Quantum Field Theory

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1.1 Preliminaries: why QFT?

We know that quantum mechanics and the special theory of relativity are both properties of nature. The validity of quantum mechanics is experimentally demonstrated by the photoelectric effect, atomic spectra etc. Similarly that of special relativity is demonstrated by experiments showing constancy of speed of light, time dilation etc.

The question then is: how do we extend quantum mechanics to incorporate relativity? Or putting it the other way, how do we extend relativity to incorporate quantum mechanics?

The basic equation of quantum mechanics is the Schrodinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\vec{\nabla}^2\psi(x,t) + V(\vec{x})\psi(x,t)$$

Clearly this is not relativistic, since it treats space and time differently.

Now, a basic equation of relativity is:

$$E^2 = \vec{p}^2c^2 + m^2c^4$$

which in units where $c = 1$ (which we use henceforth) becomes:

$$E^2 = \vec{p}^2 + m^2$$

Of course this equation is classical rather than quantum.

In quantum mechanics, $|\psi(x,t)|^2$ is the probability of finding a particle at some point in space and time. As we will see, extending the Schrödinger equation to a relativistic wave equation is not merely a technicality but forces us to change interpretation.

Where does the Schrödinger equation come from? We start with the non-relativistic classical relation:

$$E = \frac{\vec{p}^2}{2m} + V(\vec{x})$$

and then make the replacements, motivated by wave mechanics:

$$E \to i\hbar\frac{\partial}{\partial t}, \qquad \vec{p} \to -i\hbar\vec{\nabla}$$

For a free particle ($V(\vec{x}) = 0$) it is easy to guess a relativistic equation. For a relativistic particle we have:

$$E = \sqrt{\vec{p}^2 + m^2} \sim m + \frac{\vec{p}^2}{2m}$$

where the RHS holds (after dropping the constant m from the definition of energy) in the non-relativistic limit.

So we may just replace $E = \dfrac{\vec{p}^2}{2m}$ by $E^2 = \vec{p}^2 + m^2$ and then try to convert it to a Schrödinger-like equation for a wave function $\psi(x^\mu)$ where $x^\mu = (t, \vec{x})$. In what follows we will write $\psi(x^\mu)$ as $\psi(x)$ for simplicity. The result is:

$$-\hbar^2 \frac{\partial^2}{\partial t^2}\psi = -\hbar^2 \vec{\nabla}^2 \psi + m^2 \psi$$

Introducing:

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

and $\partial^\mu = \eta^{\mu\nu}\partial_\nu$, we have:

$$(E, \vec{p}) \rightarrow i\hbar\partial^\mu$$

Then the above equation can be written:

$$\hbar^2 \partial_\mu \partial^\mu \psi(x) + m^2 \psi(x) = 0$$

which is the *free Klein-Gordon equation*. To check that the dimensions are correct, note that in units where $c = 1$ we have:

$$[\hbar] = ML, \qquad [\partial_\mu] = L^{-1}, \qquad [m] = M$$

The general solution to this equation is:

$$\psi(t, \vec{x}) = e^{-ik_\mu x^\mu}$$

under the condition that:

$$-\hbar^2 k_\mu k^\mu + m^2 = 0$$

This can be solved for k_0:

$$k_0 = \pm\frac{1}{\hbar}\sqrt{\hbar^2 \vec{k}^2 + m^2}$$

Since energy is represented by $i\partial/\partial t$, the energy eigenvalue follows from:

$$i\frac{\partial}{\partial t} e^{-ik_\mu x^\mu} = k_0\, e^{-ik_\mu x^\mu} = \pm\frac{1}{\hbar}\sqrt{\hbar^2 \vec{k}^2 + m^2}\; e^{-ik_\mu x^\mu}$$

Therefore *negative energy solutions are allowed.*

Classically we could restrict to positive energy solutions. Even in relativistic quantum mechanics, we could try to do the same as long as we have a free wave equation. But the laws of quantum mechanics make it clear that once interactions are included, there will be transitions between positive and negative energy states (the energy difference between a positive and negative energy state is finite). Thus the physics of the Klein-Gordon equation, treated as a wave equation, is inconsistent.

1.2 Classical fields

We conclude that a relativistic wave equation as an analogue to the Schrödinger equation is not physically meaningful. An alternative approach is to consider *fields*. Let us consider the electromagnetic field. Non-relativistically it is defined via $\phi(t,\vec{x})$ (the scalar potential) and $\vec{A}(t,\vec{x})$ (the vector potential). In special relativity these two fields combine into a 4-vector potential:

$$A^\mu(x) = \Big(\phi(t,\vec{x}), \vec{A}(t,\vec{x})\Big)$$

We can write Maxwell's equations in free space (no charges or currents) as:

$$\vec{\nabla}\cdot\vec{B} = 0, \qquad \vec{\nabla}\times\vec{B} = \frac{\partial\vec{E}}{\partial t},$$

$$\vec{\nabla}\cdot\vec{E} = 0, \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

where:

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi, \qquad \vec{B} = \vec{\nabla}\times\vec{A}$$

These can be written in relativistic form as follows. Define

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

The electric and magnetic fields E_i, B_i are defined in terms of components of $F_{\mu\nu}$ by:

$$F_{0i} = E_i, \qquad F_{ij} = \frac{1}{2}\epsilon_{ijk}B_k$$

and the Maxwell equations become:

$$\partial^\mu F_{\mu\nu} = 0, \qquad \epsilon^{\mu\nu\lambda\rho}\partial_\nu F_{\lambda\rho} = 0$$

The electromagnetic field $A_\mu(x)$ is a 4-vector at each point of space and time. It can be thought of as classical for many purposes (hence the subject "classical electrodynamics"). However, as Planck taught us, one can go wrong by considering it to be classical in certain situations where the correct answer is only obtained by *quantising the field*.

A priori, quantising a field has little to do with particles or wave equations. Just think of the field as a dynamical variable, consider its equations of motion and convert them to equations for operators in a Hilbert space. This can be done for the electromagnetic field, but it is not the simplest case to consider. Instead of a 4-vector field A_μ, let us consider a simpler field

that has just one component: a *scalar field* $\phi(x)$. The obvious relativistic field equation one can write down for this, analogous to the Maxwell equation for A_μ, is:

$$\partial_\mu \partial^\mu \phi(x) = 0$$

This is the massless free Klein-Gordon equation. But now it is *not* a wave equation! $\phi(x)$ is not the wave function of a particle with $|\phi|^2$ being the probability density. That would lead us back to the problem of negative energy states. Instead, we consider $\phi(x)$ to be a field and quantise the field.

To be a little more general, we extend the field equation by putting back the mass term. It remains to be seen what is the physical interpretation of the field, as well as the mass term. We return to that later on. Thus we want to quantise $\phi(x)$ satisfying:

$$\hbar^2 \partial_\mu \partial^\mu \phi(x) + m^2 \phi(x) = 0$$

Let's assume we have never quantised a field before, but we know how to quantise coordinates $q(t)$ appearing in a harmonic-oscillator Lagrangian like:

$$L = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2$$

This is quantised by promoting $q(t)$ to an operator and imposing the canonical commutation relation:

$$[q, p] = i\hbar$$

As we now show, we can reduce $\phi(x)$ to a set of harmonic-oscillator coordinates labelled $q_{\vec{k}}(t)$, by a simple device. Each $q_{\vec{k}}(t)$ will describe a separate harmonic oscillator and can be quantised independently as above.

For this, place the system in a cubical box of side L. Thus,

$$0 \leq x, y, z \leq L$$

Now impose periodic boundary conditions across the box:

$$\phi(t, x, y, z) = \phi(t, x + L, y, z) = \phi(t, x, y + L, z) = \phi(t, x, y, z + L)$$

The functions of $\vec{x}$ that satisfy the box boundary conditions are:

$$e^{i\vec{k}\cdot\vec{x}}$$

where $\vec{k} = \frac{2\pi}{L}(m_1, m_2, m_3)$ for any three integers m_1, m_2, m_3.

Therefore an arbitrary field configuration $\phi(x)$ can be expanded as:

$$\phi(t,\vec{x}) = \mathcal{N}\sum_{\vec{k}} q_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{x}}$$

where $\mathcal{N}$ is a normalisation factor. The $q_{\vec{k}}(t)$ are *fluctuation modes* of the field $\phi(x)$.

Inserting this into the Klein-Gordon field equation above, we find:

$$\ddot{q}_{\vec{k}}(t) + \left(\vec{k}^2 + \frac{m^2}{\hbar^2}\right) q_{\vec{k}}(t) = 0$$

where $\vec{k}^2 = \dfrac{4\pi^2}{L^2}(m_1^2+m_2^2+m_3^3)$. Thus for every $\vec{k}$ we have an independent harmonic oscillator with frequency

$$\omega_{\vec{k}} = \sqrt{\vec{k}^{\,2} + \frac{m^2}{\hbar^2}}$$

Note that the $q_{\vec{k}}(t)$ are not physical coordinates of space, but because they obey the harmonic-oscillator equations of motion they can be thought of as some type of "generalised coordinates".

Since we chose $\phi(x)$ to be real, we have

$$q^\star_{\vec{k}}(t) = q_{-\vec{k}}(t)$$

Thus the quantum Klein-Gordon field has reduced to a collection of infinitely many harmonic oscillators, each decoupled from the other. This is a system we know how to quantise.

1.3 Quantising free scalar fields

We can write down a Lagrangian whose equations of motion give the equations of motion for $q_{\vec{k}}(t)$ written down in the previous section:

$$L(q,\dot{q}) = \frac{1}{2}\,\dot{q}_0^2 - \frac{1}{2}\,\omega_0^2\,q_0^2 + \sum_{\vec{k}>0}\left(\dot{q}_{-\vec{k}}\,\dot{q}_{\vec{k}} - \omega_{\vec{k}}^2\,q_{-\vec{k}}\,q_{\vec{k}}\right)$$

where we conventionally take the vector $\vec{k} = \frac{2\pi}{L}(m_1,m_2,m_3)$ to be > 0 if its first non-vanishing component is > 0. In deriving the above Lagrangian, the normalisation of the expansion of $\phi(x)$ in terms of the $q_{\vec{k}}$ was used.

The canonical momenta for this Lagrangian are:

$$\begin{aligned}
\vec{p}_0 &\equiv \frac{\partial L}{\partial \dot{q}_0} &= \dot{q}_0 \\
\vec{p}_{\vec{k}} &\equiv \frac{\partial L}{\partial \dot{q}_{-\vec{k}}} &= \dot{q}_{\vec{k}} \\
\vec{p}_{-\vec{k}} &\equiv \frac{\partial L}{\partial \dot{q}_{\vec{k}}} &= \dot{q}_{-\vec{k}}
\end{aligned}$$

where $\vec{k} > 0$. The Hamiltonian is then:

$$\begin{aligned}
H &= p_0\,\dot{q}_0 + p_{-\vec{k}}\,\dot{q}_{\vec{k}} + p_{\vec{k}}\,\dot{q}_{-\vec{k}} - L \\
&= \frac{1}{2}\,p_0^2 + \frac{1}{2}\,\omega_0^2\,q_0^2 + \sum_{\vec{k}>0}\left(p_{-\vec{k}}\,p_{\vec{k}} + \omega_{\vec{k}}^2\,q_{-\vec{k}}\,q_{\vec{k}}\right) \\
&= H_0 + \sum_{\vec{k}>0} H_{\vec{k}}
\end{aligned}$$

Now impose canonical commutation relations:

$$[q_{\vec{k}}, p_{-\vec{k}'}] = i\hbar\,\delta_{\vec{k},\vec{k}'}, \quad [p_{\vec{k}}, p_{\vec{k}'}] = [q_{\vec{k}}, q_{\vec{k}'}] = 0$$

The above equations hold for all positive and negative $\vec{k}$ as well as $\vec{k} = 0$.

This converts $q_{\vec{k}}(t)$ into quantum operators, and thereby

$$\phi(\vec{x}, t) = \mathcal{N} \sum_{\vec{k}} q_{\vec{k}}(t) e^{i\vec{k}\cdot\vec{x}}$$

also becomes a quantum operator. Note that the operators obey:

$$p_{\vec{k}}^\dagger = p_{-\vec{k}}, \qquad q_{\vec{k}}^\dagger = q_{-\vec{k}}$$

The Hamiltonian is solved by factorising each harmonic oscillator using the creation/annihilation operator method. We start by defining:

$$a_{\vec{k}} = \frac{1}{\sqrt{2\hbar\omega_{\vec{k}}}}\left(i p_{\vec{k}} + \omega_{\vec{k}}\,q_{\vec{k}}\right)$$

where $\vec{k}, \vec{k}'$ can be positive, negative or zero. It follows that:

$$a_{\vec{k}}^\dagger = \frac{1}{\sqrt{2\hbar\omega_{\vec{k}}}}\left(-i p_{-\vec{k}} + \omega_{\vec{k}}\,q_{-\vec{k}}\right)$$

It is easy to check that:

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \delta_{\vec{k},\vec{k}'}$$

Now we have:

$$H = \sum_{\vec{k}} \hbar\omega_{\vec{k}} \left(a^\dagger_{\vec{k}}\, a_{\vec{k}} + \frac{1}{2} \right)$$

where the sum is over all k.

For each harmonic oscillator, the state of lowest energy is the "Fock vacuum" defined by:

$$a_{\vec{k}}\,|0\rangle = 0, \qquad \text{all } \vec{k}$$

This is interpreted as the quantum state of the field theory which is *empty* of excitations. Other states will be interpreted as containing "field quanta", otherwise known as "elementary particles".

The ground state energy of the entire system is the constant (operator-independent) term in the Hamiltonian, namely:

$$E_0 = \frac{1}{2}\hbar\,\omega_0 + \hbar \sum_{\vec{k}>0} \omega_{\vec{k}}$$

Unfortunately the sum is infinite! This is the first of many potential difficulties in quantum field theory.

On the other hand, all excited states have strictly positive energy compared to the ground state. Negative energy states simply do not exist when we quantise a field!

To see this more explicitly, we first re-define the Hamiltonian by subtracting off the infinite constant. This simply means we measure all energies relative to the vacuum state, which seems quite reasonable physically. In a true harmonic oscillator we measure the ground state energy by comparing it with the energy when the oscillator is absent. For a field, we cannot do this – the field is always present, whether or not it is excited. In the absence of gravitation (which couples to all energy including vacuum energy) we can therefore just set the vacuum energy of a field to zero[1].

Hence the quantum Hamiltonian of the free real Klein-Gordon field is simply:

$$H = \sum_{\vec{k}} \hbar\omega_{\vec{k}}\, a^\dagger_{\vec{k}}\, a_{\vec{k}}$$

[1]When gravity is present we would instead set the vacuum energy equal to the value measured experimentally. However a fundamental theory of quantum gravity could possibly allow us to *calculate* the vacuum energy.

The finite size is only a convenience. If we take the limit $L \to \infty$, we get continuous allowed values of $\vec{k}$ and:

$$H = \int \frac{d^3k}{(2\pi)^3}\, \hbar\omega_{\vec{k}}\, a^\dagger_{\vec{k}}\, a_{\vec{k}}$$

where

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = (2\pi)^3\, \delta^3(\vec{k} - \vec{k}')$$

Now that we have quantised the free Klein-Gordon field, we need to interpret the result. The energy eigenstates are

$$a^\dagger_{\vec{k}_1}\, a^\dagger_{\vec{k}_2} \cdots a^\dagger_{\vec{k}_n} |0\rangle$$

As we saw, the state $|0\rangle$ is chosen to have zero energy (this amounts to a choice of the origin of energy). Now consider the state $a^\pm_{\vec{k}}|0\rangle$. This state satisfies:

$$H(a^\dagger_{\vec{k}}|0\rangle) = \hbar\, \omega_{\vec{k}}(a^\dagger_{\vec{k}}|0\rangle)$$

and therefore its energy is:

$$E_{\vec{k}} = \hbar\, \omega_{\vec{k}} = \sqrt{\hbar^2\vec{k}^2 + m^2}$$

This state fits perfectly with our expectations of a state containing a single particle of momentum $\vec{p} = \hbar\vec{k}$. *We therefore interpret it as a one-particle state.*

When we study conserved currents, we will show that the operator:

$$\vec{P} = \sum_{\vec{k}} \hbar\vec{k}\, a^\dagger_{\vec{k}}\, a_{\vec{k}}$$

measures the 3-momentum of the state. We have

$$\vec{P}|0\rangle = 0, \qquad \vec{P}\,(\, a^\dagger_{\vec{k}}|0\rangle\,) = \hbar\vec{k}\,(\, a^\dagger_{\vec{k}}|0\rangle\,)$$

which is exactly what we expect with our interpretation.

If we consider the most general state:

$$a^\dagger_{\vec{k}_1} \cdots a^\dagger_{\vec{k}_n} |0\rangle$$

we have

$$H\left(a^\dagger_{\vec{k}_1} \cdots a^\dagger_{\vec{k}_n} |0\rangle\right) = \left(\hbar\sum_{i=1}^{n}\omega_{\vec{k}_i}\right)\, \left(a^\dagger_{\vec{k}_1} \cdots a^\dagger_{\vec{k}_n} |0\rangle\right)$$

as well as:

$$\vec{P}\left(a^\dagger_{\vec{k}_1} \cdots a^\dagger_{\vec{k}_n} |0\rangle\right) = \left(\sum_{i=1}^{n}\vec{k}_i\right)\, \left(a^\dagger_{\vec{k}_1} \cdots a^\dagger_{\vec{k}_n} |0\rangle\right)$$

as expected for a multiparticle state with n non-interacting particles.

Also, because the $a^\dagger$ all commute among themselves, the state $a^\dagger_{\vec{k}_1} a^\dagger_{\vec{k}_2}|0\rangle$ is the same as the state $a^\dagger_{\vec{k}_2} a^\dagger_{\vec{k}_1}|0\rangle$. This means the particles corresponding to this field satisfy Bose statistics!

We also see that there is a *single* state for fixed particle number and momenta. The absence of another label giving a degeneracy indicates there is no *spin* in this case.

Thus we see that quantising the free real Klein-Gordon field gives us states containing arbitrary numbers of free spinless particles, all of a common mass m and obeying Bose statistics.

Let us now see what quantisation of the $a_{\vec{k}}, a^\dagger_{\vec{k}}$ tells us about the original field ϕ. We have:

$$\phi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, q_{\vec{k}}(t)\, e^{i\vec{k}\cdot\vec{x}}$$

where again we are using language appropriate for infinite space rather than a box. Since:

$$q_{\vec{k}} = \frac{1}{\sqrt{2\hbar\omega_{\vec{k}}}}\left(a_{\vec{k}} + a^\dagger_{-\vec{k}}\right)$$

we have

$$\phi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\hbar\omega_{\vec{k}}}}\left(a_{\vec{k}}\, e^{i\vec{k}\cdot\vec{x}} + a^\dagger_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{x}}\right)$$

Since we also have

$$p_{\vec{k}} = -i\sqrt{\frac{\omega_{\vec{k}}}{2\hbar}}\,(a_{\vec{k}} - a^\dagger_{-\vec{k}})$$

we can construct:

$$\pi(\vec{x}) = -i\hbar^2 \int \frac{d^3k}{\sqrt{(2\pi)^3}}\sqrt{\frac{\omega_{\vec{k}}}{2\hbar}}\left(a_{\vec{k}}\, e^{i\vec{k}\cdot\vec{x}} - a^\dagger_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{x}}\right)$$

This operator $\pi(\vec{x})$ is called the canonical momentum conjugate to $\phi(\vec{x})$. Later we will see that it has a natural definition even without going to Fourier modes.

Now it is straightforward, using

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = (2\pi)^3\,\delta^3(\vec{k} - \vec{k}')$$

to show that

$$[\phi(t,\vec{x}), \pi(t,\vec{x}')] = i\hbar\,\delta^3(\vec{x} - \vec{x}')$$
$$[\phi(t,\vec{x}), \phi(t,\vec{x}')] = [\pi(\vec{x}), \pi(\vec{x}')] = 0$$

This could be an alternate starting point for the quantisation procedure. Notice that the field commutators are taken at equal times.

Henceforth in these lectures we work in units in which $\hbar = 1$.

1.4 Comment on normalisation

How to normalise the state:

$$|\vec{k}\rangle \;\sim\; a^\dagger_{\vec{k}}\,|0\rangle$$

If we try $\langle\vec{k}|\vec{\ell}\rangle = (2\pi)^3\delta^3(\vec{k}-\vec{\ell})$, this is not Lorentz invariant.

Consider a boost along x^1 described by:

$$\begin{aligned} k'_0 &= \cosh\alpha\, k_0 + \sinh\alpha\, k_1 \\ k'_1 &= \sinh\alpha\, k_0 + \cosh\alpha\, k_1 \end{aligned}$$

Then the delta-function becomes:

$$\delta^3(\vec{k}'-\vec{\ell}') = \delta(k'_1-\ell'_1)\delta(k'_2-\ell'_2)\delta(k'_3-\ell'_3)$$

which is equal to:

$$\delta\Big(\sinh\alpha(k_0-\ell_0)+\cosh\alpha(k_1-\ell_1)\Big)\delta(k_2-\ell_2)\delta(k_3-\ell_3)$$

Since $k_0=\omega_{\vec{k}}=\sqrt{\vec{k}^2+m^2}$, we have:

$$k_0-\ell_0=\omega_{\vec{k}}-\omega_{\vec{\ell}}=\sqrt{\vec{k}^2+m^2}-\sqrt{\vec{\ell}^2+m^2}$$

Using the behaviour of the delta-function under a change of variables, one can now show that:

$$\delta\Big(\sinh\alpha(k_0-\ell_0)+\cosh\alpha(k_1-\ell_1)\Big)=\frac{\omega_{\vec{k}}}{\omega_{\vec{k}'}}\,\delta(k_1-\ell_1)$$

It follows that $\omega_{\vec{k}}\,\delta^3(\vec{k}-\vec{\ell})$ is *Lorentz invariant.*

Therefore we choose the normalisation such that:

$$\langle\vec{k}|\vec{\ell}\rangle = (2\pi)^3\,2\omega_{\vec{k}}\,\delta^3(\vec{k}-\vec{\ell})$$

which means the normalised state of momentum k should be defined as:

$$|\vec{k}\rangle=\sqrt{2\omega_{\vec{k}}}\,a^\dagger_{\vec{k}}\,|0\rangle$$

We will see that in this way, the Hamiltonian formalism recovers Lorentz invariance for physical quantities, despite non-covariant choices along the way.

1.5 Time evolution of the field

With all this, we can now go to the Heisenberg picture and explicitly determine the time evolution of $\phi(t,x)$. We have

$$\phi(t,\vec{x}) = e^{iHt}\,\phi(0,\vec{x})\,e^{-iHt}$$

This can be easily computed starting from the analogous statement for the creation and annihilation operators:

$$a_{\vec{k}}(t) = e^{iHt}a_{\vec{k}}\,e^{-iHt}, \quad a^\dagger_{\vec{k}}(t) = \; e^{iHt}a^\dagger_{\vec{k}}\,e^{-iHt}$$

where in this equation and everything that follows, $a_{\vec{k}}, a^\dagger_{\vec{k}}$ stand for $a_{\vec{k}}(t=0), a^\dagger_{\vec{k}}(t=0)$.

There is a simple trick to evaluate the above expressions. From the Hamiltonian:

$$H = \int \frac{d^3k}{(2\pi)^3}\,\omega_{\vec{k}}\,a^\dagger_{\vec{k}}\,a_{\vec{k}}$$

we find:

$$\begin{aligned}[H, a_{\vec{k}}] &= -\,\omega_{\vec{k}}\,a_{\vec{k}} \\ [H, a^\dagger_{\vec{k}}] &= \omega_{\vec{k}}\,a^\dagger_{\vec{k}}\end{aligned}$$

It follows that

$$\begin{aligned}e^{iHt}a_{\vec{k}}\,e^{-iHt} &= e^{-i\omega_{\vec{k}}t}a_{\vec{k}} \\ e^{iHt}a^\dagger_{\vec{k}}\,e^{-iHt} &= e^{i\omega_{\vec{k}}t}a^\dagger_{\vec{k}}\end{aligned}$$

Then from

$$\phi(0,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\left(a_{\vec{k}}\,e^{i\vec{k}\cdot\vec{x}} + a^\dagger_{\vec{k}}\,e^{-i\vec{k}\cdot\vec{x}}\right)$$

we get

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\left(a_{\vec{k}}\,e^{-ik\cdot x} + a^\dagger_{\vec{k}}\,e^{ik\cdot x}\right)$$

where $k\cdot x = k_\mu x^\mu$, with $k^\mu = (\omega_{\vec{k}}, \vec{k})$ and $x^\mu = (t,\vec{x})$.

Notice that $a_{\vec{k}}$ multiplies $e^{-i\omega_{\vec{k}}t}$ while $a^\dagger_{\vec{k}}$ multiplies $e^{i\omega_{\vec{k}}t}$ (by definition, $\omega_{\vec{k}} > 0$ always). We call $e^{-i\omega_{\vec{k}}t}$ a *positive frequency mode* (because acting on it with $i\hbar\partial/\partial t$ would give a positive energy eigenvalue) and $e^{i\omega_{\vec{k}}t}$ a *negative frequency mode*.

We see that the positive frequency mode is multiplied by an operator which *destroys a particle of energy* $\omega_{\vec{k}}$ while a negative frequency mode occurs with an operator that *creates a particle of energy* $\omega_{\vec{k}}$. The energy is always positive.

For complex ϕ, things are different. We then have two fields $\phi(x), \phi^\dagger(x)$. In what follows we will show that to a complex field ϕ one can associate a *charge* such that ϕ has charge +1 in some units while $\phi^\dagger$ has charge -1.

In terms of the mode expansion, we find that there are twice as many oscillator modes, and the mode expansion is:

$$\begin{aligned}\phi(\vec{x},t) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\left(b_{\vec{k}}\, e^{-ik\cdot x} + a^\dagger_{\vec{k}}\, e^{ik\cdot x}\right)\\ \phi^\dagger(\vec{x},t) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\left(a_{\vec{k}}\, e^{-ik\cdot x} + b^\dagger_{\vec{k}}\, e^{ik\cdot x}\right)\end{aligned}$$

We $a^\dagger_{\vec{k}}$ as the operator that creates a particle of momentum $\vec{k}$, as before, while $b^\dagger_{\vec{k}}$ creates a particle of the same momentum but opposite charge. We call the latter an *antiparticle*.

From the mode expansion we see that ϕ contains the modes $a^\dagger$ that create particles as well as the modes b that destroy antiparticles. This makes sense because in both cases the charge of the state *increases* by 1 unit. Similarly $\phi^\dagger$ contains modes $b^\dagger$ that create antiparticles and modes a that destroy particles. In both cases the charge of the state *decreases* by 1 unit.

1.6 Causality

In this section we examine whether, in quantum field theory, physical signals can propagate over space-like intervals. For this we should compute the vacuum expectation value of the commutator $[\phi(x), \phi(y)]$. We will see that this is zero over space-like intervals, which by the laws of quantum mechanics guarantees that measurement of one field cannot affect measurement of the other. Therefore signals cannot propagate over such intervals, exactly as expected based on the structure of classical special relativity. On the other hand, the above expectation value will turn out to be nonzero over time-like intervals indicating that signals can propagate over such intervals.

Let us work with real scalar fields $\phi(x)$. Consider the state:

$$\phi(x)|0\rangle$$

Given that

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\left(a_{\vec{k}}\, e^{-ik\cdot x} + a^\dagger_{\vec{k}}\, e^{ik\cdot x}\right)$$

we have:

$$\phi(x)|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\, e^{ik\cdot x} a^\dagger_{\vec{k}}|0\rangle$$

since the first term $a_{\vec{k}}|0\rangle$ vanishes. This can be interpreted as the state corresponding to one particle localised at position $\vec{x}$, and evolving in time t. The corresponding adjoint state is:

$$\langle 0|\phi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}e^{-ik\cdot x}\langle 0|a_{\vec{k}}$$

Now we already know that

$$[\phi(t,\vec{x}),\phi(t,\vec{x}')] = 0$$

as long as $\vec{x} \neq \vec{x}'$. If we consider the apparently more general commutator $[\phi(x),\phi(y)]$, we realise right away that this is also zero for $(x-y)^2 < 0$, i.e. for space-like intervals. The reason is that in this case, as we will show below, by a Lorentz transformation we can bring the two space-time points into the form $(t,\vec{x})$ and $(t,\vec{x}')$.

The answer is hard to compute explicitly but simplifies in the limit of large (space-like or time-like) separation. Let us define:

$$D(x,y) = \langle 0|\phi(x)\phi(y)|0\rangle$$

Note that by translation invariance, $D(x,y)$ depends only on the separation between the two space-time points. Henceforth we denote it $D(x-y)$.

Inserting the mode expansion for ϕ, we have:

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\frac{1}{\sqrt{2\omega_{\vec{k}'}}}e^{-ik'\cdot x}e^{+ik\cdot y}\langle 0|a_{\vec{k}'}a^\dagger_{\vec{k}}|0\rangle$$

The expectation value is evaluated as follows:

$$\langle 0|a_{\vec{k}'},a^\dagger_{\vec{k}}|0\rangle = \langle 0|\left[a_{\vec{k}'},a^\dagger_{\vec{k}}\right]|0\rangle = (2\pi)^3\delta^3(\vec{k}-\vec{k}')$$

where we have used that $a_{\vec{k}}|0\rangle = \langle 0|a^\dagger_{\vec{k}} = 0$.

The result is that:

$$D(x-y) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}e^{-ik\cdot(x-y)}$$

This will be computed for large separations in an Appendix to this section.

In terms of this quantity we have:

$$\langle 0|[\phi(x),\phi(y)]|0\rangle = D(x-y) - D(y-x)$$

As we show in the Appendix, in the space-like case

$$D(x-y) \sim e^{-m|r|}$$

for large r. Therefore at least in this limit, $D(x,y) - D(y,x) = 0$. In the time-like case, we have instead:

$$D(x-y) \;\sim\; e^{-imt}$$

for large t. Therefore $D(x,y) - D(y,x) \sim \sin mt \neq 0$. Therefore signals can, and do, propagate only over time-like intervals.

We have:

$$D(x-y) - D(y-x) = \langle 0|[\phi(x),\phi(y)]|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}\left(e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)}\right)$$

For $x^0 > y^0$ this is equal to:

$$\int \frac{d^3k}{(2\pi)^3}\int \frac{dk^0}{2\pi i}\,\frac{-1}{k^2-m^2}\,e^{-ik\cdot(x-y)}$$

where the contour passes above the poles $k_0 = \pm\omega_{\vec{k}}$. To show the above, note that:

$$\frac{1}{k^2-m^2} = \frac{1}{2\omega_{\vec{k}}}\left[\frac{1}{k_0-\omega_{\vec{k}}} - \frac{1}{k_0+\omega_{\vec{k}}}\right]$$

Now since $x^0 > y^0$, the contour can be closed in the lower half-plane and we get:

$$\int \frac{dk^0}{2\pi i}\frac{-i}{k^2-m^2}e^{-ik\cdot(x-y)} = \frac{1}{2\omega_{\vec{k}}}\left(e^{ik\cdot(x-y)}\big|_{k^0=\omega_{\vec{k}}} - e^{-ik\cdot(x-y)}\big|_{k^0=\omega_{\vec{k}}}\right)$$

Inserting this in the above expression we recover

$$D(x-y) - D(y-x) = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}\left(e^{-ik\cdot(x-y)} - e^{+ik\cdot(x-y)}\right)\big|_{k^0=\omega_{\vec{k}}}$$

as desired.

If instead $x_0 - y_0 < 0$ then with the same pole prescription we can close the contour in the upper half-plane and we get 0. Therefore we have shown that:

$$\begin{aligned}\int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2}\,e^{-ik\cdot(x-y)} &= \langle 0|[\phi(x),\phi(y)]|0\rangle, \quad x^0-y^0>0\\ &= 0 \quad \text{otherwise}\end{aligned}$$

This motivates us to define a propagator called the *retarded propagator* by:

$$D_R(x-y) = \theta(x^0-y^0)\langle 0|[\phi(x),\phi(y)]|0\rangle = \int_{\mathcal{C}_R}\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2}\,e^{-ik\cdot(x-y)}$$

where the contour $\mathcal{C}_R$ passes *above* the poles. Here $\theta(x^0 - y^0)$ is the step function which is $+1$ for positive arguments and 0 for negative arguments.

Correspondingly the *advanced propagator* is:

$$D_A(x-y) = \theta(y^0 - x^0)\langle 0|[\phi(x), \phi(y)]|0\rangle = \int_{\mathcal{C}_A} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} e^{-ik\cdot(x-y)}$$

where the contour passes *below* the poles.

Exercise: show that

$$(\partial^2 + m^2)D_{R,A}(x-y) = -i\delta^4(x-y)$$

showing that the retarded and advanced propagators are *Green's functions* for the Klein-Gordon equation.

Appendix to Section 1.6

We wish to compute the dependence of

$$D(x,y) \equiv \langle 0|\phi(x)\phi(y)|0\rangle$$

on $x - y$. As we have seen,

$$D(x,y) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} e^{-ik\cdot(x-y)}$$

This is *divergent* for $x^\mu = y^\mu$!

Now we consider two special cases:

$$\begin{aligned} &\text{(i)} \quad x^0 - y^0 = t, \quad \vec{x} - \vec{y} = 0 \\ &\text{(ii)} \quad x^0 - y^0 = 0, \quad \vec{x} - \vec{y} = (r, 0, 0) \end{aligned}$$

The first case corresponds to time-like separation while the second corresponds to space-like separation.

All time-like separations can be brought to the first form, and all space-like separations to the second form, by a Lorentz transformation.

(A quick sketch of the proof is as follows. By a spatial rotation, any vector a^μ can be brought to the form $(a^0, a^1, 0, 0)$. Now consider

$$a^\pm = a^0 \pm a^1$$

and notice that $a \cdot a \equiv a_\mu a^\mu = a_+ a_-$. A Lorentz transformation acts as:

$$a_+ \to e^\beta a_+, \qquad a_- \to e^{-\beta} a_-$$

for some parameter β.

Now if $a \cdot a > 0$ (time-like separation) then a_+ and a_- have the same sign and by a Lorentz transformation we can make $a_+ = a_-$, i.e. a^μ is of the form $(a_0, 0, 0, 0)$. If instead $a \cdot a < 0$ (space-like separation) then a_+ and a_- have opposite signs and by a Lorentz transformation we can make $a_+ = -a_-$, i.e. a^μ is of the form $(0, a^1, 0, 0)$.)

Let us now evaluate $D(x, y)$ in the two special cases. In the first one,

$$D(x,y) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\sqrt{\vec{k}^2 + m^2}} e^{-i\sqrt{\vec{k}^2+m^2}\, t}$$

Since the integrand depends only on the magnitude of $\vec{k}$ we use $d^3k = 4\pi \vec{k}^2 d|\vec{k}|$ and get:

$$D(x,y) = \frac{4\pi}{2(2\pi)^3} \int_0^\infty d|\vec{k}| \frac{\vec{k}^2}{\sqrt{\vec{k}^2 + m^2}} e^{-i\sqrt{\vec{k}^2+m^2}\, t} = \frac{1}{8\pi^2} \int_m^\infty dE \sqrt{E^2 - m^2}\, e^{-iEt}$$

where in the last step we have made the substitution $E = \sqrt{\vec{k}^2 + m^2}$.

One can easily check by saddle-point methods that for large t, $E \sim m$ and so the integral in the time-like case goes like:

$$\sim e^{-imt},\ t \to \infty$$

Next consider the spacelike case:

$$D(x-y) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\sqrt{\vec{k}^2 + m^2}} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}$$

This time the integrand depends on the angle made by $\vec{k}$ with the x-axis, so we use $d^3k = 2\pi \sin\theta\, d\theta\, \vec{k}^2 d|\vec{k}|$. Then:

$$\begin{aligned} D(x,y) &= \int \frac{2\pi \sin\theta\, d\theta\, \vec{k}^2\, d|\vec{k}|}{(2\pi)^3} \frac{1}{\sqrt{\vec{k}^2 + m^2}} e^{i|k||r|\cos\theta} \\ &= \frac{1}{(2\pi)^2} \int_0^\infty d|\vec{k}| \frac{\vec{k}^2}{2\sqrt{\vec{k}^2 + m^2}} \frac{e^{i|\vec{k}||r|} - e^{-i|\vec{k}||r|}}{i|\vec{k}||r|} \\ &= \frac{1}{(2\pi)^2} \cdot \frac{-i}{2|r|} \int_{-\infty}^{+\infty} dp \frac{p\, e^{ip|r|}}{\sqrt{p^2 + m^2}} \end{aligned}$$

This has square-root branch cuts at $p = \pm im$. So we set $p = i\rho$ and find:

$$D(x,y) = \frac{2}{8\pi^2|r|} \int_m^\infty d\rho \frac{\rho\, e^{-\rho|r|}}{\sqrt{\rho^2 - m^2}} \sim e^{-m|r|}$$

1.7 The Feynman propagator

There is another propagator, also a Green's functions of the Klein-Gordon equation, that corresponds to an important physical quantity. Define:

$$D_F(x-y) = \int_{\mathcal{C}_F} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} e^{-ik\cdot(x-y)}$$

where the contour $\mathcal{C}_F$ is taken to pass *below* the first pole (at $k_0 = -\omega_{\vec{k}}$) and *above* the second pole (at $k_0 = \omega_{\vec{k}}$). Now if $x^0 > y^0$ then we can close the contour below, capturing only the pole at $k_0 = \omega_{\vec{k}}$. In the reverse case $y^0 > x^0$ we close the contour above and capture the pole at $k_0 = -\omega_{\vec{k}}$.

In the first case, we find:

$$D_F(x-y) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} e^{-ik\cdot(x-y)}|_{k^0=\omega_{\vec{k}}}$$

which is the same as $D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$. In the second case we get $D(y-x)$. Therefore we have:

$$\begin{aligned} D_F(x,y) &= D(x-y), \qquad x^0 > y^0, \\ &= D(y-x), \qquad y^0 > x^0 \end{aligned}$$

or more concisely,

$$D_F(x,y) = \theta(x^0 - y^0)D(x-y) + \theta(y^0 - x^0)D(y-x)$$

Physically this is a propagator that takes a particle from earlier times to later times. It is called the *Feynman propagator* and can be expressed as:

$$D_F(x-y) = \langle 0|T(\phi(x)\phi(y))|0\rangle$$

where T is the time-ordering symbol, defined by:

$$T(\phi(x)\phi(y)) = \theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x)$$

The field at the later time is put to the left of the one at the earlier time.

We now derive an expression for the Feynman propagator that will be useful later on. Let us decompose ϕ into its creation and annihilation pieces:

$$\begin{aligned} \phi_+(t,\vec{x}) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} a_{\vec{k}} e^{-ik\cdot x} \\ \phi_-(t,\vec{x}) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} a^\dagger_{\vec{k}} e^{ik\cdot x} \end{aligned}$$

Because ϕ_+ always annihilates on the right, and ϕ_- always annihilates on the left, we have:

$$D_F(x-y) = \theta(x^0-y^0)\langle 0|\phi_+(x)\,\phi_-(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi_+(y)\,\phi_-(x)|0\rangle$$

But for the same reason, we are allowed to replace each bilinear with a commutator, getting:

$$D_F(x-y) = \theta(x^0-y^0)\langle 0|\,[\phi_+(x),\phi_-(y)]\,|0\rangle + \theta(y^0-x^0)\langle 0|\,[\phi_+(y),\phi_-(x)]\,|0\rangle$$

Now the objects inside the vacuum expectation value are c-numbers, so we can drop the expectation value altogether, getting:

$$D_F(x-y) = \theta(x^0-y^0)\,[\phi_+(x),\phi_-(y)] + \theta(y^0-x^0)\,[\phi_+(y),\phi_-(x)]$$

As we will see, the Feynman propagator and the concept of time-ordering are fundamental in Quantum Field Theory.

1.8 Actions and field equations

Let us now study some basic properties of the Klein-Gordon equation and the Lagrangian from which it is obtained. Our considerations will be classical for a while. The Klein-Gordon equation is:

$$(\partial_\mu\partial^\mu + m^2)\phi = 0$$

This can be thought of as a variational equation obtained from the action:

$$S = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2\right)$$

To show this, we must calculate $\dfrac{\delta S}{\delta\phi(y)}$ which is a *functional derivative*. For this, the basic relation is:

$$\frac{\delta\phi(x)}{\delta\phi(y)} = \delta^4(x-y)$$

which can be understood as an extension to the continuous case of the differentiation rule for variables depending on a discrete index, namely:

$$\frac{\delta x_i}{\delta x_j} = \delta_{ij}$$

where the RHS is the Kronecker delta.

Applying the functional derivative rule we find:

$$\begin{aligned}\frac{\delta S}{\delta\phi(y)} &= \int d^4x \left[\partial_\mu\phi(x)\partial^\mu\delta^4(x-y) - m^2\phi(x)\delta^4(x-y)\right] \\ &= \int d^4x \left[-\partial_\mu\partial^\mu\phi(x) - m^2\phi(x)\right]\delta^4(x-y) \\ &= -\left(\partial_\mu\partial^\mu\phi(y) + m^2\phi(y)\right)\end{aligned}$$

Thus the equation of motion is:

$$\frac{\delta S}{\delta\phi(y)} = 0 \quad \Rightarrow \quad (\partial_\mu\partial^\mu + m^2)\phi(y) = 0$$

which is the free Klein-Gordon equation as desired.

We will later encounter more general actions, which will always be integrals over space-time of a local Lagrangian density depending on ϕ and (usually) first derivatives of ϕ:

$$S = \int d^4x\, \mathcal{L}(\phi, \partial_\mu\phi)$$

It is a straightforward exercise to show that in terms of the Lagrangian density, the variational equation $\delta S/\delta\phi(x) = 0$ is equivalent to:

$$\partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi(x))} - \frac{\delta\mathcal{L}}{\delta\phi(x)} = 0$$

To generalise the Klein-Gordon Lagrangian to a complex field ϕ, we simply write:

$$S(\phi, \phi^\star) = \int d^4x \left(\partial_\mu\phi^\star\partial^\mu\phi - m^2\phi^\star\phi\right)$$

whose equations of motion are:

$$(\partial_\mu\partial^\mu + m^2)\phi = (\partial_\mu\partial^\mu + m^2)\phi^\star = 0.$$

as desired. Note that when a field is complex, we vary ϕ and ϕ^* independently.

Given a Lagrangian density $\mathcal{L}$, the Hamiltonian density is defined by first defining the canonical momentum conjugate to the field variable ϕ:

$$\pi(x) = \frac{\delta\mathcal{L}}{\delta\dot{\phi}(x)}$$

and then writing

$$\mathcal{H} = \pi(x)\dot{\phi}(x) - \mathcal{L}$$

and finally eliminating $\dot{\phi}$ in favour of π.

For the Klein-Gordon case we easily find that $\pi = \dot{\phi}$ and

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2$$

1.9 Interactions

To introduce interactions, we must add terms to the Lagrangian of Klein-Gordon theory which are of higher than quadratic power in the field ϕ. The resulting field equations will then be *nonlinear*.

Thus, consider:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - V(\phi)$$

where $V(\phi)$ is some function (with powers higher than quadratic) of ϕ, called the "potential". Note that it is a potential in field space, not a potential $V(x)$ in ordinary space.

With this, the Hamiltonian density is

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 + V(\phi)$$

What is a reasonable choice for $V(\phi)$? Let us take a fourth order polynomial:

$$V(\phi) = a + b\phi + c\phi^2 + d\phi^3 + e\phi^4$$

We will see later that scalar field theories with higher than four powers of ϕ are "non-renormalisable" and therefore inconsistent.

In this potential the constant term is irrelevant for physics so we drop it. The linear term can removed by shifting ϕ. The quadratic term just modifies the value of m^2 so we can ignore it. That leaves the cubic and quartic term. It is common to take just the quartic term, i.e. set $a = b = c = d = 0$. Conventionally the potential is then written as:

$$V(\phi) = \frac{\lambda}{4!}\phi^4$$

The corresponding equation of motion is:

$$(\partial_\mu\partial^\mu + m^2)\phi + \frac{\lambda}{3!}\phi^3 = 0$$

To study the resulting quantum theory, we are going to assume that the effect of interactions is small and answers can be expressed as a power series in λ, with $\lambda = 0$ giving the usual "free" answers that we have already obtained.

The Hamiltonian of the interacting theory can be written:

$$\begin{aligned} H &= \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right) \\ &= H_0 + H_{\text{int}} \end{aligned}$$

where $H_{\rm int} = \dfrac{\lambda}{4!}\int d^3x\,\phi(x)^4$.

We already know how to treat H_0. We showed that H_0 could be converted into:

$$H_0 = \int \frac{d^3k}{(2\pi)^3}\,\omega_{\vec{k}}\,a^\dagger_{\vec{k}}a_{\vec{k}}, \qquad [a_{\vec{k}}, a^\dagger_{\vec{k}'}] = (2\pi)^3\delta^3(\vec{k}-\vec{k}')$$

with $|0\rangle$ defined by $a_{\vec{k}}|0\rangle = 0$ for all $\vec{k}$.

Now we can write $\int d^3x\,\phi(x)^4$ in terms of $a_{\vec{k}}$, $a^\dagger_{\vec{k}}$ and therefore

$$H = \int \frac{d^3k}{(2\pi)^3}\,\omega_{\vec{k}}\,a^\dagger_{\vec{k}}a_{\vec{k}} + H_{\rm int}$$

where $H_{\rm int}$ is a function of $a^4, a^\dagger a^3, a^{\dagger 2}a^2, a^{\dagger 3}a, a^{\dagger 4}$. It is clear that generically, $H_{\rm int}|0\rangle \neq 0$.

The state Ω of minimum energy, or the "true vacuum"

$$(H_0 + H_{\rm int})|\Omega\rangle = \text{minimum}$$

is therefore very complicated and cannot be calculated exactly. However, in terms of this state we know what we would like to compute: the probability for a particle to propagate from one point to another at a later time in the true vacuum of the interacting theory:

$$\langle\Omega|T(\phi(x)\phi(y))|\Omega\rangle$$

We will now see how to "expand" $|\Omega\rangle$ and $\phi(x)$ in terms of their free versions $|0\rangle$, $\phi_0(x)$. We have already encountered the definitions:

$$\begin{aligned} |0\rangle &: H_0|0\rangle = 0 \\ |\Omega\rangle &: (H_0 + H_{\rm int})|\Omega\rangle = E_0|\Omega\rangle \end{aligned}$$

where E_0 is the lowest energy eigenvalue of $H_0 + H_{\rm int}$.

Similarly, given a field configuration $\phi(0,\vec{x})$ at time 0, the free field $\phi_0(t,\vec{x})$ is the one that evolves via H_0, while the full $\phi(t,\vec{x})$ evolves via H:

$$\begin{aligned} \phi_0(t,\vec{x}) &= e^{iH_0t}\,\phi(0,\vec{x})\,e^{-iH_0t} \\ \phi(t,\vec{x}) &= e^{iHt}\,\phi(0,\vec{x})\,e^{-iHt} \end{aligned}$$

We worked out the time dependence of $\phi_0(t,\vec{x})$ when we studied free fields. What we now need is the time evolution of the full $\phi(t,\vec{x})$, which is not simple due to the interactions. We have:

$$\begin{aligned} \phi(t,\vec{x}) &= e^{iHt}\,\phi(0,\vec{x})\,e^{-iHt} \\ &= e^{iHt}\,e^{-iH_0t}\,\phi_0(t,\vec{x})\,e^{iH_0t}\,e^{-iHt} \end{aligned}$$

Now since $[H, H_0] \neq 0$, we *cannot* write $e^{iHt}\, e^{-iH_0 t}$ as $e^{i(H-H_0)t} = e^{iH_{\rm int}}$.

To overcome this problem, define the operator

$$U(t) \equiv e^{iH_0 t} e^{-iHt}$$

Then:

$$\phi(\vec{x}, t) = U(t)^\dagger\, \phi_0(\vec{x}, t)\, U(t)$$

We would now like to express $U(t)$ purely in terms of $H_{\rm int}$. To do this, we use the trick of first obtaining a differential equation for $U(t)$:

$$\begin{aligned}
\frac{\partial U}{\partial t} &= i\, e^{iH_0 t}(H_0 - H)\, e^{-iHt} \\
&= -i\, e^{iH_0 t} H_{\rm int}\, e^{-iHt} \\
&= -i\, e^{iH_0 t} H_{\rm int}\, e^{-iH_0 t}\, e^{iH_0 t} e^{-iHt} \\
&= -i H_I(t) U(t)
\end{aligned}$$

where we have defined:

$$H_I(t) = e^{iH_0 t} H_{\rm int}\, e^{-iH_0 t}$$

called the "interaction-picture Hamiltonian". Note that this Hamiltonian is constructed in terms of the field ϕ_0, which is sometimes known as the "interaction picture field".

The differential equation we have derived:

$$i\frac{\partial U}{\partial t} = H_I(t) U$$

is like a time-dependent Schrödinger equation (with a time-dependent Hamiltonian) for U! We can write a formal solution for it as:

$$U(t) = T\left\{ e^{-i\int_0^t dt' H_I(t')} \right\}$$

where T is our friend the "time-ordering" symbol. Without it, the exponential would not be a solution because $H_I(t_1)$ and $H_I(t_2)$ do not commute. The T symbol makes sure that when we differentiate, we pull out an $H_I(t)$ at the left-most point as desired.

Now consider the state $|\Omega\rangle$ defined by $(H_0 + H_{\rm int})|\Omega\rangle = E_0|\Omega\rangle$ where E_0 is the minimum eigenvalue of $H = H_0 + H_{\rm int}$. We would like to find a relation between $|\Omega\rangle$ and $|0\rangle$. For this, note that if $|n\rangle$ are the eigenstates of the full Hamiltonian H and E_n are the corresponding

eigenvalues, then:

$$\begin{aligned} e^{-iHt}|0\rangle &= \sum_n e^{-iE_n t}|n\rangle\langle n|0\rangle \\ &= e^{-iE_0 t}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_n t}|n\rangle\langle n|0\rangle \\ &= e^{-iE_0 t}\left(|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-i(E_n-E_0)t}|n\rangle\langle n|0\rangle\right) \end{aligned}$$

We can remove the contribution of all terms except the first one by taking a limit $t \to \infty$ keeping a slightly negative imaginary part. Thus,

$$\lim_{T\to\infty(1-i\epsilon)} e^{-iHT}|0\rangle = \lim_{T\to\infty(1-i\epsilon)} e^{-iE_0 T}|\Omega\rangle\langle\Omega|0\rangle$$

We assume the right hand side above is nonzero, i.e. that the true vacuum $|\Omega\rangle$ has some overlap with the free vacuum $|0\rangle$.

Hence we find that:

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{e^{-iHT}|0\rangle}{e^{-iE_0 T}\langle\Omega|0\rangle}$$

We can temporarily forget the normalising factor on the RHS and restore it later. Then the RHS can be written in terms of $U(T)$. Because $H_0|0\rangle = 0$, we can write:

$$e^{-iHT}|0\rangle = e^{-iHT}e^{iH_0 T}|0\rangle = U^\dagger(-T)|0\rangle$$

Therefore

$$|\Omega\rangle \sim \lim_{T\to\infty(1-i\epsilon)} U^\dagger(-T)|0\rangle$$

upto a normalisation.

Similarly, by starting with

$$\langle 0|e^{-iHt}$$

and performing the analogous manipulations, one can show that:

$$\langle\Omega| \sim \lim_{T\to\infty(1-i\epsilon)} \langle 0|U(T)$$

We will restore the normalisation by simply dividing the final answer by $\langle\Omega|\Omega\rangle$.

Recalling that $\phi(x) = U(x^0)^\dagger\phi_0(x)U(x^0)$ and $\phi(y) = U(y_0)^\dagger\phi_0(y)U(y_0)$, and taking first $x^0 > y^0$, we write:

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle \sim \lim_{T\to\infty(1-i\epsilon)} \langle 0|U(T)\,U^\dagger(x^0)\phi_0(x)U(x^0)\,U(y^0)^\dagger\phi_0(y)U(y^0)\,U^\dagger(-T)|0\rangle$$

Now use:

$$U(T)U^\dagger(x^0) = T\left(e^{-i\int_{x_0}^{T} H_I(t')dt'}\right)$$

which holds because T is a much later time than x^0 (at the end we will take $T \to \infty$). Indeed all fields are in time order, so we can write the RHS above as:

$$\lim_{T\to\infty(1-i\epsilon)} \langle 0|T\Big(U(T)\,U^\dagger(x^0)\phi_0(x)U(x^0)\,U(y^0)^\dagger\phi_0(y)U(y^0)\,U^\dagger(-T)\Big)|0\rangle$$

If we started with $x^0 > y^0$, we would get exactly the *same* result, since the RHS has time-ordering built into it.

Therefore we have shown that:

$$\langle\Omega|T\left(\phi(x)\phi(y)\right)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \langle 0|T\Big(U(T)\,U^\dagger(x^0)\phi_0(x)U(x^0)\,U(y^0)^\dagger\phi_0(y)U(y^0)\,U^\dagger(-T)\Big)|0\rangle$$

Inside the time ordering we can move around operators as we like. Therefore we can write the RHS as:

$$\begin{aligned} & \lim_{T\to\infty(1-i\epsilon)} \langle 0|T\Big(\phi_0(x)\phi_0(y)U(T)U^\dagger(x^0)U(x^0)U^\dagger(y^0)U(y^0)U^\dagger(-T)\Big)|0\rangle \\ = & \lim_{T\to\infty(1-i\epsilon)} \langle 0|T\Big(\phi_0(x)\phi_0(y)U(T)U^\dagger(-T)\Big)|0\rangle \end{aligned}$$

Now

$$\begin{aligned} U(T)U^\dagger(-T) &= T\left(e^{-i\int_0^T dt' H_I(t')}\right) T\left(e^{i\int_0^{-T} dt' H_I(t')}\right) \\ &= T\left(e^{-i\int_{-T}^{T} dt' H_I(t')}\right) \end{aligned}$$

So, after restoring the normalisation, we have finally arrived at a result of great importance:

$$\frac{\langle\Omega|T\left(\phi(x)\phi(y)\right)|\Omega\rangle}{\langle\Omega|\Omega\rangle} = \lim_{T\to(1-i\epsilon)} \frac{\langle 0|T\left(\phi_0(x)\phi_0(y)e^{-\int_{-T}^{T} dt' H_I(t')}\right)|0\rangle}{\langle 0|T\left(e^{-\int_{-T}^{T} dt' H_I(t')}\right)|0\rangle}$$

It is easy to convince oneself that this result generalises to a product of any number of fields.

1.10 Wick's theorem

Having deriving the general result:

$$\frac{\langle\Omega|T\left(\phi(x_1)\cdots\phi(x_n)\right)|\Omega\rangle}{\langle\Omega|\Omega\rangle} = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\left(\phi_0(x_1)\cdots\phi_0(x_n)e^{-i\int_{-T}^{T} H_I dt}\right)|0\rangle}{\langle 0|e^{-i\int_{-T}^{T} H_I dt}|0\rangle}$$

we now need some rules to manipulate the RHS. This is made up entirely out of free fields. Therefore we essentially need a rule to compute:

$$\langle 0|T\left(\phi_0(x_1)\cdots\phi_0(x_m)\right)|0\rangle$$

for all possible m, in free field theory.

In deriving these rules we temporarily go back to free field theory and drop the "0" index on ϕ. The rule we are trying to derive is called "Wick's theorem".

Let us now define the concept of "normal ordering". This puts creation operators to the left and annihilation operators to the right. In terms of $\phi_\pm$, this just puts ϕ_+ to the right of ϕ_-. The key property of normal ordered products is that their vacuum expectation value vanishes.

Wick's theorem is a relation between time ordering and normal ordering. We now derive it for a few special cases and then state the general result without proof.

We start by considering the time-ordered product of two fields:

$$\begin{aligned}
T\left(\phi(x_1)\phi(x_2)\right) &= \theta(t_1-t_2)\phi(x_1)\phi(x_2)+\theta(t_2-t_1)\phi(x_2)\phi(x_1)\\
&= \theta(t_1-t_2)\Big(\phi_+(x_1)\phi_+(x_2)+\phi_+(x_1)\phi_-(x_2)+\phi_-(x_1)\phi_+(x_2)+\phi_-(x_1)\phi_-(x_2)\Big)\\
&\quad+\theta(t_2-t_1)\Big(\phi_+(x_2)\phi_+(x_1)+\phi_+(x_2)\phi_-(x_1)+\phi_-(x_2)\phi_+(x_1)+\phi_-(x_2)\phi_-(x_1)\Big)\\
&= \phi_+(x_1)\phi_+(x_2)+\phi_-(x_1)\phi_-(x_2)\\
&\quad+\theta(t^1-t^2)\Big(\phi_+(x_1)\phi_-(x_2)+\phi_-(x_1)\phi_+(x_2)\Big)\\
&\quad+\theta(t^2-t^1)\Big(\phi_+(x_2)\phi_-(x_1)+\phi_-(x_2)\phi_+(x_1)\Big)
\end{aligned}$$

On the other hand, the normal ordered product of the two fields is:

$$:\phi(x_1)\,\phi(x_2):\;=\;\phi_+(x_1)\phi_+(x_2)+\phi_-(x_1)\phi_+(x_2)+\phi_-(x_2)\phi_+(x_1)+\phi_-(x_1)\phi_-(x_2)$$

Thus:

$$\begin{aligned}
T(\phi(x_1)\phi(x_2))-:\phi(x_1)\phi(x_2): &= \theta(t_1-t_2)(\phi_+(x_1)\phi_-(x_2)+\phi_-(x_1)\phi_+(x_2))\\
&\quad+\theta(t_2-t_1)(\phi_+(x_2)\phi_-(x_1)+\phi_-(x_2)\phi_+(x_1))\\
&\quad-(\theta(t_1-t_2)+\theta(t_2-t_1))[\phi_-(x_1)\phi_+(x_2)+\phi_-(x_2)\phi_+(x_1)]\\
&= \theta(t_1-t_2)[\phi_+(x_1),\phi_-(x_2)]+\theta(t_2-t_1)[\phi_+(x_2),\phi_-(x_1)]\\
&= \langle 0|T(\phi(x_1)\phi(x_2))|0\rangle
\end{aligned}$$

where the last equality was shown when we discussed the Feynman propagator.

Thus we have shown that:

$$T(\phi(x_1)\phi(x_2)) = :\phi(x_1)\phi(x_2): + \langle 0|T(\phi(x_1)\phi(x_2))|0\rangle$$

This is the first example of Wick's theorem. If we take the vacuum expectation value on both sides, we get a trivial identity because the vev of the normal ordered term vanishes. The content of the theorem becomes nontrivial if we look at higher point functions $T(\phi(x_1)\cdots\phi(x_n))$, $n > 2$. Applying the same steps as above to the four-point function, and using the shorthand notation $\phi_i \equiv \phi(x_i)$, we find:

$$\begin{aligned} T(\phi_1\phi_2\phi_3\phi_4) &= :\phi_1\phi_2\phi_3\phi_4: \\ &\quad + :\phi_1\phi_2: \langle 0|T(\phi_3\phi_4)|0\rangle + :\phi_1\phi_3: \langle 0|T(\phi_2\phi_4)|0\rangle + :\phi_1\phi_4: \langle 0|T(\phi_2\phi_3)|0\rangle \\ &\quad + :\phi_2\phi_3: \langle 0|T(\phi_1\phi_4)|0\rangle + :\phi_2\phi_4: \langle 0|T(\phi_1\phi_3)|0\rangle + :\phi_3\phi_4: \langle 0|T(\phi_1\phi_2)|0\rangle \\ &\quad + \langle 0|T(\phi_1\phi_2)|0\rangle\langle 0|T(\phi_3\phi_4)|0\rangle + \langle 0|T(\phi_1\phi_3)|0\rangle\langle 0|T(\phi_2\phi_4)|0\rangle \\ &\qquad + \langle 0|T(\phi_1\phi_4)|0\rangle\langle 0|T(\phi_2\phi_3)|0\rangle \end{aligned}$$

This time we get a nontrivial result by taking the vev on both sides:

$$\begin{aligned} \langle 0|T(\phi_1\phi_2\phi_3\phi_4)|0\rangle &= \langle 0|T(\phi_1\phi_2)|0\rangle\langle 0|T(\phi_3\phi_4)|0\rangle + \langle 0|T(\phi_1\phi_3)|0\rangle\langle 0|T(\phi_2\phi_4)|0\rangle \\ &\quad + \langle 0|T(\phi_1\phi_4)|0\rangle\langle 0|T(\phi_2\phi_3)|0\rangle \\ &= D_F(x_{12})D_F(x_{34}) + D_F(x_{13})D_F(x_{24}) + D_F(x_{14})D_F(x_{23}) \end{aligned}$$

where we have introduced the shorthand $x_{ij} = x_i - x_j$. This reduces the calculation of four-point functions to products of pairs of Feynman propagators in all possible ways.

The above result is easily extended to all even-point functions in free field theory, which reduce to products of two-point functions in all possible distinct ways. The total number of possible "two-point" contractions among $2n$ points is easily seen to be $(2n-1)!! = (2n-1)(2n-3)\cdots$. For the 4-point function we have 3 independent contractions, as above, while for the 6-point function we would get 15 independent contractions. Odd n-point functions vanish.

1.11 Interactions via Wick's theorem

Now we turn to the interacting theory. The new feature is that we encounter fields coming from expanding the exponential of H_I. Thus, consider for example the leading correction to the propagator in the theory with $H_{\rm int} = \dfrac{\lambda}{4!}\displaystyle\int d^3x\,\phi^4$.

We find $\langle\Omega|T(\phi(x_1)\phi(x_2))|\Omega\rangle$

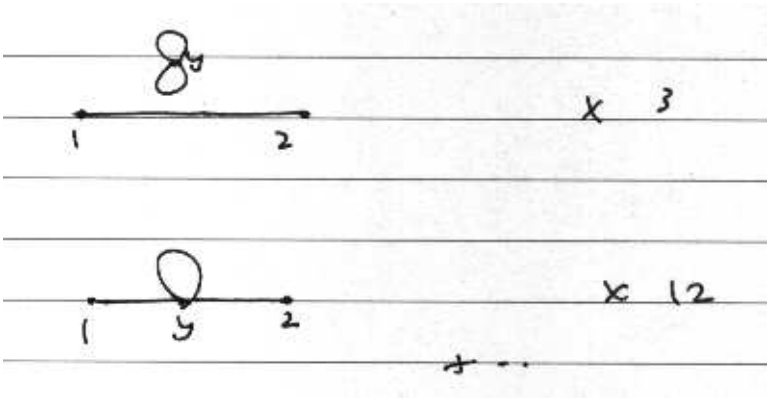
$$= \langle 0|T(\phi(x_1)\phi(x_2))|0\rangle - i\,\frac{\lambda}{4!}\int d^4y\,\langle 0|T(\phi(x_1)\phi(x_2)\phi^4(y))|0\rangle$$

The second term is a 6-point function, which we certainly know how to compute using Wick's theorem. The new feature is that four of the fields are at the *same* space-time point, and there is an integral over all possible locations of that point.

Wick's theorem applied to this 6-point function gives:

$$\begin{aligned}\langle 0|T(\phi(x_1)\phi(x_2)\phi^4(y))|0\rangle &= 3\,D_F(x_1-x_2)\,D_F(y-y)\,D_F(y-y)\\ &\quad + 12\,D_F(x_1-y)\,D_F(x_2-y)\,D_F(y-y)\end{aligned}$$

Diagrammatically we can represent this by:



In this diagram the points x_1 and x_2 are labelled by "1" and "2". Each line represents a Feynman propagator from the starting point to the end-point of the line. Since we are working to first order in the interaction, the first term in the expansion of $e^{-i\int H_I}$ is represented by a single four-point vertex. In general the number of four-point vertices in the diagram counts the order of perturbation theory. Such diagrams with vertices and lines, depicting the possible terms arising from Wick contraction in perturbation theory, are called *Feynman diagrams*.

We see a variety of potential problems with this result. For the second term, corresponding to the connected diagram, we get a factor of $D_F(y-y) = D_F(0)$. Now,

$$D_F(0) = \int d^4p\,\frac{i}{p^2-m^2}$$

which is divergent. The divergence comes from the region of large $|\vec{p}|$, so it is an ultraviolet divergence.

However, in the first term, which diagrammatically corresponds to a loop disconnected from the freely propagating particle, the integrand is y-independent! This integral therefore appears to give a divergence proportional to the volume of space-time. This is multiplied by a factor $(D_F(0))^2$ which as we have seen above is also divergent.

In general we should always imagine cutting off ultraviolet divergences using a large momentum cutoff, $|\vec{p}| < \Lambda$. This is called a UV cutoff. For the volume divergence we can put the system in a finite box in space-time, which is called an IR cutoff. The problem is then how to remove the cutoffs. We will return to this later.

Fortunately we can easily dispose of the problem of disconnected diagrams. The "figure of 8" diagram above seems rather unphysical, after all: a virtual interaction takes place at an arbitrary point y while the external particle propagates freely from x_1 to x_2. In fact, this term is cancelled by a similar term coming from the denominator, where we have to expand:

$$\langle 0|T(e^{-i\int H_I})|0\rangle = 1 - 3\frac{i\lambda}{4!}\int d^4y\, D_F(y-y)D_F(y-y) + \cdots$$

Thus we have, to order λ,

$$\begin{aligned}
&\frac{\langle\Omega|T(\phi(x)\phi(y))|\Omega\rangle}{\langle\Omega|\Omega\rangle} \\
&= \frac{D_F(x_1-x_2)(1-3\frac{i\lambda}{4!}\int d^4y\, D_F(0)^2) - 12\frac{i\lambda}{4!}\int d^4y\, D_F(0)D_F(x_1-y)D_F(x_2-y) + \cdots}{1-3\frac{i\lambda}{4!}\int d^4y\, D_F(0)^2 + \cdots} \\
&= D_F(x_1-x_2)\left(1-3\tfrac{i\lambda}{4!}\textstyle\int d^4y\, D_F(0)^2\right)\left(1+3\tfrac{i\lambda}{4!}\textstyle\int d^4y\, D_F(0)^2\right) \\
&\qquad\qquad - 12\tfrac{i\lambda}{4!}\textstyle\int d^4y\, D_F(x_1-y)\, D_F(x_2-y)\, D_F(0) + \mathcal{O}(\lambda^2)
\end{aligned}$$

Since we are working to order λ, we see that:

$$\left(1-3\tfrac{i\lambda}{4!}\textstyle\int d^4y\, D_F(0)^2\right)\left(1+3\tfrac{i\lambda}{4!}\textstyle\int d^4y\, D_F(0)^2\right) = 1 + \mathcal{O}(\lambda^2)$$

and the disconnected part has cancelled! This is an example of a general phenomenon. All disconnected diagrams can be shown to be cancelled by denominator contributions. Moreover this uses up all denominator contributions, so we can consistently ignore disconnected diagrams as well as denominators.

Therefore, to calculate any correlation function:

$$\langle 0|T\left(\phi(x_1)\phi(x_2)\cdots\phi(x_n)\right)|0\rangle$$

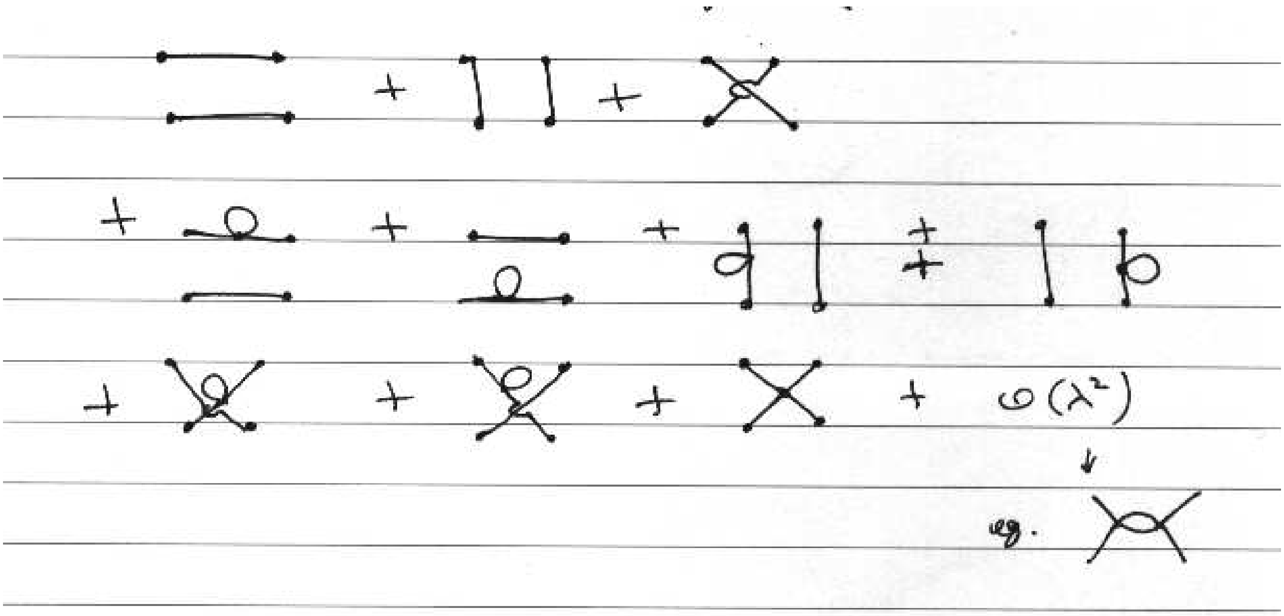
we only need to keep *connected* diagrams, by which we mean all parts of the diagram are connected to at least one of the external legs $x_1, x_2, \cdots, x_n$ (there is another meaning of

"connected" in which the diagram is required to have all points connected to each other. This may arise in later discussions).

As an example, the connected diagrams for the four-point function:

$$\langle 0|T\left(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\right)|0\rangle$$

are given to order λ by:



1.12 Momentum space

Feynman diagrams are shorthand symbols for numbers depending on external positions as well as the coupling constant λ and mass m. But in practice we rarely produce particles at fixed positions. Instead they are produced at fixed momenta so that they can scatter. Therefore it is more appropriate to consider the Fourier transform of the position space n-point function that we have studied up to now. In fact we will see that momentum space is more natural and simpler than position space.

We have seen that some diagrams give divergent answers. Among other things we will show in what follows, using momentum space, that only diagrams with closed loops can have UV divergences.

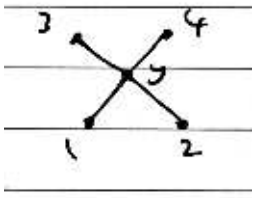
Define:

$$\tilde{D}_F(k) = \frac{i}{k^2 - m^2 + i\epsilon}$$

which is the Fourier transform of the Feynman propagator by virtue of the relation:

$$D_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \tilde{D}_F(k) e^{-ik\cdot(x-y)}$$

Now consider the diagram:



This can be evaluated as:

$$-i\lambda \int d^4y D_F(x_1-y) D_F(x_2-y) D_F(x_3-y) D_F(x_4-y)$$

$$= \quad -i\lambda \int d^4y \prod_{i=1}^{4} \left(\int \frac{d^4k_i}{(2\pi)^4} \frac{1}{(k_i^2 - m^2 + i\epsilon)} \right) e^{-i\sum_i k_i\cdot(x_i-y)}$$

The integral over y is:

$$\int d^4y \; e^{iy\cdot\sum_i k_i} = (2\pi)^4 \delta^4\Big(\sum_i k_i\Big)$$

Thus the above expression is equal to:

$$-i\lambda \int \prod_{i=1}^{4} \left(\frac{d^4k_i}{(2\pi)^4} \frac{1}{(k_i^2 - m^2 + i\epsilon)} \right) (2\pi)^4 \delta\Big(\sum_i k_i\Big) e^{-i\sum_i k_i\cdot x_i}$$

Thus the contribution of this diagram to $G(x_1, \cdots, x_4) = \langle\Omega|T(\phi(x_1)\cdots\phi(x_4))|\Omega\rangle$ is:

$$\int \prod_i \frac{d^4k_i}{(2\pi)^4} \, e^{-i\sum k_i\cdot x_i} \, (2\pi)^4 \delta\Big(\sum_i k_i\Big) \, \tilde{G}(k_1, \cdots, k_4)$$

where:

$$\tilde{G}(k_1, \cdots, k_4) = -i\lambda \prod_{i=1}^{4} \frac{1}{k_i^2 - m^2 + i\epsilon}$$

We refer to $\tilde{G}(k_1, \cdots, k_4)$ as the "momentum space correlation function" although it is not $\tilde{G}(k_1, \cdots, k_4)$ but rather $(2\pi)^4\delta(\Sigma k_i)\tilde{G}(k_1, \cdots, k_4)$ that is the Fourier transform of $G(x_1 \cdots x_4)$. An overall momentum-conserving delta-function occurs automatically for every Feynman diagram.

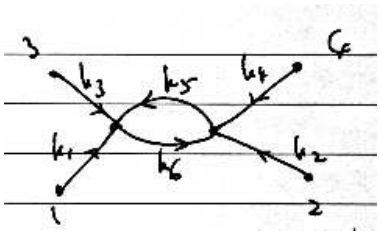
In fact, the above diagram could occur as *part* of a Feynman diagram, so we actually find a momentum-conserving δ-function at *each* vertex of a Feynman diagram.

Now, apart from the combinatoric factors (which are the same as in position space) we can describe momentum-space Feynman diagrams by giving each leg an independent momentum, then assigning a propagator

$$\tilde{D}_F(k) = \frac{1}{k^2 - m^2 + i\epsilon}$$

to each line, putting in a momentum-conserving δ-function at each vertex and then integrating over each momentum except the ones directly connected to external points. All momenta should have arrows describing their orientation. The arrows are arbitrary but must be assigned once and for all at the start. The signs of the momenta appearing in momentum-conserving δ-functions will be determined by these arrows.

Example: Consider a particular one-loop correction to the four-point function:



We must integrate over k_5, k_6 as these are not any of the external momenta. Now we have the δ-functions:

$$\delta^4(k_1 + k_3 + k_5 - k_6)\,\delta^4(k_2 + k_4 + k_6 - k_5)$$

A δ-function contributes only when its argument vanishes, so we can substitute the vanishing of the argument of the second δ-function into the first δ-function, to get:

$$= \delta^4(k_1 + k_2 + k_3 + k_4)\,\delta^4(k_2 + k_4 + k_6 - k_5)$$

As expected, there is the overall δ-function that conserves external momenta. The other one fixes, say, k_6 in terms of k_5 and external momenta, so we can write $k_6 = k_5 - k_2 - k_4$ and forget the d^4k_6 integration. That leaves only an integral over k_5.

In this way it is easy to see that there is one momentum integral for every closed loop. The physical interpretation is that there is a "virtual particle" circulating in the loop. This particle does not need to satisfy $k^2 = m^2$, nor does it need to have a positive value for k^0. However, external momenta should satisfy $k^2 = m^2$ and this property is referred to as being "on-shell". Internal momenta such as loop momenta, by contrast, are "off-shell".

A puzzle here is that if the external momenta $k_1, \cdots, k_4$ are placed on-shell then the external propagators all diverge. Therefore temporarily we will allow even external momenta to be off-shell. Keeping the external legs of correlation functions off-shell can be very useful to develop recursive formulae among them. When we connect the n-point function to physically measured quantities, specifically the scattering matrix or S-matrix, we will show that external propagators should be dropped and *then* the external momenta put on-shell.

Returning to the diagram we drew above, the corresponding loop integral is:

$$\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \frac{i}{(k - k_2 - k_2)^2 - m^2}$$

We see that as $k \to \infty$, the numerator and denominator both scale like $|k|^4$. Thus the integral is *logarithmically divergent* in the ultraviolet. We will discuss how to deal with such divergent integrals at a later stage.

2 Vector fields

2.1 Definition of vector field

A Lorentz transformation acts on space-time coordinates as:

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

The matrix Λ satisfies:

$$\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \, \eta_{\mu\nu} = \eta_{\alpha\beta}, \qquad \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \, \eta^{\alpha\beta} = \eta^{\mu\nu}$$

Suppose we now consider a field $A^\mu(x)$ which transforms in the same way:

$$A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x)$$

Such a field would be called a *vector field* rather than a scalar field. If we lower the index of this field via:

$$A_\mu(x) \equiv \eta_{\mu\nu} A^\nu(x)$$

then we find that it transforms as:

$$A'_\mu(x') \equiv \eta_{\mu\alpha} A'^\alpha(x') = \eta_{\mu\alpha} \Lambda^\alpha{}_\nu A^\nu(x) = \eta_{\mu\alpha} \Lambda^\alpha{}_\nu \eta^{\nu\beta} A_\beta(x) = (\Lambda^{-1\,T})_\mu{}^\beta A_\beta(x)$$

where we have used the result:

$$\eta_{\mu\alpha} \Lambda^\alpha{}_\nu \eta^{\nu\beta} = (\Lambda^{-1\,T})_\mu{}^\beta$$

which is straightforward to prove and is left as an exercise.

If we are given a scalar field $\phi(x)$, then $\partial_\mu \phi(x)$ provides an example of a vector field:

$$\partial'_\mu \phi'(x') = (\Lambda^{-1\,T})_\mu{}^\nu \partial_\nu \phi(x)$$

However our main interest is to consider vector fields A_μ that are fundamental and not necessarily of the form $\partial_\mu \phi$ for some scalar field.

2.2 Lagrangian for vector fields

What sort of Lagrangian can we write for such a field? By analogy with Klein-Gordon theory we might guess:

$$\mathcal{L} = \frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu$$

whose equation of motion is:

$$\partial_\mu \partial^\mu A_\nu = 0$$

In this case, A_ν ($\nu = 0, 1, 2, 3$) behaves like 4 copies of a Klein-Gordon field (we will return to the possibility of mass terms later).

However, this immediately leads to a problem. The canonical formalism gives

$$\pi^\mu = \frac{\delta \mathcal{L}}{\delta \dot{A}_\mu} = \dot{A}^\mu$$

Then the canonical commutator should be

$$[A_\mu(t, \vec{x}), \pi^\nu(t, \vec{x}')] = i\delta^\mu{}_\nu \delta^3(\vec{x} - \vec{x}')$$

This in turn means that

$$[A_\mu(t, \vec{x}), \dot{A}_\nu(t, \vec{x}')] = i\eta_{\mu\nu} \delta^3(\vec{x} - \vec{x}')$$

which leads to oscillators $a_{\mu\vec{k}}$, $a^\dagger_{\mu,\vec{k}}$ satisfying

$$[a_{\mu\vec{k}}, a^\dagger_{\nu\vec{k}'}] = (2\pi)^3 \eta_{\mu\nu} \delta^3(\vec{k} - \vec{k}')$$

This means

$$[a_{0,\vec{k}}, a^\dagger_{0,\vec{k}'}] = (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

while

$$[a_{i,\vec{k}}, a^\dagger_{j,\vec{k}'}] = -\delta_{ij} (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

Thus three of the four oscillators seem to have an unconventional – sign in their commutator!

We would be better off taking a – sign in the original canonical commutator. This can be achieved by flipping the Lagrangian to

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu \\ &= -\frac{1}{2} \partial_\mu A_0 \partial^\mu A_0 + \frac{1}{2} \partial_\mu A_i \partial^\mu A_i \end{aligned}$$

Now $\pi^\mu = -\dot{A}^\mu$ and

$$[A_\mu(\vec{x}), \dot{A}_\nu(\vec{x}')] = -i\eta_{\mu\nu} \delta^3(\vec{x} - \vec{x}')$$

which in turn means that

$$[a_{\mu,\vec{k}}, a^\dagger_{\nu,\vec{k}'}] = -\eta_{\mu\nu} (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

Now the space-like oscillators have the correct sign, however for the time-like oscillator we find:

$$[a_{0,\vec{k}}, a^\dagger_{0,\vec{k}'}] = -(2\pi)^3\delta^3(\vec{k}-\vec{k}')$$

An oscillator with a minus sign in its commutator is very badly behaved. To see this, consider the state:

$$a^\dagger_{0,\vec{k}}|0\rangle$$

Its norm is given by

$$\langle 0|a_{0,\vec{k}}\, a^\dagger_{0,\vec{k}'}|0\rangle = \langle 0|[a_{0,\vec{k}}, a^\dagger_{0,\vec{k}'}]|0\rangle = -(2\pi)^3\delta^3(\vec{k}-\vec{k}')$$

which means that this is a *negative norm state*. This leads to negative probabilities and a physically meaningless theory.

We conclude that the Lagrangian above is wrong. In the process, we see that copying the Klein-Gordon Lagrangian for a field with vector indices simply cannot work. The reason is that a Lorentz covariant commutation relation inevitably involves $\eta_{\mu\nu}$ and somewhere there is bound to be a negative sign.

Recall that Maxwell's equations, abstracted from experimental observations, are:

$$\partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) = 0$$

which can be more compactly written:

$$\partial^\mu F_{\mu\nu} = 0 \quad \text{where} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Notice that $F_{\mu\nu} = -F_{\nu\mu}$.

The above equations of motion come from the Lagrangian:

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu \\
&= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad (1)
\end{aligned}$$

In the second line, the first term is what we had tried without success but now there is a second term with a different arrangement of indices. Does this Lagrangian manage to avoid the problem of negative norm states?

Right away we see an encouraging sign that the problem of negative-norm states might be absent. When we compute canonical momenta, we find:

$$\pi^\mu = \frac{\delta\mathcal{L}}{\delta\dot{A}_\mu} = -F^{0\mu}$$

By antisymmetry $F^{00} = 0$, so we have $\pi^0 = 0$. Thus the potentially troublesome commutator is absent. In fact there is *no* canonical commutation relation for A_0! It turns out that, for this reason, A_0 is not a dynamical variable at all.

To understand the source of this unusual property, we note that the Lagrangian of electrodynamics is unchanged under the transformation:

$$A_\mu \to A_\mu + \partial_\mu \lambda(x)$$

for an arbitrary function $\lambda(x)$. This is called a *gauge transformation*. To demonstrate the invariance, simply note that under the transformation,

$$F_{\mu\nu} \to \partial_\mu A_\nu + \partial_\mu \partial_\nu \lambda - \partial_\nu A_\mu - \partial_\nu \partial_\mu \lambda = F_{\mu\nu}$$

so $F_{\mu\nu}$ is gauge invariant and consequently so is any Lagrangian made out of it.

Gauge invariance means that the space of gauge field configurations is "degenerate". Any configuration $A_\mu(x)$ and another one $A_\mu(x) + \partial_\mu \lambda(x)$ have the same Lagrangian. And this is not just a finite parameter degeneracy but an infinite parameter one, parametrised by the whole function $\lambda(x)$.

The physical interpretation for this is that the configurations A_μ, $A_\mu + \partial_\mu \lambda$ are *physically equivalent*. This means that the theory has *less* physical content than it originally seems. In fact we can simplify it down to its physical degrees of freedom, but at the cost of manifest Lorentz invariance.

Gauge invariance seems like a complicated and undesirable feature because it introduces a redundancy in the field configurations. However it is necessary in order to reconcile Lorentz invariance with unitarity (the fact that all physical states have positive norm). Note that with gauge invariance, there is no longer the possibility of adding a mass term $m^2 A_\mu A^\mu$ to this theory. Such a term would violate gauge invariance and ultimately bring back negative-norm states.

As a matter of terminology, we remark that vector fields with gauge invariance are often referred to as "gauge fields". The simple type of gauge invariance discussed here, where A_μ is a single field and $\lambda(x)$ is a single function, has the property that the gauge transformations commute with each other. Indeed, under two successive gauge transformations with parameters $\lambda_1(x), \lambda_2(x)$, we have:

$$A_\mu \to (A_\mu + \partial_\mu \lambda_1) + \partial_\mu \lambda_2 = (A_\mu + \partial_\mu \lambda_2) + \partial_\mu \lambda_1$$

Therefore it is referred to as "Abelian gauge invariance". Later we will discuss a generalisation called "non-Abelian gauge invariance" in which two different gauge transformations will not commute with each other. Both types of gauge invariance are relevant in nature.

2.3 Coulomb gauge

We now discuss ways to "fix" the gauge invariance. In this way the physical content of the theory gets revealed. We start by choosing a gauge called Coulomb gauge. For this, given any configuration $A_\mu(t, \vec{x})$, start by choosing a gauge parameter:

$$\lambda(t, \vec{x}) = -\int^t A_0(t', \vec{x})\, dt'$$

Under a gauge transformation with this parameter we see that:

$$A_0 \to A_0 + \partial_0 \lambda = 0$$

In this way we can make A_0 disappear from the theory altogether. We see that it was a "gauge artifact", rather than a physical configuration to be quantised.

Now let us perform a further gauge transformation involving $\lambda(\vec{x})$ that is independent of time. For such λ, we have:

$$\begin{aligned} A_0 &\to A_0 + \partial_0 \lambda(\vec{x}) = 0 \\ A_i &\to A_i + \partial_i \lambda(\vec{x}) \end{aligned}$$

So the $A_0 = 0$ condition is preserved. We see that there remains a redundancy in the A_i.

Next consider the equations of motion: $\partial^\mu F_{\mu\nu} = 0$. If we set the free index ν to be 0, we get:

$$\partial^i F_{0i} = 0$$

Since we have set $A_0 = 0$, this implies

$$\partial_0(\partial^i A_i) = 0$$

so we see that the spatial divergence of A_i is time independent.

If instead we set $\nu = j$ in the equation of motion, we find:

$$\partial^0 F_{0j} + \partial^i F_{ij} = 0$$

and therefore:

$$\partial^0 \partial_0 A_j + \partial^i(\partial_i A_j - \partial_j A_i) = 0$$

Without the last term, this equation is quite simple, namely a Klein-Gordon equation for the three oscillators A_i. To remove the last term we notice that:

$$\partial^i A_i \to \partial^i A_i + \partial^i \partial_i \lambda$$

Therefore if we choose $\lambda(\vec{x})$ to solve the Poisson equation

$$\partial^i \partial_i \lambda = -\partial^j A_j$$

then making a gauge transformation with this λ, we find:

$$\partial^i A_i \;\rightarrow\; \partial^i A_i + \partial^i \partial_i \lambda = 0$$

Importantly, once we have set $\partial^i A_i = 0$ in this way, the A_0 equation of motion $\partial_0(\partial^i A_i) = 0$ guarantees that $\partial^i A_i$ remains 0 for all time.

Now the equations of motion simplify to:

$$\Box A_j = 0$$

with the constraint $\delta^j A_j = 0$.

The free vector field theory no longer looks like four copies of the Klein-Gordon field, but like three copies with one constraint. Since we eliminated A_0 using gauge transformations, it is clear that there are no negative norm states. The remaining fields A_i can be canonically quantised in terms of oscillators, all of which have positive norm.

The constraint reduces the physical Hilbert space built out of the oscillators to that generated by only two independent oscillators.

The spin of the particle associated to a vector field A_μ is 1 in units of $\hbar$. This can be demonstrated formally by constructing the angular momentum operator and acting with it on a one-particle state created by this field. Since a one-particle state cannot carry orbital angular momentum, whatever angular momentum we find in this way must necessarily correspond to the spin.

Thus the particle associated to the Abelian gauge field is a massless spin-1 particle. This is identified with the photon. The two independent oscillators surviving after we gauge-fix and impose constraints are associated with the two transverse polarisations of the photon.

2.4 Lorentz gauge

Let us briefly consider another method of gauge-fixing. Suppose we do not want to break manifest Lorentz invariance (which happens in the Coulomb gauge $A_0 = 0$, $\partial^i A_i = 0$). A simple covariant gauge condition is $\partial^\mu A_\mu(x) = 0$, called the Lorentz gauge. In this gauge, the Lagrangian becomes:

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)^2 = -\frac{1}{2}(\partial_\mu A_0)^2 + \frac{1}{2}\partial_\mu A_i \partial^\mu A_i$$

So A_i behave like three massless Klein-Gordon fields. But in this gauge there is also A_0 to worry about. In fact, now it has a canonical momentum! We see that:

$$\pi^0 = \frac{\delta \mathcal{L}}{\delta \dot{A}_0} = -\dot{A}_0$$

We will treat this field along with the three others as a valid dynamical variable even though it gives negative-norm states. At the end, the constraint $\partial^\mu A_\mu$ will be seen to eliminate these states.

The equation of motion reduces in this gauge to:

$$\partial^\mu \partial_\mu A_\nu = 0$$

As usual, we must check what residual gauge invariance is present after fixing $\partial^\mu A_\mu = 0$. Sending $A_\mu \to A_\mu + \partial_\mu \lambda$, we see that this preserves the gauge if $\partial^\mu \partial_\mu \lambda = 0$. This is just the equation of motion of a free massless scalar field.

The situation simplifies considerably in momentum space. Let $\tilde{A}_\mu(k)$ be the Fourier transform of $A_\mu(x)$. The gauge condition becomes $k^\mu \tilde{A}_\mu(k) = 0$, and the residual gauge invariance becomes

$$\tilde{A}_\mu(k) \;\to\; \tilde{A}_\mu(k) + k_\mu \tilde{\lambda}(k)$$

where $\tilde{\lambda}$ is any function of k, subject to the condition that $k^\mu k_\mu \tilde{\lambda}(k) = 0$. Therefore we must restrict to k_μ satisfying $k^\mu k_\mu = 0$, which is the on-shell condition for a massless particle.

Thus we see that the Lorentz gauge has two features:

(i) a *condition*: $k^\mu \tilde{A}_\mu(k) \;\sim\; 0$

(ii) a *redundancy*: $\tilde{A}_\mu(k) \sim \tilde{A}_\mu(k) + k_\mu \tilde{\lambda}(k)$

where the $\sim$ symbol means the configurations $\tilde{A}_\mu(k)$ and $\tilde{A}_\mu(k) + k_\mu \tilde{\lambda}(k)$ are *identified*.

The above features guarantee that two of the four possible polarisations decouple from the theory. To see this, consider one-particle states in the free theory. and write:

$$A_\mu(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left(a_{\mu,\vec{k}}\, e^{-ik\cdot x} + a^\dagger_{\mu,\vec{k}}\, e^{ik\cdot x} \right)$$

Imagine that $a_{\mu,\vec{k}}, a^\dagger_{\mu,\vec{k}}$ are quantised as usual. Now consider the four states $a^\dagger_{\mu,\vec{k}}|0\rangle$ and take the most general linear combination:

$$\xi^\mu a^\dagger_{\mu,\vec{k}}|0\rangle$$

The gauge conditions are:

$$\begin{aligned} k^\mu a^\dagger_{\mu,\vec{k}}|0\rangle &\sim 0 \\ a^\dagger_{\mu,\vec{k}}|0\rangle &\sim a^\dagger_{\mu,\vec{k}}|0\rangle + k_\mu\, \lambda_{\vec{k}}|0\rangle \end{aligned}$$

where $\lambda_{\vec{k}}$ is a mode of λ.

Now choose a Lorentz frame in which $k_\mu = k(1,0,0,1)$. Then $k^\mu = k(1,0,0,-1)$ and the first condition gives:

$$a^\dagger_{0,\vec{k}}|0\rangle \;\sim\; a^\dagger_{3,\vec{k}}|0\rangle$$

where again $\sim$ means we *identify* the two states.

The second gives:

$$\xi^\mu k_\mu = 0 \;\Rightarrow\; \xi^0 + \xi^3 = 0$$

Therefore the states we are considering become

$$\begin{aligned}\xi^0 a^\dagger_{0,\vec{k}}|0\rangle + \xi^1 a^\dagger_{1,\vec{k}}|0\rangle + \xi^2 a^\dagger_{2,\vec{k}}|0\rangle + \xi^3 a^\dagger_{3,\vec{k}}|0\rangle &= \xi^0\left(a^\dagger_{0,\vec{k}}|0\rangle - a^\dagger_{3,\vec{k}}|0\rangle\right) + \xi^1 a^\dagger_{1,\vec{k}}|0\rangle + \xi^2 a^\dagger_{2,\vec{k}}|0\rangle \\ &\sim \xi^1 a^\dagger_{1,\vec{k}}|0\rangle + \xi^2 a^\dagger_{2,\vec{k}}|0\rangle\end{aligned}$$

since $a^\dagger_{0,\vec{k}}|0\rangle - a^\dagger_{3,\vec{k}}|0\rangle \sim 0$.

This leaves behind the two linearly independent states

$$a^\dagger_{1,\vec{k}}|0\rangle,\; a^\dagger_{2,\vec{k}}|0\rangle$$

describing a transverse photon.

The new thing is that the gauge condition did not break Lorentz invariance. Only in *implementing* it did we choose a Lorentz frame for k_μ. This fits with our intuitive picture of a light ray that it oscillates transverse to its direction of propagation.

2.5 Scalar electrodynamics

Now let us consider coupling gauge fields A_μ to complex scalars ϕ, $\phi^\star$. First recall that for complex scalars, a typical Lagrangian would be:

$$\mathcal{L} = \partial_\mu\phi^\star\partial^\mu\phi - m^2\phi^\star\phi - \frac{\lambda}{6}(\phi^\star\phi)^2$$

The corresponding equations of motion are:

$$\begin{aligned}\partial^\mu\partial_\mu\phi + m^2\phi + \frac{\lambda}{3}\phi^*\phi^2 &= 0 \\ \partial^\mu\partial_\mu\phi^* + m^2\phi^* + \frac{\lambda}{3}\phi\,\phi^{*2} &= 0\end{aligned}$$

Notice that this Lagrangian is invariant under

$$\begin{aligned}\phi(x) &\rightarrow e^{i\alpha}\phi(x)\\ \phi^\star(x) &\rightarrow e^{-i\alpha}\phi^\star(x)\end{aligned}$$

but only as long as α is constant. The infinitesimal version of the above transformation, assuming α to be small and keeping only the lowest order term, can be written:

$$\delta\phi(x) = i\alpha\phi(x), \quad \delta\phi^*(x) = -i\alpha\phi^*(x)$$

Transformations with constant parameters are called *global symmetries*.

Related to the above fact, this theory has a *current* that is conserved due to the equations of motion. A current is simply any 4-vector made out of fields and their derivatives. Conserved currents are associated to symmetries of the Lagrangian. The relevant current here turns out to be[2]:

$$j_\mu = i\left(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*\right)$$

To show that it is conserved, we compute its 4-divergence:

$$\begin{aligned}\partial^\mu j_\mu &= i\left(\partial^\mu\phi^*\partial_\mu\phi + \phi^*\partial^\mu\partial_\mu\phi - \partial^\mu\phi\,\partial_\mu\phi^* - \phi\,\partial^\mu\partial_\mu\phi^*\right)\\ &= i\left(-m^2\phi^*\phi - \frac{\lambda}{3}\phi^{*2}\phi^2 + m^2\phi^*\phi + \frac{\lambda}{3}\phi^{*2}\phi^2\right) = 0\end{aligned}$$

To this current is associated a conserved charge:

$$Q = \int d^3x\, j_0 = i\int d^3x\left(\phi^*\dot{\phi} - \phi\,\dot{\phi}^*\right)$$

We now show that this charge treated as an operator (and multiplied by the parameter α) generates the same symmetry transformation via the canonical commutation relations. One may think of $e^{-iQ\alpha}$ as the generator of the symmetry transformation in much the same way as e^{-iHt} generates time translations in quantum mechanics. Then the transformation of a field under a finite symmetry transformation is:

$$\phi(x) \rightarrow e^{-iQ\alpha}\phi(x)\,e^{iQ\alpha}, \quad \phi^*(x) \rightarrow e^{-iQ\alpha}\phi^*(x)e^{iQ\alpha},$$

The infinitesimal version of this is the change of ϕ to first order in α, which is easily seen to be:

$$\begin{aligned}\delta\phi(x) &= -i\alpha[Q,\phi(x)] &= i\alpha\phi(x)\\ \delta\phi^*(x) &= -i\alpha[Q,\phi^*(x)] &= -i\alpha\phi^*(x)\end{aligned}$$

[2]It can be derived using the Noether procedure.

where we have used:

$$[\phi^*(t,\vec{x}),\dot{\phi}(t,\vec{y})] = [\phi(t,\vec{x}),\dot{\phi}^*(t,\vec{y})] = i\delta^3(\vec{x}-\vec{y})$$

If we allow α to depend on x, which makes it a *local* transformation, the kinetic term in the Lagrangian changes as follows:

$$\begin{aligned}\partial_\mu\phi^\star\partial^\mu\phi &\rightarrow \partial_\mu(e^{-i\alpha}\phi^\star)\partial^\mu(e^{i\alpha}\phi)\\ &= \partial_\mu\phi^\star\partial^\mu\phi - i\partial_\mu\alpha\,\phi^\star\partial^\mu\phi + i\partial^\mu\alpha\,\partial_\mu\phi^\star\phi + \partial_\mu\alpha\,\partial^\mu\alpha\,\phi^\star\phi\end{aligned}$$

Thus by itself the scalar field theory is not invariant under local transformations.

However we can now couple a gauge field $A_\mu(x)$ to the complex scalar field such that the combined system has local (gauge) invariance. We simply make the replacement

$$\partial_\mu\phi \rightarrow (\partial_\mu - ieA_\mu)\phi$$

in the action, where e is an arbitrary constant which will turn out to be a coupling constant of the theory. Then, under a gauge transformation that simultaneously acts as:

$$\begin{aligned}\phi(x) &\rightarrow e^{i\alpha(x)}\phi(x)\\ \phi^\star(x) &\rightarrow e^{-i\alpha(x)}\phi^\star(x)\\ A_\mu(x) &\rightarrow A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x)\end{aligned}$$

one can easily check that:

$$(\partial_\mu - ieA_\mu)\phi \rightarrow e^{i\alpha(x)}(\partial_\mu - ieA_\mu)\phi$$

The generalised kinetic term:

$$(\partial_\mu + ieA_\mu)\phi^*(\partial_\mu - ieA_\mu)\phi$$

is then gauge invariant, as are the potential terms that depend only on the scalar field.

After adding a kinetic term for the gauge field, the full theory with Lagrangian:

$$\mathcal{L} = (\partial_\mu + ieA_\mu)\phi^\star(\partial^\mu - ieA_\mu)\phi - m^2\phi^\star\phi - \frac{\lambda}{6}(\phi^\star\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

is a gauge theory coupled to (scalar) matter. It is called scalar electrodynamics.

Note that in the coupled theory, the original conserved current is coupled to the gauge field by a coupling of the form:

$$\sim e\int d^4x\, j^\mu A_\mu$$

2.6 Feynman rules for scalar electrodynamics

We close this discussion by providing a summary of the Feynman rules for calculating correlation functions in scalar electrodynamics. For this we need to separate the Lagrangian into the "free" part which is treated exactly, and the "interacting" part which is treated in perturbation theory. The free part consists of terms quadratic in the fields:

$$\mathcal{L}_0 = \partial_\mu \phi^\star \partial^\mu \phi - m^2 \phi^\star \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

while the interaction is everything else:

$$\mathcal{L}_{\rm int} = ieA_\mu(\phi^* \partial^\mu \phi - \phi\, \partial^\mu \phi^*) + e^2 A_\mu A^\mu \phi^* \phi - \frac{\lambda}{6}(\phi^\star \phi)^2$$

Now the free part gives rise to Feynman propagators for the different fields ϕ, ϕ^*, A_μ. We have:

$$\langle 0|T\left(\phi^*(x)\phi(y)\right)|0\rangle = D_F(x-y)$$

which is similar to what we saw for real Klein-Gordon fields, except that for complex fields the propagator connects ϕ to ϕ^*.

For the A_μ propagator things are more subtle because we must first fix the gauge. The easiest gauge for this purpose is Lorentz gauge, for which as we have seen, we should replace the gauge field kinetic term by:

$$-\partial_\mu A_\nu\, \partial^\mu A^\nu$$

In this gauge the four components of A_μ behave like independent massless Klein-Gordon fields but the time component has an opposite sign. Thus:

$$\langle 0|T\left(A_\mu(x)A_\nu(y)\right)|0\rangle = -\eta_{\mu\nu} D_F(x-y)\Big|_{m=0}$$

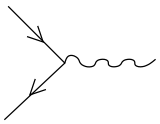
When we draw Feynman diagrams we need to be careful with two points. One is that the propagator for the scalar field comes with an *arrow* depicting the direction of charge flow. The other is that the gauge field propagator needs a different symbol so we depict it with a wavy line.

For the interaction Hamiltonian, we have:

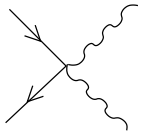
$$\mathcal{H}_{\rm int} = -\mathcal{L}_{\rm int} = -ieA_\mu(\phi^* \partial^\mu \phi - \phi\, \partial^\mu \phi^*) - e^2 A_\mu A^\mu \phi^* \phi + \frac{\lambda}{6}(\phi^\star \phi)^2$$

We notice that scalar electrodynamics has two coupling constants, e and λ. There is a familiar quartic self-coupling of ϕ with coefficient proportional to λ, as well as a new quartic

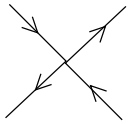
coupling $-A_\mu A^\mu \phi^* \phi$ with a coefficient proportional to e^2. Note that the sign of this term is such that the interaction for the *space* components A_i is positive as a sensible potential should be. The potential for A_0 is negative and therefore unbounded below but, as we have seen earlier, A_0 is not a propagating field so we anticipate that the gauge conditions will remove any problem with that. Finally there is a cubic potential which is new and involves derivatives. The three interaction vertices can be depicted as:



$-iek_\mu$



$-\,e^2\eta_{\mu\nu}$



$\frac{\lambda}{6}$

Scalar electrodynamics describes the physics of an electrically charged spinless particle (and its antiparticle) coupled to a photon. Since most particles in nature are spin-$\frac{1}{2}$ fermions, we will now move on to describe fermions and study electrodynamics further in that context.

3 Fermions

3.1 Lorentz algebra and Clifford algebra

Lorentz transformations are a set of three rotations (in the xy, yz and zx planes) and three boosts (with respect to x, y and z). Just as rotations are implemented by three generators J_x, J_y, J_z, the Lorentz transformations can be implemented by means of six generators labelled $M^{\mu\nu}$ where M is antisymmetric in μ and ν. The correspondence is:

$$\begin{aligned} M^{01} &\rightarrow \text{boost in } x \text{ direction} \\ M^{02} &\rightarrow \text{boost in } y \text{ direction} \\ M^{03} &\rightarrow \text{boost in } z \text{ direction} \\ M^{23} &\rightarrow \text{rotation about } x \text{ direction} \\ M^{31} &\rightarrow \text{rotation about } y \text{ direction} \\ M^{12} &\rightarrow \text{rotation about } z \text{ direction} \end{aligned}$$

So in fact the last three are the same as the usual rotation generators J_x, J_y, J_z.

Now we know that rotation generators obey commutation relations like:

$$[J_x, J_y] = J_z$$

and cyclic (we use conventions where the generators are anti-Hermitian so there is no i on the RHS). In non-relativistic quantum mechanics, any specific particle state (wave-function) transforms under rotations by some matrices which obey the above algebra. For example, spin-$\frac{1}{2}$ particles are transformed by the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In the relativistic theory there must be an algebra of commutation relations among all the $M^{\mu\nu}$ which includes the rotation algebra but also specifies the commutators of boosts with rotations and boosts with boosts. These commutation relations define the Lorentz algebra and look as follows:

$$[M^{\mu\nu}, M^{\lambda\rho}] = -\left(\eta^{\mu\lambda}M^{\nu\rho} + \eta^{\nu\rho}M^{\mu\lambda} - \eta^{\mu\rho}M^{\nu\lambda} - \eta^{\nu\lambda}M^{\mu\rho}\right)$$

As a simple exercise one can check that if we label $M^{12} = -J_z$ and cyclic, then the above commutator implies:

$$[J_x, J_y] = J_z$$

Just as J_x, J_y, J_z are abstract rotation operators that are realised on spin-$\frac{1}{2}$ fields through Pauli matrices (and on spin-1 fields in a different way), the Lorentz generators $M^{\mu\nu}$ can also be thought of as abstract operators which are realised in different ways on different fields. These different ways are called "representations". The general action is given, in terms of the matrix $m^{\mu\nu}$ defining the specified representation, as:

$$e^{\frac{1}{2}\omega_{\mu\nu}m^{\mu\nu}}$$

Here $\omega_{\mu\nu}$ are the parameters of the transformation, which specify the amount of rotation and boost. This is analogous to the generators of time translations or gauge transformations being $\exp(-iHt)$, $\exp(-iQ\lambda)$ respectively.

We are already familiar with one representation of the $M^{\mu\nu}$, called the "vector representation". Suppose we take 4×4 matrices:

$$(V^{\mu\nu})^{\alpha}{}_{\beta} = \eta^{\mu\alpha}\delta^{\nu}{}_{\beta} - \eta^{\nu\alpha}\delta^{\mu}{}_{\beta}$$

One can check that these satisfy the commutation relations of the Lorentz algebra. Because each $V^{\mu\nu}$ has a pair of vector indices, it acts on vector fields. The transformation is:

$$A'^{\alpha}(x') = \left(e^{\frac{1}{2}\omega_{\mu\nu}V^{\mu\nu}}\right)^{\alpha}{}_{\beta} A^{\beta}(x) = \left(e^{\eta\,\omega}\right)^{\alpha}{}_{\beta} A^{\beta}(x)$$

Now we have the identity:

$$e^{\eta\,\omega}\eta\left(e^{\eta\,\omega}\right)^{T} = \eta$$

which is precisely the defining relation of the usual Lorentz transformation matrix Λ. Therefore we can write:

$$\Lambda^{\alpha}{}_{\beta} = \left(e^{\eta\,\omega}\right)^{\alpha}{}_{\beta}$$

and we recover the familiar Lorentz-transformation of vector fields:

$$A'^{\alpha}(x') = \Lambda^{\alpha}{}_{\beta}A^{\beta}(x)$$

We have seen that a vector field carries 1 unit of spin. By allowing multiple space-time indices, we can find more general fields called *tensor fields* that transform in more complicated representations of the Lorentz algebra. Being built out of the vector representation, these are associated to particles of various different *integer* spins.

However, experimentally it was determined long ago that the spin of an electron is $\frac{1}{2}$ in units of $\hbar$. Therefore electrons, and more generally fermions, cannot be described by scalar, vector or tensor fields. Are there any fields that carry spin $\frac{1}{2}$ and if so, how do they transform under Lorentz transformations?

It turns out that mathematically there is an additional class of representations of the Lorentz group, called spinor representations. These arise from the following theorem. Consider the following algebra that is quite distinct from the Lorentz algebra:

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$$

This is called the *Clifford algebra*.

A state that transforms under the Clifford algebra will be acted on by matrices γ^μ that satisfy:

$$(\gamma^\mu)_{ab}(\gamma^\nu)_{bc} + (\gamma^\nu)_{ab}(\gamma^\mu)_{bc} = 2\eta^{\mu\nu}\delta_{ac}$$

or for short,

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

The reason we have introduced the Clifford algebra is that the γ^μ provide a new representation of the Lorentz algebra, the *spinor representation*. This arises as follows. Let

$$S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$$

Using the fact that γ^μ satisfy the Clifford algebra, one can easily check that $S^{\mu\nu}$ satisfy the Lorentz algebra:

$$[S^{\mu\nu}, S^{\lambda\rho}] = -[\eta^{\mu\nu}S^{\nu\rho} + \eta^{\nu\rho}S^{\mu\lambda} - \eta^{\mu\rho}S^{\nu\lambda} - \eta^{\nu\lambda}S^{\mu\rho}]$$

The Clifford algebra is simpler than it looks. It simply says:

$$\begin{aligned}
\gamma^\mu\gamma^\nu &= -\gamma^\nu\gamma^\mu \text{ if } \mu \neq \nu \\
(\gamma^\mu)^2 &= +1 \quad \mu = 0 \\
&= -1 \quad \mu = 1, 2, 3.
\end{aligned}$$

Also, we have $S^{\mu\mu} = 0$ while for $\mu \neq \nu$, we simply find:

$$S^{\mu\nu} = \frac{1}{2}\gamma^\mu\gamma^\nu$$

From what we discussed above, a relativistic field that transforms in the spinor representation of the Lorentz algebra will take the form ψ_a with the transformation law being:

$$\psi'_a(x') = \left(e^{\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}}\right)_{ab}\psi_b$$

To determine the range of values of a, b, called *spinor indices*, we need to determine the dimension of the γ-matrices. For this, note first that the Pauli matrices σ^i themselves

satisfy a Clifford algebra, but only with respect to space directions (hence we have δ^{ij} rather than $\eta^{\mu\nu}$ on the RHS):

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}$$

Our strategy will be to construct γ-matrices using tensor products of the Pauli matrices.

As an example consider $\sigma^1 \otimes \sigma^2$:

$$\sigma^1 \otimes \sigma^2 = \begin{pmatrix} 0 & -i\sigma^1 \\ i\sigma^1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

Tensor products satisfy the multiplication rule:

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

From this it is easy to see that the four 4×4 matrices:

$$\sigma^1 \otimes \sigma^1$$

$$\sigma^1 \otimes \sigma^2$$

$$\sigma^1 \otimes \sigma^3$$

$$\sigma^2 \otimes \mathbb{1}$$

all mutually anticommute and all of them square to 1.

To make a set of γ-matrices that obey the Clifford algebra, we simply need to multiply three of them by i, for example:

$$\begin{aligned} \gamma^0 &= \sigma^1 \otimes \sigma^1 \\ \gamma^1 &= i\sigma^1 \otimes \sigma^2 \\ \gamma^2 &= i\sigma^1 \otimes \sigma^3 \\ \gamma^3 &= i\sigma^2 \otimes \mathbb{1} \end{aligned}$$

There are two important representations that we will discuss later on. All the representations of interest to us are 4×4, which is the minimum allowed value. Therefore the fields that transform under them will be 4-component fields.

The full γ-matrix algebra is made up of the identity matrix $\mathbb{1}$ and all γ^μ as well as their products. There are altogether 16 independent matrices. A particularly useful matrix in this set is:

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

where the coefficient i has been introduced so that:

$$\gamma_5^\dagger = \gamma_5, \qquad (\gamma_5)^2 = 1$$

The important identity:

$$\gamma_5 \gamma^\mu = -\gamma^\mu \gamma_5$$

is easy to derive. In the representation listed above, we find that:

$$\gamma_5 = \sigma_3 \otimes \mathbb{1}$$

and can see explicitly that it verifies the above identities.

In terms of the above matrices, the γ-matrix algebra consists of the 16 matrices:

$$\mathbb{1},\ \gamma^\mu,\ \gamma_5,\ \gamma_5\gamma^\mu,\ S^{\mu\nu}$$

where as we have seen, $S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$.

It is easy to see that all other combinations of the γ-matrices are linearly dependent on the ones listed above. For example $\gamma_5 S^{\mu\nu}$ can be re-expressed in terms of $S^{\mu\nu}$, $\gamma^\mu\gamma^\nu\gamma^\lambda$ can be re-expressed in terms of $\gamma_5\gamma^\rho$ etc.

3.2 Spinor fields and Dirac equation

With the above results, we now define a *spinor field* $\psi_a(x)$, $a = 1, 2, 3, 4$, as a field that transforms as:

$$\psi'_a(x') = \left(e^{\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}}\right)_{ab} \psi_b$$

under Lorentz transformations.

To write a field equation, we need to ask ourselves what could be the possible building blocks of the equation. For a free field, the equation will be linear in the field ψ_a. As before we may then guess that each component of a spinor should satisfy a Klein-Gordon equation (of second-order in derivatives).

However there is a new possibility with spinors and it turns out that nature makes use of it. This possibility arises because the differential operator:

$$\gamma^\mu_{ab}\partial_\mu$$

when acting on a spinor, gives another quantity that also transforms as a spinor. In other words, under

$$\psi \to e^{\frac{1}{2}\omega_{\lambda\rho}S^{\lambda\rho}}\psi$$

one can show that:

$$\gamma^\mu \partial_\mu \psi \to e^{\frac{1}{2}\omega_{\lambda\rho} S^{\lambda\rho}} \gamma^\mu \partial_\mu \psi$$

We will prove this below. Assuming it to be true for a moment, we see that the equation:

$$\gamma^\mu \partial_\mu \psi = 0$$

is a Lorentz-covariant equation. We can easily generalise the equation by adding a mass term:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

This clearly retains Lorentz covariance, and is called the free (massive) Dirac equation. A commonly used terminology is to represent:

$$\gamma^\mu \partial_\mu \to \not{\partial}$$

One should remember that $\not{\partial}$ is not only a differential operator but also a matrix in spinor space.

To prove the above result, note that γ^μ is a collection of constants, therefore under a Lorentz transformation we have:

$$\gamma^\mu \partial_\mu \psi \to \gamma^\mu \left(\left(e^{-\frac{1}{2}\omega_{\lambda\rho} V^{\lambda\rho}} \right)^\nu{}_\mu \partial_\nu \right) \left(e^{\frac{1}{2}\omega_{\lambda\rho} S^{\lambda\rho}} \psi \right)$$

Equating this to the desired result:

$$e^{\frac{1}{2}\omega_{\lambda\rho} S^{\lambda\rho}} \gamma^\mu \partial_\mu \psi$$

tells us that the result will be true if γ^μ satisfies the identity:

$$\left(e^{-\frac{1}{2}\omega_{\lambda\rho} V^{\lambda\rho}} \right)^\mu{}_\nu \left(e^{-\frac{1}{2}\omega_{\lambda\rho} S^{\lambda\rho}} \right)_{ac} \left(e^{\frac{1}{2}\omega_{\lambda\rho} S^{\lambda\rho}} \right)_{db} \gamma^\nu_{cd} = \gamma^\mu_{ab}$$

Such an identity indeed holds. It is easy to check it at the level of infinitesimal transformations, for which we have:

$$\begin{aligned} LHS &= \left(\delta^\mu{}_\nu - \tfrac{1}{2}\omega_{\lambda\rho} (V^{\lambda\rho})^\mu{}_\nu \right) \left(\delta_{ac} - \tfrac{1}{2}\omega_{\lambda\rho} S^{\lambda\rho}_{ac} \right) \left(\delta_{db} + \tfrac{1}{2}\omega_{\lambda\rho} S^{\lambda\rho}_{db} \right) \gamma^\nu_{cd} \\ &= \gamma^\mu_{ab} - \tfrac{1}{2}\omega_{\lambda\rho} (V^{\lambda\rho})^\mu{}_\nu \gamma^\nu_{ab} - \tfrac{1}{2}\omega_{\lambda\rho} S^{\lambda\rho}_{ac} \gamma^\mu_{cb} + \tfrac{1}{2}\omega_{\lambda\rho} S^{\lambda\rho}_{db} \gamma^\mu_{ad} \end{aligned}$$

So the last three terms should add up to zero for any arbitrary ω, in other words:

$$(V^{\lambda\rho})^\mu{}_\nu \gamma^\nu_{ab} + S^{\lambda\rho}_{ac} \gamma^\mu_{cb} - S^{\lambda\rho}_{db} \gamma^\mu_{ad} = 0$$

This identity can be written in matrix notation as:

$$[\gamma^\mu, S^{\lambda\rho}] = (V^{\lambda\rho})^\mu{}_\nu \gamma^\nu$$

and this is easily verified using properties of the γ-matrices.

To understand the Dirac equation better, we act on it with $(-i\gamma^\nu\partial_\nu - m)$ and find:

$$(-i\gamma^\nu\partial_\nu - m)(i\gamma^\mu\partial_\mu - m)\psi = (\gamma^\nu\partial_\nu\gamma^\mu\partial_\mu + m^2)\psi = (\tfrac{1}{2}\{\gamma^\nu, \gamma^\mu\}\partial_\nu\partial_\mu + m^2)\psi = (\partial^\mu\partial_\mu + m^2)\psi = 0$$

Therefore if ψ satisfies the Dirac equation then each component of it also satisfies the Klein-Gordon equation! However the reverse is not necessarily true: every solution of the Klein-Gordon equation need not be a solution of the Dirac equation. In fact unlike the Klein-Gordon equation, the Dirac equation relates different components of the four-component spinor ψ_a to each other.

Since $i\gamma^\mu\partial_\mu$ is complex in our chosen representation, the solutions of this equation must in general be taken to be complex. However we will see that there is a particular representation, the Majorana representation, in which $i\gamma^\mu\partial_\mu$ is real and so in that representation it makes sense to consider real solutions.

3.3 Dirac Lagrangian

We now seek a Lagrangian whose variation gives rise to the free massive Dirac equation. A free Lagrangian must be bilinear in fermions and Lorentz invariant. It is tempting to contract the LHS of the Dirac equation, $\chi_a = (i\gamma^\mu\partial_\mu - m)_{ab}\psi_b$ with another fermion ψ_a. However $\psi_a\chi_a$ is not a Lorentz scalar. It is equal to:

$$\psi_a\psi_a \to \left(e^{\frac{1}{2}\omega_{\lambda\rho}S^{\lambda\rho}}\right)_{ab}\psi_b\left(e^{\frac{1}{2}\omega_{\lambda\rho}S^{\lambda\rho}}\right)_{ac}\psi_c$$

The first term in brackets is the transpose of the second, so it would only cancel the second term in brackets if we had $S^{\lambda\rho\,T} = -S^{\lambda\rho}$. Unfortunately this is not the case.

We might hope to improve things if we carry out complex conjugation of ψ_a. Then the first term depends on $S^{\lambda\rho\,\dagger}$ but unfortunately this too is not equal to $-S^{\lambda\rho}$. In fact,

$$\begin{aligned}(S^{\lambda\rho})^\dagger &= \left(\tfrac{1}{4}[\gamma^\lambda, \gamma^\rho]\right)^\dagger \\ &= \tfrac{1}{4}[\gamma^{\rho\dagger}, \gamma^{\lambda\dagger}] \\ &= -\tfrac{1}{4}[\gamma^{\lambda\dagger}, \gamma^{\rho\dagger}]\end{aligned}$$

We already know that $\gamma^{0\dagger} = \gamma^0$ while $\gamma^{i\dagger} = -\gamma^i$, which implies that $S^\dagger \neq -S$.

This suggests that the operation that reverses S should combine Hermitian conjugation with something that treats γ^0 differently from the other γ-matrices. This operation, called "bar", is defined by :

$$\psi_a \ \rightarrow \ \overline{\psi}_a \equiv \psi_b^\dagger \gamma^0_{ba}$$

The corresponding operation on γ-matrices is:

$$\overline{\gamma^\mu} \equiv \gamma^0 \gamma^{\mu\dagger} \gamma^0$$

One easily shows that $\overline{\gamma}^\mu = \gamma^\mu$, from which it follows that:

$$\overline{S^{\mu\nu}} \equiv \gamma^0 S^{\mu\nu\dagger} \gamma^0 = -S^{\mu\nu}$$

It follows that for any two spinors ψ, χ, the combination

$$\overline{\psi}\chi = \overline{\psi}_a \chi_a$$

is Lorentz invariant.

Thus the general rule is that we always multiply a spinor by a barred spinor from the left in order to get a Lorentz invariant. Thus it is clear that the Dirac action should be:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

which is a good Lorentz scalar. Since a lot of indices are implicit, just for once we write the above expression explicitly:

$$\mathcal{L} = \psi_a^\dagger \gamma^0_{ab} \left(i\gamma^\mu_{bc} \partial_\mu - m\delta_{bc} \right) \psi_c$$

Since ψ is complex, in finding the equations of motion we can vary ψ, $\psi^\dagger$ independently, to get:

$$\frac{\delta}{\delta\psi^\dagger} \ \Rightarrow \ \gamma^0(i\gamma^\mu\partial_\mu - m)\psi = 0 \ \Rightarrow \ (i\gamma^\mu\partial_\mu - m)\psi = 0$$

$$\frac{\delta}{\delta\psi} \ \Rightarrow \ \bar{\psi}(-i\gamma^\mu \overleftarrow{\partial_\mu} - m) = 0$$

We can also exhibit the canonical momenta following from the Dirac equation:

$$\pi_a = \frac{\delta\mathcal{L}}{\delta\dot{\psi}_a} = i\overline{(}\psi\gamma^0)_a = i\psi_a^\dagger$$

3.4 Weyl basis

An alternate basis of the γ-matrices gives us some useful physical information about the nature of fermions.

$$\begin{aligned}
\gamma^0 &= \mathbb{1} \otimes \sigma^1 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \\
\gamma^1 &= i\sigma^1 \otimes \sigma^2 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} \\
\gamma^2 &= i\sigma^2 \otimes \sigma^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \\
\gamma^3 &= i\sigma^3 \otimes \sigma^2 = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}
\end{aligned}$$

Again we can check explicitly, using the Pauli matrix algebra, that the above matrices form a representation of the Clifford algebra. This is called the Weyl representation.

A key feature of the Weyl representation is that the matrix γ_5 is diagonal:

$$\gamma_5 = i(i)^3\sigma^1\sigma^2\sigma^3 \otimes \sigma^1\sigma^2\sigma^2\sigma^2 = i\mathbb{1} \otimes i\sigma^3 = -\mathbb{1} \otimes \sigma^3 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

In this representation we also have the following properties for the Lorentz generators $S^{\mu\nu}$:

$$S^{0i} = \tfrac{1}{2}\gamma^0\gamma^i = \tfrac{1}{2}(\mathbb{1} \otimes \sigma^1)(i\sigma^i \otimes \sigma^2) = -\tfrac{1}{2}\sigma^i \otimes \sigma^3 = -\tfrac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

Similarly, when $i \neq j$ we have:

$$S^{ij} = \tfrac{1}{2}\gamma^i\gamma^j = \tfrac{1}{2}(i\sigma^i \otimes \sigma^2)(i\sigma^j \otimes \sigma^2) = -\tfrac{1}{2}\epsilon^{ijk}\sigma^k \otimes \mathbb{1} = -\tfrac{i}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

We see that in the Weyl representation the Lorentz generators $S^{\mu\nu}$ are *block diagonal*. Therefore the two upper components of a spinor never mix with the two lower components. This shows that a 4-component spinor is a reducible representation of the Lorentz algebra.

To reduce the representation we simply define:

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

where each of ψ_L, ψ_R is a 2-component spinor. We see that:

$$\gamma_5 \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} -\psi_L \\ \psi_R \end{pmatrix}$$

The upper part ψ_L, corresponding to the negative eigenvalue of γ_5, is called a "left-handed" spinor, while the lower part corresponding to the positive eigenvalue of γ_5 is called "right-handed".

Now consider the action of a Lorentz transformation on a spinor:

$$\psi_a \to \left(e^{\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}}\right)_{ab}\psi_b$$

Using the form of $S^{\mu\nu}$ in the Weyl representation, derived above, it is easy to show that:

$$\begin{aligned}\tfrac{1}{2}\omega_{\mu\nu}S^{\mu\nu} &= \omega_{0i}S^{0i} + \tfrac{1}{2}\omega_{ij}S^{ij} = -\tfrac{\omega_{0i}}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix} - \tfrac{i}{4}\epsilon^{ijk}\omega_{ij}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}\\ &= -\frac{1}{2}\begin{pmatrix}\omega_{0i}\sigma^i + \frac{i}{2}\epsilon^{ijk}\omega_{ij}\sigma^k & 0\\ 0 & -\omega_{0i}\sigma^i + \frac{i}{2}\epsilon^{ijk}\omega_{ij}\sigma^k\end{pmatrix}\end{aligned}$$

Let $\theta^k = \frac{1}{2}\epsilon^{ijk}\omega_{ij}$. These are the rotation parameters around the k-axis ($k = 1, 2, 3$). Similarly $\omega_{0i} = \beta_i$ are the boost parameters along $i = 1, 2, 3$.

Recalling the definitions of the two-component spinors ψ_L, ψ_R, we find that under Lorentz transformations:

$$\psi_L \to \left(1 - \beta_i\frac{\sigma^i}{2} - i\theta_i\frac{\sigma^i}{2}\right)\psi_L$$

and

$$\psi_R \to \left(1 + \beta_i\frac{\sigma^i}{2} - i\theta_i\frac{\sigma^i}{2}\right)\psi_R$$

We learn the important fact that ψ_L, ψ_R transform the some way under rotations, but oppositely under boosts.

In the Weyl representation, the LHS of the Dirac equation is:

$$(i\gamma^\mu\partial_\mu - m)\psi = \begin{pmatrix}-m & i(\partial_0 + \sigma^i\partial_i)\\ i(\partial_0 - \sigma^i\partial_i) & -m\end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}$$

So the Dirac equation becomes:

$$\begin{aligned}i(\partial_0 + \sigma^i\partial_i)\psi_R - m\,\psi_L &= 0\\ i(\partial_0 - \sigma^i\partial_i)\psi_L - m\,\psi_R &= 0\end{aligned}$$

Defining:

$$\sigma^\mu = (1, \sigma^i), \qquad \bar{\sigma}^\mu = (1, -\sigma^i)$$

the Dirac equation reduces to:

$$\begin{aligned} i\sigma^\mu \partial_\mu \psi_R - m\,\psi_L &= 0 \\ i\bar{\sigma}^\mu \partial_\mu \psi_L - m\,\psi_R &= 0 \end{aligned}$$

Note that when $m = 0$, the two equations *decouple*. This is an extremely important property. The left-handed and right-handed Weyl spinors are in fact parity conjugates of each other. In the massless limit we see that the two types of spinors do not mix. It is therefore possible to assign different quantum numbers (generalised charges) to left and right-handed spinors, leading to a parity-violating theory. This is implemented in the Standard Model.

3.5 Majorana basis

One more interesting basis for the γ-matrices is:

$$\begin{aligned} \gamma^0 &= \sigma^1 \otimes \sigma^2 \\ \gamma^1 &= i\sigma^1 \otimes \sigma^3 \\ \gamma^2 &= i\sigma^1 \otimes \sigma^1 \\ \gamma^3 &= i\sigma^3 \otimes \mathbb{1} \end{aligned}$$

Note that here, all γ^μ are pure imaginary. This is called the Majorana basis. The benefit of purely imaginary γ-matrices is that the Dirac equation becomes a *real* equation. Therefore we can consider real spinors if we like. Such spinors, which are real in the Majorana basis, are called *Majorana spinors*.

In particle physics, real fields represent particles which are their own antiparticles. In this sense a Majorana spinor is analogous to a real scalar field. Though no examples are known with certainty in nature, it is thought that (some) neutrinos might be Majorana particles.

Note that in this representation,

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \sigma_2 \otimes \mathbb{1}$$

which is also imaginary. Since we know that γ^0 is Hermitian as well as pure imaginary, it must also be antisymmetric. This is easy to check.

The existence of the Majorana basis highlights a problem with classical spinors. The mass term in the Dirac Lagrangian is:

$$m\overline{\psi}\psi = m\psi_a(\gamma^0)_{ab}\psi_b$$

Since γ^0 is antisymmetric, this suggests that the mass term vanishes!

By the same argument one can also show that the kinetic term would be a total derivative. Since $\gamma^0\gamma^\mu$ is symmetric in the Majorana basis, we apparently have:

$$\overline{\psi}\partial\!\!\!/\psi = \psi_a(\gamma^0\gamma^\mu)_{ab}\partial_\mu\psi_b = \tfrac{1}{2}\partial_\mu\left(\psi_a(\gamma^0\gamma^\mu)_{ab}\psi_b\right)$$

If this were correct there would be no Lagrangian for Majorana spinors at all!

However these statements are incorrect. Instead, we learn a fundamental fact: that classical spinor fields should *anticommute* with each other instead of commuting:

$$\psi_a\psi_b = -\psi_b\psi_a$$

Then the mass term is nonzero and the kinetic term also is nontrivial.

Because the spinors classically anticommute, the quantum mechanical treatment of them will impose canonical *anticommutators* rather than commutators, as we will see.

3.6 Free-particle solutions of Dirac equation

We have seen that solutions of wave equations are generically of the form $e^{-ik\cdot x}$. However now our fields are spinors, so this factor can only be a building block and must be multiplied by some (possibly k-dependent) spinor. Hence we take:

$$\psi_a(x) = u_a(k)e^{-ik\cdot x}$$

where $k^2 = m^2$.

This certainly solves the position-space Klein-Gordon equation. However it does not necessarily solve the Dirac equation. Acting on the above by $i\partial_\mu$ we pull down a factor of k_μ, so the Dirac equation imposes the constraint on $u_a(k)$ that:

$$(\gamma^\mu k_\mu - m)u(k) = 0$$

We assume the mass m is nonzero (we can treat the massless case later by taking the limit $m \to 0$). In this case we can boost to the rest frame of the particle: $k_\mu = (m, \vec{0})$. In this frame the Dirac equation becomes:

$$(m\gamma^0 - m)u(k) = 0$$

or equivalently:

$$(1 - \gamma^0)u(k) = 0.$$

To find zero eigenvectors of $(1-\gamma^0)$, we go to the Weyl representation where:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

Then,

$$1-\gamma^0 = \begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

This has two zero eigenvectors:

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Thus the most general zero eigenvector is:

$$a\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ a \\ b \end{pmatrix} = \begin{pmatrix} \chi \\ \chi \end{pmatrix}$$

where $a, b \in \mathbb{C}$ and χ is an arbitrary two-component spinor. We see that any spinor $u(k)$ with equal left and right-handed components solves the Dirac equation.

We choose to normalise the solution as follows:

$$u(k) = \sqrt{m}\begin{pmatrix} \chi \\ \chi \end{pmatrix}, \qquad \chi^\dagger\chi = 1$$

As a result, one can easily show that $\overline{u}u = u^\dagger\gamma^0 u = 2m$.

Note that after applying Dirac equation, a spinor has only 2 independent (complex) degrees of freedom. These will later be identified with the spin up and spin down modes of a fermionic particle.

It is a straightforward exercise to show that if we go away from the rest frame to a frame where $k^\mu = (E, 0, 0, k_3)$, the free-particle solution becomes:

$$u(k) = \begin{pmatrix} \sqrt{\sigma^\mu k_\mu}\,\chi \\ \\ \sqrt{\bar{\sigma}^\mu k_\mu}\,\chi \end{pmatrix}$$

Explicitly, we have:

$$\begin{aligned} \sqrt{\sigma^\mu k_\mu} &= \sqrt{E+k_3}\,\frac{1-\sigma^3}{2} + \sqrt{E-k_3}\,\frac{1+\sigma^3}{2} \\ \sqrt{\bar{\sigma}^\mu k_\mu} &= \sqrt{E+k_3}\,\frac{1+\sigma^3}{2} + \sqrt{E-k_3}\,\frac{1-\sigma^3}{2} \end{aligned}$$

Taking a basis for the two-component spinor χ as:

$$\begin{pmatrix}1\\0\end{pmatrix}, \qquad \begin{pmatrix}0\\1\end{pmatrix}$$

we find the basis spinors:

$$u^1(k) = \begin{pmatrix}\sqrt{E-k_3}\\0\\\sqrt{E+k_3}\\0\end{pmatrix}, \qquad u^2(k) = \begin{pmatrix}0\\\sqrt{E+k_3}\\0\\\sqrt{E-k_3}\end{pmatrix}$$

Denoting the pair $u^1(k), u^2(k)$ as $u^s, s = 1, 2$ we find that $\overline{u}^s = u^{s\dagger}\gamma^0$ are given by:

$$\begin{aligned}\overline{u}^1(k) &= \begin{pmatrix}\sqrt{E+k_3} & 0 & \sqrt{E-k_3} & 0\end{pmatrix}\\ \overline{u}^2(k) &= \begin{pmatrix}0 & \sqrt{E-k_3} & 0 & \sqrt{E+k_3}\end{pmatrix}\end{aligned}$$

Therefore:

$$\begin{aligned}\overline{u}^1(k)u^1(k) &= \left(\sqrt{E^2-k_3^2}+\sqrt{E^2-k_3^2}\right) = 2m\\ \overline{u}^1(k)u^2(k) &= 0\end{aligned}$$

which can be summarised as:

$$\overline{u}^r(k)u^s(k) = 2m\,\delta^{rs}$$

Similarly if we look for solutions of the conjugate Dirac equation:

$$(\gamma^\mu k_\mu + m)v(k) = 0$$

of the form:

$$\psi(k) = v(k)e^{ik\cdot x}$$

we find a basis to be:

$$v^1(k) = \begin{pmatrix}\sqrt{E-k_3}\\0\\-\sqrt{E+k_3}\\0\end{pmatrix}, \qquad v^2(k) = \begin{pmatrix}0\\\sqrt{E+k_3}\\0\\-\sqrt{E-k_3}\end{pmatrix}$$

and

$$\begin{aligned}\overline{v}^1(k) &= \begin{pmatrix} -\sqrt{E+k_3} & 0 & \sqrt{E-k_3} & 0 \end{pmatrix} \\ \overline{v}^2(k) &= \begin{pmatrix} 0 & -\sqrt{E-k_3} & 0 & \sqrt{E+k_3} \end{pmatrix}\end{aligned}$$

Hence we find:

$$\overline{v}^r v^s = -2m\,\delta^{rs}$$

The inner products between u's and v's vanish:

$$\overline{u}^r v^s = \overline{v}^s u^s = 0$$

We also have a pair of completeness relations:

$$\begin{aligned}\sum_{s=1,2} u^s_a(k)\overline{u}^s_b(k) &= (\gamma^\mu k_\mu + m)_{ab} \\ \sum_{s=1,2} v^s_a(k)\overline{v}^s_b(k) &= (\gamma^\mu k_\mu - m)_{ab}\end{aligned}$$

3.7 Quantisation of the Dirac field

Now that we have the positive and negative frequency free-particle solutions, we can expand the spinor field $\psi(x)$ over them as follows:

$$\begin{aligned}\psi_a(x) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\sum_{s=1}^{2}\left[a^s_{\vec{k}}\,u^s(k)\,e^{-ik\cdot x} + b^{\dagger s}_{\vec{k}}\,v^s(k)\,e^{ik\cdot x}\right] \\ \overline{\psi}_a(x) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\sum_{s=1}^{2}\left[b^s_{\vec{k}}\,\overline{v}^s(k)\,e^{-ik\cdot x} + a^{\dagger s}_{\vec{k}}\,\overline{u}^s(k)\,e^{ik\cdot x}\right]\end{aligned}$$

As we mentioned earlier, spinor fields are quantised via canonical *anticommutation* relations. We therefore impose:

$$\{\pi_a(t,\vec{x}),\psi_b(t,\vec{y})\} = i\delta_{ab}\delta^3(\vec{x}-\vec{y})$$

since $\pi_a = i\psi^\dagger_a$, we get

$$\{\psi^\dagger_a(t,\vec{x}),\psi_b(t,\vec{y})\} = \delta_{ab}\delta^3(\vec{x}-\vec{y})$$

Note that this relation does not involve derivatives of the fields, and is symmetric between ψ and $\psi^\dagger$. These features arise because the Dirac equation is first-order in time. We also have:

$$\{\psi_a(t,\vec{x}),\psi_b(t,\vec{y})\} = \{\psi^\dagger_a(t,\vec{x}),\psi^\dagger_b(t,\vec{y})\}$$

The above relations imply the following anticommutator brackets for the oscillators:

$$\{a^r_{\vec{k}}, a^{\dagger s}_{\vec{k}'}\} = \{b^r_{\vec{k}}, b^{\dagger s}_{\vec{k}'}\} = (2\pi)^3\delta^3(\vec{k}-\vec{k}')\delta^{rs}$$

and, with any choice of momenta and labels,

$$\{a, a\} = \{b, b\} = \{a, b\} = \{a, b^\dagger\} = 0.$$

Now just as we did for scalar fields, we define the vacuum state by:

$$a^s_{\vec{k}}|0\rangle = b^s_{\vec{k}}|0\rangle = 0, \qquad s = 1, 2$$

The Hamiltonian is:

$$H = \int d^3x(\pi^\dagger_a\dot{\psi}_a - \mathcal{L}) = \int d^3x(-i\bar{\psi}\gamma^i\partial_i\psi + m\bar{\psi}\psi) = \int \frac{d^3k}{(2\pi)^3}\sum_{s=1}^{2}\omega_{\vec{k}}(a^{s\dagger}_{\vec{k}}a^s_{\vec{k}} + b^{\dagger s}_{\vec{k}}b^s_{\vec{k}})$$

where we have dropped an infinite additive constant. The physical interpretation of the oscillators is as follows. $a^{\dagger s}_{\vec{k}}$ creates a fermion of polarisation s, momentum $\vec{k}$. On the other hand $b^{\dagger s}_{\vec{k}}$ creates the *anti-particle* of this fermion, with polarisation s and momentum $\vec{k}$.

We choose a Lorentz invariant norm:

$$\langle\vec{k}, r|\vec{k}', s\rangle = 2\omega_{\vec{k}}(2\pi)^3\delta^3(\vec{k}-\vec{k}')\delta^{rs}$$

which means the particle state $|\vec{k}, s\rangle$ is given in terms of oscillators by $\sqrt{2\omega_{\vec{k}}}\,a^{\dagger s}_{\vec{k}}|0\rangle$. Of course there will be a similar antiparticle state involving $b^\dagger$.

In a subsequent section we will show that the particles created by the spinor field have angular momentum $\frac{1}{2}$ in units of $\hbar$. Therefore all quarks and leptons in nature can, at least at the free level, be described by spinor fields. We will consider interactions later on.

3.8 Dirac propagator

Because fermions are quantised via anticommutators, we must define time-ordering for them as follows:

$$\begin{aligned} T(\psi(x)\,\overline{\psi}(y)) &= \psi(x)\,\overline{\psi}(y) \qquad \text{if } x^0 > y^0 \\ &= -\overline{\psi}(y)\,\psi(x) \qquad \text{if } x^0 < y^0 \end{aligned}$$

With this, we can compute

$$\langle 0|T\left(\psi_a(x)\overline{\psi}_b(y)\right)|0\rangle$$

This time we use a shortcut. Recall that for scalars,

$$\langle 0|T\left(\phi(x)\phi(y)\right)|0\rangle = D_F(x-y)$$

was the Feynman propagator which solves:

$$-(\partial^\mu\partial_\mu + m^2)D_F(x-y) = i\delta^4(x-y)$$

Thereby we can determine that

$$D_F(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{-ik\cdot(x-y)}$$

where the $i\epsilon$ determines the contour.

The fermion propagator likewise should be a Green's function for the Dirac equation, satisfying:

$$(i\gamma^\mu\partial_\mu - m)_{ab}(S_F(x-y))_{bc} = i\delta_{ac}\delta^4(x-y)$$

We have seen that acting once more with the conjugate Dirac operator gives us the Klein-Gordon operator. Using this in the above equation we get:

$$\begin{aligned}(i\gamma^\nu\partial_\nu + m)(i\gamma^\mu\partial_\mu - m)S_F(x-y) &= -(\partial^2+m^2)S_F(x-y)\\ &= i(i\gamma^\nu\partial_\nu + m)\delta^4(x-y)\end{aligned}$$

This can be solved by writing

$$(S_F(x-y))_{ab} = (i\gamma^\nu\partial_\nu + m)_{ab}D_F(x-y).$$

Using the momentum representation of $D_F(x-y)$, we have:

$$S_F(x-y) = \int \frac{d^4k}{(2\pi)^4}\,\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}\,e^{-ik\cdot(x-y)}$$

where we have used the notation $\not{k} = \gamma^\mu k_\mu$. It can be directly checked that this satisfies the desired equation and boundary conditions for a Feynman propagator.

The momentum factor in S_F is sometimes written:

$$\frac{i(\not{k}+m)}{k^2-m^2} = \frac{i}{\not{k}-m}$$

but the RHS just means what is written on the LHS.

The Dirac propagator is used in Feynman diagrams whenever there is an internal fermion line. For external lines, as we will see in a subsequent section, the propagator is to be amputated and in its place, an external polarisation (which can be one of $u^s, \overline{u}^s, v^s, \overline{v}^s$) is to be inserted instead.

3.9 Spin of the field $\psi_a(x)$

Recall the Lorentz algebra

$$[M^{\mu\nu}, M^{\lambda\rho}] = -[\eta^{\mu\lambda}M^{\nu\rho} + \eta^{\nu\rho}M^{\mu\lambda} - \eta^{\mu\rho}M^{\nu\lambda} - \eta^{\nu\lambda}M^{\mu\rho}]$$

We know that M^{0i} generates boosts and M^{ij} generates rotations. But we have not taken account of translations. These are generated by the momentum operator P^μ, which satisfies:

$$\begin{aligned} [P^\mu, P^\nu] &= 0 \\ [P^\mu, J^{\nu\lambda}] &= \eta^{\mu\nu}P^\lambda - \eta^{\mu\lambda}P^\nu \end{aligned}$$

The Lorentz algebra $[M, M]$ together with the commutators $[P, P]$ and $[P, M]$ is called the *Poincare algebra*.

Now there is a theorem that representations of the Poincare algebra are classified by the values of $P^2 = P_\mu P^\mu$ and $W^2 = W_\mu W^\mu$ where

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^\sigma$$

is called the Pauli-Lubanski vector.

We have $P^2 = m^2$, while

$$W^2 = -m^2 s(s+1)$$

where $s =$ half-integer or integer is called the spin.

In the rest frame, $P_\mu = (m, 0, 0, 0)$ and:

$$W_0 = 0, \qquad W_i = -\tfrac{1}{2}m\,\epsilon_{ijk}M^{jk}$$

so W_i is proportional to the ordinary angular momentum that we define in nonrelativistic quantum mechanics. In fact W_i is what we normally call J_i (apart from a factor m) and we know that:

$$J_i J_i = j(j+1).$$

Now, for spinors we have

$$\psi'(x') = \left(1 + \frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\psi(x)$$

or

$$\delta\psi = \left(1 + \frac{1}{2}\omega_{\mu\nu}S^{\mu\nu} - \omega_{\mu\nu}x^\mu\partial^\nu\right)\psi$$

Defining the RHS to be $\left(1+\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}\right)\psi$, we have

$$M^{\mu\nu} = S^{\mu\nu} - (x^\mu\partial^\nu - x^\nu\partial^\nu)$$

which expresses the angular momentum as the sum of a spin part and an orbital part.

Thus as an operator the Pauli-Lubanski vector is:

$$\begin{aligned} W_\mu &= -\frac{1}{2}\epsilon_{\mu\nu\lambda\rho}\left(S^{\nu\lambda} - (x^\nu\partial^\lambda - x^\lambda\partial^\nu)\right) i\partial^\rho \\ &= -\frac{i}{2}\epsilon_{\mu\nu\lambda\rho}S^{\nu\lambda}\partial^\rho \end{aligned}$$

Note that the orbital part has dropped out, justifying the claim that W_μ measures the *intrinsic* angular momentum.

Now:

$$\begin{aligned} W_\mu W^\mu &= -\frac{i}{2}\epsilon_{\mu\nu\lambda\rho}S^{\nu\lambda}\partial^\rho \cdot -\frac{i}{2}\epsilon^\mu{}_{\alpha\beta\gamma}S^{\alpha\beta}\partial^\gamma \\ &= -\frac{1}{4}\left(\eta_{\nu\alpha}\eta_{\lambda\beta}\eta_{\rho\gamma} + \cdots\right)\left(\frac{1}{4}[\gamma^\nu,\gamma^\lambda]\right)\left(\frac{1}{4}[\gamma^\alpha,\gamma^\beta]\right)\partial^\rho\partial^\gamma \\ &= -\frac{3}{4}(i\gamma^\mu\partial_\mu)^2 = -\frac{3}{4}m^2 \end{aligned}$$

The last steps above are left as an exercise.

Thus we find that $s(s+1) = \frac{3}{4}$ and therefore $s = \frac{1}{2}$. This proves that the spinor field $\psi_a(x)$ describes a field of spin half.

3.10 Charge of a state

Just like complex scalars, spinors too have a global phase invariance:

$$\psi \to e^{i\alpha}\psi, \qquad \psi^\dagger \to e^{-i\alpha}\psi^\dagger$$

The Dirac Lagrangian is easily seen to be invariant under this transformation as long as α is constant. The corresponding conserved current, which can be deduced using Noether's theorem, is:

$$J_\mu = \overline{\psi}\gamma_\mu\psi$$

and it is easily checked that this is conserved on using the equations of motion.

The conserved charge is:

$$Q = \int d^3x\, J_0 = \int d^3x\, \overline{\psi}\gamma^0\psi = \int d^3x\, \psi^\dagger\psi$$

At the quantum level the charge operator has to be defined with *normal ordering* so that the expectation value of charge in the vacuum is zero. Thus after quantisation we must write:

$$Q = \int d^3x\ :\psi^\dagger\psi:$$

In terms of the oscillators $a^s_{\vec{k}}, b^s_{\vec{k}}$ we get:

$$Q = \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{k}}}\,\left(a^{\dagger s}_{\vec{k}}a^s_{\vec{k}} - b^{\dagger s}_{\vec{k}}b^s_{\vec{k}}\right)$$

Using the anti-commutation relations for the a and b, we can now check that:

$$Q\,\left(a^{\dagger s}_{\vec{k}}|0\rangle\right) = a^{\dagger s}_{\vec{k}}|0\rangle, \qquad Q\,\left(b^{\dagger s}_{\vec{k}}|0\rangle\right) = -b^{\dagger s}_{\vec{k}}|0\rangle$$

demonstrating that $a^\dagger$ creates particles while $b^\dagger$ creates antiparticles.

This charge will become a physical quantity when we couple the theory to the vector field A_μ. In that situation the physical charge depends on the value of the coupling constant e which always multiplies Q. If e is a negative number equal to the charge of the electron, then the charge of particles is $-|e|$ and of antiparticles is $+|e|$, which is what we expect for electrons and positrons respectively.

The free Dirac theory described here can be used to describe all charged fermions in nature: electrons, muons, tau leptons and quarks. For the neutrinos there is a slight subtlety. It is possible that one or more of the neutrinos is a Majorana particle, which means it would be described by a field that is real in the Majorana representation. Such a particle would be its own antiparticle. This question is not yet settled and remains under investigation.

4 The S-matrix

We now have all the machinery needed to compute physical quantities in QFT. But we still need to make a connection between the correlation functions or n-point functions

$$\langle 0|T(\phi(x_1)\phi(x_2)\cdots\phi(x_n))|0\rangle$$

and physically measured quantities. This is done via the S-matrix.

In this section we do not want to be very specific about which field we are referring to (scalar, spinor or vector) because the discussion applies to all of them. However whenever specific formulae are needed, we will work with a real scalar field with a $\lambda\phi^4$ interaction, since it provides the simplest example.

The S-matrix, or scattering matrix, of a QFT is given by a collection of matrix elements between "in" states and "out" states in a scattering process. To understand these states, recall how we defined the interacting in-vacuum:

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} e^{-iHT}|0\rangle$$

and the out-vacuum:

$$\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)} \langle 0|e^{-iHT}$$

(both upto normalisation).

We have seen that the overlap between the two vacua is nontrivial:

$$\langle\Omega|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \langle 0|T\left(e^{-i\int_{-T}^{T} H_I(t')dt'}\right)|0\rangle$$

The physical interpretation of the operator appearing above is that it is a unitary operator that evolves us from the free vacuum in the far past to the free vacuum in the far future. The S-matrix will be a similar unitary operator, but it will take us from a noninteracting *multi-particle state* in the past to another one in the future.

Let us take the "in" state to be $|\vec{p}_1,\vec{p}_2\rangle_{\text{in}}$ associated to two particles of 4-momenta $(\omega_{\vec{p}_1},\vec{p}_1)$ and $(\omega_{\vec{p}_2},\vec{p}_2)$. In the interacting theory it would be hard to define this state. However in the free theory we know that the corresponding state is:

$$|\vec{p}_1,\vec{p}_2\rangle_0 = \sqrt{2\omega_{\vec{p}_1}}\sqrt{2\omega_{\vec{p}_2}}\; a^\dagger_{\vec{p}_1}a^\dagger_{\vec{p}_2}|0\rangle$$

Accordingly we define the state:

$$|\vec{p}_1,\vec{p}_2\rangle_{\text{in}} = \lim_{T\to\infty(1-i\epsilon)} e^{-iHT}|\vec{p}_1,\vec{p}_2\rangle_0$$

and likewise:

$$ {}_{\text{out}}\langle \vec{k}_1, \cdots, \vec{k}_n| = \lim_{T\to\infty(1-i\epsilon)} {}_0\langle \vec{k}_1, \cdots, \vec{k}_n| e^{-iHT} $$

Now the S-matrix is defined as the overlap, or inner product, of the "in" state with the "out" state:

$$ \boldsymbol{S}_{(\vec{k}_1,\cdots,\vec{k}_n|\vec{p}_1,\vec{p}_2)} = {}_{\text{out}}\langle \vec{k}_1, \cdots, \vec{k}_n|\vec{p}_1, \vec{p}_2\rangle_{\text{in}} $$

From the definition of in and out states, we can guess a formula that provides a useful calculational technique to evaluate the S-matrix in terms of free fields in perturbation theory, namely:

$$ \boldsymbol{S}_{(\vec{k}_1,\cdots,\vec{k}_n|\vec{p}_1,\vec{p}_2)} = \lim_{T\to\infty(1-i\epsilon)} {}_0\langle \vec{k}_1, \cdots, \vec{k}_n| T\left(e^{-i\int_{-T}^{T} H_I(t')dt'}\right)|\vec{p}_1, \vec{p}_2\rangle_0 $$

It is not easy to justify the above formula very precisely as we did for the vacuum expectation value earlier. Here we will simply assume it.

Consider the special case of $2 \to 2$ scattering, which means the final state has two particles in it. Expanding the above formula, we have:

$$ \begin{aligned} \boldsymbol{S}_{(\vec{k}_1,\vec{k}_2|\vec{p}_1,\vec{p}_2)} &= \lim_{T\to\infty(1-i\epsilon)} {}_0\langle \vec{k}_1, \vec{k}_2| T\left(e^{-i\int_{-T}^{T} H_I(t')dt'}\right)|\vec{p}_1, \vec{p}_2\rangle_0 \\ &= {}_0\langle \vec{k}_1, \vec{k}_2|\vec{p}_1, \vec{p}_2\rangle_0 + \mathcal{O}(\lambda) \end{aligned} \tag{2} $$

where for definiteness we can imagine an interaction of the form $H_{\text{int}} = \frac{\lambda}{4!}\int \phi^4$. Now, the first term in the last line is just:

$$ \begin{aligned} &\sqrt{2\omega_{\vec{k}_1}}\sqrt{2\omega_{\vec{k}_2}}\sqrt{2\omega_{\vec{p}_1}}\sqrt{2\omega_{\vec{p}_2}}\ \langle 0|a_{\vec{k}_1}a_{\vec{k}_2}a^\dagger_{\vec{p}_1}a^\dagger_{\vec{p}_2}|0\rangle \\ &\qquad = \ 2\omega_{\vec{p}_1}\cdot 2\omega_{\vec{p}_2}\cdot(2\pi)^6\left(\delta^3(\vec{p}_1-\vec{k}_1)\delta^3(\vec{p}_2-\vec{k}_2)+\delta^3(\vec{p}_1-\vec{k}_2)\delta^3(\vec{p}_2-\vec{k}_1)\right) \end{aligned} $$

This term simply sets the initial state equal to the final state. So it does not take into account interactions. The corresponding Feynman diagrams are "disconnected", not in the sense of bubble diagrams that we saw earlier but in the sense that particles pass through the process without interacting with each other.

We would like to define a matrix $\mathbf{T}$ that is the "nontrivial" part of S. For this purpose we need to drop all terms which do not contribute to scattering. Schematically we have:

$$ \boldsymbol{S} = \mathbf{1} + i\boldsymbol{T} $$

As we go along, we will see precisely how $\boldsymbol{T}$ should be defined.

Consider now the terms of order λ:

$$ -i\frac{\lambda}{4!}\ {}_0\langle \vec{k}_1, \vec{k}_2|\, T\Big(\int \phi^4(y)d^4y\Big)|\vec{p}_1, \vec{p}_2\rangle_0 $$

(here ϕ refers to the free field in the interaction picture, which should really be denoted ϕ_0 but we drop the 0 hoping no confusion will arise). Notice that this time we do not have a vacuum to vacuum amplitude so we cannot just expand the time-ordered product using Wick's theorem. However we can use the result, derived in a previous lecture, that:

$$T(\phi^4(y)) = :\phi^4(y): + 6D_F(0)\,:\phi^2(y): + 3D_F(0)^2$$

Now, recalling that $\phi = \phi_+ + \phi_-$ with:

$$\begin{aligned}\phi_+(t,\vec{x}) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\,a_{\vec{k}}\,e^{-ik\cdot x}\\ \phi_-(t,\vec{x}) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\,a^\dagger_{\vec{k}}\,e^{ik\cdot x}\end{aligned}$$

we see that $\phi_+(x)|\vec{p}_1\rangle \neq 0$ (unlike $\phi_+|0\rangle$). In fact,

$$\phi_+(x)|\vec{p}_1\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\,a_{\vec{k}}\,e^{-ik\cdot x}\sqrt{2\omega_{\vec{k}_1}}\,a^\dagger_{\vec{p}_1}|0\rangle = e^{-ip_1\cdot x}|0\rangle$$

It follows that $\langle\vec{k}_1,\vec{k}_2|:\phi^4(y):|\vec{p}_1,\vec{p}_2\rangle \neq 0$. We evaluate it as follows. Notice that terms in $:\phi^4(y):$ of the form $\phi^4_+(y)$ give a vanishing contribution to the matrix element:

$$\langle\vec{k}_1,\vec{k}_2|\phi^4_+(y)|\vec{p}_1,\vec{p}_2\rangle \sim \langle 0|a_{\vec{k}_1}a_{\vec{k}_2}\,(a)^4\,a^\dagger_{\vec{p}_1}a^\dagger_{\vec{p}_2}|0\rangle = 0$$

Extending this argument to the other terms, the only term we need to keep from $:\phi(y)^4:$ is $6\,\phi^2_-(y)\phi^2_+(y)$. Thus we have:

$$6\cdot -\frac{i}{4!}\lambda\int d^4y\langle\vec{k}_1,\vec{k}_2|\,\phi^2_-(y)\phi^2_+(y)\,|\vec{p}_1,\vec{p}_2\rangle$$

Now it's easy to see that:

$$\phi_+(y)\,|\vec{p}_1,\vec{p}_2\rangle = e^{-ip_1\cdot y}|\vec{p}_2\rangle + e^{-ip_2\cdot y}|\vec{p}_1\rangle$$

So

$$\phi^2_+(y)\,|\vec{p}_1,\vec{p}_2\rangle = 2\,e^{-i(p_1+p_2)\cdot y}|0\rangle$$

and similarly,

$$\langle\vec{k}_1,\vec{k}_2|\,\phi^2_-(y) = 2\,\langle 0|e^{i(k_1+k_2)\cdot y}$$

Thus we finally get:

$$-i\lambda\int d^4y\,e^{i(k_1+k_2-p_1-p_2)\cdot y} = -i\lambda(2\pi)^4\delta^4(k_1+k_2-p_1-p_2)$$

This is clearly a contribution to $\langle \boldsymbol{T} \rangle$. Let us define $\mathcal{M}$ by:

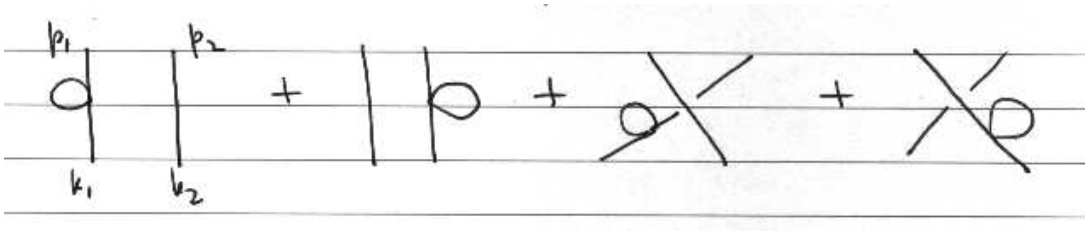
$$\langle \vec{k}_1, \vec{k}_2 | i\boldsymbol{T} | \vec{p}_1, \vec{p}_2 \rangle = i\mathcal{M} \cdot (2\pi)^4 \delta^4(k_1 + k_2 - p_1 - p_2)$$

So this term, coming from $:\phi^4(y):$, contributes $\mathcal{M} = -\lambda$.

Now we consider the other terms. For example, $D_F(0) :\phi^2(y):$ gives:

$$D_F(0) \, \langle \vec{k}_1, \vec{k}_2 | \phi_-(y)\phi_+(y) | \vec{p}_1, \vec{p}_2 \rangle$$

This can be represented as 4 terms:



Since all these represent non-interacting propagation, we include these in the "1" part of $\boldsymbol{S} = \mathbb{1} + i\boldsymbol{T}$.

Finally terms with $(D_F(0))^2$ give "completely disconnected" pieces which are therefore dropped.

Thus to order λ, we have:

$$\mathcal{M} = -\lambda$$

Here we finally see the physical meaning of λ: it is just (minus) the leading contribution to the $\mathcal{M}$-matrix, which in turn contains all the physical scattering data contained in the $\boldsymbol{S}$-matrix.

We also see that $\mathcal{M}$ differs in an important way from the four-point correlation function:

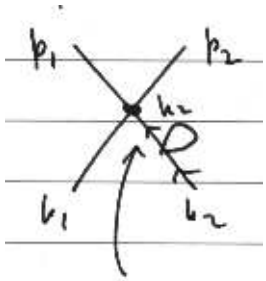
$$\langle 0 | T\Big(\phi(x_1) \cdots \phi(x_4) \int d^4y \, \phi^4(y)\Big) | 0 \rangle.$$

The latter in momentum space has four propagators and is equal to:

$$-\lambda \prod_{i=1}^{4} \frac{1}{k_i^2 - m^2 + i\epsilon} \, (2\pi)^4 \delta^4(k_i + \cdots + k_r)$$

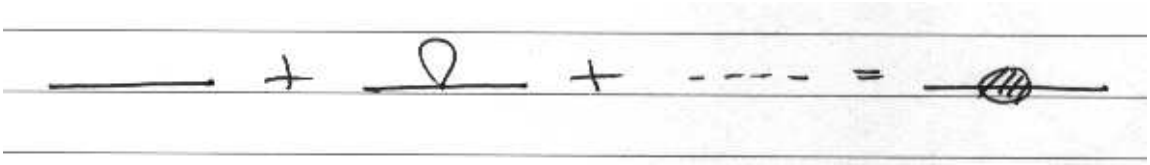
Thus $\mathcal{M}$ is an "amputated" version of the four-point function, i.e. with the external legs chopped off. This is just as well, because with external on-shell particles satisfying $k^2 = m^2$, the propagators $\dfrac{1}{k^2 - m^2 + i\epsilon}$ would have diverged!

But now we see a problem. In the next order we would find



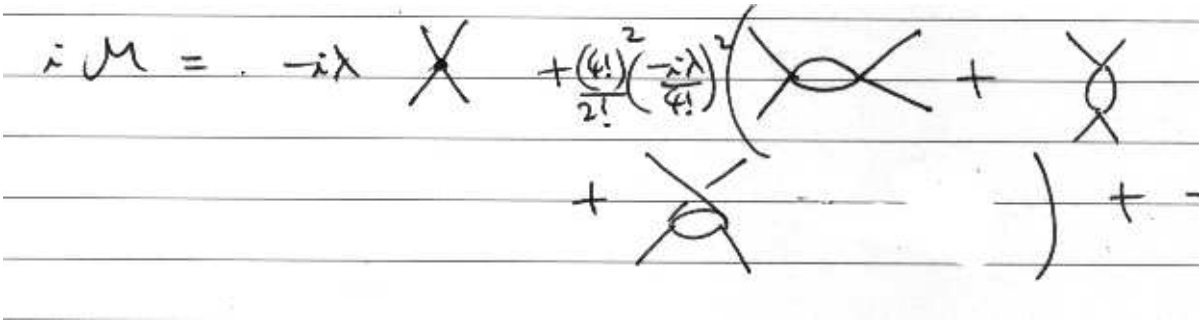
In this diagram, even after chopping off the external propagators, we find there is an *internal* propagator (marked by the arrow) of momentum k_2! This would diverge on-shell.

The solution is to understand that the true propagator is



and it is the whole thing that should be chopped off. In fact the "blob" diagram on the RHS encodes how $|\vec{p}\rangle_0$ turns into $|\vec{p}\rangle$.

Therefore the prescription to compute the $\boldsymbol{T}$ or $\mathcal{M}$ matrix is to keep only completely connected diagrams, amputate the full (corrected) full external legs, and then place the external particles on-shell.



5 Quantum Electrodynamics

We are now ready to compute physical quantities using QFT. We will work with the theory of quantum electrodynamics, namely a single charged fermion (electron) $\psi(x)$ coupled to the photon $A_\mu(x)$. This theory is obtained by starting with the Dirac action:

$$i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi$$

and promoting the phase invariance under $\psi \to e^{i\alpha}\psi$, with α constant, to a gauge invariance where $\alpha = \alpha(x)$. This requires the introduction of a vector field A_μ and the generalisation of the derivative:

$$\partial_\mu \to \partial_\mu - ieA_\mu$$

We also add a kinetic term for A_μ. The Lagrangian density is therefore:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\overline{\psi}\gamma^\mu(\partial_\mu - ieA_\mu)\psi - m\overline{\psi}\psi$$

We split the above Lagrangian into its free (quadratic) part and its interacting (cubic) part. The free part is then quantised via anticommutators for the fermions. For the gauge field we choose Lorentz gauge and then perform quantisation via the usual commutators. The next step is to make a mode expansion for ψ as we have done in a previous section, and a mode expansion for A_μ, in Lorentz gauge as follows:

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\sum_{r=0}^{3}\left(a^r_{\vec{k}}\epsilon^r_\mu(k)e^{-ik\cdot x} + a^{\dagger r}_{\vec{k}}\epsilon^{r\star}_\mu(k)e^{ik\cdot x}\right)$$

where $\epsilon^r_\mu(k), r = 1, 2$ are the polarisation vectors, analogous to the free-particle spinors $u_a(x), v_a(x)$ for spinor fields $\psi_a(x)$. If the photon momentum is chosen to be $(k, 0, 0, k)$ then a basis for $\epsilon^r_\mu(k)$ is $(0, 1, 0, 0)$ and $(0, 0, 1, 0)$ (plane polarised) or $(0, 1, i, 0)$ and $(0, 1, -i, 0)$ (circularly polarised). Finally the interaction $e\overline{\psi}\gamma^\mu A_\mu\psi$ is treated as a perturbation.

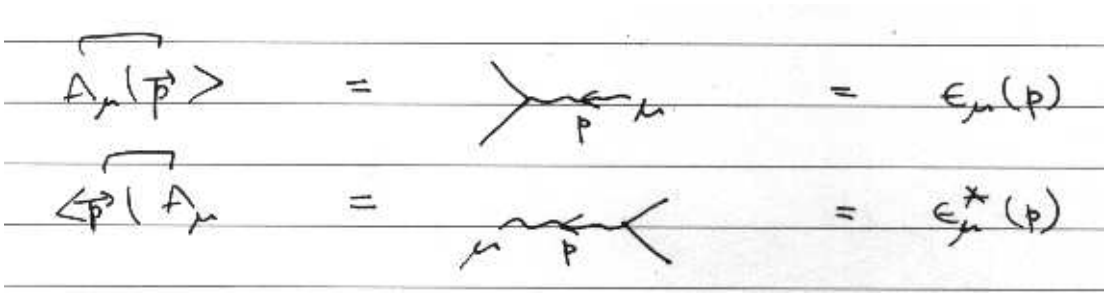
5.1 Feynman rules

We start by formulating the Feynman rules for this theory. The momentum-space propagators are:

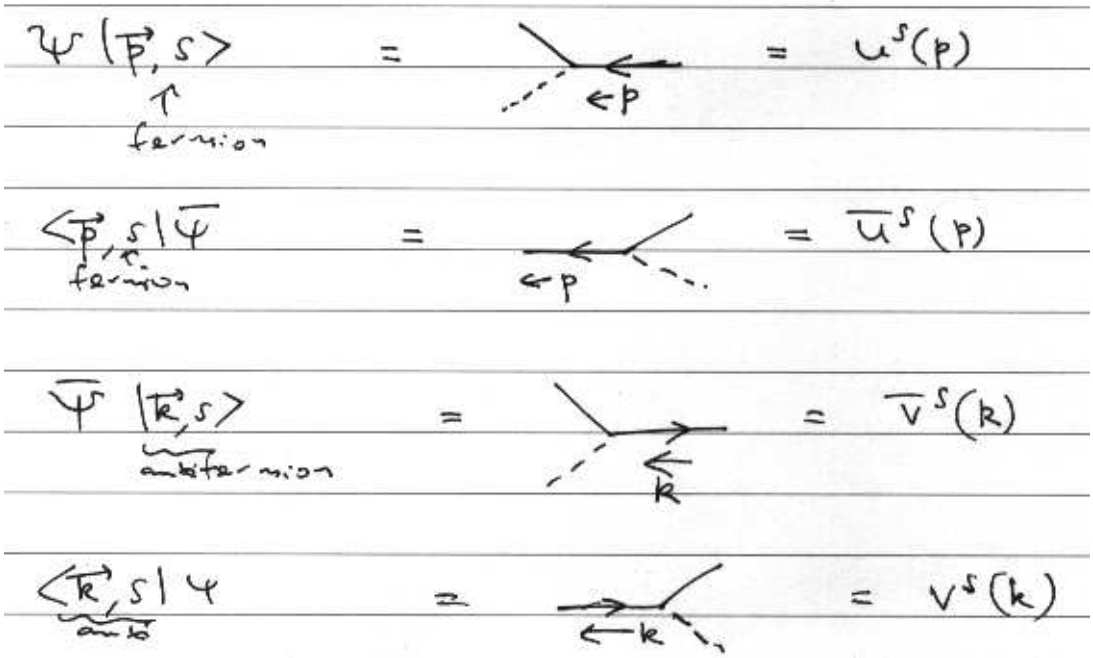
$$\psi: \qquad \frac{i(\not{k} + m)}{k^2 - m^2 + i\epsilon}$$

$$A_\mu: \qquad \frac{-i\,\eta_{\mu\nu}}{k^2 + i\epsilon}$$

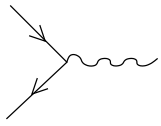
We need to put polarisation factors on the external photon lines as follows:



Similarly we need to put in external polarisation factors for spinors:



Finally, the interaction vertex of QED is just the γ-matrix:



$-ie\gamma^\mu$

Depending on where this appears in a diagram, it can describe an electron emitting a photon, or an electron absorbing a photon, or a positron emitting a photon, or a positron absorbing a photon, or an electron-positron pair annihilating into a photon, or a photon dissociating into an electron-positron pair. We see that the vertex is rather a versatile object! It must

be noted that for kinematical reasons, none of the above six processes can actually occur as an on-shell processes with external particles – for example, it is well-known that an electron-positron pair cannot give rise to a single photon because energy-momentum conservation along cannot be satisfied for this process along with the on-shell conditions for each particle.

This is not a problem, because all the particles in the interaction vertex are allowed to be off-shell. The vertex is then used as a building block for more complicated physical processes that do satisfy all the kinematical conditions, such as electron-positron annihilation into a pair of photons or scattering of electrons with positrons or photons or with themselves.

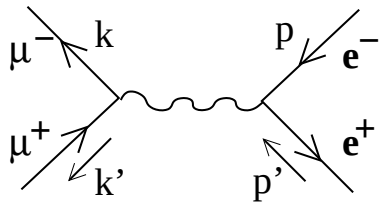
To add more "matter" particles in the theory, we simply add one copy of the Dirac action (minimally coupled to the gauge field) for each species of particle, with the mass parameter appearing in the mass term being fixed to its experimental value for each species. Each species will give its own interaction vertex $-ie\gamma^\mu$ coupling the photon to the particle and antiparticle of that species.

5.2 $e^+e^- \to \mu^+\mu^-$

To illustrate how QED calculations of physical quantities are carried out, consider the process:

$$e^+e^- \to \mu^+\mu^-$$

where the external polarisations are labelled s, s' for e^-, e^+ and r, r' for μ^-, μ^+.



Time is taken to flow *from right to left* in this diagram. Note that the momentum flow and particle number flow (the latter is the same as negative charge flow) are in the same direction for particles. Hence a single arrow on the line labels both. However they are in opposite directions for antiparticles. In this case the arrow on the line indicates the particle number flow while the separate arrow labels the momentum flow.

The corresponding $\boldsymbol{S}$-matrix element is

$$\boldsymbol{S}_{(\vec{k},r,\vec{k}',r'|\vec{p},s,\vec{p}',s')} = 1 + (2\pi)^4\delta^4(p + p' - k - k') \cdot i\mathcal{M}$$

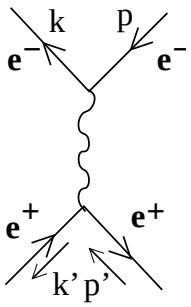
with

$$\mathcal{M} = \overline{v}^{s'}(p')(-ie\gamma^\mu)u^s(p)\,\frac{-ig_{\mu\nu}}{(p+p')^2}\,\overline{u}^r(k)(-ie\gamma^\nu)v^{r'}(k')$$

where we have used the fact that the 4-momentum carried by the internal photon is $(p+p')_\mu$. The above process is of order e^2 in the $\boldsymbol{S}$-matrix. Note that no other diagrams contribute at this order.

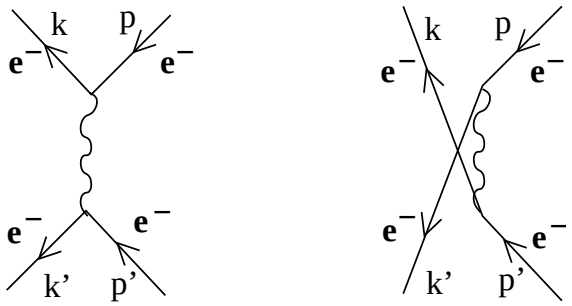
A diagram like the above where the incoming particles first annihilate into a virtual particle and then the final state particles are produced, is called an "s-channel" diagram.

If we were doing $e^+e^- \to e^+e^-$ there would be another diagram in addition to the s-channel one. This is:



and is called a "t-channel" diagram.

On the other hand if we were doing $e^-e^- \to e^-e^-$ we would have the two diagrams:



where the first one is t-channel and the second is "u-channel". However there is no s-channel diagram – because e^-e^- cannot annihilate into a photon. We see that all conservation laws are already respected by the Feynman diagram expansion in QFT.

Returning to our original process $e^+e^- \to \mu^+\mu^-$, the quantity $\mathcal{M}$ can be simplified to:

$$\mathcal{M} = \frac{ie^2}{(p+p')^2}\overline{v}^{s'}(p')\gamma^\mu u^s(p)\,\overline{u}^r(k)\gamma_\mu v^{r'}(k')$$

Now we need to extract a physical quantity out of this. This quantity is the *differential cross-section*, which can be shown to be:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm cm} = \frac{1}{2\omega_{\vec{p}}\,2\omega_{\vec{p}'}\,|\vec{v}-\vec{v}\,'|}\,\frac{|\vec{k}|}{16\pi^2 E_{\rm CM}}|\mathcal{M}(k,k';p,p')|^2$$

where $\vec{v} = \frac{\vec{p}}{\omega_{\vec{p}}}$ and $\vec{v}\,' = \frac{\vec{p}\,'}{\omega_{\vec{p}\,'}}$, so $\vec{v} - \vec{v}'$ is the relative velocity of the two incoming beams. A derivation of the above formula is provided in Appendix I to this Section.

The above formula is general for any $2 \to 2$ scattering. Now in principle we must plug in the value of $\mathcal{M}$ for $e^+e^- \to \mu^+\mu^-$, given above. But we actually need $|\mathcal{M}|^2$ which turns out to be simpler! We find:

$$|\mathcal{M}|^2 = \frac{e^4}{(p+p')^4} \left(\overline{v}^{s'}(p')\gamma^\mu u^s(p)\, \overline{u}^s(p)\gamma^\nu v^{s'}(p')\right) \left(\overline{u}^r(k)\gamma_\mu v^{r'}(k')\, \overline{v}^{r'}(k')\gamma_\nu u^r(k)\right)$$

where we have used $(\overline{u}\gamma^\mu v)^\star = (\overline{v}\gamma^\mu u)$, an identity that is easily proved by going to a fixed γ-matrix basis.

Notice that in the present process since there is only a single diagram that contributes to leading order, $|\mathcal{M}|^2$ is just the square of the contribution of that diagram. However for processes where there are many diagrams, for example when both s and t channels contribute (or when we go to the next order in perturbation theory), $\mathcal{M}$ is the sum of contributions from all the diagrams and therefore $|\mathcal{M}|^2$ involves *both* the squares of individual diagrams *and* the cross-terms that arise in squaring. The latter can be thought of as "interference terms". One should not make the mistake of squaring each diagram separately and then adding the result, which would miss the important cross-terms.

Returning to the present calculation, in principle r, r', s, s' (the last two corresponding to incoming polarisations and the former to outgoing polarisations) are all arbitrary. However, in experiments we often scatter unpolarised beams, so we must *average* over s and s'. Also we often don't measure the final spins, in which case we must *sum* over final-state spins r, r'. In such situations we therefore need to calculate:

$$\tfrac{1}{2}\sum_s \tfrac{1}{2}\sum_{s'}\sum_r\sum_{r'} |\mathcal{M}|^2$$

In this situation the completeness relations, derived in a previous lecture, help us to simplify the expression. Using:

$$\begin{aligned}\sum_s u^s(p)\overline{u}^s(p) &= \not{p} + m \\ \sum_s v^s(p)\overline{v}^s(p) &= \not{p} - m\end{aligned}$$

we find that:

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{4(p+p')^4}\ \text{tr}\left[(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\right]\ \text{tr}\left[(\not{k} + m_\mu)\gamma_\mu(\not{k}' - m_\mu)\gamma_\nu\right]$$

As shown in Appendix II to this Section, this reduces to:

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{8e^4}{(p+p')^4}\left[p\cdot k\, p'\cdot k' + p\cdot k'\, p'\cdot k + m_\mu^2\, p\cdot p' + m_e^2\, k\cdot k' + 2m_\mu^2 m_e^2\right]$$

We now set $m_e \to 0$ since $m_e^2 \ll p\cdot p'$. The error will be smaller than that introduced by neglecting higher-order terms in the perturbation series.

In the CM frame, we have:

$$p = (E,0,0,E),\ p' = (E,0,0,-E)$$

with $2E = E_{\text{cm}}$. Also, $k = (E,\vec{k})$ and $k' = (E,-\vec{k})$, where $E = \sqrt{\vec{k}^2 + m_\mu^2}$.

Now,

$$\begin{aligned}
(p+p')^2 &= 4E^2 \\
p\cdot p' &= 2E^2 \\
p\cdot k &= E^2 - E|k|\cos\theta = p'\cdot k' \\
p\cdot k' &= E^2 + E|k|\cos\theta = p'\cdot k
\end{aligned}$$

where θ is the angle of the outgoing $\mu^+\mu^-$ pair with respect to the incoming e^+e^- pair. So,

$$\begin{aligned}
\frac{1}{4}|\mathcal{M}|^2 &= \frac{8e^4}{16E^4}\left[E^2(E-|k|\cos\theta)^2 + E^2(E+|k|\cos\theta)^2 + 2m_\mu^2E^2\right] \\
&= \frac{e^4}{2E^2}\left[2E^2 + 2|k|^2\cos^2\theta + 2m_\mu^2\right] \\
&= \frac{e^4}{E^2}\left[E^2 + \cos^2\theta(E^2 - m_\mu^2) + m_\mu^2\right] \\
&= e^4\left[\left(1+\frac{m_\mu^2}{E^2}\right) + \left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]
\end{aligned}$$

Finally,

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\,E^2|\vec{v}-\vec{v}'|}\frac{|\vec{k}|}{16\pi^2E_{\text{cm}}}e^4\left[\left(1+\frac{m_\mu^2}{E^2}\right) + \left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$$

Using

$$|\vec{v}-\vec{v}'| = \left|\frac{p_z}{E} - \frac{p_z'}{E}\right| = 2,\ E = \frac{E_{\text{cm}}}{2}$$

and

$$\alpha = \frac{e^2}{4\pi},\ |k| = E\sqrt{1-\frac{m_\mu^2}{E^2}}$$

we can write down our final answer for the differential cross-section:

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{1}{2E_{\rm cm}^2}\frac{E_{\rm cm}}{2}\sqrt{1-\frac{4m_\mu^2}{E_{\rm cm}^2}}\,\frac{1}{16\pi^2 E_{\rm cm}}\,e^4\left[\left(1+\frac{4m_\mu^2}{E_{\rm cm}^2}\right)+\left(1-\frac{4m_\mu^2}{E_{\rm cm}^2}\right)\cos^2\theta\right] \\ &= \frac{\alpha^2}{4E_{\rm cm}^2}\sqrt{1-\frac{4m_\mu^2}{E_{\rm cm}^2}}\left[\left(1+\frac{4m_\mu^2}{E_{\rm cm}^2}\right)+\left(1-\frac{4m_\mu^2}{E_{\rm cm}^2}\right)\cos^2\theta\right]\end{aligned}$$

For the total cross-section, we simply integrate:

$$\sigma = \int d\Omega\,\frac{d\sigma}{d\Omega} = \int d(\cos\theta)\,d\phi\,\frac{d\sigma}{d\Omega}$$

The differential cross-section is independent of ϕ and has a dependence of the form $A + B\cos^2\theta$. Then the integral is given by:

$$\begin{aligned}\sigma &= 2\pi\int d(\cos\theta)[A+B\cos^2\theta] \\ &= 2\pi\left(2A+\frac{2B}{3}\right) \\ &= \frac{4\pi}{3}\,(3A+B)\end{aligned}$$

Inserting the value of A and B we get:

$$\begin{aligned}\sigma &= \frac{4\pi}{3}\frac{\alpha^2}{4E_{\rm cm}^2}\sqrt{1-\frac{4m_\mu^2}{E_{\rm cm}^2}}\left(4+\frac{8m_\mu^2}{E_{\rm cm}^2}\right) \\ &= \frac{4\pi\alpha^2}{3E_{\rm cm}^2}\sqrt{1-\frac{4m_\mu^2}{E_{\rm cm}^2}}\left(1+\frac{2m_\mu^2}{E_{\rm cm}^2}\right)\end{aligned}$$

In the limit $E_{\rm cm} \gg m_\mu$, we get

$$\begin{aligned}\frac{d\sigma}{d\Omega} &\to \frac{\alpha^2}{4E_{\rm cm}^2}(1+\cos^2\theta) \\ \sigma &\to \frac{4\pi\alpha^2}{3E_{\rm cm}^2}\end{aligned}$$

Appendix I to Section 5.2: Formula for $\frac{d\sigma}{d\Omega}$

For this, note that in an experiment, rather than the idealised process:

$$\vec{p}, s; \quad \vec{p}', s' \;\rightarrow\; \vec{k}, r; \quad \vec{k}', r'$$

we actually start with wave packets:

$$|\phi_{e^-}, \phi_{e^+}\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}'}}}\,\phi(\vec{p})\phi'(\vec{p}')|\vec{p}, \vec{p}'\rangle$$

where $\phi(\vec{p})$, $\phi(\vec{p}')$ are the Fourier transforms of the spatial wave-function:

$$\int \frac{d^3p}{(2\pi)^3}|\phi(\vec{p})|^2 = 1 = \int \frac{d^3\vec{p}'}{(2\pi)^3}|\phi'(p')|^2$$

More precisely, one of the wave-packets is allowed to have a <u>transverse</u> spread with an impact parameter $\vec{b}$. Thus we use

$$|\phi_{e^-}, \phi_{e^+}\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}'}}}\phi(\vec{p})\phi'(\vec{p}')e^{-i\vec{p}'\cdot\vec{b}}|\vec{p}, s; \vec{p}', s'\rangle$$

For the final state, we assume the momenta k, k' are distributed in a region with invariant volumes

$$\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}, \;\; \frac{d^3k'}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}'}}.$$

Then the infinitesimal cross-section $d\sigma$ is defined as:

$$d\sigma = \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}\frac{d^3k'}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}'}}\int d^2b\,|\langle\vec{k}, r; \vec{k}', r'|\phi_{e^-}, \phi_{e^+}\rangle|^2$$

This appears to depend sensitively on $\phi(\vec{p})$, $\phi'(\vec{p}')$ but we will see that under suitable assumptions, it does not.

Now, suppressing the spin labels r, r', s, s' we have:

$$\int d^2b\,|\langle\vec{k}, \vec{k}'|\phi_{e^-}, \phi_{e^+}\rangle|^2 =$$

$$\int d^2b \int \frac{d^2p}{(2\pi)^3}\frac{\phi(\vec{p})}{\sqrt{2\omega_{\vec{p}}}}\int \frac{d^3p'}{(2\pi)^3}\frac{\phi'(\vec{q}')}{\sqrt{2\omega_{\vec{p}'}}}\int \frac{d^3q}{(2\pi)^3}\frac{\phi^\star(\vec{q})}{\sqrt{2\omega_{\vec{q}}}}\int \frac{d^3q'}{(2\pi)^3}\frac{\phi'^\star(\vec{q}')}{\sqrt{2\omega_{\vec{q}'}}}$$

$$\times\; \langle\vec{k}, \vec{k}'|\vec{p}, \vec{p}'\rangle\; \langle\vec{k}, \vec{k}'|\vec{q}, \vec{q}'\rangle^\star\; e^{i\vec{b}\cdot(\vec{q}'-\vec{p}')}$$

Now from the definition,

$$\begin{aligned}\langle\vec{k}, \vec{k}'|\vec{p}, \vec{p}'\rangle &= i\mathcal{M}(\vec{k}, \vec{k}'; \vec{p}, \vec{p}')\,(2\pi)^4\delta^4(p + p' - k - k')\\ \langle\vec{k}, \vec{k}'|\vec{q}, \vec{q}'\rangle^* &= -i\mathcal{M}^\star(\vec{k}, \vec{k}'; \vec{q}, \vec{q}')\,(2\pi)^4\delta^4(q + q' - k - k')\end{aligned}$$

The $\int d^2b$ integral can now be done, giving $(2\pi)^2\delta^2(\vec{q}'_\perp - \vec{p}'_\perp)$.

Next we would like to evaluate the integrals $\int d^3q$, $\int d^3q'$, using the δ-functions:

$$\delta^2(\vec{q}'_\perp - \vec{p}'_1)\,\delta^4(q + q' - k - k')$$

If the scattering direction is z then $\delta^2(\vec{q}'_\perp - \vec{p}'_\perp)$ just sets $q'_x = p'_x$ and $q'_y = p'_y$. That leaves integrals over q'_z, q_x, q_y, q_z. But the q_x and q_y integrals are also easy – they set:

$$\begin{aligned} q_x &= k_x + k'_x - p'_x = p_x \\ q_y &= k_y + k'_y - p'_y = p_y \end{aligned}$$

The integral over q'_z is also easy:

$$\int dq'_z\,\delta(q_z + q'_z - k_z - k'_z)$$

sets $q'_z = k_z + k'_z - q_z$.

Finally we have:

$$\begin{aligned} \int dq_z\,\delta(q_0 + q'_0 - k_0 - k'_0) &= \int dq_z\,\delta\left(\sqrt{\vec{q}^2 + m^2} + \sqrt{\vec{q}'^2 + m^2} - k_0 - k'_0\right)\Big|_{q'_z = k_z + k'_z - q_z} \\ &= \frac{1}{\left|\frac{q_z}{q_0} - \frac{q'_z}{q'_0}\right|} \end{aligned}$$

Now we also set $q_0 + q'_0 - k_0 - k'_0 = 0$ and $q_z + q'_z = k_z + k'_z = p_z + p'_z$. Therefore $q_0 + q'_0 = p_0 + p'_0$ which implies that $q_z = p_z,\ q'_z = p'_z$.

Notice that:

$$\frac{q_z}{q_0} - \frac{q'_z}{q'_0} = v - v'$$

where the RHS is the relative velocity of the two beams in the lab frame. Thus the δ-functions set $q_\mu = p_\mu$, $q'_\mu = p'_\mu$ and we have

$$\begin{aligned} \int d^2b\,|\langle \vec{k}, \vec{k}'|\phi_{e^-}, \phi_{e^+}\rangle|^2 = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\int \frac{d^3p'}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}'}}\frac{|\mathcal{M}(\vec{k}, \vec{k}'; \vec{p}, \vec{p}')|^2}{|v - v'|} \\ \times\ |\phi(\vec{p})|^2|\phi'(\vec{p}')|^2(2\pi)^4\delta^4(p + p' - k - k') \end{aligned}$$

At this stage there is some simplification because $\dfrac{1}{(2\pi)^6}$ from $\int d^3q, \int d^3q'$ multiplied by a $(2\pi)^2$ from $\int d^2b$, times a $(2\pi)^4$ from one of the $\mathcal{M}$'s gives 1.

Now if $\phi(\vec{p})$, $\phi'(\vec{p}')$ are sufficiently peaked at their central values (which we also call $\vec{p}$, $\vec{p}'$ respectively) then we can evaluate

$$\int \frac{d^3p}{(2\pi)^3}|\phi(\vec{p})|^2 = \int \frac{d^3p'}{(2\pi)^3}|\phi(\vec{p}')|^2 = 1$$

and take everything else outside:

$$\int d^2b|\langle\ |\ \rangle|^2 = \frac{|\mathcal{M}(k,k';p,p')|^2}{2\omega_{\vec{p}}\cdot 2\omega_{\vec{p}'}|v-v'|}(2\pi)^4\delta^4(p+p'-k-k')$$

Thus the infinitesimal cross-section $d\sigma$ is the above multiplied by:

$$\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}\frac{d^3k'}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}'}}$$

We have chosen the z direction as the scattering direction. Therefore the answer is not expected to be rotation- or Lorentz-invariant, except under z-boosts. This is indeed the case:

$$\frac{1}{\omega_{\vec{p}}\omega_{\vec{p}'}|v-v'|} = \frac{1}{|p_z\omega_{\vec{p}'}-p'_z\omega_{\vec{p}}|} = \frac{1}{|p_zp'_t-p_tp'_z|} = \frac{4}{|p_-p'_+-p_+p'_-|}$$

with $p_\pm = p_t \pm p_z$. And this manifestly exhibits $2d$ Lorentz invariance.

The differential cross-section per unit solid angle is obtained by writing out the k, k' integral and carrying out the k' integration:

$$\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}\int \frac{d^3k'}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}'}}(2\pi)^4\delta^4(p+p'-k-k')$$

If we are in the centre-of-mass frame then $\vec{p}+\vec{p}'=0$. Thus $\int d^3k'$ sets $\vec{k}'=-\vec{k}$ and the above expression becomes:

$$\int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2E}\,\frac{1}{2E'}\,2\pi\,\delta(E_{\rm CM}-E-E')$$

where $E=\sqrt{\vec{k}^2+m^2}$ and $E'=\sqrt{\vec{k}'^2+m^2}$. Now writing $d^3k = d|k|\,|k|^2\,d\Omega$, the integral becomes:

$$\int \frac{d|\vec{k}|\ |\vec{k}|^2d\Omega}{16\pi^2}\,\frac{\delta(E_{\rm CM}-E-E')}{EE'} = \int \frac{|\vec{k}|^2d\Omega}{16\pi^2}\,\frac{1}{EE'}\frac{1}{\left(\frac{|\vec{k}|}{E}+\frac{|\vec{k}|}{E'}\right)}$$

where now $|\vec{k}|$ is no longer an integration variable but is determined by:

$$\sqrt{\vec{k}^2+m^2}+\sqrt{\vec{k}\,'^2+m^2} = E_{\rm CM}$$

Thus the above expression is equal to:

$$\int d\Omega \frac{|k|}{16\pi^2} \cdot \frac{1}{E+E'} = \int \frac{d\Omega}{16\pi^2} \frac{|k|}{E_{\rm cm}}$$

and:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm cm} = \frac{1}{2\omega_{\vec{p}}\, 2\omega_{\vec{p}'}\, |\vec{v}-\vec{v}'|} \frac{|\vec{k}|}{16\pi^2 E_{\rm cm}} |\mathcal{M}(k,k';p,p')|^2$$

Appendix II to Section 5.2: Evaluation of a trace

In this Appendix we wish to evaluate:

$$\frac{1}{4}\sum_{\rm spins} |\mathcal{M}|^2 = \frac{e^4}{4(p+p')^4}\ {\rm tr}\ \left[(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\right]\ {\rm tr}\ \left[(\not{k} + m_\mu)\gamma_\mu(\not{k}' - m_\mu)\gamma_\nu\right]$$

For this we need to know the traces of several matrix products. It is easy to establish that:

$$\begin{aligned}
&{\rm tr}\, \mathbb{1} = 4 \\
&{\rm tr}\, \gamma^\mu = 0 \\
&{\rm tr}\, \gamma^\mu\gamma^\nu = 4\,\eta^{\mu\nu} \\
&{\rm tr}\, \gamma^\mu\gamma^\nu\gamma^\lambda = 0 \\
&{\rm tr}\, \gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho = 4(\eta^{\mu\nu}\eta^{\lambda\rho} - \eta^{\mu\lambda}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\lambda})
\end{aligned}$$

From the above,

$$\begin{aligned}
(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu &= p'_\alpha p_\beta\, {\rm tr}\, \gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu - m_e^2\, {\rm tr}\, \gamma^\mu\gamma^\nu \\
&= 4\, p'_\alpha p_\beta\, (\eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\beta}\eta^{\mu\nu} + \eta^{\alpha\nu}\eta^{\mu\beta}) - 4m_e^2\eta^{\mu\nu} \\
&= 4\,(p'^\mu p^\nu + p'^\nu p^\mu - pp'\eta^{\mu\nu}) - 4m_e^2\eta^{\mu\nu} \\
&= 4\,(p'^\mu p^\nu + p'^\nu p^\mu - \eta^{\mu\nu}(p\cdot p' + m_e^2))
\end{aligned}$$

Similarly,

$${\rm tr}\,(\not{k} + m_\mu)\gamma_\mu(\not{k} - m_\mu)\gamma_\nu = 4\,(k_\mu k'_\nu + k_\nu k'_\mu - \eta_{\mu\nu}(k\cdot k' + m_\mu^2))$$

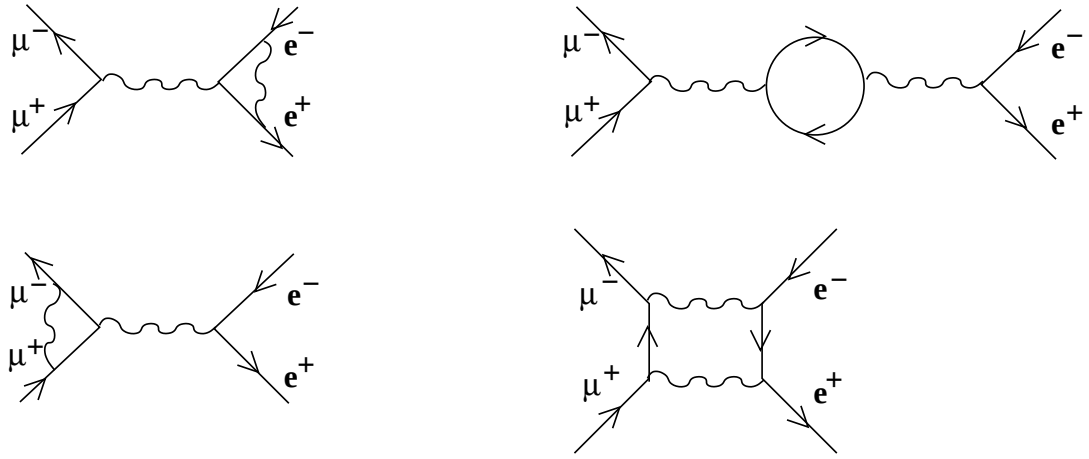
With these results we find that:

$$\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 &= \frac{16\,e^4}{4(p+p')^4}\,[p'^{\mu}p^{\nu}+p'^{\nu}p^{\mu}-\eta^{\mu\nu}(p\cdot p'+m_e^2)][k'_{\mu}k_{\nu}+k'_{\nu}k_{\mu}-\eta_{\mu\nu}(k\cdot k'+m_{\mu}^2)] \\
&= \frac{4e^4}{(p+p')^4}\Big[2p\cdot k\,p'\cdot k'+2p\cdot k'\,p'\cdot k-2(p\cdot p'+m_e^2)(k\cdot k') \\
&\qquad\qquad -2(k\cdot k'+m_{\mu}^2)p\cdot p'+4(p\cdot p'+m_e^2)(k\cdot k'+m_{\mu}^2)\Big] \\
&= \frac{8e^4}{(p+p')^4}\,\big[p\cdot k\,p'\cdot k'+p\cdot k'\,p'\cdot k+m_{\mu}^2\,p\cdot p'+m_e^2\,k\cdot k'+2m_{\mu}^2m_e^2\big]
\end{aligned}$$

which is the desired formula.

6 Radiative corrections and renormalisation

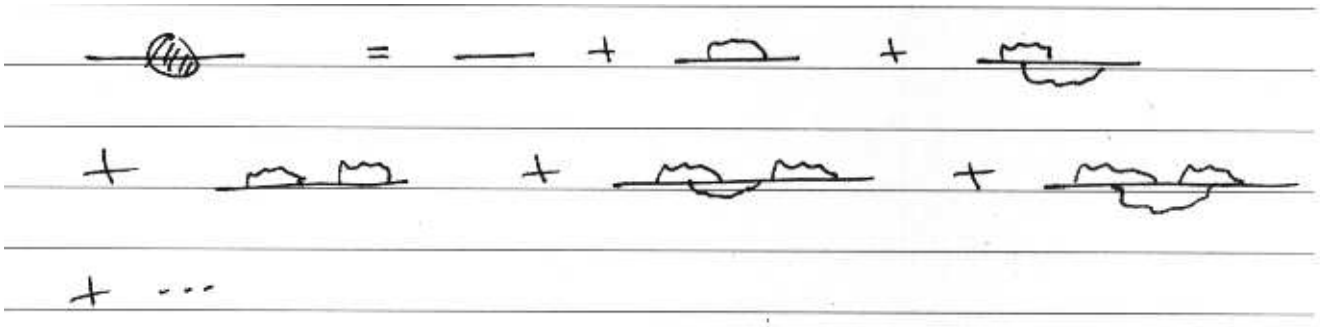
We must now understand how to deal with corrections that are higher order in $e^2 \sim \alpha$. For example, in $e^+e^- \to \mu^+\mu^-$ we have:



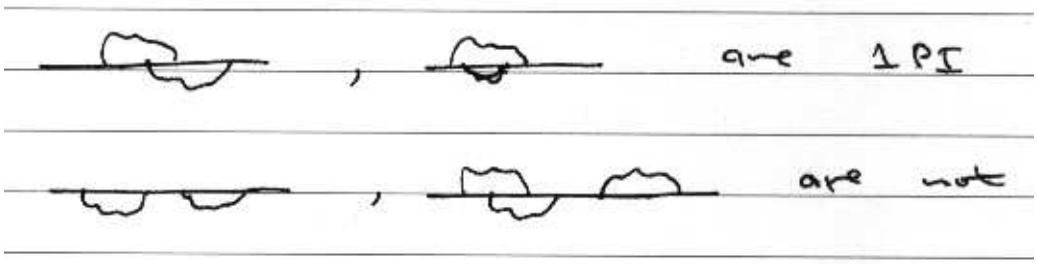
These corrections include a diagram where the photon, while propagating, emits a virtual electron-positron pair and re-absorbs it. This is considered to be a contribution to the "photon self-energy". Also there are two diagrams where the cubic interaction vertex of QED gets corrected by a virtual photon. This is called the "vertex correction". Finally there could have been diagrams where the electron or positron had a self-energy correction due to a virtual photon, but these have been dropped as part of the prescription of amputating full external legs.

The photon self-energy and vertex correction are associated to two- and three-point functions respectively. So we can study them in isolation and then plug them back into the 4-point function later. However, the last diagram is not of this type. It is intrinsically a correction to a four-point function.

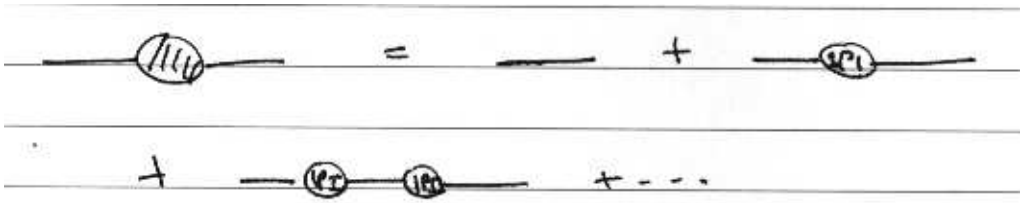
Let us therefore examine each one in turn. We start by studying the electron propagator correction. To all orders in perturbation theory, one can represent it by:



Now let us we define a "one-particle irreducible" diagram (1PI) by saying it does not fall apart on cutting one line:



Then the entire expansion can be organised as a collection of 1PI contributions strung in a row:



Note that since we are attaching them to each other with Feynman propagators, the 1PI pieces themselves should be amputated. Let this amputated 1PI contribution, calculated to any order in perturbation theory, be called $-i\Sigma(p)$. Note that it is a matrix in spinor space.

Now the full propagator can be written:

$$\begin{aligned} & \frac{i}{\not{p} - m_0} + \frac{i}{\not{p} - m_0}(-i\Sigma)\frac{i}{\not{p} - m_0} + \cdots \\ = & \frac{i}{\not{p} - m_0}\left(1 + \Sigma\frac{1}{\not{p} - m_0} + \Sigma\frac{1}{\not{p} - m_0}\Sigma\frac{1}{\not{p} - m_0} + \cdots\right) \\ = & \frac{i}{\not{p} - m_0 - \Sigma(p)} \end{aligned}$$

Here we have denoted the mass that appears in the Lagrangian by m_0. We see that the full propagator $\dfrac{i}{\not{p} - m_0 - \Sigma(p)}$ has a pole *not* at $\not{p} = m_0$ (since $\Sigma(p)|_{\not{p}=m_0} \neq 0$), but at some shifted value which we can write $\not{p} = m$.

Accordingly we define the physical mass of the electron to be the parameter m satisfying:

$$(\not{p} - m_0 - \Sigma(p))_{\not{p}=m} = 0$$

To lowest order $\Sigma_2(p)$ is of order e^2, so to the same order, inside it we can set $\not{p} = m \simeq m_0$, thus:

$$\not{p} - m_0 - \Sigma_2(\not{p} = m_0) = 0$$

which tells us that to this order,

$$m = m_0 + \Sigma_2(\not{p} = m_0).$$

Now using Feynman diagram techniques, the lowest-order contribution to Σ can be computed. It involves a divergent momentum integral but if we cut off the integral at a UV scale Λ then we get:

$$\Sigma_2(\not{p} = m_0) = \frac{3\alpha}{4\pi} m_0 \log\frac{\Lambda^2}{m_0^2}$$

where as usual $\alpha = e^2/4\pi$ and we ignore higher-order corrections in α.

Thus there is a mass *shift* or *renormalization* so that the physical mass m (the pole in the full propagator) is related to the "bare mass" m_0 (the parameter in the Lagrangian) by:

$$m = m_0\left(1 + \frac{3\alpha}{4\pi}\log\frac{\Lambda^2}{m_0^2}\right)$$

This tells us that once quantum corrections are introduced, the parameter m_0 in the Lagrangian is <u>not</u> the physically observed mass $m = 0.5110$ MeV. Rather, m_0 is tuned as a function of the UV cutoff *so that* the physically observed mass m becomes 0.5110 MeV.

The physical picture that emerges is that parameters in the classical QFT Lagrangian do not correspond to experimentally measured quantities! Instead, one first computes quantum

corrections and then sets the quantum-corrected parameter (in this case, the mass) to the experimentally measured value. The same is done with the coupling constant. The bare constant is called e_0 and the vertex correction diagram allows one to define a renormalised constant e in terms of e_0 (as well as the cutoff). The experimentally measured fine structure constant α is not $e_0^2/4\pi$ but rather $e^2/4\pi$. Besides the masses and coupling constants, fields are also renormalised.

Notice that the relation between the bare parameters $m_0, e_0 \cdots$ and the renormalised ones $m, e, \cdots$ is divergent as the cutoff goes to ∞. In the present interpretation, this simply means that the bare parameters $m_0, e_0, \cdots$ are themselves infinite, while the renormalised parameters $m, e, \cdots$ are finite. This causes no conceptual problem since by definition, bare parameters are physically unobservable.

Once the coupling constants of a theory have been renormalised, one can compute loop diagrams for scattering processes. For example in QED we can calculate loop corrections to $e^-e^+ \to \mu^-\mu^+$. These will initially be expressed in terms of m_0, e_0. The cutoff is kept finite during the calculation. At the end, m_0, e_0 are eliminated in terms of the physical (renormalised) couplings (m, e). If the resulting expression is finite in the limit that the cutoff is taken to infinity then we obtain an unambiguous and physically meaningful answer for a physically observable scattering cross-section (say) in terms of the physically observed parameters of the theory. Theories satisfying this requirement for all possible processes are said to be *renormalisable*.

Renormalisation can be carried out to any arbitrary order in perturbation theory (with increasing difficulty). However if the theory has any interaction of dimension > 4, perturbation theory generates more and more couplings that need to be independently renormalised, leading to a loss of predictive power. Such theories are called "non-renormalisable" and gravity is among them. Theories which satisfy the dimensional argument for renormalisability need not necessarily be renormalisable (the argument is necessary but not sufficient). Typically, very hard work is required to prove renormalisability for any field theory.

7 The Higgs mechanism

In this section we will describe a remarkable phenomenon that occurs in a rather simple quantum field theory. We will couple a complex scalar to a vector field, as we already did in a previous section. This is the theory that we called scalar electrodynamics.

First we re-examine the complex scalar theory by itself. Let us make a change of field variables and write the complex scalar field as:

$$\phi(x) = \frac{1}{\sqrt{2}} R(x) e^{i\theta(x)}.$$

In these variables, the Lagrangian

$$\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - \frac{\lambda}{6} (\phi^* \phi)^2$$

becomes:

$$\mathcal{L} = \frac{1}{2} \partial_\mu R \partial^\mu R - \frac{1}{2} m^2 R^2 - \frac{\lambda}{4!} R^4 + \frac{1}{2} R^2 \, \partial_\mu \theta \, \partial^\mu \theta$$

In the new variables, the phase transformation $\phi \to e^{i\alpha} \phi$ that we studied earlier becomes:

$$\theta(x) \to \theta(x) + \alpha$$

The variables we have introduced do not seem particularly convenient and in fact they make it rather tricky to quantise the theory. For example the kinetic term for the field θ is multiplied by the field R^2 so it vanishes at $R = 0$ (where polar coordinates are always singular). However these variables have some amusing features. θ is an angle-valued field variable and the global symmetry transformation is therefore a constant shift around a circle. There is no mass term for θ and indeed no potential at all – such terms would break the shift symmetry. Meanwhile the θ-independent part of the action is just that of a real scalar field $R(x)$.

If there were some way to "stabilise" the classical value of R at a finite value, then the θ kinetic term would indeed take a conventional form when R was expanded about that value. This would make θ a true massless field.

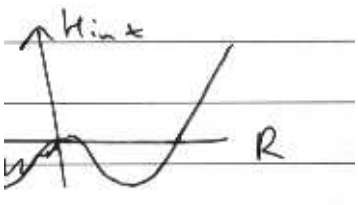
Such a stabilisation is easy to arrange. We choose a potential such that the field R gets a minimum at a finite value. Replace m^2 in the Lagrangian by $-m^2$. The theory then becomes:

$$\mathcal{L} = \frac{1}{2} \partial_\mu R \partial^\mu R + \frac{1}{2} m^2 R^2 - \frac{\lambda}{4!} R^4 + \frac{1}{2} R^2 (\partial_\mu \theta)^2$$

The potential term is now:

$$V(R) = \frac{\lambda}{4!} R^4 - \frac{1}{2} m^2 R^2$$

which has a "well" shape (recall that $R(x)$ is always positive, as it is a radial variable):



It is strange to consider a field with a negative value of m^2 (imaginary mass!). However this merely tells us we have a bad choice of variables. The potential has a local maximum at $R = 0$, and a minimum at:

$$\frac{dV}{dR} = 0 \quad \Rightarrow \quad \frac{\lambda}{6}R^3 - m^2 R = 0$$

$$\Rightarrow \quad R = \sqrt{\frac{6m^2}{\lambda}}$$

If we study the theory around the local maximum, the negative m^2 simply indicates the instability of the field to "roll down" to the minimum.

Therefore we are motivated to define a new field:

$$\tilde{R} = R - \sqrt{\frac{6m^2}{\lambda}}$$

The field $\tilde{R}$ takes its minimum at 0 and therefore this shifted field should have decent properties including a normal positive value of $(\text{mass})^2$. Performing this shift in the Lagrangian, we find after a short calculation:

$$\begin{aligned}\mathcal{L} \;=\; & \frac{1}{2}\partial_\mu \tilde{R}\,\partial^\mu \tilde{R} - m^2\tilde{R}^2 - \frac{m}{2}\sqrt{\frac{3\lambda}{2}}\,\tilde{R}^3 - \frac{\lambda}{4!}\tilde{R}^4 \\ & + \frac{3m^2}{\lambda}\,(\partial_\mu\theta)^2 + \sqrt{\frac{6m^2}{\lambda}}\,\tilde{R}(\partial_\mu\theta)^2 + \frac{1}{2}\tilde{R}^2\,(\partial_\mu\theta)^2\end{aligned}$$

Notice now that $\tilde{R}$ has a sensible mass $\sqrt{2}\,m$, and also that θ has a sensible kinetic term. Therefore θ is now a physical field and it is truly massless – as we saw earlier, it has no potential energy at all!

In this theory, a continuous symmetry was "spontaneously broken". In the original variables the symmetry was the phase rotation $\phi \to e^{i\alpha}\phi$ and because the potential was minimised by nonzero ϕ, we had to give the field a definite nonzero value in the vacuum e.g. $R = \sqrt{\frac{6m^2}{\lambda}}, \theta = 0$. Thus the vacuum configuration, whichever one we choose, is not invariant under the symmetry. In the radial variables the corresponding fact is that $\theta =$ constant is a vacuum configuration for any constant.

At the end, the theory has one real massive scalar and one "axionic" scalar with vanishing potential. The latter is an illustration of Goldstone's theorem, which says that spontaneous breaking of a continuous symmetry leads to an axionic scalar. In the (R, θ) variables Goldstone's theorem becomes almost obvious.

Now let us promote α to a local parameter $\alpha(x)$. As we have seen, the Lagrangian will not be invariant under such a phase transformation. In the new variables the θ-independent terms are all invariant under the local transformation (since it does not affect R) but the last term $\frac{1}{2}R^2\partial_\mu\theta\partial^\mu\theta$ changes to:

$$\frac{1}{2}R^2\,\partial_\mu(\theta+\alpha)\,\partial^\mu(\theta+\alpha)$$

We know that coupling a vector field $A_\mu(x)$ to a complex scalar field can preserve local gauge invariance. In the new variables we achieve this by making the replacement $\partial_\mu\theta \to (\partial_\mu\theta - A_\mu)$ in the Lagrangian, which becomes:

$$\mathcal{L} = \frac{1}{2}\partial_\mu R\partial^\mu R - \frac{1}{2}m^2R^2 - \frac{\lambda}{4!}R^4 + \frac{1}{2}R^2(\partial_\mu\theta - A_\mu)^2$$

Now if we perform the transformation $\theta \to \theta + \alpha(x)$ together with $A_\mu \to A_\mu + \partial_\mu\alpha$, it is manifest that this action is invariant. Then, to the above Lagrangian we add:

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

and we end up with a coupled theory that is gauge invariant. In fact it is just scalar electrodynamics in new variables.

Now we again consider the negative m^2 type of potential that stabilises R at a finite value. With a gauge field coupled in the system, something rather different happens compared to what we saw above. The key difference arises from the θ-dependent term, which in R, θ variables is now:

$$\frac{1}{2}R^2(\partial_\mu\theta - A_\mu)^2$$

Once we take account of the nontrivial minimum away from $R = 0$ and expand the field about it, this term becomes:

$$\frac{1}{2}\left(\tilde{R} + \sqrt{\frac{6m^2}{\lambda}}\right)^2(\partial_\mu\theta - A_\mu)^2$$

from which we can extract the quadratic term:

$$\frac{3m^2}{\lambda}(\partial_\mu\theta - A_\mu)^2$$

This term is not a "good" quadratic term as it stands, since it has a cross term between A_μ and $\partial_\mu\theta$. This motivates us to perform a field redefinition:

$$A_\mu \to A_\mu + \partial_\mu\theta$$

The effect of this is that *the field θ disappears completely from the theory* and the term becomes:

$$\frac{3m^2}{\lambda}A_\mu A^\mu$$

which is a *mass term* for A_μ. Meanwhile in the vector field kinetic term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ the field redefinition just looks like a gauge transformation and therefore that term remains invariant.

Recall that a mass term for a gauge field was prohibited by gauge invariance. However here we have managed to introduce such a term in a gauge invariant theory! In fact the theory above has a real massive scalar $\tilde{R}$ of mass $\sqrt{2}\,m$ and a *massive* gauge field of mass $\mu = \sqrt{\dfrac{3m^2}{\lambda}}$.

This process is called the Higgs mechanism, and $\tilde{R}$ is the Higgs field. We learn that when we try to implement spontaneous symmetry breaking in the presence of a gauge symmetry, Goldstone's theorem does *not* apply but instead the Higgs mechanism takes place. The would-be Goldstone boson disappears from the theory and the gauge field acquires a mass.

8 Non-abelian gauge theories

8.1 Non-abelian gauge invariance

Symmetries play an important role in quantum field theory, to which we have not been able to do full justice in this short course. What we have seen is the simplest class of continuous symmetries, which act as phase transformations on scalar and fermion fields. We also saw that these transformations can be given an arbitrary space-time dependence and thereby converted[3] from constant ("global") to space-time dependent ("local").

We now extend the phase transformations to transformations that act as a matrix on multi-component fields. Then we will consider the local version of such transformations.

Start with a set of complex scalar fields ϕ_I where $I = 1, 2, \cdots N$. The natural generalisation of a phase transformation on a single field would be a *unitary* transformation on this set of fields:

$$\phi_I(x) \to U_{IJ}\phi_J(x)$$

Here U_{IJ} is a constant matrix that satisfies $\boldsymbol{U}^\dagger \boldsymbol{U} = \mathbb{1}$. Clearly the Lagrangian:

$$\partial_\mu \phi_I^* \partial^\mu \phi_I - m^2 \phi_I \phi_I - V(\phi_I \phi_I)$$

is invariant under the above transformations, where we have taken the potential to be an arbitrary function of $\phi_I \phi_I$.

Choosing the unitary matrix to be:

$$\boldsymbol{U} = \begin{pmatrix} e^{i\alpha_1} & & \\ & e^{i\alpha_2} & \\ & & e^{i\alpha_3} \end{pmatrix}$$

we see that an independent phase rotation on each ϕ_I is included in the possible unitary matrices. However there are many more transformations included in $\boldsymbol{U}$, and the non-diagonal ones mix the different components of ϕ_I. It is easy to show that a unitary $N \times N$ matrix has N^2 independent components. There are ways of parametrising the matrix in terms of these N^2 variables but we will not need this explicitly for the moment.

Let us now consider promoting the unitary symmetry to a local one, which means letting $\boldsymbol{U}$ be an arbitrary function $\boldsymbol{U}(x)$. The Lagrangian above fails to be invariant because:

$$\partial_\mu \phi^\dagger \partial^\mu \phi \to \partial_\mu \left(\phi^\dagger \boldsymbol{U}^{-1}\right) \partial^\mu \left(\boldsymbol{U}\phi\right)$$

[3] If we consider a completely general local transformation $\alpha(x)$ then it can be decomposed into a part that is constant everywhere and another part that is space-time dependent but falls off at ∞. In this sense $\alpha(x)$ contains distinct global and local parts. Therefore it is not accurate to say that "local" is a *generalisation* of "global", though it is often said nonetheless.

where now we have suppressed the index I on the scalar field, as well as the corresponding indices of U. Expanding out the right side, we get:

$$\left(\partial_\mu \phi^\dagger \boldsymbol{U}^{-1} + \phi^\dagger \, \partial_\mu \boldsymbol{U}^{-1}\right) \left(\partial^\mu \boldsymbol{U} \, \phi + \boldsymbol{U} \, \partial_\mu \phi\right) = \partial_\mu \phi^\dagger \partial^\mu \phi + \partial_\mu \phi^\dagger \left(\boldsymbol{U}^{-1} \partial^\mu \boldsymbol{U}\right) \phi + \phi^\dagger (\partial_\mu \boldsymbol{U}^{-1} \boldsymbol{U}) \partial^\mu \phi + \phi^\dagger (\partial_\mu \boldsymbol{U}^{-1} \partial^\mu \boldsymbol{U}) \phi$$

This time the "unwanted" terms are matrices made out of the U_{IJ} and sandwiched between $\phi_I^\dagger$ and ϕ_J. It is reasonable to guess that they could be cancelled by introducing a *matrix-valued* vector field $A_{\mu\, IJ}$ and generalising the derivative to:

$$\partial_\mu \phi_I \to (D_\mu \phi)_I \equiv (\partial_\mu \delta_{IJ} - ie A_{\mu\, IJ}) \phi_J$$

which we henceforth write in more implicit notation as

$$\partial_\mu \phi \to \boldsymbol{D}_\mu \, \phi \equiv (\partial_\mu - ie \boldsymbol{A}_\mu) \phi$$

Now we would like to arrange that under $\phi \to \boldsymbol{U}(x) \, \phi$ we get:

$$\boldsymbol{D}_\mu \phi \to \boldsymbol{U}(x) \boldsymbol{D}_\mu \phi$$

so that the generalised kinetic term $\boldsymbol{D}_\mu \phi^\dagger \boldsymbol{D}_\mu \phi$ remains invariant. This will be so if $\boldsymbol{A}_\mu$ transforms to $\boldsymbol{A}'_\mu$ in such a way that:

$$(\partial_\mu - ie \boldsymbol{A}'_\mu) \, \boldsymbol{U} \phi = \boldsymbol{U} \, (\partial_\mu - ie \boldsymbol{A}_\mu) \, \phi$$

Expanding both sides we have:

$$\partial_\mu \boldsymbol{U} \, \phi + \boldsymbol{U} \, \partial_\mu \phi - ie \boldsymbol{A}'_\mu \, \boldsymbol{U} \phi = \boldsymbol{U} \, \partial_\mu \phi - ie \boldsymbol{U} \, \boldsymbol{A}_\mu \, \phi$$

Since this must be true for every ϕ, we have:

$$\boldsymbol{A}'_\mu \, \boldsymbol{U} = \boldsymbol{U} \, \boldsymbol{A}_\mu - i \partial_\mu \boldsymbol{U}$$

or

$$\boldsymbol{A}'_\mu = \boldsymbol{U} \, \boldsymbol{A}_\mu \, \boldsymbol{U}^{-1} - \frac{i}{e} \partial_\mu \boldsymbol{U} \, \boldsymbol{U}^{-1}$$

This is the required generalised gauge-transformation law for the matrix-valued vector field.

As a check, we may specialise to the case of a single-component scalar, whereupon $\boldsymbol{U}$ becomes the "1×1 unitary matrix" $e^{i\alpha}$ and $\boldsymbol{A}_\mu$ reduces to a single vector field satisfying:

$$A'_\mu = A_\mu + \frac{1}{e} \partial_\mu \alpha$$

This is the familiar Abelian gauge transformation that appears in electrodynamics.

To understand the general case better, note that any unitary matrix can be written as:

$$\boldsymbol{U} = e^{i\boldsymbol{\Lambda}}$$

where $\boldsymbol{\Lambda}$ is a Hermitian matrix[4]. We can derive the formula for an infinitesimal gauge transformation by assuming the matrix $\boldsymbol{\Lambda}$ has small entries and dropping terms of second-order and higher. The result is:

$$\delta \boldsymbol{A}_\mu = \frac{1}{e}\partial_\mu \boldsymbol{\Lambda} + i[\boldsymbol{\Lambda}, \boldsymbol{A}_\mu]$$

We see that the *commutator* of the vector field and the gauge parameter enters into the transformation law.

Finally we need to consider adding a kinetic term that will make the matrix $\boldsymbol{A}_\mu$ into a propagating field, analogous to $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ for the abelian case. To find this one may use trial and error. However a more reliable method is to notice that in the Abelian case we have the following identity:

$$D_\mu D_\nu \phi - D_\nu D_\mu \phi = [D_\mu, D_\nu]\phi = [\partial_\mu - ieA_\mu, \partial_\nu - ieA_\nu]\phi = -ieF_{\mu\nu}\,\phi$$

We learn the fact (which has a deep mathematical significance) that the field strength $F_{\mu\nu}$ arises as the commutator of two covariant derivatives. Notice that in this case the gauge invariance of $F_{\mu\nu}$ is guaranteed by the fact that:

$$D'_\mu \phi' = e^{i\alpha} D_\mu\, \phi$$

where $\phi' = e^{i\alpha}\phi$. The above relation implies:

$$[D'_\mu, D'_\nu]\,\phi' = e^{i\alpha}[D_\mu, D_\nu]\,\phi$$

The LHS can be evaluated to give $-ieF'_{\mu\nu}\, e^{i\alpha}\phi$ while the RHS gives $-ie\, e^{i\alpha}F_{\mu\nu}\phi$. It follows that $F'_{\mu\nu} = F_{\mu\nu}$.

Returning to the general case, the commutator of two covariant derivatives can be evaluated and gives:

$$[\boldsymbol{D}_\mu, \boldsymbol{D}_\nu]\,\phi = -ie\boldsymbol{F}_{\mu\nu}\,\phi$$

where now $\boldsymbol{F}_{\mu\nu}$ is a matrix acting on the column vector ϕ, and is given by:

$$\boldsymbol{F}_{\mu\nu} = \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu - ie[\boldsymbol{A}_\mu, \boldsymbol{A}_\nu]$$

Repeating the above steps we find that the transform of this F under matrix gauge transformations satisfies:

$$\boldsymbol{U}\boldsymbol{F}'_{\mu\nu} = \boldsymbol{F}_{\mu\nu}\boldsymbol{U}$$

[4]There should be no confusion with our previous use of the symbol Λ for Lorentz transformations.

from which we see that:

$$\boldsymbol{F}'_{\mu\nu} = \boldsymbol{U}^{-1}\boldsymbol{F}_{\mu\nu}\boldsymbol{U}$$

It follows that the Lagrangian $-\frac{1}{4}\operatorname{tr}\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}$ is gauge invariant[5].

To summarise, we have found a "non-Abelian" analogue of scalar electrodynamics by starting out with the assumption that the scalars ϕ form a column vector and promoting the unitary global symmetry to a gauge symmetry. The resulting Lagrangian is:

$$(\boldsymbol{D}_\mu\phi^\dagger)_I(\boldsymbol{D}_\mu\phi)^I - V(\phi^\dagger\phi) - \frac{1}{4}\operatorname{tr}\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}$$

The last term in this Lagrangian is called the "Yang-Mills Lagrangian". It is a beautiful mathematical generalisation of the Maxwell Lagrangian and we see that it is required by local unitary gauge invariance[6]. Expanding it out, we see a key difference between the Yang-Mills and Maxwell Lagrangians:

$$-\frac{1}{4}\operatorname{tr}\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu} = -\frac{1}{4}\operatorname{tr}(\partial_\mu\boldsymbol{A}_\nu - \partial_\nu\boldsymbol{A}_\mu)^2 + \frac{ie}{2}(\partial_\mu\boldsymbol{A}_\nu - \partial_\nu\boldsymbol{A}_\mu)[\boldsymbol{A}_\mu,\boldsymbol{A}_\nu] + \tfrac{1}{4}e^2[\boldsymbol{A}_\mu,\boldsymbol{A}_\nu][\boldsymbol{A}_\mu,\boldsymbol{A}_\nu]$$

We see that the single term $-\frac{1}{4}\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}$ dictated by gauge invariance contains *both* free terms *and* interactions. Moreover the interactions are very precisely dictated: a 3-point and a 4-point interaction with related coefficients.

This means we can drop the scalar fields altogether and just study the "pure" Yang-Mills theory, since it is an interacting field theory all on its own. Or we can add fermions.

8.2 Quantum chromodynamics

Today it is believed that the strong interactions are described by a non-Abelian gauge theory very close to the one described above. First of all, instead of scalars we have a triplet of fermions $\psi_I, I = 1,2,3$ that are supposed to describe the 3 "colours" associated to each quark. Because there are six types (flavours) of quarks (u,d,c,s,t,b) we should introduce six such triplets, but for a conceptual understanding of the key features we can restrict ourselves to a single flavour of quark.

Because there are three colours of quark, we might expect a unitary gauge invariance based on the group $U(3)$, however the gauge invariance of this theory is actually based on $SU(3)$. In general the group $SU(N)$ is the subgroup of $U(N)$ built from unitary matrices with

[5] In general we need to be careful about the normalisation of the trace, here we simply assume it is normalised to unity.

[6] Unitary transformations are not essential: one can derive a Yang-Mills theory with gauge invariance based on any non-Abelian Lie group.

determinant 1. It can be parametrised as the exponential of a Hermitian *traceless* matrix, where the tracelessness reduces from N^2 to $N^2 - 1$ generators. Thus this theory, called "quantum chromodynamics" or QCD, has 8 gauge bosons, called "gluons".

The Lagrangian of QCD with a single flavour is:

$$i\overline{\psi}(\partial_\mu - ie\boldsymbol{A}_\mu)\psi - m\overline{\psi}\psi - \frac{1}{4}\operatorname{tr}\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}$$

where we must be careful to keep in mind that colour indices have been suppressed in the fermionic terms. Making them explicit we have:

$$i\overline{\psi}_I(\partial_\mu\delta_{IJ} - ieA_{\mu\,IJ})\psi_J - m\overline{\psi}_I\psi_I, \quad I, J = 1, 2, 3$$

Unlike QED with deals with electrons and photons, QCD deals with quarks and gluons which have never been observed as free particles. This posed a puzzle for a long time. Quarks were experimentally observed via "hard" scattering off nucleons, but could not be liberated, while gluons were proposed essentially just for the sake of having non-Abelian gauge symmetry. Today we understand the resolution to the puzzle of why they cannot be liberated.

The effective strength of a force is given by the renormalised rather than bare value of the corresponding coupling constant. In QCD the relevant coupling constant is the strong interaction coupling constant, denoted g_s rather than e as we have done above. Now unlike the bare coupling, the renormalised coupling involves subtraction of the cutoff-dependent piece at some energy scale. This effectively imposes an energy-dependence on the coupling constant which is then thought of as $g_s(\mu)$ where μ is the typical energy scale at which renormalisation is performed.

Now as we work at higher and higher energies E, the strength of the interaction is defined by $g_s(\mu \sim E)$. In pure non-Abelian gauge theory as well as in the same theory after coupling to a small number of quarks, it has been shown that

$$g_s(E) \sim \frac{1}{\log E}$$

Therefore the strong interactions become *weaker* at high energies. In fact one needs to work at sufficiently high energy to be able to use perturbation theory to study QCD.

The converse of this statement, which is on a less rigorous footing, is that at lower and lower energies, corresponding to long distances, the coupling constant g_s becomes *large*. Hence if a quark or gluon tries to escape from a nucleon then the force pulling it back is effectively very large. This is believed to lead to the phenomenon of "permanent confinement" according to which the coloured quarks and gluons never appear as external physical states in the theory.

Instead, only "colour singlets", bound states of quarks and gluons into colour-neutral objects, are seen.

Studying QCD at low energies, in particular proving the confinement hypothesis, cannot be done in perturbation theory. Therefore one has to use methods like lattice gauge theory or string theory.

8.3 Electro-weak interactions

We do not have time to discuss the electro-weak interactions in any detail here. So we will limit ourselves to the fact that they are described by a non-Abelian gauge theory with gauge group $SU(2) \times U(1)$, i.e. a non-Abelian and an Abelian factor, coupled to all the fermions in specific ways as well as to a scalar field that forms an $SU(2)$ doublet.

There are therefore four gauge bosons in the theory. Introducing a negative $(\text{mass})^2$ in the scalar potential, we find at the end that the Higgs mechanism renders three of the gauge bosons massive. The remaining massless one is identified with the photon. The massive ones are named $W^{\pm}, Z$ and are the mediators of the weak interactions. This accounts for the experimentally observed fact that the weak interactions are short-ranged. In fact this theory, called the Glashow-Salam-Weinberg theory, can be well treated in perturbation theory and is spectacularly successful. Moreover in some way it "unifies" the weak and electromagnetic interactions.