Completing MHV rules via Equivalence Theorem evasion

James H. Ettle[1] Jonathan P. Fudger[1] Chih-Hao Fu[2]
Paul R.W. Mansfield[2] Tim R. Morris[1]

[1]School of Physics and Astronomy,
University of Southampton

[2]Department of Mathematical Sciences, University of Durham

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Higher precision and high multiplicity processes:
LHC and beyond

For the LHC as well as for the linear collider program, the computation of higher precision and high multiplicity processes has become a must.

Higher precision and high multiplicity processes:

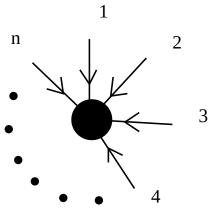
Les Houches 2005. The NLO target list. $V \in \{Z, W, \gamma\}$, VBF = vector boson fusion

process	relevant for
1. $pp \to V\,V\,\mathrm{jet}$	$t\bar{t}H$, new physics
2. $pp \to H + 2\,\mathrm{jets}$	H production by VBF
3. $pp \to t\bar{t}\,b\bar{b}$	$t\bar{t}H$
4. $pp \to t\bar{t} + 2\,\mathrm{jets}$	$t\bar{t}H$
5. $pp \to V\,V\,b\bar{b}$	VBF$\to H \to VV$, $t\bar{t}H$, new physics
6. $pp \to V\,V + 2\,\mathrm{jets}$	VBF$\to H \to VV$
7. $pp \to V + 3\,\mathrm{jets}$	various new physics signatures
8. $pp \to V\,V\,V$	SUSY trilepton

Colour ordered Amplitudes A_n

$$A_n(1, \cdots, n) = \sum_{perms} \mathrm{Tr}(T^{a_1} \cdots T^{a_n}) \mathcal{A}_n(1, \cdots, n)$$

- Order gluons fixed
- Have nice properties



Massless momenta

$$\sigma_{\mu\ \alpha\dot{\alpha}} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

$$p_{\alpha\dot{\alpha}} = p^{\mu}\, \sigma_{\mu\ \alpha\dot{\alpha}} = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix}$$

$$p^2 = 0 \quad \Longrightarrow \quad \text{only one eigenvector}$$

$$\Longrightarrow \quad p_{\alpha\dot{\alpha}} = \lambda_{\alpha}\tilde{\lambda}_{\dot{\alpha}}$$

p^{μ} in terms of positive and negative chirality spinors.

Lorentz Invariants. Gluon helicities

- Massless on-shell: $p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$
- Dot product: $2p_{i\mu}\, p_j^\mu = \langle i\, j\rangle [i\, j]$

$$\langle \lambda_i\, \lambda_j \rangle \equiv \langle i\, j\rangle = \epsilon^{\alpha\beta} \lambda_{i\,\alpha} \lambda_{j\,\beta}, \qquad [\lambda_i\, \lambda_j] \equiv [i\, j] = \epsilon^{\dot{\alpha}\dot{\beta}} \lambda_{i\,\dot{\alpha}} \lambda_{j\,\dot{\beta}}$$

- Gluon helicities:

$$\epsilon^+_{\alpha\dot{\alpha}} = \frac{\eta_\alpha \tilde{\lambda}_{\dot{\alpha}}}{\langle \eta\, \lambda\rangle} \quad \text{and} \quad \epsilon^-_{\alpha\dot{\alpha}} = \frac{\lambda_\alpha \tilde{\eta}_{\dot{\alpha}}}{[\lambda\, \eta]}.$$

- $\mu_{\alpha\dot{\alpha}} = \eta_\alpha \tilde{\eta}_{\dot{\alpha}}$ is arbitrary gauge choice

Maximal Helicity Violating Amplitudes

Amplitudes with mostly same such helicity have surprisingly nice properties. (Parke, Taylor '85; Berends, Giele '88)

$$\mathcal{A}(1^+, \cdots, n^+) = 0$$

$$\mathcal{A}(1^-, 2^+, \cdots, n^+) = 0$$

$$\mathcal{A}(1^+, \cdots, i^-, \cdots, j^-, \cdots, n^+) = \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle \ldots \langle n\, 1\rangle} \qquad \textbf{MHV}$$

$$\mathcal{A}(1^-, \cdots, i^+, \cdots, j^+, \cdots, n^-) = \frac{[i\, j]^4}{[1\, 2][2\, 3] \cdots [n\, 1]} \qquad \overline{\textbf{MHV}}$$

MHV rules
or Cachazo Svrček Witten rules

- Inspired by an understanding through string theory on twistor space. (Witten '04)
- Non-Lagrangian method proceeding by, conjecture, confirmation, proof.

MHV rules

or Cachazo Svrček Witten rules

The colour-ordered amplitudes can be made by joining MHV amplitudes together as though 'vertices' from a 'field theory'

- Amplitudes sewn together in all possible ways preserving cyclic colour order using *scalar propagators*,

$$\frac{i}{p^2}.$$

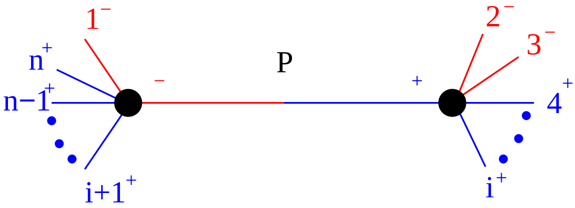
- Off-shell propagator momenta: use *arbitrary reference spinor* η:

$$\lambda_\alpha \propto p_{\alpha\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\eta}_{\dot{\beta}}$$

Total colour-ordered amplitudes are η independent.

MHV rules

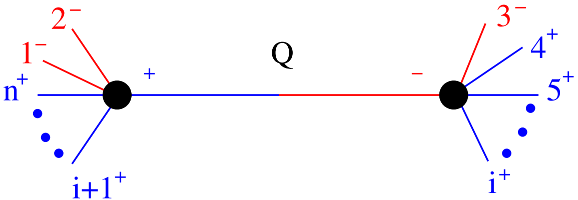
NMHV amplitudes



$$\mathcal{A}_n(- - - + \cdots +) = \sum_{i=3}^{n-1} \frac{\langle 1\ P\rangle^3}{\langle P\ i{+}1\rangle\langle i{+}1\ i{+}2\rangle\cdots\langle n\ 1\rangle} \frac{1}{P^2} \frac{\langle 2\ 3\rangle^3}{\langle P\ 2\rangle\langle 3\ 4\rangle\cdots\langle i\ P\rangle}$$
$$+ \sum_{i=4}^{n} \frac{\langle 1\ 2\rangle^3}{\langle 2\ Q\rangle\langle Q\ i{+}1\rangle\cdots\langle n\ 1\rangle} \frac{1}{Q^2} \frac{\langle Q\ 3\rangle^3}{\langle 3\ 4\rangle\cdots\langle i{-}1\ i\rangle\langle i\ Q\rangle}$$

MHV rules

NMHV amplitudes



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MHV rules

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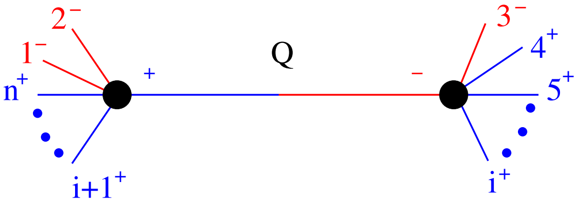
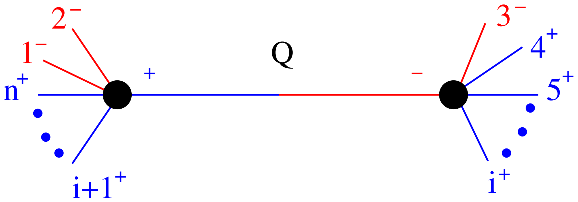


- Connecting two MHVs → 3 negative helicities **NMHV**
- Connecting three MHVs → 4 negative helicities **N^{2}MHV**

MHV rules

NNMHV amplitudes



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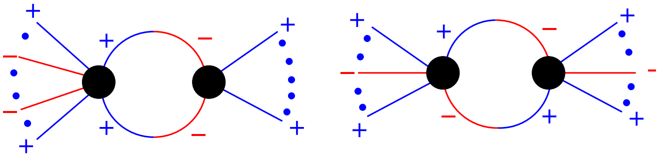
MHV rules at one loop
Queen Mary Approach

It works at one loop also for the parts with cuts:



- Connecting two MHVs → 2 negative helicities
- Connecting three MHVs → 3 negative helicities

MHV rules at one loop
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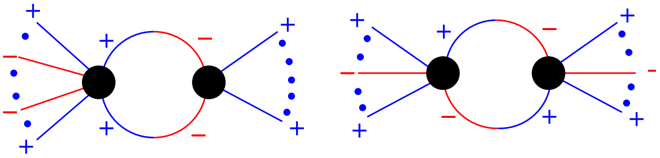

- Connecting two MHVs → 2 negative helicities
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Summary

- Connecting MHVs at tree level and one loop gives the amplitudes with two or more negative helicities.

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Puzzles

- Connecting MHVs at tree level and one loop gives the amplitudes with two or more negative helicities.
- Where is the tree level $++-$ amplitude?

$$\mathcal{A}[++-] = \frac{[1\,2]^3}{[2\,3][3\,1]}$$

Summary

Puzzles

- Connecting MHVs at tree level and one loop gives the amplitudes with two or more negative helicities.
- Where is the tree level $++-$ amplitude?

$$\mathcal{A}[++-] = \frac{[1\,2]^3}{[2\,3][3\,1]}$$

- Where are the one loop $++\cdots+$ amplitudes?

$$\mathcal{A}[++++] = -\frac{i}{3(4\pi)^2}\frac{[1\,2][3\,4]}{\langle 1\,2\rangle\langle 3\,4\rangle}$$

- Where are the missing rational pieces at one-loop?

MHV rules follow from a Lagrangian

- The Lagrangian follows from a canonical change of variables.

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Light cone coordinates

- Define light-cone time by $x^0 = \mu \cdot x$

Light cone coordinates $\mu = \eta\tilde{\eta} \propto (1,0,0,1)$

$$x^0 = \frac{1}{\sqrt{2}}(t - x^3) \quad x^{\bar{0}} = \frac{1}{\sqrt{2}}(t + x^3) \quad z = \frac{1}{\sqrt{2}}(x^1 + ix^2) \quad \bar{z} = \frac{1}{\sqrt{2}}(x^1 - ix^2)$$

$$\check{p} \qquad \hat{p} \qquad \tilde{p} \qquad \bar{p}$$

$$p \cdot q = \check{p}\hat{q} + \hat{p}\check{q} - \tilde{p}\bar{q} - \bar{p}\tilde{q}$$

On-shell massless $p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$ where

$$\lambda = 2^{1/4} \begin{pmatrix} -\tilde{p}/\sqrt{\hat{p}} \\ \sqrt{\hat{p}} \end{pmatrix} \quad \text{and} \quad \tilde{\lambda} = 2^{1/4} \begin{pmatrix} -\bar{p}/\sqrt{\hat{p}} \\ \sqrt{\hat{p}} \end{pmatrix}.$$

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Makes sense off-shell, where now

$$\lambda\tilde{\lambda} = p - \frac{p^2}{2p\cdot\mu}\mu$$

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$$\lambda\tilde{\lambda} = p - \frac{p^2}{2p\cdot\mu}\mu \implies \lambda_\alpha \propto p_{\alpha\dot{\alpha}} \epsilon^{\dot{\alpha}\dot{\beta}} \tilde{\eta}_{\dot{\beta}}$$

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  Total colour-ordered amplitudes are η independent.

Light cone Yang-Mills

Light-cone gauge $\hat{A} = 0$. Integrate out $\check{A}$ and get

$$S = \mathrm{tr}\int dx^0 \int_\Sigma dx^{\bar{0}} dz\, d\bar{z}\, \mathcal{L}$$

$$\mathcal{L} = \mathcal{L}^{-+}[A] + \mathcal{L}^{++-}[A] + \mathcal{L}^{--+}[A] + \mathcal{L}'^{--++}[A]$$

$$\begin{aligned}
\mathcal{L}^{-+}[A] &= -\bar{A}\Box A \\
\mathcal{L}^{++-}[A] &\sim \mathcal{A}^{++-}\bar{A}AA \\
\mathcal{L}^{--+}[A] &\sim \mathcal{A}^{--+}A\bar{A}\bar{A} \\
\mathcal{L}'^{--++}[A] &\sim V^{--++}\bar{A}\bar{A}AA
\end{aligned}$$

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Change variables so that

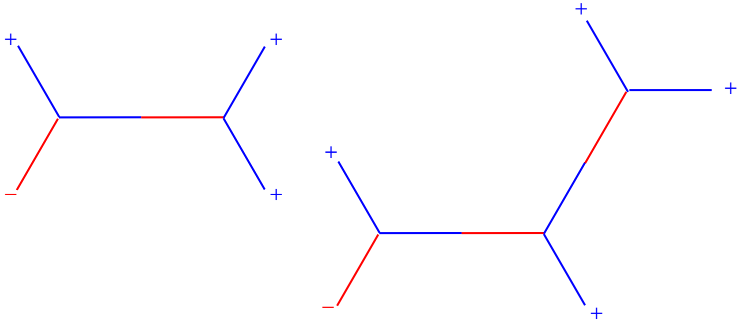
$$\mathcal{L}^{-+}[A] + \mathcal{L}^{++-}[A] = \mathcal{L}^{-+}[B] \quad i.e. \quad -\bar{B}\Box B$$

$$\begin{aligned} \mathcal{L}^{-+}[A] &= -\bar{A}\Box A \\ \mathcal{L}^{++-}[A] &\sim \mathcal{A}^{++-}\bar{A}AA \\ \mathcal{L}^{--+}[A] &\sim \mathcal{A}^{--+}A\bar{A}\bar{A} \\ \mathcal{L}'^{--++}[A] &\sim V^{--++}\bar{A}\bar{A}AA \end{aligned}$$

Self-dual Yang-Mills: $\mathcal{L}^{-+} + \mathcal{L}^{++-}$

Chalmers & Siegel

Classically a free theory:



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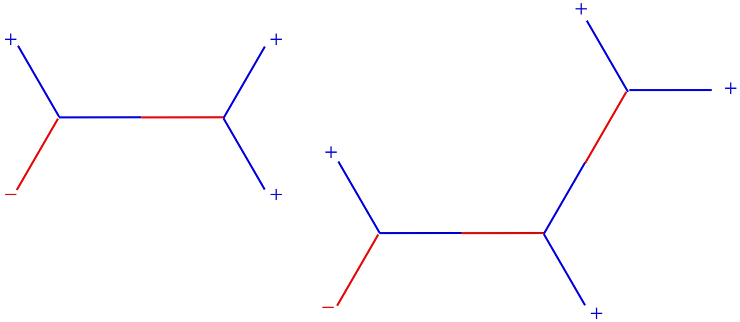

But at one-loop or with complex momenta or $2, 2$ signature or off shell . . .

Solving $\mathcal{L}^{-+}[A] + \mathcal{L}^{++-}[A] = \mathcal{L}^{-+}[B]$

Rough sketch...

$$-\bar{A}\Box A + \mathcal{A}^{++-}\bar{A}AA = -\bar{B}\Box B$$

Solving $\mathcal{L}^{-+}[A] + \mathcal{L}^{++-}[A] = \mathcal{L}^{-+}[B]$

Rough sketch...

$$\hat{\partial}\bar{A}\check{\partial}A + \bar{A}\tilde{\partial}\bar{\partial}A + \mathcal{A}^{++-}\bar{A}AA = \hat{\partial}\bar{B}\check{\partial}B + \bar{B}\tilde{\partial}\bar{\partial}B$$

expanding kinetic terms

Solving $\mathcal{L}^{-+}[A] + \mathcal{L}^{++-}[A] = \mathcal{L}^{-+}[B]$

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Require canonical transformation in quantization surface:

$$\hat{\partial}\bar{B} = \hat{\partial}\bar{A}\frac{\delta A}{\delta B} \qquad \text{and thus} \qquad \hat{\partial}\bar{A}\check{\partial}A = \hat{\partial}\bar{B}\check{\partial}B$$

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$$A_1 = B_1 + \sum_{n=3}^{\infty} \int_{2\cdots n} \Upsilon_{12\cdots n} \, B_{\bar{2}} \cdots B_{\bar{n}}$$

$$A_1 \equiv A(x^0, \mathbf{p}_1), \qquad B_{\bar{2}} \equiv B(x^0, -\mathbf{p}_2)$$

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$$\Upsilon_{123} \sim \frac{\mathcal{A}^{++-}}{\sum_{i=1}^{3} \tilde{p}_i \bar{p}_i / \hat{p}_i} \quad \text{and} \quad \Upsilon_{1\cdots n} \sim \frac{1}{\sum_{i=1}^{n} \tilde{p}_i \bar{p}_i / \hat{p}_i} \sum \cdots \Upsilon\Upsilon$$

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$\Upsilon_{1\cdots n}$ has the structure of a tree expansion in $\mathcal{A}^{++-}$

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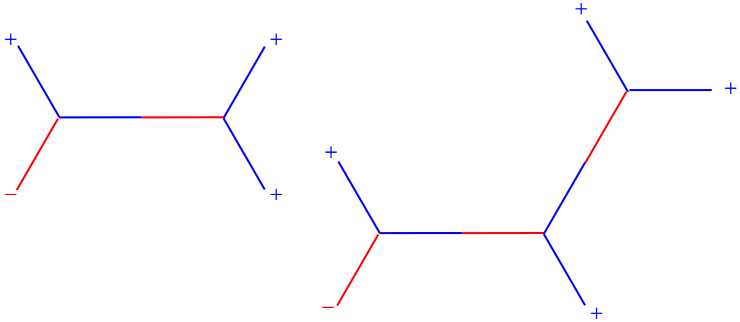

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$$\sum_i \tilde{p}_i \bar{p}_i / \hat{p}_i = \sum_i (\tilde{p}_i \bar{p}_i - \hat{p}_i \check{p}_i) / \hat{p}_i = -\frac{1}{2} \sum_i p_i^2 / \hat{p}_i$$

Solving $\mathcal{L}^{-+}[A] + \mathcal{L}^{++-}[A] = \mathcal{L}^{-+}[B]$

Rough sketch...

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Solving $\mathcal{L}^{-+}[A] + \mathcal{L}^{++-}[A] = \mathcal{L}^{-+}[B]$

Rough sketch...

$$\hat\partial \bar A = \hat\partial \bar B \frac{\delta B}{\delta A} \Longrightarrow \bar A_1 = \bar B_1 - \sum_{m=3}^{\infty} \sum_{s=2}^{m} \int_{2\cdots m} \frac{\hat s}{\hat 1}\, \Xi^{s-1}_{12\cdots m}\, B_{\bar 2} \cdots \bar B_{\bar s} \cdots B_{\bar m}$$

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$$\Xi_{123} \sim \Upsilon_{123} \quad \text{and} \quad \Xi_{1\cdots n} \sim \Upsilon_{1\cdots n} + \sum \Xi\Upsilon$$

Solving $\mathcal{L}^{-+}[A] + \mathcal{L}^{++-}[A] = \mathcal{L}^{-+}[B]$

Summary

$$A_1 = \sum_{n=2}^{\infty} \int_{2\cdots n} \Upsilon_{12\cdots n}\, B_{\bar{2}} \cdots B_{\bar{n}}$$

$$\bar{A}_1 = -\sum_{m=2}^{\infty} \sum_{s=2}^{m} \int_{2\cdots m} \frac{\hat{s}}{\hat{1}}\, \Xi^{s-1}_{12\cdots m}\, B_{\bar{2}} \cdots \bar{B}_{\bar{s}} \cdots B_{\bar{m}}$$

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Solving $\mathcal{L}^{-+}[A] + \mathcal{L}^{++-}[A] = \mathcal{L}^{-+}[B]$

Summary

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$$\Upsilon(1\cdots n) = -\frac{\hat{1}}{\hat{s}}\Xi^{s-1}(1\cdots n) = -\frac{g^{n-2}}{\sqrt{\hat{2}\hat{n}}} \frac{\hat{1}}{\langle 2\,3\rangle\langle 3\,4\rangle\cdots\langle n{-}1\; n\rangle}$$

Canonical $A \to B$

$$A_1 = \sum_{n=2}^{\infty} \int_{2\cdots n} \Upsilon_{12\cdots n} B_{\bar{2}} \cdots B_{\bar{n}}$$

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$$\mathcal{L} = \mathcal{L}^{-+}[A] + \mathcal{L}^{++-}[A] + \mathcal{L}^{--+}[A] + \mathcal{L}'^{--++}[A]$$

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$$\mathcal{L}^{--+}[A] \sim \mathcal{A}^{--+}\bar{A}\bar{A}A$$

$$\mathcal{L}'^{--++}[A] \sim V^{--++}\bar{A}\bar{A}AA$$

Canonical $A \to B \implies$ MHV Lagrangian

$$A_1 = \sum_{n=2}^{\infty} \int_{2\cdots n} \Upsilon_{12\cdots n}\, B_{\bar{2}} \cdots B_{\bar{n}}$$

$$\bar{A}_1 = -\sum_{m=2}^{\infty} \sum_{s=2}^{m} \int_{2\cdots m} \frac{\hat{s}}{\hat{1}}\, \Xi^{s-1}_{12\cdots m}\, B_{\bar{2}} \cdots \bar{B}_{\bar{s}} \cdots B_{\bar{m}}$$

$$\mathcal{L} = \mathcal{L}^{-+}[B] + \mathcal{L}^{--+}[B] + \mathcal{L}^{--++}[B] + \mathcal{L}^{--+++}[B] + \cdots$$

$$\mathcal{L}^{--+}[A] \sim \mathcal{A}^{--+}\bar{A}\bar{A}A$$

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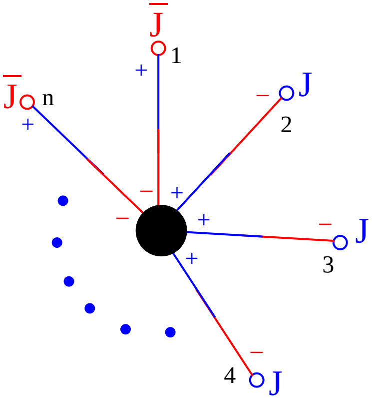
LSZ Reduction → Amputated Amplitude

Source terms: $J\bar{A} + \bar{J}A$



J couples to A & $\bar{J}$ couples to $\bar{A}$ because propagators removed by

$$\lim_{p_j^2 \to 0} \prod_{j=1}^{n} -ip_j^2$$

Equivalence Theorem

Source terms: $J\bar{A} + \bar{J}A$

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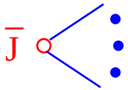

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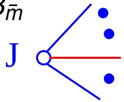
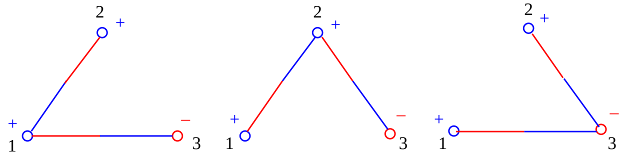


Equivalence Theorem: only $J\bar{B} + \bar{J}B$ survive amputation.

$+ + -$



$$\mathcal{A}(1^+ 2^+ 3^-) \sim \frac{p_1^2}{\hat{1}} \Xi^2_{1\,2\,3} + \frac{p_2^2}{\hat{2}} \Xi^1_{2\,3\,1} + \frac{p_3^2}{\hat{3}} \Upsilon_{1\,2\,3}$$

$$\sim \Upsilon_{123} \left\{ \frac{p_1^2}{\hat{1}} + \frac{p_2^2}{\hat{2}} + \frac{p_3^2}{\hat{3}} \right\}$$

Solving $\mathcal{L}^{-+}[A] + \mathcal{L}^{++-}[A] = \mathcal{L}^{-+}[B]$

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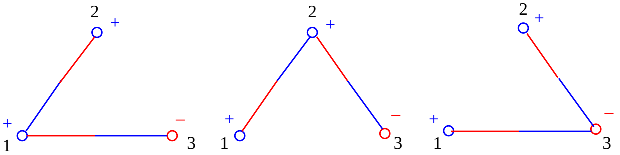


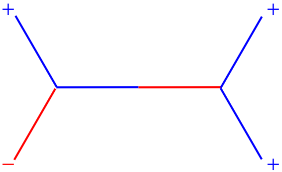
$$\begin{aligned}\mathcal{A}(1^+ 2^+ 3^-) &\sim \frac{p_1^2}{\hat{1}} \Xi^2_{1\,2\,3} + \frac{p_2^2}{\hat{2}} \Xi^1_{2\,3\,1} + \frac{p_3^2}{\hat{3}} \Upsilon_{1\,2\,3} \\ &\sim \Upsilon_{123} \left\{ \frac{p_1^2}{\hat{1}} + \frac{p_2^2}{\hat{2}} + \frac{p_3^2}{\hat{3}} \right\} \\ &= \mathcal{A}^{++-}(123)\end{aligned}$$

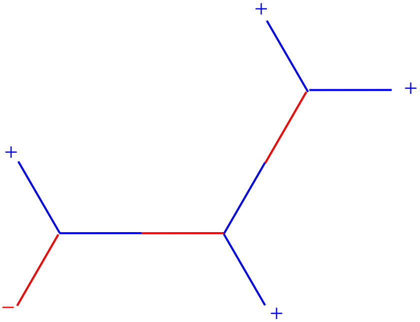
exact off shell relation

Self-dual Yang-Mills: $\mathcal{L}^{-+} + \mathcal{L}^{++-}$

Chalmers & Siegel

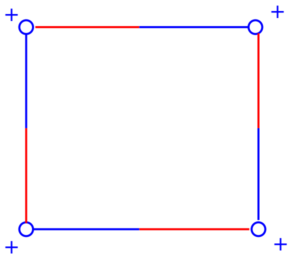
Classically a free theory:



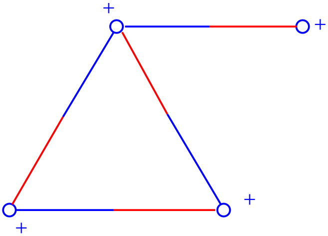


But at one-loop or with complex momenta or $2, 2$ signature or off shell . . .

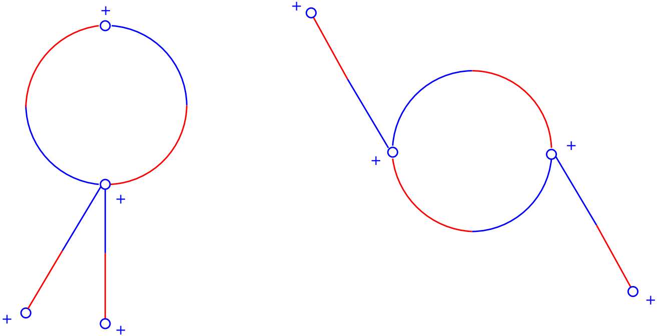
$++++$ Box contributions



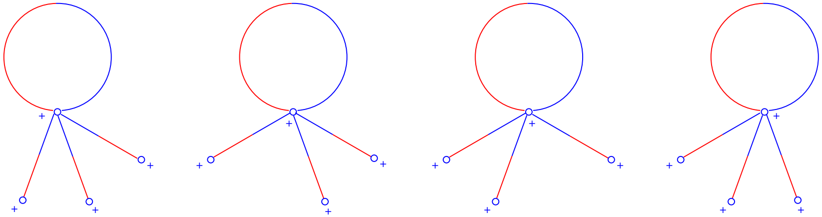
$++++$
Triangle contribution (example)



$+ + + +$
Bubble contributions (examples)



$+ + + +$
Tadpole contributions



Solving $\mathcal{L}^{-+}[A] + \mathcal{L}^{++-}[A] = \mathcal{L}^{-+}[B]$

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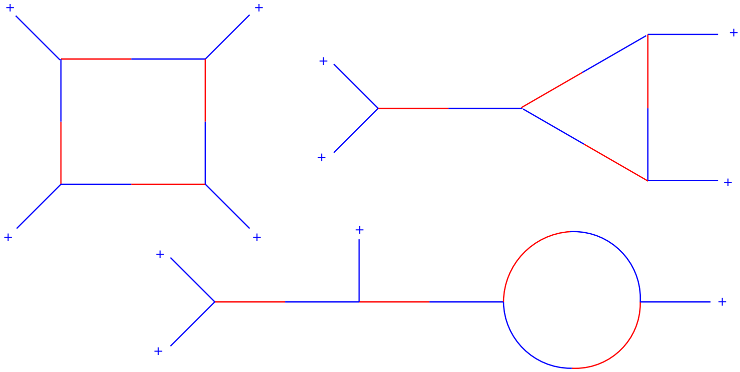
Υ and Ξ have tree expansions $\cdots$

+ + + +
Self-dual Yang-Mills topologies (examples)



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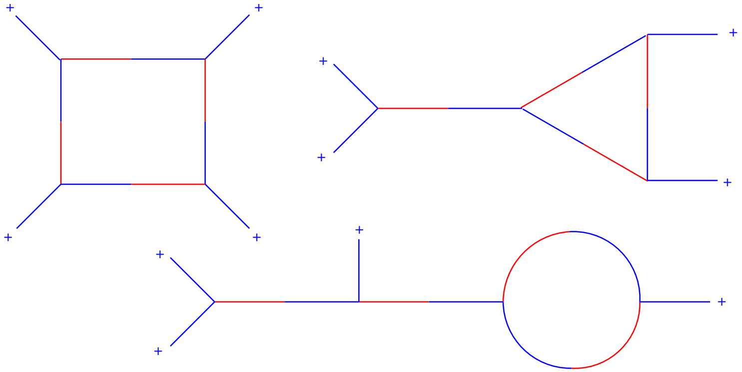
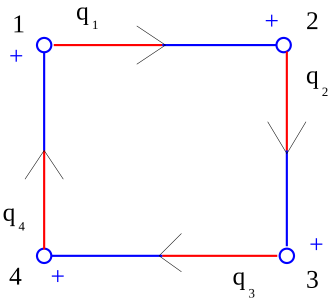


Summing over all contributions gives exactly all such LCYM contributions

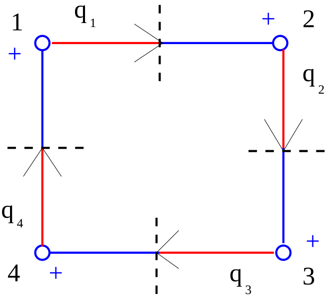
+ + + +
Conventional 4-cut contribution



$$\frac{p_1^2 p_2^2 p_3^2 p_4^2}{\hat{1}\hat{2}\hat{3}\hat{4}} \int \frac{d^D q}{(2\pi)^D} \; \frac{\hat{q}_1 \hat{q}_2 \hat{q}_3 \hat{q}_4}{q_1^2 q_2^2 q_3^2 q_4^2} \Big\{$$

$$\Upsilon(-q_4, 1, q_1)\Upsilon(-q_1, 2, q_2)\Upsilon(-q_2, 3, q_3)\Upsilon(-q_3, 4, q_4)\Big\}$$

+ + + +
Conventional 4-cut contribution



$$\frac{p_1^2 p_2^2 p_3^2 p_4^2}{\hat{1}\hat{2}\hat{3}\hat{4}} \int \frac{d^D q}{(2\pi)^D} \; \frac{\hat{q}_1 \hat{q}_2 \hat{q}_3 \hat{q}_4}{q_1^2 q_2^2 q_3^2 q_4^2} \Big\{$$

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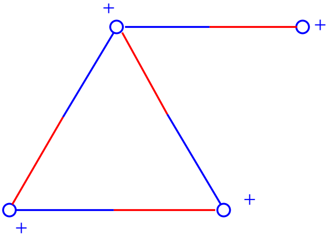
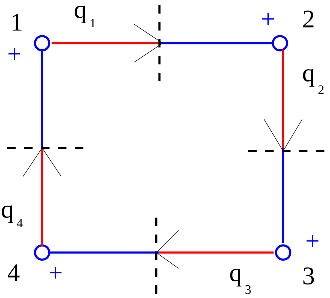


Triangles and lower topologies don't have any

$++++$
Conventional 4-cut contribution



$$\frac{p_1^2 p_2^2 p_3^2 p_4^2}{\hat{1}\hat{2}\hat{3}\hat{4}} \int \frac{d^D q}{(2\pi)^D} \; \frac{\hat{q}_1 \hat{q}_2 \hat{q}_3 \hat{q}_4}{q_1^2 q_2^2 q_3^2 q_4^2} \Big\{$$

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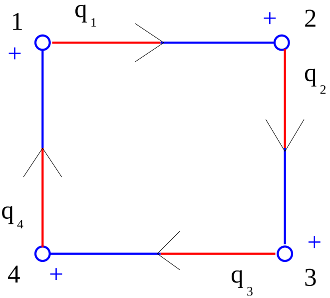
$+ + + +$

Let external momenta go on shell first $\cdots$



$$\frac{p_1^2 p_2^2 p_3^2 p_4^2}{\hat{1}\hat{2}\hat{3}\hat{4}} \int \frac{d^D q}{(2\pi)^D} \; \frac{\hat{q}_1 \hat{q}_2 \hat{q}_3 \hat{q}_4}{q_1^2 q_2^2 q_3^2 q_4^2} \Big\{$$

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+ + + +
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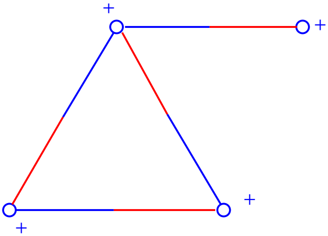
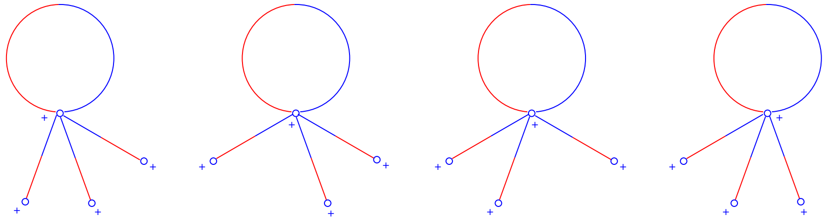


All diagrams vanish except · · ·

$+ + + +$
Let external momenta go on shell first $\cdots$



$$\propto \left\{ \frac{p_1^2}{\hat{1}} + \frac{p_2^2}{\hat{2}} + \frac{p_3^2}{\hat{3}} + \frac{p_4^2}{\hat{4}} \right\}$$

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Loop corrections not well defined: need regularisation

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$$\check{p} \qquad \hat{p} \qquad p_\alpha \qquad \bar{p}_\alpha$$

$4 - 2\epsilon$ dimensional

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$$\check{p} \qquad \hat{p} \qquad p_\alpha \qquad \bar{p}_\alpha$$

$1-\epsilon$ dimensional complex

Light cone Yang-Mills

$$
\begin{aligned}
\mathcal{L}^{-+}[A] &= -\bar{A}_\alpha \Box A_\alpha \\
\mathcal{L}^{++-}[A] &= \bar{\partial}_\alpha \hat{\partial}^{-1} A_\alpha \, [\hat{\partial}\bar{A}_\beta, \, A_\beta] + \bar{\partial}_\alpha \bar{A}_\beta [A_\alpha, \, A_\beta] \\
\mathcal{L}^{--+}[A] &= \partial_\alpha \hat{\partial}^{-1} \bar{A}_\alpha \, [\hat{\partial} A_\beta, \, \bar{A}_\beta] + \partial_\alpha A_\beta [\bar{A}_\alpha, \, \bar{A}_\beta] \\
\mathcal{L}'^{--++}[A] &= [A_\alpha, \, A_\beta][\bar{A}_\alpha, \, \bar{A}_\beta] + [A_\alpha, \, \bar{A}_\beta][\bar{A}_\alpha, \, A_\beta] \\
&\quad -[\hat{\partial} A_\alpha, \, \bar{A}_\alpha] \, \hat{\partial}^{-2} \, [\hat{\partial} A_\beta, \, \bar{A}_\beta] - [\hat{\partial}\bar{A}_\alpha, \, A_\alpha] \, \hat{\partial}^{-2} \, [\hat{\partial}\bar{A}_\beta, \, A_\beta] \\
&\quad -2[\hat{\partial} A_\alpha, \, \bar{A}_\alpha] \, \hat{\partial}^{-2} \, [\hat{\partial}\bar{A}_\beta, \, A_\beta]
\end{aligned}
$$

Equivalence Theorem Evasion

Source terms: $J\bar{A} + \bar{J}A$

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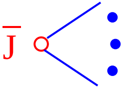
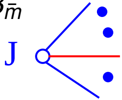


$$\bar{A}_1 = -\sum_{m=2}^{\infty} \sum_{s=2}^{m} \int_{2\cdots m} \frac{\hat{s}}{\hat{1}} \Xi^{s-1}_{12\cdots m} B_{\bar{2}} \cdots \bar{B}_{\bar{s}} \cdots B_{\bar{m}}$$



These extra terms complete the MHV rules

Conclusions

- They put back all removed (parts of) amplitudes
- leading to full equivalence off shell.
- Regularised theory has similar properties but not as neat.
- There are no anomalies in the canonical transformation!
- The *completed* MHV Lagrangian is renormalisable.
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