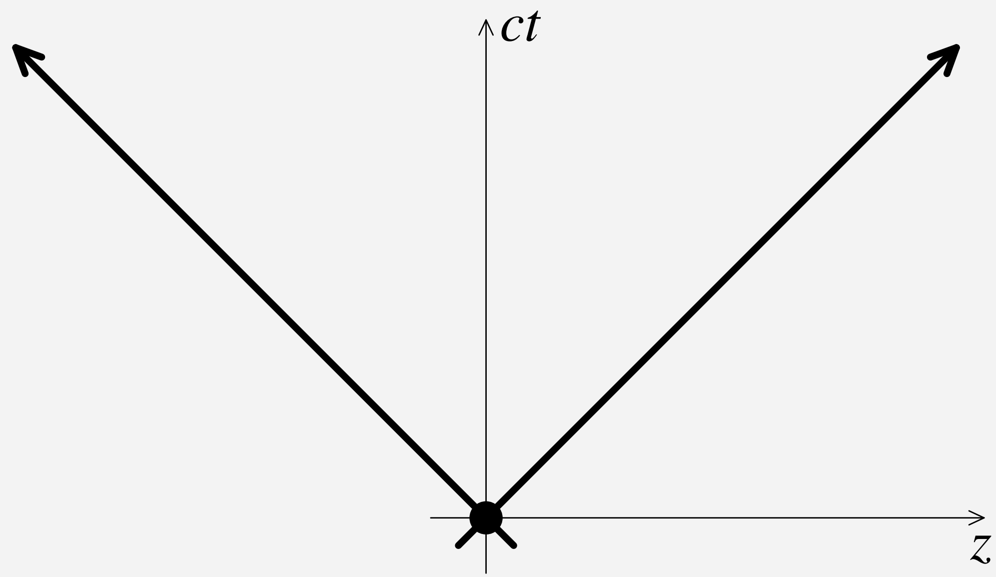
- Heavy ion collisions
- Overview of the Color Glass Condensate
- Calculation of particle multiplicities
- Gluon production at leading order
- Quark production
- Instabilities ?
- Conclusions and perspectives

**Based on :**

Krasnitz, Venugopalan, hep-ph/9809433, hep-ph/0007108
Baltz, FG, McLerran, Peshier, nucl-th/0101024
Lappi, hep-ph/0303076
FG, Kajantie, Lappi, hep-ph/0409048, hep-ph/0508229
Romatschke, Venugopalan, hep-ph/0510121

# Heavy ion collisions

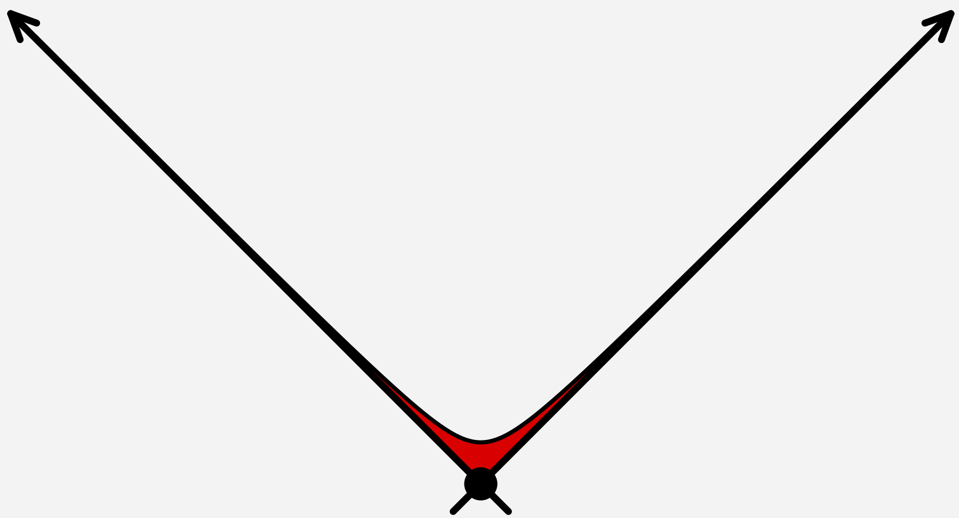





- $\tau \sim 0$ fm/c
- Production of hard particles
- calculable with the tools of perturbative QCD

# Heavy ion collisions

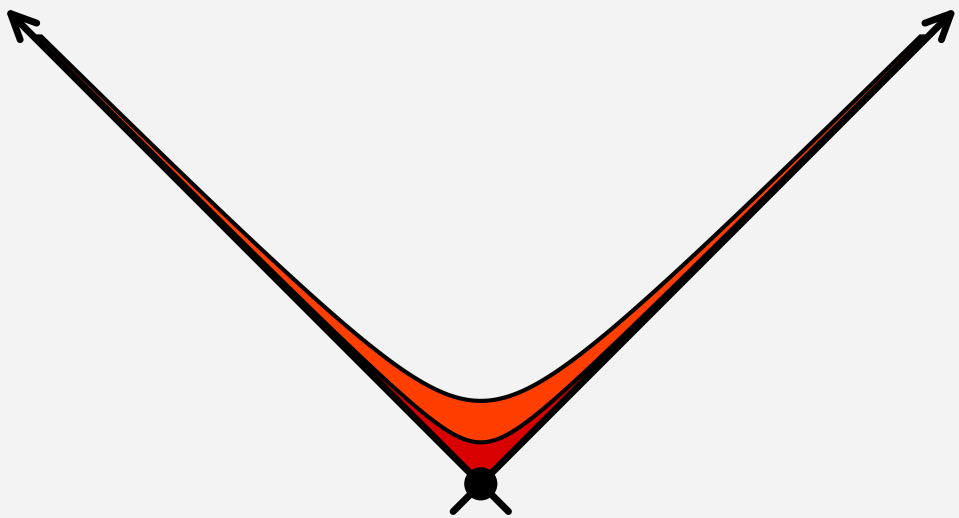




- $\tau \sim 0.2$ fm/c
- Production of semi-hard particles
- relatively small momentum : $p_\perp \lesssim 1$–$2$ GeV
- make up for most of the multiplicity
- sensitive to the physics of saturation (CGC)

# Heavy ion collisions

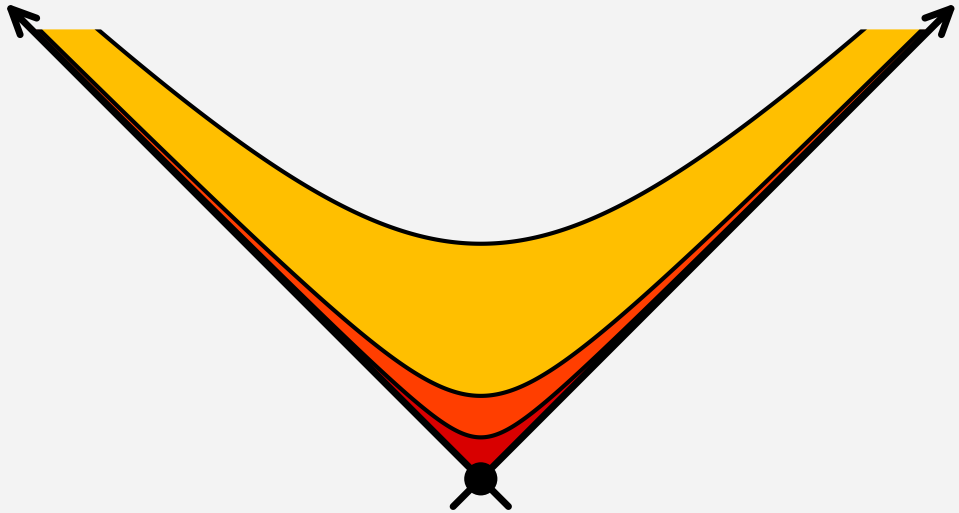




- Thermalization
  - experiments tend to point towards a fast thermalization (?)
  - but this is still not fully understood from QCD

# Heavy ion collisions

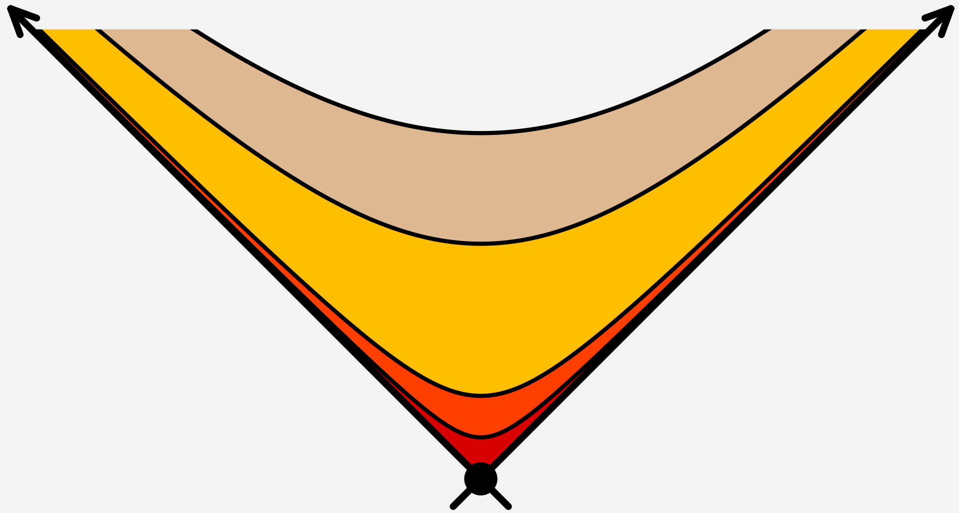




- Quark gluon plasma

# Heavy ion collisions

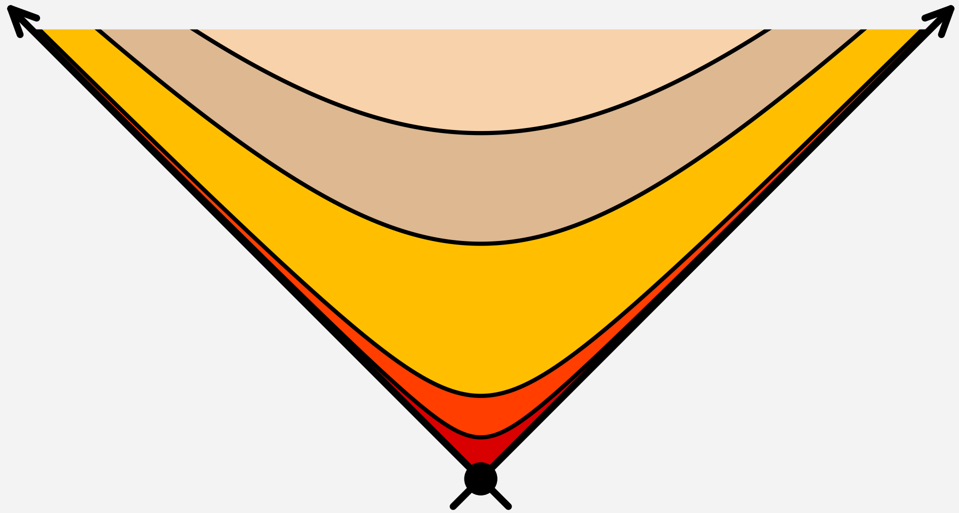




- Hot hadron gas

# Heavy ion collisions

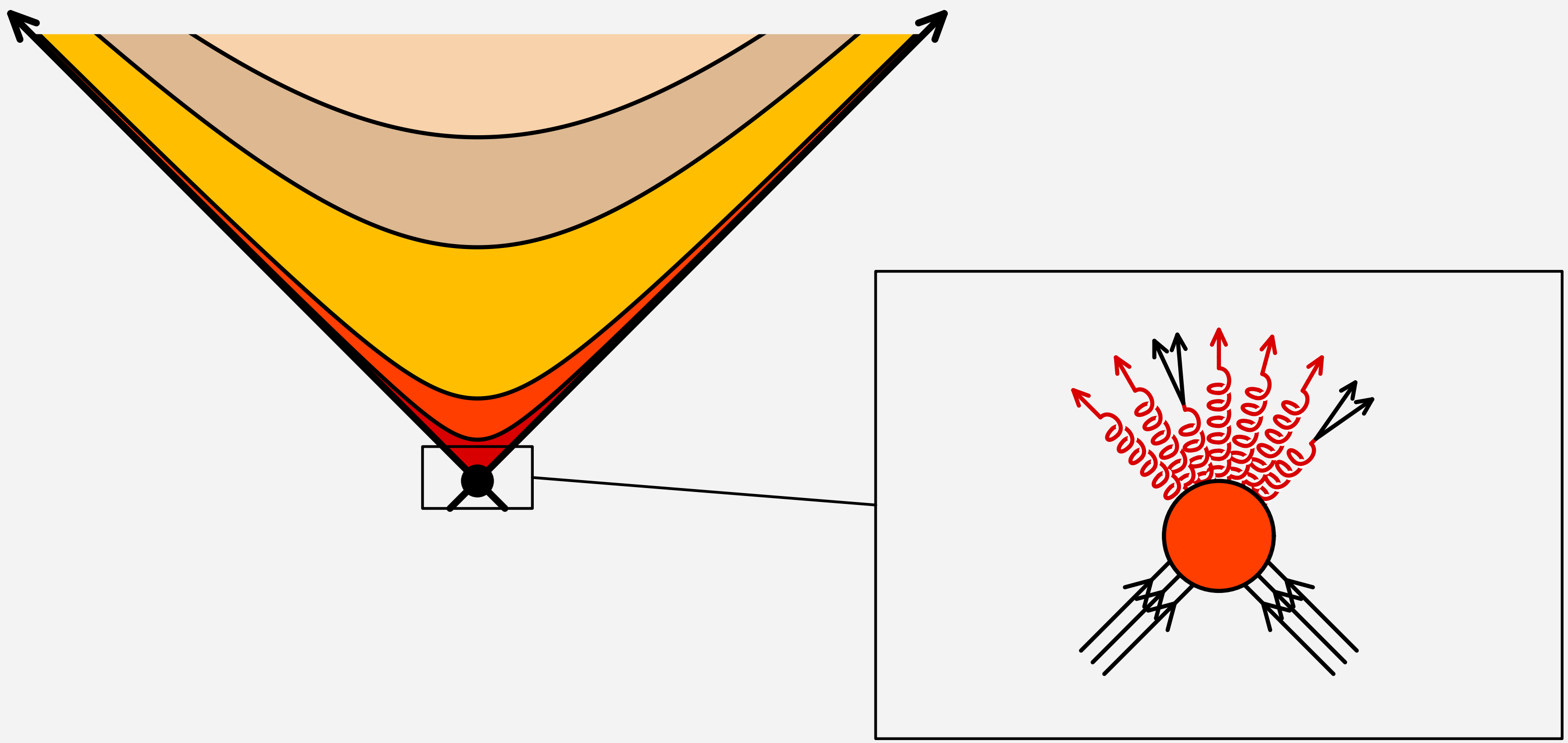




- Chemical freeze-out :
  density too small to have inelastic interactions
- Kinetic freeze-out :
  density too small to have elastic interactions

# What is the CGC good for?

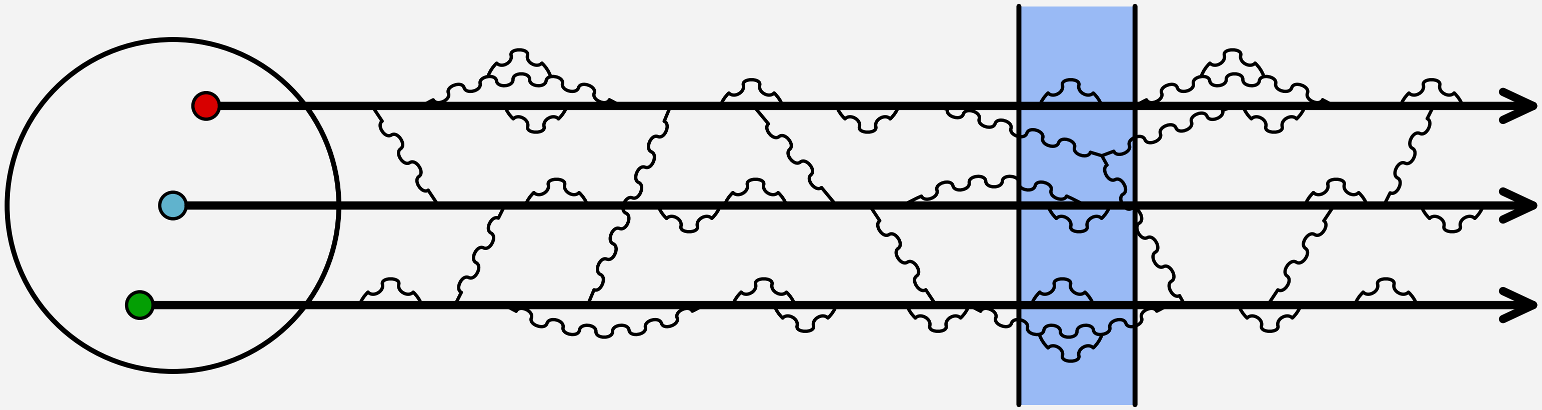




- describes the content of nucleons and nuclei at small $x$
- framework to calculate the production of semi-hard particles
- provides initial conditions for the subsequent evolution

# Nucleon at rest

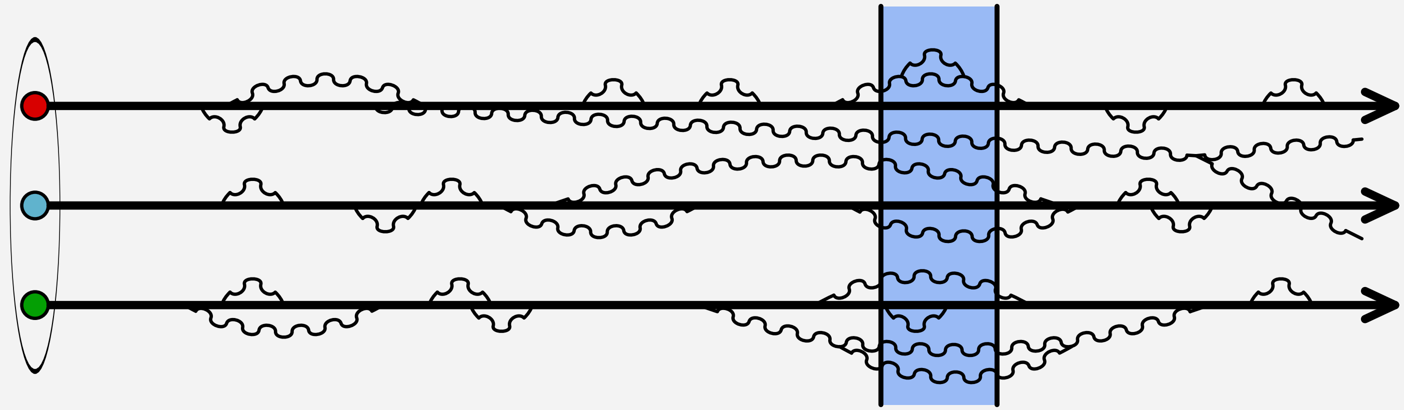




- Very complicated non-perturbative object...
- Contains lots of fluctuations at all space-time scales smaller than $\Lambda_{QCD}^{-1}$
- Only the fluctuations that live longer than the external probe are relevant in the interaction process
- All the effect of the shorter-lived fluctuations is to renormalize the coupling and masses
- Interaction processes are very complicated if the nucleon constituents have non-trivial interactions over the characteristic timescale seen by the probe

# Nucleon at high energy

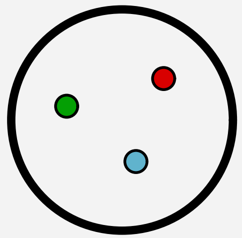




- Time dilation of all the internal timescales of the nucleon
- The interactions among the constituents now occur over timescales much larger than the interaction with the external probe ▷ the constituents behave as if they were free
- Some of the fluctuations now live long enough to be seen. The nucleon appears denser at high energy. The new partons have a smaller momentum fraction $x$
- Previous fluctuations are now frozen over the timescale of the probe, and merely act as static sources of new partons

# Evolution and saturation

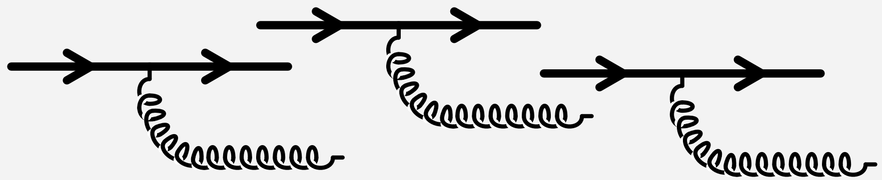


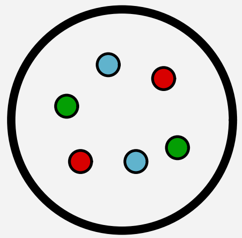


▷ at low energy, only valence quarks are present in the hadron wave function

# Evolution and saturation







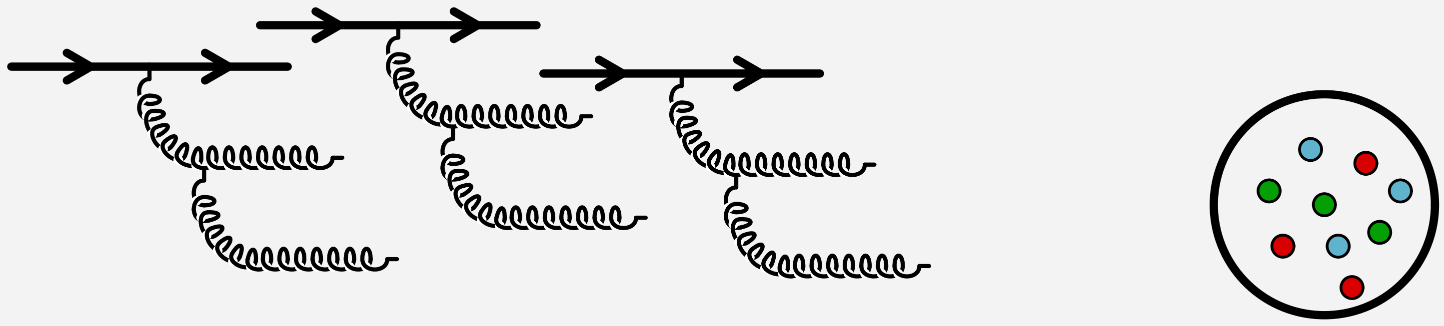
▷ when energy increases, new partons are emitted

▷ the emission probability is $\alpha_s \int \frac{dx}{x} \sim \alpha_s \ln(\frac{1}{x})$, with $x$ the longitudinal momentum fraction of the gluon

▷ at small-$x$ (i.e. high energy), these logs need to be resummed

# Evolution and saturation

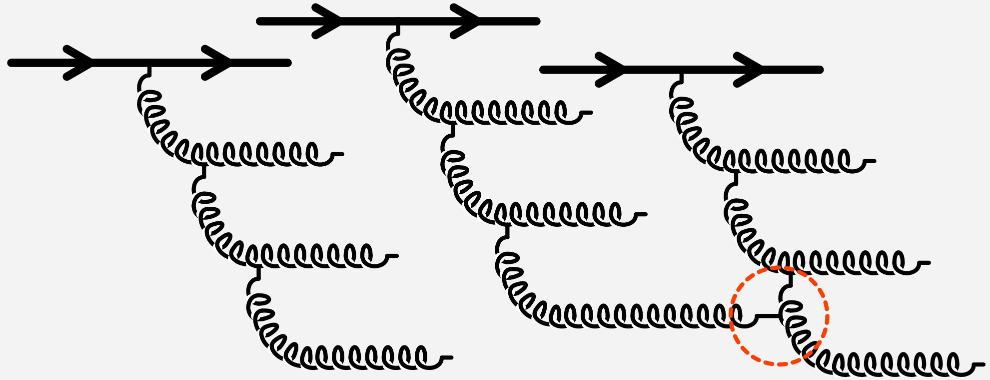


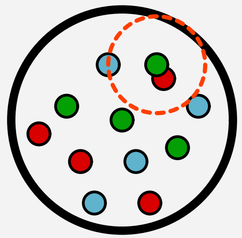


▷ as long as the density of constituents remains small, the evolution is linear: the number of partons produced at a given step is proportional to the number of partons at the previous step (BFKL) Kuraev, Lipatov, Fadin (1977), Balitsky, Lipatov (1978)

# Evolution and saturation







▷ eventually, the partons start overlapping in phase-space
▷ parton recombination becomes favorable
▷ after this point, the evolution is non-linear:

the number of partons created at a given step depends non-linearly on the number of partons present previously

Balitsky (1996), Kovchegov (1996,2000)
Jalilian-Marian, Kovner, Leonidov, Weigert (1997,1999)
Iancu, Leonidov, McLerran (2001)

# Saturation criterion



Gribov, Levin, Ryskin (1983), Mueller, Qiu (1986)

- Number of partons per unit area:

$$\rho \sim \frac{xG(x,Q^2)}{\pi R^2}$$

- Recombination cross-section:

$$\sigma_{gg\to g} \sim \frac{\alpha_s}{Q^2}$$

- Recombination if $\rho\sigma_{gg\to g} \gtrsim 1$, or $Q^2 \lesssim Q_s^2$, with:

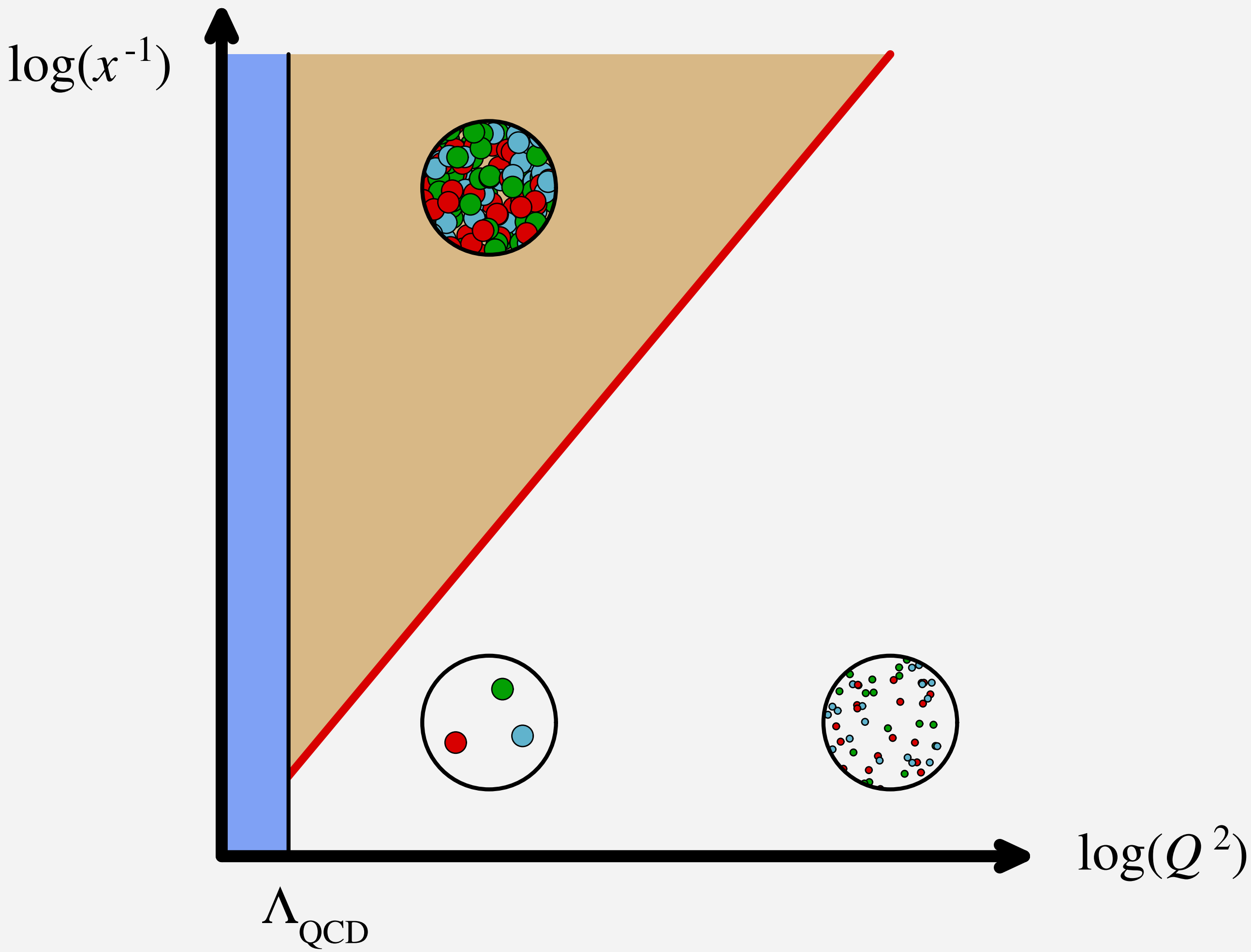
$$Q_s^2 \sim \frac{\alpha_s xG(x,Q_s^2)}{\pi R^2} \sim A^{1/3}\frac{1}{x^{0.3}}$$

- At saturation, the gluon phase-space density is:

$$\frac{dN_g}{d^2\vec{\boldsymbol{x}}_\perp d^2\vec{\boldsymbol{p}}_\perp} \sim \frac{\rho}{Q^2} \sim \frac{1}{\alpha_s}$$

# Saturation domain






- Boundary defined by $Q^2 = Q_s^2(x)$

# Degrees of freedom and their interplay



McLerran, Venugopalan (1994)
Iancu, Leonidov, McLerran (2001)

- Small-$x$ modes have a large occupation number
  - ▷ they are described by a classical color field $A^\mu$
- The large-$x$ modes, slowed down by time dilation, are described as frozen color sources $\rho_a$. They act as sources for the modes at lower values of $x$
- The classical field obeys Yang-Mills's equation:

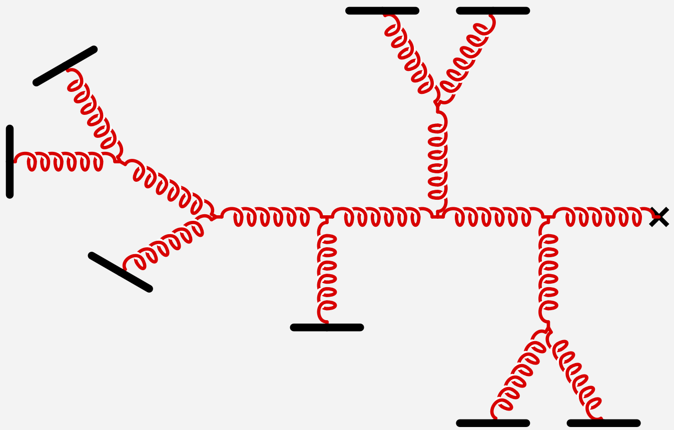
$$[D_\nu, F^{\nu\mu}]_a = \delta^{\mu +}\delta(x^-)\rho_a(\vec{\boldsymbol{x}}_\perp)$$

- The color sources $\rho_a$ are random, and described by a distribution functional $W_{x_0}[\rho]$, with $x_0$ the separation between "small-$x$" and "large-$x$".

# Yang-Mills equation and saturation



- The solution of the classical Yang-Mills equation is the sum of all the tree diagrams that connect the point where the gauge field is evaluated to an arbitrary number of sources :



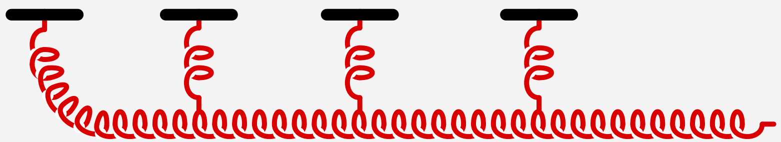
▷ this solution incorporates all the diagrams responsible for saturation

# Evolution with x



Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner

- The distribution $W_{x_0}[\rho]$ evolves with $x_0$ (more modes are included in $W$ as $x_0$ decreases)
- In a high density environment, the newly created gluons can interact with all the sources already present:



- The evolution is governed by the JIMWLK equation:

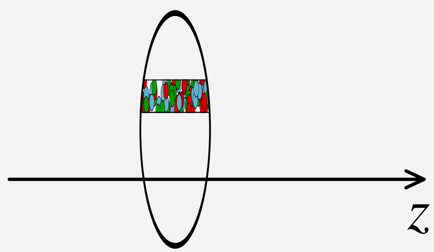
$$\frac{\partial W_{x_0}[\rho]}{\partial \ln(1/x_0)} = \frac{1}{2}\int_{\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp} \frac{\delta}{\delta\rho_a(\vec{\boldsymbol{x}}_\perp)} \chi_{ab}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp) \frac{\delta}{\delta\rho_b(\vec{\boldsymbol{y}}_\perp)} W_{x_0}[\rho]$$

  - ▷ $\chi_{ab}$ depends on $\rho$ to all orders
  - ▷ reduces to BFKL in the low density regime
  - ▷ diffusion equation in a functional space

# Initial condition - MV model



- The JIMWLK equation must be completed by an initial condition, given at some moderate $x_0$
- As with DGLAP, the problem of finding the initial condition is in general non-perturbative
- The McLerran-Venugopalan model is often used as an initial condition at moderate $x_0$ for a large nucleus :




  - partons distributed randomly
  - many partons in a small tube
  - no correlations at different $\vec{x}_\perp$

- The MV model assumes that the density of color charges $\rho(\vec{x}_\perp)$ has a Gaussian distribution :

$$W_{x_0}[\rho] = \exp\left[-\int d^2\vec{x}_\perp \frac{\rho_a(\vec{x}_\perp)\rho_a(\vec{x}_\perp)}{2\mu^2(\vec{x}_\perp)}\right]$$

# Calculation of observables



- Required steps:
  - Solve the classical Yang-Mills equations for arbitrary sources $\rho_{1,2}$. For the collision of two nuclei at high energy, the current in the YM equations reads

$$J^\mu = \delta^{\mu +}\delta(x^-)\rho_1(\vec{\boldsymbol{x}}_\perp) + \delta^{\mu -}\delta(x^+)\rho_2(\vec{\boldsymbol{x}}_\perp)$$

    ▷ so far, analytical solutions are known only if the source of one of the projectiles is treated at lowest order
    ▷ the full solution (all orders in the two sources) has been determined numerically
  - Calculate the relevant matrix element $\mathcal{M}$ with the previously obtained gauge field in the background
    Note : the background field is now time-dependent, and transitions from the vacuum to populated states are non zero
  - Perform the average over the sources of each projectile, with the weights $W_{x_1}[\rho_1]$ and $W_{x_2}[\rho_2]$

# Particle multiplicity



- Particle multiplicities are more relevant in situations where the external sources are large, and produce a copious amount of particles. One method would be to calculate all the probabilities $P_n$ for producing $n$ particles, and to get $\overline{n}$ as $\overline{n} = \sum_{n=1}^{+\infty} n P_n$
- The transition probabilities are hard to calculate in the presence of strong time dependent sources (vacuum-vacuum diagrams must be included, because their sum is not a pure phase)
- Instead, one can bypass these difficulties by using the following equivalent formula :

$$E_p \frac{d\overline{n}}{d^3\vec{\boldsymbol{p}}} = \frac{1}{16\pi^3} \left\langle 0_{\rm in} \middle| a^\dagger_{\rm out}(\vec{\boldsymbol{p}}) a_{\rm out}(\vec{\boldsymbol{p}}) \middle| 0_{\rm in} \right\rangle$$

- For simplicity, consider here a scalar theory coupled to a strong external source (such that $gj(x) \sim \mathcal{O}(1)$) :

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{g}{3!}\phi^3 + j\phi$$

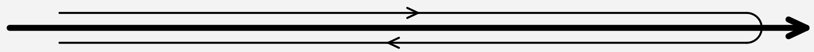
# Reduction formula



- The reduction formula for this correlator is :

$$\langle 0_{\rm in} | a^\dagger_{\rm out}(\vec{\boldsymbol p}) a_{\rm out}(\vec{\boldsymbol p}) | 0_{\rm in}\rangle = \left(\frac{i}{\sqrt{Z}}\right)^2 \int d^4x d^4y \; e^{-ip\cdot(x-y)}$$
$$\times (\square_x + m^2)(\square_y + m^2) \langle 0_{\rm in} | \phi(x)\phi(y) | 0_{\rm in}\rangle$$

- This can be calculated by the Schwinger-Keldysh method, with a time contour $\mathcal{C}$ that wraps around the real axis :



- This also solves the problem of the lack of time ordering between the two fields. Assign $\phi(x)$ to the lower branch of $\mathcal{C}$ and $\phi(y)$ to the upper branch: the two fields are now "path ordered". In other words :

$$\langle 0_{\rm in} | \phi(x)\phi(y) | 0_{\rm in}\rangle =$$
$$= \langle 0_{\rm in} | \mathcal{P} \phi^{(-)}_{\rm in}(x) \phi^{(+)}_{\rm in}(y) e^{i\int_{\mathcal{C}} d^4x \mathcal{L}_{\rm int}(\phi_{\rm in})} | 0_{\rm in}\rangle$$

# Perturbative expansion



- Feynman rules in momentum space :
  - Assign a $+$ or $-$ index to each vertex, and sum over this index
  - $+$ vertices come with a $-ig$, and $-$ vertices have a $+ig$
  - Connect a vertex $i$ to a vertex $j$ by a propagator $G^0_{ij}(p)$
  - Free propagators :

$$G^0_{++}(p) = \frac{i}{p^2 - m^2 + i\epsilon} \qquad G^0_{--}(p) = \frac{-i}{p^2 - m^2 - i\epsilon}$$

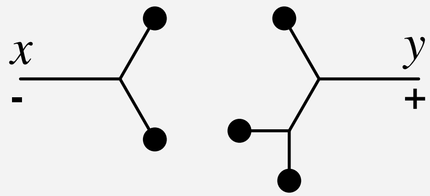
$$G^0_{-+}(p) = 2\pi\theta(p^0)\delta(p^2 - m^2) \qquad G^0_{+-}(p) = 2\pi\theta(-p^0)\delta(p^2 - m^2)$$

- Note : the vacuum-vacuum diagrams are all zero
- The particle spectrum is obtained from the propagator $G_{-+}$, via the formula :

$$E_p \frac{d\overline{n}}{d^3\vec{\boldsymbol{p}}} = \frac{1}{16\pi^3} \int d^4x d^4y e^{-ip\cdot(x-y)} (\square_x + m^2)(\square_y + m^2) G_{-+}(x,y)$$

# Classical approximation

- The classical approximation is the sum of the leading terms in $g$, at fixed $gj(x)$. Diagrammatically, this corresponds to a product of two disconnected tree diagrams :




- Thus, it sums all the terms of order $g^{-2}(gj)^n$
- The sum over the $\pm$ indices attached to the vertices in each of the tree diagrams can be performed by noting that :

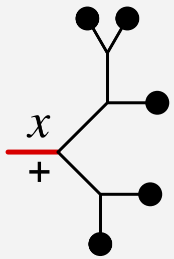
$$\text{For } i=+,- \quad , \qquad G^0_{i+}(x,y) - G^0_{i-}(x,y) = G^0_R(x,y)$$

- Using this property recursively, one can prove that the sum of these tree diagrams is nothing but the retarded solution of the classical equation of motion, with the boundary condition $\phi(x^0 = -\infty) = 0$

# Classical approximation



- Consider a generic tree diagram whose root is at the point $x$, with an index $+$



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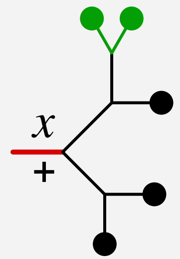


- Start with the leaves of the tree that are the farthest away from the root, and sum over the index $i = \pm$ of each leaf. The result is a set of factors $\int d^4y G_R^0(\cdots, y) j(y)$, which are independent of the indices carried by the vertices on the layer immediately below

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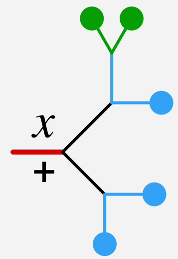


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- Sum over the indices of the vertices at the next layer. Again, we obtain retarded propagators. Proceed until the end

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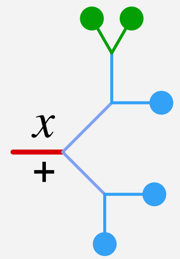


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- Sum over the indices of the vertices at the next layer. Again, we obtain retarded propagators. Proceed until the end
- Finally, one gets $\overline{n}$ in terms of the retarded solution $\phi_c$ of the EOM :

$$E_p \left. \frac{d\overline{n}}{d^3\vec{\boldsymbol{p}}} \right|_{\text{classical}} = \frac{1}{16\pi^3} \left| (p^2 - m^2)\phi_c(p) \right|^2$$

# Classical approximation



- $(p^2 - m^2)\phi_c(p)$ is given by a 4-dim Fourier transform :

$$(p^2 - m^2)\phi_c(p) = -\int d^4x \; e^{ip\cdot x} \; (\Box_x + m^2)\phi_c(x)$$

- This formula is cumbersome in practice because it requires to store the solution of the EOM at all times
- Instead, notice that :

$$\phi_c(x) = \int d^4z \; G_R^0(x,z) \, (\Box_z + m^2)\phi_c(z) \; ,$$

from which one can obtain :

$$(p^2 - m^2)\phi_c(p) = \lim_{x^0 \to +\infty} \int d^3\vec{\boldsymbol{x}} \; e^{ip\cdot x} \; [\partial_{x_0} - iE_p] \, \phi_c(x)$$

- With the latter formula, one needs only a spatial Fourier transform of the classical field at late times. It is only necessary to keep the classical field at the current time when solving the EOM

# Gluon production

- At the classical level, the gluon spectrum is given directly by the retarded solution of Yang-Mills equations:

$$E_p \frac{d\overline{n}_{\rm g}}{d^3\vec{\boldsymbol{p}}} = \frac{1}{16\pi^3} \sum_\lambda \left| \lim_{x^0 \to +\infty} \int d^3\vec{\boldsymbol{x}} \; e^{ip\cdot x} \; [\partial_{x_0} - iE_p] \, \epsilon_\mu^{(\lambda)}(\vec{\boldsymbol{p}}) \, A^\mu(x) \right|^2$$

- The calculation is usually done in the gauge :

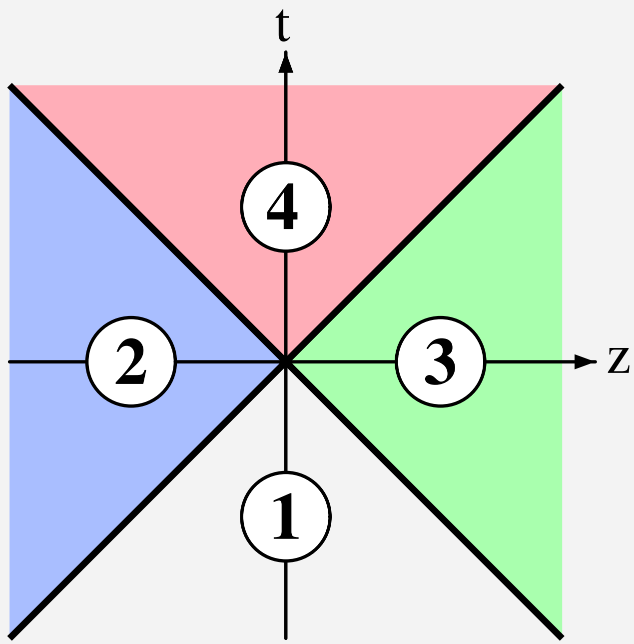
$$A^\tau = x^+ A^- + x^- A^+ = 0$$

  - This gauge interpolates between two light-cone gauges : $A^- = 0$ on the trajectory $z = t$ and $A^+ = 0$ on the trajectory $z = -t$
  - This implies that the produced gauge field does not make the currents $J^+, J^-$ precess
- In this gauge, it is easy to find the field at $\tau = 0^+$, and then let it evolve according to the vacuum Yang-Mills equations (because the currents are zero at $\tau > 0$)

# Classical color field



- Space-time structure of the classical color field:

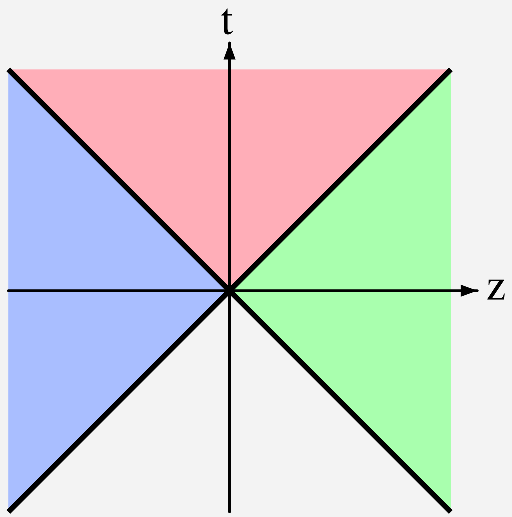



  - Region 1 : no causal relation to either nuclei
  - Region 2 : causal relation to the 1st nucleus only
  - Region 3 : causal relation to the 2nd nucleus only
  - Region 4 : causal relation to both nuclei

# Classical color field



- Propagation through region 1:



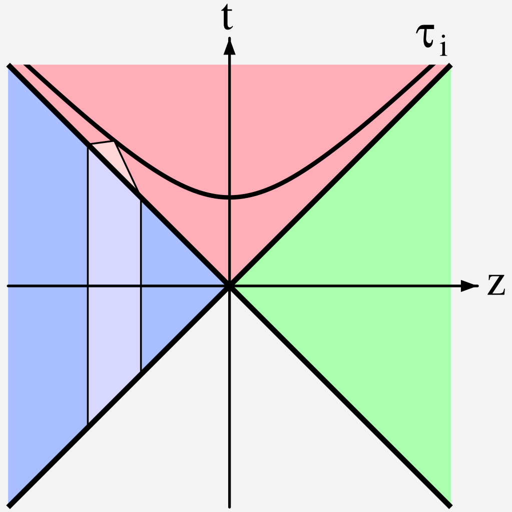

▷ trivial : the classical field is entirely determined by the initial condition, i.e.

$$A^\mu = 0$$

# Classical color field



- Propagation through region 2:




▷ the Yang-Mills equation can be solved analytically when there is only one nucleus :

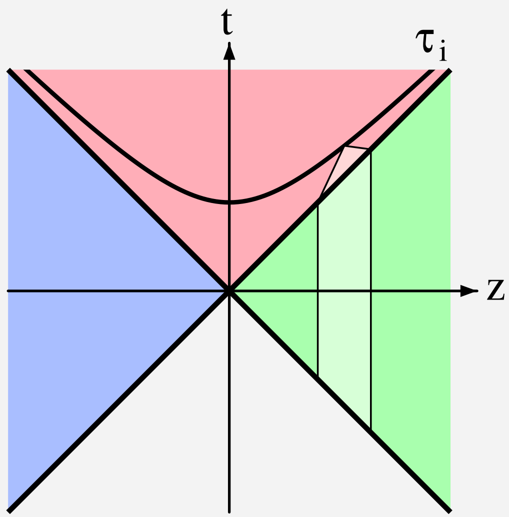
$$A^+ = A^- = 0 \quad , \quad A^i = \theta(x^-)\,\frac{i}{g} U_1(\vec{\boldsymbol{x}}_\perp)\partial^i U_1^\dagger(\vec{\boldsymbol{x}}_\perp)$$

with $\quad U_1(\vec{\boldsymbol{x}}_\perp) = T_+ \exp\; ig \int dx^+ T^a \frac{1}{\boldsymbol{\nabla}_\perp^2} \rho_1^a(x^+, \vec{\boldsymbol{x}}_\perp)$

# Classical color field



- Propagation through region 3:




▷ the Yang-Mills equation can be solved analytically when there is only one nucleus :

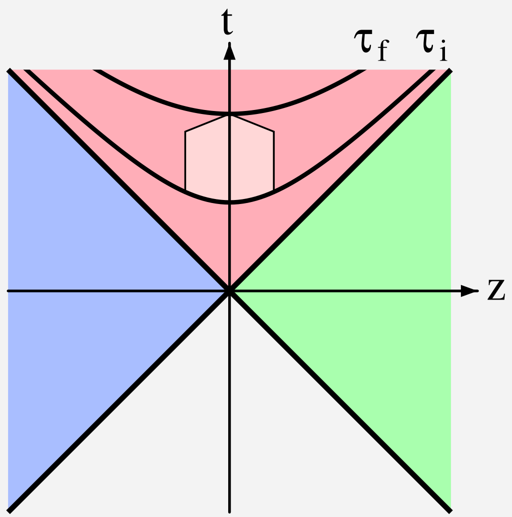
$$A^+ = A^- = 0 \quad , \quad A^i = \theta(x^+)\,\frac{i}{g} U_2(\vec{\boldsymbol{x}}_\perp)\partial^i U_2^\dagger(\vec{\boldsymbol{x}}_\perp)$$

with $$U_2(\vec{\boldsymbol{x}}_\perp) = T_-\ \exp\ ig \int dx^- T^a \frac{1}{\boldsymbol{\nabla}_\perp^2}\rho_2^a(x^-, \vec{\boldsymbol{x}}_\perp)$$

# Classical color field



- Propagation through region 4:




▷ one must solve numerically the Yang-Mills equations with the following initial condition at $\tau_i = 0^+$ :

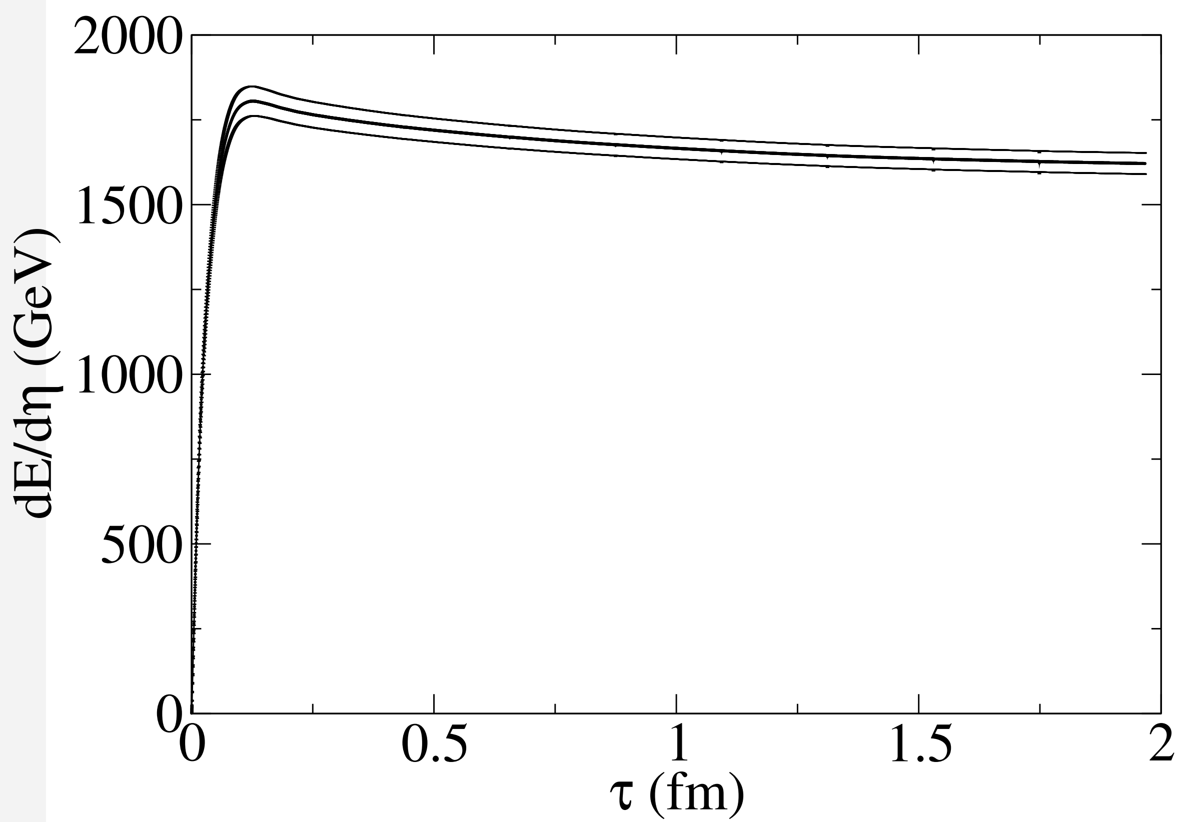
$$A^i(\tau = 0, \vec{x}_\perp) = \frac{i}{g}\Big(U_1(\vec{x}_\perp)\partial^i U_1^\dagger(\vec{x}_\perp) + U_2(\vec{x}_\perp)\partial^i U_2^\dagger(\vec{x}_\perp)\Big)$$

$$A^\eta(\tau = 0, \vec{x}_\perp) = -\frac{i}{2g}\Big[U_1(\vec{x}_\perp)\partial^i U_1^\dagger(\vec{x}_\perp)\ ,\ U_2(\vec{x}_\perp)\partial^i U_2^\dagger(\vec{x}_\perp)\Big]$$

# Energy per unit rapidity



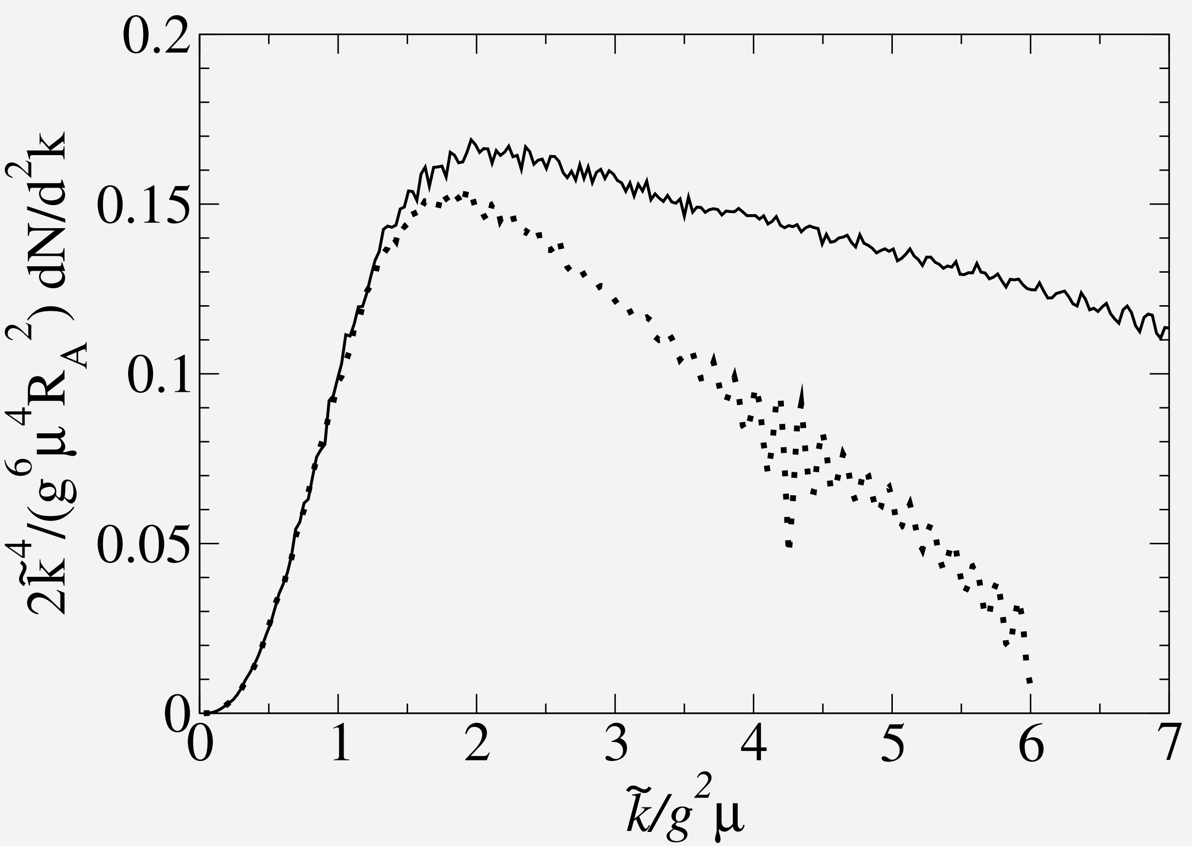
- Time dependence of $dE/d\eta$:



# Gluon spectrum



- Gluon spectra for $512^2$ and $256^2$ lattices:




  - Lattice artifacts at large momentum (does not affect much the overall number of gluons)
  - Important softening at small $k_\perp$ compared to pQCD

# Anisotropy



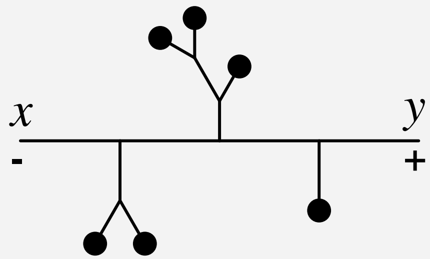
- The boost invariance of the sources implies that the distribution of the produced gluons depends only on the difference $\eta - y$
- This by itself does not say anything about the local isotropy of the particle distribution. Indeed, it is also satisfied by Bjorken hydrodynamics...
- What is specific to the MV model is that, at leading order, this function of $\eta - y$ is close to a delta function :

$$\frac{d\overline{n}_{\rm g}}{d^3\vec{\boldsymbol{x}}d^3\vec{\boldsymbol{p}}} \sim \delta(\eta - y)$$

▷ the gluon distribution is extremely anisotropic

# Production of pairs

- The production of pairs arises at the order $(gj)^n$. It is obtained by summing all the simply connected tree diagrams :



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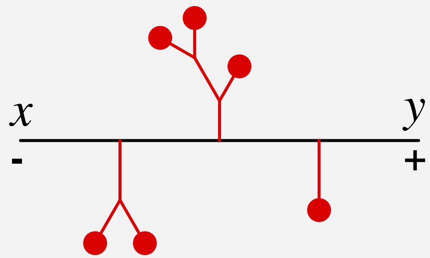
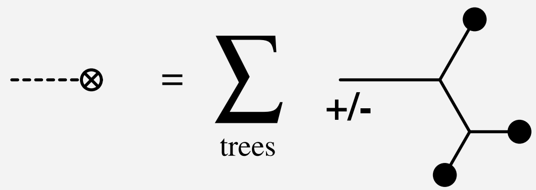



- One can perform a partial resummation of all the sub-diagrams that correspond to the classical solution :

$\phi_c =$



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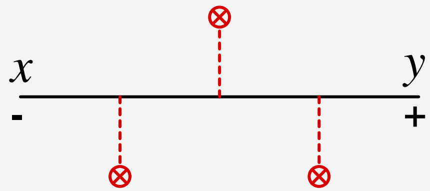
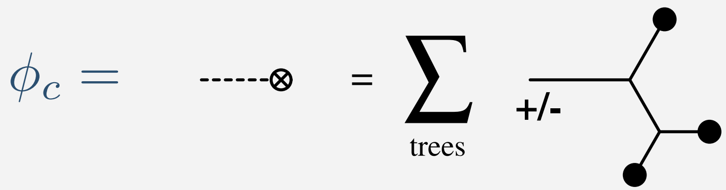



- One can perform a partial resummation of all the sub-diagrams that correspond to the classical solution :




- Thus, we need the tree level propagator $G_{-+}(x,y)$ with the retarded field $\phi_c$ in the background. (The classical field insertion is the same for the $+$ and $-$ indices)

# Production of pairs



- The summation is done by a Lippmann-Schwinger equation :

$$G_{ij}(x,y) = G^0_{ij}(x,y) - ig \sum_{k=\pm} \int d^4z\, G^0_{ik}(x,z)\phi_c(z)(-1)^k G_{kj}(z,y)$$

- After some work, one gets :

$$E_p \left.\frac{d\overline{n}}{d^3\vec{\boldsymbol{p}}}\right|_{\text{pairs}} = \frac{1}{16\pi^3}\int \frac{d^3\vec{\boldsymbol{q}}}{(2\pi)^3 2E_q} \left|T_{_R}(p,-q)\right|^2$$

where $G_{_R} \equiv G^0_{_R} + G^0_{_R} * T_{_R} * G^0_{_R}$ $\triangleright$ One must get the retarded propagator in the classical field $\phi_c$, amputate the external legs, square and integrate over the (on-shell) momentum at one end

- After more work, one obtains :

$$T_{_R}(p,-q) = \lim_{x^0\to+\infty}\int d^3\vec{\boldsymbol{x}}\; e^{ip\cdot x}\left[\partial_{x_0} - iE_p\right]\eta(x)$$

with $\left(\square + m^2 + g\phi_c(x)\right)\eta(x) = 0$ and $\eta(x) = e^{iq\cdot x}$ when $x^0 \to -\infty$

# Quark production



FG, Kajantie, Lappi (2004, 2005)

- The inclusive quark spectrum can be obtained from the retarded propagator of the quark in the classical color field:

$$E_p \frac{d\overline{n}_{\rm q}}{d^3\vec{\boldsymbol{p}}} = \frac{1}{16\pi^3} \int \frac{d^3\vec{\boldsymbol{q}}}{(2\pi)^3 2E_q} \left|\overline{u}(\vec{\boldsymbol{p}}) T_R(p,-q) v(\vec{\boldsymbol{q}})\right|^2$$

- Alternate representation of the retarded amplitude:

$$\overline{u}(\vec{\boldsymbol{p}}) T_R(p,-q) v(\vec{\boldsymbol{q}}) = \lim_{x^0 \to +\infty} \int d^3\vec{\boldsymbol{x}}\; e^{ip\cdot x}\, u^\dagger(\vec{\boldsymbol{p}}) \psi_{\boldsymbol{q}}(x)$$

$$(i\partial\!\!\!/_x - gA\!\!\!/(x) - m)\psi_{\boldsymbol{q}}(x) = 0\;,\;\; \psi_{\boldsymbol{q}}(x^0, \vec{\boldsymbol{x}}) \underset{x^0 \to -\infty}{\longrightarrow} v(\vec{\boldsymbol{q}}) e^{iq\cdot x}$$

- On a surface of constant proper time:

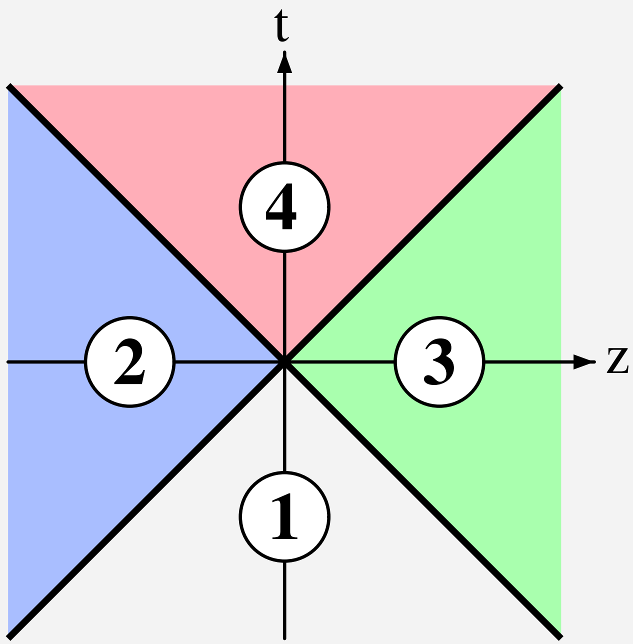
$$\overline{u}(\vec{\boldsymbol{p}}) T_R(p,-q) v(\vec{\boldsymbol{q}}) = \lim_{\tau \to +\infty} \tau \int d\eta d^2\vec{\boldsymbol{x}}_\perp\; e^{ip\cdot x}\, u^\dagger(\vec{\boldsymbol{p}})\; e^{-\eta\gamma^0\gamma^3} \psi_{\boldsymbol{q}}(x)$$

$$t = \tau \cosh(\eta)\;,\;\; z = \tau \sinh(\eta)$$

# Background field



- Space-time structure of the classical color field:

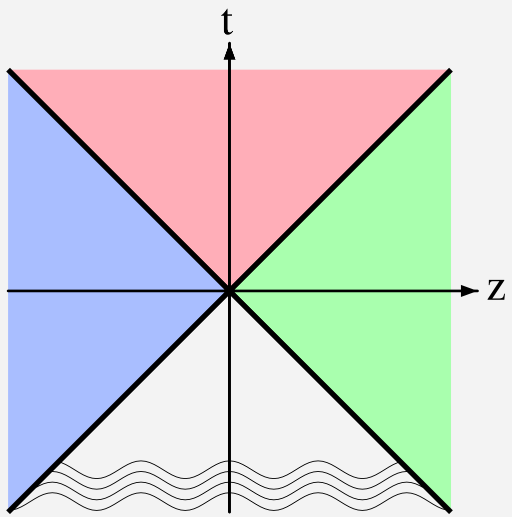



- Region 1: $A^\mu = 0$
- Region 2: $A^\pm = 0$, $A^i = \frac{i}{g} U_1 \nabla_\perp^i U_1^\dagger$
- Region 3: $A^\pm = 0$, $A^i = \frac{i}{g} U_2 \nabla_\perp^i U_2^\dagger$
- Region 4: $A^\mu \neq 0$

- **Notes:**
  - In the region 4, $A^\mu$ is known only numerically
  - We will have to solve the Dirac equation numerically as well

# Quark propagation

- Propagation through region 1:



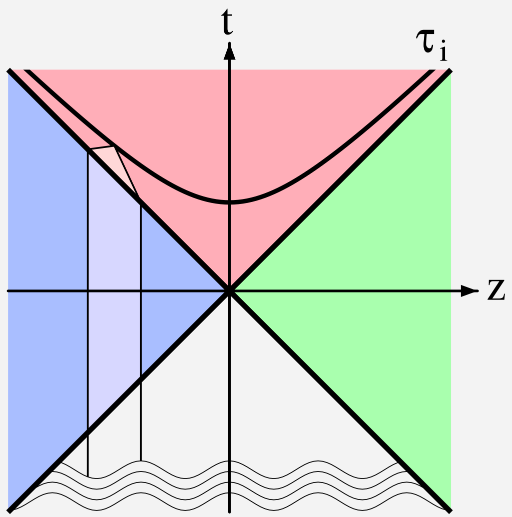

▷ trivial because there is no background field

$$\psi_{\boldsymbol{q}}(x) = v(\vec{\boldsymbol{q}})e^{iq\cdot x}$$

# Quark propagation



- Propagation through region 2:



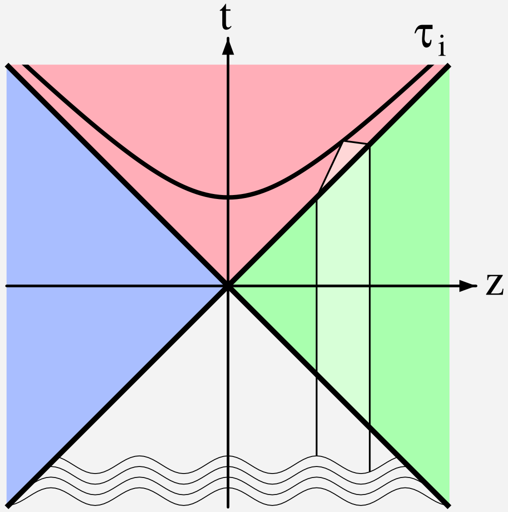

▷ Pure gauge background field

▷ $\psi_{\boldsymbol{q}}^{-}(\tau_i)$ can be obtained analytically

# Quark propagation



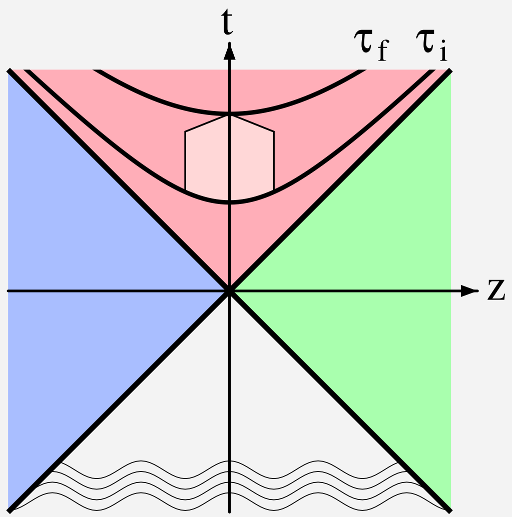
- Propagation through region 3:




  - ▷ Pure gauge background field
    - ▷ $\psi^+_{\boldsymbol{q}}(\tau_i)$ can be obtained analytically

# Quark propagation

- Propagation through region 4:




▷ One must solve the Dirac equation :

$$\left[i\partial\!\!\!/ - gA\!\!\!/ - m\right] \psi_{\boldsymbol{q}}(\tau, \eta, \vec{\boldsymbol{x}}_\perp) = 0$$

▷ initial condition: $\psi_{\boldsymbol{q}}(\tau_i) = \psi^+_{\boldsymbol{q}}(\tau_i) + \psi^-_{\boldsymbol{q}}(\tau_i)$
($\tau_i = 0^+$ in practice)

# Main difficulties



- The Boost invariance (i.e. the $\eta$-independence of the background color field) only implies that :

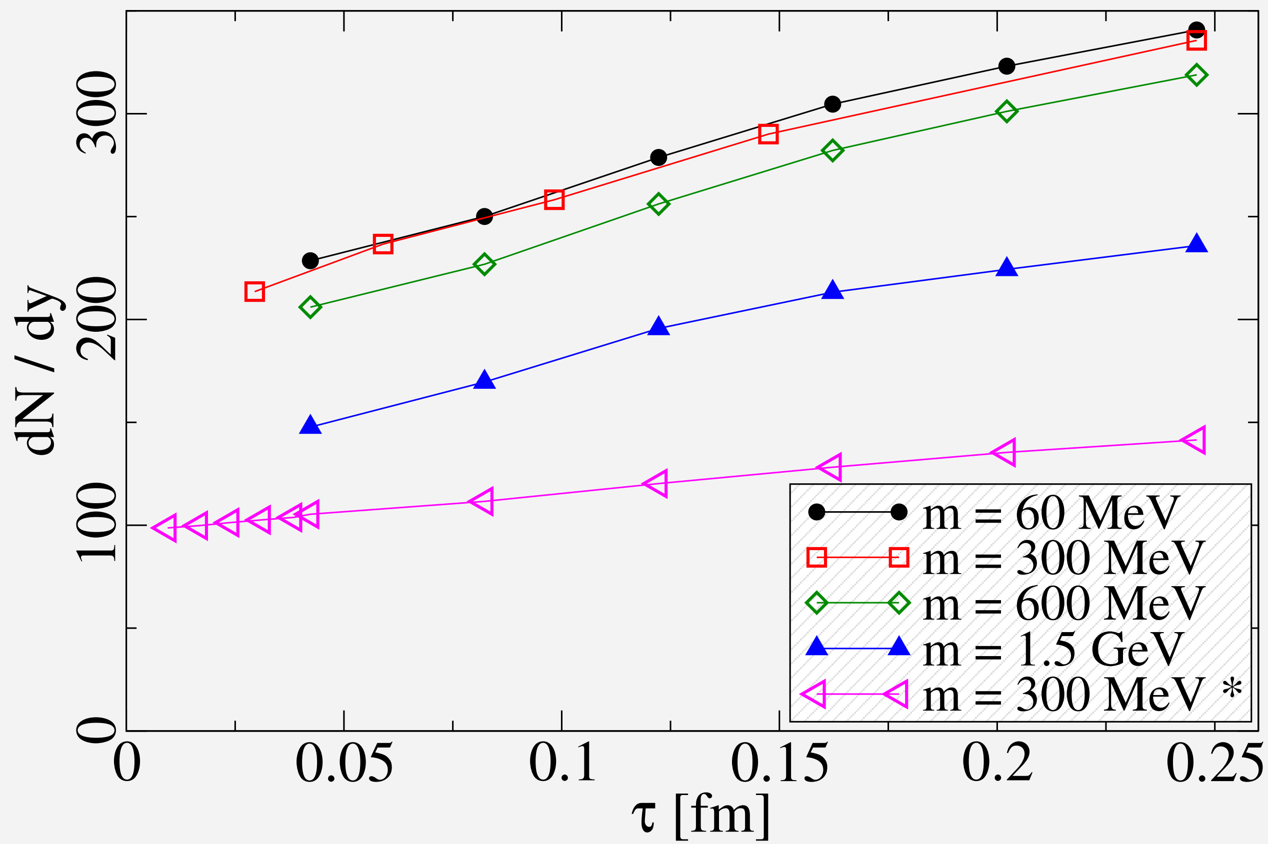
$$e^{-\frac{\eta}{2}\gamma^0\gamma^3}\psi_{\boldsymbol{p}}(\tau, \eta, \vec{\boldsymbol{x}}_\perp) \qquad \text{depends only on} \quad \eta - y_p$$

- The final projection will lead to an amplitude $T_R(p, -q)$ that depends on the difference of the rapidities of the quark and the antiquark, $y_q - y_p$. But the rapidity $\eta$ does not go away in intermediate stages of the calculation

- Moreover, the rapidity $\eta$ is not a good variable to represent the initial condition at $\tau_i = 0$. We use $z$ instead

# Time dependence

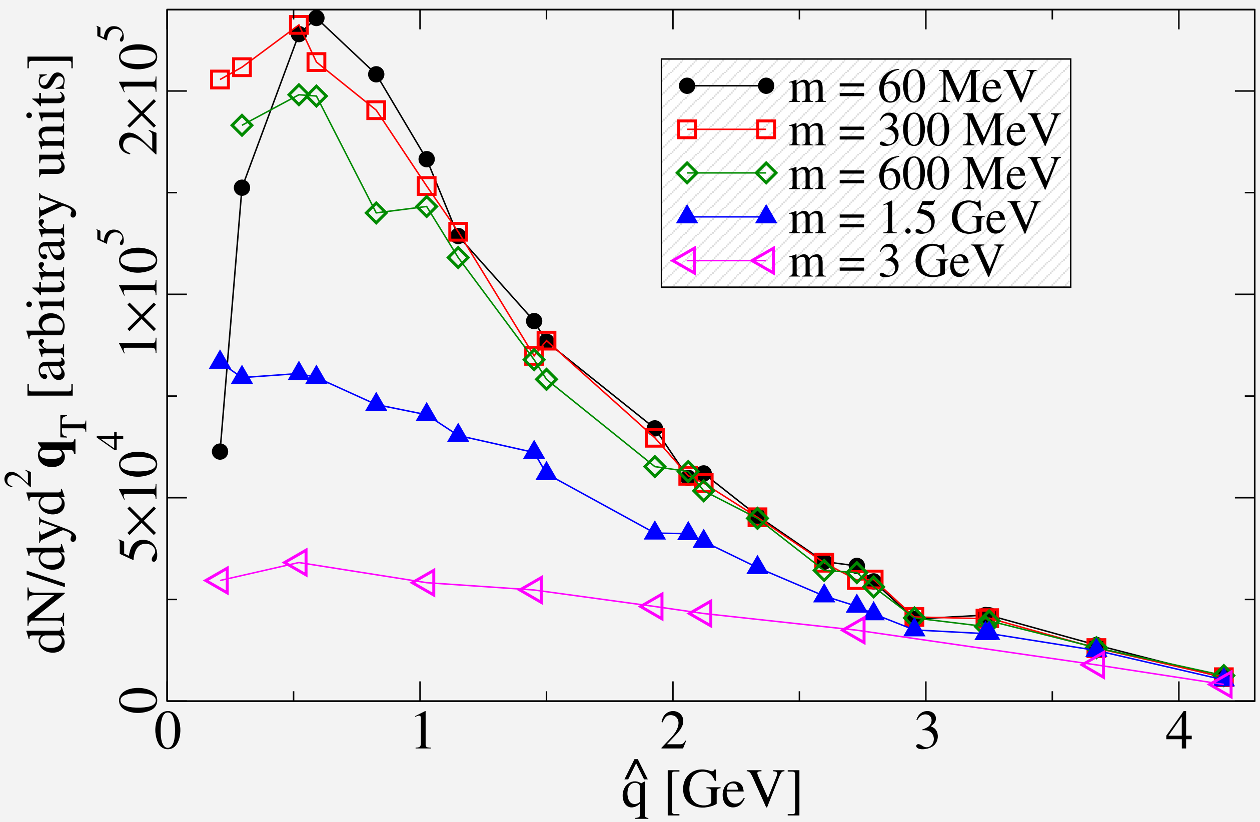
- $g^2\mu = 2$ GeV , (*) $g^2\mu = 1$ GeV :



# Spectra for various quark masses



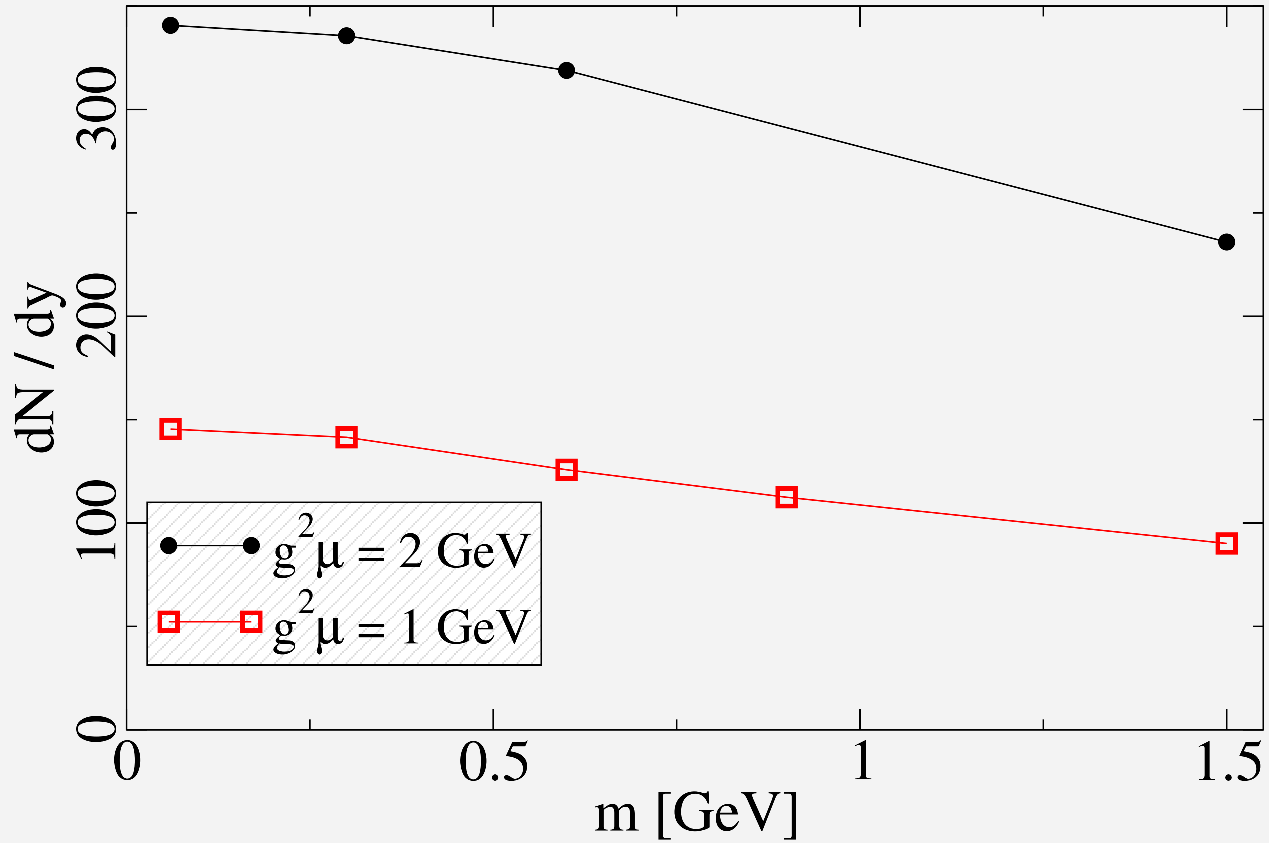
- $g^2\mu = 2$ GeV , $\tau = 0.25$ fm :



# Mass dependence of dN/dy



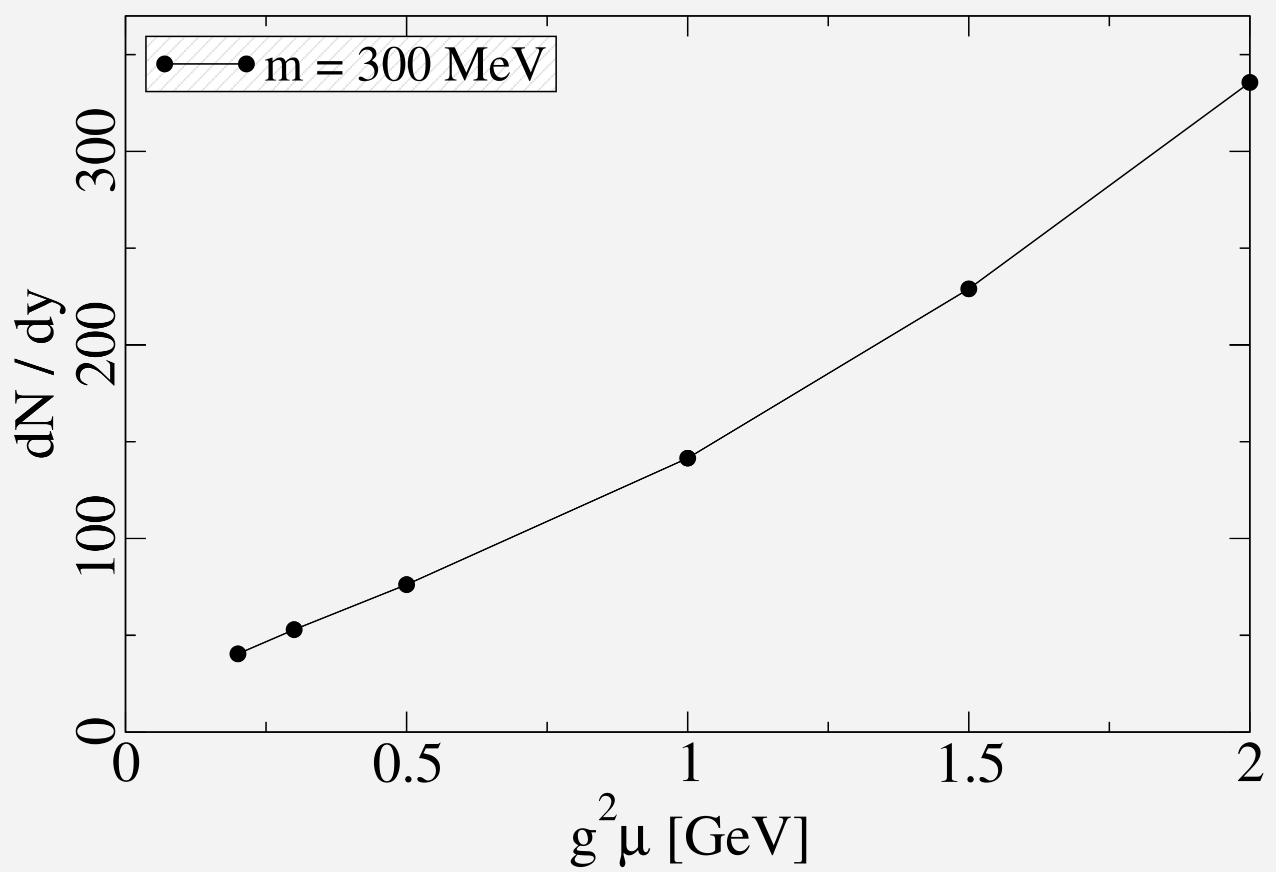
- Number of quarks at $\tau = 0.25$ fm :



# g2mu dependence of dN/dy



- Number of quarks at $\tau = 0.25$ fm :



# Phenomenological implications



- Quarks seem to be produced early
- For $g^2\mu = 2$ GeV, one gets about 1000 gluons per unit rapidity, and 900 pairs of light quarks (3 flavors)
- This is roughly consistent with approximate chemical equilibration: $n_{\rm quarks}/n_{\rm gluons} = 9N_f/24 \sim 1$ for 3 flavors
- If this is true, we must revise our additions :
  - RHIC experiments observe about $600$ charged particles per unit rapidity at $\sqrt{s}_{_{NN}} = 200$ GeV, i.e. about $1000$ particles (charged + neutral)
  - To get a total of $1000$ partons, one would have to take a somewhat smaller saturation momentum $g^2\mu \sim 1.2$ GeV, so that $n_{\rm gluons} \sim n_{\rm quarks} \sim n_{\rm antiquarks} \sim 330$

# Stability of the boost-invariant solution

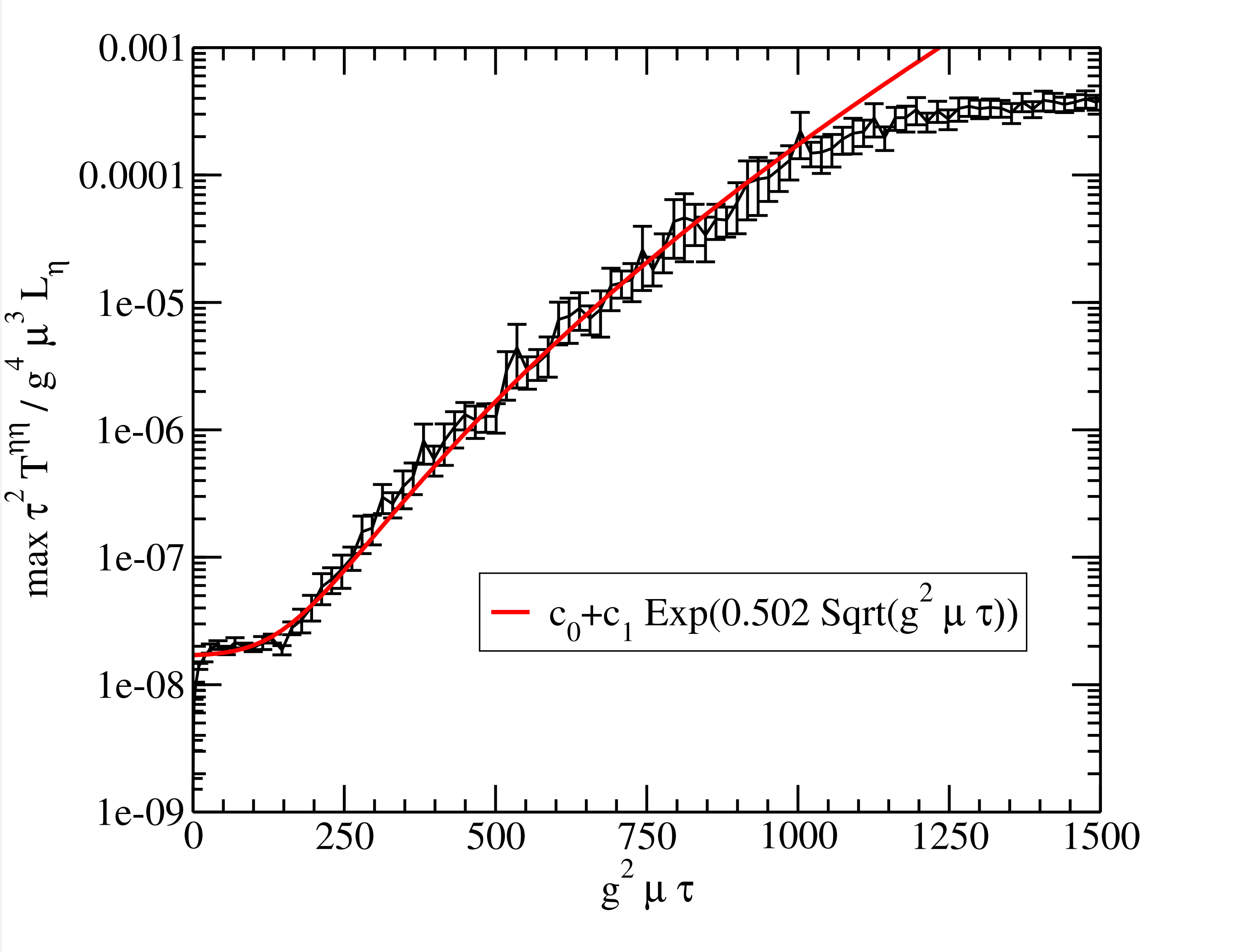


Romatschke,Venugopalan (2005)

- One obvious question at this point is whether the boost invariant solution for the classical color field is stable.
  In other words, how do rapidity dependent perturbations of the solution evolve in time ?
- Model :
  - at some very small $\tau$, perturb the boost invariant solution by a rapidity-dependent additional term
  - solve the 3+1 dimensional classical Yang-Mills equations
  - compute the component $\tau^2 T^{\eta\eta}$ of the energy-momentum tensor (which would be zero in the boost invariant case)

# Growth of the longitudinal pressure






- Exponential growth, followed by a saturation

# Plasma instabilities



- This instability in the classical Yang-Mills equations can be related to the Weibel instabilities known in the physics of anisotropic plasmas
- In an anisotropic plasma, the Hard Thermal Loop gluon polarization tensor leads to a "Debye mass" squared that depends on the direction of the gluon momentum
- In certain directions, one can have $m_D^2 < 0$
  $\triangleright$ anti-screening instead of the usual screening!
- The collective phenomena in such a plasma will blow the system apart
- In the linear regime, the growth of the fields due to this instability is of the form $\exp(\Gamma t)$. $\Gamma$ is determined by the most unstable mode, and is related to the plasmon mass by :

$$\Gamma = \frac{\sqrt{3}}{2}\omega_{\rm plasmon}$$

# Plasma instabilities

- From the solution of the boost-invariant Yang-Mills equation, one can determine the "dispersion relation" $\omega(\vec{\boldsymbol{p}}_\perp)$ by doing a plane wave decomposition of the gauge field. This gives :

$$\omega_{\rm plasmon} = \omega(\vec{\boldsymbol{p}}_\perp = \vec{0}) \sim \sqrt{\frac{g^2\mu}{\tau}}$$

  Note: the square of the plasmon mass is usually linear in the particle density, which is decreasing as $\tau^{-1}$. Hence the $\tau^{-1/2}$ behavior of $\omega_{\rm plasmon}$

- Translating the value of $\omega_{\rm plasmon}$ into a value of $\Gamma$, we expect the longitudinal part of the energy-momentum tensor to grow as :

$$\tau^2 T^{\eta\eta} \sim e^{2\Gamma\tau} = e^{\gamma\sqrt{g^2\mu\tau}}$$

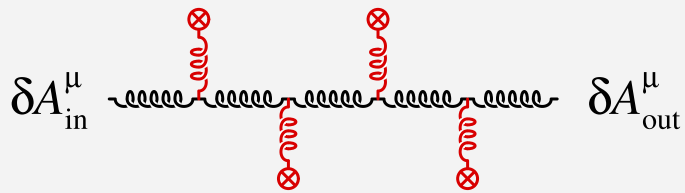
  with a coefficient $\gamma$ of order unity

  ▷ the instability observed in the Yang-Mills equation may have the same origin as the instability obtained from the HTL calculation

# NLO gluon production



- As long as the $\eta$-dependent perturbation is small, it can be treated as a perturbation by linearizing the Yang-Mills equations
  $\triangleright$ one obtains a linear evolution equation for the perturbation, in a background provided by the boost-invariant solution




- In fact, this linear approximation explains the exponential growth. The instability ends when the linear approximation breaks down
- This diagram is also what is needed in order to calculate the correction to the number of gluons due to the production of pairs of gluons

# NLO gluon production



- The instability may also affect NLO gluon production
- Normally, one expects a correction of order $1$ to the number of gluons, compared to $\alpha_s^{-1}$ for the leading contribution
- If the instability kicks in, this might be enhanced by a factor

$$e^{\gamma\sqrt{g^2\mu\tau}}$$

- This means that the "subleading" correction would be as large as the leading contribution after a time

$$\tau \sim (g^2\mu)^{-1} \ln^2\left(\frac{1}{\alpha_s}\right)$$

For $g^2\mu \sim 1$ GeV and $\alpha_s \sim 0.2$, this gives $\tau \sim 0.5$ fm

- After that, the classical approximation would be invalid
- Note also that the gluons produced in pairs have a momentum distribution which much less anisotropic than that of gluons produced at leading order

# Conclusions



- Initial gluon and quark production can be studied by solving the Yang-Mills and Dirac equations with retarded boundary conditions
- The number of quarks is as large as the number of gluons (not very surprising when the coupling is $g \sim 2$)
- The boost invariant solution of the Yang-Mills equations is unstable, with a growth which is compatible with what one expects from plasma instabilities
- Outstanding issues :
  - Is the instability affecting subleading gluon production ?
  - Are there other quantities, besides $\overline{n}$, that are accessible from retarded solutions of the classical EOM?
  - Numerical solution of the JIMWLK equation (see Rummukainen, Weigert (2003) for a pionneer work on this)
  - Pomeron loops

# Production of pairs



- The summation of all the classical field insertions can be done via a Lippmann-Schwinger equation :

$$G_{ij}(x,y) = G^0_{ij}(x,y) - ig \sum_{k,l=\pm} \int d^4z\, G^0_{ik}(x,z)\phi_c(z)\boldsymbol{\sigma}^3_{kl} G_{lj}(z,y)$$

- This equation is quite non-trivial to solve in this form, because the 4 components of the propagator mix. Perform a rotation on the $\pm$ indices :

$$G_{ij} \quad \rightarrow \quad \boldsymbol{G}_{\alpha\beta} \equiv \sum_{i,j=\pm} U_{\alpha i} U_{\beta j} G_{ij}$$

$$[\boldsymbol{\sigma}^3\phi_c(z)]_{ij} \quad \rightarrow \quad \phi_c(x)\boldsymbol{\Sigma}^3_{\alpha\beta} \equiv \sum_{i,j=\pm} U_{\alpha i} U_{\beta j} [\boldsymbol{\sigma}^3\phi_c(z)]_{ij}$$

- A useful choice for the rotation matrix $U$ is $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

# Production of pairs



- Under this rotation, the matrix propagator and field insertion become :

$$\boldsymbol{G}_{\alpha\beta} = \begin{pmatrix} 0 & G_{_A} \\ G_{_R} & G_{_S} \end{pmatrix} \quad , \qquad \phi_c(x)\boldsymbol{\Sigma}^3_{\alpha\beta} = \phi_c(x) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $G^0_{_S}(p) = 2\pi\delta(p^2 - m^2)$

- The main simplification comes from the fact that $\boldsymbol{G\Sigma}^3$ is the sum of a diagonal matrix and a nilpotent matrix
- One finds that $G_{_R}$ and $G_{_A}$ do not mix, i.e. they obey the equations :

$$G_{_{R,A}}(x,y) = G^0_{_{R,A}}(x,y) - ig\int d^4z\, G^0_{_{R,A}}(x,z)\phi_c(z)G_{_{R,A}}(z,y)$$

- One can express $G_{_S}$ in terms of $G_{_R}$ and $G_{_A}$ :

$$G_{_S} = G_{_R} * {G^0_{_R}}^{-1} * G^0_{_S} * {G^0_{_A}}^{-1} * G_{_A}$$

# Production of pairs



- In order to go back to $G_{-+}$, invert the rotation :

$$G_{-+} = \frac{1}{2}\left[G_A - G_R + G_S\right]$$

- Split $G_{R,A}$ into free propagators and a scattering matrix :

$$G_{R,A} \equiv G^0_{R,A} + G^0_{R,A} * T_{R,A} * G^0_{R,A}$$

Note : the retarded/advanced scattering matrices $T_{R,A}$ obey :

$$T_R - T_A = T_R * \left[G^0_R - G^0_A\right] * T_A$$

- Wrapping up everything, in momentum space, gives :

$$E_p \left.\frac{d\overline{n}}{d^3\vec{\boldsymbol{p}}}\right|_{\rm pairs} = \frac{1}{16\pi^3}\int \frac{d^3\vec{\boldsymbol{q}}}{(2\pi)^3 2E_q}\left|T_R(p,-q)\right|^2$$

▷ At this order, one has to obtain the retarded propagator in the classical field $\phi_c$, amputate the external legs, square and integrate over the (on-shell) momentum at one end

# Production of pairs



- $T_R(p,-q)$ can be obtained from retarded solutions of the linearized EOM with the classical field in the background

$$\left(\square + m^2 + g\phi_c(x)\right)\eta(x) = 0$$

- Start from Green's formula for the retarded solution $\eta(x)$ :

$$\eta(x) = \int d^3\vec{\boldsymbol{y}}\; G_R(x,y)\;\overleftrightarrow{\partial}_{y_0}\;\eta(y)$$

- The scattering matrix $T_R$ is related to the propagator $G_R$ by :

$$G_R(x,y) = G_R^0(x,y) + \int_{z_1,z_2} G_R^0(x,z_1)T_R(z_1,z_2)G_R^0(z_2,y)$$

- From there, it is straightforward to verify that :

$$T_R(p,-q) = \lim_{x^0\to+\infty}\int d^3\vec{\boldsymbol{x}}\; e^{ip\cdot x}\left[\partial_{x_0} - iE_p\right]\eta(x)$$

with $\eta(x) = e^{iq\cdot x}$ when $x^0 \to -\infty$

# Less inclusive quantities



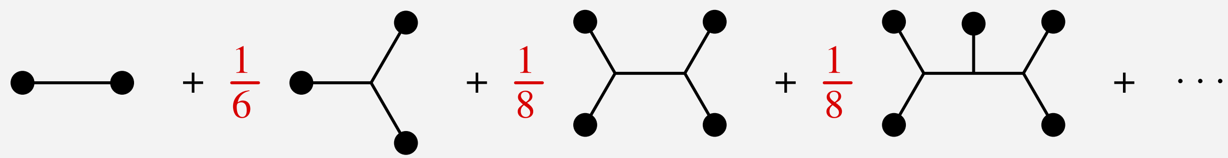
- What do we know about the multiplicity distribution?
  Can we calculate the probability $P_n$ of producing $n$ particles?

- Each transition amplitude contains a disconnected factor which is the sum of all the vacuum-vacuum diagrams
- This sum is the exponential of the sum of connected vacuum-vacuum diagrams :

$$i\sum_{\rm all} V = e^{i\sum_{\rm conn} V}$$

- At the classical level, the sum of connected vacuum-vacuum diagrams is :

$$i\sum_{\rm conn} V = \frac{1}{2}\,[\ldots] + \frac{1}{6}\,[\ldots] + \frac{1}{8}\,[\ldots] + \frac{1}{8}\,[\ldots] + \cdots$$



# Cutting rules

- Each transition probability contains $\exp -2\,\mathrm{Im}\sum_{\rm conn} V$
- $2\,\mathrm{Im}\sum_{\rm conn} V$ can be obtained from Cutkosky's rules, but they need to be modified because of the coupling to the external source
- Decompose the free time ordered propagator, $G^0_{++}$, as :

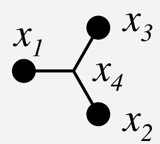
$$G^0_{++}(x,y) = \theta(x^0-y^0)G^0_{-+}(x,y) + \theta(y^0-x^0)G^0_{+-}(x,y)$$

- Define also :

$$G^0_{--}(x,y) \equiv \theta(x^0-y^0)G^0_{+-}(x,y) + \theta(y^0-x^0)G^0_{-+}(x,y)$$

- Consider a diagram in $i\sum_{\rm conn} V$ before performing the space-time integrations : $iV(x_1\cdots x_n)$. The $x_i$ are the locations of the sources $j$, or of the vertices $g$. For instance :

$$iV(x_1,x_2,x_3,x_4) \quad =$$



# Cutting rules



- The diagrams $iV$ are made only of the propagator $G^0_{++}$
- For each diagram $iV(x_1 \cdots x_n)$, construct $2^n$ diagrams $iV_{\epsilon_1\cdots\epsilon_n}(x_1 \cdots x_n)$ where $\epsilon_i$ is a sign attached to the vertex $i$ :
  - Connect a vertex of type $i$ to a vertex of type $j$ by $G^0_{ij}$
  - For vertices of type $-$, substitute $ig \to -ig$, $ij \to -ij$
- Largest time equation : if $x_i^0$ is the largest time in the diagram :

$$iV_{\cdots\epsilon_i\cdots}(x_1 \cdots x_n) + iV_{\cdots -\epsilon_i\cdots}(x_1 \cdots x_n) = 0$$

(the indices hidden in the dots are the same for both terms)

- Therefore, the sum of the $2^n$ $iV_{\epsilon_1\cdots\epsilon_n}$ is zero :

$$\sum_{\{\epsilon_i=\pm\}} iV_{\epsilon_1\cdots\epsilon_n}(x_1 \cdots x_n) = 0$$

(group the terms in pairs, and use the previous result)

# Cutting rules



- In momentum space, the propagators $G^0_{ij}$ read :

$$G^0_{++}(p) = \frac{i}{p^2 - m^2 + i\epsilon} \qquad G^0_{--}(p) = \left[G^0_{++}(p)\right]^*$$

$$G^0_{-+}(p) = 2\pi\theta(p^0)\delta(p^2 - m^2) \qquad G^0_{+-}(p) = 2\pi\theta(-p^0)\delta(p^2 - m^2)$$

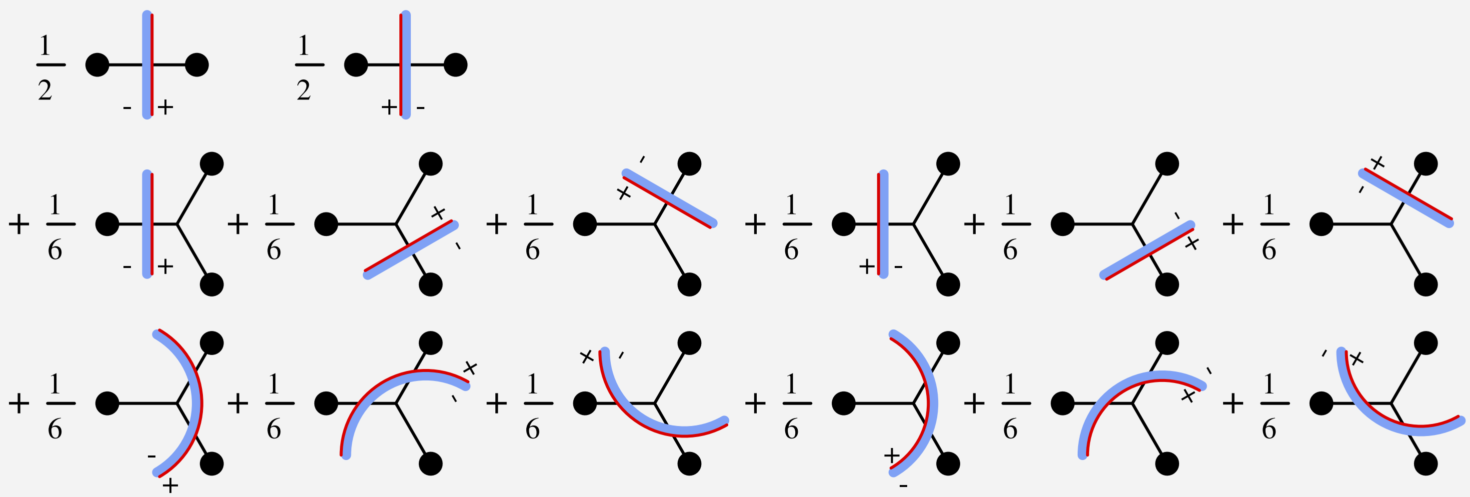
- Therefore, in momentum space, $iV_{++\cdots+}$ is the original diagram and $iV_{--\cdots-}$ is its complex conjugate
- By isolating these two terms from the sum over the $2^n$ terms, we get :

$$2\,\mathrm{Im}\, v = \sum_{\{\epsilon_i = \pm\}'} iV_{\epsilon_1\cdots\epsilon_n}$$

where the prime indicates that the sum over the $\epsilon_i$'s does not have the $+ + \cdots +$ and $- - \cdots -$ terms

# Cutting rules

- For each term in $\sum_{\{\epsilon_i=\pm\}}' iV_{\epsilon_1\cdots\epsilon_n}$, draw a line ("cut") separating the $+$ from the $-$ vertices
- The lowest order terms in $2\,\mathrm{Im}\,\sum_{\rm conn} V$ are given by :



- Cuts through vacuum-vacuum diagrams are non-zero because of the source $j$
- A cut going through $r$ propagators will be called a $r$-particle cut

# Some examples



- Let us denote :

$$2\,\mathrm{Im}\sum_{\mathrm{conn}} V \equiv \frac{a}{g^2}$$

$\frac{b_r}{g^2}$ the sum of all $r-$ particle cuts

- Probability of producing 1 particle :

$$P_1 = e^{-a/g^2}\,\frac{b_1}{g^2}$$

- Probability of producing 2 particles :

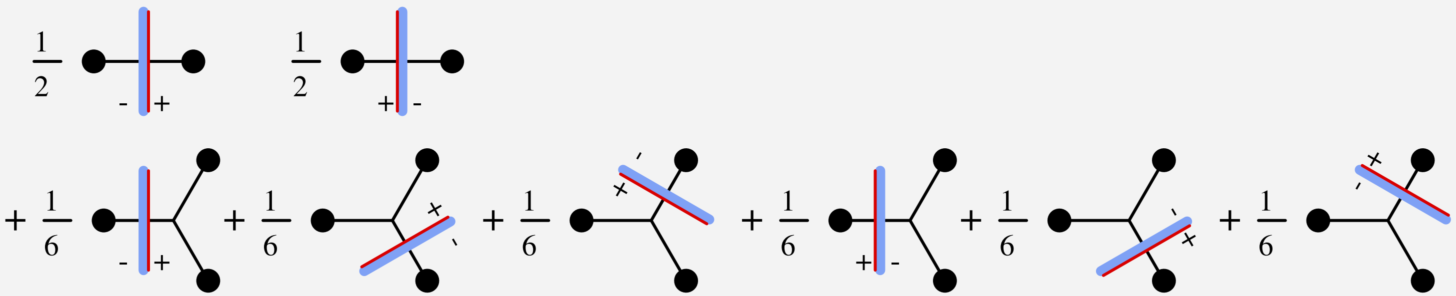
$$P_2 = e^{-a/g^2}\,\left[\frac{1}{2!}\frac{b_1^2}{g^4}+\frac{b_2}{g^2}\right]$$

  - $(b_1/g^2)^2$ if the 2 particles are produced in disconnected diagrams
  - $b_2/g^2$ if they are produced in the same connected diagram

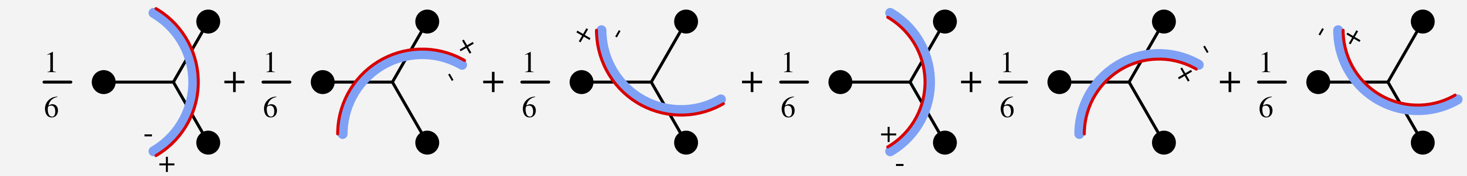
# Some examples

- Lowest order diagrams in $b_1/g^2$, $b_2/g^2$, $b_3/g^2$ :

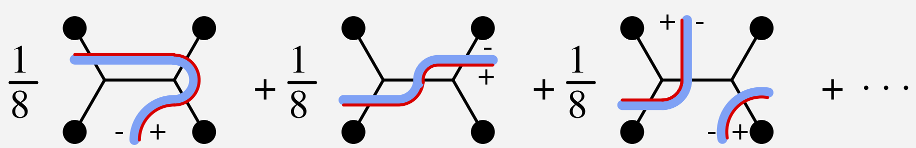
$$\frac{b_1}{g^2} =$$



$$\frac{b_2}{g^2} =$$



$$\frac{b_3}{g^2} =$$



# General case



- For an arbitrary number of particles, $n$, one must sum over all the possibilities to produce them in various numbers of disconnected subdiagrams :

$$P_n = e^{-a/g^2} \sum_{p=1}^{n} \frac{1}{p!} \sum_{\alpha_1+\cdots+\alpha_p=n} \frac{b_{\alpha_1}\cdots b_{\alpha_p}}{g^{2p}}$$

  - In this formula, $p$ is the number of disconnected subdiagrams producing the $n$ particles

- Unitarity :

$$\sum_{n=0}^{\infty} P_n = 1$$

thanks to

$$a = \sum_{r=1}^{\infty} b_r$$

# Moments of the distribution



- Generating function :

$$G(x) \equiv \sum_{n=0}^{\infty} P_n x^n = e^{-a/g^2} \exp\left[\frac{1}{g^2}\sum_{r=1}^{\infty} b_r x^r\right]$$

- By calculating derivatives of $G(x)$ at $x = 1$, it is easy to obtain moments of the distribution :

$$\overline{n} = \sum_{n=1}^{\infty} n P_n = \frac{1}{g^2}\sum_{r=1}^{\infty} r b_r$$

$$\sigma \equiv \overline{n^2} - \overline{n}^2 = \frac{1}{g^2}\sum_{r=1}^{\infty} r^2 b_r$$

- Note : since there are non-zero cuts with $r \geq 2$ particles, we have $\overline{n} \neq \sigma$. Hence the distribution is not Poissonian

# AGK cancellations



- This approach requires to calculate the $r$-particle cuts $b_r/g^2$ of the vacuum-vacuum diagrams ▷ very intricate
- In the case of the first moment of the distribution, $\overline{n}$, we have previously seen a much simpler method to calculate it
- The equivalence between the two methods of calculating $\overline{n}$ implies the following identity :

$$\frac{1}{g^2}\sum_{r=1}^{\infty} r b_r = \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_p} \left|(p^2-m^2)\phi_c(p)\right|^2$$

  between weighted cuts through time-ordered diagrams and retarded diagrams
  Note: the l.h.s is much more complicated than the r.h.s
- The cancellations needed in the l.h.s in order for that to work are known as the AGK cancellations
  Abramovsky, Gribov, Kancheli (1973)