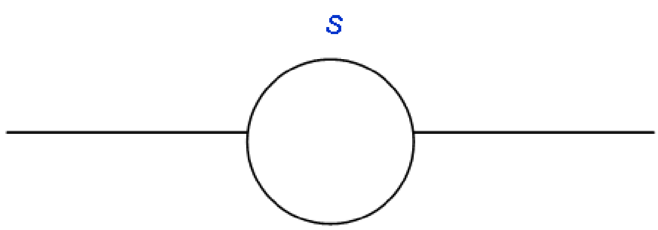
Perturbative relations between gauge theory and gravity

Sudarshan Ananth
Albert Einstein Institute

Divergences and Dynkin indices

$$\beta = -\frac{11}{3}\, C_a^{(2)} + \frac{2}{3}\, C_f^{(2)} + \frac{1}{3}\, C_s^{(2)}$$



$$(-1)^{2s}\ \frac{1}{3}\left(1 - 12\, s^2\right)$$

Hughes, 1980

see also: background field method

$\mathcal{N} = 4$ Yang-Mills theory

- reduction from $(\mathcal{N} = 1, d = 10)$ Yang-Mills
- scale invariant

Mandelstam, 1983 ; Brink-Lindgren-Nilsson, 1983

- ultra-violet properties determined by $SO(8)$

Curtright, 1981

Triality and finiteness

Dynkin index for a representation R

$$I^{(p)}[R] = \sum_{w \text{ in } R} (w \cdot w)^{\frac{p}{2}}$$

Divergent part of one-loop two-point function is

$$\propto (-1)^f \left\{ I_R^{(0)} - I_R^{(2)} \right\} (p^\mu p^\nu - \delta^{\mu\nu} p^2)\, \mathcal{G}$$

Curtright, 1981

SO(8) triality

implies that

$$I^{(p)}_{vector} = I^{(p)}_{spinor}$$

- It is important to identify efficient tools that will make loop computations in (super) gravity tractable
- Perturbative relations to gauge theory are crucial to unitarity based investigations of $\mathcal{N} = 8$ supergravity

Bern-Carrasco-Dixon-Johansson-Kosower-Roiban, 2007

SO(9) Dynkin indices

Index	$G_{\mu\nu}$	$A_{\mu\nu\rho}$	Ψ_{μ}
$I^{(0)}$	44	84	128
$I^{(2)}$	88	168	256
$I^{(4)}$	232	408	640
$I^{(6)}$	712	1080	1792
$I^{(8)}$	2440	3000	5248

Gravity and Yang-Mills theory

Based on local symmetries - behave differently

Tree-level closed string amplitudes factorize into products of open string tree amplitudes

Kawai-Lewellen-Tye, 1986

In the field theory limit this relates gravity tree amplitudes to Yang-Mills tree amplitudes

$$\sqrt{-g}\, R \qquad\qquad F^{a}_{\mu\nu}\, F^{a\,\mu\nu}$$

No obvious factorization property that might explain the origin of these relations

Yang-Mills action has up to four-point interactions while the gravity action involves infinitely many

- Can we derive KLT relations from field theory?
- gravity $\sim$ (gauge theory) $\times$ (gauge theory)

Outline

Part I

- Conventions and notation
- MHV Lagrangian for Yang-Mills theory
- Proposal

Part II

- Gravity in light-cone gauge
- Shifted gravity Lagrangian
- KLT relations

Conventions and notation

Tree amplitudes are simple in a helicity basis

Xu-Zhang-Chang, 1986 ; Berends-Giele, 1987

Natural to work in light-cone gauge

$$x^{\pm} = \frac{1}{\sqrt{2}} (x^0 \pm x^3) \, , \quad \partial_{\pm} = \frac{1}{\sqrt{2}} (\partial_0 \pm \partial_3)$$

$$x^+ : \text{ light-cone time} \qquad \partial_+ : \text{ Hamiltonian}$$

Transverse coordinates and derivatives read

$$x = \frac{1}{\sqrt{2}} (x^1 + i\, x^2) \, , \quad \bar{\partial} \equiv \frac{\partial}{\partial x} = \frac{1}{\sqrt{2}} (\partial_1 - i\, \partial_2)$$

$$\bar{x} = \frac{1}{\sqrt{2}} (x^1 - i\, x^2) \, , \quad \partial \equiv \frac{\partial}{\partial \bar{x}} = \frac{1}{\sqrt{2}} (\partial_1 + i\, \partial_2)$$

A vector can be represented as a bi-spinor

$$p_{a\dot{a}} \equiv p_\mu \, (\sigma^\mu)_{a\dot{a}} = \begin{pmatrix} -p_- & \overline{p} \\ p & -p_+ \end{pmatrix}$$

With metric signature $(-,+,+,+)$ this implies

$$\det\left(p_{a\dot{a}} \right) = -2 \left(p\overline{p} - p_+ p_- \right) = -p^\mu p_\mu$$

Conventions and notation

When the vector p_μ is light-like

$$p_+ \;=\; \frac{p\overline{p}}{p_-}$$

which is the on-shell condition

The holomorphic and anti-holomorphic spinors

$$\lambda_a = \frac{2^{\frac{1}{4}}}{\sqrt{p_-}} \begin{pmatrix} p_- \\ -p \end{pmatrix} \;, \quad \tilde{\lambda}_{\dot{a}} = -(\lambda_a)^* = -\frac{2^{\frac{1}{4}}}{\sqrt{p_-}} \begin{pmatrix} p_- \\ -\overline{p} \end{pmatrix}$$

On-shell, the product $\lambda_a \tilde{\lambda}_{\dot{a}}$ is equal to $p_{a\dot{a}}$

Introduce the off-shell spinor products

$$\langle k\, l\rangle \;=\; \sqrt{2}\; \frac{k\, l_- - l\, k_-}{\sqrt{k_-\, l_-}}$$

$$[k\, l] \;=\; \sqrt{2}\; \frac{\bar{k}\, l_- - \bar{l}\, k_-}{\sqrt{k_-\, l_-}}$$

When on-shell, these spinor products satisfy

$$\langle k\, l\rangle\, [l\, k] = s_{kl} \equiv -(k+l)^2$$

All particle momenta are taken to be outgoing

MHV amplitudes

Strip gauge group dependence in a tree amplitude

$$\prod_{\sigma} \mathrm{Tr}(T^{a_1} \ldots T^{a_\sigma})$$

Mangano-Parke-Xu, 1988

to obtain the partial amplitudes (kinematic)

$$A_n^{\mathrm{tree}}(1, 2, \ldots, n)$$

Helicity tree amplitudes where all, or all but one, of the external particles have the same helicity vanish

$$A_n^{\mathrm{tree}}(+, +, \ldots, +) = 0$$
$$A_n^{\mathrm{tree}}(-, +, \ldots, +) = 0$$
$$A_n^{\mathrm{tree}}(-, -, \ldots, +) \neq 0 \quad \text{(MHV)}$$

Grisaru-Pendleton-van Nieuwenhuizen, 1977

To quartic order, all non-vanishing scattering amplitudes in Yang-Mills and gravity are MHV

MHV amplitudes are remarkably simple

$$A_n^{\mathrm{tree}}(1^-, 2^-, 3^+, \ldots, n^+) = \frac{\langle 1\, 2 \rangle^4}{\langle 1\, 2 \rangle \langle 2\, 3 \rangle \ldots \langle n\, 1 \rangle}$$

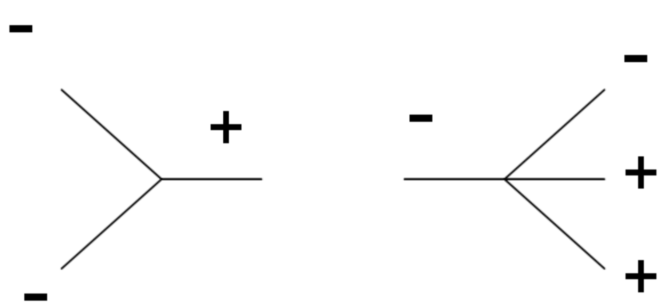
Parke-Taylor, 1986

MHV vertices

Off-shell continuations of MHV amplitudes may be used as fundamental vertices in a Lagrangian for Yang-Mills theory

Vertices are connected using scalar propagators

Cachazo-Svrcek-Witten, 2004



This form of the Lagrangian involves an infinite number of vertices

Each vertex has exactly two negative helicity legs

The (off-shell) structure of the MHV amplitudes is manifest in this Lagrangian

Yang-Mills theory

The gauge field is $A_\mu = (A_- \, , \, A_+ \, , \, A \, , \, \bar{A})$

$$A_- \; = \; 0 \;\; : \quad \text{light-cone gauge}$$

A_+ is eliminated in favor of A and $\bar{A}$

Structurally, the Yang-Mills Lagrangian is then

$$L \; \sim \; L_{+-} + L_{++-} + L_{+--} + L_{++--}$$

L_{++-} does not occur in the MHV Lagrangian

Identify a field redefinition that maps the first two terms of the Lagrangian into a purely kinetic term

Gorsky-Rosly, 2005 ; Mansfield, 2005

This results in an infinite series

$$L_{YM} \; \sim \; L_{+-} + L_{+--} + \ldots + L_{(+)^n--} + \ldots$$

which is the MHV Lagrangian

Ettle-Morris, 2006 ; Ettle-Fu-Fudger-Mansfield-Morris, 2007

Brandhuber-Spence-Travaglini-Zoubos, 2007

Proposal

The KLT relations, in string theory and field theory, are on-shell statements

The MHV form of the Lagrangian has advantages

- ∘ straightforward to compute MHV amplitudes
- ∘ the Ward identity for amplitudes is built-in

This form for the Lagrangian is closely allied to on-shell physics

→ A similar re-formulation of the gravity Lagrangian ought to make the KLT relations manifest

Based on work with S. Theisen, arXiv:0706.1778 [hep-th]

- ∘ Are the relations valid only for on-shell amplitudes or, more generally, at the level of the action ?

Outline

Part I

- Conventions and notation
- MHV Lagrangian for Yang-Mills theory
- Proposal

Part II

- Gravity in light-cone gauge
- Shifted gravity Lagrangian
- KLT relations

Gravity in light-cone gauge

The Einstein-Hilbert action reads

$$S_{EH} = \frac{1}{2\,\kappa^2} \int d^4x\,\sqrt{-g}\,R$$

Light-cone gauge is chosen by setting

$$g_{--} = g_{-i} = 0 \ , \qquad i = 1, 2$$

The metric is parameterized as follows

$$g_{+-} = -\,e^{\frac{\psi}{2}} \ , \quad g_{ij} = e^{\psi}\,\gamma_{ij}$$

ψ is real, γ_{ij} is symmetric and unimodular

The $R_{-i} = 0$ constraint eliminates g^{-i}

$$R_{--} = 0 \ \Rightarrow \ \ \psi = \frac{1}{4}\,\frac{1}{\partial_-^2}\,(\partial_-\gamma^{ij}\partial_-\gamma_{ij})$$

Scherk-Schwarz, 1974

Perturbative expansion:

$$\gamma_{ij} = (e^{\kappa\,H})_{ij} \ , \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} h + \bar{h} & -i(h - \bar{h}) \\ -i(h - \bar{h}) & -h - \bar{h} \end{pmatrix}$$

h and $\bar{h}$ represent gravitons of helicity $+2$ and -2

Bengtsson-Cederwall-Lindgren, 1983

Gravity in light-cone gauge

To quartic order we have

$$
\begin{aligned}
&\bar{h}\,\Box\,h \\
&+\bar{h}\,\partial_-^2\left(\frac{\bar\partial}{\partial_-}h\frac{\bar\partial}{\partial_-}h-h\frac{\bar\partial^2}{\partial_-^2}h\right)+h\,\partial_-^2\left(\frac{\partial}{\partial_-}\bar{h}\frac{\partial}{\partial_-}\bar{h}-\bar{h}\frac{\partial^2}{\partial_-^2}\bar{h}\right) \\
&+\frac{1}{\partial_-^2}(\partial_-h\partial_-\bar{h})\left\{\frac{\partial\bar\partial}{\partial_-^2}(\partial_-h\partial_-\bar{h})-\frac{\partial\bar\partial}{\partial_-}h\,\partial_-\bar{h}-\partial_-h\frac{\partial\bar\partial}{\partial_-}\bar{h}\right\} \\
&-\frac{1}{\partial_-^2}(\partial_-h\partial_-\bar{h})\left\{2\,\partial\bar\partial h\,\bar{h}+2\,h\partial\bar\partial\bar{h}+9\,\bar\partial h\partial\bar{h}+\partial h\bar\partial\bar{h}\right\} \\
&-\frac{1}{\partial_-}(2\bar\partial h\partial_-\bar{h}+h\partial_-\bar\partial\bar{h}-\partial_-\bar\partial h\bar{h})\frac{1}{\partial_-}(2\partial_-h\partial\bar{h}+\partial_-\partial h\bar{h}-h\partial_-\partial\bar{h}) \\
&-h\,\bar{h}\left(2\,\bar\partial h\partial\bar{h}+\partial\bar\partial h\,\bar{h}+h\partial\bar\partial\bar{h}+3\frac{\partial\bar\partial}{\partial_-}h\,\partial_-\bar{h}+3\partial_-h\frac{\partial\bar\partial}{\partial_-}\bar{h}\right) \\
&+\frac{1}{\partial_-^3}(\partial_-h\partial_-\bar{h})\left(\partial\bar\partial h\,\partial_-\bar{h}+\partial_-h\partial\bar\partial\bar{h}\right) \\
&-2\frac{1}{\partial_-}(2\bar\partial h\,\partial_-\bar{h}+h\partial_-\bar\partial\bar{h}-\partial_-\bar\partial h\bar{h})\,h\,\partial\bar{h} \\
&-2\frac{1}{\partial_-}(2\partial_-h\,\partial\bar{h}+\partial_-\partial h\,\bar{h}-h\partial_-\partial\bar{h})\,\bar\partial h\,\bar{h}
\end{aligned}
$$

Bengtsson-Cederwall-Lindgren, 1983; Ananth-Brink-Heise-Svendsen, 2006

The three-vertices are $(-,+,+)$ and $(+,-,-)$

The field redefinition

We seek a transformation such that

$$\bar{h}\,\Box\,h + \kappa\,\bar{h}\,\partial_-^2\left(\frac{\bar{\partial}}{\partial_-}h\,\frac{\bar{\partial}}{\partial_-}h - h\,\frac{\bar{\partial}^2}{\partial_-^2}h\right) = \bar{C}\,\Box\,C$$

We choose h to be a function of C alone

$$h(p) = \sum_{n=1}^{\infty}\int\prod_{i=1}^{n} d^3k_i\; Z^{(n)}(p_1, k_1, \ldots, k_n)\,C(k_1)\ldots C(k_n)$$

By requiring that the transformation be canonical
we find that $\bar{h}$ is a function of C and linear in $\bar{C}$

$$Z^{(1)}(p,k) = \delta^{(3)}(p-k)$$

$$Z^{(2)}(p,k,l) = -\frac{\kappa}{2}\,\frac{p_-^2}{k_-l_-}\,\frac{[k\,l]}{\langle k\,l\rangle}\,\delta^{(3)}(p-k-l)$$

$$\bar{h}(p) = \bar{C}(p) + \kappa\int d^3k\,d^3l\,\frac{k_-^3}{p_-^2\,l_-}\,\frac{[k\,l]}{\langle k\,l\rangle}\,\bar{C}(k)C(l) + \ldots$$

Ananth-Theisen, 2007

It is straightforward to write down a recursion
relation which can be used to solve for any $Z^{(n)}$

Shifted Lagrangian to order κ^2

$$\int d^4p\ \bar{C}(-p)\, p^2\, C(p)$$

$$+\,\kappa\int d^4p\ d^4k\ d^4l\ \delta^{(4)}(p+k+l)\ \frac{\langle k\,l\rangle^6}{\langle l\,p\rangle^2\,\langle p\,k\rangle^2}\ C(p)\bar{C}(k)\bar{C}(l)$$

$$+\,\kappa^2\int d^4p\ d^4q\ d^4k\ d^4l\ \delta^{(4)}(p+q+k+l)$$
$$\times\,\frac{\langle k\,l\rangle^8\,[k\,l]}{\langle k\,l\rangle\,\langle k\,p\rangle\,\langle k\,q\rangle\,\langle l\,p\rangle\,\langle l\,q\rangle\,\langle p\,q\rangle^2}\ C(p)C(q)\bar{C}(k)\bar{C}(l)$$

Ananth-Theisen, 2007

The redefinition produces some interaction vertices proportional to the free equations of motion

Suitable field redefinitions eliminate these vertices

Salam-Strathdee, 1970 ; Georgi, 1991 ; Arzt, 1993

The four-vertex, for example, now factorizes

$$\frac{\langle k\,l\rangle^4}{\langle k\,l\rangle\,\langle l\,p\rangle\,\langle p\,q\rangle\,\langle q\,l\rangle}\ \langle k\,l\rangle\,[k\,l]\ \frac{\langle k\,l\rangle^4}{\langle k\,l\rangle\,\langle l\,q\rangle\,\langle q\,p\rangle\,\langle p\,k\rangle}$$

Relevant results

The Yang-Mills three-point MHV amplitude is

$$A_3^{\rm tree}(k^-, l^-, p^+) = \frac{\langle k\, l\rangle^3}{\langle l\, p\rangle \, \langle p\, k\rangle}$$

The Yang-Mills four-point MHV amplitude reads

$$A_4^{\rm tree}(k^-, l^-, p^+, q^+) = \frac{\langle k\, l\rangle^4}{\langle k\, l\rangle \, \langle l\, p\rangle \, \langle p\, q\rangle \, \langle q\, l\rangle}$$

The off-shell spinor products read

$$\langle k\, l\rangle = \sqrt{2}\, \frac{k\, l_- - \, l\, k_-}{\sqrt{k_-\, l_-}} \, , \qquad [k\, l] = \sqrt{2}\, \frac{\bar{k}\, l_- - \, \bar{l}\, k_-}{\sqrt{k_-\, l_-}}$$

on-shell, they satisfy

$$\langle k\, l\rangle \, [l\, k] = s_{kl} \equiv -(k+l)^2$$

Dixon, 1996 (TASI Lectures) ; Bern, 2002 (Living reviews)

KLT relations

The off-shell gravity vertices factorize into products of off-shell Yang-Mills vertices

In particular, on-shell, this implies

$$M_3^{\rm tree}(1,2,3) = A_3^{\rm tree}(1,2,3)\, A_3^{\rm tree}(1,2,3)$$
$$M_4^{\rm tree}(1,2,3,4) = -\, s_{12}\, A_4^{\rm tree}(1,2,3,4)\, A_4^{\rm tree}(1,2,4,3)$$

where the gravity amplitudes are

$$\mathcal{M}_n^{\rm tree}(1,2,\ldots,n) = \left(\frac{\kappa}{2}\right)^{(n-2)} M_n^{\rm tree}(1,\ldots,n)$$

Unlike Yang-Mills, these MHV vertices in gravity appear only after the second field redefinition

This was to be expected: gravity is more complicated + amplitudes are non-holomorphic

Berends-Giele-Kuijf, 1988

At the loop level, Jacobians from the field redefinitions must be handled with care

Salam-Strathdee, 1970

Comments

This procedure needs to be verified for higher-point vertices where non-MHV structures appear

In particular, studying $M^{\rm tree}(+,+,-,-,-)$ and $M^{\rm tree}(+,+,+,-,-)$ will prove instructive

- Is there an MHV Lagrangian for gravity ?
- Are there variables in which gravity stops at κ^2

The $\mathcal{N}=8$ supergravity action is known to order κ^2 in light-cone superspace

Ananth-Brink-Ramond, 2005 ; Ananth-Brink-Heise-Svendsen, 2006

- Is the $\mathcal{N}=8$ supergravity action the "square" of the $\mathcal{N}=4$ action ?
- What implications, if any, does this have for the ultra-violet properties of the $\mathcal{N}=8$ model ?

Light-cone gauge: defining $\frac{1}{\partial_-}$

We define $\frac{1}{\partial_-}$ using the Green function relation

$$\partial_- G(x^-) = \delta(x^-)$$

Determined up to a zero mode

$$\partial_- h = 0$$

Information in the form of boundary conditions

$$G(x^-) \propto \theta(x^-) - \theta(-x^-)$$

Kogut-Soper, 1970

The Fourier transform of the first θ-function

$$\int_0^\infty e^{-i\,p_- x^-}\, dx^-$$

has problems as $x^- \to \infty$ so we set

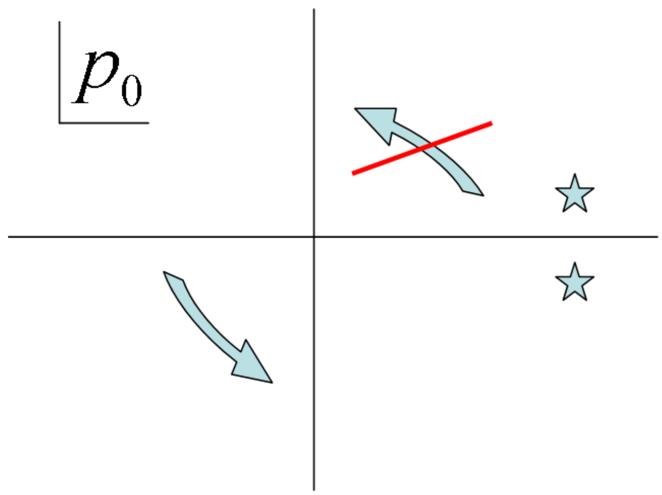
$$p_- \to p_- - i\,\epsilon$$

This yields the principal value prescription

$$iG(p_-) = \mathcal{P}\frac{1}{p_-} = \frac{1}{2}\left(\frac{1}{p_- + i\epsilon} + \frac{1}{p_- - i\epsilon}\right)$$

Wick rotation

The usual prescription for $\frac{1}{\partial_-}$ leads to two poles



Choose instead the Green function

$$G(x^-) \,=\, 2\pi\, i\, \theta(-p_+)\theta(x^-) \,-\, 2\pi\, i\, \theta(p_+)\theta(-x^-)$$

Mandestam, 1983 ; Leibbrandt, 1984

which leads to the prescription

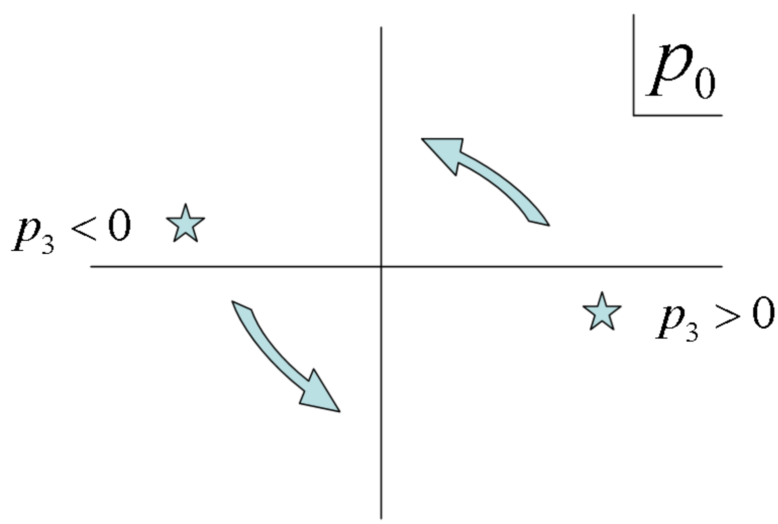
$$p_- \;\rightarrow\; \frac{1}{p_- \,+\, i\,\epsilon\, p_+}$$

To see this start with

$$\int \frac{1}{p_- \,+\, i\,\epsilon\, p_+}\, e^{i\, p_-\, x^-}\, dp_-$$

- $p_+ > 0\,,\, x^- > 0$: This is zero since in the p_- plane, the pole lies on the negative y-axis and we close the contour in the UHP
- $p_+ > 0\,,\, x^- < 0$: This is $-2\pi\, i$ since the contour encloses the pole (the exponential tends to 1 as $\epsilon \to 0$)
- $p_+ < 0\,,\, x^- > 0$: This is $-2\pi\, i$
- $p_+ < 0\,,\, x^- < 0$: This is zero

Depending on the sign of p_3, we have poles as shown below



Now rotate the Re p_0 axis into the Im p_0 axis

Light-cone gauge: the LC_2 boson

The functional integral reads

$$I = \int \mathbf{D}\, A \quad e^{i\int d^4x\; (-\frac{1}{4}\, F_{\mu\nu}{}^{a} F^{\mu\nu\; a})}$$

The Lagrangian density is

$$L = -\frac{1}{4}\, F_{\mu\nu}{}^{a} F^{\mu\nu\; a}$$

where

$$F_{\mu\nu}{}^{a} = \partial_\mu A_\nu{}^{a} - \partial_\nu A_\mu{}^{a} - g f^{abc} A_\mu{}^{b} A_\nu{}^{c}$$

Light cone gauge is chosen by setting

$$A_-{}^{a} = 0$$

The non-zero $F^{\mu\nu}$'s are

$$\begin{aligned}
F_{+-}{}^{a} &= -F^{+-a} = -\partial_- A_+{}^{a} \\
F_{+k}{}^{a} &= -F^{-k a} = \partial_+ A_k{}^{a} - \partial_k A_+{}^{a} - igf^{abc} A_+{}^{b} A_k{}^{c} \\
F_{-k}{}^{a} &= -F^{+k a} = \partial_- A_k{}^{a} \\
F_{k\,l}{}^{a} &= F^{\,k\,l a} = \partial_k A_l{}^{a} - \partial_l A_k{}^{a} - igf^{abc} A_k{}^{b} A_l{}^{c}
\end{aligned}$$

The equations of motion are

$$\partial_\nu F^{\mu\nu\; a} + g f^{abc} F^{\mu\nu\; b} A_\nu{}^{c} = 0$$

We choose the $\mu = +$ component and solve for A_+

$$\partial_- {A_+}^a = (\nabla\cdot \mathbf{A}^a) + g f^{abc} \frac{1}{\partial_-} (\partial_- {A_k}^b {A_k}^c)$$

so

$$ {A_+}^a = \frac{1}{\partial_-} (\nabla\cdot \mathbf{A}^a) + g f^{abc} \frac{1}{{\partial_-}^2} (\partial_- {A_k}^b {A_k}^c)$$

The action has four parts

$$F_{\mu\nu} F^{\mu\nu} = 2F_{+-}F^{+-} + 2F_{+k}F^{+k} + 2F_{-k}F^{-k} + F_{kl}F^{kl}$$

Substituting for A_+^a we obtain

$$-g\, f^{abc} (\partial_i {A_i}^a) \frac{1}{\partial_-} (\partial_- {A_j}^b {A_j}^c) + g\, f^{abc} (\partial_k {A_l}^a) {A_k}^b {A_l}^c$$
$$-\frac{g^2}{2}\, f^{abc} f^{amn} \frac{1}{\partial_-} (\partial_- {A_k}^b {A_k}^c) \frac{1}{\partial_-} (\partial_- {A_l}^m {A_l}^n)$$
$$-\frac{g^2}{4}\, f^{abc} f^{amn} {A_k}^b {A_l}^c {A_k}^m {A_l}^n - \frac{1}{2}\, {A_i}^a \partial_l \partial_l {A_i}^a$$

Light-cone gauge: the LC_2 fermion

We start with

$$\mathcal{L}_0 = i\,\overline{\lambda}\,\gamma^\mu \partial_\mu\,\lambda$$

For any Weyl spinor λ, we define

$$\lambda^{(\pm)} = \frac{1}{2}\,\gamma^\pm\,\gamma^\mp\,\lambda$$

The equations of motion read

$$\gamma^+\,\partial_+\,\lambda + \gamma^-\,\partial_-\,\lambda + \gamma^k\,\partial_k\,\lambda = 0$$

Multiply this by $\gamma^+\,\gamma^-\,\gamma^+$ to obtain

$$0 + 2\,\gamma^+\,\gamma^-\,\partial_-\,\lambda^{(+)} + 2\,\gamma^+\,\gamma^k\,\partial_k\,\lambda^{(-)} = 0$$

where we have used the following relations

$$\gamma^{(\pm)^2} = 0\ , \quad \{\gamma^{(\pm)}, \gamma^k\} = 0\ , \quad \{\gamma^+, \gamma^-\} = 2$$

So the equation of motion tells us that

$$4\partial_-\lambda^{(+)} = -2\,\sqrt{2}\,(\frac{\gamma^+ + \gamma^-}{\sqrt{2}})\,\gamma^k\,\partial_k\,\lambda^{(-)}$$

Since $\gamma^-\,\lambda^{(-)} = 0$, the final result for $\lambda^{(+)}$ reads

$$\lambda^{(+)} = -\frac{1}{\sqrt{2}}\,\gamma^0\,\gamma^k\,\frac{\partial_k}{\partial_-}\,\lambda^{(-)}$$

The field redefintion

- Does the field redefinition map an interacting theory into a free theory ?

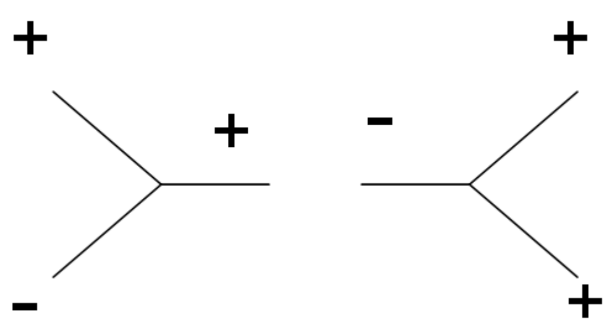
Helicity tree amplitudes where all, or all but one, of the external particles have the same helicity vanish

$$A^{\text{tree}}(\pm, +, +, \ldots, +) = 0$$

Grisaru-Pendleton-van Nieuwenhuizen, 1977

Terms affected by the shift are $L_{+-} + L_{++-}$

Graphs generated by these terms involve a single negative helicity external leg and hence vanish



At tree-level, this fact is simply being made explicit

Gorsky-Rosly, 2005 ; Mansfield, 2005

Supersymmetric Ward identity

The supercharge Q annihilates the vacuum

$$0 = \langle 0|[Q, \Phi_1\Phi_2\cdots\Phi_n]|0\rangle = \sum_{i=1}^{n}\langle 0|\Phi_1\cdots[Q,\Phi_i]\cdots\Phi_n|0\rangle$$

$\Phi(k) = g^{\pm}(k), \Lambda^{\pm}(k)$ create gluon and gluino states of momentum k ($k^2 = 0$) and helicity $\pm$

Multiply Q by a Grassmann spinor parameter $\bar{\eta}$, defining $Q(\eta) \equiv \bar{\eta}^{\alpha}Q_{\alpha}$: commutators take the form

$$\begin{aligned}\Big[Q(\eta), g^{\pm}(k)\Big] &= \mp\Gamma^{\pm}(k,\eta)\,\Lambda^{\pm}(k),\\ \Big[Q(\eta), \Lambda^{\pm}(k)\Big] &= \mp\Gamma^{\mp}(k,\eta)\,g^{\pm}(k)\end{aligned}$$

$\Gamma(k,\eta)$ is linear in η and constrained by

$$\begin{aligned}0 = &\ [[Q(\eta), Q(\zeta)], \Phi(k)] + [[Q(\zeta), \Phi(k)], Q(\eta)]\\ + &\ [[\Phi(k), Q(\eta)], Q(\zeta)]\end{aligned}$$

Since $[Q(\eta), Q(\zeta)] = -2i\bar{\eta}\not{P}\zeta$

$$\Gamma^{+}(k,\eta)\Gamma^{-}(k,\zeta) + \Gamma^{-}(k,\eta)\Gamma^{+}(k,\zeta) = -2i\bar{\eta}\not{k}\zeta$$

with the solution

$$\Gamma^{+}(k,\eta) = \bar{\eta}u_{-}(k), \quad \Gamma^{-}(k,\eta) = \bar{\eta}u_{+}(k) = \bar{u}_{-}(k)\eta$$

η is chosen to be a Grassmann parameter θ multiplied by an arbitrary massless spinor q

$$\Gamma^+(k,q) = \theta \langle q^+|k^-\rangle = \theta \ [q\, k]$$
$$\Gamma^-(k,q) = \theta \langle q^-|k^+\rangle = \theta \ \langle q\, k\rangle$$

The simplest case is

$$\begin{aligned} 0 = \ & \Big\langle 0|[Q(\eta(q)), \Lambda_1^+ g_2^+ g_3^+ \cdots g_n^+]|0\Big\rangle \\ = \ & -\Gamma^-(k_1,q) A_n(g_1^+, g_2^+, \ldots, g_n^+) \\ & +\Gamma^+(k_2,q) A_n(\Lambda_1^+, \Lambda_2^+, g_3^+, \ldots, g_n^+) \\ & + \cdots + \Gamma^+(k_n,q)\, A_n(\Lambda_1^+, g_2^+, \ldots, g_{n-1}^+, \Lambda_n^+) \end{aligned}$$

Since massless gluinos have only helicity-conserving interactions $A_n(g_1^+, g_2^+, \ldots, g_n^+)$ vanishes

With one negative helicity we get

$$\begin{aligned} 0 &= \Big\langle 0|[Q(\eta(q)), \Lambda_1^+ g_2^- g_3^+ \cdots g_n^+]|0\Big\rangle \\ &= -\Gamma^-(k_1,q)\, A_n(g_1^+, g_2^-, g_3^+, \ldots, g_n^+) \\ & \quad -\Gamma^-(k_2,q)\, A_n(\Lambda_1^+, \Lambda_2^-, g_3^+, \ldots, g_n^+) \end{aligned}$$

$q = k_1 \ \Rightarrow$ second amplitude vanishes
$q = k_2 \ \Rightarrow$ first amplitude vanishes

Color ordering

The generators of the gauge group satisfy

$$\mathrm{Tr}(T^a T^b) = \delta^{ab}$$

Eliminate the structure constants f^{abc}

$$f^{abc} \;=\; -\frac{i}{\sqrt{2}}\Big(\mathrm{Tr}(T^a T^b T^c) - \mathrm{Tr}(T^a T^c T^b)\Big)$$

Reduce the number of traces using

$$(T^a)_{i_1}^{\;\bar{\jmath}_1}\,(T^a)_{i_2}^{\;\bar{\jmath}_2} = \delta_{i_1}^{\;\bar{\jmath}_2}\delta_{i_2}^{\;\bar{\jmath}_1} - \frac{1}{N_c}\,\delta_{i_1}^{\;\bar{\jmath}_1}\delta_{i_2}^{\;\bar{\jmath}_2}$$

This leads to the color decomposition of the the n-gluon tree amplitude $\mathcal{A}_n^{\mathrm{tree}}\left(\{k_i, \lambda_i, a_i\}\right)$

$$g^{n-2} \sum_{\sigma \; in \; S_n/Z_n} \mathrm{Tr}\left(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}\right)\; A_n^{\mathrm{tree}}(\sigma(1^{\lambda_1}), \ldots, \sigma(n^{\lambda_n}))$$

g : coupling $\quad k_i\,,\;\lambda_i$: momenta and helicities

$A_n^{\mathrm{tree}}(1^{\lambda_1}, \ldots, n^{\lambda_n})$ are the partial amplitudes which contain all the kinematic information

S_n : all permutations, Z_n : cyclic permutations

The field redefinition

We seek a transformation $(h, \bar{h}) \to (C, \bar{C})$ such that

$$-\bar{h}\partial_+\partial_- h + \bar{h}\left\{\partial\bar{\partial}h + \kappa\,\partial_-^2\left(\frac{\bar{\partial}}{\partial_-}h\,\frac{\bar{\partial}}{\partial_-}h - h\,\frac{\bar{\partial}^2}{\partial_-^2}h\right)\right\}$$

$$= -\bar{C}\partial_+\partial_- C + \bar{C}\partial\bar{\partial}C$$

Consider the generating function

$$G(C, \pi_h) = \int g(C)\,\pi_h$$

We have

$$\pi_C \equiv \frac{\partial L}{\partial(\partial_+ C)} = \partial_-\bar{C} = \frac{\delta G}{\delta C} = \int \frac{\delta g}{\delta C}\pi_h$$

and

$$h = \frac{\delta G}{\delta \pi_h} = g(C)$$

Since $\pi_h = \partial_-\bar{h}$

$$\partial_-\bar{C}(y) = \int d^3x\,\partial_-\bar{h}(x)\,\frac{\delta h(x)}{\delta C(y)}$$

integral performed on surface of constant x^+

The Lagrangian density

$$\mathcal{L} = -\bar{C}\partial_+\partial_- C + \bar{C}\partial\bar{\partial}C = \partial_-\bar{C}\,\partial_+ C - \partial_-\bar{C}\,\frac{\partial\bar{\partial}}{\partial_-}C$$

becomes

$$\begin{aligned} L = & \int d^3x\,\partial_-\bar{h}(x)\partial_+ h(x) \\ - & \int d^3x \int d^3y\,\partial_-\bar{h}(y)\frac{\partial\bar{\partial}}{\partial_-}C(x)\,\frac{\delta h(y)}{\delta C(x)} \end{aligned}$$

We want this to equal

$$L = \int d^3x\left(\partial_-\bar{h}(x)\partial_+ h(x) - \partial_-\bar{h}(x)\frac{1}{\partial_-}V(h(x))\right)$$

In momentum space, this requirement reads

$$\frac{p\bar{p}}{p_-}h(p_-) - \int d^3m\,\frac{m\bar{m}}{m_-}C(m)\,\frac{\delta h(p)}{\delta C(m)} =$$
$$-\,\kappa\int d^3k\,d^3l\;\delta^{(3)}(p-k-l)\;(k_- + l_-)\left(\frac{\bar{k}\bar{l}}{k_- l_-} - \frac{\bar{l}^2}{l_-^2}\right)h(k)\,h(l)$$

Dimensional reduction

In four dimensions, the spin $\frac{1}{2}$ fermion loop reads

$$-g^2\, tr[t^a t^b] \int \frac{d^{2\,\omega}\, k}{(2\pi)^{2\omega}}\ \mathbf{Tr}[\,\gamma_\mu\, \frac{1}{\not{k}+\not{q}}\, \gamma_\nu\, \frac{1}{\not{k}}\,]$$

For fermions transforming in a representation f

$$tr[t^a t^b] = C_2(f)\delta^{ab}$$

Introducing the Feynman parameter we get

$$-g^2 n_f C_2(f)\delta^{ab}\!\int\! dx \int \frac{d^{2\,\omega}\, k}{(2\pi)^{2\omega}}\ \{\, k_\rho k_\sigma - q_\rho q_\sigma x(1-x)\}$$
$$\frac{1}{[k^2+q^2 x(1-x)]^2}\ \mathbf{Tr}[\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\sigma]$$

where n_f is the number of species of fermions

This integral is done in $d = 4$

$$g^2 n_f C_2(f)\delta^{ab}\frac{1}{(16\pi^2)}\ \frac{1}{6}\ \frac{1}{\epsilon}\ [q^2\frac{\delta_{\rho\sigma}}{2} + q_\rho q_\sigma]\ \mathbf{Tr}\{\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\sigma\}$$

In four dimensions, the trace satisfies

$$\mathbf{Tr}\{\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\sigma\} = 4\{\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\nu}\delta_{\rho\sigma}\}$$

giving the divergent piece

$$g^2 n_f C_2(f)\delta^{ab}\frac{1}{(16\pi^2)}\,\frac{4}{3}\,\frac{1}{\epsilon}\,[q_\mu q_\nu - q^2\delta_{\mu\nu}]$$

Dimensionally reduced case

Start again at

$$-g^2 n_f C_2(f)\delta^{ab}\!\int\! dx \int \tfrac{d^{\,2\,\omega}\,k}{(2\pi)^{2\omega}}\;\{\,k_\rho k_\sigma - q_\rho q_\sigma x(1-x)\}$$
$$\tfrac{1}{[k^2+q^2x(1-x)]^2}\;\mathbf{Tr}[\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\sigma]$$

and evaluate the integrals in $d = 4$ to get

$$g^2 n_f C_2(f)\delta^{ab}\frac{1}{(16\pi^2)}\,\frac{1}{6}\,\frac{1}{\epsilon}\,[q^2\frac{\delta_{\rho\sigma}{}^{(d)}}{2} + q_\rho q_\sigma]\,\mathbf{Tr}\{\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\sigma\}$$

In D dimensions

$$\mathbf{Tr}\{\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\sigma\} = 2^{\frac{D}{2}}\{\delta^{(D)}_{\mu\rho}\delta^{(D)}_{\nu\sigma} + \delta^{(D)}_{\mu\sigma}\delta^{(D)}_{\nu\rho} - \delta^{(D)}_{\mu\nu}\delta^{(D)}_{\rho\sigma}\}$$

This gives us the divergent piece as

$$g^2 n_f C_2(f)\delta^{ab}\frac{1}{16\pi^2}\frac{2^{\frac{D}{2}}}{6}\,\frac{1}{\epsilon}[(q^2\delta^{(d)}_{\mu\nu} - q_\mu q_\nu) - 3(q^2\delta^{(D)}_{\mu\nu} - q_\mu q_\nu)]$$

ET evasion

"We demonstrate that the canonical change of variables that yields the MHV lagrangian, also provides contributions to scattering amplitudes that evade the equivalence theorem. This 'ET evasion' in particular provides the tree-level $(-++)$ amplitude, which is non-vanishing off shell, or on shell with complex momenta or in (2,2) signature, and is missing from the MHV rules"

Ettle-Fu-Fudger-Mansfield-Morris, 2007

"It has been known for some time that the standard MHV diagram formulation of perturbative Yang-Mills theory is incomplete, as it misses rational terms in one-loop scattering amplitudes of pure Yang-Mills. We propose that certain Lorentz violating counterterms, when expressed in the field variables which give rise to standard MHV vertices, produce precisely these missing terms"

Brandhuber-Spence-Travaglini-Zoubos, 2007