Introduction to Supersymmetry and Supergravity

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Outline

Lorentz Algebra and Fields

- The Lorentz algebra $SO(3,1)$ has 6 generators:

$$J_1, J_2, J_3 \text{ (rotations)}; \quad K_1, K_2, K_3 \text{ (boosts)}$$

  with

$$[J_1, J_2] = iJ_3, \quad [J_1, K_2] = iK_3, \quad [K_1, K_2] = -iJ_3$$

  and cyclic permutations.

- Taking the combinations:

$$\mathcal{J}_i = \tfrac{1}{2}(J_i + iK_i), \qquad \overline{\mathcal{J}}_i = \tfrac{1}{2}(J_i - iK_i),$$

  we find:

$$[\mathcal{J}_1, \mathcal{J}_2] = i\mathcal{J}_3, \quad [\overline{\mathcal{J}}_1, \overline{\mathcal{J}}_2] = i\overline{\mathcal{J}}_3, \quad [\mathcal{J}, \overline{\mathcal{J}}] = 0$$

- Thus we have two decoupled $SU(2)$ algebras, but one is the complex conjugate of the other.
- This is a remarkable situation. It means we can understand representations of $SO(3,1)$ entirely in terms of familiar $SU(2)$ representations ("spin" J, dimension $2J+1$).
- But we must always keep in mind that one $SU(2)$ is the conjugate of the other:

$$SO(3,1) = SU(2) \times \overline{SU(2)}$$

- The simplest $SO(3,1)$ representation is the singlet of both $SU(2)$'s, described by a scalar field $\phi(x)$.
- Since all other representations of $SU(2)$ are products of the doublet representation, the first nontrivial representations of the Lorentz algebra are:

$$\begin{aligned} \xi_A(x),\ \ A = 1,2: &\qquad SU(2)\ \text{doublet} \\ \xi^A(x) \equiv \epsilon^{AB}\xi_B(x): &\qquad SU(2)\ \text{conjugate doublet} \end{aligned}$$

- Complex-conjugating these gives us:

$$\begin{aligned} \xi^{\dagger\dot{A}}(x) = (\xi^A(x))^\dagger,\ \dot{A} = \dot{1},\dot{2}: &\qquad \overline{SU(2)}\ \text{doublet} \\ \xi^\dagger{}_{\dot{A}}(x) \equiv \epsilon_{\dot{A}\dot{B}}\,\xi^{\dot{B}}(x): &\qquad \overline{SU(2)}\ \text{conjugate doublet} \end{aligned}$$

where:

$$\epsilon^{AB} = \epsilon^{\dot{A}\dot{B}} = -\epsilon_{AB} = -\epsilon_{\dot{A}\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- Finally the simplest mixed representation, transforming under both $SU(2)$ factors, is:

$$A_{A\dot{B}}$$

where A transforms as a doublet of $SU(2)$ and $\dot{B}$ transforms as a conjugate doublet of $\overline{SU(2)}$.

- This is just another description of a vector field.

- Physical interpretation of scalar field: spin $= 0$ and

$$\begin{aligned} \phi(x) : &\quad \text{particle} \\ \phi^\dagger(x) : &\quad \text{antiparticle} \end{aligned}$$

- Physical interpretation of spinor fields: spin $= \frac{1}{2}$ and

$$\begin{aligned} \xi_A(x) : &\quad \text{left-handed fermion} \\ {\xi^\dagger}_{\dot{A}}(x) \equiv (\xi_A(x))^\dagger : &\quad \text{right-handed antifermion} \\ \bar{\xi}_A(x) : &\quad \text{left-handed antifermion} \\ {\bar{\xi}^\dagger}_{\dot{A}}(x) \equiv (\bar{\xi}_A(x))^\dagger : &\quad \text{right-handed fermion} \end{aligned}$$

- Physical interpretation of vector fields: spin $= 1$ and

$$\begin{aligned} A_{A\dot{B}}(x) : &\quad \text{particle} \\ (A_{A\dot{B}})^\dagger(x) : &\quad \text{antiparticle} \end{aligned}$$

- Note that in our conventions, a $\dagger$ denotes complex conjugation but a $\bar{\ }$ denotes an independent field.

- For a particle to be its own antiparticle, special constraints need to be imposed.
- For ϕ and $A_{A\dot{B}}$ the constraint is that they are respectively real and Hermitian. This constraint is preserved by Lorentz transformations.
- For spinors the constraint is as follows. A general ("Dirac") fermion is described by an independent pair of spinors:

$$\xi_A(x), \quad \bar{\xi}^{\dagger\dot{A}}(x)$$

The desired constraint is then $\bar{\xi}_A = \xi_A$. As a result the Dirac spinor reduces to:

$$\xi_A(x), \quad \xi^{\dagger\dot{A}}(x) = \epsilon^{\dot{A}\dot{B}}\, \xi^{\dagger}{}_{\dot{B}}(x)$$

This is called a "Majorana" fermion when referring to the pair, or a "Weyl" fermion when referring to just the left-handed part. Either way, the degrees of freedom are the same.

- To write an action we need to make Lorentz invariants from these fields.
- On fermions, the Lorentz transformations have a complex action:

$$\xi_A(x) \to \xi'_A(x) = U_A^{\ B}\, \xi_B(x)$$
$$\xi^\dagger{}_{\dot{A}}(x) \to \xi^{\dagger\prime}{}_{\dot{A}}(x) = U_{\dot{A}}^{\ \dot{B}}\, \xi^\dagger{}_{\dot{B}}(x)$$

where $\det U = 1$.
- Now $\epsilon^{AB} U_A^{\ C} U_B^{\ D} = (\det U)\, \epsilon^{CD}$, hence the combinations:

$$\xi\xi \equiv \xi^A \xi_A = -\epsilon^{AB} \xi_A \xi_B$$
$$\xi^\dagger \xi^\dagger \equiv \xi^\dagger{}_{\dot{A}} \xi^{\dagger \dot{A}} = \epsilon^{\dot{A}\dot{B}}\, \xi^\dagger{}_{\dot{A}} \xi^\dagger{}_{\dot{B}}$$

are Lorentz invariant.

- Now introduce the matrices:

$$\sigma^{\mu}{}_{A\dot{B}} \equiv (1, \sigma^1, \sigma^2, \sigma^3)$$
$$\overline{\sigma}^{\mu\dot{A}B} \equiv (1, -\sigma^1, -\sigma^2, -\sigma^3)$$

which have the property that:

$\sigma^{\mu}{}_{A\dot{B}}\,\xi^{\dagger\dot{B}}$ transforms as undotted spinor $\otimes$ vector

$\overline{\sigma}^{\mu\dot{A}B}\xi_B$ transforms as dotted spinor $\otimes$ vector

- Finally we define:

$$\sigma^{\mu\nu}{}_A{}^B = \frac{i}{4}(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu)_A{}^B$$
$$\overline{\sigma}^{\mu\nu\dot{A}}{}_{\dot{B}} = \frac{i}{4}(\overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu)^{\dot{A}}{}_{\dot{B}}$$

- Now the action for a Dirac fermion $\xi_A(x), \bar{\xi}^{\dagger\dot{A}}(x)$ is:

$$\mathcal{L}_{\text{Dirac}} = i\,\xi^\dagger\,\bar{\sigma}^\mu\partial_\mu\xi + i\,\bar{\xi}\sigma^\mu\partial_\mu\bar{\xi}^\dagger - m\,\xi\,\bar{\xi} - m\,\xi^\dagger\,\bar{\xi}^\dagger$$

- For a Majorana fermion the identification $\xi = \bar{\xi}$ is made and an extra factor of $\frac{1}{2}$ introduced, giving:

$$\mathcal{L}_{\text{Majorana}} = \frac{i}{2}\,\xi\sigma^\mu\overleftrightarrow{\partial}_\mu\,\xi^\dagger - \frac{m}{2}(\xi\xi + \xi^\dagger\xi^\dagger\,)$$

Outline

Supersymmetry - free fields

- Consider the free massive action for a complex scalar and a Majorana fermion of the same mass:

$$\mathcal{L} = \partial_\mu\phi\,\partial^\mu\phi^\dagger - m^2\phi\,\phi^\dagger + \frac{i}{2}\,\xi\sigma^\mu\overleftrightarrow{\partial}_\mu\xi^\dagger - \frac{m}{2}(\xi\xi + \xi^\dagger\,\xi^\dagger\,)$$

  This action is invariant upto a total derivative under the infinitesimal variations:

$$\begin{aligned} \delta\phi &= \epsilon\,\xi \\ \delta\xi &= -i\sigma^\mu\partial_\mu\phi\,\epsilon^\dagger - m\phi^\dagger\,\epsilon \end{aligned}$$

  where ϵ is a (constant) anticommuting spinor.

- This is a (free) supersymmetric theory with the physical fields ϕ, ξ of a chiral multiplet.
- Note that the propagating degrees of freedom are 2 bosonic + 2 fermionic, equality being a precondition for supersymmetry.

- As another example consider the action for a free massless Weyl fermion and a vector field:

$$\frac{i}{2}\,\lambda\sigma^\mu \overleftrightarrow{\partial}_\mu \lambda^\dagger - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

- This action too is invariant under:

$$\begin{aligned}\delta\lambda &= i\sigma^{\mu\nu}F_{\mu\nu}\epsilon \\ \delta A_\mu &= i(\epsilon\sigma^\mu\lambda^\dagger - \lambda\sigma^\mu\epsilon^\dagger)\end{aligned}$$

- This free supersymmetric action has the physical fields λ, A_μ of a vector multiplet.
- And again, the propagating degrees of freedom are 2 bosonic $+\ 2$ fermionic.

- By repeated application of the supersymmetry variations we find in these simple cases that:

$$(\delta_1\delta_2 - \delta_2\delta_1)\phi = -i(\epsilon_2\sigma^\mu\epsilon_1^\dagger - \epsilon_1\sigma^\mu\epsilon_2^\dagger)\partial_\mu\phi$$

  and similarly on the other fields.
- Thus the commutator of two supersymmetries is a translation in spacetime.
- We will see that abstractly this takes the form:

$$\{Q_A, Q^\dagger{}_{\dot{B}}\} = 2\sigma^\mu{}_{A\dot{B}}P_\mu$$

  where P_μ is the 4-momentum.

Outline

Superspace and superfields

- To introduce interactions, we introduce the superfield formalism where supersymmetry is manifest.
- For this, adjoin to the spacetime coordinates x^μ the anticommuting coordinates $\theta_A, \theta^{\dagger\dot{A}}$.
- A general superfield is a field $f(x, \theta, \theta^\dagger)$ depending on all the coordinates (commuting and anticommuting) of superspace.
- Anticommutation implies $(\theta_A)^2 = (\theta^{\dagger\dot{A}})^2 = 0$. Hence the superfield can be Taylor-expanded with a finite number of terms:

$$
\begin{aligned}
f(x, \theta, \theta^\dagger) &= \phi(x) + \theta\xi(x) + \theta^\dagger\chi^\dagger(x) + \theta\theta M(x) + \theta^\dagger\theta^\dagger N(x) \\
&\quad + \theta\sigma^\mu\theta^\dagger A_\mu(x) + \theta\theta\,\theta^\dagger\zeta^\dagger(x) + \theta^\dagger\theta^\dagger\theta\lambda(x) + \theta\theta\,\theta^\dagger\theta^\dagger D(x)
\end{aligned}
$$

- We see that a superfield encodes a finite number of ordinary fields. These will form a supermultiplet.
- In superspace, a supersymmetry transformation acts as:

$$\begin{aligned} \theta &\to \theta + \epsilon \\ \theta^\dagger &\to \theta^\dagger + \epsilon^\dagger \\ x^\mu &\to x^\mu + i(\epsilon\sigma^\mu\theta^\dagger - \theta\sigma^\mu\epsilon^\dagger) \end{aligned}$$

- On superfields, this action is generated by:

$$\delta f(x,\theta,\theta^\dagger) = i(\epsilon Q + \epsilon^\dagger Q^\dagger) f(x,\theta,\theta^\dagger)$$

where

$$Q_A \equiv -i\left(\frac{\partial}{\partial\theta^A} + i\sigma^\mu_{A\dot{B}}\theta^{\dagger\dot{B}}\partial_\mu\right), \; Q^\dagger{}_{\dot{A}} \equiv i\left(\frac{\partial}{\partial\theta^{\dagger\dot{A}}} + i\theta^B\sigma^\mu_{B\dot{A}}\partial_\mu\right)$$

satisfying

$$\{Q_A, Q^\dagger{}_{\dot{B}}\} = 2i\sigma^\mu_{A\dot{B}}\partial_\mu$$

- The supermultiplet $f(x, \theta, \theta^\dagger)$ turns out to be reducible.
- Hence we must impose a covariant constraint (i.e. preserving Lorentz invariance and supersymmetry).
- One such constraint requires the introduction of operators that anticommute with the supercharges $Q_A, Q^\dagger{}_{\dot{A}}$:

$$\mathcal{D}_A \equiv \frac{\partial}{\partial \theta^A} - i\sigma^\mu{}_{A\dot{B}}\theta^{\dagger\dot{B}}\partial_\mu, \quad \mathcal{D}^\dagger{}_{\dot{A}} \equiv -\frac{\partial}{\partial \theta^{\dagger\dot{A}}} + i\theta^B\sigma^\mu{}_{B\dot{A}}\partial_\mu$$

with

$$\{\mathcal{D}_A, Q_B\} = \{\mathcal{D}^\dagger{}_{\dot{A}}, Q_B\} = \{\mathcal{D}_A, Q^\dagger{}_{\dot{B}}\} = \{\mathcal{D}^\dagger{}_{\dot{A}}, Q^\dagger{}_{\dot{B}}\} = 0$$

- We define:

$$\mathcal{D}^\dagger{}_{\dot{A}}\Phi(x, \theta, \theta^\dagger) = 0 \text{ (chiral)}, \quad \mathcal{D}_A\Phi^\dagger(x, \theta, \theta^\dagger) = 0 \text{ (anti-chiral)}$$

- A chiral superfield contains the fields $\phi(x), \xi(x)$ of a chiral multiplet, along with a complex auxiliary field $F(x)$.

- The other possible covariant constraint is:

$$V(x,\theta,\theta^\dagger) = V^\dagger(x,\theta,\theta^\dagger) \;\rightarrow\; V \text{ is a vector superfield}$$

- A vector superfield contains the fields $\lambda(x), A_\mu(x)$ of a vector multiplet along with several auxiliary fields.
- A suitable choice of gauge freedom using:

$$V \rightarrow V + i(\Lambda - \Lambda^\dagger)$$

where Λ is a chiral superfield, puts it in the Wess-Zumino gauge in which there is a single real auxiliary field $D(x)$.

- The Taylor expansion of a chiral superfield is:

$$\Phi(y, \theta) = \phi(y) + \sqrt{2}\,\theta\xi(y) + \theta\theta F(y)$$

  where $y^\mu = x^\mu - i\theta\sigma^\mu\theta^\dagger$.
- If ξ represents a conventional fermion (quark/lepton) then ϕ will be a sfermion (squark/slepton).
- Similarly the Taylor expansion of a vector superfield in the Wess-Zumino gauge is:

$$\begin{aligned} V(x, \theta, \theta^\dagger) &= \theta\sigma^\mu\theta^\dagger A_\mu(x) + \theta\theta\,\theta^\dagger\lambda^\dagger(x) + \theta^\dagger\theta^\dagger\theta\lambda(x) \\ &\quad + \tfrac{1}{2}\,\theta\theta\,\theta^\dagger\theta^\dagger D(x) \end{aligned}$$

- Since A_μ is a gauge field, λ is called a gaugino.

Outline

Supersymmetric actions

- Now we can write actions for interacting supersymmetric field theories.
- For this we need the rules for integrating over anticommuting coordinates:

$$\int d\theta_1\, \theta_1 = \int d\theta_2\, \theta_2 = 1$$

$$\int d\theta_1 = \int d\theta_2 = 0$$

and similarly for the dotted coordinates.
- Define $d^2\theta = \frac{1}{2}d\theta_1 d\theta_2$. Then:

$$\int d^2\theta\ \theta\theta = \int \tfrac{1}{2}d\theta_1 d\theta_2\,.\,2\theta_2\theta_1 = 1$$

- Under the supersymmetry transformations we described, the F-term of a chiral superfield and the D-term of a vector superfield transform as total derivatives. Therefore their variations vanish under $\int d^4x$.
- The proof runs as follows:

$$
\begin{aligned}
\delta V \;\sim\; \epsilon Q V(x,\theta,\theta^\dagger) \;&\sim\; \epsilon\left(\partial_\theta + i\sigma^\mu\theta^\dagger\partial_\mu\right)\left(\theta\sigma^\mu\theta^\dagger A_\mu\right.\\
&\quad \left. + \theta\theta\,\theta^\dagger\lambda^\dagger + \theta^\dagger\theta^\dagger\theta\lambda + \tfrac{1}{2}\,\theta\theta\,\theta^\dagger\theta^\dagger D\right)\\
&= \cdots + (i\epsilon\sigma^\mu\theta^\dagger)\theta\theta\,\theta^\dagger\partial_\mu\lambda^\dagger
\end{aligned}
$$

- Thus the D-term transforms as a total derivative. A similar proof holds for the F-term.

- Now $\int d^2\theta$ on a chiral superfield, and $\int d^2\theta d^2\theta^\dagger$ on a vector superfield, pick out the "last" terms (F-term and D-term respectively).
- Combining the above facts, general actions of the form:

$$\int d^4x \, d^2\theta \ \text{(chiral superfield)}, \ \int d^4x \, d^2\theta d^2\theta^\dagger \ \text{(vector superfield)}$$

are automatically supersymmetric.

- As the canonical dimension of θ is $-\frac{1}{2}$, and that of a chiral superfield Φ is $+1$, one possible free (quadratic) action is:

$$S_D = \int d^4x\, d^2\theta\, d^2\theta^\dagger \quad \Phi^\dagger(y^\dagger, \theta^\dagger)\, \Phi(y, \theta)$$

  where recall that $y^\mu = x^\mu - i\theta\sigma^\mu\theta^\dagger$.
- Another supersymmetric quadratic action for a chiral superfield is:

$$S_F = \int d^4x\, d^2\theta \quad \Phi(y, \theta)^2 \; + \; \text{c.c.}$$

- Note that under $\int d^4x$ we can first shift $y^\mu \to x^\mu$ in the integrand and thereby remove the $\theta^\dagger$ dependence. After this, $\int d^2\theta$ picks out the term proportional to $\theta\theta$ as desired.

- It is straightforward to show that:

$$\int d^4\theta\; \Phi^\dagger(y)\,\Phi(y) \;+\; \tfrac{1}{2}\left(m\int d^2\theta\; \Phi(y)^2 \;+\; \text{c.c.}\right) =$$

$$\partial_\mu\phi\,\partial^\mu\phi^\dagger + F\,F^\dagger \;+\; \tfrac{i}{2}\,\xi\sigma^\mu\overleftrightarrow{\partial}_\mu\xi^\dagger$$

$$+\;\tfrac{1}{2}\,m\,(2\phi F - \xi\xi) \;+\; \tfrac{1}{2}\,m^*(2\phi^\dagger F^\dagger - \xi^\dagger\,\xi^\dagger)$$

  where all fields on the RHS depend on x.

- Eliminating the auxiliary field F by its equation of motion:

$$F = -m^*\phi^\dagger$$

  we finally arrive at the familiar free action:

$$S = \int d^4x\left\{\partial_\mu\phi\,\partial^\mu\phi^\dagger - |m|^2\phi\,\phi^\dagger + \tfrac{i}{2}\,\xi\sigma^\mu\overleftrightarrow{\partial}_\mu\xi^\dagger - \tfrac{m}{2}(\xi\xi + \xi^\dagger\,\xi^\dagger\,)\right\}$$

- The payoff is that we can now generalise this to an interacting supersymmetric action.
- For a set of chiral superfields $\Phi^i, i = 1, 2, \cdots, n$, the most general action is:

$$S = \int d^4x\, d^4\theta\, K(\Phi^i, \Phi^{\dagger i}) + \int d^4x \Big(d^2\theta\, \mathcal{W}(\Phi^i) + \text{c.c.} \Big)$$

- Here K is an arbitrary real function of $\Phi^i, \Phi^{\dagger i}$, called the Kähler potential, and $\mathcal{W}$ is an arbitrary analytic function of Φ^i alone, called the superpotential.
- For renormalisability, $K = \sum_i \Phi^{\dagger i}\Phi^i$ and $\mathcal{W}$ is a cubic polynomial. But one often talks of supersymmetric effective actions, for which renormalisability is not a requirement.

- Instead of writing the full interacting action in terms of ordinary fields, we just note that the Kähler potential and superpotential have the following effects:

$$\begin{aligned} \partial_\mu\phi\,\partial^\mu\phi^\dagger &\rightarrow G_{i\bar{j}}\,\partial_\mu\phi^i\partial^\mu\phi^{\dagger j} \\ -|m|^2\phi\,\phi^\dagger &\rightarrow -G^{i\bar{j}}\,\mathcal{W}_{,i}\mathcal{W}^\dagger{}_{,\bar{j}} \\ -\frac{m}{2}(\xi\xi+\xi^\dagger\xi^\dagger) &\rightarrow -\frac{1}{2}(\mathcal{W}_{,ij}\,\xi^i\xi^j+\text{c.c.}) \end{aligned}$$

where $G_{i\bar{j}}(\phi,\phi^\dagger)=K_{,i\bar{j}}$ is the Kähler metric on field space, and $G^{i\bar{j}}$ is its inverse.

- Now consider a single vector superfield $V(x, \theta, \theta^\dagger)$. We need an action that is gauge invariant under:

$$V \to V + i(\Lambda - \Lambda^\dagger)$$

- This has the effect of transforming the vector field A_μ in V as:

$$A_\mu \to A_\mu + \partial_\mu \beta$$

where β is real and $\frac{1}{2}\beta$ is the lowest component of Λ.
- It is convenient to define the field-strength superfield:

$$W_A = -\tfrac{1}{4}\mathcal{D}^\dagger \mathcal{D}^\dagger \mathcal{D}_A V$$

which is gauge invariant.

- W_A is a chiral superfield and has the expansion:

$$W_A(y,\theta) = \lambda_A(y)+\theta_A D(y)-(\sigma^{\mu\nu}\theta)_A F_{\mu\nu}(y)+i\theta\theta\,(\sigma^\mu\partial_\mu\lambda^\dagger(y))_A$$

- The free super-electrodynamics Lagrangian can now be written in terms of W:

$$\begin{aligned} S &= \frac{1}{16\pi}\left(-i\tau\int d^2\theta\, W^A W_A + \text{c.c.}\right) \\ &= \frac{1}{g^2}\left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{i}{2}\lambda\sigma^\mu\overleftrightarrow{\partial}_\mu\lambda^\dagger + \tfrac{1}{2}D^2\right) - \frac{\vartheta}{32\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}\end{aligned}$$

where

$$\tau = \frac{\vartheta}{2\pi} + i\frac{4\pi}{g^2}$$

and this ϑ is not to be confused with the superspace coordinates θ.

- Next let us couple vector superfields to (charged) chiral superfields.
- If g is the $U(1)$ coupling constant and t_i the $U(1)$ charge of the chiral superfield Φ^i then the superfield undergoes gauge transformations as:

$$\Phi^i(y,\theta) \to e^{-2ig\,t_i\Lambda(y,\theta)}\,\Phi^i(y,\theta)$$

- Hence:

$$\Phi^{\dagger i}\Phi^i \to \Phi^{\dagger i}e^{-2ig\,t_i(\Lambda-\Lambda^\dagger)}\,\Phi^i$$

  and therefore the combination:

$$\Phi^{\dagger i}e^{2g\,t_i V}\,\Phi^i$$

  is gauge invariant. (Here we are taking the Kähler potential to be trivial.)

- Another possible coupling of vector and chiral superfields occurs via the gauge kinetic function $\mathcal{F}(\Phi)$. This function replaces the coupling constant τ in the kinetic term of the vector superfields:

$$\mathrm{Im}\left(\tau \int d^2\theta \, WW\right) \to \mathrm{Im}\left(\int d^2\theta \, \mathcal{F}(\Phi) \, WW\right)$$

- Such a coupling is always non-renormalisable, so we ignore it – along with the possibility of a nontrivial Kähler potential – until we discuss supergravity.

- Finally, there is one more possible gauge-invariant and supersymmetric interaction:

$$2\eta \int d^2\theta \, d^2\theta^\dagger \, V(x, \theta, \theta^\dagger) = \eta \, D(x)$$

  called the Fayet-Iliopoulos D-term.
- This harmless-looking term has an important effect when we eliminate the auxiliary field D after coupling vector and chiral multiplets.

- The full (Abelian) Lagrangian is then:

$$\int d^2\theta d^2\theta^\dagger \;\; \Phi^{\dagger i} e^{2g\, t_i V}\, \Phi^i + \frac{1}{8\pi}\mathrm{Im}\left(\tau \int d^2\theta\, WW\right)$$
$$+ 2\,\mathrm{Re}\left(\int d^2\theta\; \mathcal{W}(\Phi^i)\right) + 2\,\eta \int d^2\theta d^2\theta^\dagger\; V$$

- Next we go to component fields and eliminate the auxiliary fields D, F via their equations of motion:

$$F_i = -\mathcal{W}_{,\bar{i}} \qquad D = -(\eta + g\, t_i|\phi^i|^2)$$

- Then we get:

$$-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{i}{2}\lambda\sigma^\mu \overleftrightarrow{\partial}_\mu \lambda^\dagger - \tfrac{\vartheta}{32\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} - \tfrac{1}{2}\,\left(\eta + g\, t_i|\phi^i|^2\right)^2$$
$$+ |(\partial_\mu + ig\, t_i A_\mu)\phi^i|^2 + i\xi^i\,\sigma^\mu(\partial_\mu - ig\, t_i A_\mu)\,\xi^{\dagger\,\bar{i}}$$
$$- |\mathcal{W}_{,i}|^2 - \left(\tfrac{1}{2}\xi^i\xi^j\mathcal{W}_{,ij} + \text{c.c.}\right) - \sqrt{2}g\, t_i\left(\lambda^\dagger\,\xi^{\dagger i}\phi^i + \text{c.c.}\right)$$

- We learn the important fact that there are two sources of scalar self-couplings and two sources of Yukawa couplings.

- The most general renormalisable superpotential is cubic in fields:

$$\mathcal{W}(\Phi_i) = a + b_i\Phi^i + \frac{1}{2}c_{ij}\Phi^i\Phi^j + \frac{1}{3!}d_{ijk}\Phi^i\Phi^j\Phi^k$$

- With this, the full scalar potential of the theory is:

$$\mathcal{V}(\phi^i, \phi^{\dagger i}) = \left|b_i + c_{ij}\phi^j + \frac{1}{2}d_{ijk}\phi^j\phi^k\right|^2 + \frac{1}{2}\left(\eta + g t_i|\phi^i|^2\right)^2$$

- For a non-Abelian theory based on a group G, we generalise the vector superfield to a matrix-valued field:

$$\mathbf{V}_{IJ} = 2g \sum_{a=1}^{\text{dim } G} V^a \mathbf{T}^a_{IJ}$$

  where $\mathbf{T}^a$ are matrices of the algebra in some representation $\mathcal{R}$. They satisfy:

$$[\mathbf{T}^a, \mathbf{T}^b] = i f^{abc} \mathbf{T}^c$$

- The gauge transformation superfield must be similarly generalised:

$$\mathbf{\Lambda}_{IJ} = 2g \sum_{a=1}^{\text{dim } G} \Lambda^a \mathbf{T}^a_{IJ}$$

- Then a gauge transformation acts on $\mathbf{V}$ as:

$$e^{\mathbf{V}} \rightarrow e^{-i\mathbf{\Lambda}^\dagger} e^{\mathbf{V}} e^{i\mathbf{\Lambda}}$$

- To show this has the right action on the gauge fields in $\mathbf{V}$, we must restrict the gauge parameter superfield $\mathbf{\Lambda}$ to be of the Wess-Zumino gauge-preserving form:

$$\mathbf{\Lambda} = \frac{1}{2}\boldsymbol{\beta} - \frac{i}{2}\,\theta\sigma^\mu\theta^\dagger\partial_\mu\boldsymbol{\beta} - \frac{1}{8}\theta\theta\theta^\dagger\theta^\dagger\,\Box\boldsymbol{\beta}$$

- In this case one finds:

$$\begin{aligned} \delta\mathbf{A}_\mu &= \partial_\mu\boldsymbol{\beta} + ig[\mathbf{A}_\mu, \boldsymbol{\beta}] \\ \delta\boldsymbol{\lambda}_A &= ig[\boldsymbol{\lambda}_A, \boldsymbol{\beta}] \end{aligned}$$

- Not surprisingly, the gaugino is in the adjoint representation of the gauge group.

- Now we must define the field strength superfield, generalising $W_A = -\frac{1}{4}\mathcal{D}^\dagger\mathcal{D}^\dagger\mathcal{D}_A V$ in the Abelian case.
- It turns out the right generalisation is:

$$\mathbf{W}_A = -\tfrac{1}{4}\mathcal{D}^\dagger\mathcal{D}^\dagger\, e^{-\mathbf{V}}\, \mathcal{D}_A\, e^{\mathbf{V}}$$

- Under gauge transformations on $\mathbf{V}$, this transforms as:

$$\mathbf{W}_A \to e^{-i\boldsymbol{\Lambda}}\, \mathbf{W}_A\, e^{i\boldsymbol{\Lambda}}$$

- With this definition we find that the component expansion of $\mathbf{W}_A$ is:

$$\mathbf{W}_A(y,\theta) = \boldsymbol{\lambda}_A(y) + \mathbf{D}(y)\theta_A - (\sigma^{\mu\nu}\theta)_A \mathbf{F}_{\mu\nu}(y) + i\theta\theta(\sigma^\mu D_\mu \boldsymbol{\lambda}^\dagger_A(y))$$

where

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + ig[\mathbf{A}_\mu, \mathbf{A}_\nu]$$

- The gauge invariant, supersymmetric action for matter coupled to non-Abelian gauge fields is then:

$$\int d^2\theta d^2\theta^\dagger\, \Phi^{\dagger I}\left(e^{\mathbf{V}}\right)_{IJ}\Phi^J + \frac{1}{32\pi T(\mathcal{R})}\mathrm{Im}\ \mathrm{tr}\left(\tau\int d^2\theta\ \mathbf{W}\mathbf{W}\right) + 2\,\mathrm{Re}\left(\int d^2\theta\ \mathcal{W}(\Phi^I)\right)$$

  where now the indices I, J on chiral superfields incorporate not only representation labels but also all other labels such as flavour.

- Note that we can add gauge-invariant D-terms only if there are $U(1)$ factors in the gauge group. If the gauge group is

  $$G = G_{(1)} \otimes G_{(2)} \otimes \cdots G_{(n)} \otimes U(1)_{(1)} \otimes U(1)_{(2)} \otimes \cdots U(1)_{(m)}$$

  where the G_i are all simple, then the D-term is:

$$\sum_{h=1}^{m}\eta^{(h)}\int d^2\theta d^2\theta^\dagger\ D^{(h)}$$

- In ordinary space many of the interactions are what we would expect, for example gauge-covariant kinetic terms for the scalars and fermions in Φ^I and for the gauge fields and gauginos in V_{IJ}.
- The Yukawa couplings, as for the Abelian case, are of two types:

$$\tfrac{1}{2}(\xi^I \xi^J \mathcal{W}_{,IJ} + \text{c.c.}) - \sqrt{2}\, g(\xi^{\dagger I} \lambda^{\dagger}_{IJ} \phi^J + \text{c.c.})$$

- Finally, the scalar potential is:

$$F^I F^{\dagger I} + \tfrac{1}{2}(D^a D^a + D^h D^h)$$

where a runs over a non-Abelian gauge group and h runs over Abelian factors.

- Here,

$$F^I = -\mathcal{W}^{,\bar{I}} \quad D^a = -g\phi^{\dagger I} T^a_{IJ} \phi^{\dagger J} \quad D^h = -(\eta^h + g_h t_I \phi^{\dagger I} \phi^I)$$

Outline

Non-renormalisation theorems

- Having developed the classical formalism of supersymmetry, we may ask what consequences this has for the quantum theory.
- Although a detailed discussion is outside the scope of the present lectures, here we will summarise some relevant results.
- Supersymmetric perturbation theory can be carried out via "super-Feynman rules" for superfields.
- Examining general Feynman diagrams, it is possible to demonstrate the following "non-renormalisation" theorem:

  "Perturbative corrections to the quantum effective action in a supersymmetric theory are always given as $\int d^2\theta d^2\theta^\dagger$ of a local function of superfields and their derivatives."

- One consequence of this theorem is that the minimum value of the effective potential on a classically supersymmetric configuration is unchanged by (perturbative) quantum corrections.
- A special case of this is that the vacuum energy is not renormalised in perturbation theory.
- Because the supersymmetry algebra in the rest frame is:

$$\{Q_A, {Q^\dagger}_{\dot{B}}\} = 2\delta_{A\dot{B}}\, E$$

it follows that:

$$Q_A|0\rangle,\ {Q^\dagger}_{\dot{B}}|0\rangle \quad \Leftrightarrow \quad E = 0$$

- The LHS vanishes if and only if supersymmetry is not spontaneously broken. Hence we learn that if supersymmetry is unbroken at tree level, a supersymmetry-breaking F-term cannot be generated in perturbation theory.

- Another application is to the issue of naturalness.
- In the SM, corrections to the Higgs mass are quadratically divergent. This is the root of the naturalness problem in grand unified theories.
- With supersymmetry, the non-renormalisation theorem says that perturbatively, superpotential terms receive no quantum corrections.
- As an example, $\int d^2\theta \, m \, \Phi^2$ is uncorrected.
- This does not mean there is no mass renormalisation in these theories! Rather, it means that the mass renormalisation is equal and opposite to the wave-function renormalisation, which comes from D-terms and is logarithmic.
- This explains the absence of quadratic divergences, at least from F-terms.
- For suitably "soft" explicit supersymmetry breaking, the dependence of the Higgs mass on heavy particle masses can be made to remain logarithmic rather than quadratic.

Outline

The MSSM

- We can now introduce the Minimal Supersymmetric Standard Model (MSSM).
- The motivations for this lie in the quantum properties of supersymmetric field theories, namely solution or amelioration of the naturalness problem.
- In the MSSM, all left-handed fermions and left-handed antifermions of the SM are promoted to chiral superfields:

$$L_1 = \begin{pmatrix} L_{\nu_e} \\ L_e \end{pmatrix}, \overline{E}_1, \overline{N}_1;\ L_2 = \begin{pmatrix} L_{\nu_\mu} \\ L_\mu \end{pmatrix}, \overline{E}_2, \overline{N}_2;\ L_3 = \begin{pmatrix} L_{\nu_\tau} \\ L_\tau \end{pmatrix}, \overline{E}_3, \overline{N}_3$$

$$Q_1 = \begin{pmatrix} Q_u \\ Q_d \end{pmatrix}, \overline{U}_1, \overline{D}_1;\ Q_2 = \begin{pmatrix} Q_c \\ Q_s \end{pmatrix}, \overline{U}_2, \overline{D}_2;\ Q_3 = \begin{pmatrix} Q_t \\ Q_b \end{pmatrix}, \overline{U}_3, \overline{D}_3$$

- We will henceforth ignore the existence of left-handed anti-neutrinos $\overline{N}_i$ since technically they are not part of the MSSM.

- The component expansions of these superfields are, for example:

$$
\begin{aligned}
L_e(y,\theta) &= \tilde{e}(y) + \sqrt{2}\theta\, e(y) + \cdots \\
\overline{E}_1(y,\theta) &= \tilde{\bar{e}}(y) + \sqrt{2}\theta\, \bar{e}(y) + \cdots
\end{aligned}
$$

- The scalar components of these superfields are the left selectron $\tilde{e}$ and the left spositron $\tilde{\bar{e}}$.
- Being scalars they do not have a handedness. The term left refers not to their chirality but to that of their superpartners.
- Also, clearly $\tilde{\bar{e}}$ is not the antiparticle of $\tilde{e}$.
- The antiparticles are found in the anti-chiral superfields:

$$
\begin{aligned}
L_e^\dagger(y^\dagger,\theta^\dagger) &= \tilde{e}^\dagger(y^\dagger) + \sqrt{2}\theta^\dagger\, e^\dagger(y^\dagger) + \cdots \\
\overline{E}_1^\dagger(y^\dagger,\theta^\dagger) &= \tilde{\bar{e}}^\dagger(y^\dagger) + \sqrt{2}\theta^\dagger\, \bar{e}^\dagger(y^\dagger) + \cdots
\end{aligned}
$$

with $\tilde{e}^\dagger$ being the right spositron and $\tilde{\bar{e}}^\dagger$ the right selectron.

- Similarly the quark chiral superfields contain left squarks and left anti-squarks:

$$\begin{aligned} Q_u(y,\theta) &= \tilde{u}(y) + \sqrt{2}\theta\, u(y) + \cdots \\ \overline{U}_1(y,\theta) &= \tilde{\bar{u}}(y) + \sqrt{2}\theta\, \bar{u}(y) + \cdots \end{aligned}$$

- The corresponding anti-chiral superfields contain the right anti-squarks and the right squarks.

$$\begin{aligned} Q_u^\dagger(y^\dagger,\theta^\dagger) &= \tilde{u}^\dagger(y^\dagger) + \sqrt{2}\theta^\dagger\, u^\dagger(y^\dagger) + \cdots \\ \overline{U}_1^\dagger(y^\dagger,\theta^\dagger) &= \tilde{\bar{u}}^\dagger(y^\dagger) + \sqrt{2}\theta^\dagger\, \bar{u}^\dagger(y^\dagger) + \cdots \end{aligned}$$

- Next we extend the gauge fields of the SM to vector superfields:

$$\begin{aligned} B_\mu(x) &\rightarrow V_Y(x,\theta,\theta^\dagger) \\ W^a_\mu(x) &\rightarrow V^a_W(x,\theta,\theta^\dagger), \quad a=1,2,3 \\ g^{a'}_\mu(x) &\rightarrow V^{a'}_g(x,\theta,\theta^\dagger), \quad a'=1,2,\cdots,8 \end{aligned}$$

- The $SU(2)$ and $SU(3)$ vector superfields are converted to matrices $\mathbf{V}_W, \mathbf{V}_g$ as described earlier.
- The component expansions of these superfields contain fermion partners called gauginos:

$$\begin{aligned} V_Y(x,\theta,\theta^\dagger) &= \theta\sigma^\mu\theta^\dagger\, B_\mu(x) + \theta\theta\,\theta^\dagger\lambda^\dagger_B(x) + \theta^\dagger\theta^\dagger\,\theta\lambda_B(x) + \cdots \\ \mathbf{V}_W(x,\theta,\theta^\dagger) &= \theta\sigma^\mu\theta^\dagger\, \mathbf{W}_\mu(x) + \theta\theta\,\theta^\dagger\boldsymbol{\lambda}^\dagger_W(x) + \theta^\dagger\theta^\dagger\,\theta\boldsymbol{\lambda}_W(x) + \cdots \\ \mathbf{V}_g(x,\theta,\theta^\dagger) &= \theta\sigma^\mu\theta^\dagger\, \mathbf{g}_\mu(x) + \theta\theta\,\theta^\dagger\boldsymbol{\lambda}^\dagger_g(x) + \theta^\dagger\theta^\dagger\,\theta\boldsymbol{\lambda}_g(x) + \cdots \end{aligned}$$

- $\lambda_B, \boldsymbol{\lambda}_W, \boldsymbol{\lambda}_g$ are known as binos, winos and gluinos respectively.

- Finally, we need to introduce chiral superfields for the Higgs boson.
- Here we encounter a novel feature of supersymmetry: while in the SM a single Higgs doublet gives masses to all quarks and leptons, in the MSSM that is not possible.
- The reason is that chiral superfields couple through a complex analytic superpotential.
- Thus out of the SM couplings:

$$
\begin{aligned}
-f^{(u)}(q \cdot \phi)\ \bar{u} - f^{*\,(u)}(q^\dagger \cdot \phi^\dagger)\ \bar{u}^\dagger \\
-f^{(d)}(q \cdot \phi^\dagger)\ \bar{d} - f^{*\,(d)}(q^\dagger \cdot \phi)\ \bar{d}^\dagger
\end{aligned}
$$

  where q is the first-generation quark doublet and ϕ is the Higgs doublet, only the first line can be promoted to a superfield coupling.
- The second line, where $\phi^\dagger$ couples to left-handed fermions, cannot be made into a superfield coupling.

- For this reason there have to be two Higgs doublets in the MSSM (of course there could be more than two, but that would not be the MSSM).
- One gives mass to the down quark (and down squark) while the other gives mass to the up quark (and up squark).
- Thus:

  $$-f^{(u)}(q \cdot \phi)\,\bar{u} \;\rightarrow\; -\,f^{(u)}(Q_1 \cdot H_2)\,\overline{U}_1$$

  where H_2 is a Higgs chiral superfield of hypercharge $+1$ (like the SM Higgs). It is also called the "up-type" Higgs.
- Next, introduce a Higgs H_1 of hypercharge -1 and couple it by replacing:

  $$-f^{(d)}(q \cdot \phi^{\dagger})\,\bar{d} \;\rightarrow\; -\,f^{(d)}(Q_1 \cdot H_1)\,\overline{D}_1$$

  H_1 is also called the "down-type" Higgs.

- The component expansion of the Higgs superfields is:

$$H_I(y,\theta) = h_I(y) + \sqrt{2}\theta\,\tilde{h}_I(y) + \cdots\,, \quad I = 1,2$$

  and the fermions $\tilde{h}_I$ are called Higgsinos.
- The Higgsinos $\tilde{h}_1$ and $\tilde{h}_2$ are chiral fermions of hypercharges $-1, +1$ respectively.
- Being chiral, they contribute to the triangle anomaly for the hypercharge, but because they carry equal and opposite hypercharge, the anomalies cancel out:

$$\sum_{\tilde{h}} Y_{\tilde{h}}^3 = 0$$

- This provides an independent reason for two Higgs doublets, and puts a constraint on extensions of the MSSM.

- Anticipating the Higgs mechanism, the charge assignments of the two Higgs doublets have to be:

$$H_1 = \begin{pmatrix} H_1^0 \\ H_1^- \end{pmatrix}, \quad H_2 = \begin{pmatrix} H_2^+ \\ H_2^0 \end{pmatrix}$$

- Let the scalar components of the superfields H_1, H_2 be $h_1(x), h_2(x)$. The possible VEV's they can develop, consistent with charge conservation, are:

$$\langle h_1 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} v_1 \\ 0 \end{pmatrix}, \quad \langle h_2 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_2 \end{pmatrix}$$

- It is possible to rotate the fields so that both VEV's are real. Then we define:

$$\tan\beta = \frac{v_2}{v_1}$$

and this is the important β-parameter of the MSSM.

- We have discussed the results of the Higgs mechanism, but not yet written down the Higgs potential that achieves this.
- Actually, except for a few terms, we haven't yet written down any interactions of the MSSM!
- Partly this is because of supersymmetry breaking. Because supersymmetry must be broken in nature, we need to consider interactions that break it spontaneously or explicitly.
- Supersymmetry breaking will have some interplay with the rest of the MSSM and in particular the Higgs mechanism.

- Keeping that in mind, we start by writing down the action of the supersymmetric part of the MSSM:

$$S_{\text{susy}} = S_{\text{gauge}} + S_{\text{matter}} + S_{\text{Higgs}}$$

- Here,

$$S_{\text{gauge}} = \frac{1}{8\pi}\text{Im}\int d^4x\, d^2\theta \Big(\tau_Y\, W_Y W_Y + \tfrac{1}{4T(\mathcal{R}_W)}\text{tr}\,\tau_W\, \mathbf{W}_W \mathbf{W}_W + \tfrac{1}{4T(\mathcal{R}_g)}\text{tr}\,\tau_g\, \mathbf{W}_g \mathbf{W}_g\Big)$$

while

$$S_{\text{matter}} = \int d^4x\, d^2\theta\, d^2\theta^\dagger \Big(L_i^\dagger e^{g_Y V_Y Y + \mathbf{V}_W} L_i + \overline{E}_i^\dagger e^{g_Y V_Y Y}\overline{E}_i + Q_i^\dagger e^{g_Y V_Y Y + \mathbf{V}_W + \mathbf{V}_g} Q_i + \overline{U}_i^\dagger e^{g_Y V_Y Y + \mathbf{V}_g}\overline{U}_i + \overline{D}_i^\dagger e^{g_Y V_Y Y + \mathbf{V}_g}\overline{D}_i\Big)$$

- To conclude the SUSY part of the action we write down:

$$S_{\text{Higgs}} = \int d^4x\, d^2\theta\, d^2\theta^\dagger \left(H_1^\dagger e^{-g_Y V_Y + \mathbf{V}_W} H_1 + H_2^\dagger e^{g_Y V_Y + \mathbf{V}_W} H_2 \right) + \int d^4x\, d^2\theta\, \mathcal{W}_{\text{MSSM}}$$

where

$$\mathcal{W}_{\text{MSSM}} = \mu H_1 \cdot H_2 - f_{ij}^{(e)} (L_i \cdot H_1)\, \overline{E}_j - f_{ij}^{(d)} (Q_i \cdot H_1)\, \overline{D}_j - f_{ij}^{(u)} (Q_i \cdot H_2)\, \overline{U}_j$$

- The last three terms in $\mathcal{W}_{\text{MSSM}}$ are evident from our previous discussions.
- The bilinear interaction between the Higgs fields is new. This is the famous μ-term.

- In principle there could be additional terms of the form:

$$L_i \cdot H_2, \quad (L_i \cdot L_j)\,\overline{E}_k, \quad (L_i \cdot Q_j)\,\overline{D}_k, \quad \overline{U}_i \overline{D}_j \overline{D}_k$$

- However they are forbidden by R-parity, a discrete symmetry under which $\theta \to -\theta$.
- We can assign an R-parity to a superfield. This is equal to the R-parity of the bosonic components, and minus the R-parity of the fermionic components.
- In the MSSM it is natural to choose the following R-parity assignments:

$$\begin{array}{rl} L_i,\ \overline{E}_i,\ Q_i,\ \overline{U}_i,\ \overline{E}_i: & \text{odd} \\ H_1,\ H_2,\ V_Y,\ \mathbf{V}_W,\ \mathbf{V}_g: & \text{even} \end{array}$$

- This has the pleasant effect that all known particles have $R = +1$ while all superpartners have $R = -1$.
- It is easy to check that all the terms on the previous slide are allowed by R-parity while all the ones at the top of this slide are forbidden.

Outline

Supersymmetry breaking and soft parameters

- We now briefly discuss supersymmetry breaking.
- It is possible to break supersymmetry spontaneously at tree level using either F-terms or D-terms.
- Let us give an example of each one. The classic model of F-term breaking is the O'Raifeartaigh model:

$$\mathcal{W}(\Phi_0, \Phi_1, \Phi_2) = \Phi_1 f_1(\Phi_0) + \Phi_2 f_2(\Phi_0)$$

- Here, f_1, f_2 are two polynomials chosen such that

$$f_1(\Phi_0) = 0, \;\; f_2(\Phi_0) = 0,$$

do not have a common solution for Φ_0.

- The potential for the scalar components of these superfields is:

$$\sum_{i=0,1,2} \left| \frac{\partial \mathcal{W}}{\partial \phi_i} \right|^2 = |\phi_1 f_1'(\phi_0) + \phi_2 f_2'(\phi_0)|^2 + |f_1(\phi_0)|^2 + |f_2(\phi_0)|^2$$

- For the vacuum energy to vanish, all three terms in this potential must vanish.
- However the last two cannot vanish together since the relevant polynomials have no common zero.
- It follows that the vacuum energy is > 0 and supersymmetry is spontaneously broken. There is a massless fermion ("goldstino") in the spectrum.
- Note that the auxiliary fields, by their equations of motion, are set equal to:

$$F_i = -\frac{\partial \mathcal{W}^\dagger}{\partial \phi_i^\dagger}, \qquad i = 0, 1, 2$$

  and therefore term by term, the potential above can be written:

$$|F_0|^2 + |F_1|^2 + |F_2|^2$$

- Now we have just shown that at least one of F_1, F_2 develops a vev, thereby illustrating the general principle that spontaneous supersymmetry breaking is signalled by a vev of the auxiliary field.

- Next we consider a model of D-term breaking.
- Consider a theory with an Abelian gauge field and a single chiral superfield Φ of charge $t = 1$. Because of its charge, there can be no gauge-invariant superpotential for Φ.
- Now the scalar potential comes from the D-term, and is:

$$\frac{1}{2}(\eta + g\phi^\dagger\phi)^2$$

- If $\eta > 0$, the vacuum configuration is $\phi = 0$ and the vacuum energy is positive, so supersymmetry is spontaneously broken.
- On the other hand if $\eta < 0$, the vacuum configuration is any ϕ satisfying $g\phi^\dagger\phi = |\eta|$.
- In this case the vacuum energy is zero, and supersymmetry is unbroken. However, since ϕ gets a vev, the gauge symmetry is spontaneously broken.
- This illustrates an amusing, and generally complex, interplay between supersymmetry and gauge symmetry.

- Unfortunately, it can be shown that spontaneous supersymmetry breaking within the MSSM at tree level leads to sum rules on particle masses that are in conflict with experiment.
- Therefore the current attitude is to assume supersymmetry breaking takes place by some other mechanism, usually involving extensions beyond the MSSM.
- Without knowing the details of this mechanism, we can still discuss the impact of the breaking on the MSSM, in the form of soft supersymmetry breaking terms.

- The basic idea is to allow non-supersymmetric interactions of scaling dimension ≤ 3, moreover restricted so that they do not induce dimension 4 operators via loop diagrams.
- This ensures that the non-renormalisation theorems of supersymmetry continue to apply in an approximate, but still useful, form.
- These are called "soft terms". The following types of terms qualify as soft:

$$
\begin{aligned}
S_{\text{soft}} &= \int d^4x\Big(-\phi^{\dagger I}(m^2)_{IJ}\phi^J \\
&\quad +(C_I\,\phi^I - \tfrac{1}{2}B_{IJ}\,\phi^I\phi^J + \tfrac{1}{6}A_{IJK}\,\phi^I\phi^J\phi^K + \text{c.c.}) \\
&\quad -\tfrac{1}{2}(M_\alpha\lambda^\alpha\lambda^\alpha + \text{c.c.})\Big)
\end{aligned}
$$

where we have used the generic notation ϕ^I for all scalars, and λ^α for all gauginos, in the theory.

- The following terms have dimension ≤ 3 but are either equivalent to terms above or do not qualify as soft because they generically induce non-soft operators via loops:

$$\phi^{\dagger I}\phi^J\phi^K, \quad \xi^I\xi^J$$

- For the MSSM we note that the linear term $C_I\phi^I$ is absent since there is no gauge-invariant scalar in the theory.
- This restriction actually makes the term $\phi^{\dagger I}\phi^J\phi^K$ a soft term.
- And it also makes it possible to absorb the $\xi^I\xi^J$ by re-defining the superpotential as well as the scalar soft breaking terms.

- Before writing down the above terms in detail for the MSSM, let us introduce the spurion formalism.
- This makes it possible to write non-supersymmetric interactions in terms of superfields at the cost of introducing a fixed ("spurion") superfield:

$$X = \theta\theta F$$

where F is a constant parametrising the supersymmetry breaking scale.

- Then we can write the soft terms as:

$$\begin{aligned} S_{\text{soft}} \;=\; & \int d^4x \Big(- \int d^2\theta \, d^2\theta^\dagger \; X^\dagger X \; \Phi^{\dagger I} (m^2)_{IJ} \big(e^V\big)_{IJ} \Phi^J \\ & + \int d^2\theta \; X \; \big(-\tfrac{1}{2} B_{IJ} \Phi^I \Phi^J + \tfrac{1}{6} A_{IJK} \Phi^I \Phi^J \Phi^K + \text{c.c.}\big) \\ & -\tfrac{1}{2} \int d^2\theta \; X \; \text{tr}\, \big(M_\alpha W^\alpha W^\alpha + \text{c.c.}\big) \Big) \end{aligned}$$

- Let us make the soft terms more explicit in terms of MSSM fields. The first term becomes:

$$
\begin{aligned}
-\phi^{\dagger I}(m^2_{IJ})\phi^J \quad \rightarrow \quad & -\Big(\tilde{\ell}^\dagger_i\,(M^2_{\tilde{\ell}})_{ij}\,\tilde{\ell}_j + \tilde{\bar{e}}^\dagger_i\,(M^2_{\tilde{\bar{e}}})_{ij}\,\tilde{\bar{e}}_j \\
& + \tilde{q}^\dagger_i\,(M^2_{\tilde{q}})_{ij}\,\tilde{q}_j + \tilde{\bar{u}}^\dagger_i\,(M^2_{\tilde{\bar{u}}})_{ij}\,\tilde{\bar{u}}_j + \tilde{\bar{d}}^\dagger_i\,(M^2_{\tilde{\bar{d}}})_{ij}\,\tilde{\bar{d}}_j \\
& + m^2_1\,h^\dagger_1 h_1 + m^2_2\,h^\dagger_2 h_2\Big)
\end{aligned}
$$

- The remaining terms become:

$$
\begin{aligned}
-\tfrac{1}{2}B_{IJ}\phi^I\phi^J \quad &\rightarrow \quad B\mu\,h_1\cdot h_2 + \text{c.c.} \\
\tfrac{1}{6}A_{IJK}\phi^I\phi^J\phi^K \quad &\rightarrow \quad (h_1\cdot\tilde{\ell}_i)(f^e A^e)_{ij}\tilde{\bar{e}}_j + (h_1\cdot\tilde{q}_i)(f^d A^d)_{ij}\tilde{\bar{d}}_j \\
& \qquad + (\tilde{q}_i\cdot h_2)(f^u A^u)_{ij}\tilde{\bar{u}}_j \\
\tfrac{1}{2}M_\alpha\lambda^\alpha\lambda^\alpha + \text{c.c.} \quad &\rightarrow \quad \tfrac{1}{2}(M_1\lambda_B\lambda_B + M_2\lambda^a_W\lambda^a_W + M_3\lambda^{a'}_g\lambda^{a'}_g)
\end{aligned}
$$

- This is a substantial proliferation of parameters from the SM and even the MSSM!

Outline

Supergravity

- So far we have discussed supersymmetric field theory in the absence of gravity.
- This is reasonable if we want to consider physics at energy scales much lower than 10^{19} GeV, the Planck scale, as the effects of gravity are then very weak.
- However, if we want to embed the MSSM in a supersymmetric unified theory near the Planck scale, then we need to consider the supersymmetric form of gravity, called supergravity.
- This is also necessary if we want to embed the MSSM in superstring theory, which for slowly varying fields reduces to a theory of supergravity.

- Supergravity is an extension of ordinary gravity in which the graviton field $e^a_{\ \mu}(x)$ is part of a supermultiplet along with a fermionic field.
- Because the metric describes a graviton of spin 2, the fermion superpartner must have spin $\frac{3}{2}$.
- This has a desirable consequence. In ordinary field theory a global symmetry can be rendered local by coupling the conserved Noether current to a gauge field:

$$\partial^\mu J_\mu = 0 \Rightarrow \int \delta(J^\mu A_\mu) = \int J^\mu \, \partial_\mu \epsilon(x) = -\int \partial_\mu J^\mu \, \epsilon(x) = 0$$

- A similar procedure with supersymmetry would start with the conserved spin-$\frac{3}{2}$ fermionic supercurrent $S_{A,\mu}$ and couple it to a spin-$\frac{3}{2}$ fermion $\Psi_{A,\mu}$ transforming as $\delta\Psi_{A,\mu} \sim \partial_\mu \epsilon_A$:

$$\int \delta(S^{A,\mu}\Psi_{A,\mu}) = \int S^{A,\mu} \, \partial_\mu \epsilon_A(x) = -\int \partial_\mu S^{A,\mu} \, \epsilon_A(x) = 0$$

- Thus we see that a spin-$\frac{3}{2}$ field is suggestive of local supersymmetry.
- In fact, just as fundamental spin-1 fields must always be gauge fields to avoid inconsistencies, spin-$\frac{3}{2}$ fields must always be gauge fields of local supersymmetry.
- Thus supergravity requires local supersymmetry.
- Now the circle closes, for the supersymmetry algebra:

  $$\{Q_A, {Q^\dagger}_{\dot{B}}\} = 2{\sigma^\mu}_{A\dot{B}} P_\mu$$

  tells us that local supersymmetry implies local translation invariance, which amounts to reparametrisation invariance – the hallmark of gravitation.
- Thus it is sometimes said that supergravity is the square-root of general relativity.

- The field variables of supergravity are the vierbein e^a_μ and the gravitino $\Psi_{A,\mu}$.
- The metric of spacetime is:

$$g_{\mu\nu}(x) = e^a{}_\mu(x) e_{a\,\nu}(x)$$

  but (as in any gravitational theory involving fermions) the vierbein is taken to be fundamental.
- We also need the inverse vierbein $E^{a\mu}$, defined by:

$$\eta_{ab}\, E^{a\mu} e^b{}_\nu = \delta^\mu_\nu$$

- The gravitino $\Psi_{A,\mu}$ is Majorana, like the gaugino. It also satisfies a constraint that projects it to a spin-$\frac{3}{2}$ state.

- To write the Lagrangian, we first define the spin connection:

$$\omega_\mu^{ab} = \frac{1}{2}\left[E^{a\nu}(\partial_\nu e^b{}_\mu - \partial_\mu e^b{}_\nu) + \frac{1}{2}E^{a\lambda}E^{b\rho}(\partial_\lambda e_{c\,\rho} - \partial_\rho e_{c\,\lambda})e^c{}_\mu\right] - [a \leftrightarrow b]$$

- This is the gauge field in terms of which the Riemann curvature tensor is written:

$$R_{\mu\nu}{}^{ab}(\omega) = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + \omega_\mu^{ac}\omega_{\nu c}{}^b - \omega_\nu^{ac}\omega_{\mu c}{}^b$$

- Now we can define the scalar curvature by:

$$R(e,\omega) = E_a^{\,\mu} E_b^{\,\nu} R_{\mu\nu}{}^{ab}(\omega)$$

- The final ingredients in the supergravity action are the special derivative:

$$(\hat{D}_\mu)_A^{\;B} = \partial_\mu - \frac{1}{2}\omega_\mu^{ab}(\sigma_{ab})_A^{\;B}, \quad (\sigma_{ab})_A^{\;B} \equiv e_{a\,\mu} e_{b\,\nu}(\sigma^{\mu\nu})_A^{\;B}$$

and the determinant of the vierbein:

$$e = ||e^a{}_\mu||$$

- To write gravity actions we must introduce a dimensional parameter, the Planck mass M_p defined in terms of other familiar parameters by:

$$M_p = \frac{1}{8\pi G_N} = \frac{1}{\kappa}$$

- The supergravity action is then:

$$S_{\mathsf{sugra}} = -\frac{M_p^2}{2}\, e\, R(e,\omega) - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\Psi_\mu \sigma_\nu \hat{D}_\rho \Psi^\dagger_\sigma + \cdots$$

  where the $\cdots$ refers to 4-fermion terms that are rarely, if ever, needed in detail.

- The above action is invariant under:

$$\begin{aligned} \delta e^a_\mu &= \frac{i}{M_p}(\epsilon\sigma^a\Psi^\dagger_\mu - \Psi_\mu\sigma^a\epsilon^\dagger) \\ \delta\Psi_\mu &= 2M_p\hat{D}_\mu\epsilon + \cdots \end{aligned}$$

  where this time the $\cdots$ refers to $\Psi\Psi\epsilon$ terms that, again, we do not usually need.

- Next we discuss the action for a matter and gauge system coupled to supergravity. We temporarily choose units where $M_p = 1$.
- Since this is a non-renormalisable theory, we are tempted to re-instate the non-renormalisable couplings within the matter-gauge system and write:

$$\int d^2\theta d^2\theta^\dagger \; K\big(\Phi^{\dagger\, I}\, e^{2g\, t_i V}_{IJ}, \Phi^J\big) + \frac{1}{8\pi}\mathrm{Im}\left(\int d^2\theta \; \mathcal{F}_{ab}(\Phi^I) W^a W^b\right) + 2\,\mathrm{Re}\left(\int d^2\theta \; \mathcal{W}(\Phi^I)\right) + 2\sum_{U(1)\ \text{factors}} \eta_{(h)} \int d^2\theta d^2\theta^\dagger \; V_{(h)}$$

- However, the requirement of local supersymmetry puts a constraint on the above terms depending on three independent functions $K, \mathcal{F}, \mathcal{W}$. The action can depend on only one combination of $K, \mathcal{W}$:

$$\mathcal{G}(\Phi^{\dagger I}, \Phi^I) = K(\Phi^{\dagger I}, \Phi^I) + \log \mathcal{W}(\Phi^I) + \log \mathcal{W}^\dagger(\Phi^\dagger_I)$$

- Because the superpotential is analytic, the Kähler metric following from $\mathcal{G}$ is the same as that from K:

$$G_{I\bar{J}} = K_{,I\bar{J}} = \mathcal{G}_{,I\bar{J}}$$

- Now the bosonic part of the matter-gauge-supergravity action has the following terms:

$$\begin{aligned}
e^{-1}S_{\text{sugra}} &= -\tfrac{1}{2}\,R(e,\omega) \\
e^{-1}S_{\text{matter}} &= G_{I\bar{J}}\,D_\mu\phi^I D^\mu\phi^{\dagger\bar{J}} - e^{\mathcal{G}}\left(G^{I\bar{J}}\mathcal{G}_{,I}\mathcal{G}_{,\bar{J}} - 3\right) \\
e^{-1}S_{\text{gauge}} &= -\tfrac{1}{4}\mathrm{Re}(\mathcal{F}_{ab})F^a_{\mu\nu}F^{b\,\mu\nu} - \tfrac{1}{4}\mathrm{Im}(\mathcal{F}_{ab})F^a_{\mu\nu}\tilde{F}^{b\,\mu\nu} \\
&\quad -\tfrac{g^2}{2}\mathrm{Re}(\mathcal{F}^{-1}_{ab})\left(\mathcal{G}^{,I}T^a_{IJ}\phi^J\right)\left(\mathcal{G}^{,K}T^b_{KL}\phi^L\right)
\end{aligned}$$

- We recognise the second term in S_{matter} as the F-term and the last term in S_{gauge} as the D-term.

- For the matter and gauge sectors, the action has supergravity modifications compared with its globally-supersymmetric version.
- These all involve powers of M_p, not visible in our conventions where $M_p = 1$.
- It is a straightforward, but illuminating, exercise to re-instate the correct powers of M_p in the above formulae by dimensional analysis.
- One can then check that in the limit $M_p \to \infty$ the effects of supergravity decouple and the action reduces to the familiar one for the globally supersymmetric matter-gauge system.

- Note that in supergravity, the scalar potential is no longer positive-semidefinite as it was for global supersymmetry!
- Moreover, positivity of the vacuum energy is no longer the condition for supersymmetry breaking. But auxiliary fields getting a vev is still the right condition.
- As a result it is possible in principle to break supersymmetry and yet have a vanishing or small cosmological constant. But that requires a high degree of fine-tuning.

- Let us also mention here a special case of the above system where we take the Kähler potential and the gauge kinetic function to be trivial:

$$K(\Phi^{\dagger I}, \Phi^I) = \Phi^{\dagger I}\Phi^I, \quad \mathcal{F}_{ab} = \delta_{ab}$$

- This is called "minimal supergravity", though it should be distinguished from any specific model which also carries that name.

- Once a matter-gauge system has been coupled to supergravity, the supersymmetry transformation laws (of pure supergravity or of pure global supersymmetry) are modified.
- In particular, we have (among many others) the following terms:

$$\begin{aligned}\delta\xi^I &\sim e^{\frac{\mathcal{G}}{2}} G^{I\bar{J}} \mathcal{G}_{,\bar{J}}\, \epsilon + \cdots \\ \delta\lambda^a &\sim \mathcal{G}^{,I} T^a_{IJ} \phi^J\, \epsilon\end{aligned}$$

- Again we recognise the appearance of F- and D-terms.
- Local supersymmetry is broken when the RHS of either of these variations is non-vanishing:

$$e^{\frac{\mathcal{G}}{2}} G^{I\bar{J}} \mathcal{G}_{,\bar{J}} \neq 0 \quad \text{or} \quad \mathcal{G}^{,I} T^a_{IJ} \phi^J \neq 0$$

- Under these conditions we expect the super-Higgs mechanism to take place. Instead of a massless goldstino appearing in the spectrum, the gravitino becomes massive.
- While the full analysis requires some work, one relevant observation is that in the fermionic part of the coupled matter-gauge-supergravity system, there is a term:

$$e^{\frac{\mathcal{G}}{2}}\Psi_\mu \sigma^{\mu\nu}\Psi_\nu$$

- This is a gravitino mass term for any finite value of the vev $\langle \mathcal{G} \rangle$ (remember that because of the log term in $\mathcal{G}$, the natural value for $\langle \mathcal{G} \rangle$ is $-\infty$).
- Thus (re-instating M_p in an obvious way):

$$m_{\Psi_\mu} = e^{\frac{\mathcal{G}}{2}} M_p$$

- In the minimal supergravity case it is quite easy to write down the full scalar potential:

$$\begin{aligned} V(\phi^{\dagger I}, \phi^I) &= e^{\mathcal{G}}\left(G^{I\bar{J}}\mathcal{G}_{,I}\mathcal{G}_{,\bar{J}} - 3\right) \\ &\quad + \tfrac{g^2}{2}\left(\mathcal{G}^{,I}T^a_{IJ}\Phi^J\right)\left(\mathcal{G}^{,K}T^b_{KL}\Phi^L\right) \\ &= e^{\phi^{\dagger I}\phi^I}\left(\left|\mathcal{W}^{,I} + \phi^{\dagger I}\mathcal{W}\right|^2 - 3|\mathcal{W}|^2\right) \\ &\quad + \tfrac{g^2}{2}\left(\phi^{\dagger I}T^a_{IJ}\phi^J\right)\left(\phi^{\dagger K}T^a_{KL}\phi^L\right) \end{aligned}$$

- As a very simple example, the Polonyi superpotential:

$$\mathcal{W}(\Phi) = m^2(\Phi + \beta)$$

for a single neutral superfield gives rise to supersymmetry breaking for some values of β.

- The scalar potential is:

$$V(\phi^\dagger, \phi) = m^4 e^{\phi^\dagger\phi}\left(|1 + \phi^\dagger(\phi+\beta)|^2 - 3|\phi+\beta|^2\right)$$

- In minimal supergravity, the F-type breaking condition reduces to:

$$\mathcal{W}^{,I} + \phi^{\dagger I}\mathcal{W} \neq 0$$

which in this model is:

$$1 + \phi^{\dagger}(\phi + \beta) \neq 0$$

- Now this is true for all ϕ as long as $-2 < \beta < 2$.
- One can make the potential vanish at its minimum for $\beta = 2 - \sqrt{3}$, so the cosmological constant vanishes despite supersymmetry breaking – but this amounts to fine-tuning!

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