Membrane Gravity correspondence

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By

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DECLARATION

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

The work was done under the guidance of Professor Shiraz Minwalla, at the Tata Institute of Fundamental Research, Mumbai.

Yogesh Dandekar

In my capacity as supervisor of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

Prof. Shiraz Minwalla

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List of Publications

Publications relevant to the thesis

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2. S. Bhattacharyya, P. Biswas, B. Chakrabarty, Y. Dandekar and A. Dinda, "The large D black hole dynamics in AdS/dS backgrounds," JHEP **1810**, 033 (2018). [arXiv:1704.06076 [hep-th]].

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4. S. Bhattacharyya, P. Biswas and Y. Dandekar, "Black holes in presence of cosmological constant: second order in $\frac{1}{D}$," JHEP **1810**, 171 (2018). [arXiv:1805.00284 [hep-th]].

Other publications

1. Y. Dandekar, M. Mandlik and S. Minwalla, "Poles in the S-Matrix of Relativistic Chern-Simons Matter theories from Quantum Mechanics," JHEP **1504**, 102 (2015). [arXiv:1407.1322 [hep-th]].

2. Y. Dandekar, S. Mazumdar, S. Minwalla and A. Saha, "Unstable 'black branes' from scaled membranes at large D," JHEP **1612**, 140 (2016). [arXiv:1609.02912 [hep-th]].

To

my family.

Contents

Chapter 1

Introduction

Einstein equations naively look very simple. But they are not very easy to solve analytically or even numerically for the complicated dynamical processes, in particular for the case of recent interest - black hole mergers. An option might be to hope for simplifications by doing a perturbation theory. In this thesis we use the number of spacetime dimensions D as a perturbative parameter. Building on previous initial works, we construct an effective theory for black hole dynamics in large number of spacetime dimensions. We then construct an improved form of this effective theory which is well defined and consistent even at finite D. We hope that the equations of motion for this effective theory could be used as a simpler model for numerical simulations. We also hope that we might gain new structural understanding about Gravity, as we will discuss in this thesis.

The idea of introducing number spacetime dimensions D as a perturbative parameter in Einstein equations was introduced in [1]. In subsequent papers [2, 3, 4, 5, 6, 7] the authors observed that taking D large simplifies the dynamics of black holes for simple cases. They also observed that the answers for Quasinormal modes (for particular subsets that could be compared) of Schwarzschild black holes are in reasonable agreement with the answers at $D = 4$. At large D, there are in fact two length scales for a black hole spacetime. These are the Schwarzschild radius r_0 and the thickness δr of black hole's gravitational tail. And at large D, $\delta r \sim r_0/D$. Beyond distances of order δr away from horizon, the spacetime metric effectively reduces to its unperturbed value. The spectrum of quasinormal modes (QNMs) of Schwarzschild black holes also gets affected. At each angular momentum number, there are a finite number of 'light' modes with frequencies of order $1/r_0$, lifetimes of order r_0, and have support only over the gravitational tail. Rest of the infinite number of modes are 'heavy' with frequencies of order $\sim 1/\delta r$, lifetimes of order δr, and are nontrivial even outside the gravitational tail.

Now consider a complicated dynamical event like black hole collision and merger. The dynamics is expected to be complicated during time of order δr after the event. But after time $t \gg \delta r$, the heavy modes will get decayed away in the form of gravitational radiation, and the dynamics would be governed by a nonlinear effective theory of the light modes. One could expect that the effective nonlinear equations of motion for the light modes would admit a power series expansion in the ratio $\delta r/r_0 \sim 1/D$. Hence these effective nonlinear equations of motion would be a reformulation of black hole dynamics at large D.

Membrane paradigm at large D is such an effective theory for the dynamics of black hole at large D. At first nontrivial order in $1/D$, the equations of motion that govern the dynamics of black holes were derived in the papers [8, 9]. In these papers, the dynamics of black holes in flat spacetime background was considered. A zeroth order ansatz metric was suitably chosen as a starting point of large D perturbation theory. By adding unknown subleading metric corrections and by requiring that this total metric solves Einstein equations at the next nontrivial order, those metric corrections were found. Solving Einstein constraint equations for evolution along the normal to the horizon implies that the 'membrane equations of motion' are obeyed. These equations describe the dynamics of a codimension-1 timelike 'membrane' which is embedded in the unperturbed background spacetime. The dynamical variables are the arbitrary shape of the membrane and the locally varying velocity field parallel to membrane surface. The membrane equations are dynamical relations between various quantities constructed out of the shape and velocity field for the membrane embedded in unperturbed background spacetime. The number of membrane equations is equal to number of variables. Thus the problem of black hole dynamics becomes dual to the problem of dynamics of a codimension-one non-gravitational timelike 'membrane' embedded in the unperturbed background spacetime.

We end this introduction with a brief overview of each chapter in this thesis.

(Chapter 2) Membrane paradigm at higher orders

In previous works [8, 9], the perturbation theory of the membrane paradigm was implemented to leading order in large D for flat spacetime background, as described above. That is, the $\mathcal{O}(1/D)$ corrections to ansatz metric were determined and the leading order membrane equations of motion was found.

In Chapter 2 (based on [10]), we show that this correspondence extends to all higher orders as well. We formulate a systematic procedure to implement the perturbation series to any arbitrary order in $1/D$ expansion. In particular we find the integral expressions for the metric corrections at any n^{th} order in terms of sources at that order. We show that the membrane equations at any n^{th} order can be obtained by solving the Einstein constraint equations considered for the evolution along the normal direction to the membrane. We also show that for finding the membrane equations at n^{th} order we do not need to solve for the unknown metric corrections at n^{th} order, only the knowledge of metric corrections upto $(n-1)^{th}$ order is sufficient.

As an illustration of the general procedure described above, we then explicitly implement the perturbation theory to first subleading order. That is, we find the $1/D$ corrections to the leading order membrane equations of motion. We also find the $\mathcal{O}(1/D^2)$ correction to the metric ansatz. We observe a new feature which was absent at leading order calculations of [8, 9], which is that at first subleading order, the divergence of velocity field is nonzero and is manifestly positive definite, it is similar result to Entropy current of hydrodynamics.

(Chapter 3) Introducing cosmological constant: First order calculation

As mentioned above, in previous works, the membrane paradigm at large D was developed for the dynamics of black holes in flat spacetime background. In Chapter 3 (based on [11]), we extend the membrane paradigm at large D to any background metric which satisfies Einstein equations with a possible cosmological constant, of any sign. We explicitly implement the perturbation theory to leading order. That is, we determine the $\mathcal{O}(1/D)$ corrections to metric ansatz and also find the leading order membrane equations.

As a check for the correctness of the membrane equations, we calculate the spectrum of linearized fluctuations about the spherical membrane in AdS/dS background and the planar membrane in AdS. We find that they match with the light QNM spectrum for the Schwarzschild black hole in AdS/dS and AdS black brane respectively at large D, upto leading order.

In a parallel work, the authors of [12] have developed a large D effective theory for the dynamics of small fluctuations about AdS black branes, by focusing on length scales of order $1/\sqrt{D}$, unlike order unity as we have done. To make contact with this approach, we take the special case of small fluctuations about a planar membrane in AdS and consider a particular scaling limit of our membrane equations. We then show that they reduce to the effective equations as developed in [12]. This is an instance where we could successfully apply the large D membrane paradigm outside its originally intended scope of validity.

(Chapter 4) Introducing cosmological constant: Second order calculation

In Chapter 4 (based on [13]), we extend the calculations of Chapter 3 to second order in large D perturbation theory. We find the explicit answers for the membrane equations and the metric corrections at second order. At this order we find that there are terms in membrane equations which have explicit multiplicative factor of cosmological constant, unlike the case at first order of Chapter 3. As a check for the correctness of the membrane equations, we calculate the spectrum of linearized fluctuations about the spherical membrane in AdS/dS background and the planar membrane in AdS. We find that they match with the light QNM spectrum for the Schwarzschild black hole in AdS/dS and AdS black brane respectively at large D, upto the relevant order.

(Chapter 5) Improved Large D membrane

In [14], the membrane localized Stress tensor and Entropy current was defined and then calculated at leading order for the Membrane paradigm at large D with flat background. Stress tensor conservation implies the membrane equations at large D. Local form of second law of thermodynamics is obeyed at large D. For the consistency of this construction, large D was necessary.

In Chapter 5 (based on [15]), we construct the Stress tensor and Entropy current for an 'improved' membrane embedded in arbitrary D (where D is not necessarily taken large) dimensional background which solves Einstein's equations with a possible cosmological constant. As before, we consider the membrane to be a timelike codimension-1 hypersurface that doesn't backreact on the background spacetime. This 'improved' formulation is consistent even at finite D. At large D, the improved Stress tensor and Entropy current reduce to the leading order Stress tensor and Entropy current for the large D membrane as derived in [14]. Hence it follows that at large D, this formulation correctly reproduces the results of the leading order large D membrane paradigm. The improved formulation is a consistent finite D completion of the leading order large D membrane equations. Local form of second law is exactly obeyed.

There exist exact stationary solutions for the improved membrane, and the membrane shape satisfies a simple equation. In the Stationary case, we construct an Action from which we can recover the membrane shape equation and the Stress tensor of Stationary solutions. This action, when evaluated onshell reduces to $-\ln Z$ where Z denotes thermal Partition function for the stationary membrane configuration. It turns out that the thermodynamics of a static spherical membrane in any maximally symmetric spacetime matches exactly with the Schwarzschild black hole in that maximally symmetric spacetime, even at finite D.

We then consider long wavelength dynamics of a planar membrane in AdS and calculate the linearized metric fluctuations sourced by the membrane. By using the standard AdS/CFT prescription, we construct the boundary conformal Stress tensor upto 2nd order in derivatives. The form of this boundary Stress tensor matches exactly with the respective Fluid-gravity answer upto first order in derivatives as found in [16], even at finite D. At second order, the agreement is only at large D.

Chapter 2

Membrane paradigm at higher orders

2.1 Introduction

It has recently been noted that the classical dynamics of black holes simplifies in the limit of a large number of dimensions. The key observation - first made by Emparan, Suzuki, Tanabe and collaborators in [1, 2, 3, 4, 5, 6, 7] - is that black holes at large D have two effective length scales. The first of these, r_0, is the size of the black holes. The second is the thickness of the black hole's gravitational tail, i.e. the distance beyond the black hole event horizon after which the gravitational potential rapidly decays to zero. In four dimensions the black hole size and thickness are comparable. In the large D limit, however, the thickness of the gravitational tail turns out to scale like r_0/D [1] and so is much smaller than the the black hole size.

This observation suggests the possibility of an effective 'dimensional reduction' of black hole dynamics to the membrane region; a slab of spacetime of thickness $1/D$ centered around the codimension one event horizon. In work done over the last few years, this expectation has been borne out in various contexts. In this chapter we will focus on black holes propagating in an otherwise unperturbed flat space. Assuming that r_0 (see above) and the length scale of variation along the horizon are both of order unity, the dimensional reduction described above was worked out to leading nontrivial order in the $1/D$ expansion for the most general nonlinear dynamical context in [8, 9]; the special case of stationary solutions and their small fluctuations has also been studied at higher orders in the $1/D$ expansion in [17, 18, 19, 20]. In addition the dimensional reduction of small horizon ripples at length scale $1/\sqrt{D}$ about particular solutions (black strings or black branes in flat, AdS or dS space) has been studied in [21, 22, 23, 12, 24]. Further developments were presented in [25, 26, 27, 28, 29, 30, 31].

In this chapter we further develop the general nonlinear dynamical construction of [8, 9]. In particular we demonstrate that the reduction of black hole dynamics to membrane dynamics, worked out to leading nontrivial order in the $1/D$ expansion in [8, 9], can be systematically generalized to every order in $1/D$. As an application of this systematic framework we explicitly work out the first subleading corrections to the membrane equations of motion in the $1/D$ expansion, and also determine the spacetimes dual to any particular membrane solution at next subleading order in the $1/D$ expansion. In this introduction we first review the leading order construction presented in [8, 9] and then present our explicit higher order results.

2.1.1 Review of earlier work

Consider a class of D dimensional metrics of the form

$$g_{MN} = \eta_{MN} + \frac{(n_M - u_M)(n_N - u_N)}{\psi^{D-3}} \tag{2.1}$$

The metrics (2.1) are parameterized by a smooth D dimensional function ψ and a smooth oneform field u_M. n_M in (2.1) is the normal field to surfaces of constant ψ, (i.e. $n_M = \frac{\partial_M \psi}{\sqrt{\partial_P \psi \partial_Q \psi \eta^{PQ}}}$). The oneform field u_M is assumed to be unit normalized (i.e. $u_N u_M \eta^{MN} = -1$) and tangent to surfaces of constant ψ (i.e. $u_M n_N \eta^{MN} = 0$).

In order to gain intuition for spacetimes of the form (2.1) it is useful to first consider a special case. Working with coordinates in which the metric on Minkowski space takes the form

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{D-2}^2,$$

the choice $u = -dt$ and $\psi = \frac{r}{r_0}$ turns (2.1) into the metric of a Schwarzschild black hole of radius r_0 in the so called Kerr Schild coordinates.

Note $\psi = 1$ is the event horizon of the Schwarzschild black hole. More generally the surface $\psi = 1$ is easily verified to be a null submanifold of (2.1) for every choice of ψ and u. This null manifold coincides with the event horizon of the (2.1) provided that ψ and u are chosen such that the metric (2.1) settles down into a collection of stationary black holes at late times. Following [8, 9] we refer to the submanifold $\psi = 1$ as the membrane world volume. [1]

Note that as ψ increases past unity $\frac{1}{\psi^{D-3}}$ decays to zero very rapidly. This decay is exponential in D once $\psi - 1 \gg \frac{1}{D}$. It follows that (2.1) represents a class of asymptotically flat spacetimes with the following property; the spacetime outside the event horizon deviates significantly from flat space only in a slab of thickness $\frac{1}{D}$ around the event horizon. We will refer to this as the membrane region.

[8, 9] set out to characterize solutions of the vacuum Einstein equations, $R_{MN} = 0$, that reduce to metrics of the form (2.1) in the large D limit, with corrections in a power series in $\frac{1}{D}$. As we have reviewed above, when $\psi - 1 \gg \frac{1}{D}$ the spacetimes (2.1) reduce to flat space. Deviations from flatness are nonperturbatively small in the $\frac{1}{D}$ expansion. Thus Einstein's equations are automatically solved at all order in $1/D$ outside the membrane region. In order to obtain a true solution of Einstein's equations, the solution (2.1) needs to be corrected order by order in the $\frac{1}{D}$ expansion only in the membrane region.

Consider a region of size $\frac{1}{D}$ centered around any point x_0 on the event horizon of (2.1). It may be shown that the metric of this ball is closely approximated by the metric in an

[1] Through this chapter we assume that ψ in (2.1) is chosen to ensure that the membrane surface is a smooth codimension one surface that is timelike when viewed as a submanifold of flat space (we have emphasized above that this surface is a null submanifold of the metric (2.1)). We also assume that ψ is chosen to ensure that $\frac{1}{\psi^{D-3}}$ decays at spatial infinity.

equivalent small region centered around the appropriate event horizon point of *some* boosted Schwarzschild black hole provided that

$$\nabla^2 \left(\frac{1}{\psi^{D-3}} \right) = 0, \quad \nabla . u = 0, \tag{2.2}$$

(the contraction of all indices is achieved by use of the metric η_{MN} in the equations above) [2]. These equations need only be satisfied at leading order in D and can be violated at subleading orders. As Schwarzschild black holes are exact solutions to Einstein's equations, it follows as a consequence that the spacetimes (2.1) *almost* solve Einstein's equations in the membrane region, provided that (2.2) is satisfied at every point on the membrane.

The statement that Einstein's equations are 'almost' solved in the membrane region has the following precise meaning. When evaluated in the membrane region the four derivative scalar $R_{AB}R^{AB}$ is in general of order D^4. This estimate follows immediately from the fact that the metric varies on a length scale of order $1/D$ in the membrane region. Once we impose (2.2), on the other hand, $R_{AB}R^{AB}$ turns out to be of order D^2, i.e. In a coordinate system in which all components of the metric are of order unity, R_{AB} is of order D; one order lower than the generic order suggested by a dimensional estimate. In other words (2.2) ensures that Einstein's equations are obeyed to leading order - but are generically violated at first subleading order. Consequently the metrics (2.1) - with the conditions (2.2) imposed at leading order- are plausible starting points for the construction of true solutions of Einstein's equations in a power series in $\frac{1}{D}$.

The authors of [8, 9] were able to carry out this perturbative expansion to first subleading order in $\frac{1}{D}$ (see below for a review). Interestingly they discovered that arbitrary metrics of the form (2.1) could *not* be corrected to yield regular solutions to Einstein's equations at next order in $\frac{1}{D}$. It turns out to be possible to correct (2.1) at first order in $1/D$ only when the fields ψ and u obey an integrability constraint - a membrane equation of motion - that we will describe in considerable detail below. Whenever this condition is obeyed, a regular correction (of order $1/D$) to the metric (2.1) was found in [8, 9]. The corrected metric obeys $R_{AB} = \mathcal{O}(1)$ [3]; i.e. once the corrections are taken into account, Einstein's equations are solved at leading *and first subleading order* in $\frac{1}{D}$.

We now turn to a description of the integrability constraints mentioned in the previous paragraph. Consider the surface $\psi = 1$, viewed as a submanifold of flat space with metric η_{MN}; we refer to this submanifold as the membrane. Let K_{MN} represent the extrinsic

[2] When an expression like ∇^2 acts on $\frac{1}{\psi^{D-3}}$ we get two distinct terms of order D^2 in two ways. The first term is $\propto (D-3)(D-2)\frac{(\nabla\psi)^2}{\psi^{D-1}}$. The second term is $\propto (D-3)\frac{\nabla^2\psi}{\psi^{D-2}}$. Though the second term has one less explicit factor of D than the first, it actually contributes at the same order in the $1/D$ expansion - i.e. at leading order - because of the contraction of indices in ∇^2. This is the reason that (2.1) solves the leading order equations only if $\nabla^2\psi$ takes the same value as it does in a Schwarzschild black hole, leading to the first requirement listed in (2.2). In a similar manner worldvolume derivatives of the horizon shape and velocity field - which are of order unity - compete with derivatives acting on $\frac{1}{\psi^{D-3}}$ only if their order is enhanced by the contraction of a worldvolume index. The only first derivative expression involving the black hole velocity that has such a contraction is $\nabla.u$. It follows that (2.1) satisfies the leading order equations only if $\nabla.u$ takes the same value as it does on a Schwarzschild black hole. This leads to the second of (2.2).

[3] More precisely, $R_{AB} = \mathcal{O}(1)$ in coordinates in which all metric components are of order unity. More generally, $R_{AB}R^{AB}$ is of order unity.

curvature of this (generically timelike) submanifold. Recall also that the velocity oneform field u_M on the membrane surface is tangent to the membrane and so may be regarded as a oneform field in the membrane world volume. The authors of [8, 9] found that the metric (2.1) could be corrected to a regular [4] solution of Einsteins equations at first order if and only if the following constraints are obeyed

$$\left(\frac{\nabla^2 u_A}{\mathcal{K}} - \frac{\nabla_A \mathcal{K}}{\mathcal{K}} + u_C K^C_A - u.\nabla u_A\right)\mathcal{P}^A_B = 0 \tag{2.3}$$

where $\mathcal{P}^A_B = \delta^A_B + u^A u_B$ is the projector orthogonal to the velocity vector on the membrane world volume, and all covariant derivatives are taken with respect to the induced metric on the membrane. The quantity $\mathcal{K}$ is the trace of the extrinsic curvature of the membrane worldvolume.

The integrability conditions (2.3) have an interesting interpretation. They may be thought of as a set of $D-2$ equations for $D-2$ variables (one of these variables is the shape of the membrane, and the other $D-3$ variables are the components of the unit normalized, divergence free velocity field). In other words the equations (2.3) define an initial value problem for membrane dynamics. As every configuration that obeys (2.3) gives rise to a metric that obeys Einstein's equations to the appropriate order in $1/D$, it follows that solutions of the membrane equations (2.3) are in one to one correspondence with asymptotically flat dynamical black hole configurations that solve Einstein's equations to first subleading order in $1/D$.

2.1.2 The membrane paradigm at higher orders in $1/D$

In this chapter we demonstrate that first order perturbative procedure outlined above extends systematically to arbitrary orders in the expansion in $\frac{1}{D}$. We will now very briefly outline our inductive argument. We assume that the perturbative procedure has been implemented upto n^{th} order, i.e. that corrections to the metric (2.1) have been determined upto n^{th} order in the $1/D$ expansion in such a manner that R_{MN} evaluated on the corrected solution is of order D^{1-n}. We then add further corrections of order $1/D^{n+1}$ to the metric (see (2.12) and (2.15)). At order D^{n-1} we demonstrate that the Einstein constraint equations are independent of the new unknown correction functions when evaluated on the event horizon $\psi = 1$. These equations determine the correction to the membrane equations (and the divergence condition on the velocity) at order $1/D^{n+1}$. Moving away from the horizon we argue that the order D^{1-n} part of R_{MN} takes the form listed in table 2.2. Setting the expressions in this table yields a set of inhomogeneous linear differential equations that can be used to determine order $1/D^{n+1}$ corrections to the metric. Explicit expressions for the sources in these differential equations can only be obtained by grinding through the perturbative procedure, but we use a contracted Bianchi identity to demonstrate that the sources that occur in these equations are not all independent, but obey certain relations (see (2.30)) at every order of perturbation theory. Using these relations we are able to integrate the inhomogeneous differential equations for any source functions and obtain an explicit

[4]By a regular solution we mean a solution with a smooth event horizon that is regular everywhere outside the event horizon.

and unique expressions for the metric corrections at order $1/D^{n+1}$ (see Section 2.3) that are manifestly regular and obey all required boundary conditions.

As an illustration of the general method outlined above we explicitly implement the perturbative procedure to second subleading order in $\frac{1}{D}$. We find that the modified membrane equations take the form

$$\begin{aligned}
& \left[\frac{\nabla^2 u_A}{\mathcal{K}} - \frac{\nabla_A \mathcal{K}}{\mathcal{K}} + u^B K_{BA} - u\cdot\nabla u_A\right]\mathcal{P}^A_C \\
+ & \left[\left(-\frac{u^C K_{CB} K^B_A}{\mathcal{K}}\right) + \left(\frac{\nabla^2\nabla^2 u_A}{\mathcal{K}^3} - \frac{u\cdot\nabla\mathcal{K}\nabla_A\mathcal{K}}{\mathcal{K}^3} - \frac{\nabla^B\mathcal{K}\nabla_B u_A}{\mathcal{K}^2} - 2\frac{K^{CD}\nabla_C\nabla_D u_A}{\mathcal{K}^2}\right)\right. \\
+ & \left(-\frac{\nabla_A\nabla^2\mathcal{K}}{\mathcal{K}^3} + \frac{\nabla_A\left(K_{BC}K^{BC}\mathcal{K}\right)}{\mathcal{K}^3}\right) + 3\frac{(u\cdot K\cdot u)(u\cdot\nabla u_A)}{\mathcal{K}} - 3\frac{(u\cdot K\cdot u)(u^B K_{BA})}{\mathcal{K}} \\
- & \left. 6\frac{(u\cdot\nabla\mathcal{K})(u\cdot\nabla u_A)}{\mathcal{K}^2} + 6\frac{(u\cdot\nabla\mathcal{K})(u^B K_{BA})}{\mathcal{K}^2} + \frac{3}{(D-3)}u\cdot\nabla u_A - \frac{3}{(D-3)}u^B K_{BA}\right]\mathcal{P}^A_C = 0
\end{aligned} \tag{2.4}$$

while the divergence free condition on the velocity field is modified, at second subleading order, to the equation

$$\nabla\cdot u = \frac{1}{2\mathcal{K}}\left(\nabla_{(A}u_{B)}\nabla_{(C}u_{D)}\mathcal{P}^{BC}\mathcal{P}^{AD}\right) \tag{2.5}$$

Note that the first line in (2.4) is simply a rewriting of (2.3); the 2nd-4th lines of this equations represent corrections to (2.3). There is a well defined sense (see below) in which each of these correction terms is of order $\frac{1}{D}$ relative to the leading order terms in the first line. It follows that the equations (2.4) represent small corrections to the leading order equations (2.3). The first order corrected membrane equation of motion (2.4) and (2.5) are the main result of this chapter.

We then present explicit expressions for the second order sources for all the inhomogeneous differential equations (see table A.1). Plugging these sources into the general equations for the metric corrections at any order we obtain explicit results for the second order correction to the spacetime metric dual to any particular solution of the membrane equations of motion.

2.2 Perturbation theory: general structure

2.2.1 A more detailed description of the starting ansatz

As we have explained in the introduction, the starting point of our perturbative construction of large D solutions to Einstein's equations is the metric (2.1). In the introduction we noted that the metrics (2.1) are parameterized by the D dimensional function ψ and the oneform field u. We assume these fields have a good large D limit, i.e. that the length scale of variation in ψ and u is of order unity. Following [8, 9], however, consider two different functions ψ with the same membrane surface (i.e. with coincident zero sets for $\psi - 1$).

These two functions define metrics (2.1) that coincide (outside the event horizon) at leading order in $1/D$ but differ at subleading orders in $1/D$. Similarly u functions that agree on the membrane but differ off it lead to metrics (2.1) that differ only at subleading order in $1/D$.

Any two metrics (2.1) that differ only at subleading orders in $1/D$ constitute equivalent starting points for the perturbative construction of solutions in the following sense: the end result of perturbation theory starting from the two different starting points will be the same. In order to construct all distinct final metrics we need only consider one member of each 'equivalence class' of metrics (2.1). As explained above the equivalence classes are labeled by the zero set of the function $\psi - 1$ (the membrane world volume) and the value of the velocity field on the membrane world volume. In order to pick a representative from each equivalence class that we can use to set up our perturbation theory we invent an arbitrary way of constructing the full function ψ from its zero set, and the full velocity field u from its values on the membrane. Following [8, 9] we refer to the (essentially arbitrary) rule for achieving this construction as a subsidiary condition on the functions ψ and u.

For technical reasons, in this chapter we utilize the subsidiary conditions of [8] rather than that of [9]. We now describe these conditions in detail.

Consider a given timelike membrane submanifold in flat space. At each point on the manifold consider a geodesic that shoots outwards from the manifold along its normal vector. The resultant collection of curves [5] is a spacefilling congruence of spacelike geodesics; caustics of this congruence, if any, only occur at distances of order unity (rather than $1/D$) away from the membrane. [6] We define the scalar function B in the neighborhood of the membrane as follows; B at any point is defined to be the signed proper distance, along the geodesic that passes through it, to the membrane. This distance is defined to be positive outside the membrane and negative inside the membrane. Note that B vanishes on the membrane. We define

$$n_M = \nabla_M B \tag{2.6}$$

It follows from our construction above that

$$n.n = 1 \tag{2.7}$$

n_A is the normal oneform to surfaces of constant B. We use the symbol K_{MN} denote the extrinsic curvature of surfaces of constant B. Note of course that $n^A K_{AB} = 0$. We also define $\mathcal{K} = K^A_A$. We then proceed to define the function ψ as

$$\psi = 1 + \frac{\mathcal{K} B}{D-3} \tag{2.8}$$

In a similar manner we use the velocity function on the membrane to define a velocity oneform field in spacetime simply by parallel transport along our congruence of geodesics. It follows from our definitions above that

$$\begin{aligned} n.\nabla n_A = 0 \\ n.\nabla u_A = 0 \end{aligned} \tag{2.9}$$

[5]These 'curves' are actually straight lines as they are all geodesics in flat space. We use the term 'curve' to bring to mind the obvious generalization of this construction when the membrane is embedded in a curved spacetime.

[6]The quantity $\frac{D}{\mathcal{K}}$ gives a rough estimate for the distance away from the membrane at which the geodesics caustic. Below we explain that $\mathcal{K}$ is of order D so that this caustic length scale is of order unity.

The first line of (2.9) follows upon differentiating 0 (2.7), using (2.6) and interchanging derivatives. This equation is in fact simply the geodesic equations for the congruence of geodesics that defines B. The equation on the second line of (2.9) follows from the fact that u is defined off the membrane by parallel transport. It follows from (2.9) that

$$K_{AB} = (\eta_A^C - n_A n^C)(\nabla_C n_D)\left(\eta_B^D - n^D n_B\right) = (\nabla_A - n_A(n.\nabla))\, n_B = \nabla_A n_B = \nabla_A \nabla_B B \tag{2.10}$$

Note that our definition of n_A in this section, and the rest of this chapter, differs slightly from the definition given in the introduction. The two definitions agree at leading order (which was all that was required in the discussion around (2.1)) but differ at subleading orders in $1/D$. The vector n_A defined in this section - rather than the normal vector defined in the introduction - will be used through the rest of this chapter.

Using (2.8) it is easily verified that on the submanifold $B = 0$

$$\begin{aligned} \psi \nabla^2 \psi &= \frac{\mathcal{K}^2}{D-3} + 2\frac{n.\nabla \mathcal{K}}{D-3} \\ (D-2)\nabla\psi.\nabla\psi &= \frac{D-2}{D-3}\frac{\mathcal{K}^2}{D-3} \end{aligned} \tag{2.11}$$

As we explain below, in the large D limit taken in this chapter $2\frac{n.\nabla\mathcal{K}}{D-3}$ is of order unity while $\frac{\mathcal{K}^2}{D-3}$ is order D. It follows that to leading order in D

$$(D-2)\nabla\psi.\nabla\psi = \psi\nabla^2\psi, \quad i.e.\nabla^2\left(\frac{1}{\psi^{D-3}}\right) = 0$$

In other words our construction satisfies the first equation of (2.2). We satisfy the second equation in (2.2) by construction; we simply choose our u oneform on the membrane such that its divergence vanishes at leading order in D. The divergence of u will turn out not to vanish at a subleading order.

2.2.2 Coordinate Choice for the correction metric

In this chapter we search for solutions of Einstein's equations in a power series expansion in $\frac{1}{D}$

$$\begin{aligned} G_{MN} &= \eta_{MN} + h_{MN}, \\ h_{MN} &= \sum_{n=0}^{\infty} \frac{h^{(n)}_{MN}}{(D-3)^n}, \\ \text{with, } h^{(0)}_{MN} &= \frac{O_M O_N}{\psi^{D-3}}, \end{aligned} \tag{2.12}$$

Here

$$O_M = n_M - u_M \tag{2.13}$$

We fix coordinate redefinition ambiguities by demanding

$$h_{MN} O^N = 0, \tag{2.14}$$

Consider any point in the metric (2.1). The tangent space built about this point has two special vectors; the vector n and the vector u. All the other $D-2$ directions orthogonal to n and u are equivalent and can be rotated into each other. It is thus useful to parameterize the most general fluctuation field h_{MN} (subject to the gauge condition (2.14)) in the form

$$h^{(n)}_{MN} = H^{(S,n)} O_M O_N + O_{(M} H^{(V,n)}_{N)} + H^{(T,n)}_{MN} + \frac{1}{D-3} H^{(Tr,n)} \mathcal{P}_{MN},$$

where,

$$\begin{aligned} \mathcal{P}_{MN} =& \eta_{MN} - O_M n_N - O_N n_M + O_M O_N, \\ O^N H^{(V,n)}_N = 0, \quad n^N H^{(V,n)}_N = 0, \quad & O^M H^{(T,n)}_{MN} = 0, \quad n^M H^{(T,n)}_{MN} = 0, \quad \mathcal{P}^{MN} H^{(T,n)}_{MN} = 0, \end{aligned} \tag{2.15}$$

The superscripts S, V and T stand for scalar, vector and tensor respectively, and denote the transformation properties of the relevant symbol under the $SO(D-2)$ rotations in tangent space that leave n and u fixed. The superscript Tr stands for trace, and labels a second scalar.

2.2.3 Orders of D

As we have explained above, in this chapter we solve Einstein's equations in a systematic expansion in $\frac{1}{D}$. In order for this process to be well defined, we need to be able to unambiguously estimate the scaling with D of various terms that appear in the metric and in the membrane equation of motion. Such an estimation is only unambiguous within subclasses of solutions, as we will now explain with an example.

Consider a membrane whose world volume is a $D-2$ sphere (of radius R) times time. The trace of extrinsic curvature, $\mathcal{K}$, of this surface is easily shown to be $\frac{D-2}{R}$ and so is of order D (assuming R is of order unity). On the other hand the surface $S^p \times R^{D-2-p}$ times time has $\mathcal{K} = \frac{p}{R}$. If p and R are both held fixed as D is taken to infinity, $\mathcal{K}$ is of order unity for this surface. It follows that $\mathcal{K}$ cannot unambiguously be assigned a scaling with D without making further assumptions. The same holds true of various other quantities (e.g. $\nabla^2 u_M$) that enter the metric and equation of motion.

In this chapter we follow [8, 9] and estimate the D scalings of all terms as follows. We assume that

- Our starting ansatz is constructed by sewing together bits of the event horizon of black holes of radii R and timelike velocity u^M where R and u^M are everywhere finite and of order unity.
- Our starting configuration (and so our full solution) preserves an $SO(D-p-2)$ rotational invariance with p held fixed as D is taken to infinity

As explained in [9], these assumptions unambiguously specify the scaling with D of all quantities of interest (in particular they force $\mathcal{K}$ to be of order D).

We emphasize that in this chapter we use the assumptions listed above only to estimate the scalings of D of various quantities. When the assumptions listed in the previous paragraph are obeyed, the membrane equations and metrics listed in this chapter certainly apply.

However the formulae of this chapter apply more generally to any spacetime whose variables scale with D in the same manner in which they would if the assumptions above were obeyed - a much larger class of configurations.

2.2.4 All orders definition of the membrane surface and velocity

As explained in subsection 2.2.1, the metric (2.1) - the starting point of our perturbative expansion - is completely determined by the shape of a membrane and a velocity field on the membrane. To what precision can this procedure be reversed? In other words if we are given a solution to Einstein's equations of the appropriate kind, how precisely can we read off the corresponding 'shape' and 'velocity' of the membrane?

We could attempt to identify the membrane shape and velocity field by simply expanding the exact solution in powers of $1/D$ and focusing attention on the leading order term. By comparing with (2.1) we could then read off the membrane shape and velocity field. While this procedure is simple, a moment's thought will convince the reader that it is ambiguous at all orders in $1/D$ save the leading order. [7] In other words the requirement that our solution reduce to (2.1) defines the membrane shape and velocity only at leading order, leaving the subleading corrections to these quantities ambiguous. In this subsection we will fix this ambiguity by adopting a more precise definition of the shape and velocity field. This definition agrees with that of (2.1) at leading order, but is precise at all orders. We use this precise definition in the computations presented in the rest of this chapter.

We define the membrane shape to be the location of the event horizon of our spacetime, and will choose higher order corrections to the metric (2.1) to ensure that this event horizon coincides with the surface $\psi = 1$.

Turning to the velocity field, let G^{AB} denote the full spacetime inverse metric. Let n_A be the oneform normal to the event horizon. We define the velocity field on the membrane by the requirement that

$$u^A = G^{AB} n_B \tag{2.16}$$

on the event horizon (i.e. at $\psi = 1$). In other words the velocity field is a tangent vector to the generators of the event horizon. It is easily verified that (2.16) is a true equation for the starting point of perturbation theory (2.1). We will choose corrections to the perturbative ansatz to ensure that (2.16) holds at all orders in $1/D$.

The requirement (2.16) together with the requirement that $\psi = 1$ is the exact event horizon of our spacetime are easily seen to be satisfied provided that

$$\begin{aligned} H^{(S)}(\psi = 1) &= 0 \\ H^{(V)}_M(\psi = 1) &= 0 \end{aligned} \tag{2.17}$$

The first condition ensures that $G^{MN}\partial_M\psi\partial_N\psi = 0$, i.e. $d\psi$ is null at $\psi = 1$ while the second condition then ensures that the full spacetime metric on the event horizon takes the form

$$\eta_{MN} + O_M O_N + H^{(T)}_{MN} + \frac{1}{D-3} H^{Tr} \mathcal{P}_{MN}$$

[7] For instance, the velocity redefinition $u^\mu \to u^\mu + \delta u^\mu / D$ does not change the metric at leading order in $1/D$.

Let us write this metric in a the local basis of oneforms (n, u, Y_a) where Y_a is any $D-2$ dimensional basis of oneforms chosen orthogonal to n and u. In this basis the metric takes a block diagonal form with a 2×2 block (with basis n and u) and a $D-2 \times D-2$ block (with basis Y_a). It follows that the inverse metric also has this block diagonal structure. Note that the 2×2 block is universal, i.e. it is the same at every order in perturbation theory. This block is the only one that contributes in (2.16). As (2.16) holds at leading order, it follows that the conditions (2.17) ensure that (2.16) holds at every order in perturbation theory.

Recall that according to (2.2) the velocity field used in (2.1) is divergence free at leading order in $\frac{1}{D}$. As we will see below, the divergence of the velocity field defined in this subsection will not, in general, vanish at subleading orders in $1/D$.

2.2.5 Structure of the equations of perturbation theory

Our perturbative procedure proceeds as follows. We assume that our solution takes the form (2.12) together with (2.14) and (2.15). The Ricci tensor of this metric - evaluated in a slab of spacetime of thickness $1/D$ around $\psi = 1$ - takes the schematic form

$$R_{MN} = \sum_n D^{2-n} R^n_{MN} \tag{2.18}$$

Let us imagine that we have implemented our perturbative procedure to order $n-1$, i.e. that we have determined $h^{(m)}_{MN}$ for $m = 1 \ldots n-1$ in a manner that ensures that $R^{(m)}_{MN} = 0$ for $m = 0 \ldots n-1$. In order to go to one higher order in perturbation theory we must solve for $h^{(n)}_{MN}$ to ensure that R^n_{MN} also vanishes.

Schematically

$$R^{(n)}_{MN} = C^{PQ}_{MN} h^{(n)}_{PQ} + \mathcal{S}^{(n)}_{MN}$$

where C^{PQ}_{MN} is a linear differential operator with derivatives only in the ψ direction and $\mathcal{S}^{(n)}_{MN}$ is a source function. As $h^{(n)}_{PQ}$ is already of order n, the differential operator C^{PQ}_{MN} is built entirely out of the zero order background metric (2.1), and so is the same at every order. On the other hand the source function $\mathcal{S}^{(n)}_{MN}$ is proportional to expressions of n^{th} order in $1/D$ built out of derivatives of the membrane velocity and shape function, and is different at every order.

At every point of the event horizon of the ansatz metric (2.1) there are two distinguished vectors; n^A and u^A. Let

$$\mathcal{P}_{AB} = \eta_{AB} - n_A n_B + u_A u_B$$

denote the projector orthogonal to these two vectors (all dot products taken in flat space). Instead of dealing directly with the components of R_{MN} we find it more convenient to use a basis adopted to u^A and n^A listed in table 2.1.

By explicit computation (plugging (2.12) into the formula for the Ricci tensor) we find that the linear combinations listed in Table 2.1 of the curvature components R^n_{MN} (see (2.18)) are given by the expressions listed in Table 2.2.

Table 2.1: Basis of components of R_{MN}

Scalar sector	Vector sector	Tensor sector
$R^{S_1} = O^M R_{MN} O^N$ $R^{S_2} = O^M R_{MN} u^N$ $R^{S_3} = u^M R_{MN} u^N$ $R^{S_4} = R_{MN}\mathcal{P}^{MN}$	$R^{V_1}_L = O^M R_{MN}\mathcal{P}^N_L$ $R^{V_2}_L = u^M R_{MN}\mathcal{P}^N_L$	$R^T_{AB} = \mathcal{P}^M_A R_{MN} P^N_B - \frac{\mathcal{P}_{AB}}{D-2}\mathcal{P}^{MN} R_{MN}$

Table 2.2: Expressions for basis of R_{MN}

Scalar sector
$R^{S_1} = \left(\frac{-\mathcal{K}^2}{2(D-3)^2}\right)\frac{d^2 H^{(Tr)}}{dR^2} + \mathcal{S}^{S_1}(R)$
$R^{S_2} = \left(\frac{\mathcal{K}^2}{2(D-3)^2}\right) e^{-R}\frac{d}{dR}\left(e^R \frac{d}{dR} H^{(S)}\right) - \frac{\mathcal{K}^2}{4(D-3)^2} e^{-R}\frac{d}{dR}H^{(Tr)} + \frac{\mathcal{K}}{2(D-3)}\nabla^M H^{(V)}_M + \mathcal{S}^{S_2}(R) + \frac{\mathcal{K}}{2(D-3)} e^{-R}\ \nabla.u$
$R^{S_3} = \left(\frac{\mathcal{K}^2}{2(D-3)^2}\right) e^{-2R}(1-e^R)\frac{d}{dR}(e^R \frac{dH^{(S)}}{dR}) - \left(\frac{\mathcal{K}^2}{4(D-3)^2}\right) e^{-2R}(1-e^R)\frac{dH^{(Tr)}}{dR} - \frac{\mathcal{K}}{2(D-3)} e^{-R}\nabla^M H^{(V)}_M + \mathcal{S}^{S_3}(R) + \frac{\mathcal{K}}{2(D-3)}\ e^{-2R}\ \nabla.u$
$R^{S_4} = \left(\frac{\mathcal{K}^2}{(D-3)^2}\right) e^{-R}\frac{d}{dR}(e^R H^{(S)}) + \left(\frac{\mathcal{K}^2}{2(D-3)^2}\right) e^{-2R}(1-e^R)\frac{d}{dR}\left(e^R \frac{d}{dR}H^{(Tr)}\right) - \left(\frac{\mathcal{K}^2}{2(D-3)^2}\right)\frac{dH^{(Tr)}}{dR} + \frac{\mathcal{K}}{D-3}\nabla^M H^{(V)}_M + \frac{2\mathcal{K}}{D-3}\frac{d}{dR}\nabla^M H^{(V)}_M + \nabla^M\nabla^N H^{(T)}_{MN} + \mathcal{S}^{S_4}(R) - \frac{\mathcal{K}}{(D-3)} e^{-R}\nabla.u$
Vector sector
$R^{V_1}_M = \left(\frac{\mathcal{K}^2}{2(D-3)^2}\right) e^{-R}\frac{d}{dR}(e^R \frac{d}{dR} H^{(V)}_M) + \frac{1}{2}\frac{\mathcal{K}}{(D-3)}\frac{d}{dR}\left(\nabla^N H^{(T)}_{NM}\right) + \mathcal{S}^{V_1}_M(R)$
$R^{V_2}_M = \left(\frac{\mathcal{K}^2}{2(D-3)^2}\right) e^{-2R}(1-e^R)\frac{d}{dR}(e^R \frac{d}{dR} H^{(V)}_M) + \mathcal{S}^{V_2}_M(R)$
Tensor sector
$R^T_{AB} = \left(\frac{-\mathcal{K}^2}{2(D-3)^2}\right) e^{-R}\frac{d}{dR}\left((e^R - 1)\frac{dH^{(T)}_{AB}}{dR}\right) + \mathcal{S}^T_{AB}(R)$

In table 2.2, fluctuation fields H^S, H^{Tr} H^V_A and H^T_{MN} are taken to be of n^{th} order and all source functions (e.g. $\mathcal{S}^{S_1}$) also understood to be n^{th} order sources. All appearances of $\nabla.u$ [8] in the table 2.2 should also be understood as follows. Naively $\nabla.u$ is of order D. For that reason we expand

$$\nabla.u = (D-3)\left(\sum_{n=0}^{\infty}\frac{(\nabla.u)_n}{(D-3)^n}\right) \tag{2.19}$$

Every appearance of $\nabla.u$ in table 2.2 should actually be replaced by $(\nabla.u)_n$. We have already seen in the introduction that $(\nabla.u)_0 = 0$. We will see below that $(\nabla.u)_1$ also vanishes, but that $(\nabla.u)_2$ is nonzero.

In order to obtain Table 2.2 we have worked in the neighbourhood of the surface $\psi = 1$ and the variable R is defined by $R = (D-3)(\psi - 1)$. [9]

[8] $\nabla.u$ is the divergence of the velocity field thought of as a vector field in $R^{D-1,1}$. On the surface $\psi = 1$, however, $\nabla.u$ coincides with the membrane worldvolume divergence of velocity field (this follows upon using the second of (2.9)).

[9] We will explain below that the sources listed in Table 2.2 are not completely independent, but are

2.2.6 The Einstein Constraint Equations

In the process of solving for the fluctuation fields $h^{(n)}_{MN}$ we will find the Einstein constraint equations (relevant to the foliation of our spacetime in slices of constant ψ) particularly useful. We will now provide a careful definition of these equations.

Let us define

$$E_{MN} \equiv R_{MN} - \tilde{R}\frac{G_{MN}}{2} \tag{2.21}$$

where $\tilde{R}$ is the Ricci scalar. The constraint equations are defined by the relations

$$E^{(ec)}_M = E_{MN} G^{NL} n_L \tag{2.22}$$

We have a total of D constraint equations. These equations decompose into two scalars and one vector under local $SO(D-2)$ rotations.

Let us imagine we have solved for our membrane metric at $(n-1)^{th}$ order in perturbation theory, and are now attempting to solve for the metric correction at n^{th} order. If, in this process, we evaluate the constraint equation (2.22) and retain terms only up to n^{th} order then we need use G^{NL} on the RHS of (2.22) only at zero order (i.e. from the metric (2.1)), because E_{MN} is already of n^{th} order. It follows that the n^{th} order scalar and vector constraint equations are simply linear combinations of the n^{th} order scalars and vectors listed in table 2.1. We will now determine the relevant linear combinations. In order to to this we first determine the n^{th} order Ricci scalar $\tilde{R}$ as a linear combination of the scalars in table 2.1.

$$\tilde{R} = R_{AB}G^{AB} = \left(R^{AB}P_{AB} + O.R.O(1-e^{-R}) + 2O.R.u\right) = (R^{S_4} + (1-e^{-R})R^{S_1} + 2R^{S_2}) \tag{2.23}$$

Using this equation we find

$$\begin{aligned} E^{(ec)}_M &= \left(R_{MN} - \frac{\tilde{R}}{2}G_{MN}\right) G^{NL} n_L \\ &= R_{MN}O^N(1-e^{-R}) + R_{MN}u^N - \frac{1}{2}\tilde{R}\ n_M \end{aligned} \tag{2.24}$$

By dotting (2.24) with n and u or by projecting it orthogonal to these vectors we finally obtain the n^{th} order constraint equations written as linear combinations of the scalars and vectors in table 2.1.

$$\begin{aligned} E^{S_1} &= E^{(ec)}_M u^M = (1-e^{-R})R^{S_2} + R^{S_3} \\ E^{S_2} &= E^{(ec)}_M O^M = \frac{1}{2}\left((1-e^{-R})R^{S_1} - R^{S_4}\right) \\ E^V_L &= E^{(ec)}_N \mathcal{P}^N_L = (1-e^{-R})R^{V1}_L + R^{V_2}_L \end{aligned} \tag{2.25}$$

The explicit form of the n^{th} order constraint equations is listed in table 2.3 below

---

constrained by the well known relation

$$\nabla^M \left(R_{MN} - \frac{\tilde{R}}{2}G_{MN}\right) = 0 \tag{2.20}$$

.

Table 2.3: Listing of constraint equations

Vector constraint
$E^V_M = E^{(ec)}_N \mathcal{P}^N_M = (1-e^{-R})R^{V_1}_M + R^{V_2}_M$ $= \frac{1}{2}\frac{\mathcal{K}}{(D-3)}(1-e^{-R})\frac{d}{dR}\left(\nabla^A H^{(T)}_{AM}\right) + \mathcal{V}^V_M(R)$
Scalar constraint 1
$E^{S_1} = E^{(ec)}_M u^M = (1-e^{-R})R^{S_2} + R^{S_3}$ $= \frac{\mathcal{K}}{2(D-3)}(1-e^{R})\frac{d}{dR}\left(\nabla^M H^{(V)}_M\right) - \frac{\mathcal{K}}{2(D-3)}e^{-R}\nabla^M H^{(V)}_M + \mathcal{V}^{S1}(R) + \frac{\mathcal{K}}{2(D-3)}\ e^{-R}\ \nabla.u$
Scalar constraint 2
$E^{S_2} = E^{(ec)}_M O^M = \frac{1}{2}\left((1-e^{-R})R^{S_1} - R^{S_4}\right) = -\frac{\mathcal{K}}{2(D-3)}\frac{d}{dR}\left(\nabla^M H^{(V)}_M\right) - \frac{\mathcal{K}}{(D-3)}\nabla^M H^{(V)}_M$ $+\frac{\mathcal{K}^2}{4(D-3)^2}(2-e^{-R})\frac{d}{dR}H^{(Tr)} - \frac{\mathcal{K}^2}{2(D-3)^2}\left(\frac{d}{dR}H^{(S)} + H^{(S)}\right) - \frac{1}{2}\nabla_M\nabla_N H^{(T)}_{MN} + \mathcal{V}^{S2}(R) + \frac{\mathcal{K}}{2(D-3)}e^{-R}\ \nabla.u$

As in table 2.1, all fluctuation fields in table 2.3 should be taken to be of n^{th} order. The source functions in table 2.3 are also of n^{th} order and are given in terms of the sources in table 2.1 and the as yet unknown quantity $\nabla.u$ by

$$
\begin{aligned}
\mathcal{V}^{S_1}(R) &= (1-e^{-R})\mathcal{S}^{S_2}(R) + \mathcal{S}^{S_3}(R) \\
\mathcal{V}^{S_2}(R) &= \frac{1}{2}\left[(1-e^{-R})\mathcal{S}^{S_1}(R) - \mathcal{S}^{S_4}(R)\right] \\
\mathcal{V}^V_L(R) &= (1-e^{-R})\mathcal{S}^{V_1}_L(R) + \mathcal{S}^{V_2}_L(R)
\end{aligned}
\tag{2.26}
$$

Now it is well known that the Einstein tensor obeys the identity

$$
\nabla_M E^{MN} = 0 \tag{2.27}
$$

It is also well known (and easy to see) that this identity ensures that the 'normal' derivative of the constraint equations is a linear combination of the 'in plane' derivatives of Einstein's equations. [10] Within the perturbation theory of interest to this chapter the equation (2.27) may be evaluated and projected onto its scalar and vector sectors and shown to be equivalent to the following relations

$$
\begin{aligned}
&\frac{d}{dR}E^V_M + E^V_M + \frac{(D-3)}{\mathcal{K}}\nabla^N R^T_{NM} = 0 \\
&\frac{d}{dR}E^{S_1} + E^{S_1} + \frac{(D-3)}{\mathcal{K}}\nabla^N R^{V_2}_N = 0 \\
&\frac{d}{dR}E^{S_2} + E^{S_2} + \left(\frac{1}{2}R^{S_1} + R^{S_2} + \frac{1}{2}R^{S_4}\right) + \frac{(D-3)}{\mathcal{K}}\nabla^N R^{V_1}_N = 0
\end{aligned}
\tag{2.28}
$$

[10] This is the fact that ensures that if all Einstein constraint equations are solved on one 'time' slice then they are automatically solved on the next 'time' slice. In other words, in order to solve Einstein's equations you need only solve the constraint equations on one time slice provided you solve the other equations - lets call them the dynamical equations - everywhere.

Using (2.25) the RHS of these relations may be recast in the equivalent form

$$\begin{aligned}
&\frac{d}{dR}E^V_M + (1-e^{-R})R^{V_1}_M + R^{V_2}_M + \frac{(D-3)}{\mathcal{K}}\nabla^N R^T_{NM} = 0\\
&\frac{d}{dR}E^{S_1} + (1-e^{-R})R^{S_2} + R^{S_3} + \frac{(D-3)}{\mathcal{K}}\nabla^N R^{V_2}_N = 0\\
&\frac{d}{dR}E^{S_2} + \frac{1}{2}e^{-R}R^{S_1} + (1-e^{-R})R^{S_1} + R^{S_2} + \frac{(D-3)}{\mathcal{K}}\nabla^N R^{V_1}_N = 0
\end{aligned} \tag{2.29}$$

In either form these equations express the R derivatives of the Einstein constraint equations (2.25) in terms of linear combinations of the Einstein equations. Using the explicit expressions in tables 2.2 and 2.3, it is possible to verify that the equations (2.28) are indeed obeyed, provided that the scalar and vector sources in table 2.2 and 2.3 are not all independent but are constrained by the following relations

$$\begin{aligned}
&\frac{d}{dR}\mathcal{V}^V_M + \mathcal{V}^V_M + \frac{(D-3)}{\mathcal{K}}\nabla^N \mathcal{S}^T_{NM} = 0\\
&\frac{d}{dR}\mathcal{V}^{S_1} + \mathcal{V}^{S_1} + \frac{(D-3)}{\mathcal{K}}\nabla^N \mathcal{S}^{V_2}_N = 0\\
&\frac{d}{dR}\mathcal{V}^{S_2} + \mathcal{V}^{S_2} + \left[\frac{1}{2}\mathcal{S}^{S_1} + \left(\mathcal{S}^{S_2} + \frac{\mathcal{K}}{2(D-3)}e^{-R}\ \nabla.u\right) + \frac{1}{2}\left(\mathcal{S}^{S_4} - \frac{\mathcal{K}}{(D-3)}e^{-R}\nabla.u\right)\right]\\
&+\frac{(D-3)}{\mathcal{K}}\nabla^N \mathcal{S}^{V_1}_N = 0
\end{aligned} \tag{2.30}$$

Note that we have two relations between the four scalar sources and one relation between the two vector sources in table 2.2. Note that the relations also involve the as yet unknown quantity $\nabla.u$. Later in this chapter we will explicitly verify that the sources that appear in the first and second order calculation obey the relations (2.30). However we would like to emphasize here that these relations are necessarily obeyed at every order in perturbation theory.

2.2.7 Choice of basis for the constraint and dynamical equations

Because we have the linear relationship between constraint and dynamical equations we use the following basis for solving the scalar, vector and tensor fluctuations

$$\begin{aligned}
&\text{Tensor:}\quad R^T_{AB}\\
&\text{Vector:}\quad R^{V_2}_M,\ \ E^V_M\\
&\text{Scalar:}\quad R^{S_1},\ \ R^{S_2},\ \ E^{S_1},\ \ E^{S_2}
\end{aligned} \tag{2.31}$$

From now on we write every expression in this basis. The expressions that we get from Bianchi identities i.e. equations (2.28),(2.29) can be converted to the basis (2.31) as

$$
\begin{aligned}
&\frac{d}{dR}E^V_M + E^V_M + \frac{(D-3)}{\mathcal{K}}\nabla^N R^T_{NM} = 0 \\
&\frac{d}{dR}E^{S_1} + E^{S_1} + \frac{(D-3)}{\mathcal{K}}\nabla^N R^{V_2}_N = 0 \\
&\frac{d}{dR}E^{S_2} + (1-\frac{1}{2}e^{-R})R^{S_1} + R^{S_2} + \frac{1}{1-e^{-R}}\frac{(D-3)}{\mathcal{K}}\nabla^M \left(E^V_M - R^{V_2}_M\right) = 0
\end{aligned}
\tag{2.32}
$$

The corresponding relationship between the sources is given by

$$
\begin{aligned}
&\frac{d}{dR}\mathcal{V}^V_M + \mathcal{V}^V_M + \frac{(D-3)}{\mathcal{K}}\nabla^N \mathcal{S}^T_{NM} = 0 \\
&\frac{d}{dR}\mathcal{V}^{S_1} + \mathcal{V}^{S_1} + \frac{(D-3)}{\mathcal{K}}\nabla^N \mathcal{S}^{V_2}_N = 0 \\
&\frac{d}{dR}\mathcal{V}^{S_2} + (1-\frac{1}{2}e^{-R})\mathcal{S}^{S_1} + \mathcal{S}^{S_2} + \frac{1}{1-e^{-R}}\frac{(D-3)}{\mathcal{K}}\nabla^N \left(\mathcal{V}^V_N - \mathcal{S}^{V_2}_N\right) = 0
\end{aligned}
\tag{2.33}
$$

2.3 Perturbation theory at first order

In this section we will explicitly solve for the first order correction metric $h^{(1)}_{MN}$. However we will perform our analysis in a manner that makes the generalization to higher orders obvious.

2.3.1 Listing first order source functions

As we have explained in the previous section, the components of R^1_{MN} are given in terms of $h^{(1)}_{MN}$ by the expressions in Table 2.2 with particular values for the source functions in that table. By explicit calculation at first order we find that these source functions are given by the values listed in the table 2.4.

Table 2.4: Sources of R_{MN} equations at 1st order

Scalar sector
$\mathcal{S}^{S_1}(R) = 0$ $\mathcal{S}^{S_2}(R) = \frac{\mathcal{K}}{2(D-3)}e^{-R}u.K.u - \frac{e^{-R}(-1+R)}{2}\frac{u.\nabla\mathcal{K}}{(D-3)} - \frac{\mathcal{K}^2}{2(D-3)^2}e^{-R}(-3+2R)$ $\mathcal{S}^{S_3}(R) = \frac{1}{2\mathcal{K}(D-3)}Re^{-R}\nabla^2\mathcal{K} - \frac{e^{-2R}(-2+2e^R+R)}{2}\frac{u.\nabla\mathcal{K}}{(D-3)} + \frac{\mathcal{K}^2}{2(D-3)^2}e^{-2R}\left(3e^R(R-1) - 2R + 3\right)$ $\mathcal{S}^{S_4}(R) = e^{-R}(-1+R)\frac{u.\nabla\mathcal{K}}{(D-3)} + \frac{\mathcal{K}^2}{(D-3)^2}e^{-R}(-1+2R)$
Vector sector
$\mathcal{S}^{V_1}_A(R) = \frac{\mathcal{K}}{2(D-3)}e^{-R}\left(u^M K_{MN} - u^M\nabla_M u_N\right)\mathcal{P}^N_A$ $\mathcal{S}^{V_2}_A(R) = \frac{\mathcal{K}}{2(D-3)}e^{-2R}\left(u^M K_{MN} - u^M\nabla_M u_N\right)\mathcal{P}^N_A + \frac{e^{-R}}{2}\left(\frac{\nabla^2 u_A}{(D-3)} - \frac{\nabla_A\mathcal{K}}{(D-3)}\right)$
Tensor sector
$\mathcal{S}^T_{AB}(R) = 0$

Table 2.5: Sources to constraint equations at 1st order

Vector constraint source
$\mathcal{V}^V_M(R) = \frac{e^{-R}}{2}\left(\frac{\nabla^2 u_M}{(D-3)} - \frac{\nabla_M \mathcal{K}}{(D-3)} + \frac{\mathcal{K}}{(D-3)}(u^A K_{AM} - u.\nabla u_M)\right)$
Scalar constraint 1 source
$\mathcal{V}^{S_1}(R) = \frac{1}{2\mathcal{K}(D-3)} Re^{-R}\nabla^2\mathcal{K} - \frac{-e^{-2R}+e^{-R}(1+R)}{2}\frac{u.\nabla\mathcal{K}}{(D-3)}$ $+\frac{\mathcal{K}}{2(D-3)}e^{-R}(1-e^{-R})u.K.u + Re^{-R}\frac{\mathcal{K}^2}{2(D-3)^2}$
Scalar constraint 2 source
$\mathcal{V}^{S_2}(R) = \frac{e^{-R}}{2}\left(\frac{\mathcal{K}^2}{(D-3)^2}(1-2R) + \frac{u.\nabla\mathcal{K}}{(D-3)}(1-R)\right)$

Moreover the constraint equations take the form listed in Table 2.3 with first order source functions listed in Table 2.5. We list the corresponding sources to the constraint equations at 1st order in table 2.5. We have verified that our explicit expressions for the sources obey the constraints (2.30).

We now proceed to solve the metric corrections at 1st order i.e. $h^{(1)}_{MN}$. We impose the conditions (2.17) as discussed in section 2.2.4.

2.3.2 Tensor sector

In this sector we have a single equation for the single variable $H^{(T)}_{MN}$. This equation is obtained by equating the last line of Table 2.2 to zero and takes the form

$$R^T_{AB} = e^{-R}\frac{d}{dR}\left(\left(e^R - 1\right)\frac{dH^{(T)}_{AB}}{dR}\right)\left(\frac{-\mathcal{K}^2}{2(D-3)^2}\right) + \mathcal{S}^T_{AB}(R) = 0 \tag{2.34}$$

where $\mathcal{S}^T_{AB}(R)$ is the source for the tensor sector. At first order it turns out that $\mathcal{S}^T_{AB}(R) = 0$ (see Table 2.5). In order to facilitate generalizations to higher orders however, in this subsection we will solve (2.34) for an arbitrary source function, and substitute $\mathcal{S}^T_{AB}(R) = 0$ only at the end of the calculation.

Integrating (2.34) once we find

$$\frac{d}{dR}(H^{(T)}_{AB}) = \left(\frac{-2(D-3)^2}{\mathcal{K}^2}\right)\frac{-1}{e^R - 1}\int_0^R e^x \mathcal{S}^T_{AB}(x)dx \tag{2.35}$$

The condition that $H^{(T)}_{AB}$ (and so RHS of (2.35)) is regular at $R = 0$ fixes the lower limit of the integral in (2.35). Integrating a second time we find

$$\begin{aligned} H^{(T)}_{AB} &= \left(\frac{-2(D-3)^2}{\mathcal{K}^2}\right)\int_R^\infty \frac{dy}{e^y - 1}\int_0^y e^x \mathcal{S}^T_{AB}(x)dx \\ &= \left(\frac{2(D-3)^2}{\mathcal{K}^2}\right)\left[\log(1-e^{-R})\int_0^R e^x S^T_{AB}(x)dx + \int_R^\infty \log(1-e^{-x})e^x S^T_{AB}(x)\right] \end{aligned} \tag{2.36}$$

where the upper limit in the outer integral in (2.36) is fixed by the requirement that $H^{(T)}_{AB}$ decay at large R.

In summary, the tensor fluctuation $H^{(T)}_{AB}$ is given at any order, in terms of the tensor source function $\mathcal{S}^T_{AB}(x)$ at that order, by the expression (2.36). Note that $H^{(T)}_{AB}$ is uniquely determined by its source function; requirements of regularity at $R = 0$ and decay at infinity unambiguously fix all integration constants in (2.34).

As we have mentioned above, at first order $\mathcal{S}^{T,1}_{AB}(R) = 0$. It follows from (2.36) that the first order tensor fluctuation $H^{(T)}_{AB}$ also vanishes and so

$$H^{(T,1)}_{AB} = 0 \tag{2.37}$$

2.3.3 Vector Sector

Constraint Equation and the Membrane Equation of Motion

In the vector sector we have two equations for the single variable $H^{(V)}_M$. The two equations may be chosen to be the vector constraint equation E^V_M (see the first line of Table 2.3) and the equation $R^{V_2}_L = 0$ (see Table 2.2).

One cannot, of course, solve two equations for a single variable unless one linear combination of the two equations is an identity. Indeed the first equation of (2.32)

$$\frac{d}{dR}E^V_M + E^V_M + \frac{(D-3)}{\mathcal{K}}\nabla^N R^T_{NM} = 0 \tag{2.38}$$

asserts that the vector constraint equation is automatically solved at all values of R if its solved at one value of R (we use here that we have already solved the tensor equation so that $R^T_{AB} = 0$).

We will find it convenient to solve the vector constraint equation at $R = 0$. From Table 2.3 we see that

$$E^V_M = \frac{1}{2}\frac{\mathcal{K}}{(D-3)}(1-e^{-R})\frac{d}{dR}\left(\frac{\nabla^M H^{(T)}_{MN}}{(D-3)}\right) + \mathcal{V}^V_M(R)$$

At $R = 0$

$$E^V_M = \mathcal{V}^V_M(0)$$

It follows that the constraint equation is solved at $R = 0$ if and only if $\mathcal{V}^V_M(0)$ vanishes (here we use the fact that $H^{(T)}_{MN}$ is regular at $R = 0$; see the previous subsection) . This requirement is a statement of the membrane equations of motion.

We would like to reemphasize that the membrane equations of motion at n^{th} order are obtained simply by evaluating the n^{th} order vector constraint equation at $R = 0$. At $R = 0$ this equation is independent of all the unknown n^{th} order fluctuation fields. As a consequence the membrane equations of motion may be obtained at n^{th} order *before* solving for the fluctuation fields at n^{th} order, as in studies of the fluid gravity correspondence.

The analysis presented in this subsection so far has been valid at every order in perturbation theory. Specializing now to the first order, we read off the value of $\mathcal{V}^V_M(0)$ from Table

2.5. Equating this expression to zero we find the first order membrane equation of motion

$$\left(\frac{\nabla^2 u_A}{\mathcal{K}} - \frac{\nabla_A \mathcal{K}}{\mathcal{K}} + u_C K^C_A - u.\nabla u_A\right) \mathcal{P}^A_B = 0 \tag{2.39}$$

While all fields in (2.39) live in the full bulk spacetime $R^{D-1,1}$, and all derivatives in that equation are bulk spacetime derivatives, the equation (2.39) itself holds only on the membrane surface $\psi = 1$. Using the subsidiary conditions (2.9) it is possible to rewrite (2.39) as an equation restricted to the membrane. As demonstrated in [9] the equation of motion of motion turns out to take exactly the same form as (2.39) in this language. In other words (2.39) also holds true if we think of K_{MN} and u_M as membrane world volume fields, and regard every derivative in that equation as a covariant derivative on the membrane world volume.

Solving for the vector fluctuation

As we have explained in the previous subsubsection, the constraint vector equation is automatically solved at every R provided the membrane equation is obeyed. Assuming this is the case, we have already solved one of the two vector equations.

In order to solve for the unknown function, $H^{(V)}_M$, in the vector sector, we now turn to the second vector equation $R^{V_2}_L = 0$. This equation takes the form

$$\left(\frac{-\mathcal{K}^2}{2(D-3)^2}\right) e^{-2R}(-1+e^R)\frac{d}{dR}(e^R \frac{d}{dR} H^{(V)}_M) + \mathcal{S}^{V_2}_M(R) = 0 \tag{2.40}$$

As in the previous subsection we will proceed to solve (2.40) for an arbitrary source function, plugging in the first order result for the source

$$\mathcal{S}^{V_2,1}_A(R) = -\frac{\mathcal{K}}{2(D-3)} e^{-2R}(-1+e^R)\left(u^M K_{MN} - u^M \nabla_M u_N\right) \mathcal{P}^N_A \tag{2.41}$$

only at the end of the computation.

Notice that the LHS of (2.40) vanishes at $R=0$. It follows that (2.40) admits regular solutions if and only if $\mathcal{S}^{V_2}_M(R)$ also vanishes at $R=0$. It would naively seem that this requirement imposes a new constraint on membrane data, independent of (2.39). [11] However it turns out that the vanishing of $\mathcal{S}^{V_2}_M(R)$ is automatic; indeed it follows from (2.25) that $R^{V_2}_M$ is simply identical to the vector constraint equation E^V_M at $R=0$. It follows as a consequence that $\mathcal{S}^{V_2}_M(R)$ is proportional to the LHS of (2.39) at $R=0$. [12].

Using the fact that $\mathcal{S}^{V_2,1}_M(0)$ vanishes, we integrate (2.40) once to find

$$e^R \frac{d}{dR} H^{(V)}_M = \left(\frac{-2(D-3)^2}{\mathcal{K}^2}\right)\left[\int_0^R \left(\frac{-e^y}{1-e^{-y}}\right) \mathcal{S}^{V_2}_M(y)dy + C^{V_2}_M\right] \tag{2.42}$$

[11] Had this step of the programme imposed a new constraint, we would have obtained a new membrane equation - and so obtained more membrane equations than membrane variables, leading to an inconsistent dynamical system.

[12] To see this we note that (2.40) reduces to $\mathcal{S}^{V_2}_M(R)$ at $R=0$ while E^V_M reduces to the LHS of (2.39) at $R=0$.

where $C_M^{V_2}$ is an as yet undetermined integration constant. Integrating a second time we find

$$H_M^{(V)} = \left(\frac{2(D-3)^2}{\mathcal{K}^2}\right) \int_R^\infty e^{-x} \left[\int_0^x \left(\frac{-e^y}{1-e^{-y}}\right) \mathcal{S}_M^{V_2}(y)dy\right] dx - C_M^{V_2} e^{-R} \tag{2.43}$$

The upper limit on the the outer integral of (2.43) has been determined from the requirement that $H_M^{(V)}$ vanishes at large R. The expression for H_M^V may be simplified by integrating by parts; we find

$$H_M^{(V)}(R) = \left(\frac{2(D-3)^2}{\mathcal{K}^2}\right) \left(e^{-R} \int_0^R \left(\frac{-e^x}{1-e^{-x}}\right) \mathcal{S}_M^{V_2}(x)dx - \int_R^\infty \frac{\mathcal{S}_M^{V_2}(x)}{1-e^{-x}}\right) - C_M^{V_2} e^{-R} \tag{2.44}$$

In particular that

$$H_M^{(V)}(0) = -\left(\frac{2(D-3)^2}{\mathcal{K}^2}\right) \int_0^\infty \frac{\mathcal{S}_M^{V_2}(x)}{1-e^{-x}} - C_M^{V_2} \tag{2.45}$$

It follows (see (2.17)) that

$$C_M^{V_2} = -\left(\frac{2(D-3)^2}{\mathcal{K}^2}\right) \int_0^\infty \frac{\mathcal{S}_M^{V_2}(x)}{1-e^{-x}} \tag{2.46}$$

so that

$$H_M^{(V)}(R) = \left(\frac{2(D-3)^2}{\mathcal{K}^2}\right) \left(e^{-R} \int_0^R \left(\frac{-e^x}{1-e^{-x}}\right) \mathcal{S}_M^{V_2}(x)dx - \int_R^\infty \frac{\mathcal{S}_M^{V_2}(x)}{1-e^{-x}} + e^{-R} \int_0^\infty \frac{\mathcal{S}_M^{V_2}(x)}{1-e^{-x}}\right) \tag{2.47}$$

The expression (2.47) is our final expression for $H_M^{(V)}(R)$ at any order in perturbation theory in terms of the source function at that order. Note that $H_M^{(V)}(R)$ is uniquely determined in terms of its source function; the integration constants in (2.40) are uniquely determined by the requirement that $H_M^{(V)}(R)$ vanish at infinity and that (2.17) is obeyed at $R = 0$.

Plugging the first order expression for the source (2.41) into (2.47), at first order we find

$$H_M^{(V,1)} = \frac{(D-3)}{\mathcal{K}} R e^{-R} \left(u^A K_{AN} - u^A \nabla_A u_N\right) P_M^N \tag{2.48}$$

2.3.4 Scalar sector

In the scalar sector we have four equations for the two variables $H^{(Tr)}$ and $H^{(S)}$. As a basis for the four equations we find it convenient to use the two scalar constraint equations E^{S_1} and E^{S_2} (see Table 2.3) together with the two additional equations $R^{S_1} = 0$ and $R^{S_2} = 0$ (see Table 2.1).

Constraint Equations and $\nabla.u$

As in the previous subsection it is consistent to have four equations for two variables only if two of the four equations are identities. The last two equations in (2.32)

$$\begin{aligned} &\frac{d}{dR}E^{S_1} + E^{S_1} + \frac{(D-3)}{\mathcal{K}} \nabla^N R_N^{V_2} = 0 \\ &\frac{d}{dR}E^{S_2} + (1 - \frac{1}{2}e^{-R})R^{S_1} + R^{S_2} + \frac{(D-3)}{\mathcal{K}} \frac{1}{1-e^{-R}} \nabla^M \left(E_M^V - R_M^{V_2}\right) = 0 \end{aligned} \tag{2.49}$$

assert that this is indeed the case. As we have already solved the vector sector at n^{th} order $R_N^{V_2}$ vanishes. It follows that the first equation in (2.49) asserts that if E^{S_1} is solved at any R it is automatically solved at every R. When evaluated at $R = 0$ this equation reduces to the condition

$$\mathcal{V}^{S_1}(0) + \frac{\mathcal{K}}{2(D-3)}\ \nabla.u = 0 \tag{2.50}$$

Recall that at leading order $\nabla.u = 0$. (2.50) determines the correction to this statement at subleading orders.

As in the previous subsection we emphasize that the expression for $\nabla.u$ at n^{th} order is determined simply by evaluating the n^{th} order constraint equation E^{S_1} at $R = 0$. In order to obtain this correction we do not need to solve for any of the n^{th} order fluctuation fields, all of which drop out in E^{S_1} evaluated at $R = 0$.

The analysis of this subsection has, so far, been valid at every order in perturbation theory. Specializing to first order it is easily verified from Table 2.5 that $\mathcal{V}^{S_1}(0) = 0$. It follows that the zero order relation $\nabla.u = 0$ is uncorrected at first order (since $(\nabla.u)_0 = \mathcal{V}^{S_1}(0) = 0$). As we will see in the next section, the situation is different at second order.

The constraint equation E^{S_2} plays a distinct logical role from E^{S_1} in our perturbative programme. Once the tensor and vector equations had been solved, (2.49) assured us that $E^{S_1}(R)$ obeys a homogeneous differential equation in R (see (2.28) which makes no reference to any of the other equations in the scalar sector. On the other hand the differential equation obeyed by E^{S_2} involves the other scalar equations (see the last equation in (2.29)). The most useful way to view the last equation in (2.29) is as follows. It might, a priori, have seemed that we have 4 equations in the scalar sector. We have already dealt with E^{S_1} above leaving behind a three dimensional space of equations. A useful basis for this space is given by E^{S_2}, R^{S_1} and R^{S_2}. The last equation in (2.29) allows us to eliminate R^{S_2} from this basis. In order to complete solving in the scalar sector we need only solve the equations E^{S_2}, R^{S_1}. In other words the constraint equation E^{S_2} does not constrain data: instead it may be used to solve for the scalar fluctuation. We turn to this task in the next subsubsection.

Solving for the scalar fluctuations

The equation R^{S_1}

$$R^{S_1} = \left(\frac{-\mathcal{K}^2}{2(D-3)^2}\right)\frac{d^2H^{(Tr)}}{dR^2} + \mathcal{S}^{S_1}(R) = 0 \tag{2.51}$$

is easily solved. Integrating the above equation once we get

$$\frac{dH^{(Tr)}}{dR} = \left(\frac{-2(D-3)^2}{\mathcal{K}^2}\right)\int_R^\infty dx\ \mathcal{S}^{S_1}(x) \tag{2.52}$$

Where we have fixed the boundary condition from the requirement that $H^{(Tr)}$ and so its derivative $\frac{dH^{(Tr)}}{dR} = 0$ vanish at large R. Integrating this equation once again we have

$$\begin{aligned} H^{(Tr)} &= \left(\frac{2(D-3)^2}{\mathcal{K}^2}\right)\int_R^\infty dy\int_y^\infty dx\ \mathcal{S}^{S_1}(x) \\ &= \left(\frac{2(D-3)^2}{\mathcal{K}^2}\right)\left[-R\int_R^\infty dx\ \mathcal{S}^{S_1}(x) + \int_R^\infty dx\ x\ \mathcal{S}^{S_1}(x)\right] \end{aligned} \tag{2.53}$$

where, once again we have fixed the integration constant from the requirement that $H^{(Tr)} = 0$ at large R.

Specializing now to first order we note $\mathcal{S}^{S_1,1} = 0$ so that

$$H^{(Tr,1)} = 0 \tag{2.54}$$

The equation E^{S_2} takes the form

$$\begin{split} &\frac{d}{dR}(H^{(S)}e^R) = \frac{2(D-3)^2}{\mathcal{K}^2} e^R \mathcal{S}_S(R) \quad \text{where,} \\ &\mathcal{S}_S(R) = -\frac{\mathcal{K}}{2(D-3)}\frac{d}{dR}\left(\nabla^M H_M^{(V)}\right) - \frac{\mathcal{K}}{(D-3)}\nabla^M H_M^{(V)} \\ &+ \frac{\mathcal{K}^2}{4(D-3)^2}(2-e^{-R})\frac{d}{dR}H^{(Tr)} - \frac{1}{2}\nabla^M\nabla^N H_{MN}^{(T)} + \mathcal{V}^{S_2}(R) + \frac{\mathcal{K}}{2(D-3)}e^{-R}\ \nabla.u \end{split} \tag{2.55}$$

Plugging in the already obtained expressions of $H_M^{(V)}$, $H_{MN}^{(T)}$, $H^{(Tr)}$ (see (2.47),(2.53) and (2.36)) and using (2.33), the source function $\mathcal{S}_S(R)$ can be rewritten as a linear functional of the elementary sources $\mathcal{S}^{S_1}$, $\mathcal{S}^{S_2}$ and $\mathcal{V}^{S_1}$ [13]. Upon simplifying (by integrating by parts on several occasions) we find

$$\begin{split} \mathcal{S}_S(R) = &\int_R^\infty \mathcal{S}^{S_2}(x)dx + \frac{1}{2}\int_R^\infty (2-e^{-x})\mathcal{S}^{S_1}(x)dx - \frac{1}{2}(2-e^{-R})\int_R^\infty \mathcal{S}^{S_1}(x)dx \\ &- \left(1-e^{-R}\right)\int_R^\infty \left(\frac{e^x\left({\mathcal{V}^{S_1}}'(x) + \mathcal{V}^{S_1}(x)\right)}{(e^x-1)}dx\right)dy - \mathcal{V}^{S_1}(R) + e^{-R}\mathcal{V}^{S_1}(0) \\ &+ \log(1-e^{-R})\left({\mathcal{V}^{S_1}}'(0) + \mathcal{V}^{S_1}(0)\right) + (\nabla\cdot u)\frac{\mathcal{K}e^{-R}}{2(D-3)} \end{split} \tag{2.56}$$

We note that $\mathcal{S}_S$ is analytic at $R=0$ if and only if

$${\mathcal{V}^{S_1}}'(0) + \mathcal{V}^{S_1}(0) = 0 \tag{2.57}$$

This condition is, in fact, automatic. It follows from the second of (2.33) that the LHS of (2.57) is proportional to $\nabla^N \mathcal{S}_N^{V_2}(0)$. We have already argued, however, that $\mathcal{S}_N^{V_2}$ vanishes at $R=0$. Since this condition holds at every point on the membrane, it follows also that $\nabla^N \mathcal{S}_N^{V_2}(0) = 0$ establishing (2.57). [14]

Plugging (2.56) into (2.55), integrating (and simplifying using integration by parts) we

[13] It turns out that all dependence on the fourth independent scalar source, $\mathcal{V}^{S_2}$ cancels.

[14] In studies of the fluid gravity correspondence a derivative of the equation of the n^{th} order equation contributes to sources only at $(n+1)^{th}$ order in the derivative expansion. In the large D expansion of this chapter, however, the suppression in order resulting from using an extra derivative can be compensated for by an enhancement in order resulting from the contraction of a spacetime index. Consequently the equation of motion and its contracted derivatives are of the same order in the large D expansion.

find

$$H_S(R) = \frac{2(D-3)^2}{\mathcal{K}^2} e^{-R} \Bigg(\frac{(\mathcal{K}(\nabla\cdot u))R}{2(D-3)} + e^R \int_R^\infty \mathcal{S}^{S_2}(x)dx - \int_0^\infty \mathcal{S}^{S_2}(x)dx + \int_0^R e^x \mathcal{S}^{S_2}(x)dx$$
$$+ \frac{e^R}{2}\int_R^\infty (2-e^{-x})\mathcal{S}^{S_1}(x)dx + \frac{1}{2}\int_0^R e^x(2-e^{-x})\mathcal{S}^{S_1}(x)dx - \frac{1}{2}\int_0^\infty (2-e^{-x})\mathcal{S}^{S_1}(x)dx$$
$$- \frac{1}{2}(2e^R - R)\int_R^\infty \mathcal{S}^{S_1}(x)dx + \int_0^\infty \mathcal{S}^{S_1}(x)dx - \frac{1}{2}\int_0^R (2e^y - y)\mathcal{S}^{S_1}(x)dx$$
$$- \int_0^R (e^y - 1)\int_y^\infty \left(\frac{e^x\left(\mathcal{V}^{S_1\prime}(x) + \mathcal{V}^{S_1}(x)\right)}{(e^x-1)} dx \right) dy - \int_0^R e^x \mathcal{V}^{S_1}(x)dx + R\mathcal{V}^{S_1}(0) \Bigg) \tag{2.58}$$

Explicitly at first order

$$H^{(S,1)} = \frac{D-3}{\mathcal{K}} R e^{-R} \left(R\left(-\frac{\mathcal{K}}{D-3} - \frac{u\cdot\nabla\mathcal{K}}{\mathcal{K}} + \frac{u\cdot K\cdot u}{2} \right) + \left(\frac{\mathcal{K}}{D-3} + u\cdot K\cdot u \right) \right) \tag{2.59}$$

2.3.5 Final Result for the first order metric

After integrating the ordinary differential equations corresponding to Einstein's equations and imposing the condition that the metric is regular at the horizon, matches flat space at the end of the membrane region and (2.17), we get the following solutions for the various components of the metric correction.

$$\begin{aligned}
H^{(T,1)}_{MN} &= 0 \\
H^{(Tr,1)} &= 0 \\
H^{(V,1)}_M &= \frac{(D-3)}{\mathcal{K}} R e^{-R} \left(u^A K_{AL} - u^A \nabla_A u_L \right) \mathcal{P}^L_M \\
H^{(S,1)} &= \frac{D-3}{\mathcal{K}} R e^{-R} \left(R\left(-\frac{\mathcal{K}}{D-3} - \frac{u\cdot\nabla\mathcal{K}}{\mathcal{K}} + \frac{u\cdot K\cdot u}{2} \right) + \left(\frac{\mathcal{K}}{D-3} + u\cdot K\cdot u \right) \right)
\end{aligned} \tag{2.60}$$

Thus we can write the 1st order corrected metric as

$$\begin{aligned}
&g_{MN} = \eta_{MN} + \frac{O_M O_N}{\psi^{D-3}} \\
&+ \frac{1}{D-3}\Bigg[\frac{D-3}{\mathcal{K}} R e^{-R} \left(R\left(-\frac{\mathcal{K}}{D-3} - \frac{u\cdot\nabla\mathcal{K}}{\mathcal{K}} + \frac{u\cdot K\cdot u}{2} \right) + \left(\frac{\mathcal{K}}{D-3} + u\cdot K\cdot u \right) \right) O_M O_N \\
&+ \frac{(D-3)}{\mathcal{K}} R e^{-R} \left(u^A K_{AL} - u^A \nabla_A u_L \right) P^L_{(M} O_{N)} \Bigg]
\end{aligned} \tag{2.61}$$

2.4 2nd order solution

The metric (2.61) solves Einstein equation to first subleading order. In this section we implement the perturbative procedure to one higher order. In other words we determine the correction $H^{(2)}_{MN}$ in a way that ensures that R_{AB} evaluated on the corrected metric is of order $1/D$ (more precisely that $R_{AB}R^{AB}$ is of order $1/D^2$).

The procedure we follow is exactly that of the previous section: in fact second order corrections to the metric are given directly by the formulae of the previous subsection with one modification: we need to use the second order rather than first order source functions. In other words the computation at second order boils down entirely to determining the second order sources.

In order to determine the sources at second order we plug the first order corrected metric (2.61) together with an as yet undetermined second order correction h^2_{MN} into Einstein's equations. We use the fact that the shape and velocity functions in the first order corrected metric obey the equation of motion

$$\left(\frac{\nabla^2 u_A}{\mathcal{K}} - \frac{\nabla_A \mathcal{K}}{\mathcal{K}} + u_C K^C_A - u.\nabla u_A\right)\mathcal{P}^A_B + \frac{1}{D}\mathcal{E}_A\mathcal{P}^A_B = 0 \tag{2.62}$$

where $\mathcal{E}_B$ is an as yet undetermined '2nd order' correction to the equations of motion. As in the previous section we solve the equations in the neighbourhood of a particular point on the event horizon. In our analysis, however, we use the fact that the membrane equations of motion (2.62) are obeyed not just at the particular point we are expanding about but everywhere on the membrane. In other words we use the fact that the derivative of (2.62) vanishes at the point of interest. Finally we also use the fact that $\nabla.u$ is an as yet undetermined quantity of order $1/D$.

We find by explicit computation that the curvature components listed table 2.1 do indeed take the form listed in table 2.2,2.3 once all metric fluctuation fields in that table are identified with second order fluctuations. Our explicit computations also yield explicit expressions for all the second order source functions. We present an explicit listing of these source functions in Tables A.1 and A.2 in the Appendix.

In the rest of this section we obtain the second order correction to the metric by inserting the second order sources listed above into the general integral formulae of the previous section and performing all integrals.

2.4.1 Constraints on membrane data

Correction to the membrane equations from the vector sector

As in the previous subsection (2.38) guarantees that the vector constraint equation $E^V_M = 0$ is solved at any R if the equation is obeyed at $R = 0$. As in the previous subsection the constraint equation at $R = 0$ is independent of the second order fluctuation fields. From table A.2 we see that this constraint equation at $R = 0$ determines $-\frac{1}{D}\mathcal{E}_A\mathcal{P}^A_B$ - the second order correction to the membrane equation of motion - in terms of appropriate expressions involving the membrane extrinsic curvature and velocity fields. Adding these correction terms to the first order membrane equation (2.3) we recover the second order corrected membrane equation

$$
\begin{aligned}
&\left[\frac{\nabla^2 u}{\mathcal{K}} - \frac{\nabla\mathcal{K}}{\mathcal{K}} + u\cdot K - (u\cdot\nabla)u\right]\cdot\mathcal{P} + \left[\frac{\nabla^2\nabla^2 u}{\mathcal{K}^3} - \frac{\nabla(\nabla^2\mathcal{K})}{\mathcal{K}^3}\right.\\
+&\quad 3\frac{(u\cdot K\cdot u)(u\cdot\nabla u)}{\mathcal{K}} - 3\frac{(u\cdot K\cdot u)(u\cdot\nabla n)}{\mathcal{K}} - 6\frac{(u\cdot(\nabla^2 n))(u\cdot\nabla u)}{\mathcal{K}^2}\\
+&\quad 6\frac{(u\cdot(\nabla^2 n))(u\cdot\nabla n)}{\mathcal{K}^2} + \left.\frac{3}{D-3}u\cdot\nabla u - \frac{3}{D-3}u\cdot\nabla n\right]\cdot\mathcal{P} = 0
\end{aligned}
\tag{2.63}
$$

where

$$
\mathcal{P}^{AB} = \eta^{AB} - n^A n^B + u^A u^B \tag{2.64}
$$

The 1st square bracket in (2.63) is simply the 1st order equation of motion while the 2nd square bracket represents subleading corrections. [15]

We would like, however, to emphasize an important technical point. All the fields in (2.63) are assumed to live in all of the embedding flat spacetime; they are extended off the surface of the membrane by the subsidiary conditions listed earlier in this chapter. While all covariant derivatives listed in (2.63) are evaluated on the surface of the membrane, they act on fields defined in all of spacetime.

As the membrane equations of motion are intrinsic to the membrane, it is clearly unnatural to write them in terms of spacetime derivatives of an essentially arbitrary extension of membrane fields into the embedding spacetime. The equation of motion (2.63) can be rewritten so that all fields in that equation are purely membrane world volume fields, and every derivative in the equation is a covariant derivative on the membrane world volume. We now explain how this is done.

The relationship between the bulk covariant derivatives of tensors (e.g. u_M) and membrane worldvolume derivatives of the same quantities is quite straightforward when no more than one derivative acts on the same object. The spacetime covariant derivative is obtained from the corresponding bulk quantity by projecting every index (not just the derivative indices) onto the membrane world volume. However this relationship is more complicated when we have two or more derivatives acting on the same object; the reason for the additional complication is that the formula for multiple worldvolume covariant derivatives involves inserting projectors at each step (when you define the first derivative in terms of bulk derivatives, then again when you define the second derivative in terms of bulk derivatives etc); when such expressions are opened out, outer derivatives act on projectors used to define the inner derivatives. Tracing through the required algebra we find that the corrected second order membrane equation of motion, written in terms of fields and covariant derivatives that

[15] Note that we can write the equation (2.63) in a nicer looking form by using the subsidiary conditions (2.9), divergence of first order membrane equation of motion (2.3) and divergence of velocity condition (2.5). The form is

$$
\left(\frac{\nabla^2 O}{\nabla.O} + O.\nabla O\right)\cdot\mathcal{P} + \left(\frac{\nabla^2\nabla^2 O}{(\nabla.O)^3} + 3\frac{\nabla^2(\nabla.O)}{(\nabla.O)^3}O.\nabla O\right)\cdot\mathcal{P} = 0 \tag{2.65}
$$

live purely on the membrane world volume, takes the form

$$
\begin{aligned}
&\left[\frac{\nabla^2 u_A}{\mathcal{K}} - \frac{\nabla_A \mathcal{K}}{\mathcal{K}} + u^B K_{BA} - u\cdot\nabla u_A\right]\mathcal{P}^A_C \\
+ &\left[\left(-\frac{u^C K_{CB} K^B_A}{\mathcal{K}}\right) + \left(\frac{\nabla^2\nabla^2 u_A}{\mathcal{K}^3} - \frac{u\cdot\nabla\mathcal{K}\nabla_A\mathcal{K}}{\mathcal{K}^3} - \frac{\nabla^B\mathcal{K}\nabla_B u_A}{\mathcal{K}^2} - 2\frac{K^{CD}\nabla_C\nabla_D u_A}{\mathcal{K}^2}\right)\right. \\
+ &\left(-\frac{\nabla_A\nabla^2\mathcal{K}}{\mathcal{K}^3} + \frac{\nabla_A\left(K_{BC}K^{BC}\mathcal{K}\right)}{\mathcal{K}^3}\right) + 3\frac{(u\cdot K\cdot u)(u\cdot\nabla u_A)}{\mathcal{K}} - 3\frac{(u\cdot K\cdot u)(u^B K_{BA})}{\mathcal{K}} \\
- &\left. 6\frac{(u\cdot\nabla\mathcal{K})(u\cdot\nabla u_A)}{\mathcal{K}^2} + 6\frac{(u\cdot\nabla\mathcal{K})(u^B K_{BA})}{\mathcal{K}^2} + \frac{3}{D-3}u\cdot\nabla u_A - \frac{3}{D-3}u^B K_{BA}\right]\mathcal{P}^A_C = 0
\end{aligned}
\tag{2.66}
$$

The projector $\mathcal{P}^{AB}$ used in this equation

$$\mathcal{P}^{AB} = \tilde{g}^{AB} + u^A u^B \tag{2.67}$$

where $\tilde{g}^{AB}$ is the induced metric on the world volume of the membrane.

The equation (2.66) can be slightly simplified as follows. Let us first note that (2.66) takes the schematic form

$$F^A + \frac{S^A}{\mathcal{K}} = 0 \tag{2.68}$$

where F^A is the first order contribution to the equation of motion (the first line of (2.66)) while $\frac{S^A}{\mathcal{K}}$ is the second order contribution (the second-fourth lines of (2.66)). F^A and S^A are each vector fields of order unity.

Let us now consider the modified equation of motion

$$F^A + \frac{S^A}{\mathcal{K}} + \nabla.F\frac{\zeta^A}{\mathcal{K}^2} = 0 \tag{2.69}$$

where ζ^A is any vector field of order unity. As $\nabla.F$ is naively of order D, the difference between the equations (2.69) and (2.68) is naively of order $\frac{1}{D}$ suggesting that (2.68) and (2.69) differ at first subleading order. This is not the case. By taking a divergence of either (2.68) or (2.69), the reader can easily convince herself that, onshell, $\nabla.F$ is of order unity (rather than the naive estimate of order D). If follows that (2.69) and (2.68) actually differ only at second subleading order ($\frac{1}{D^2}$) and are equivalent at first subleading order. We are thus allowed to simplify (2.66) by adding any expression of the form $\nabla.F\frac{\zeta^A}{\mathcal{K}^2}$ to it.

Now it was demonstrated in [9] that

$$\frac{\nabla.F}{\mathcal{K}} = \frac{\nabla^2\mathcal{K}}{\mathcal{K}^2} - 2\,\frac{u.\nabla\mathcal{K}}{\mathcal{K}} + u.K.u \tag{2.70}$$

Using this relation and making the choice

$$\zeta^A = -3\left((u.\nabla u)_A - u_B K^B_A\right) \tag{2.71}$$

we find that (2.66) is equivalent to (2.69) whose explicit form is

$$
\begin{aligned}
&\left[\frac{\nabla^2 u_A}{\mathcal{K}} - \frac{\nabla_A \mathcal{K}}{\mathcal{K}} + u^B K_{BA} - u\cdot\nabla u_A\right]\mathcal{P}^A_C \\
+ &\left[\left(-\frac{u^C K_{CB} K^B_A}{\mathcal{K}}\right) + \left(\frac{\nabla^2\nabla^2 u_A}{\mathcal{K}^3} - \frac{u\cdot\nabla\mathcal{K}\nabla_A\mathcal{K}}{\mathcal{K}^3} - \frac{\nabla^B\mathcal{K}\nabla_B u_A}{\mathcal{K}^2} - 2\frac{K^{CD}\nabla_C\nabla_D u_A}{\mathcal{K}^2}\right)\right. \\
+ &\left(-\frac{\nabla_A\nabla^2\mathcal{K}}{\mathcal{K}^3} + \frac{\nabla_A\left(K_{BC}K^{BC}\mathcal{K}\right)}{\mathcal{K}^3}\right) - 3\frac{\nabla^2\mathcal{K}\ u\cdot\nabla u_A}{\mathcal{K}^3} + 3\frac{\nabla^2\mathcal{K}\ u^B K_{BA}}{\mathcal{K}^3} \\
+ &\left.\frac{3}{D-3}u\cdot\nabla u_A - \frac{3}{D-3}u^B K_{BA}\right]\mathcal{P}^A_C = 0
\end{aligned}
\tag{2.72}
$$

Divergence of velocity from a scalar constraint

As we have explained in the previous section, the Einstein constraint equation E^{S_1} is satisfied at all R if it is satisfied at $R=0$. As explained in the previous subsection, the equation at $R=0$ simply asserts that

$$\nabla.u_2 = -\frac{2(D-3)}{\mathcal{K}}\mathcal{V}^{S_1}(0)$$

Reading off the value of $\mathcal{V}^{S_1}(0)$ from the table A.2 we find

$$\nabla\cdot u = \frac{(\nabla.u)_2}{D-3} = \frac{1}{2\mathcal{K}}\left(\nabla_{(A}u_{B)}\nabla_{(C}u_{D)}\mathcal{P}^{BC}\mathcal{P}^{AD}\right) \tag{2.73}$$

2.4.2 Second order corrections to the metric

Tensor Sector

The metric correction in the tensor sector is given by (2.36)

$$
\begin{aligned}
H^{(T)}_{AB} &= \left(\frac{-2(D-3)^2}{\mathcal{K}^2}\right)\int_R^\infty \frac{dy}{e^y-1}\int_0^y e^x \mathcal{S}^T_{AB}(x)dx \\
&= \left(\frac{2(D-3)^2}{\mathcal{K}^2}\right)\left[\log(1-e^{-R})\int_0^R e^x S^T_{AB}(x)dx + \int_R^\infty \log(1-e^{-x})e^x S^T_{AB}(x)\right]
\end{aligned}
\tag{2.74}
$$

where $\mathcal{S}^T_{AB}$ is the second order source listed in table A.1. All the integrals that appear in the final answer can easily be performed analytically, but the final results (given in terms of polylogs) are not very illuminating; we prefer to leave our final result in terms of an explicit integral.

Vector Sector

The solution for $H_M^{(V)}(R)$ at second order is given by (2.47)

$$H_M^{(V)}(R) = \left(\frac{2(D-3)^2}{\mathcal{K}^2}\right)\left(e^{-R}\int_0^R \left(\frac{-e^x}{1-e^{-x}}\right)\mathcal{S}_M^{V2}(x)dx - \int_R^\infty \frac{\mathcal{S}_M^{V2}(x)}{1-e^{-x}} + e^{-R}\int_0^\infty \frac{\mathcal{S}_M^{V2}(x)}{1-e^{-x}}\right) \tag{2.75}$$

with all sources read off at 2nd order from table A.1. As in the tensor sector, all integrals that appear in (2.75) can be explicitly performed in terms of polylogs, but we find the expression (2.75) in terms of explicit integrals more illuminating.

Scalar Sector

Equation R^{S_1} is decoupled equation for $H^{(Tr)}$. The integrated form is given by (2.53) which we write again

$$\begin{aligned} H^{(Tr)} &= \left(\frac{2(D-3)^2}{\mathcal{K}^2}\right)\int_R^\infty dy \int_y^\infty dx\ \mathcal{S}^{S_1}(x) \\ &= \left(\frac{2(D-3)^2}{\mathcal{K}^2}\right)\left[-R\int_R^\infty dx\ \mathcal{S}^{S_1}(x) + \int_R^\infty dx\ x\ \mathcal{S}^{S_1}(x)\right] \end{aligned} \tag{2.76}$$

The source $\mathcal{S}^{S_1}$ for 2nd order is given in table A.1. Substituting this we get the final form of the metric correction

$$H^{(Tr,2)} = -\left(\frac{2(D-3)^2}{\mathcal{K}^2}\right)e^{-R}(1+R)\left((u\cdot K - u\cdot\nabla u)\cdot\mathcal{P}\cdot(u\cdot K - u\cdot\nabla u)\right) \tag{2.77}$$

In a similar manner the fluctuation H^S can is given by (2.58) upon plugging in the explicit values of the second order sources from Tables A.1,A.2.

2.5 Discussion

In this chapter we have worked out the duality between the dynamics of black holes in a large number of dimensions and the motion of a non gravitational membrane in flat space to second subleading order in $1/D$. Our work generalizes the analysis of [8, 9]. The concrete new results of this chapter are

- The second order corrected membrane equations of motion listed in (2.4).
- The formula (2.5) for the divergence of the velocity field (which vanished at first order).
- The explicit form of the second order corrected metric dual to any given membrane motion (see subsection 2.4.2)

In addition to obtaining the new results listed above we have also achieved an improved understanding of the structure of the perturbative expansion in $1/D$. We have demonstrated that the perturbative programme, implemented to first nontrivial order in [8, 9], can systematically be extended to every order in the $1/D$ expansion. In particular we have shown that the algebraically nontrivial 'integrability' properties that allowed for the existence of a first order solution in [8, 9] are actually automatic at all orders as as a consequence of the well known equation (2.27).

We have also explained that the membrane equations may directly be obtained by evaluating the Einstein constraint equation on the event horizon. In particular the membrane equations at $(n+1)^{th}$ order in $1/D$ are obtained by evaluating the constraint equations on n^{th} order metric, without needing to solve for the $(n+1)^{th}$ order metric. We have also explained that the assumption of $SO(D-p-2)$ isometry, made in [9], is not necessary; the membrane equations can be derived under much more general conditions

The fact our membrane equations arise from the Einstein constraint equations at the event horizon is strongly reminiscent of the 'traditional' membrane paradigm of black hole physics. It would be very interesting to better understand the relationship between the the large D membrane and the traditional membrane paradigm. [32, 33, 34].

As black holes are thermodynamical objects, the black hole membrane studied in [8, 9] and this chapter should carry an entropy current. At leading order in $1/D$ it turns out (see [14]) that this entropy current is given simply by a constant times u^M. The divergence of this entropy current is thus proportional to $\nabla.u$. It follows that the RHS of the formula (2.5) gives an expression for the rate of entropy production on the membrane.

On a related note, it would be interesting to derive the most general stationary solution of the second order corrected equations of motion derived in this chapter and compare our results with those of [18].

In this chapter we have focused our attention on black holes propagating in an otherwise perfectly flat spacetime. It would be interesting to generalize our study to the motion of black holes propagating in any vacuum solution of Einstein's equations, e.g. a gravity wave. Such a generalization would allow us, for instance, to study the absorption of gravity waves by black holes at large D. At first order, we expect the generalized effective membrane equation to be given simply by covariantizing first order flat space equations of motion. At second order, however, the equations of motion could receive genuinely new contributions from the background Riemann tensor of the space in which the black hole propagates [16]. In chapters 3 and 4 of this thesis, which are based on works [11, 13] we have worked on these questions. We have extended the membrane paradigm for the case of any background metric which solves Einstein equations with a cosmological constant. We have done the calculations upto second order and verified the expectations stated above.

Finally, it would be interesting to put the membrane equations derived in this chapter to practical use to allow us to learn new things about black holes. One possible direction would be to test out how well the large D expansion does in astrophysical contexts (i.e. when $D = 4$). Another direction would be to use the formalism developed herein to address interesting unanswered structural questions about gravity, e.g. questions about the second law of thermodynamics in higher derivative gravity. We leave such investigations for the

[16]Something similar happens in the study of forced fluids in the fluid gravity correspondence [35]

future.

Chapter 3

Introducing cosmological constant: First order calculation

3.1 Introduction

In previous works [8, 9, 10], the Membrane paradigm at large D was formulated for the dynamics of black holes in flat spacetime background. In this chapter we extend this formulation for the dynamics of black holes in background spacetimes which solve Einstein equations with a cosmological constant. We explicitly implement the $1/D$ perturbation theory to first order. In the next Chapter we implement the perturbation theory to second order.

In the Introductory sections to this Chapter and Chapter 4, we will briefly summarize the setup, procedure followed and the final results. The detailed derivations of these final results of this Chapter and Chapter 4 - in particular the membrane equations and the metric corrections - will appear in the thesis of Parthajit Biswas, who is one of the collaborators in these works. In this Chapter and the next Chapter, our main focus is rather on performing nontrivial checks for the correctness of these membrane equations, and making contact with another approach to large D effective theory for the dynamics of AdS black branes.

We work with the following two derivative action of gravity in D spacetime dimensions

$$\mathcal{S} = \int \sqrt{-G}\,[R - \Lambda] \tag{3.1}$$

Hence we get Einstein equations

$$\mathcal{E}_{AB} \equiv R_{AB} - \left(\frac{R-\Lambda}{2}\right) G_{AB} = 0 \tag{3.2}$$

We choose to set the large D scaling of the cosmological constant in the following way

$$\Lambda = [(D-1)(D-2)]\,\lambda, \qquad \lambda \sim \mathcal{O}(1) \tag{3.3}$$

We will describe the motivation for this choice later in the introduction. We proceed in a very similar way as the case for flat spacetime background as explained in Chapter 2. Only here we consider any background metric which solves (3.2) - instead of the flat background metric

- and we maintain manifest covariance with respect to this background metric throughout the calculation. In the rest of the Introduction, we will describe the setup and the results briefly and referring to Chapter 2 for detailed explanations.

We choose to implement the large D perturbation theory about a zeroth order ansatz metric of the form

$$G^0_{MN} = g_{MN} + \frac{O_M O_N}{\psi^D}, \quad O_M = n_M - u_M \tag{3.4}$$

with

$$n_M = \frac{\partial_M \psi}{\sqrt{\partial_P \psi \partial_Q \psi g^{PQ}}}, \quad u_N u_M g^{MN} = -1, \quad u_M n_N g^{MN} = 0, \quad O_M O_N g^{MN} = 0 \tag{3.5}$$

where ψ and u are arbitrary smooth scalar and vector fields in a D dimensional background spacetime g_{MN} which solves Einstein equations (3.2). We assume these fields have a good large D limit, i.e. the length scale of variation of ψ and u is of order unity. Note $\psi = 1$ is the horizon surface for the metric (3.4). Also note $\psi = 1$ is timelike surface when viewed as embedded in metric g_{MN}. This ansatz is similar form as (2.1) of Chapter 2.

We add subleading metic corrections to the zeroth order ansatz metric and then solve the Einstein equations (3.2) order by order in $1/D$ perturbation theory and then find the spacetime metric solution of the form

$$\begin{gathered} G_{MN} = g_{MN} + h_{MN} = G^0_{MN} + \sum_{n=1}^{k} \left(\frac{1}{D}\right)^n h^{(n)}_{MN}, \\ \text{where,} \quad h_{MN} = \sum_{n=0}^{k} \left(\frac{1}{D}\right)^n h^{(n)}_{MN} \quad \text{with,} \quad h^{(0)}_{MN} = \frac{O_M O_N}{\psi^D}, \end{gathered} \tag{3.6}$$

after implementing the perturbation theory to k^{th} order.

As mentioned in Chapter 2, to estimate the large D scalings of various quantities we assume that the full spacetime metric solution G_{MN} preserves an $SO(D-p-2)$ rotational invariance, with $p \sim \mathcal{O}(1)$. Using such symmetry, it can be shown that Ricci scalar evaluated for a generic smooth metric would be $\mathcal{O}(D^2)$, and this motivates our choice of scaling of Λ as done in (3.3).

The zeroth order ansatz metric solves Einstein equations at the leading order only when the following conditions are satisfied on $\psi = 1$ surface [1]

$$\nabla^2 \psi^{-D} = \mathcal{O}(D), \quad \nabla . u = \mathcal{O}(1), \tag{3.7}$$

where ∇ is covariant derivative with respect to the background metric g_{MN}.

As it was argued in section 2.2.1 of Chapter 2, we need to specify how to construct the scalar field ψ in full spacetime from the $\psi = 1$ surface and how to construct vector field u in full spacetime from its value on the $\psi = 1$ surface. Here we choose to work with the subsidiary conditions for the fields ψ and u of the following form

$$\nabla^2 \psi^{-D} = 0, \quad P^B_A (O.\nabla) O^A = 0 \tag{3.8}$$

[1] By assuming the $SO(D-p-2)$ rotational invariance, for generic ψ and u we have the large D scalings $\nabla^2 \psi^{-D} = \mathcal{O}(D^2)$ and $\nabla . u = \mathcal{O}(D)$. These are the similar conditions as in (2.2) of Chapter 2.

which we impose everywhere in spacetime. Where, $P_A^B = \delta_A^B - n_A O^B - n^B O_A + O_A O^B$. Note that we choose a different set of subsidiary conditions than (2.9) used in Chapter 2.

We fix the coordinate redefinition ambiguity by imposing the same condition as (2.14) in Chapter 2, namely

$$h_{MN} O^N = 0 \tag{3.9}$$

We parameterize the most general fluctuation field $h_{MN}^{(n)}$ in the same form (2.15) of Chapter 2, i.e.

$$\begin{aligned}
h_{MN}^{(n)} = H^{(S,n)} O_M O_N + O_{(M} H_{N)}^{(V,n)} + H_{MN}^{(T,n)} + \frac{1}{D} H^{(Tr,n)} \mathcal{P}_{MN},
\end{aligned}$$

where,

$$\begin{aligned}
\mathcal{P}_{MN} =& \eta_{MN} - O_M n_N - O_N n_M + O_M O_N, \\
O^N H_N^{(V,n)} = 0, \quad n^N H_N^{(V,n)} = 0, &\quad O^M H_{MN}^{(T,n)} = 0, \quad n^M H_{MN}^{(T,n)} = 0, \quad \mathcal{P}^{MN} H_{MN}^{(T,n)} = 0,
\end{aligned} \tag{3.10}$$

As discussed in section 2.2.4 of Chapter 2, we need to fix field redefinition ambiguities in ψ and u. We choose to fix this using the same conditions used in Chapter 2.

$$\begin{aligned}
H^{(S,n)}(\psi = 1) = 0 \\
H_M^{(V,n)}(\psi = 1) = 0
\end{aligned} \tag{3.11}$$

Now consider

$$\begin{aligned}
\mathcal{E}_{MN} &\equiv R_{MN} - \frac{1}{2} R \; G_{MN} + \frac{1}{2} \Lambda \; G_{MN} \\
&= R_{MN} - (D-1)\lambda G_{MN}
\end{aligned} \tag{3.12}$$

We want to solve $\mathcal{E}_{MN} = 0$ order by order in $1/D$ expansion. We can rewrite

$$\mathcal{E}_{MN} = \sum_n D^{2-n} \mathcal{E}_{MN}^{(n)} \tag{3.13}$$

where the index n denotes the order of the perturbation theory in $1/D$. After implementing the perturbation theory to k^{th} order we would have $\mathcal{E}_{MN}^{(n)} = 0$ for $n = 0, 1, ..., k$. Similar as in Chapter 2, $\mathcal{E}_{MN}^{(n)}$ at every order n can be divided into a 'homogeneous' part and a 'source' part

$$\mathcal{E}_{MN}^{(n)} \sim H_{MN}^{(n)} + S_{MN}^{(n)} \tag{3.14}$$

And at each order we have to solve the ordinary linear differential equations $\mathcal{E}_{MN}^{(n)} = 0$ and find the unknown metric corrections $h_{MN}^{(n)}$ at n^{th} order in perturbation theory in terms of sources $S_{MN}^{(n)}$ at n^{th} order. The integration constants are set by the requirements of proper boundary conditions. Namely, we impose regularity of the metric corrections on the horizon. We impose that the metric corrections vanish at infinity. In addition we also use (3.11) which came from fixing the all order definitions of ψ and u. In this Chapter we explicitly implement the perturbation theory to first order.

Similar as in Chapter 2, the Einstein constraint equations play an important role here as well. When we systematically solve all the Einstein equations at n^{th} order, we choose to solve the Einstein constraint equation on $\psi = 1$. It turns out that the constraint equations evaluated on $\psi = 1$ surface are independent of the unknown metric corrections. Hence they imply certain relations between the quantities constructed from shape and velocity fields. These equations describe the dynamics of a codimension-1 timelike non-gravitational membrane, in unperturbed background spacetime. The membrane equations at first order turn out to be

$$\begin{aligned} \left[\frac{\hat{\nabla}^2 u_\alpha}{\mathcal{K}} - \frac{\hat{\nabla}_\alpha \mathcal{K}}{\mathcal{K}} + u^\beta \mathcal{K}_{\beta\alpha} - u\cdot\hat{\nabla} u_\alpha\right]\mathcal{P}^\alpha_\gamma = \mathcal{O}\left(\frac{1}{D}\right) \\ \hat{\nabla}\cdot u = \mathcal{O}\left(\frac{1}{D}\right) \end{aligned} \tag{3.15}$$

Note the equation (3.15) is now written as a membrane worldvolume equation. We define covariant derivatives with respect to the induced metric on the membrane by $\hat{\nabla}$. Greek indices denote the $D-1$ dimensional membrane worldvolume coordinates. u_μ and $\mathcal{K}_{\mu\nu}$ are the pullbacks of velocity and Extrinsic curvature on membrane and $\mathcal{P}^\alpha_\beta = \delta^\alpha_\beta + u^\alpha u_\beta$. Notice that (3.15) have the same form as in the case of flat background considered in Chapter 2, but the covariant derivatives in (3.15) are with respect to the induced metric on the membrane as embedded in the backgrounds considered in this Chapter. Notice that (3.15) reduce to the answers of Chapter 2, when we put $\Lambda = 0$. We solve the remaining Einstein equations to first order and find the unknown metric corrections. It turns out that there are no metric corrections upto first order.

In the rest of this Chapter, we will perform nontrivial checks for the correctness of (3.15) by matching with known spectrum of light quasinormal modes of Schwarzschild black hole in AdS/dS backgrounds and AdS black brane at large D. We also make contact with other approach of [12] for an effective theory of small fluctuations about AdS black brane using the large D techniques.

3.2 Quasinormal Modes for Schwarzschild black hole in background AdS/dS spacetime

In this section, using our membrane equations (3.15) we shall compute the spectrum of light quasi normal modes for Schwarzschild Black hole with horizon topology $S^{D-2} \times \mathrm{R}$ in background AdS/dS spacetimes. We will find that the QNM frequencies that we get match exactly with the earlier obtained result from purely gravitational analysis done in [7].

We do this calculation in Global AdS/dS coordinates. The background AdS/dS in global coordinates can be written as

$$ds^2_{(bgd)} = g_{AB}dX^A dX^B = -\left(1-\sigma\frac{r^2}{L^2}\right)dt^2 + \frac{dr^2}{\left(1-\sigma\frac{r^2}{L^2}\right)} + r^2 d\Omega^2_{D-2}. \tag{3.16}$$

Where

$$\begin{aligned}\Lambda &= \frac{\sigma}{L^2}(D-1)(D-2)\\ L &= \text{AdS/dS radius}\\ \sigma &= 0 \quad \text{for Flat}\\ &= 1 \quad \text{for dS}\\ &= -1 \quad \text{for AdS}\end{aligned} \tag{3.17}$$

The black hole solution in this coordinate system has the following form

$$ds^2_{(BH)} = -\left(1 - \sigma\frac{r^2}{L^2} - \left(\frac{r_0}{r}\right)^{D-3}\right)dt^2 + \frac{dr^2}{\left(1 - \sigma\frac{r^2}{L^2} - \left(\frac{r_0}{r}\right)^{D-3}\right)} + r^2 d\Omega^2_{D-2}. \tag{3.18}$$

In equation (3.18) r_0 is an arbitrary constant. From now on we choose $r_0 = 1$ for convenience. After we find the answers for QNM frequencies we can reinstate these factors easily from dimensional analysis.

A static black hole corresponds to a spherical membrane with a velocity field purely in the time direction. Here we introduce a small fluctuation around this spherical membrane along with a small fluctuation in the velocity field. The amplitude of the fluctuation will be denoted by ϵ - the linearization parameter for our analysis.

$$\begin{aligned} r &= 1 + \epsilon\ \delta r(t,a)\\ u &= u_0\ dt + \epsilon\ \delta u_\mu(t,a) dx^\mu \end{aligned} \tag{3.19}$$

Here a indices denote the angle coordinates along the $(D-2)$ dimensional sphere and the coordinates along the membrane (i.e., time t and angles a) are denoted by μ indices. The induced metric on the membrane worldvolume upto linear order in ϵ is given by (with the components denoted by $g^{(ind)}_{\mu\nu}$)

$$ds^2_{(ind)} = g^{(ind)}_{\mu\nu} dy^\mu dy^\nu = -\left(1 - \sigma\frac{1+2\epsilon\delta r}{L^2}\right)dt^2 + (1+2\epsilon\delta r)d\Omega^2_{D-2} \tag{3.20}$$

Normalization of the velocity field ($u_\mu g^{\mu\nu}_{(ind)} u_\nu = -1$) implies

$$u_0 = -\left(1 - \frac{\sigma}{L^2}\right)^{\frac{1}{2}} \quad \text{and} \quad \delta u_t(t,a) = \left(1 - \frac{\sigma}{L^2}\right)^{-\frac{1}{2}}\left(\frac{\sigma}{L^2}\right)\delta r(t,a) \tag{3.21}$$

The membrane equations have the following form

$$\hat{\nabla}\cdot u = 0, \qquad \mathcal{P}^\nu_\mu\left\{\frac{\hat{\nabla}^2 u_\nu}{\mathcal{K}} - \frac{\hat{\nabla}_\nu\mathcal{K}}{\mathcal{K}} + u^\alpha\mathcal{K}_{\alpha\nu} - (u\cdot\hat{\nabla})u_\nu\right\} = 0 \tag{3.22}$$

Here $\hat{\nabla}$ denotes the covariant derivative with respect to the metric (3.20). $\mathcal{K}_{\mu\nu}$ is the extrinsic curvature viewed as a symmetric tensor along the membrane world volume. $\mathcal{K}$ is the trace of

$\mathcal{K}_{\mu\nu}$ and the projector (again as a tensor along the membrane world volume) perpendicular to u_μ is denoted as

$$\mathcal{P}_{\mu\nu} \equiv g^{(ind)}_{\mu\nu} + u_\mu u_\nu$$

Different components of this projector are given by

$$\mathcal{P}^t_t = 0, \quad \mathcal{P}^a_t = -\left(1 - \frac{\sigma}{L^2}\right)^{\frac{1}{2}} (\epsilon \delta u^a), \quad \mathcal{P}^t_a = \left(1 - \frac{\sigma}{L^2}\right)^{-\frac{1}{2}} (\epsilon \delta u_a), \quad \mathcal{P}^a_b = \delta^a_b \tag{3.23}$$

Now, for convenience we rewrite the vector membrane equation (second equation in (3.22)) as

$$E^{tot}_\mu \equiv \mathcal{P}^\nu_\mu E_\nu$$

where,

$$E_\mu \equiv \frac{\hat{\nabla}^2 u_\mu}{\mathcal{K}} - \frac{\hat{\nabla}_\mu \mathcal{K}}{\mathcal{K}} + u^\nu \mathcal{K}_{\nu\mu} - u^\nu \hat{\nabla}_\nu u_\mu$$

Hence we have

$$\begin{aligned} E^{tot}_t &= E_t \mathcal{P}^t_t + E_b \mathcal{P}^b_t \\ E^{tot}_a &= E_t \mathcal{P}^t_a + E_b \mathcal{P}^b_a \end{aligned} \tag{3.24}$$

Note that because of the spherical symmetry of the background E_a(remember 'a' denotes the angle coordinates along the sphere) will be nonzero only if fluctuations are present or in other words $E_a \sim \mathcal{O}(\epsilon)$. Also $\mathcal{P}^t_t = 0$ and $\mathcal{P}^a_t \sim \mathcal{O}(\epsilon)$. It follows that the time component E^{tot}_t identically vanishes at the linear order. Now since $\mathcal{P}^t_a = \mathcal{O}(\epsilon)$, we see that only $\mathcal{O}(\epsilon^0)$ pieces of E_t matter in the computation of E^{tot}_a. We keep these facts in mind and evaluate only those terms in E_μ that will be important for our linearized analysis.

First note that $\mathcal{K}_{\mu\nu}$ is generically not equal to the $(\mu\nu)$ component of K_{AB}, the extrinsic curvature viewed as a tensor in the full background space-time. $\mathcal{K}_{\mu\nu}$ is given by the pullback of the extrinsic curvature on the membrane surface as

$$\mathcal{K}_{\mu\nu} = \left(\frac{\partial X^M}{\partial y^\mu}\right)\left(\frac{\partial X^N}{\partial y^\nu}\right) K_{MN}|_{r=1+\epsilon\delta r} \tag{3.25}$$

where we have denoted the set (r, t, θ^a) by X^M and the set (t, θ^a) by y^μ. The space-time form of extrinsic curvature K_{AB} is given by

$$K_{AB} = \Pi^C_A \nabla_C n_B, \quad \text{where } \Pi_{AC} = g_{AC} - n_A n_C \tag{3.26}$$

Now if we apply equation (3.25) for our case, we find

$$\mathcal{K}_{\mu\nu} = \epsilon(\partial_\mu \delta r) K_{r\nu} + \epsilon(\partial_\nu \delta r) K_{r\mu} + K_{\mu\nu} + \mathcal{O}(\epsilon^2) \tag{3.27}$$

From explicit computation we know (see appendix B.1) that $K_{rN} = \mathcal{O}(\epsilon)$. It follows that for this linearized analysis $\mathcal{K}_{\mu\nu}$ is just the 'truncation' of the K_{MN} evaluated on the membrane surface and is given by (see appendix B.1 for details).

$$\begin{aligned}
\mathcal{K}_{tt} &= \left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}(-\epsilon\partial_t^2\delta r) + \left(1-\frac{\sigma}{L^2}\right)^{\frac{1}{2}}\left(\frac{\sigma}{L^2}\right)\left(1+\epsilon\delta r - \frac{\sigma\epsilon\delta r}{L^2-\sigma}\right)\\
\mathcal{K}_{ta} &= \left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}(-\epsilon\partial_t\bar{\nabla}_a\delta r)\\
\mathcal{K}_{ab} &= \left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}(-\epsilon\bar{\nabla}_a\bar{\nabla}_b\delta r) + \left(1-\frac{\sigma}{L^2}\right)^{\frac{1}{2}}\left(1+\epsilon\delta r - \frac{\sigma\epsilon\delta r}{L^2-\sigma}\right)\hat{g}_{ab}
\end{aligned} \tag{3.28}$$

Here $\bar{\nabla}_a$ denotes the covariant derivative with respect to the metric on a $(D-2)$ dimensional unit sphere.
The trace of the Extrinsic curvature is given by

$$\begin{aligned}
\mathcal{K} = &\left(1-\frac{\sigma}{L^2}\right)^{-\frac{3}{2}}(\epsilon\partial_t^2\delta r) - \left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}\left(\frac{\sigma}{L^2}\right)\left(1+\frac{\epsilon L^2\delta r}{L^2-\sigma}\right)\\
&+\left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}(-\epsilon\bar{\nabla}^2\delta r) + \left(1-\frac{\sigma}{L^2}\right)^{\frac{1}{2}}\left(1-\frac{\epsilon L^2\delta r}{L^2-\sigma}\right)(D-2)
\end{aligned} \tag{3.29}$$

Thus the components that would be relevant for the linearized membrane equation are given by

$$\begin{aligned}
u^\nu\mathcal{K}_{\nu t} &= \frac{\sigma}{L^2} + \mathcal{O}(\epsilon)\\
u^\nu\mathcal{K}_{\nu a} &= \left(1-\frac{\sigma}{L^2}\right)^{-1}(-\epsilon\partial_t\bar{\nabla}_a\delta r) + \left(1-\frac{\sigma}{L^2}\right)^{\frac{1}{2}}(\epsilon\delta u_a)\\
u^\nu\hat{\nabla}_\nu u_t &= 0\\
u^\nu\hat{\nabla}_\nu u_a &= \left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}(\epsilon\partial_t\delta u_a) - \left(1-\frac{\sigma}{L^2}\right)^{-1}\frac{\sigma}{L^2}(\epsilon\bar{\nabla}_a\delta r)\\
\hat{\nabla}_t\mathcal{K} &= \mathcal{O}(\epsilon)\\
\hat{\nabla}_a\mathcal{K} &= \left(1-\frac{\sigma}{L^2}\right)^{-\frac{3}{2}}(\epsilon\partial_t^2\bar{\nabla}_a\delta r) - \left(1-\frac{\sigma}{L^2}\right)^{-\frac{3}{2}}\frac{\sigma}{L^2}(\epsilon\bar{\nabla}_a\delta r)\\
&\quad+\left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}(-\epsilon\bar{\nabla}_a\bar{\nabla}^2\delta r) - (D-2)\left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}(\epsilon\bar{\nabla}_a\delta r)\\
\hat{\nabla}^2 u_t &= \mathcal{O}(\epsilon)\\
\hat{\nabla}^2 u_a &= -\left(1-\frac{\sigma}{L^2}\right)^{-1}(\epsilon\partial_t^2\delta u_a) + \left(1-\frac{\sigma}{L^2}\right)^{-\frac{3}{2}}\frac{\sigma}{L^2}(\epsilon\partial_t\bar{\nabla}_a\delta r)\\
&\quad+\epsilon\bar{\nabla}^2\delta u_a + \left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}(\epsilon\partial_t\bar{\nabla}_a\delta r)
\end{aligned} \tag{3.30}$$

As before in equations (3.30) $\hat{\nabla}$ denotes the covariant derivative with respect to the the induced metric as given in equation (3.20) and $\bar{\nabla}$ denotes the covariant derivative with respect to the metric on a $(D-2)$ dimensional unit sphere.

Using equations (3.30) the linearized vector membrane equation in the angular directions evaluates to

$$
\begin{aligned}
E_a^{tot} \equiv & \left(\frac{\sigma}{L^2}\right)\left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}(\epsilon\delta u_a) + \left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}\epsilon\frac{\bar{\nabla}^2\delta u_a}{D-2} + \left(1-\frac{\sigma}{L^2}\right)^{-1}\epsilon\frac{\bar{\nabla}_a\bar{\nabla}^2\delta r}{D-2} \\
& + \left(1-\frac{\sigma}{L^2}\right)^{-1}(\epsilon\bar{\nabla}_a\delta r) + \left(1-\frac{\sigma}{L^2}\right)^{-1}(-\epsilon\partial_t\bar{\nabla}_a\delta r) + \left(1-\frac{\sigma}{L^2}\right)^{\frac{1}{2}}(\epsilon\delta u_a) \\
& - \left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}(\epsilon\partial_t\delta u_a) + \left(1-\frac{\sigma}{L^2}\right)^{-1}\left(\frac{\sigma}{L^2}\right)(\epsilon\bar{\nabla}_a\delta r)
\end{aligned}
\tag{3.31}
$$

In writing (3.31) we have also neglected the terms which are subleading in $1/D$.
We also need to process the first equation of (3.22). We have to evaluate the divergence of the velocity field. It comes out to be

$$
\hat{\nabla}.u = 0 = \epsilon\bar{\nabla}^a\delta u_a + \epsilon\left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}(\partial_t\delta r)(D-2) \tag{3.32}
$$

Now, similar to calculation done in Section (5) of [9] we divide the fluctuation δu_a in two parts

$$
\delta u_a = \delta v_a + \bar{\nabla}_a\Phi\ , \quad \text{with} \quad \bar{\nabla}^a\delta v_a = 0 \tag{3.33}
$$

Substituting (3.33) into (3.32) we get

$$
\bar{\nabla}^2\Phi = -\left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}(\partial_t\delta r)(D-2) \tag{3.34}
$$

Now consider $\bar{\nabla}^a E_a^{tot}$. Using the identity $\bar{\nabla}^a\bar{\nabla}^2 V_a = ((D-2)+\bar{\nabla}^2)\bar{\nabla}^a V_a$ and simplifying we get

$$
-2(D-2)\partial_t\delta r - 2\partial_t\bar{\nabla}^2\delta r + \frac{\bar{\nabla}^2\bar{\nabla}^2\delta r}{D-2} + \bar{\nabla}^2\delta r + (D-2)\partial_t^2\delta r + \frac{\sigma}{L^2}\bar{\nabla}^2\delta r = 0 \tag{3.35}
$$

Note that compared to the flat case (refer to Eq. (5.16) in [9]), it is easy to see that the only term extra in equation (3.35) is the last term which is crucial.

We expand the fluctuation as

$$
\delta r = \sum_{l,m} a_{lm} Y_{lm} e^{-i\omega_l^s t} \tag{3.36}
$$

where the scalar spherical harmonics Y_{lm} on S^{D-2} obey

$$
\bar{\nabla}^2 Y_{lm} = -l(D+l-3)Y_{lm}. \tag{3.37}
$$

After substituting (3.36) in (3.35) and solving we get the scalar QNM frequencies

$$
\omega^s = \pm\sqrt{l\left(1-\frac{\sigma}{L^2}\right)-1} - i(l-1) \tag{3.38}
$$

Reinstating the factors of r_0 we get

$$
\omega^s r_0 = \pm\sqrt{l\left(1-\frac{\sigma r_0^2}{L^2}\right)-1} - i(l-1) \tag{3.39}
$$

Upto the required order, this answer agrees with the corresponding answer given in expressions (D.3),(D.4) of [7].

Now we find the vector QNM frequencies. Since we have solved (3.35) the δr and Φ terms in (3.31) drop out and the equation reduces to

$$\frac{\bar{\nabla}^2 \delta v_a}{D-2} + \delta v_a - \partial_t \delta v_a = 0 \tag{3.40}$$

We expand the fluctuation as

$$\delta v_a = \sum_{l,m} b_{lm} Y_a^{lm} e^{-i\omega_l^v t} \tag{3.41}$$

where the vector spherical harmonics Y_a^{lm} on S^{D-2} obey

$$\bar{\nabla}^2 Y_a^{lm} = -[(D+l-3)l-1] Y_a^{lm} \tag{3.42}$$

Substituting (3.41) in (3.40) and solving we get the vector QNM frequency as

$$\omega^v = -i(l-1) \tag{3.43}$$

Reinstating the factors of r_0 we have

$$\omega^v r_0 = -i(l-1) \tag{3.44}$$

Upto the required order, this answer agrees with the corresponding answer given in expression (D.2) of [7].

3.3 Nonlinear effective equations for AdS Black brane dynamics from scaled membrane equations

In a parallel development, the authors of [12] have developed an effective theory for black brane dynamics in background AdS spacetime. They focus on the length scales of order $\frac{1}{\sqrt{D}}$ and derive a pair of nonlinear effective differential equations that govern the dynamics of fluctuations which are suppressed for large D by appropriate inverse powers of D given in [12]. In this section we show that by doing appropriate scalings in our membrane equations (3.15), we are able to reduce our membrane equations to the form that matches the effective equations given in [12] under appropriate field redefinition. This analysis is very similar to the one done for the case of black p-brane in flat spacetime in [36]. In this section, we first do the linearized analysis of fluctuations without any scalings and get idea about how the various quantities need to be rescaled. Then we do the nonlinear analysis by applying the scalings and show the correspondence to effective equations of [12].

3.3.1 Linearized fluctuation analysis and hints for scalings

Now we do the linearized fluctuation analysis for a planar membrane in AdS. This membrane corresponds to a Schwarzschild black brane in AdS with horizon topology $R^{D-2} \times R$ in

Poincare patch metric. We will consider the fluctuations in shape and velocity field in time plus all the $D-2$ brane directions.

The background metric in Poincare patch (with AdS radius $L=1$) is given by

$$ds^2 = -\hat{r}^2 d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^2 d\hat{x}^a d\hat{x}_a \tag{3.45}$$

Where the indices a, b take $D-2$ number of values.
Let $\hat{r} = r_0$ be the position of static unperturbed membrane in Poincare patch coordinates. We choose to scale the coordinates with r_0 in the following way

$$\hat{r} = r_0 r, \quad \hat{t} = \frac{t}{r_0}, \quad \hat{x}^a = \frac{x^a}{r_0} \tag{3.46}$$

In these scaled coordinates the background metric takes the following form

$$ds^2_{(bgd)} = g_{AB} dX^A dX^B = -r^2 dt^2 + \frac{dr^2}{r^2} + r^2 dx^a dx_a \tag{3.47}$$

And also the position of the membrane is now at $r=1$.
When we introduce the fluctuations on this membrane we will consider the time dependence as

$$e^{-i\hat{\omega}\hat{t}} = e^{-i\omega t}, \quad \text{with } \hat{\omega} = r_0 \omega$$

We work with this choice from now on. Note that with this choice all the new (non-hatted) coordinates are dimensionless.

As mentioned before, in this section we shall consider small fluctuations around a static membrane solution. The fluctuations will be of the form

$$\begin{aligned} r &= 1 + \epsilon \delta r(t, a) \\ u &= u_0 dt + \epsilon \delta u_t(t, a) dt + \epsilon \delta u_b(t, a) dx^b \end{aligned} \tag{3.48}$$

ϵ is the linearization parameter.

To the leading order in ϵ, the induced metric on the membrane worldvolume is

$$ds^2 = g^{(ind)}_{\mu\nu} dy^\mu dy^\nu = -(1 + 2\epsilon\delta r) dt^2 + (1 + 2\epsilon\delta r) dx^a dx_a \tag{3.49}$$

As before, we use the notation $\hat{\nabla}$ for denoting the covariant derivative constructed from the induced metric (3.49) and ∇ for denoting the covariant derivative constructed from the background metric (3.47). In this notation the membrane equation will have the same form in as in equation (3.22), where $\mathcal{P}_{\mu\nu}$, $\mathcal{K}_{\mu\nu}$ and $\mathcal{K}$ are the projector, extrinsic curvature and its trace exactly as in previous section.
$\mathcal{K}_{\mu\nu}$ is given by the pullback of the space-time extrinsic curvature K_{MN} on the membrane. Here also using the same reasoning as in the previous section one could show that $\mathcal{K}_{\mu\nu}$ is just the 'truncation' of the K_{MN} evaluated on the membrane surface. The nonzero components of $\mathcal{K}_{\mu\nu}$ are given by

$$\mathcal{K}_{tt} = -\epsilon \partial_t^2 \delta r - (1 + 2\epsilon\delta r), \quad \mathcal{K}_{ta} = -\epsilon \partial_t \partial_a \delta r, \quad \mathcal{K}_{ab} = -\epsilon \partial_a \partial_b \delta r + (1 + 2\epsilon\delta r)\delta_{ab} \tag{3.50}$$

Thus the trace of Extrinsic curvature becomes

$$\mathcal{K} = (D-1) + \epsilon \partial_t^2 \delta r - \epsilon \partial_a \partial^a \delta r \tag{3.51}$$

where the index a in (3.51) is raised with δ^{ab}.

Normalization of the velocity field fixes u_0 and δu_t, defined in (3.48), in terms of the radial fluctuation

$$u_t = u_0 + \epsilon \delta u_t = -(1 + \epsilon \delta r) \tag{3.52}$$

Given the velocity field, different components of the projectors $\mathcal{P}^\mu_\nu = \delta^\mu_\nu + u^\mu u_\nu$ are given by

$$\mathcal{P}^a_b = \delta^a_b, \quad \mathcal{P}^t_t = 0, \quad \mathcal{P}^t_a = \epsilon \delta u_a, \quad \mathcal{P}^a_t = -\epsilon \delta u_a \tag{3.53}$$

Now, following the same trick as in the previous section we denote the vector membrane equation (the 2nd equation in (3.22)) as

$$E^{tot}_\mu \equiv \mathcal{P}^\nu_\mu E_\nu$$

where,

$$E_\mu \equiv \frac{\hat{\nabla}^2 u_\mu}{\mathcal{K}} - \frac{\hat{\nabla}_\mu \mathcal{K}}{\mathcal{K}} + u^\nu \mathcal{K}_{\nu\mu} - u^\nu \hat{\nabla}_\nu u_\mu$$

Then we have

$$\begin{aligned} E^{tot}_t &= E_t \mathcal{P}^t_t + E_b \mathcal{P}^b_t \\ E^{tot}_a &= E_t \mathcal{P}^t_a + E_b \mathcal{P}^b_a \end{aligned} \tag{3.54}$$

The background has a translational symmetry along the x_a directions that is broken by the fluctuations. Hence $E_b \sim \mathcal{O}(\epsilon)$. Now using the facts that $\mathcal{P}^t_t = 0$, $\mathcal{P}^a_t \sim \mathcal{O}(\epsilon)$ and $E_b \sim \mathcal{O}(\epsilon)$ we can see that the time component E^{tot}_t vanishes at the linear order. Similarly for E^{tot}_a, since $\mathcal{P}^t_a \sim \mathcal{O}(\epsilon)$, we see that only $\mathcal{O}(\epsilon^0)$ pieces of E_t contributes. We keep these facts in mind and calculate only those terms that are important.

$$\frac{\hat{\nabla}_t \mathcal{K}}{\mathcal{K}} = \mathcal{O}(\epsilon), \quad \frac{\hat{\nabla}^2 u_t}{\mathcal{K}} = \mathcal{O}(\epsilon), \quad u^\mu \mathcal{K}_{\mu t} = -1, \quad (u \cdot \hat{\nabla}) u_t = \mathcal{O}(\epsilon) \tag{3.55}$$

and (with the notation $\partial^2 = \partial_a \partial^a$)

$$\begin{aligned} &\frac{\hat{\nabla}_a \mathcal{K}}{\mathcal{K}} = \frac{\epsilon \partial_a \partial_t^2 \delta r}{D} - \frac{\epsilon \partial_a \partial^b \partial_b \delta r}{D}, \quad \frac{\hat{\nabla}^2 u_a}{\mathcal{K}} = \frac{-\epsilon \partial_t^2 \delta u_a}{D} - \frac{\epsilon \partial_a \partial_t \delta r}{D} + \frac{\epsilon \partial^b \partial_b \delta u_a}{D} + \frac{\epsilon \partial_a \partial_t \delta r}{D}, \\ &u^\mu \mathcal{K}_{\mu a} = -\epsilon \partial_t \partial_a \delta r + \epsilon \delta u_a, \quad (u \cdot \hat{\nabla}) u_a = \epsilon \partial_t \delta u_a + \epsilon \partial_a \delta r \end{aligned} \tag{3.56}$$

Using equation (3.54) the expression for linearized E^{tot}_a is given by

$$-\partial_a \delta r - \partial_t \partial_a \delta r - \partial_t \delta u_a + \frac{1}{D}\left(-\partial_a \partial_t^2 \delta r - \partial_t^2 \delta u_a\right) + \frac{1}{D}(\partial^b \partial_b \delta u_a + \partial^b \partial_b \partial_a \delta r) = 0 \tag{3.57}$$

The $\hat{\nabla}.u = 0$ equation becomes

$$\hat{\nabla}.u = 0 = \epsilon \partial^a \delta u_a + \epsilon (D-2) \partial_t \delta r \tag{3.58}$$

Note that if we assume all spatial and temporal frequencies are of order $\mathcal{O}(1)$, then in equation (3.57) the last two terms in parenthesis are suppressed compared to the first three terms by a factor of $\left(\frac{1}{D}\right)$. However, it turns out, that the temporal and the spatial frequencies are related by a factor of $\left(\frac{1}{\sqrt{D}}\right)$ even if we ignore the last two terms in equation (3.57), mentioned above. This happens because in the scalar sector the divergence of the velocity fluctuation couples to the shape fluctuation (i.e., δr) and the coupling is through equation (3.58) which involves a relative factor of D.
This simply says that it is inconsistent to assume both the temporal and spatial frequencies to be of order $\mathcal{O}(1)$. Now we shall demand that the temporal frequency is of order $\mathcal{O}(1)$, but we shall not restrict the spatial frequencies. In that case the terms in the first parenthesis in equation (3.57) are certainly suppressed compared to the first three terms, but the terms in the last parenthesis need not be. Thus for our purpose E_a^{tot} is given by

$$-\partial_a \delta r - \partial_t \partial_a \delta r - \partial_t \delta u_a + \frac{1}{D}(\partial^b \partial_b \delta u_a + \partial^b \partial_b \partial_a \delta r) = 0 \tag{3.59}$$

One might wonder that since we are considering the fluctuations with $k \sim \mathcal{O}(\sqrt{D})$, there might be instances where the subleading correction terms in the membrane equations of motion will contribute at the same order as the terms present in (3.59) and (3.58). But one can carefully think that this will not happen. The only type of potentially dangerous terms are where we have $(\hat{\nabla}^2)^n$ (with $n > 0$) in the numerator. But since action of each $\hat{\nabla}^2$ raises the order of the term by D, there would be a corresponding factor of D^n in the denominator, and thus the order of this term will be still subleading compared to terms present in (3.59) and (3.58), even if $k \sim \mathcal{O}(\sqrt{D})$.

Now we find the scalar and vector QNMs of the membrane. Finding $\partial^a E_a^{tot}$ and substituting (3.58) we get

$$-\partial^b \partial_b \delta r - \partial_t \partial^b \partial_b \delta r + D \partial_t^2 \delta r - \partial_t \partial^b \partial_b \delta r + \frac{1}{D} \partial^a \partial_a \partial^b \partial_b \delta r = 0 \tag{3.60}$$

We can compare (3.60) with the analogous equation that can be derived for black p-brane in flat space as was done in [36] and we note that the equation remains the same as in [36] except that now the first term has negative sign. As we will see this sign difference implies that there is no instability in shape fluctuations unlike in the case of black p-brane in flat space.

Now we consider the plane wave expansion of the fluctuations as

$$\delta r = \delta r^0 e^{-i\omega t} e^{i k_a x^a} \tag{3.61}$$

Thus substituting (3.61) into (3.60) and solving we get the scalar QNM frequencies

$$\omega_s = \pm \frac{k}{\sqrt{D}} - \frac{i k^2}{D}, \quad \text{where} \;\; k^2 = k_a k^a \;\; \text{and} \;\; k = \sqrt{k^2} \tag{3.62}$$

Thus the most general solution to (3.60) is given by

$$\delta r = \delta r_1^0 e^{-i\omega_1 t} e^{i k_a x^a} + \delta r_2^0 e^{-i\omega_2 t} e^{i k_a x^a} \tag{3.63}$$

where,

$$\omega_1 = \frac{k}{\sqrt{D}} - \frac{ik^2}{D}, \quad \omega_2 = -\frac{k}{\sqrt{D}} - \frac{ik^2}{D} \tag{3.64}$$

Now we can take the form of the most general solution of δu_a which solves (3.58) and (3.59) as (Note there is only one vector QNM frequency as (3.59) has at max one time derivative acting on δu_a)

$$\delta u_a = \delta r_1^0 V_a^1 e^{-i\omega_1 t} e^{ik_a x^a} + \delta r_2^0 V_a^2 e^{-i\omega_2 t} e^{ik_a x^a} + v_a e^{-i\omega_v t} e^{ik_a x^a} \tag{3.65}$$

where V_a^1 and V_a^2 are vectors in the direction of k_a and v_a is any vector such that $v_a k^a = 0$. Putting (3.65) into (3.58) and (3.59) we get

$$\omega_v = -\frac{ik^2}{D}, \quad V_a^1 = \left(-i + \frac{\sqrt{D}}{k}\right) k_a, \quad V_a^2 = \left(-i - \frac{\sqrt{D}}{k}\right) k_a \tag{3.66}$$

Thus we see that there is no instability in AdS case. We write again the most general solution to the equations (3.59) and (3.58)

$$\begin{aligned} \delta r &= \delta r_1^0 e^{-i\omega_1 t} e^{ik_a x^a} + \delta r_2^0 e^{-i\omega_2 t} e^{ik_a x^a} \\ \delta u_a &= \delta r_1^0 V_a^1 e^{-i\omega_1 t} e^{ik_a x^a} + \delta r_2^0 V_a^2 e^{-i\omega_2 t} e^{ik_a x^a} + v_a e^{-i\omega_3 t} e^{ik_a x^a} \end{aligned} \tag{3.67}$$

where,

$$\begin{aligned} &\omega_1 = \frac{k}{\sqrt{D}} - \frac{ik^2}{D}, \quad \omega_2 = -\frac{k}{\sqrt{D}} - \frac{ik^2}{D}, \quad \omega_3 = -\frac{ik^2}{D}, \quad v_a k^a = 0 \\ &V_a^1 = \left(-i + \frac{\sqrt{D}}{k}\right) k_a, \quad V_a^2 = \left(-i - \frac{\sqrt{D}}{k}\right) k_a \end{aligned} \tag{3.68}$$

From (3.67) and (3.68) we see that the interesting length scale along the x^a directions is $\frac{1}{\sqrt{D}}$, rather than the scale of order unity we take. Next we work in the scaled limit adapted to capture the physics at length scale $\frac{1}{\sqrt{D}}$.

3.3.2 Scaled nonlinear analysis and derivation of effective equations

Now, similar to previous subsection, here we will consider a membrane configuration which has fluctuations about a uniform planar membrane in AdS. Taking hints from the linear analysis of the previous subsection, we consider a particular scaling limit of our membrane equations, like [12]. Like [12], we will consider fluctuations that depend only on time and p number of brane directions (with $p \sim \mathcal{O}(1)$), in the sense that the shape and velocity fluctuations will be function of time and some p spacial coordinates. And also the velocity fluctuations will be only along time and the same p directions. By considering this setup, we then reduce our membrane equations to a pair of nonlinear effective equations, which we then match with those of [12].

We rewrite the background metric (3.47) in the following form

$$ds^2_{(bgd)} = g_{AB}dX^AdX^B = -r^2dt^2 + \frac{dr^2}{r^2} + r^2(dx^a dx_a + dx^i dx_i) \tag{3.69}$$

Where the indices a, b now take p number of values. We will only consider fluctuations in these directions as mentioned above. The indices i, j take $D - p - 2$ number of values.

From the analysis in the previous subsection we could see that if we want the frequency along the time direction to be of order $\mathcal{O}(1)$, then the spatial frequency k has to be very high, of the order of $\mathcal{O}\left(\sqrt{D}\right)$. To zoom into this regime, in this subsection we work with the scaled spatial coordinates $x^a \to y^a = \sqrt{D}\, x^a$. The spatial frequencies in the new coordinate will scale as $\tilde{k}_a = \frac{k_a}{\sqrt{D}}$. Therefore in the regime of interest the frequency along the new space coordinates will be of order one $\tilde{k}^a \sim \mathcal{O}(1)$. So, the AdS Poincare patch metric (3.69) takes the form

$$ds^2 = -r^2dt^2 + \frac{dr^2}{r^2} + r^2(\frac{dy^a dy_a}{D} + dx^i dx_i) \tag{3.70}$$

Now suppose we repeat the linearized analysis of the previous subsection in these new coordinates. We shall normalize the fluctuations such that the components of the velocity vector field in the directions of ∂_{y^a} are of order $\mathcal{O}(1)$. It follows that $[u \cdot dy^a] \sim \mathcal{O}\left(\frac{1}{D}\right)$. In old x^a coordinate we already know the solution of $[u \cdot dx^a]$ (see equation (3.65) and the second equation of (3.67)). Solution in new coordinates will simply be the coordinate transform of the old solution. In other words if we expand the velocity field as

$$u = u_0\ dt + \left(\frac{1}{D}\right) u_1(t, y^b)\ dt + \left(\frac{1}{D}\right) U_a(t, y^b)\ dy^a$$

then it follows that

$$\left(\frac{1}{D}\right) U_a = \frac{1}{\sqrt{D}} \left(\delta r_1^0\ V_a^1 e^{-i\omega_1 t} e^{i\tilde{k}_a y^a} + \delta r_2^0\ V_a^2 e^{-i\omega_2 t} e^{i\tilde{k}_a y^a} + v_a e^{-i\omega_v t} e^{i\tilde{k}_a y^a}\right)$$

In the RHS of the previous equation the extra factor of $\left(\frac{1}{\sqrt{D}}\right)$ comes as a consequence of coordinate transformation. From equation (3.68) we could see V_a^1 and V_a^2 are of order $\mathcal{O}\left(\sqrt{D}\right)$. It follows that in these coordinates if we normalize our fluctuations such that $u \cdot \partial_{y^a}$ is of order $\mathcal{O}(1)$, then for consistency we must have δr_1^0 and δr_2^0 and therefore δr to be of order $\mathcal{O}\left(\frac{1}{D}\right)$. This further implies that $u_1 \sim \mathcal{O}(1)$ (see equation (3.52)).
Now we shall scale the radial coordinate so that in the new coordinate the amplitude of the radial perturbation is of order $\mathcal{O}(1)$.

$$r = 1 + \frac{\rho}{D} \tag{3.71}$$

With this coordinate redefinition the metric (3.70) becomes

$$ds^2_{(bgd)} = g_{AB}dX^AdX^B = -\left(1 + \frac{\rho}{D}\right)^2 dt^2 + \frac{d\rho^2}{D^2\left(1 + \frac{\rho}{D}\right)^2} + \left(1 + \frac{\rho}{D}\right)^2 (\frac{dy^a dy_a}{D} + dx^i dx_i) \tag{3.72}$$

In this scaled coordinate system the fluctuations are of the form

$$\rho = Y(t,y^b), \quad u = u_0 dt + \frac{1}{D}u_1(t,y^b)dt + \frac{1}{D}U_a(t,y^b)dy^a \tag{3.73}$$

where Y, u_1 and U_a are all of order $\mathcal{O}(1)$ in terms of $(1/D)$ expansion. Now we shall substitute these fluctuations in our 'membrane equation' and evaluate it at very leading order in $(1/D)$ expansion. However now we shall not consider any linearization with respect to the amplitude of the fluctuations.
The procedure for evaluating the equation of motion is very similar as in the previous two subsections. So we shall be very brief here.

The metric induced on the membrane worldvolume is given by

$$\begin{aligned} ds^2_{(ind)} &= g^{(ind)}_{\mu\nu} dz^\mu dz^\nu \\ &= -\left(1+\frac{Y}{D}\right)^2 dt^2 + \left(1+\frac{Y}{D}\right)^2 \left(\frac{dy^a dy_a}{D} + dx^i dx_i\right) \\ &\text{where } z^\mu \equiv \text{Coordinates along the membrane} \equiv \{t, y^a, x^i\} \end{aligned} \tag{3.74}$$

We use the notation $\tilde{\nabla}$ for covariant derivative with respect metric (3.74) and $\bar{\nabla}$ for covariant derivative constructed from metric (3.72). In this notation the membrane equation is given by

$$\begin{aligned} &\mathcal{P}^\nu_\mu \left\{ \frac{\tilde{\nabla}^2 u_\nu}{\mathcal{K}} - \frac{\tilde{\nabla}_\nu \mathcal{K}}{\mathcal{K}} + u^\alpha \mathcal{K}_{\alpha\nu} - (u\cdot\tilde{\nabla})u_\nu \right\} = 0 \quad \text{and} \quad \tilde{\nabla}\cdot u = 0 \\ &\text{where} \quad \mathcal{P}_{\mu\nu} = g^{(ind)}_{\mu\nu} + u_\mu u_\nu \end{aligned} \tag{3.75}$$

As before $\mathcal{K}_{\mu\nu}$ is given by the pullback of the extrinsic curvature tensor K_{MN} expressed as a tensor in the full background metric.

$$\begin{aligned} \mathcal{K}_{\mu\nu} &= \left(\frac{\partial X^M}{\partial z^\mu}\right)\left(\frac{\partial X^N}{\partial z^\nu}\right) K_{MN} \\ &\text{where } K_{AB} = \Pi_{AC}\bar{\nabla}_C n_B \\ &\text{where } \Pi_{AC} = g_{AC} - n_A n_C \end{aligned} \tag{3.76}$$

Here we have denoted the set (ρ, t, a, i) by X^M and the set (t, a, i) by z^μ. Now from explicit calculation we see that $K_{\rho N} \sim \mathcal{O}(D^{-2})$ (see appendix B.2.2). Hence here also $\mathcal{K}_{\mu\nu}$ is just the 'truncation' of the K_{MN} evaluated on the membrane surface and the nonzero components are given by

$$\begin{aligned} \mathcal{K}_{tt} &= -\frac{\partial_t^2 Y}{D} - \left(1 + \frac{2Y}{D} - \frac{\partial_a Y \partial^a Y}{2D}\right), \quad \mathcal{K}_{ta} = -\frac{\partial_t \partial_a Y}{D} \\ \mathcal{K}_{ab} &= -\frac{\partial_a \partial_b Y}{D} + \frac{\delta_{ab}}{D}, \quad \mathcal{K}_{ij} = \delta_{ij}\left(1 + \frac{2Y}{D} - \frac{\partial_a Y \partial^a Y}{2D}\right) \\ \mathcal{K} &= -\partial^b \partial_b Y + (D-1) - \frac{\partial_a Y \partial^a Y}{2} \end{aligned} \tag{3.77}$$

where the rest of the components are zero.

Normalization fixes the time component of the velocity field in terms of its space-component.

$$u_t = u_0 + \frac{u_1}{D} = -\left(1 + \frac{Y}{D} + \frac{U^a U_a}{2D}\right) \tag{3.78}$$

The answer for the membrane projector $\mathcal{P}^\mu_\nu = \delta^\mu_\nu + u^\mu u_\nu$ is given by

$$\begin{split} &\mathcal{P}^t_t = -\frac{U^a U_a}{D}, \quad \mathcal{P}^t_a = \frac{U_a}{D}, \quad \mathcal{P}^a_t = -U^a\left(1+\frac{Y}{D}\right)^{-1} - \frac{U^a U^b U_b}{2D}, \\ &\mathcal{P}^a_b = \delta^a_b - \frac{U_a U_b}{D}, \quad \mathcal{P}^i_j = \delta^i_j, \quad \mathcal{P}^i_t = \mathcal{P}^t_i = \mathcal{O}(D^{-2}), \quad \mathcal{P}^i_a = \mathcal{P}^a_i = \mathcal{O}(D^{-2}) \end{split} \tag{3.79}$$

Now, we denote the vector membrane equation (the 1st equation in (3.75)) as

$$\begin{split} &E^{tot}_\mu \equiv \mathcal{P}^\nu_\mu E_\nu \\ \text{where} \quad &E_\mu \equiv \frac{\tilde{\nabla}^2 u_\mu}{\mathcal{K}} - \frac{\tilde{\nabla}_\mu \mathcal{K}}{\mathcal{K}} + u^\nu K_{\nu\mu} - u^\nu \tilde{\nabla}_\nu u_\mu \end{split}$$

Note that E_i and E^{tot}_i won't contribute in the leading order analysis as both the background and the fluctuations satisfy translational symmetry along x^i directions. Also we know from explicit computation that the leading terms in E_a are of order $\mathcal{O}\left(\frac{1}{D}\right)$. Thus it is easy to see that we only need to evaluate the quantities E_t to order $\mathcal{O}(1)$.

At leading order it is easy to see that $E^{tot}_t = -U^a E^{tot}_a$. Thus the only independent components of the membrane equation are along y^a directions. The relevant terms are given by (keeping terms of leading order in $1/D$)

$$\frac{\tilde{\nabla}_t \mathcal{K}}{\mathcal{K}} = \mathcal{O}(D^{-1}), \quad \frac{\tilde{\nabla}^2 u_t}{\mathcal{K}} = \mathcal{O}(D^{-1}), \quad (u\cdot\mathcal{K})_t = -1, \quad u.\tilde{\nabla} u_t = \mathcal{O}(D^{-1}) \tag{3.80}$$

$$\begin{split} &\frac{\tilde{\nabla}_a \mathcal{K}}{\mathcal{K}} = -\frac{\partial_a \partial^b \partial_b Y}{D} - \frac{\partial^b Y \partial_a \partial_b Y}{D}, \quad \frac{\tilde{\nabla}^2 u_a}{\mathcal{K}} = \frac{\partial^b \partial_b U_a}{D} + \frac{\partial^b Y \partial_b U_a}{D}, \\ &(u\cdot\mathcal{K})_a = -\frac{\partial_t \partial_a Y}{D} - \frac{U^b \partial_b \partial_a Y}{D} + \frac{U_a}{D}, \quad u.\tilde{\nabla} u_a = \frac{\partial_t U_a}{D} + \frac{\partial_a Y}{D} + \frac{U^b \partial_b U_a}{D} \end{split} \tag{3.81}$$

Thus we now can evaluate E^{tot}_a and is given by

$$\begin{split} E^v_a \equiv\ &\partial^b \partial_b U_a - \partial_a Y - U^b \partial_b U_a + \partial^b Y \partial_b U_a - U^b \partial_b \partial_a Y + \partial^b Y \partial_a \partial_b Y \\ &+ \partial_a \partial^b \partial_b Y - \partial_t U_a - \partial_t \partial_a Y = 0 \end{split} \tag{3.82}$$

Also the equation $\tilde{\nabla}.u$ evaluates to

$$E^s \equiv \partial_b U^b + \partial_t Y + U_b \partial^b Y = 0 \tag{3.83}$$

Equations (3.82) and (3.83) are the two effective equations for the membrane variables (membrane's shape and the velocity field on it) at leading order. Now we shall perform a variable redefinition which will recast these equation exactly as given in [12]. Consider the change of variables of the form

$$\begin{aligned} Y(t,a) &= \log m(t,a) \\ U_a(t,a) &= \frac{p_a(t,a) - \partial_a m(t,a)}{m(t,a)} \end{aligned} \tag{3.84}$$

After performing this change in the effective equations (3.82) and (3.83), it can be shown that the linear combinations

$$E_t \equiv m(t,a)E^s \quad E_a \equiv p_a(t,a)E^s - m(t,a)E^v_a \tag{3.85}$$

of the equations (3.82) and (3.83) are just the effective equations derived in [12] which are (in the notations used in this chapter)

$$\begin{aligned} &\partial_t m - \partial^b\partial_b m + \partial_a p^a = 0 \\ &\partial_t p_a - \partial^b\partial_b p_a + \partial_a m + \partial^b\left(\frac{p_a p_b}{m}\right) = 0 \end{aligned} \tag{3.86}$$

Thus we see from this procedure that the effective equations derived in [12] are just a particular scaling limit of our membrane equations. Note that our leading order membrane equations were derived in systematic expansion in $1/D$. But in this section we have rescaled our membrane equation to consider the length scales of order $\frac{1}{\sqrt{D}}$. Strictly speaking, this is beyond the regime of validity of our membrane equations. But it can be argued that this scaling is a consistent thing to do, in particular, the most general terms that can be written as subleading corrections to our membrane equations do not get so much enhanced so as they become comparable to the leading order terms in the membrane equations. The argument goes just like given in the Discussion of [36] for black p-brane in flat spacetime and we won't rewrite here.

3.4 Discussion

In this chapter, we have used 'large D' techniques to find new dynamical 'black hole' solution to pure Einstein equations in presence of cosmological constant. The solutions are determined in an expansion in $\left(\frac{1}{D}\right)$ and are in 'one-to-one' correspondence with a dynamical membrane (characterized by its shape and a velocity field on it) embedded in the asymptotic geometry (which could be AdS or dS).
The method we have used is manifestly covariant with respect to this asymptotic geometry (which we have referred to as 'background'). We do not need to choose any coordinate system for the background geometry at any point of our derivation. The same calculation works for both global AdS and Poincare patch. The form of the final answer also remains invariant. However, they are different solutions with different asymptotic geometries and horizon topologies and this fact is encoded in the various covariant derivatives that appear in the final solution. These covariant derivatives are always defined with respect to the background.

We have applied this method to calculate the metric and the governing equation for the dual dynamical membrane upto the first subleading correction. We have performed several checks for our universal coordinate independent answer, by specializing to different coordinate systems.

- We have linearized our membrane equations and matched them against the known spectrum of black hole/brane QNMs in AdS space and black hole QNMs in dS space.
- We have taken a special scaling limit of our equations and recovered the dual effective equations that was determined in [12] for the AdS black-branes in large number of dimensions.

One could generalize this analysis to Einstein-Maxwell system in presence of cosmological constant. This was eventually done in [37]. It would be interesting to calculate the stress tensor and the entropy current for this system following the method developed in [14].

For AdS space we know there exists another perturbative technique to construct new gravity solution that are dual to fluid dynamics in one lower dimension [38]. This duality works in any dimension [16] and therefore in particular large number dimensions where we can also apply $\left(\frac{1}{D}\right)$ expansion. It is interesting to see how these two perturbative techniques could be patched together. Such a comparison was eventually done in [39].

Chapter 4

Introducing cosmological constant: Second order calculation

4.1 Introduction

In this Chapter, we extend the calculation of the Chapter 3 to the second order. The key motivation is two-fold. Firstly from the result of Chapter 3 we know that at the first order, the background curvature does not appear explicitly in any of the equation or the solution. However it can appear explicitly at second order (which, very roughly speaking, captures the effect of two derivatives on the background). Secondly from the experience of the 'flat space computation' of Chapter 2, it is expected that at this order we should see the entropy production from a dynamical black hole. We shall confine ourselves only to the computation of the membrane equations of motion upto the second order in $\left(\frac{1}{D}\right)$ expansion. We leave the 'study of entropy production' for future.

The setup of the perturbation theory is same as described in Introduction to Chapter 3. The procedure at the second order is straightforward extension of first order. Following the perturbative procedure we systematically solve all the Einstein equations upto the second order and find the metric corrections and membrane equations of motion. We find that the membrane equations of motion at second order are given as

$$\begin{split}
&\left[\frac{\hat{\nabla}^2 u_\alpha}{\mathcal{K}} - \frac{\hat{\nabla}_\alpha \mathcal{K}}{\mathcal{K}} + u^\beta \mathcal{K}_{\beta\alpha} - u\cdot\hat{\nabla} u_\alpha\right]\mathcal{P}^\alpha_\gamma + \Bigg[-\frac{u^\beta \mathcal{K}_{\beta\delta}\mathcal{K}^\delta_\alpha}{\mathcal{K}} + \frac{\hat{\nabla}^2\hat{\nabla}^2 u_\alpha}{\mathcal{K}^3} - \frac{(\hat{\nabla}_\alpha\mathcal{K})(u\cdot\hat{\nabla}\mathcal{K})}{\mathcal{K}^3} \\
&- \frac{(\hat{\nabla}_\beta\mathcal{K})(\hat{\nabla}^\beta u_\alpha)}{\mathcal{K}^2} - \frac{2\mathcal{K}^{\delta\sigma}\hat{\nabla}_\delta\hat{\nabla}_\sigma u_\alpha}{K^2} - \frac{\hat{\nabla}_\alpha\hat{\nabla}^2\mathcal{K}}{\mathcal{K}^3} + \frac{\hat{\nabla}_\alpha(\mathcal{K}_{\beta\delta}\mathcal{K}^{\beta\delta}\mathcal{K})}{K^3} + 3\frac{(u\cdot\mathcal{K}\cdot u)(u\cdot\hat{\nabla}u_\alpha)}{\mathcal{K}} \\
&- 3\frac{(u\cdot\mathcal{K}\cdot u)(u^\beta\mathcal{K}_{\beta\alpha})}{\mathcal{K}} - 6\frac{(u\cdot\hat{\nabla}\mathcal{K})(u\cdot\hat{\nabla}u_\alpha)}{\mathcal{K}^2} + 6\frac{(u\cdot\hat{\nabla}\mathcal{K})(u^\beta\mathcal{K}_{\beta\alpha})}{\mathcal{K}^2} + 3\frac{u\cdot\hat{\nabla}u_\alpha}{D-3} \\
&- 3\frac{u^\beta\mathcal{K}_{\beta\alpha}}{D-3} - \frac{(D-1)\lambda}{\mathcal{K}^2}\left(\frac{\hat{\nabla}_\alpha\mathcal{K}}{\mathcal{K}} - 2u^\sigma\mathcal{K}_{\sigma\alpha} + 2(u\cdot\hat{\nabla})u_\alpha\right)\Bigg]\mathcal{P}^\alpha_\gamma = \mathcal{O}\left(\frac{1}{D}\right)^2
\end{split} \tag{4.1}$$

$$\hat{\nabla}\cdot u - \frac{1}{2\mathcal{K}}\left(\hat{\nabla}_{(\alpha}u_{\beta)}\hat{\nabla}_{(\gamma}u_{\delta)}\mathcal{P}^{\beta\gamma}\mathcal{P}^{\alpha\delta}\right) = \mathcal{O}\left(\frac{1}{D}\right)^2 \tag{4.2}$$

We observe that the vector equation (4.1) contains terms with explicit factors of the cosmological constant of the background spacetime. The covariant derivatives in (4.1) and (4.2) are with respect to the induced metric on the membrane surface considered as embedded in the background metric which solves Einstein equations with a cosmological constant. Note that (4.1) and (4.2) reduce to the results of the Chapter 2 for flat background in the $\Lambda = 0$ case.

In the rest of this Chapter, we check the correctness of the membrane equations (4.1) and (4.2) by linearizing around appropriate static solutions and comparing the spectrum with that of the low lying QNMs (already determined in [7]) for Schwarzschild black hole in AdS/dS and for black brane in AdS. We shall find a perfect match upto the relevant order.

4.2 Quasinormal Modes for Schwarzschild black hole in background AdS/dS spacetime

As a check for our membrane equations (4.1) and (4.2), we will calculate the light quasinormal mode frequencies for Schwarzschild black hole in AdS/dS background. As expected, we find that the answers for the frequencies of light quasinormal modes match exactly with those derived in [7] from gravitational analysis.

We shall follow [11] for the computation. Many steps and arguments are exactly same as in [11]. For such steps we shall simply refer to [11] or quote them in the appendix. And here we shall present only those parts of computation where we have to do some extension of what has been done in [11].

We shall write the background AdS/dS in global coordinates as

$$ds^2_{(bgd)} = g_{AB}dX^A dX^B = -\left(1-\sigma\frac{r^2}{L^2}\right)dt^2 + \frac{dr^2}{\left(1-\sigma\frac{r^2}{L^2}\right)} + r^2 d\Omega^2_{D-2}. \tag{4.3}$$

Where

$$\begin{aligned}
\Lambda &= \frac{\sigma}{L^2}(D-1)(D-2)\\
L &= \text{AdS/dS radius}\\
\sigma &= 0 \quad \text{for Flat}\\
&= 1 \quad \text{for dS}\\
&= -1 \quad \text{for AdS}
\end{aligned} \tag{4.4}$$

And the Schwarzschild black hole in this coordinate system is

$$ds^2_{(BH)} = -\left(1-\sigma\frac{r^2}{L^2}-\left(\frac{r_0}{r}\right)^{D-3}\right)dt^2 + \frac{dr^2}{\left(1-\sigma\frac{r^2}{L^2}-\left(\frac{r_0}{r}\right)^{D-3}\right)} + r^2 d\Omega^2_{D-2}. \tag{4.5}$$

Where, r_0 is an arbitrary constant. Note that the position of horizon is $r = r_H$ where r_H is the zero of the function $f(r) = \left(1 - \sigma\frac{r^2}{L^2} - \left(\frac{r_0}{r}\right)^{D-3}\right)$.

$$r_H = r_0\left(1 - \frac{1}{D}\ln\left(1 - \frac{\sigma r_0^2}{L^2}\right) + \mathcal{O}(D^{-2})\right) \tag{4.6}$$

From now on we choose $r_H = 1$ or in other words r_0 will be set to

$$r_0 = \left(1 + \frac{1}{D}\ln\left(1 - \frac{\sigma}{L^2}\right) + \mathcal{O}(D^{-2})\right)$$

for convenience. We will later reinstate the factors of r_0 from dimensional analysis.

A small fluctuation around a static black hole corresponds to a small fluctuation around a spherical membrane along with a small fluctuation in the velocity field, which is purely in the time direction at zeroth order. We will work upto linear order in the amplitude of fluctuations, which we denote by ϵ.

$$\begin{aligned} r &= 1 + \epsilon\ \delta r(t,a) \\ u &= u_0\ dt + \epsilon\ \delta u_\mu(t,a) dx^\mu \end{aligned} \tag{4.7}$$

Here, we denote the angle coordinates along $(D-2)$ dimensional sphere by a and the coordinates μ on the membrane worldvolume contain time t and angles a. The induced metric on the membrane worldvolume (viewed as a hypersurface embedded in the background metric (4.3)) upto linear order in ϵ is (where we denote the metric components by $g^{(ind)}_{\mu\nu}$)

$$ds^2_{(ind)} = g^{(ind)}_{\mu\nu} dy^\mu dy^\nu = -\left(1 - \sigma\frac{1+2\epsilon\delta r}{L^2}\right)dt^2 + (1+2\epsilon\delta r)d\Omega^2_{D-2} \tag{4.8}$$

Also, $u_\mu g^{\mu\nu}_{(ind)} u_\nu = -1$ implies

$$u_0 = -\left(1 - \frac{\sigma}{L^2}\right)^{\frac{1}{2}} \quad \text{and} \quad \delta u_t(t,a) = \left(1 - \frac{\sigma}{L^2}\right)^{-\frac{1}{2}}\left(\frac{\sigma}{L^2}\right)\delta r(t,a) \tag{4.9}$$

The membrane equations (4.1) and (4.2) are

$$\hat{\nabla}\cdot u = \frac{1}{2\mathcal{K}}\left(\hat{\nabla}_{(\alpha}u_{\beta)}\hat{\nabla}_{(\gamma}u_{\delta)}\mathcal{P}^{\beta\gamma}\mathcal{P}^{\alpha\delta}\right) \tag{4.10}$$

$$\begin{aligned} E^{tot}_\mu \equiv &\left[\frac{\hat{\nabla}^2 u_\alpha}{\mathcal{K}} - \frac{\hat{\nabla}_\alpha\mathcal{K}}{\mathcal{K}} + u^\beta\mathcal{K}_{\beta\alpha} - u\cdot\hat{\nabla}u_\alpha\right]\mathcal{P}^\alpha_\gamma + \left[-\frac{u^\beta\mathcal{K}_{\beta\delta}\mathcal{K}^\delta_\alpha}{\mathcal{K}} + \frac{\hat{\nabla}^2\hat{\nabla}^2 u_\alpha}{\mathcal{K}^3}\right. \\ &- \frac{(\hat{\nabla}_\alpha\mathcal{K})(u\cdot\hat{\nabla}\mathcal{K})}{\mathcal{K}^3} - \frac{(\hat{\nabla}_\beta\mathcal{K})(\hat{\nabla}^\beta u_\alpha)}{\mathcal{K}^2} - \frac{2\mathcal{K}^{\delta\sigma}\hat{\nabla}_\delta\hat{\nabla}_\sigma u_\alpha}{K^2} - \frac{\hat{\nabla}_\alpha\hat{\nabla}^2\mathcal{K}}{\mathcal{K}^3} + \frac{\hat{\nabla}_\alpha(\mathcal{K}_{\beta\delta}\mathcal{K}^{\beta\delta}\mathcal{K})}{K^3} \\ &+ 3\frac{(u\cdot\mathcal{K}\cdot u)(u\cdot\hat{\nabla}u_\alpha)}{\mathcal{K}} - 3\frac{(u\cdot\mathcal{K}\cdot u)(u^\beta\mathcal{K}_{\beta\alpha})}{\mathcal{K}} - 6\frac{(u\cdot\hat{\nabla}\mathcal{K})(u\cdot\hat{\nabla}u_\alpha)}{\mathcal{K}^2} + 6\frac{(u\cdot\hat{\nabla}\mathcal{K})(u^\beta\mathcal{K}_{\beta\alpha})}{\mathcal{K}^2} \\ &\left.+ 3\frac{u\cdot\hat{\nabla}u_\alpha}{D-3} - 3\frac{u^\beta\mathcal{K}_{\beta\alpha}}{D-3} - \frac{(D-1)\lambda}{\mathcal{K}^2}\left(\frac{\hat{\nabla}_\alpha\mathcal{K}}{\mathcal{K}} - 2u^\sigma\mathcal{K}_{\sigma\alpha} + 2(u\cdot\hat{\nabla})u_\alpha\right)\right]\mathcal{P}^\alpha_\gamma = 0 \end{aligned} \tag{4.11}$$

In (4.10) and (4.11), the covariant derivative with respect to metric (4.8) is denoted by $\hat{\nabla}$. The extrinsic curvature of membrane is denoted by $\mathcal{K}_{\mu\nu}$ and its trace by $\mathcal{K}$. The projector orthogonal to u_μ is denoted by $\mathcal{P}_{\mu\nu}$.

It turns out that E_t^{tot} vanishes at linear order in ϵ. Using (C.14) and (C.15), we evaluate the vector membrane equation (4.11) in the angular directions

$$
\begin{aligned}
E_a^{tot} \equiv & \left[(D-2) - \left(1-\frac{\sigma}{L^2}\right)^{-1}\frac{\sigma}{L^2}\right]^{-1}\left[-\left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}(\epsilon\partial_t^2\delta u_a) + \left(1-\frac{\sigma}{L^2}\right)^{-1}\frac{\sigma}{L^2}(\epsilon\partial_t\bar{\nabla}_a\delta r)\right. \\
& \left. + \left(1-\frac{\sigma}{L^2}\right)^{\frac{1}{2}}\epsilon\bar{\nabla}^2\delta u_a + \epsilon\partial_t\bar{\nabla}_a\delta r\right] - \left[(D-2) - \left(1-\frac{\sigma}{L^2}\right)^{-1}\frac{\sigma}{L^2}\right]^{-1}\times \\
& \left[\left(1-\frac{\sigma}{L^2}\right)^{-1}(\epsilon\partial_t^2\bar{\nabla}_a\delta r) - \left(1-\frac{\sigma}{L^2}\right)^{-1}\frac{\sigma}{L^2}(\epsilon\bar{\nabla}_a\delta r) - \epsilon\bar{\nabla}_a\bar{\nabla}^2\delta r - (D-2)(\epsilon\bar{\nabla}_a\delta r)\right] \\
& + \left[\left(1-\frac{\sigma}{L^2}\right)^{-1}(-\epsilon\partial_t\bar{\nabla}_a\delta r) + \left(1-\frac{\sigma}{L^2}\right)^{\frac{1}{2}}(\epsilon\delta u_a) + \left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}\frac{\sigma}{L^2}\delta u_a\right] \\
& - \left[\left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}(\epsilon\partial_t\delta u_a) - \left(1-\frac{\sigma}{L^2}\right)^{-1}\frac{\sigma}{L^2}(\epsilon\bar{\nabla}_a\delta r)\right] \\
& - \frac{1}{D-2}\left[\left(1-\frac{\sigma}{L^2}\right)^{-2}\frac{\sigma}{L^2}\epsilon\partial_t\bar{\nabla}_a\delta r - \left(1-\frac{\sigma}{L^2}\right)^{-1}\epsilon\partial_t\bar{\nabla}_a\delta r + \left(1-\frac{\sigma}{L^2}\right)^{\frac{1}{2}}\epsilon\delta u_a\right. \\
& \left. - \left(\frac{\sigma}{L^2}\right)^2\left(1-\frac{\sigma}{L^2}\right)^{-\frac{3}{2}}\delta u_a\right] + \frac{1}{(D-2)^3}\left(1-\frac{\sigma}{L^2}\right)^{-\frac{3}{2}}\left[\bar{\nabla}^2\bar{\nabla}^2\delta u_a\right] \\
& - \frac{2}{(D-2)^2}\left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}\epsilon\hat{\nabla}^2\delta u_a - \frac{1}{(D-2)^3}\left(1-\frac{\sigma}{L^2}\right)^{-2}\left[-\hat{\nabla}_a\hat{\nabla}^2\hat{\nabla}^2\delta r - (D-2)\hat{\nabla}_a\hat{\nabla}^2\delta r\right] \\
& + \frac{1}{(D-2)^3}\left(1-\frac{\sigma}{L^2}\right)^{-1}\left[-3(D-2)\epsilon\left(\hat{\nabla}_a\hat{\nabla}^2\delta r + (D-2)\hat{\nabla}_a\delta r\right)\right] \\
& + \frac{3}{D-2}\left[\left(1-\frac{\sigma}{L^2}\right)^{-\frac{3}{2}}\frac{\sigma}{L^2}\epsilon\partial_t\delta u_a - \left(1-\frac{\sigma}{L^2}\right)^{-2}\left(\frac{\sigma}{L^2}\right)^2\epsilon\hat{\nabla}_a\delta r\right] \\
& - \frac{3}{D-2}\left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}\left[-\left(1-\frac{\sigma}{L^2}\right)^{-\frac{3}{2}}\left(\frac{\sigma}{L^2}\right)\epsilon\partial_t\hat{\nabla}_a\delta r + \frac{\sigma}{L^2}\epsilon\delta u_a + \left(\frac{\sigma}{L^2}\right)^2\left(1-\frac{\sigma}{L^2}\right)^{-1}\delta u_a\right] \\
& + \frac{3}{D-2}\left[\left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}(\epsilon\partial_t\delta u_a) - \left(1-\frac{\sigma}{L^2}\right)^{-1}\frac{\sigma}{L^2}(\epsilon\bar{\nabla}_a\delta r)\right] \\
& - \frac{3}{D-2}\left[\left(1-\frac{\sigma}{L^2}\right)^{-1}(-\epsilon\partial_t\bar{\nabla}_a\delta r) + \left(1-\frac{\sigma}{L^2}\right)^{\frac{1}{2}}(\epsilon\delta u_a) + \frac{\sigma}{L^2}\left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}\delta u_a\right] \\
& - \frac{2}{D-2}\frac{\sigma}{L^2}\left[\left(1-\frac{\sigma}{L^2}\right)^{-\frac{3}{2}}(\epsilon\partial_t\delta u_a) - \left(1-\frac{\sigma}{L^2}\right)^{-2}\frac{\sigma}{L^2}(\epsilon\bar{\nabla}_a\delta r)\right] \\
& + \frac{2}{D-2}\frac{\sigma}{L^2}\left[\left(1-\frac{\sigma}{L^2}\right)^{-2}(-\epsilon\partial_t\bar{\nabla}_a\delta r) + \left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}(\epsilon\delta u_a) + \frac{\sigma}{L^2}\left(1-\frac{\sigma}{L^2}\right)^{-\frac{3}{2}}\delta u_a\right] \\
& - \frac{1}{D-2}\frac{\sigma}{L^2}\left(1-\frac{\sigma}{L^2}\right)^{-2}\left[\frac{1}{D-2}\left(-\epsilon\bar{\nabla}_a\bar{\nabla}^2\delta r - (D-2)\epsilon\bar{\nabla}_a\delta r\right)\right]
\end{aligned}
\tag{4.12}
$$

Where, in (4.12) we have neglected the terms which are order $\mathcal{O}\left(1/D^2\right)$ or higher. We denote the covariant derivative with respect to a unit sphere metric in $D-2$ dimensions by $\bar{\nabla}_a$. Similarly we evaluate the membrane equation (4.10)

$$\hat{\nabla}.u = \epsilon\bar{\nabla}^a\delta u_a + \epsilon\left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}(\partial_t\delta r)(D-2) = 0 \tag{4.13}$$

We choose to divide the fluctuation δu_a in two parts (see Section (5) of [9])

$$\delta u_a = \delta v_a + \bar{\nabla}_a\Phi\ , \quad \text{with} \quad \bar{\nabla}^a\delta v_a = 0 \tag{4.14}$$

Substituting (4.14) into (4.13) we find

$$\bar{\nabla}^2\Phi = -(D-2)\left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}(\partial_t\delta r) \tag{4.15}$$

Now we evaluate $\bar{\nabla}^a E^{tot}_a$. We use the identity $\bar{\nabla}^a\bar{\nabla}^2 V_a = ((D-3)+\bar{\nabla}^2)\bar{\nabla}^a V_a$ for simplification. We find

$$\begin{aligned}
\bar{\nabla}^a E^{tot}_a \equiv& \left[(D-2)-\left(1-\frac{\sigma}{L^2}\right)^{-1}\frac{\sigma}{L^2}\right]^{-1}\left[(D-2)\left(1-\frac{\sigma}{L^2}\right)^{-1}(\epsilon\partial_t^3\delta r)\right.\\
&\left.+\left(1-\frac{\sigma}{L^2}\right)^{-1}\frac{\sigma}{L^2}(\epsilon\partial_t\bar{\nabla}^2\delta r)-(D-2)\epsilon\partial_t(D-3+\bar{\nabla}^2)\delta r+(\epsilon\partial_t\bar{\nabla}^2\delta r)\right]\\
&-\left[(D-2)-\left(1-\frac{\sigma}{L^2}\right)^{-1}\frac{\sigma}{L^2}\right]^{-1}\left[\left(1-\frac{\sigma}{L^2}\right)^{-1}(\epsilon\partial_t^2\bar{\nabla}^2\delta r)-\left(1-\frac{\sigma}{L^2}\right)^{-1}\frac{\sigma}{L^2}\epsilon\bar{\nabla}^2\delta r\right.\\
&\left.-\epsilon\bar{\nabla}^2\bar{\nabla}^2\delta r-(D-2)\epsilon\bar{\nabla}^2\delta r\right]+\frac{2}{(D-2)}\left(1-\frac{\sigma}{L^2}\right)^{-1}\left[\epsilon(D-3+\hat{\nabla}^2)\partial_t\delta r\right]\\
&+\left[\left(1-\frac{\sigma}{L^2}\right)^{-1}(-\epsilon\partial_t\bar{\nabla}^2\delta r)-(D-2)\epsilon\partial_t\delta r-(D-2)\left(1-\frac{\sigma}{L^2}\right)^{-1}\frac{\sigma}{L^2}\partial_t\delta r\right]\\
&-\left[-(D-2)\left(1-\frac{\sigma}{L^2}\right)^{-1}(\epsilon\partial_t^2\delta r)-\left(1-\frac{\sigma}{L^2}\right)^{-1}\frac{\sigma}{L^2}(\epsilon\bar{\nabla}^2\delta r)\right]\\
&-\frac{1}{D-2}\left[\left(1-\frac{\sigma}{L^2}\right)^{-2}\frac{\sigma}{L^2}\epsilon\partial_t\bar{\nabla}^2\delta r-\left(1-\frac{\sigma}{L^2}\right)^{-1}\epsilon\partial_t\bar{\nabla}^2\delta r-(D-2)\epsilon\partial_t\delta r\right.\\
&\left.+(D-2)\left(\frac{\sigma}{L^2}\right)^2\left(1-\frac{\sigma}{L^2}\right)^{-2}\partial_t\delta r\right]\\
&-\frac{1}{(D-2)^2}\left(1-\frac{\sigma}{L^2}\right)^{-2}\epsilon\left[(D-3+\bar{\nabla}^2)(D-3+\bar{\nabla}^2)\partial_t\delta r\right]\\
&-\frac{1}{(D-2)^3}\left(1-\frac{\sigma}{L^2}\right)^{-2}\left[-\hat{\nabla}^2\hat{\nabla}^2\hat{\nabla}^2\delta r-(D-2)\hat{\nabla}^2\hat{\nabla}^2\delta r\right]\\
&-\frac{1}{(D-2)^2}\left(1-\frac{\sigma}{L^2}\right)^{-1}\left[3\epsilon\left(\hat{\nabla}^2\hat{\nabla}^2\delta r+(D-2)\hat{\nabla}^2\delta r\right)\right]\\
&+\frac{3}{D-2}\left(1-\frac{\sigma}{L^2}\right)^{-2}\left[-(D-2)\frac{\sigma}{L^2}\epsilon\partial_t^2\delta r-\left(\frac{\sigma}{L^2}\right)^2\epsilon\hat{\nabla}^2\delta r\right]
\end{aligned}$$

$$
\begin{aligned}
&-\frac{3}{D-2}\Bigg[-\left(1-\frac{\sigma}{L^2}\right)^{-2}\left(\frac{\sigma}{L^2}\right)\epsilon\partial_t\hat{\nabla}^2\delta r-(D-2)\frac{\sigma}{L^2}\left(1-\frac{\sigma}{L^2}\right)^{-1}\epsilon\partial_t\delta r\\
&-(D-2)\left(\frac{\sigma}{L^2}\right)^2\left(1-\frac{\sigma}{L^2}\right)^{-2}\partial_t\delta r\Bigg]+\frac{3}{D-2}\left(1-\frac{\sigma}{L^2}\right)^{-1}\left[-(D-2)(\epsilon\partial_t^2\delta r)-\frac{\sigma}{L^2}(\epsilon\bar{\nabla}^2\delta r)\right]\\
&-\frac{3}{D-2}\left[\left(1-\frac{\sigma}{L^2}\right)^{-1}(-\epsilon\partial_t\bar{\nabla}^2\delta r)-(D-2)(\epsilon\partial_t\delta r)-(D-2)\frac{\sigma}{L^2}\left(1-\frac{\sigma}{L^2}\right)^{-1}\partial_t\delta r\right]\\
&-\frac{2}{D-2}\frac{\sigma}{L^2}\left(1-\frac{\sigma}{L^2}\right)^{-2}\left[-(D-2)(\epsilon\partial_t^2\delta r)-\frac{\sigma}{L^2}(\epsilon\bar{\nabla}^2\delta r)\right]\\
&+\frac{2}{D-2}\frac{\sigma}{L^2}\left(1-\frac{\sigma}{L^2}\right)^{-2}\left[(-\epsilon\partial_t\bar{\nabla}^2\delta r)-(D-2)\left(1-\frac{\sigma}{L^2}\right)^{1}(\epsilon\partial_t\delta r)-(D-2)\frac{\sigma}{L^2}\partial_t\delta r\right]\\
&-\frac{1}{D-2}\frac{\sigma}{L^2}\left(1-\frac{\sigma}{L^2}\right)^{-2}\left[\frac{1}{D-2}\Big((-\epsilon\bar{\nabla}^2\bar{\nabla}^2\delta r)-(D-2)(\epsilon\bar{\nabla}^2\delta r)\Big)\right]
\end{aligned}
\tag{4.16}
$$

Now we reinstate the factors of r_H. [1] We expand the shape fluctuations

$$
\delta r=\sum_{l,m}a_{lm}Y_{lm}e^{-i\omega_l^s r_H t}
\tag{4.17}
$$

where, Y_{lm} are the scalar spherical harmonics on S^{D-2} for which

$$
\bar{\nabla}^2 Y_{lm}=-l(D+l-3)Y_{lm}.
\tag{4.18}
$$

Now, we substitute (4.17) in (4.16) and solve for the scalar QNM frequencies

$$
\begin{aligned}
\omega^s r_0=\pm\sqrt{l\left(1-\frac{\sigma r_0^2}{L^2}\right)-1}\Bigg[1+\frac{1}{2D}\frac{l-1}{l-\left(1-\frac{\sigma r_0^2}{L^2}\right)^{-1}}\Bigg(\left(2\left(1-\frac{\sigma r_0^2}{L^2}\right)^{-1}+1\right)l\\
-4\left(1-\frac{\sigma r_0^2}{L^2}\right)^{-1}+2\left(1-\frac{\sigma r_0^2}{L^2}\right)^{-1}\ln\left(1-\frac{\sigma r_0^2}{L^2}\right)\Bigg)\Bigg]\\
-i(l-1)\left[1+\frac{1}{D}\left(l-2+\ln\left(1-\frac{\sigma r_0^2}{L^2}\right)\right)\right]
\end{aligned}
\tag{4.19}
$$

Upto the required order, the answer (4.19) agrees with the respective answer given in equations (D.3),(D.4) of [7].

Similarly we now calculate the vector QNM frequencies. Note that we have solved (4.16).

[1] We use the dimensional analysis to replace L by $\frac{L}{r_H}$ and replace ω^s by $\omega^s r_H$. Where, r_H is defined in terms of r_0 in (4.6).

So, the δr and Φ terms in (4.12) will drop out and we have

$$\begin{aligned}
E_a^{tot} \equiv & \left[(D-2)\left(1-\frac{\sigma}{L^2}\right)^{\frac{1}{2}} - \left(1-\frac{\sigma}{L^2}\right)^{\frac{-1}{2}}\frac{\sigma}{L^2}\right]^{-1}\left[-\left(1-\frac{\sigma}{L^2}\right)^{-1}(\epsilon\partial_t^2\delta v_a) + \epsilon\bar{\nabla}^2\delta v_a\right] \\
& + \left[\left(1-\frac{\sigma}{L^2}\right)^{\frac{1}{2}}(\epsilon\delta v_a) + \left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}\frac{\sigma}{L^2}\delta v_a\right] - \left[\left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}(\epsilon\partial_t\delta v_a)\right] \\
& - \frac{1}{D-2}\left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}\left[\left(1-\frac{\sigma}{L^2}\right)\epsilon\delta v_a - \left(\frac{\sigma}{L^2}\right)^2\left(1-\frac{\sigma}{L^2}\right)^{-1}\delta v_a\right] \\
& + \frac{1}{(D-2)^3}\left(1-\frac{\sigma}{L^2}\right)^{-\frac{3}{2}}\left[\bar{\nabla}^2\bar{\nabla}^2\delta v_a\right] - \frac{2}{(D-2)^2}\left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}\left[\epsilon\hat{\nabla}^2\delta v_a\right] \\
& + \frac{3}{D-2}\left(1-\frac{\sigma}{L^2}\right)^{-\frac{3}{2}}\left[\frac{\sigma}{L^2}\epsilon\partial_t\delta v_a\right] + \frac{3}{D-2}\left[\left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}(\epsilon\partial_t\delta v_a)\right] \\
& - \frac{3}{D-2}\left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}\left[\frac{\sigma}{L^2}\epsilon\delta v_a + \left(\frac{\sigma}{L^2}\right)^2\left(1-\frac{\sigma}{L^2}\right)^{-1}\delta v_a\right] \\
& - \frac{3}{D-2}\left[\left(1-\frac{\sigma}{L^2}\right)^{\frac{1}{2}}(\epsilon\delta v_a) + \frac{\sigma}{L^2}\left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}\delta v_a\right] \\
& - \frac{2}{D-2}\frac{\sigma}{L^2}\left[\left(1-\frac{\sigma}{L^2}\right)^{-\frac{3}{2}}(\epsilon\partial_t\delta v_a)\right] \\
& + \frac{2}{D-2}\frac{\sigma}{L^2}\left[\left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}(\epsilon\delta v_a) + \frac{\sigma}{L^2}\left(1-\frac{\sigma}{L^2}\right)^{-\frac{3}{2}}\delta v_a\right]
\end{aligned} \tag{4.20}$$

We expand the δv_a fluctuations as

$$\delta v_a = \sum_{l,m} b_{lm} Y_a^{lm} e^{-i\omega_l^v r_H t} \tag{4.21}$$

where, Y_a^{lm} are the vector spherical harmonics on S^{D-2} for which

$$\bar{\nabla}^2 Y_a^{lm} = -[(D+l-3)l-1]Y_a^{lm} \tag{4.22}$$

We Substitute (4.21) in (4.20) and solve for vector QNM frequencies

$$\omega^v r_0 = -i(l-1)\left[1+\frac{1}{D}\left(l-1+\ln\left(1-\frac{\sigma r_0^2}{L^2}\right)\right)\right] \tag{4.23}$$

Upto the required order, the answer (4.23) agrees with the respective answer given in equation (D.2) of [7].

4.3 Quasinormal Modes for AdS Schwarzschild black brane

Now we shall repeat the above analysis for the case of uniform planar membrane in AdS. This membrane corresponds to AdS Schwarzschild black brane with horizon topology of $R^{D-2} \times R$

in Poincare patch metric. Here we consider membrane fluctuations in time and all the $D-2$ spatial brane directions.

The background metric in Poincare patch coordinates is

$$ds^2 = -\hat{r}^2 d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^2 d\hat{x}^a d\hat{x}_a \tag{4.24}$$

Where we have set AdS radius $L = 1$, i.e. $\Lambda = (D-1)(D-2)$. For our convenience we use the following notation for this section

$$n \equiv D - 1 \tag{4.25}$$

We consider a uniform planar membrane located at the location $\hat{r} = r_0$. We find it convenient to perform the following rescaling

$$\hat{r} = r_0 r, \quad \hat{t} = \frac{t}{r_0}, \quad \hat{x}^a = \frac{x^a}{r_0} \tag{4.26}$$

With this rescaling, the background metric (4.24) becomes

$$ds^2_{(bgd)} = g_{AB} dX^A dX^B = -r^2 dt^2 + \frac{dr^2}{r^2} + r^2 dx^a dx_a \tag{4.27}$$

Where now $r = 1$ is the location of the uniform membrane. We will consider the time dependence of the shape and velocity fluctuations of the form

$$e^{-i\hat{\omega}\hat{t}} = e^{-i\omega t}, \quad \text{where } \hat{\omega} = \omega r_0$$

This choice means that the new coordinates in (4.27) are all dimensionless.

We consider the fluctuations around the uniform planar membrane as

$$\begin{aligned} r &= 1 + \epsilon \delta r(t,a) \\ u &= u_0 dt + \epsilon \delta u_t(t,a) dt + \epsilon \delta u_b(t,a) dx^b \end{aligned} \tag{4.28}$$

Where ϵ is the amplitude of fluctuations and we work upto linear order in ϵ.
Upto linear order, the induced metric on the membrane worldvolume becomes

$$ds^2 = g^{(ind)}_{\mu\nu} dy^\mu dy^\nu = -(1 + 2\epsilon\delta r) dt^2 + (1 + 2\epsilon\delta r) dx^a dx_a \tag{4.29}$$

Upto linear order, $u_\mu g^{\mu\nu}_{(ind)} u_\nu = -1$ implies

$$u_t = u_0 + \epsilon \delta u_t = -(1 + \epsilon \delta r) \tag{4.30}$$

The covariant derivative with respect to induced metric (4.29) is denoted by $\hat{\nabla}$ and of the background metric (4.27) is denoted by ∇. Also $\mathcal{K}_{\mu\nu}$ and $\mathcal{K}$ are defined in the same way as the previous section. So we now again consider the membrane equations (4.10) and (4.11).

Substituting the equations (4.28) and (4.30) in the LHS of (4.11) (see appendix (C.2) for details) we find that E_t^{tot} is of order $\mathcal{O}(\epsilon^2)$, and the 'a' components of the equation becomes

$$\begin{aligned}E_a^{tot} = \epsilon\Bigg[&- \partial_t \delta u_a - \frac{\partial_t^2 \delta u_a}{n} + \frac{\partial^2 \delta u_a}{n} + \frac{\partial_t^4 \delta u_a}{n^3} - 2\frac{\partial_t^2 \partial^2 \delta u_a}{n^3} + \frac{\partial^2 \partial^2 \delta u_a}{n^3} + 2\frac{\partial_t^2 \delta u_a}{n^2} \\ &- 2\frac{\partial^2 \delta u_a}{n^2} + 2\frac{\partial_t \delta u_a}{n}\Bigg] + \Bigg[- \partial_a \delta r - \partial_t \partial_a \delta r - \frac{\partial_a \partial_t^2 \delta r}{n} + \frac{\partial^2 \partial_a \delta r}{n} + 2\frac{\partial_a \partial_t \delta r}{n} \\ &+ \frac{\partial_a \partial_t^4 \delta r - 2\partial_t^2 \partial_a \partial^2 \delta r + \partial_a \partial^2 \partial^2 \delta r}{n^3} + 3\frac{\partial_a \partial_t^2 \delta r - \partial_a \partial^2 \delta r}{n^2} + \frac{\partial_a \partial_t^2 \delta r}{n^2} - \frac{\partial_a \partial^2 \delta r}{n^2} \\ &+ 2\frac{\partial_a \partial_t \delta r}{n} + 2\frac{\partial_a \delta r}{n}\Bigg] = 0\end{aligned} \tag{4.31}$$

Similarly the expansion of equation (4.10) to linear order in fluctuation leads to the following equation

$$\hat{\nabla}.u = 0 = \epsilon \partial^a \delta u_a + \epsilon(n-1)\partial_t \delta r \tag{4.32}$$

Now to find the scalar QNM frequencies, the relevant equations are (4.32) and $\partial^a E_a^{tot}$. Finding $\partial^a E_a^{tot}$ and substituting (4.32) we get

$$\begin{aligned}&- (n-1)\epsilon\Bigg[- \partial_t^2 \delta r - \frac{\partial_t^3 \delta r}{n} + \frac{\partial_t \partial^2 \delta r}{n} + \frac{\partial_t^5 \delta r}{n^3} - 2\frac{\partial_t^3 \partial^2 \delta r}{n^3} + \frac{\partial_t \partial^2 \partial^2 \delta r}{n^3} + 2\frac{\partial_t^3 \delta r}{n^2} \\ &- 2\frac{\partial_t \partial^2 \delta r}{n^2} + 2\frac{\partial_t^2 \delta r}{n}\Bigg] + \Bigg[- \partial^2 \delta r - \partial_t \partial^2 \delta r - \frac{\partial^2 \partial_t^2 \delta r}{n} + \frac{\partial^2 \partial^2 \delta r}{n} + 2\frac{\partial^2 \partial_t \delta r}{n} \\ &+ \frac{\partial^2 \partial_t^4 \delta r - 2\partial_t^2 \partial^2 \partial^2 \delta r + \partial^2 \partial^2 \partial^2 \delta r}{n^3} + 3\frac{\partial^2 \partial_t^2 \delta r - \partial^2 \partial^2 \delta r}{n^2} + \frac{\partial^2 \partial_t^2 \delta r}{n^2} - \frac{\partial^2 \partial^2 \delta r}{n^2} \\ &+ 2\frac{\partial^2 \partial_t \delta r}{n} + 2\frac{\partial^2 \delta r}{n}\Bigg] = 0\end{aligned} \tag{4.33}$$

We consider the plane wave expansion of the shape fluctuations

$$\delta r = \delta r^0 e^{-i\omega t} e^{i k_a x^a} \tag{4.34}$$

We then substitute (4.34) into (4.33) and solve for scalar QNM frequencies (where we take $k \sim \mathcal{O}(\sqrt{n})$)[2]

$$\omega_s = \pm \frac{k}{\sqrt{n}}\left(1 + \frac{1 + 2k^2/n}{2n}\right) - \frac{ik^2}{n}\left(1 - \frac{1}{n}\right), \qquad \text{where } \; k^2 = k_a k^a \; \text{ and } \; k = \sqrt{k^2} \tag{4.35}$$

[2]It turns out, as in 1st order, that the orders of temporal and spatial frequencies are related by factor of $\left(\frac{1}{\sqrt{n}}\right)$. This can be seen from the equation (4.32), where there is a relative factor of $(n-1)$ between the divergence of velocity fluctuations and the shape fluctuations. So we cannot have both the temporal and spacial frequencies of the same order.
Here we demanded that the temporal frequency is of order $\mathcal{O}(1)$, but no restriction was put on the spatial frequencies. Such scaling is consistent with the present $\left(\frac{1}{D}\right)$ expansion. See [11] and [36] for detailed explanation.

Hence we can write the most general solution of (4.33)

$$\delta r = \delta r_1^0 e^{-i\omega_1 t} e^{ik_a x^a} + \delta r_2^0 e^{-i\omega_2 t} e^{ik_a x^a} \tag{4.36}$$

where,

$$\begin{aligned}
\omega_1 &= \frac{k}{\sqrt{n}}\left(1 + \frac{1+2k^2/n}{2n}\right) - \frac{ik^2}{n}\left(1 - \frac{1}{n}\right), \\
\omega_2 &= -\frac{k}{\sqrt{n}}\left(1 + \frac{1+2k^2/n}{2n}\right) - \frac{ik^2}{n}\left(1 - \frac{1}{n}\right)
\end{aligned} \tag{4.37}$$

Similarly, we can write the form of the most general solution of (4.32) and (4.31) (Note that there is only one vector QNM frequency)

$$\delta u_a = \delta r_1^0 V_a^1 e^{-i\omega_1 t} e^{ik_a x^a} + \delta r_2^0 V_a^2 e^{-i\omega_2 t} e^{ik_a x^a} + v_a e^{-i\omega_v t} e^{ik_a x^a} \tag{4.38}$$

where V_a^1 and V_a^2 are vectors along k_a, and v_a is any vector which satisfies $v_a k^a = 0$. Substituting (4.38) into (4.32) and (4.31) and solving we find

$$\begin{aligned}
&\omega_v = -\frac{ik^2}{n}\left(1 + \mathcal{O}(n^{-2})\right), \quad V_a^1 = \left[-i\left(1 - \frac{1}{n}\right) + \frac{\sqrt{n}}{k}\left(1 + \frac{1+2k^2/n}{2n}\right)\right] k_a, \\
&V_a^2 = \left[-i\left(1 - \frac{1}{n}\right) - \frac{\sqrt{n}}{k}\left(1 + \frac{1+2k^2/n}{2n}\right)\right] k_a
\end{aligned} \tag{4.39}$$

Thus, we see that there is no subleading correction to ω_v. Collecting the results for light QNM frequencies

$$\begin{aligned}
\omega_s &= \pm\frac{k}{\sqrt{n}}\left(1 + \frac{1+2k^2/n}{2n}\right) - \frac{ik^2}{n}\left(1 - \frac{1}{n}\right) \\
\omega_v &= -\frac{ik^2}{n}\left(1 + \mathcal{O}(n^{-2})\right)
\end{aligned} \tag{4.40}$$

Upto the required order, the answers (4.40) agree with the respective answers given in equations (4.23),(4.24),(4.25) of [7].

4.4 Discussion

In this chapter we have found new dynamical 'black-hole' type solutions of the Einstein equations in presence of cosmological constant in an expansion in the inverse powers of dimension. We have done the calculation upto second subleading order. The space-time, determined here, will necessarily possess an event horizon. The dynamics of the horizon could be mapped to the dynamics of a velocity field on a dynamical membrane, embedded in the asymptotic background. We have determined the equation for this dual dynamics of the membrane and the velocity field also in an expansion in $\left(\frac{1}{D}\right)$.

There are several directions along which we could proceed from here.
As we have mentioned in the introduction, one of our key motivation for this second subleading calculation is to have some insight in entropy production, which is expected to take

place only at this order. Calculation of this entropy production along with the effective stress tensor for the membrane (see [14] for the stress tensor at first order) could be one immediate project.

As a check we have matched the spectrum of the Quasi-Normal modes. This gives a check on the equation of motion for the membrane. Another important check would be to match the metric with the large dimension limit of known black hole solutions. Apart from just a check on our results, this exercise could also give hints to some exact but non-trivial solutions of our membrane equations. This might lead to some techniques to solve the membrane equation analytically.

Chapter 5

Improved Large D membrane

5.1 Introduction

It has recently been demonstrated that the dynamics of black holes in a large number of dimensions is 'dual' to the motion of a probe membrane [1] propagating without back reaction on any background that solves Einstein's equations. [2] The degrees of freedom of this probe membrane are its shape (one degree of freedom) and a velocity field ($D-2$ degrees of freedom) that lives on its world volume. The membrane hosts a stress tensor which is given in terms of the shape and velocity field. The equations of motion for the membrane variables are generated by the requirement that the membrane stress tensor is conserved. This requirement yields as many equations as variables - and so presumably defines well posed probe dynamics - as we now explain in more detail.

The membrane stress tensor T_{MN} - viewed as a tensor field in the background space time on which the membrane propagates - is delta function localized on the membrane world volume. The tensor indices M and N lie purely 'within' the membrane world volume (i.e. $T_{MN}n^M = 0$ where n^M is the normal to the membrane world volume), so this stress tensor is equally well characterized by its restriction, $T_{\mu\nu}$, to the membrane world volume of the membrane. The membrane is a consistent source for gravitational fluctuations about the background spacetime in which it propagates if and only if its stress tensor field is conserved in spacetime i.e. if

$$E_M = \nabla^N T_{NM} = 0 \tag{5.1}$$

[1]The development of this 'membrane-gravity' correspondence was motivated by early observations and computations [1, 2, 4, 6] by Emparan, Suzuki and Tanabe (EST) (see also [29, 30, 3]). A precise formulation of the duality between black hole motion and the solutions of an initial value problem for membrane motion was presented in [8, 9, 10, 36, 14, 11]. Parallel work developing the effective description of black hole dynamics at large D in various special limits and using it to address physical questions of interest can be found in [5, 7, 17, 18, 22, 21, 23, 19, 28, 12, 25, 26, 24, 20, 27, 31, 40, 41, 42, 43].

[2]The reason that the membrane does not correct the spacetime in which it moves is essentially kinematical . It follows from Newton's law that the 'Coulombic' fields of the membrane die off with distance away from the membrane like $1/r^{D-3} \sim e^{-(D-3)\ln r}$ and so are exponentially small at fixed distances away from the membrane. It turns out that radiation fields from the membrane die off even more rapidly - like $\frac{1}{D^D}$ [14]. Consequently the effect of the membrane on the background geometry is extremely small at distances larger than those of order $\frac{1}{D}$ away from the membrane; this is the case even though the membrane stress tensor is not small at large D.

The projection of (5.1) tangent to the membrane world volume imposes the world volume stress tensor conservation equations [3]

$$\nabla^\mu T_{\mu\nu} = 0 \tag{5.2}$$

On the other hand the normal component of the equation of motion yields

$$n_M E^M \propto T_{\mu\nu} K^{\mu\nu} = 0 \tag{5.3}$$

where $K_{\mu\nu}$ is the extrinsic curvature of the membrane.

(5.2) and (5.3) are D equations for the $D-1$ independent membrane variables. These equations nonetheless define consistent membrane dynamics at large D because it turns out that the form of the large D membrane stress tensor is such that (5.3) is obeyed as an identity order by order in the $\frac{1}{D}$ expansion. If, for instance we insert the leading order membrane stress tensor [14] into the LHS of (5.3) we find that the RHS is of a low enough order in $\frac{1}{D}$ that it can - and presumably will - cancel against the contribution of subleading terms in $T_{\mu\nu}$. In other words the conservation equations (5.1) applied to the leading order stress tensor of [14] yields consistent probe membrane dynamics in a power series expansion in $\frac{1}{D}$. However if the equations of motion are taken literally at any finite D, no matter how large, they are inconsistent and generically have no solutions.

This chapter is devoted to a study of the near equilibrium properties of our membrane. We will find it instructive to perform our analysis at finite D, even though our results are guaranteed to reproduce black hole physics only at large D. This is only possible once we have a formulation of probe membrane dynamics that is self consistent at finite D. It turns out to be not too difficult to find such a formulation. In this chapter we present an 'improved' version of the leading order membrane stress tensor of [14]. Our improved stress tensor reduces to the results of [14] at large D, but differs from it at subleading orders in $\frac{1}{D}$. The improvement is chosen to ensure that the new stress tensor obeys the equation (5.3) as an identity even at finite D. It follows that the equations of motion that follow from the conservation of this stress tensor constitute $D-1$ equations for the $D-1$ membrane variables even at finite D and so presumably define consistent membrane dynamics even at finite D. Moreover the improved stress tensor turns out also to exactly obey a local form of the second law of thermodynamics under certain assumptions. More precisely our improved stress tensor quantitatively reproduces the entropy production equation reported in [10] at leading order in large D.

In the rest of this chapter we first present our improved version of the leading order large D membrane stress tensor of [14]. We then use this stress tensor to study of the properties of the membrane in equilibrium. In particular we demonstrate that all stationary solutions of the resultant membrane equations can be obtained from the extremisation of an action functional of the shape of the membrane. We apply this formalism to simple stationary solutions. Finally, in the case of a background AdS spacetime, we proceed to study the dynamics of our membrane in near equilibrium situations and investigate relationship between our improved large D membrane equations and the equation of fluid gravity.

In the rest of this introduction we present a more detailed outline of the contents of this chapter. To end this subsection, we re emphasize that - as in previous work - the membranes

[3] In the equation below ∇_μ is the covariant derivative on the world volume of the membrane.

of this chapter reproduce black hole motion only at large D limit even though their dynamics is well defined even at finite D. The membrane equations presented in this chapter are just the first term in a systematically improvable approximation to black hole dynamics. Given this fact it is somewhat surprising that the membrane equations presented in this chapter turn out - in simple situations - to reproduce black hole physics better than we had the right to expect, getting some results exactly right even at finite values of D - as we explain below.

5.1.1 The improved membrane stress tensor and resultant equations of motion

Consider a D dimensional bulk spacetime with metric G_{MN} that obeys Einstein's equations with a cosmological constant

$$\bar{R}_{MN} + (D-1)\lambda G_{MN} = 0 \tag{5.4}$$

[4] Consider a codimension one membrane propagating in this spacetime. The membrane stress tensor obtained from the analysis of Einstein's equations at large D was reported in equation 1.10 of [14] as

$$16\pi T_{\mu\nu} = \mathcal{K} u_\mu u_\nu - 2\sigma_{\mu\nu} + K_{\mu\nu} \tag{5.5}$$

upto corrections that are subleading in $1/D$. Here u_μ is a velocity field on the membrane, $\sigma_{\mu\nu}$ is the shear tensor of this velocity field (see (5.36) for a definition), $K_{\mu\nu}$ is the extrinsic curvature of the membrane world volume (see (5.37) for a definition), $\mathcal{K}$ is the trace of the extrinsic curvature.

(5.5) may be rewritten in the form

$$16\pi T_{\mu\nu} = \mathcal{K}\mathcal{P}_{\mu\nu} - 2\sigma_{\mu\nu} + (K_{\mu\nu} - \mathcal{K} g_{\mu\nu}) \tag{5.6}$$

where $g_{\mu\nu}$ is the induced metric on the membrane world volume and $\mathcal{P}_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the projector orthogonal to the membrane velocity.

In this chapter we study the dynamics of membranes governed by the improved stress tensor

$$16\pi T_{\mu\nu} = \tilde{\mathcal{K}}\mathcal{P}_{\mu\nu} - 2\sigma_{\mu\nu} + (K_{\mu\nu} - \mathcal{K} g_{\mu\nu}) \tag{5.7}$$

where

$$\tilde{\mathcal{K}} = \frac{\mathcal{K}^2 - K^{\mu\nu}K_{\mu\nu} + 2K^{\mu\nu}\sigma_{\mu\nu}}{\mathcal{K} + u.K.u} \tag{5.8}$$

It is easily verified that $\tilde{\mathcal{K}}$ reduces to $\mathcal{K}$ in the large D limit defined in [8, 9, 11], and so it follows that (5.7) reduces to (5.6) at leading order in the large D limit. Moreover it is easily verified that the stress tensor (5.7) obeys the equation

$$K_{\mu\nu} T^{\mu\nu} = 0 \tag{5.9}$$

[4] The constant λ in (5.4) is proportional to (minus of) the usual cosmological constant. We have chosen the normalization of λ to ensure that AdS_D with radius $\frac{1}{\sqrt{\lambda}}$ is a solution the equations (5.4) when λ is positive, while de Sitter space with radius $\frac{1}{\sqrt{-\lambda}}$ solves (5.4) when λ is negative. Upon setting $\lambda = 0$ (5.4) reduces to the usual (flat space) vacuum Einstein equations.

as an exact algebraic identity (the same is not true for the stress tensor (5.5)).

We emphasize that (5.7) is the stress tensor that lives on a probe brane that does not back react on the background spacetime. [5]

Note that the stress tensor (5.7) consists of the sum of the identically conserved Brown York stress tensor

$$16\pi T^{BY}_{\mu\nu} = K_{\mu\nu} - \mathcal{K} g_{\mu\nu} \tag{5.10}$$

and the 'fluid' stress tensor

$$16\pi T^{fluid}_{\mu\nu} = \tilde{\mathcal{K}} \mathcal{P}_{\mu\nu} - 2\sigma_{\mu\nu} \tag{5.11}$$

Comparing (5.11) to the standard fluid form of the stress tensor

$$T^{fluid}_{\mu\nu} = \rho u_\mu u_\nu + p\mathcal{P}_{\mu\nu} - 2\eta\sigma_{\mu\nu} \tag{5.12}$$

(here ρ is the fluid energy density, p is its pressure and η its shear viscosity) we see that our membrane fluid has

$$\rho = 0, \quad p = \frac{\tilde{\mathcal{K}}}{16\pi}, \quad \eta = \frac{1}{16\pi} \tag{5.13}$$

It is striking that fluid energy density vanishes identically; it follows immediately that the notion of an intrinsic fluid temperature T is ambiguous and that the fluid entropy density s is a pure number. [6] However the dynamics of the membrane is defined by an interaction between the membrane 'fluid' and its shape - this interaction apparently endows any bit of the membrane with a definite temperature. Indeed the formula for the membrane pressure (5.13) - together with the vanishing of the fluid energy density plus standard thermodynamics - allows us to conclude that $Ts = \frac{\tilde{\mathcal{K}}}{16\pi}$. As we have explained above we expect the entropy density s to be a constant. Below we will see that $s = \frac{1}{4}$ so that $T = \frac{\tilde{\mathcal{K}}}{4\pi}$, where T is the local temperature of the membrane. Note that the temperature - which was left undetermined by the fluid equation of state - is determined by the membrane's local extrinsic geometry. [7] [8] Note also that the viscosity of our membrane obeys the KSS relation [44]

$$\frac{\eta}{s} = \frac{1}{4\pi} \tag{5.14}$$

[5] In other words, in working with (5.7) we multiply the full stress tensor by ϵ, work only to first order in the ϵ expansion and then set ϵ to unity at the end of the computation. The order ϵ back reaction of the membrane on the background spacetime produces an order ϵ^2 correction to the membrane equations, which we ignore.

[6] Usually, the entropy density is a function of the energy density. However our fluid has vanishing energy density. It follows that in this special case the entropy density has nothing to be a function of and so is a pure number.

[7] It is easy to cook up systems with the unusual thermodynamics of our fluid. Consider a substance consisting of $\frac{1}{4\ln 2}$ qubits per unit volume. Let the Hamiltonian of this system simply vanish. A volume V of such a system is associated with a finite dimensional Hilbert space of zero energy states whose number is given by $e^{\frac{V}{4}}$.

[8] Had our membrane fluid been less exceptional, the energy density of the fluid as a function of position would have been an additional variable of our problem. Membrane motion would then have had $D-1$ fluid variables plus one shape variable - the additional equation of motion could then have come from the equation (5.3) which would no longer have been identically obeyed. Black hole membranes are special precisely because they are described by a fluid of vanishing energy density - and so a total of $D-1$ rather than D variables, and so (for consistency) by a stress tensor that obeys (5.3) as an identity.

Simple algebraic manipulations (see the next section) reveal that

$$\nabla \cdot J_S = \frac{1}{2\tilde{\mathcal{K}}}\sigma_{\alpha\beta}\sigma^{\alpha\beta}$$
$$J_S^\mu = \frac{u^\mu}{4} \tag{5.15}$$

We identify J_S^μ as the entropy current of our membrane. This definition reduces to the entropy current of [14] at large D. In the same limit (5.15) reduces to the entropy production equation Eq. (1.5) of [10] at large D. It follows that the membrane equations of this chapter obey a local form of the second law of thermodynamics provided $\tilde{\mathcal{K}}$ is everywhere (pointwise) positive. In this chapter we simply restrict attention to those solutions - large classes of which certainly exist - that obey this condition [9] leaving the analysis of the dynamical closure of this condition to later work.

The derivation (5.15) used the conservation of the Brown York part of the membrane stress tensor. As this conservation applies only in spacetimes that obeys Einstein's equation, it follows that, in general, the local form of the second law (5.15) is valid only when the membrane probes solutions of Einstein's equations rather than general smooth manifolds.

5.1.2 Stationary Solutions and Thermodynamics

In papers written over three years ago, Emparan, Suzuki and Tanabe [17, 18] demonstrated that stationary black holes are governed by simple effective equations in a power series expansion in $\frac{1}{D}$. The formulation of [17, 18], while very convenient for the study of stationary solutions, has not previously been shown to generalize in a simple way to allow for the study of dynamical phenomena. In this chapter we rederive (suitably generalized versions of) the equations of [17, 18] starting with the membrane equations that follow from the conservation of our improved membrane stress tensor. It follows that (suitable generalizations of) the beautiful results of [17, 18] follow from the restriction of our general dynamical membrane equations to stationary situations.

Having obtained the equations of motion that govern stationary solutions we proceed to elucidate their structure. In particular we demonstrate that these equations follow from the extremization of an intriguing action, and uncover their thermodynamical significance.

In order to focus on stationary solutions, in this subsection we restrict attention to background spacetimes G_{MN} that have a timelike killing vector k^M. [10]

Let J_M^E denote the conserved 'energy current'

$$J_M^E = k^N T_{MN} \tag{5.16}$$

[9]This condition is always met in the strict large D limit. Even at finite D it is possible that this condition is stable under time evolution (configurations that obey this condition never evolve to those that do not). The investigation whether - and when - this is true is an interesting problem for the future.

[10]A large class of interesting examples of such backgrounds are the 'vacuum' solutions of Einstein's equations with a negative cosmological constant that are asymptotically locally AdS, and that tend, at small z, to the metric

$$ds^2 = \frac{dz^2 + g_{\alpha\beta}dx^\alpha dx^\beta}{z^2}$$

where $g_{\alpha\beta}$ is an arbitrary field theory metric that admits a timelike killing vector.

and let J^E_μ denote the restriction of this current to the membrane world volume. The conserved energy of the membrane is given by

$$E = \int \sqrt{h}\; q \cdot J^E \tag{5.17}$$

where the integral in (5.17) is taken over any spatial slice of the membrane world-volume, h is the determinant of the metric on this slice, and q is the unit normal to this slice within the membrane world volume.

Consider a membrane configuration in which k^M is everywhere tangent to the membrane and so defines a vector field k^μ on the membrane. If, in addition $\mathcal{L}_k u^\nu$ vanishes ($\mathcal{L}_k$ denotes Lie derivative, on the membrane world volume along k^μ) then we say that the membrane is in a stationary configuration w.r.t the killing field k^M.

As entropy production vanishes on any stationary solution $\nabla.u = 0$ and so $\sigma_{\mu\nu} = 0$ (see (5.15)). The first of (5.15) then implies that $\nabla.u = 0$. However a velocity field can be both shear and divergence free only if it is proportional to a killing vector [45]. It follows that

$$u^\mu = \frac{k^\mu}{\sqrt{-k.k}} \tag{5.18}$$

Using (5.18) it is not difficult to demonstrate that the stress tensor conservation equation projected orthogonal to the velocity u^μ reduces to

$$\mathcal{P}^\alpha_\mu \nabla_\alpha \left(\tilde{\mathcal{K}} \sqrt{-k.k} \right) = 0 \tag{5.19}$$

implying that

$$\tilde{\mathcal{K}} = \frac{4\pi T_0}{\sqrt{-k.k}} \tag{5.20}$$

where T_0 is a constant. At large D, (5.20) reduces to

$$\mathcal{K} = \frac{4\pi T_0}{\sqrt{-k.k}} \tag{5.21}$$

in agreement with the large D results of [17, 18] cited above.

We demonstrate in the main text below that the equations of motion (5.20) follow as the condition that the action

$$S = \frac{1}{16\pi} \left[-(D-1)\lambda \int_V \sqrt{-G} + \int_M \sqrt{-g}\; \left(\mathcal{K} - \frac{4\pi T_0}{\sqrt{-k.k}} \right) \right] \tag{5.22}$$

is extremized. Here $\mathcal{K}$ is the trace of the extrinsic curvature of the membrane, $g_{\mu\nu}$ is the metric on the membrane world volume M and V denotes the region of spacetime enclosed by the membrane. The variation of (5.22) w.r.t the induced metric on the world volume defines a stress tensor given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} \tag{5.23}$$

[11] It is easily verified that the stress tensor (5.23) agrees with (5.7) evaluated on the equilibrium solution (5.18), (5.20). In other words the offshell action (5.22) generates the equations of motion for the shape of stationary solutions, while variation of the value of the onshell action w.r.t. the background metric reproduces the conserved stress tensor of this solution. [12]

We will now uncover the thermodynamical significance of the action (5.22). Let t be any 'time coordinate' that obeys

$$k.dt = k_t = 1 \tag{5.24}$$

Consider the two time slices of the bulk space time $t = t_1$ and $t = t_2$. Let $\beta = t_1 - t_2$. Let dB represent that part of the membrane world volume that lies between these two times and let B denote the part of the bulk spacetime enclosed by the membrane between these two time slices. In the main text we show that provided (5.18) (but not necessarily (5.19)) holds, the membrane energy and entropy is given by

$$\begin{aligned} E &= \frac{1}{16\pi\beta}\left(\int_{dB}\sqrt{-g}\mathcal{K} - (D-1)\lambda\int_{B}\sqrt{-G}\right) \\ S_{ent} &= \frac{1}{4\beta}\int_{dB}\frac{\sqrt{-g}}{\sqrt{-k.k}} \end{aligned} \tag{5.25}$$

Note that the second term on the RHS of (5.25) is proportional to the volume of spacetime enclosed by the membrane world volume and the two time slices. The contribution of this term vanishes at $\lambda = 0$.

Comparing (5.22) with (5.25) it follows that the action in (5.22) may be rewritten as

$$S = \beta\left(E - T_0 S_{ent}\right) \tag{5.26}$$

where β is the 'length' of the time coordinate. In Euclidean space $\beta = \frac{1}{T}$ where T is the temperature of our system. It follows that the Euclidean action (5.26) is proportional to the logarithm of the partition function (as expected on general grounds)

$$S = -\ln Z = \frac{E}{T_0} - S_{ent} \tag{5.27}$$

provided we identify

$$T = T_0. \tag{5.28}$$

In other words the arbitrary constant T_0 that appears in the action (5.22) - which we have already identified with the integration constant in (5.20) - is the temperature of the stationary membrane configuration.

[11] The variation in this equation is defined as follows. We change the metric on the membrane worldvolume by changing the background solution of Einstein's equations with which we work. Note that regular solutions of Einstein's equations are completely determined - and therefore parametrized - by the induced metric on a bounding surface, which in this case is taken to be the world volume of the membrane.

[12] It may be useful to emphasize a potentially confusing point. The action (5.22) is defined as an integral over the full world volume of the membrane - and so is well defined also for time dependent membrane shape configurations. In this chapter, however, we are interested in (5.22) only for stationary membrane configurations. All variations of the action (5.22) are performed within the space of stationary membrane shapes - and with respect to killing metrics.

It then follows from (5.27) that, on shell, [13]

$$\partial_\beta S = E, \tag{5.29}$$

confirming our identification the action

$$S = -\ln Z \tag{5.30}$$

[14]

Recall that stationary solutions of the membrane equations extremize the action (5.27). Viewing $\beta = \frac{1}{T_0}$ as a Lagrange multiplier, it follows from (5.27) that stationary membrane solutions extremize membrane entropy at fixed membrane energy. This is satisfying as we expect, on physical grounds, that the equilibrium configurations in the microcanonical ensemble extremize their entropy.

It follows in particular from (5.20) that the temperature of a static spherical membrane in flat space is given by $T = \frac{\tilde{\mathcal{K}}}{4\pi}$. In a more general configuration that is not necessarily in equilibrium, we simple define the local membrane temperature to be given by

$$T(x) = \frac{\tilde{\mathcal{K}}(x)}{4\pi}, \tag{5.31}$$

We emphasize that the formula (5.31) defines the local temperature of the membrane in any dynamical configuration. The local temperature (5.31) is, in general, a function of position and is distinct from the temperature T_0 of a stationary solution of the membrane equations. In a stationary solution the relationship between T_0 and the local membrane temperature T follows from (5.20) and takes the form

$$T(x) = \frac{T_0}{\sqrt{-k.k}} \tag{5.32}$$

In words, the local temperature in equilibrium is given by the global temperature T_0 times the effective red shift factor $\frac{1}{\sqrt{-k.k}}$. See [48] for a very similar discussion in the context of hydrodynamics on a fixed background manifold.

The simplest stationary membrane solutions are those dual to Schwarzschild type black holes of arbitrary size in global AdS and global dS spaces [15]. Quite remarkably we will find below that the membrane formalism described in this subsection reproduces the thermodynamics of the dual black holes exactly - rather than only in the large D limit.

[13] Naively the action (5.27) changes when we vary the temperature for two reasons. First because (5.27) explicitly depends on β. Second because the equilibrium membrane solution - hence its energy and entropy - depends explicitly on $T = T_0$. However the second variation actually vanishes, as the onshell action is stationary w.r.t. an arbitrary variation of the membrane configuration.

[14] In order to find the partition function of our system we first had to extremize the action w.r.t the membrane shape and then evaluate this extremized action. The situation is more closely analogous to that of the superfluid partition function (see [46]) than the ordinary fluid partition function of, e.g. [47].

[15] Schwarzschild black holes in flat space and black branes in AdS space can be regarded as special limits of these solutions.

5.1.3 Fluid Gravity from Membranes

We now focus on the study of Einstein's equations with a negative cosmological constant, i.e. solutions of the equation (5.4) with $\lambda = 1$. A simple solution of these equations is unit radius AdS_D space in Poincare coordinates, i.e. the space

$$ds^2 = \frac{dz^2 + dx^\mu dx_\mu}{z^2} \tag{5.33}$$

where $\mu = 0, \ldots D-2$ and μ indices are raised and lowered using the metric $\eta_{\mu\nu}$. A simple solution of the membrane equations is the configuration

$$z = \frac{D-1}{4\pi T_{bb}}, \quad u^\mu = z v^\mu, \quad v^\mu = \text{const}, \quad \eta_{\mu\nu} v^\mu v^\nu = -1 \tag{5.34}$$

where T_{bb} is the temperature T_0 of the membrane configuration. This solution is dual to uniform black brane of temperature T_{bb}. By treating the membrane stress tensor as a linearized source for Einstein's equations, it is easy to compute the resultant backreaction. For $z < \frac{D-1}{4\pi T_{bb}}$ the resultant spacetime is a linearized normalizable perturbation about AdS space, and the (AdS/CFT) boundary stress tensor induced by this fluctuation is easily computed. It turns out that this boundary stress tensor agrees precisely (at finite D) with the exact boundary stress tensor of a uniform black brane of temperature T_{bb} and moving at a uniform velocity v^μ. The membrane entropy density also exactly matches the entropy density of the uniform black brane.

Now consider a membrane whose shape and velocity field take the form listed in (5.34) with T_{bb} and v^μ slowly varying functions of the membrane coordinates x^μ. In an expansion in derivatives it is, once again, not difficult to solve the 'dynamical' linearized Einstein equations to compute the linearized gravitational fluctuations sourced by such a membrane. [16] As in the previous paragraph one can now compute the boundary stress tensor induced by this linearized fluctuation. The fact that the boundary stress tensor is conserved follows from Einstein's constraint equations evaluated on the boundary. On the other hand the membrane equations follow from the constraint equations evaluated 'outside' the membrane (the constraint equations are identically obeyed 'inside' the membrane).

Given a solution to the dynamical Einstein equations, it is well known that the constraint equations on any slice imply the constraint equation on any other slice. It follows that the condition of conservation of the boundary stress tensor is equivalent to the requirement of conservation of membrane stress tensor. At the algebraic level, the procedure described earlier in this subsection (coupling the membrane to linearized gravity fluctuations) allows us to find a linear map from the membrane world volume stress tensor to the boundary stress tensor. The fact allows us to regard the boundary stress tensor as a linear functional of the membrane stress tensor (the precise form of this functional depends on the membrane shape in a nonlinear way). This functional has the property that it ensures that the boundary stress tensor is conserved whenever the membrane stress tensor it is obtained from is also conserved.

[16] We compute the fluctuation fields with the boundary conditions that they die off (i.e. are normalizable) towards the boundary of AdS, and also that they do not blow up as we approach the Poincare horizon.

The procedure outlined in the previous paragraph yields an expression for the boundary stress tensor in terms of membrane stress tensor, and so in terms of membrane variables (membrane shape and velocity field). It is possible, however, to perform a field redefinition to a local boundary temperature and a local boundary fluid velocity, and rewrite the boundary stress tensor in terms of these new variables. In these variables the boundary stress tensor takes the standard form for the stress tensor of a conformal fluid in the derivative expansion. Below we have evaluated this expansion to second order in the derivative expansion, and compared our results with literature on the fluid gravity correspondence in which the same expansion of the boundary stress tensor as a function of the boundary velocity and temperature has been computed in every dimension by an exact direct analysis of Einstein's equations. We find that two results (the results of this chapter and the exact results of the fluid gravity correspondence) are in perfect agreement at zero and first order in the derivative expansion even at finite D, but deviate from each other (at finite D) at second order in the derivative expansion.

This discussion of the last paragraph implies, in particular, that the spectrum of the lightest quasinormal modes around a black brane in an arbitrary number of dimensions agrees at finite D and upto first subleading order in k, with the corresponding spectrum around the uniform planar membrane solution (5.34). On the other hand these two spectra deviate at order k^3 and at finite D. We have independently verified that these predictions are borne out.

Note that traditional hydrodynamics (and so, in the gravitational context, fluid gravity) and our large D expansion are distinct expansions of bulk black brane dynamics. Fluid gravity functions order by order in an expansion in derivatives; however the coefficients of this expansion are computed exactly as functions of D. On the other hand the large D membrane equations are constructed order by order in $\frac{1}{D}$. At any given order in $\frac{1}{D}$, however, the resultant equations are exact in derivatives, and so have terms of all orders in the derivative expansion.

We have already pointed out that the leading order membrane equations presented in this chapter accurately reproduces the black brane Navier Stokes equations. In addition the membrane equations capture the contribution of infinite number of arbitrarily high derivative corrections to Navier Stokes. The membrane equations retain only the contribution of those terms that survive in the improved large D limit. From the viewpoint of a boundary observer the truncation to these terms does not appear to help much; outside the long wavelength limit the equations for boundary hydrodynamics appear to continue to be a nonlocal mess. The miracle is that there exists a field redefinition (namely the redefinition that maps boundary to the membrane world volume) that turns this nonlocal mess into local - and so tractable - hydrodynamical equations. Note that these $D-1$ dimensional equations are local only when formulated on the membrane world volume, itself a dynamical $D-1$ dimensional submanifold of the D dimensional bulk AdS space.

The fact that the membrane equations remain local even outside the traditional boundary derivative expansion potentially allows them to capture qualitatively new phenomena. If, for example, the the membrane were to fold on itself then the parametrization $z(x^\mu)$, and so the map to boundary fluid variables becomes singular. It is, however, manifest from the bulk membrane viewpoint that this singularity is a fake, an artefact of the incorrect choice of dynamical variables. We leave a serious investigation of this and other issues to future work.

5.2 Details of the formalism

As explained in the introduction, in this chapter we study a membrane that resides on a codimension one submanifold of any background spacetime that obeys the Einstein equation (5.4). For mathematical purposes it is sometimes convenient to parametrize the membrane world volume by the solutions to the equation

$$\rho - 1 = 0$$

where ρ is a suitably chosen scalar function that takes values on the background manifold. Let

$$|\partial\rho| = \sqrt{\partial_M \rho G^{MN} \partial_N \rho}, \quad n_A = \frac{\partial_A \rho}{|\partial\rho|}.$$

Note that n_A is normal to the membrane world volume and that $n_M G^{MN} n_N = 1$.

Our membrane has a stress tensor, $\mathcal{T}_{MN}$, living on its world volume. The stress tensor has the form

$$\mathcal{T}_{MN} = |\partial\rho| \delta(\rho - 1) T_{MN}, \quad n^M \mathcal{T}_{MN} = n^N T_{MN} = 0 \tag{5.35}$$

Let $T_{\mu\nu}$ [17] denote the pull back of T_{MN} onto the membrane world volume. The equation $T_{MN} n^M = 0$ ensures that there is as much information in $T_{\mu\nu}$ as T_{MN}; knowledge of one is sufficient to reconstruct the other. As explained in the introduction, the world volume stress tensor, $T_{\mu\nu}$ for the membrane studied in this chapter is taken to be given by the form (5.7) where the membrane shear and extrinsic curvature are defined by

$$\sigma_{\mu\nu} = \frac{1}{2} \mathcal{P}^{\alpha}_{\mu} \mathcal{P}^{\beta}_{\nu} \left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \mathcal{P}_{\alpha\beta} \frac{2\nabla.u}{D-2} \right) \tag{5.36}$$

$$K_{\mu\nu} = \left(\frac{\nabla_A n_B + \nabla_B n_A}{2} \right) \frac{\partial X^A}{\partial x^\mu} \frac{\partial X^B}{\partial x^\nu} \tag{5.37}$$

where, X^M are coordinates on the full spacetime and x^μ are the coordinates on the membrane.

5.2.1 Membrane Stress Tensor and equations of motion

As explained in the introduction, our membrane stress tensor is a sum of two terms, $T^{BY}_{\mu\nu}$ (see (5.10)) and $T^{fluid}_{\mu\nu}$ (see (5.11)). $T^{BY}_{\mu\nu}$ is identically conserved on the membrane world volume (provided it propagates in a background satisfying Einstein equations)

$$\nabla^\mu T^{BY}_{\mu\nu} = 0 \tag{5.38}$$

[17] In the rest of this chapter we will use the indices $M, N \ldots$ to denote spacetime coordinates and Greek indices $\mu \ldots$ to denote membrane world volume coordinates. G_{MN} denotes the metric of spacetime, while $g_{\mu\nu}$ is the metric on the world volume of the membrane.

The non-trivial part of the membrane equation of motion is the conservation of the fluid stress tensor

$$
\begin{aligned}
\nabla^\mu T^{fluid}_{\mu\nu} &= 0 \\
16\pi T^{fluid}_{\mu\nu} &= \tilde{\mathcal{K}}\ \mathcal{P}_{\mu\nu} - 2\sigma_{\mu\nu}
\end{aligned}
\tag{5.39}
$$

It is useful to decompose the membrane equations of motion into their components in the direction of and orthogonal to u^μ, i.e.

$$
u^\nu \nabla^\mu T_{\mu\nu} = 0, \quad \mathcal{P}^\nu_\alpha \nabla^\mu T_{\mu\nu} = 0, \quad \mathcal{P}^\nu_\alpha = \delta^\nu_\alpha + u^\nu u_\alpha \tag{5.40}
$$

Using

$$
\begin{aligned}
8\pi \nabla_\mu T^{\mu\nu} u_\nu &= -\frac{\tilde{\mathcal{K}}}{2}\nabla\cdot u + \mathcal{P}^{\mu\alpha}\left(\frac{\nabla_\alpha u_\beta + \nabla_\beta u_\alpha}{2}\right)\nabla_\mu u^\beta - \frac{(\nabla\cdot u)^2}{D-2} \\
&= -\frac{\tilde{\mathcal{K}}}{2}\nabla\cdot u + \mathcal{P}^{\mu\alpha}\left(\frac{\nabla_\alpha u_\beta + \nabla_\beta u_\alpha}{2}\right)\mathcal{P}^{\beta\theta}\nabla_\mu u_\theta - \frac{(\nabla\cdot u)^2}{D-2} \\
&= -\frac{\tilde{\mathcal{K}}}{2}\nabla\cdot u + \mathcal{P}^{\mu\alpha}\left(\frac{\nabla_\alpha u_\beta + \nabla_\beta u_\alpha}{2}\right)\mathcal{P}^{\beta\theta}\left(\frac{\nabla_\mu u_\theta + \nabla_\theta u_\mu}{2}\right) - \frac{(\nabla\cdot u)^2}{D-2} \\
&= -\frac{\tilde{\mathcal{K}}}{2}\nabla\cdot u + \sigma_{\alpha\beta}\sigma^{\alpha\beta}
\end{aligned}
\tag{5.41}
$$

it follows that the first equation in (5.40) can be rewritten in the form (5.15), and is a statement of a local form of the second law of thermodynamics provided $\tilde{\mathcal{K}}$ is everywhere positive.

On the other hand, the stress tensor conservation equation projected orthogonal to u^μ takes the form

$$
16\pi\ \mathcal{P}^\nu_\alpha \nabla^\mu T_{\mu\nu} = \left(\tilde{\mathcal{K}}\ u.\nabla u_\nu + \nabla_\nu \tilde{\mathcal{K}} - 2\nabla^\mu \sigma_{\mu\nu}\right)\mathcal{P}^\nu_\alpha \tag{5.42}
$$

In order to explicitly verify that (5.42) reduces to the membrane equations of motion presented in [8, 9, 11] we manipulate (5.42) as follows. Let X^A denote any space time coordinates and let $\bar{R}_{ABCD}$ be the background spacetime Riemann tensor. Let x^α denote an arbitrary set of coordinates on the membrane world volume and let $e^A_\alpha = \frac{\partial X^A}{\partial x^\alpha}$. Using the Gauss Codacci relationship

$$
R_{\mu\nu} = \mathcal{K}K_{\mu\nu} - K^\beta_\mu K_{\nu\beta} + \bar{R}_{ABCD} e^A_\sigma e^B_\nu e^C_\gamma e^D_\mu g^{\sigma\gamma} \tag{5.43}
$$

it is not difficult to show that

$$
\begin{aligned}
16\pi\ \mathcal{P}^\nu_\alpha \nabla^\mu T_{\mu\nu} = &\left(\tilde{\mathcal{K}}\ u.\nabla u_\nu + \nabla_\nu \tilde{\mathcal{K}}\right)\mathcal{P}^\nu_\alpha + \Big[-2u.\nabla u^\beta \nabla_\beta u_\nu - (\nabla.u)u.\nabla u_\nu \\
&- \nabla^2 u_\nu - u^\gamma u^\mu \nabla_\gamma \nabla_\mu u_\nu - \mathcal{K}\ u^\mu K_{\mu\nu} + u_\mu K^{\mu\gamma}K_{\gamma\nu} - \bar{R}_{ABCD} e^A_\sigma e^B_\nu e^C_\gamma e^D_\mu g^{\sigma\gamma} u^\mu \\
&+ \frac{2}{D-2}(\nabla.u)u.\nabla u_\nu + \frac{2}{D-2}\nabla_\nu(\nabla.u)\Big]\mathcal{P}^\nu_\alpha
\end{aligned}
\tag{5.44}
$$

At leading order in the large D limit, (5.44) reduces to

$$16\pi\ \mathcal{P}^{\nu}_{\alpha}\nabla^{\mu}T_{\mu\nu} = \left(\mathcal{K}\ u.\nabla u_{\nu} + \nabla_{\nu}\mathcal{K} - \nabla^2 u_{\nu} - \mathcal{K}\ u^{\mu}K_{\mu\nu}\right)\mathcal{P}^{\nu}_{\alpha} \tag{5.45}$$

in agreement with the membrane equations of motion presented in [8, 9, 11]. [18]

5.2.2 Regular Stationary solutions of Einstein's equations

We now turn our attention to the construction of stationary solutions to our membrane equations. As explained in the introduction, stationary solutions exist only when the background spacetime in which the membrane propagates has a killing direction. In this subsection and the next we assume this is the case, and denote the killing vector by k^A. We now construct a coordinate system for any such background spacetime that is adapted to this killing direction. It is useful to look at [47, 49] as the setup and construction is similar in flavour

Consider any spacetime with a timelike killing vector field k^A. The spacetime in question can be foliated by the $D-1$ parameter set of integral curves of this killing vector field, i.e. by curves that obey the equation

$$\frac{dX^A(t)}{dt} = k^A(X) \tag{5.46}$$

where X^A represents an arbitrary set of coordinates in the bulk spacetime. Note that there exists a $D-1$ parameter set of such curves which we choose to label by the $D-1$ parameters X^a. Making an arbitrary (X^a dependent) choice for the origin of the t coordinate in (5.46), it follows that the background spacetime metric takes the 'Kaluza Klein' form

$$ds^2_{ST} = G_{MN}dX^M dX^N = -e^{2\Sigma(X^a)}(dt + A_a(X^a)dX^a)^2 + W_{ab}(X^a)dX^a dX^b \tag{5.47}$$

The fact that Σ, A_a and W_{ab} are all independent of t follows from the condition that ∂_t is a killing direction. Note also that an X^a dependent shift of the origin of t preserves the form of the metric (5.47), inducing an effective a 'Kaluza Klein gauge transformation' on the 'Kaluza Klein gauge field' A_a.

We wish to study stationary membrane configurations. As explained in the introduction, this implies, in particular, that the killing field k^A - evaluated at any point on the membrane - is tangent to the membrane at that point. This requirement forces the membrane world volume to be given by a shape of the form

$$f(X^a) = 0 \tag{5.48}$$

(note that the function f does not depend on the 'time' t.). It follows that the induced metric on the membrane, in a stationary configuration, takes the form

$$ds^2 = -e^{2\sigma(x)}(dt + a_i(x)dx^i)^2 + w_{ij}(x)dx^i dx^j \tag{5.49}$$

[18]Using the large D counting described in [8, 9, 11] we find that, at leading order in the large D limit $\tilde{\mathcal{K}} \to \mathcal{K}$. Moreover $\tilde{\mathcal{K}}\ u.\nabla u_{\nu}$, $\nabla_{\nu}\tilde{\mathcal{K}}$, $\nabla^2 u_{\nu}$ and $\mathcal{K}\ u^{\mu}K_{\mu\nu}$ are all $\mathcal{O}(D)$ while $\bar{R}_{ABCD}e^A_{\sigma}e^B_{\nu}e^C_{\gamma}e^D_{\mu}g^{\sigma\gamma}u^{\mu}\mathcal{P}^{\nu}_{\alpha}$ are all $\mathcal{O}(1)$. This conclusion holds for all values of the cosmological constant.

where the variables x^i label the the $D-2$ parameter set of curves (5.46) that obey (5.48) and so lie on the membrane. [19]

As explained in the introduction, the velocity field configuration for a stationary solution takes the form (5.18). It follows from (5.18) that

$$\begin{aligned} u.\nabla u_\mu &= \frac{k^\nu}{\sqrt{-k.k}}\nabla_\nu\left(\frac{k_\mu}{\sqrt{-k.k}}\right) \\ &= \frac{k_\mu}{\sqrt{-k.k}}k^\nu\nabla_\nu\left(\frac{1}{\sqrt{-k.k}}\right) + \frac{k^\nu\nabla_\nu k_\mu}{(-k.k)} \\ &= \frac{1}{2}\frac{\nabla_\mu(-k.k)}{(-k.k)} \\ &= \nabla_\mu \ln\sqrt{-k.k} \end{aligned} \tag{5.50}$$

(5.50), together with the identity $\sigma_{\mu\nu}=0$ turns the equation of motion (5.42) into the simpler equation (5.19), which can immediately be integrated to (5.20).

We now turn to a derivation of the thermodynamical formulae (5.25). Let us begin with the second of (5.25). Recall that the entropy of a stationary configuration of the membrane is obtained by integrating the entropy current over any spacelike slice of the membrane. Consider a spacelike slice of the membrane given by the equation

$$t = t_0 \tag{5.51}$$

where t_0 is a constant. [20]

The normal oneform t to this slice - viewed as a oneform on the membrane world volume - is given by

$$q = \frac{dt}{\sqrt{e^{-2\sigma} - a_i w^{ij} a_j}} \tag{5.52}$$

Let g represent the determinant of the metric on the $D-1$ membrane world volume and let h represent the determinant of the metric on the $D-2$ dimensional membrane slice (5.51). It is easy to find an expression for g and h in terms w, the determinant of the metric w_{ij} (see (5.49)). We have

$$\begin{aligned} \sqrt{-g} &= \sqrt{w}e^{\sigma}, \\ \sqrt{h} &= \sqrt{w}\sqrt{1-e^{2\sigma}a_i w^{ij} a_j} \end{aligned} \tag{5.53}$$

Finally recall that in the coordinate system of (5.49) the velocity vector field u takes the form

$$u = e^{-\sigma}\partial_t \tag{5.54}$$

The entropy of the membrane is given by

$$S_{ent} = \int \sqrt{h}\; q_\mu J_S^\mu \tag{5.55}$$

[19] In other words the $D-2$ parameters x^i label the most general solution of (5.48). This solution is given by the schematic form $X^a(x^i)$. Recall that while a runs over $D-1$ variables, i runs over $D-2$ variables.

[20] This special choice of slice entails no loss of generality, as the most general slice of spacetime, $t = f(x^i)$, can be recast in the form (5.51) by the'Kaluza Klein' x^i dependent shift of the origin of t.

where the integral is taken over the $D-2$ dimensional slice of the membrane world volume (5.51). Using (5.54), however, it follows that

$$J_S^\mu\ q_\mu = \frac{e^{-\sigma}}{4\sqrt{e^{-2\sigma} - a_i w^{ij} a_j}}$$

Using (5.53) it then follows that

$$S_{ent} = \frac{1}{4}\int \frac{\sqrt{-g}}{\sqrt{-k.k}} \tag{5.56}$$

where, once again the integral is taken over the $D-2$ dimensional slice of the membrane world volume (5.51) and we have used the fact that $\sqrt{-k.k} = e^\sigma$. The LHS and RHS of (5.56) are both independent of time. Integrating both sides of that equation from $t = t_1$ to $t = t_2$ we obtain the second of (5.25).

We now turn to the derivation of the first of (5.25). The energy of the membrane is given by

$$16\pi E = -16\pi \int \sqrt{h}\ q^\mu T_{\mu\nu} k^\nu = -\int \sqrt{h}\ q^\mu (K_{\mu\nu} - \mathcal{K} g_{\mu\nu}) k^\nu = \int \sqrt{-g}\ (\mathcal{K} - K^t_t) \tag{5.57}$$

As above, the integral in (5.57) is taken over the $D-2$ dimensional slice of the membrane world volume (5.51). In going from the middle expression in (5.57) to the RHS we have used the fact that $k = \partial_t$ and easily verified formulae

$$\sqrt{h} k.q = \sqrt{-g}, \quad \sqrt{h} q_\mu K^\mu_\nu k^\nu = \sqrt{-g} K^t_t$$

Integrating both sides of this equation from $t = t_2$ to $t = t_1$ we obtain

$$16\pi (t_1 - t_2) E = \int_M \sqrt{-g}\ (\mathcal{K} - K^t_t) \tag{5.58}$$

where the integral on the RHS of (5.58) is taken over the part of the membrane world volume that lies between $t = t_1$ and $t = t_2$.

We will now complete our derivation of (5.25) by demonstrating that

$$\int_M \sqrt{-g}\ K^t_t = (D-1)\,\lambda \int_V \sqrt{-G} \tag{5.59}$$

The LHS of (5.59) is integrated, as in (5.58), over the part of the membrane contained between times t_1 and t_2. The RHS of (5.59), on the other hand, is integrated over the region of the *bulk* D dimensional spacetime enclosed by three codimension one surfaces: the membrane world volume, the bulk slices $t = t_1$ and the bulk slice $t = t_2$. If (5.59) holds then clearly (5.25) follows from (5.58).

In order to establish (5.59), consider

$$Q = \int_V \sqrt{-G} \nabla_M \left[(dt)_N \nabla^N k^M \right] \tag{5.60}$$

where the integral is taken over the bulk region V defined in the previous paragraph. We will establish (5.59) by evaluating (5.60) in two separate ways.

Our first evaluation uses an integration by parts to express (5.60) as

$$Q = \int_M \sqrt{-g} n_M \left[(dt)_N \nabla^N k^M \right] \tag{5.61}$$

where n_M is the normal to the membrane and the integral is taken over the region of the membrane world volume for times t that lie between t_1 and t_2. [21] Recall that k^M is tangent to the membrane, in other words $n_M k^M$ vanishes. It follows that $n_M \nabla^N k^M = -k^M \nabla^N n_M$, so that (5.61) may be rewritten as

$$\begin{aligned} Q &= -\int_M \sqrt{-g} (\nabla^N n_M)(dt)_N k^M \\ &= -\int_M \sqrt{-g} K^N_M (dt)_N k^M \\ &= -\int_M \sqrt{-g}\ K^t_t \end{aligned} \tag{5.62}$$

where the integral is, once again, taken over the part of the membrane world volume at times between t_1 and t_2. [22]. (5.62) is the final result of our first evaluation of Q.

Our second evaluation proceeds by expanding out the integrand in (5.60). We have

$$\nabla_M \left((dt)_N \nabla^N k^M \right) = (\nabla_M (dt)_N) \nabla^N k^M + dt^N [\nabla_M, \nabla_N] k^M + (dt)^N \nabla_N \nabla_M k^M$$

The first term in this expression vanishes because $(\nabla_M (dt)_N)$ is symmetric [23] whereas $\nabla_N k_M$ is antisymmetric in its indices (recall k^M is a killing vector). The third term in this equation vanishes because $\nabla_M k^M$ vanishes. The second term is non-vanishing and is easily evaluated to be

$$R_{NA} (dt)^N k^A = -(D-1)\lambda$$

where in the final equality we have used the bulk Einstein equation (5.4). It follows that

$$Q = -(D-1)\lambda \int_V \sqrt{-G} \tag{5.63}$$

(5.63) and (5.62) together establish (5.59).

Note that the last step in our derivation of (5.59) made crucial use of the fact that the membrane encloses a *regular solution* of Einstein's equations (5.4). Our derivation does not apply to a membrane propagating in an arbitrary spacetime, and also does not apply to the membrane propagating about a solution of Einstein's equations if that solution encloses either a singularity or (secretly) a second asymptotic region, as is the case for a black hole spacetime.

[21]In addition we have similar surface terms on the time slices at $t = t_1$ and $t = t_2$. However it is easily verified that the contribution of the bulk constant time slice at t_2 cancels the analogous contribution at t_1.

[22]In obtaining the first line in (5.62) starting from (5.60) we have integrated by parts and used the fact that $n.k = 0$.

[23]This follows from the symmetry of Γ matrices in our particular coordinate system.

5.3 The action and its variations

We now demonstrate that equilibrium membrane configurations are governed by the action (5.22) and establish some properties of this action.

5.3.1 Variation of the action w.r.t. the membrane shape

Consider a membrane whose world volume is given by a smooth codimension one submanifold of the ambient spacetime. Let x^μ represent a set of coordinates on the membrane. The membrane world volume can be described by specifying the spacetime coordinates X^M as functions of the membrane coordinates, i.e. by the functions $f^M(x^\mu)$ s.t.

$$X^M = f^M(x^\mu) \tag{5.64}$$

We denote the induced metric on this membrane surface by $g_{\mu\nu}(x)$. The extrinsic curvature of the membrane surface is denoted by $K_{\mu\nu}(x)$.

Now consider the slightly displaced membrane described by

$$X^M = f^M(x^\mu) + \delta z(x^\mu) n^M(x^\mu) \tag{5.65}$$

Here $n^M(x^\mu)$ is the normal vector of the membrane surface at the point x^μ and $\delta z(x^\mu)$ is an arbitrary infinitesimal displacement function on the membrane. Let the induced metric on the displaced surface (5.64) be given by $g_{\mu\nu} + \delta g_{\mu\nu}$, and let the extrinsic curvature of the displaced surface be given by $K_{\mu\nu} + \delta K_{\mu\nu}$. In Appendix D.1 we demonstrate that, to first order in δz

$$\begin{aligned}
&\delta g_{\mu\nu} = 2K_{\mu\nu}\delta z \\
&\delta g^{\mu\nu} = -2K^{\mu\nu}\delta z \\
&\delta K_{\mu\nu} = \left(R_{\mu\nu} + (D-1)\lambda G_{\mu\nu} + 2K_{\mu\alpha}K^\alpha_\nu - \mathcal{K}K_{\mu\nu}\right)\delta z - \nabla_\mu\nabla_\nu\delta z \\
&\delta\sqrt{-g} = \sqrt{-g}\ \mathcal{K}\ \delta z \\
&\delta\mathcal{K} = \left(-K_{\mu\nu}K^{\mu\nu} + (D-1)\lambda\right)\delta z - \nabla^2\delta z \\
&\delta\gamma = \gamma(u.K.u)\delta z \\
&\delta\int_V \sqrt{-G} = \int_M \sqrt{-g}\ \delta z
\end{aligned} \tag{5.66}$$

where we have used the notation

$$\gamma = \frac{1}{\sqrt{-k.k}}, \quad u = \frac{k}{\sqrt{-k.k}} = \gamma k \tag{5.67}$$

In order to obtain the formula for $\delta\gamma$ reported in (5.66) above we have used the fact that, for stationary membrane configurations, $n_A k^A = 0$ where n_A is the normal to the membrane. All of the other formulae in (5.66) are valid even without making this assumption.

In the last of (5.66) the volume integral on the LHS is taken over V, the region of spacetime enclosed by the membrane, whereas the integral on the RHS is taken over the M, the world volume of the membrane.

Using (5.66) it follows immediately that the variation of the action (5.22) under the operation (5.65) is given by [24]

$$\delta S = \frac{1}{16\pi} \int_M \sqrt{-g} \, \left(\mathcal{K}^2 - K_{\mu\nu} K^{\mu\nu} - \frac{4\pi T_0}{\sqrt{-k.k}} \left(\mathcal{K} + u.K.u \right) \right) \delta z \tag{5.68}$$

It follows that the action (5.22) is stationary under shape variations provided that

$$\frac{\mathcal{K}^2 - K_{\mu\nu} K^{\mu\nu}}{\mathcal{K} + u.K.u} = \frac{4\pi T_0}{\sqrt{-k.k}} \tag{5.69}$$

In the stationary situation under consideration $\sigma_{\mu\nu} = 0$ and so the LHS of (5.69) equals $\tilde{\mathcal{K}}$ (see (5.8)) and (5.69) is the same as (5.20). We have thus demonstrated that (5.20) follows as the condition for stationarity of the membrane action (5.22).

5.3.2 Variation of the action w.r.t. the metric

In this subsection we study the change in the membrane action as a response to a variation of the induced metric on the membrane world volume - rather than the membrane shape as in the previous subsection. We pause to explain precisely what this means.

Consider a spacetime with a boundary S. Consider the Einstein Hilbert action for the spacetime contained within S, supplemented by the Gibbon's Hawking term on the boundary S. It is well known that Einstein's equations in the interior of S follow from the variation of this functional, subject to the boundary conditions that the induced metric on S is a specified metric $g_{\mu\nu}$. Moreover it is expected to be generically true that there are at most discretely many solutions to Einstein's equations for any given boundary metric $g_{\mu\nu}$. In other words the boundary metric, on any surface surrounding a region of spacetime, labels solutions of Einstein's equations in its interior upto discrete ambiguities. [25]

Now the membrane action (5.22) is a functional of both the induced metric on the membrane as well as the extrinsic curvature of the membrane. As the extrinsic curvature depends on the normal derivative of the spacetime metric away from the membrane, (5.22) would appear to be a functional of both the induced metric on the membrane as well as its first normal derivative inwards. However the spacetimes on which the membrane propagates are not arbitrary - they are solutions to Einstein's equations. And we have just argued in the previous paragraphs that the entire solution to the interior of the membrane - hence the normal derivative of the boundary metric on the membrane - and hence the extrinsic curvature of the membrane - are all determined by the induced metric on the world volume of the membrane. It follows that the variation the extrinsic curvature $K_{\mu\nu}$ (and so the membrane action) w.r.t. the boundary membrane metric is well defined. We define the membrane stress

[24] The variation of this action w.r.t its shape can be more systematically computed using the general formalism developed in [50, 51, 52, 53, 54, 55, 56], and yields the same results as those presented below. We thank J. Armas and J. Bhattacharya for discussions on this point.

[25] These expectations are best motivated in Euclidean space - and so are expected to apply well to the equilibrium spacetimes under study in this section.

tensor in equilibrium via the equation [26]

$$\delta S = -\frac{1}{2}\int_M \sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu} \tag{5.70}$$

The variation in (5.70) is performed within the space of stationary membrane metrics (i.e. membrane metrics that admit a killing direction). The variation in (5.70) can be taken to be performed with k^μ held fixed. Though we will not need this for calculational purposes, at the conceptual level it is sometimes useful to work in the coordinate system (5.49). In this coordinate system stationary variations of $g_{\mu\nu}$ are a consequence of varying w_{ij}, a_i and σ. Note that we have enough variations to define every component of the stress tensor; note also that, with this coordinate choice, all variations are performed holding $k^\mu\partial_\mu = \partial_t$ fixed.

Although the stress tensor (5.70) is well defined, there is a catch. The variation of the extrinsic curvature w.r.t the induced metric on the membrane is, in general, a highly nonlocal function of the induced metric on the membrane. [27]. Consequently the variation of a generic action build out of Extrinsic curvatures would lead to a highly nonlocal stress tensor (5.70). However our membrane action

$$S = \frac{1}{16\pi}\left[-(D-1)\lambda\int_V \sqrt{-G} + \int_M \sqrt{-g}\ \mathcal{K} - 4\pi T_0 \int_M \sqrt{-g}\ \gamma\right] \tag{5.71}$$

is not generic. In particular the sum of the first two terms in (5.71) is precisely one half of the onshell value of the Einstein action of the region of spacetime enclosed by the membrane. [28] It follows that (5.71) may be rewritten as

$$S = \frac{1}{2}S_{in} - \frac{T_0}{4}\int_M \sqrt{-g}\ \gamma \tag{5.72}$$

[26]We emphasize that the variation in (5.70) is performed onshell. The initial membrane configuration in (5.70) is assumed to be onshell w.r.t shape variations of the membrane. Logically speaking, the final membrane configuration in (5.70) should also be taken onshell, but for the purposes of computing the stress tensor (5.70) this condition is unimportant and can be dropped. The reason for this is simply that the variation of the membrane action - due to a change in shape of the membrane - vanishes when taken around a solution to the membrane equations of motion.

[27]This is analogous to the fact - familiar from the study of electrostatics - that the 'normal component of the electric field', $n.\nabla\phi$ at a point x just outside a conductor is given by an integral of the form $\int G(x,y)\phi(y)$ where he integral is taken over the boundary of the conductor and G is a Greens function. In this analogy the boundary value of the potential ϕ plays the role of the induced metric, while the normal component of the electric field plays the role of the extrinsic curvature.

[28]More precisely, the first term in (5.71) is half of the onshell value of the bulk part of the action

$$\frac{1}{16\pi}\int_V \sqrt{-G}\,(R + \lambda(D-1)(D-2))$$

(this is easily verified by making the the onshell substitution $R = -D(D-1)\lambda$) while the second term is half of the Gibbons Hawking boundary term

$$\frac{1}{8\pi}\int_M \sqrt{-g}\ \mathcal{K}$$

.

where

$$S_{in} = \frac{1}{8\pi}\left[-(D-1)\lambda\int_V \sqrt{-G} + \int_M \sqrt{-g}\ \mathcal{K}\right] \tag{5.73}$$

S_{in} is the value of Einstein's action of the spacetime to the interior of our membrane; this can be made more explicit by using the bulk Einstein equation to rewrite (5.73) as

$$S_{in} = \frac{1}{16\pi}\int_V \sqrt{-G}\left(R + \lambda(D-1)(D-2)\right) + \frac{1}{8\pi}\int_M \sqrt{-g}\ \mathcal{K} \tag{5.74}$$

The only dependence of (5.72) on the extrinsic curvature comes from the fact that S_{in} depends on $K_{\mu\nu}$. However this dependence is very special. In particular it follows from the Hamilton Jacobi equations applied to Einstein gravity that

$$\delta S_{in} = -\frac{1}{16\pi}\int_M \sqrt{-g}\delta g_{\mu\nu}\left(K^{\mu\nu} - \mathcal{K}g^{\mu\nu}\right) \tag{5.75}$$

[29]

Consequently the variation of the first two terms in (5.71) leads to a contribution to the membrane stress tensor equal to half of the Brown York stress tensor, i.e.

$$\delta\left(\frac{S_{in}}{2}\right) = -\frac{1}{2}\int_M \sqrt{-g}\left(\frac{1}{16\pi}\left(K^{\mu\nu} - \mathcal{K}g^{\mu\nu}\right)\right)\delta g_{\mu\nu} \tag{5.76}$$

and is completely local.

The third term in (5.71) is a manifestly local functional of the induced metric on the membrane, and so its variation w.r.t. the induced metric results in an manifestly local contribution to the stress tensor.

$$\begin{aligned}
\delta\left(\sqrt{-g}\ \gamma\right) &= \frac{1}{2}\sqrt{-g}\ \gamma\ g^{\mu\nu}\ \delta g_{\mu\nu} + \frac{1}{2}\sqrt{-g}\ \gamma^3\ k^\mu k^\nu \delta g_{\mu\nu} \\
&= \frac{1}{2}\sqrt{-g}\ \gamma\left(g^{\mu\nu} + u^\mu u^\nu\right)\delta g_{\mu\nu} \\
&= \frac{1}{2}\sqrt{-g}\ \gamma \mathcal{P}^{\mu\nu}\delta g_{\mu\nu}
\end{aligned} \tag{5.77}$$

[29]Logically speaking the variation in (5.75) is performed in a completely onshell manner in the bulk - i.e. from one solution of Einstein's equations parametrized by an induced boundary metric to another solution of Einstein's equations parametrized by a slightly varied boundary metric. At the formal level, however, one could ask the following question. Suppose we start in a solution of the bulk Einstein equation, but let the bulk metric variation be offshell. This offshell variation is an arbitrary function of the full bulk, not just the boundary. We could then define the appropriately normalized coefficient of the bulk metric variation to be the bulk spacetime stress tensor resulting from the action (5.75). J. Armas has pointed out to us that provided we use (5.74) to define S_{in} then - as follows from standard textbook derivations of Einstein's equations - the bulk stress tensor that follows has the form (5.35) where the restriction of T_{MN} to the membrane is given by (5.75). Note, in particular, that with this choice of S_{in} our bulk stress tensor has no terms proportional to $\delta'(\rho-1)$. The offshell stress tensor that follows from the full action (5.72) also has the form (5.35) where the restriction of T_{MN} to the membrane is given by (5.81). It is important that, in the language of [50, 51, 52, 53, 54, 55, 56], we find $\hat{T}_2^{MNO} = 0$ (this is equivalent to the fact that the 'bulk' stress tensor has no δ' pieces). From the point of view of [50, 51, 52, 53, 54, 55, 56] it is this fact that allows - for example - the energy current of our membrane to take the simple form (5.16) rather than the more complicated form it would have taken had the bulk stress tensor also had δ' pieces. We thank J. Armas and J. Bhattacharya detailed discussions and explanations on this topic.

Consequently it follows that

$$\delta\left(-\frac{T_0}{4}\int_M \sqrt{-g}\,\gamma\right) = -\frac{1}{2}\int_M \sqrt{-g}\left(\frac{T_0}{4}\gamma\mathcal{P}^{\mu\nu}\right)\delta g_{\mu\nu} \tag{5.78}$$

Adding (5.76) and (5.78), it follows from (5.75) that

$$\delta S = -\frac{1}{2}\int_M \sqrt{-g}\left(\frac{1}{16\pi}\left(K^{\mu\nu} - Kg^{\mu\nu}\right) + \frac{T_0}{4}\gamma\mathcal{P}^{\mu\nu}\right)\delta g_{\mu\nu} \tag{5.79}$$

Comparing (5.79) with (5.70) we conclude that

$$16\pi\ T^{\mu\nu} = \frac{4\pi T_0}{\sqrt{-k.k}}\ \mathcal{P}^{\mu\nu} + \left(K^{\mu\nu} - \mathcal{K}g^{\mu\nu}\right) \tag{5.80}$$

Recall that (5.80) applies only for stationary membranes that obey the onshell condition (5.20). Using (5.20) it follows that (5.80) may be rewritten as

$$16\pi\ T^{\mu\nu} = \tilde{\mathcal{K}}\ \mathcal{P}^{\mu\nu} + \left(K^{\mu\nu} - \mathcal{K}g^{\mu\nu}\right) \tag{5.81}$$

in perfect agreement with (5.7) in the stationary case. In summary, we have demonstrated that the stress tensor that follows from the variation of our membrane action agrees with the general fluid stress tensor (5.7) evaluated on equilibrium configurations.

As (5.81) is a special case of (5.7), it follows that it obeys the condition (5.9). We end this subsection with a brief logical explanation (i.e. one that does not rely on algebraic verification) that this had to be the case.

Consider a membrane propagating in a given background solutions of Einstein's equations. There is one vary easy way to vary the induced metric on the membrane while ensuring that the spacetime inside the membrane continues to solve Einstein's equations. One can do this by simply infinitesimally displacing the membrane a little bit within the given background solution of Einstein's equations. Even though this process does not modify the background metric, it changes the induced metric on the membrane. As explained in (5.66), the change in the induced membrane metric produced by such a manoeuvre is equal to $2K_{\mu\nu}\delta z$ where δz is arbitrary. As explained in the previous subsection, however, the onshell membrane action is stationary under arbitrary variations of the membrane volume, and so we find from (5.70) that

$$0 = -\frac{1}{2}\int_M \sqrt{-g}T^{\mu\nu}(2K_{\mu\nu}\delta z) \tag{5.82}$$

As (5.82) is true for any choice of the function δz it follows that

$$T^{\mu\nu}K_{\mu\nu} = 0.$$

In other words the stress tensor defined by varying the action using (5.70) automatically obeys the equation (5.9).

5.4 Simple Static Membrane Configurations and their Thermodynamics

In this section we study simple static solutions of the membrane equations and compare their thermodynamics with that of the dual black holes. The solutions we study are Schwarzschild black holes in flat space, global AdS space and de Sitter space.

5.4.1 Coordinates and Conventions

In this section we study the maximally symmetric backgrounds

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{D-2}^2, \quad f(r) = 1 + \lambda\, r^2. \tag{5.83}$$

Of course (5.83) are exact solutions to the Einstein equations (5.4). Before proceeding with our analysis we pause to describe the coordinates employed in (5.83).

When $\lambda = 0$ (5.83) is just flat space in polar coordinates, and this case requires no further elaboration. When $\lambda > 0$ the spacetime (5.83) is Anti de Sitter space of squared radius $\frac{1}{\lambda}$ in global coordinates. Notice that the function $f(r)$ never vanishes in this case. For $r^2 \ll \frac{1}{\lambda}$ this spacetime is approximately flat; for $r^2 \gg \frac{1}{\lambda}$ the spacetime approximates Poincare Patch AdS space (i.e. AdS space with planar sections). According to the AdS/CFT correspondence, this is the spacetime dual to the vacuum of $\mathcal{N} = 4$ Yang Mills theory. Finally when $\lambda < 0$, the part of (5.83) with $r^2|\lambda| < 1$ is the static patch of de Sitter spacetime. Recall that the static patch is the causal past of a static observer in global de Sitter spacetime. The submanifold $r^2|\lambda| = 1$ is the future horizon of the causal patch. Points with $r^2|\lambda| > 1$ lie outside the static patch. While the killing vector ∂_t is timelike within the causal patch, it is spacelike outside the causal patch. As we have explained above, our construction of stationary membranes is based on a timelike killing vector field, which we will chose to be ∂_t in the case of the backgrounds (5.83). When $\lambda < 0$ the requirement that our killing vector field be timelike forces us to restrict our attention to within the static patch. At any rate below we will focus our attention principally on $\lambda = 0$ or λ positive.

5.4.2 Exact Black Hole solutions and their Thermodynamics

It is well known that following metrics are exact solutions of the Einstein equations (5.4)

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{D-2}^2, \quad f(r) = 1 - \frac{r_0^{D-3}}{r^{D-3}} + \lambda\, r^2 \tag{5.84}$$

The metrics (5.84) reduce to (5.83) at large r. They also possess an event horizon (or in the case of $\lambda < 0$ an additional event horizon) and so represent Schwarzschild black holes in flat, global AdS and static patch de Sitter space respectively.

The (additional) event horizon of the metric (5.84) is located at $r = r_H$ determined by the condition that $f(r_H) = 0$, i.e. r_H obeys the equation

$$r_0^{D-3} = \left(1 + \lambda r_H^2\right) r_H^{D-3} \tag{5.85}$$

At least for $\lambda = 0$ and $\lambda > 0$ - the cases to which we restrict attention in most of the rest of this subsection - the mass, entropy and temperature of the black hole solutions are unambiguously well defined; specializing to this case, the mass and entropy of the black holes were listed in, e.g., [57] and are given by; [30]

$$\begin{aligned} M_{bh} &= \frac{(D-2)\,(1+\lambda r_H^2)\, r_H^{D-3}\Omega_{D-2}}{16\pi}, \quad S_{bh} = \frac{r_H^{D-2}\Omega_{D-2}}{4}, \\ T_{bh} &= \frac{1}{4\pi r_H}\left[(D-3)+(D-1)\lambda r_H^2\right] \end{aligned} \tag{5.86}$$

5.4.3 Membrane solutions and their thermodynamics

We now study stationary membrane solutions in the background (5.83). We use the formalism developed in earlier sections, base our construction on the killing vector $k^\mu\partial_\mu = \partial_t$. It is easily verified that the spherical membranes $r = \tilde{r}_H$ are solutions of the stationary membrane equations (5.20). As we will see below, the thermodynamics of spherical membranes at $r = \tilde{r}_H$ exactly matches (5.86) provided we make the identification $\tilde{r}_H = r_H$, forcing us to identify $\tilde{r}_H$ with r_H. We use this foreknowledge to lighten the notation of this subsection by simply dropping the tilde on r_H in all the formulae that follow.

For the membrane shape under consideration it is not difficult to verify that

$$\begin{aligned} q_\mu = \left(1+\lambda r_H^2\right)^{1/2}(dt)_\mu, \quad k^\mu = (\partial_t)^\mu, \quad u^\mu = \left(1+\lambda r_H^2\right)^{-1/2}(\partial_t)^\mu \\ K_{tt} = -\lambda r_H(1+\lambda r_H^2)^{1/2}, \quad K_{ta} = 0, \quad K_{ab} = r_H(1+\lambda r_H^2)^{1/2}\Omega_{ab} \end{aligned} \tag{5.87}$$

and that

$$\begin{aligned} &\mathcal{K} = r_H\lambda(1+\lambda r_H^2)^{-1/2} + \frac{D-2}{r_H}(1+\lambda r_H^2)^{1/2}, \\ &K_{\mu\nu}K^{\mu\nu} = \lambda^2 r_H^2(1+\lambda r_H^2)^{-1} + \frac{D-2}{r_H^2}(1+\lambda r_H^2), \quad u.K.u = -\lambda r_H(1+\lambda r_H^2)^{-1/2}, \\ &\tilde{\mathcal{K}} = (1+\lambda r_H^2)^{-1/2}\left[\frac{D-3}{r_H} + (D-1)\lambda r_H\right], \quad \sqrt{-k.k} = (1+\lambda r_H^2)^{1/2} \end{aligned} \tag{5.88}$$

It follows from the second last and the last of (5.88) and from (5.20) that the temperature, T_0, of this membrane configuration is given by

$$T_0 = \frac{1}{4\pi r_H}\left[(D-3)+(D-1)\lambda r_H^2\right] \tag{5.89}$$

in perfect agreement with (5.86).

The energy of our membrane is given by

$$E = -\int \sqrt{h}\; q^\mu T_{\mu\nu}k^\nu = -\frac{1}{16\pi}\int \sqrt{h}\; q^\mu(K_{\mu\nu} - \mathcal{K}g_{\mu\nu})k^\nu \tag{5.90}$$

[30] In the case $\lambda = 0$ the black hole mass is its usual ADM energy. In the case $\lambda > 0$ the black hole mass is given by integrating the boundary stress tensor (Brown York stress tensor plus suitable counterterms) over the boundary sphere, and coincides with the energy of the dual field theory on S^{D-2}.

and its entropy by

$$S_{ent} = \int \sqrt{h}\ q_\mu J^\mu_S = \frac{1}{4}\int \sqrt{-g}\ \gamma \tag{5.91}$$

Substituting (5.87) into (5.90) and (5.91) we find the explicit results

$$E = \frac{(D-2)r_H^{D-3}(1+\lambda r_H^2)\Omega_{D-2}}{16\pi}, \quad S_{ent} = \frac{r_H^{D-2}\Omega_{D-2}}{4} \tag{5.92}$$

Once again (5.92) is in perfect agreement with (5.86).

As a check we note that

$$\frac{\partial E}{\partial S_{ent}} = \frac{\partial E}{\partial r_H}\left(\frac{\partial S_{ent}}{\partial r_H}\right)^{-1} = \frac{1}{4\pi r_H}\left[(D-3)+(D-1)\lambda r_H^2\right] = T_0 \tag{5.93}$$

(where we have used (5.89) in the last equality). We conclude that the thermodynamical temperature or our system is, indeed, T_0.

Finally, it is not difficult to evaluate the $S = -\ln Z$ of our spherical membrane solutions. Using (5.22) we find

$$-\ln Z = S = \frac{r_H^{D-3}\Omega_{D-2}}{16\pi T_0}\left[(D-2)\left(1+\lambda r_H^2\right) - 4\pi T_0 r_H\right] \tag{5.94}$$

The partition function (5.94) has been presented as a function of r_H; however one can, in principle, invert (5.89) to obtain r_H as a function of temperature and so view $\ln Z$ as a function of temperature.

As a check, it is not difficult to use (5.94), together with the thermodynamical relations

$$E = \partial_\beta S, \quad S_{ent} = \partial_T(TS) \tag{5.95}$$

together with the explicit formula for the temperature (5.89) to reproduce the relations (5.92).

In this section we study only the membrane duals of static Schwarzschild type black holes. We largely leave the generalization of this discussion to rotating Kerr type black holes to future work. However see Appendix D.4 for preliminary work in this direction.

5.4.4 The membrane and boundary stress tensors

It may be verified that the induced metric on the membrane and its world volume stress tensor, evaluated on the equilibrium configurations of this section are given by

$$\begin{aligned} &ds^2 = -(1+\lambda r_H^2)dt^2 + r_H^2 d\Omega_{D-2}^2 \\ &16\pi T_{tt} = \frac{D-2}{r_H}(1+\lambda r_H^2)^{3/2}, \quad T_{ta} = 0, \quad 16\pi T_{ab} = (1+\lambda r_H^2)^{-1/2}\lambda r_H^3 \Omega_{ab} \end{aligned} \tag{5.96}$$

where Ω_{ab} is the metric on the world volume of a unit sphere. As a check on this formula it may be verified that

$$\int \frac{\sqrt{-g}}{-g_{tt}} T_{tt} = M_{bh} \tag{5.97}$$

where g represents the metric on the world volume of the membrane.

It is interesting to specialize (5.96) to the case $\lambda = 1$ (in which case our solution is a spherical membrane in a unit radius AdS space) and compare (5.96) with the boundary stress tensor of the dual gravitational black hole (5.84). The stress tensor lives on the manifold on which the field theory is defined, i.e.

$$ds^2 = -dt^2 + d\Omega_{D-2}^2 \tag{5.98}$$

Its form may be read off, for instance, from section 5.3 of [48] and is given by

$$\begin{aligned} &16\pi T^B_{tt} = 2m(D-2) = (D-2)r_H^{D-3}(1+r_H^2), \qquad T^B_{ta} = 0, \\ &16\pi T^B_{ab} = 2m\Omega_{ab} = r_H^{D-3}(1+r_H^2)\Omega_{ab} \end{aligned} \tag{5.99}$$

It is also easily verified that

$$\int \frac{\sqrt{-g}}{-g_{tt}} T^B_{tt} = M_{bh} \tag{5.100}$$

where, here g represents the metric (5.98).

The fact that (5.100) and (5.97) are both true of course means that the membrane and field theory stress tensors are related to each other. However the precise relationship, while easy to state, [31] is not visually transparent.

In the large r_H limit, on the other hand, (5.96) simplifies to

$$\begin{aligned} &ds^2 \equiv \mathbf{g}_{\alpha\beta}dx^\alpha dx^\beta = r_H^2(-dt^2 + d\Omega_{D-2}^2) \\ &16\pi \mathbf{T}_{tt} = (D-2)r_H^2, \qquad 16\pi \mathbf{T}_{ab} = r_H^2\Omega_{ab} \end{aligned} \tag{5.101}$$

while (5.99) simplifies to

$$\begin{aligned} &ds^2 \equiv \mathbb{G}_{\alpha\beta}dx^\alpha dx^\beta = -dt^2 + d\Omega_{D-2}^2 \\ &16\pi \mathbb{T}_{tt} = (D-2)r_H^{D-1}, \qquad 16\pi \mathbb{T}_{ab} = r_H^{D-1}\Omega_{ab} \end{aligned} \tag{5.102}$$

In this case (5.101) and (5.102) are simply related by the Weyl scaling

$$\mathbf{g}_{\alpha\beta} = r_H^2 \mathbb{G}_{\alpha\beta}, \quad \mathbf{T}^\alpha_\beta = \frac{1}{r_H^{D-1}}\mathbb{T}^\alpha_\beta.$$

where the Field Theory metric and Stress tensor are written with 'hollow' letters ($\mathbb{G}_{\alpha\beta}$, $\mathbb{T}_{\mu\nu}$) and the Membrane metric and Stress tensor are written with 'thick bold' letters ($\mathbf{g}_{\alpha\beta}$, $\mathbf{T}_{\mu\nu}$).

5.4.5 Spectrum of linearized Excitations

In Appendix D.2 we have linearized our membrane equations around the spherical membranes dual to Schwarzschild black holes in flat space. The final results for the spectrum of linearized fluctuations is presented in (D.30), (D.31), (D.32). It is easily verified that this spectrum agrees with the results of Emparan, Suzuki and Tanabe [6] at leading order in the large d limit. It should be straightforward to generalize this fluctuation analysis to static membranes at non-zero λ, however we have not done this calculation in this chapter.

[31] The requirement that (5.100) and (5.97) be simultaneously valid determines the ratio of the membrane and boundary T_{tt} components. The requirement that the boundary stress tensor is traceless, while the membrane stress tensor obeys (5.3) then also determines the ratio of energy density to pressure, on both the boundary and the membrane.

5.5 Fluid Gravity from Membrane Dynamics

In this section we specialize to the study of the motion of a membrane in planar AdS space, and ask ourselves the following question: what does the membrane dynamics look like from the perspective of a boundary observer.

In principle this question is easily answered in the following manner. The membrane is a source for linearized gravitational fluctuations about pure AdS space. The precise form of these fluctuations may be obtained by convoluting the membrane stress tensor with the appropriate Green's function. The Green's function may be obtained along the lines of the analysis of [14] (in which the same problem was solved about flat space). This Green's function may be used to construct a linear map from the membrane to the boundary stress tensor of the schematic form

$$\mathbb{T}(x) = \int H(x,y)\mathbf{T}(y) \tag{5.103}$$

for some kernel function $H(x,y)$ (all indices have been omitted in the highly schematic equation (5.103)). As gravitational fluctuations can only be consistently sourced by a conserved bulk stress tensor, the map (5.103) is well defined only when $\mathbf{T}(y)$ is conserved in the bulk. Whenever this is the case, $\mathbb{T}(x)$ is well defined - and is automatically conserved and traceless on the boundary. In other words (5.103) maps a membrane stress tensor that is conserved in the bulk to a boundary stress tensor conserved on the boundary. We will see below that the relationship between these two conservation equations is very tight - at the algebraic level the map (5.103) converts the membrane world volume stress tensor conservation equation (5.2) into a conservation equation for the boundary stress tensor, while the equation (5.3) is mapped to the condition that the boundary stress tensor is traceless.

Restated, the membrane equations - which we have so far viewed as conservation equations on the world volume of the membrane - may be recast as conservation equations in the flat boundary spacetime $R^{d-1,1}$. If we adopt this presentation then it is very unnatural to use the membrane velocity and height function as our dynamical variables, as these variables do not naturally live on the boundary. Instead, as we explain below, it is natural for the boundary observer to use a boundary velocity field $v^\mu(x)$ and a local boundary temperature field $T(x)$ to study dynamics (we will provide precise definitions of these variables in terms of the boundary stress tensor below). [32] Note that the boundary velocity field has the same number of components as the bulk membrane velocity field while the 'location' variable of the bulk membrane is traded for the boundary temperature. Using the map (5.103), the explicit form of the membrane stress tensor (5.7) yields precise expressions for the boundary variables in terms of the bulk variables. These expressions take the schematic form

$$v^\mu = v^\mu(u^\mu, z), \qquad T = T(z, u^\mu) \tag{5.104}$$

where z denotes the location of the membrane in the radial AdS direction (see below) and u^μ is the membrane world volume velocity field. The relations (5.104) may be inverted, and may be regarded as a field redefinition from membrane to boundary variables. The boundary stress tensor may now be re expressed in terms of v^μ and T, and the condition

[32] In the long wavelength limit this choice of boundary variables is standard in the study of hydrodynamics. We emphasize, however, that these variables are well defined, and so can be utilized, even outside this limit.

that the boundary stress tensor is conserved yields a set of boundary equations of motion for these natural boundary variables. In the long wavelength limit - which we will now focus on - these are simply the equations of boundary hydrodynamics.

In general the expressions (5.104) are highly non-local; as a consequence the boundary dynamical system for the variables v^μ and T is, in general, highly non-local Consider however a limit in which the membrane is 'nearly flat' (see below for what this means) and varies slowly in the 'field theory directions'. In this limit it turns out that the map between membrane and boundary variables is approximately local. The boundary stress tensor is, thus, also an approximately local functional of boundary variables in this limit - and takes the form of a hydrodynamical stress tensor that is expressed in terms of the boundary temperature and velocity field by a set of constitutive relations that may be obtained and presented, order by order, in a derivative expansion. In this section we focus in this limit and work out the resultant boundary hydrodynamical constitutive relations at upto second order in the derivative expansion.

Of course the exact finite D expressions for the constitutive relations of the boundary stress tensor are known upto second order in the derivative expansion - the determination of these coefficients was achieved as part of the programme of the fluid gravity correspondence. Comparing our results with those of fluid gravity we find - perhaps unexpectedly - that our membrane induced constitutive relations agree exactly -at finite D - with the results of fluid gravity at zeroth and first order in the derivative expansion. At second order in derivatives, however, the membrane constitutive relations agree with the exact results of fluid gravity only at large D and deviate from the exact results in a power series in $\frac{1}{D}$.

The papers [12, 26] have previously demonstrated that the equations of 'scaled black brane dynamics' reduce - under an appropriate field redefinition - to the equations of boundary hydrodynamics at large D in an appropriate scaling limit. The analysis of this section generalizes the discussions of [12, 26] in several ways. First, in this chapter we map the full nonlinear membrane equations of motion to full nonlinear equations of boundary hydrodynamics, and do not work in a particular scaling limit. Next, the starting point of our analysis is the equations for probe membrane dynamics. Our probe membrane is defined by the improved stress tensor (5.7) and its motion is well defined at finite D. In this section we map our finite D probe membrane dynamics to the equations of finite D boundary hydrodynamics, and obtain results that agree exactly with those of fluid gravity at zero and first order in derivatives even at finite D. Finally, the method we employ in our analysis utilizes the linearized backreaction of the membrane on gravity, and - in our opinion - conceptually clarifies the relationship between membrane dynamics and boundary hydrodynamics.

Let us end these introductory comments by re emphasizing that the improved membrane stress tensor (5.7) yields the exact zero and first order constitutive relations of hydrodynamics even at finite D. Recall that our improved stress tensor (5.7) represents the sum over of what - from other points of view - would be regarded as a very particularly chosen infinite set of corrections to the leading large D stress tensor (5.5). The fact that precisely this infinite class of terms was sufficient to obtain exact results for zero and first order fluid coefficients suggests that improved membrane equations presented in this chapter represents a useful resummation of $\frac{1}{D}$ perturbation theory.

5.5.1 Equilibrium

Black Branes

In the previous section we studied the membrane solutions dual to static black holes of radius r_H in AdS_D spacetime. In the limit that $r_H \to \infty$, black hole reduce (locally) to black branes and their dual spherical membranes in global AdS space reduce locally to planar membranes in Poincare patch AdS space. We use notation

$$d = D - 1 \tag{5.105}$$

Recall that a black brane in AdS space is defined by the metric

$$ds^2 = \frac{1}{\rho^2}\left[-\left(1 - \frac{\rho^d}{\mathbf{z}^d}\right)dt^2 + \frac{d\rho^2}{\left(1 - \frac{\rho^d}{\mathbf{z}^d}\right)} + \delta_{ij}dx^i dx^j\right] \tag{5.106}$$

In (5.106) $\rho = \mathbf{z}$ is the event horizon of the black brane. Now we rewrite the metric (5.106) in Fefferman-Graham coordinates by change of ρ variable to z [33]. The metric becomes

$$ds^2 = \frac{1}{z^2}\left[-\frac{\left(1 - \frac{z^d}{4\mathbf{z}^d}\right)^2}{\left(1 + \frac{z^d}{4\mathbf{z}^d}\right)^{2-4/d}}dt^2 + dz^2 + \left(1 + \frac{z^d}{4\mathbf{z}^d}\right)^{4/d}\delta_{ij}dx^i dx^j\right] \tag{5.107}$$

Throughout this section we employ the Fefferman-Graham coordinate choice. Now expanding the metric (5.107) in power series in $\frac{z^d}{\mathbf{z}^d}$ and retaining terms upto the first subleading order in this expansion we get

$$ds^2 = -\left[1 - \left(\frac{d-1}{d}\right)\frac{z^{d-2}}{\mathbf{z}^d}\right]dt^2 + \frac{dz^2}{z^2} + \left[1 + \left(\frac{1}{d}\right)\frac{z^{d-2}}{\mathbf{z}^d}\right]\delta_{ij}dx^i dx^j \tag{5.108}$$

Now we can recover the boundary stress tensor corresponding to the black brane solution (5.108) by the prescription

$$\mathbb{T}^\mu_\nu = -\frac{1}{8\pi}\lim_{z\to 0}\frac{K^\mu_\nu - \delta^\mu_\nu}{z^d} \tag{5.109}$$

Note that to use the prescription (5.109), it is sufficient to use the metric expanded to linear order in $\frac{z^d}{4\mathbf{z}^d}$, i.e. (5.108), rather than full metric (5.107). Indeed it is easily verified that the boundary stress tensor corresponding to this solution is just the coefficient of z^{d-2} in the metric (5.108), and the boundary stress tensor [34] dual to the black brane solution is given by

$$16\pi\mathbb{T}_{\mu\nu} = \left(\frac{4\pi T_{bb}}{d}\right)^d(\eta_{\mu\nu} + d v_\mu v_\nu) \tag{5.110}$$

[33] Fefferman-Graham coordinates are defined by the requirement $G_{zz} = \frac{1}{z^2}$ and $G_{z\mu} = 0$.

[34] For this subsection, the boundary fluid-gravity metric, Stress tensor and Entropy current are written with 'hollow' letters ($\mathbb{G}_{\alpha\beta}$, $\mathbb{T}_{\mu\nu}$, $\mathbb{J}^\mu_S$) and the Membrane metric, Stress tensor and Entropy current are written with 'thick bold' letters ($\mathbf{g}_{\alpha\beta}$, $\mathbf{T}_{\mu\nu}$ $\mathbf{J}^\mu_S$).

where $v^\mu = (1, 0, 0, 0...)$ and the temperature of the black brane T_{bb} is given by

$$T_{bb} = \frac{d}{4\pi \mathbf{z}}, \tag{5.111}$$

The boundary entropy current corresponding to the black brane is given by

$$\mathbb{J}^\mu_S = \frac{1}{4}\left(\frac{4\pi T_{bb}}{d}\right)^{d-1} v^\mu \tag{5.112}$$

Flat Membranes

The membrane configuration dual to this black brane is given by the submanifold $z = \mathbf{z}$ of the pure AdS metric

$$ds^2 = \frac{dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu}{z^2} = \frac{dz^2 - dt^2 + dx^i dx_i}{z^2} \tag{5.113}$$

The membrane induced metric and stress tensor are given by

$$\begin{aligned} \mathbf{g}_{\mu\nu} &= \frac{\eta_{\mu\nu}}{\mathbf{z}^2} \\ \mathcal{T}_{MN} &= z\delta(z - \mathbf{z})T_{MN} \\ T_{zz} &= T_{\mu z} = 0, \quad T_{M=\mu, N=\nu} = \mathbf{T}_{\mu\nu} = \text{independent of } x^\mu \end{aligned} \tag{5.114}$$

[35] Until the very end of this subsection we will use no property of $\mathbf{T}_{\mu\nu}$ other than the fact that it is constant.

Let us now regard the stress tensor (5.114) as a source for gravitational fluctuations about AdS space (5.113) and compute the resultant linearized gravitational response. We consider the most general linearized correction to the background metric of the form

$$ds^2 \equiv \mathfrak{G}_{MN}dX^M dX^N = (G_{MN} + h_{MN})\, dX^M dX^N = \frac{dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu}{z^2} + h_{MN}dX^M dX^N \tag{5.115}$$

We adopt the Fefferman-Graham coordinate choice and so set

$$h_{zM} = 0 \tag{5.116}$$

The linearized Einstein equations evaluate to

$$\begin{aligned} E_{zz} &= -\frac{(d-1)}{2}z\partial_z h - (d-1)h + \frac{z^2}{2}\partial^2 h - \frac{z^2}{2}\partial_\alpha\partial_\beta h^{\alpha\beta} \\ E_{z\mu} &= \left(\frac{z^2}{2}\partial_z + z\right)(\partial_\alpha h^\alpha{}_\mu - \partial_\mu h) \\ E_{\mu\nu} &= \frac{z^2}{2}\left(\partial_\nu\partial_\alpha h^\alpha{}_\mu + \partial_\mu\partial_\alpha h^\alpha{}_\nu - \partial^2 h_{\mu\nu}\right) - \frac{z^2}{2}\partial_z^2 h_{\mu\nu} - \frac{z^2}{2}\partial_\mu\partial_\nu h + \left(\frac{d-5}{2}\right) z\partial_z h_{\mu\nu} \\ &\quad + (d-2)h_{\mu\nu} + \left[\frac{z^2}{2}\partial_z^2 h + \frac{z^2}{2}\partial^2 h - \frac{z^2}{2}\partial_\alpha\partial_\beta h^{\alpha\beta} - \frac{(d-5)}{2}z\partial_z h - (d-2)h\right]\eta_{\mu\nu} \end{aligned} \tag{5.117}$$

[35] The factor of z on the RHS of (5.114) is the factor of $|\partial\rho|$ in (5.35). In the case at hand $\rho = -z + \mathbf{z} + 1$.

In (5.117), the μ, ν indices are raised with $\eta^{\mu\nu}$ and $h \equiv h_{\mu\nu}\eta^{\mu\nu}$. Now in this case, clearly the resultant response inherits the translational invariance in the x^μ directions of the source (5.114). Away from $z = \mathbf{z}$ the response is thus a translationally invariant solution to the linearized Einstein equations about AdS space. In the Fefferman-Graham gauge it is easily verified that the most general linearized solution of this form is given by

$$h_{\mu\nu} = \frac{1}{z^2}\left(A^{(out)}_{\mu\nu} z^d + A^{(in)}_{\mu\nu}\right) \tag{5.118}$$

The requirement that our fluctuation is normalizable ensures $A^{(in)}_{\mu\nu} = 0$ outside the membrane , i.e. for $z < \mathbf{z}$. On the other hand the requirement that the fluctuation remain bounded on the Poincare horizon forces $A^{(out)}_{\mu\nu} = 0$ inside the membrane i.e. for $z > \mathbf{z}$. The requirement that the fluctuation $h_{\mu\nu}$ is continuous across the membrane implies that

$$A^{(in)}_{\mu\nu} = \mathbf{z}^d A^{(out)}_{\mu\nu} \equiv \mathbf{z}^d A_{\mu\nu}$$

So we have

$$h_{\mu\nu} = \begin{cases} A_{\mu\nu}\mathbf{z}^d z^{-2}, & \text{for} \quad z \geq \mathbf{z} \\ A_{\mu\nu} z^{d-2}, & \text{for} \quad z \leq \mathbf{z} \end{cases} \tag{5.119}$$

Finally, the junction matching condition on the membrane (refer [14] for similar calculation) relates the discontinuity of the Extrinsic curvature (as calculated in the linearized metric (5.115)) to membrane stress tensor as

$$\mathbf{T}_{\mu\nu} = -\frac{1}{8\pi}\left(\left[K^{(out)}_{\mu\nu} - K^{(in)}_{\mu\nu}\right] - \left[\mathcal{K}^{(out)} - \mathcal{K}^{(in)}\right]\mathbf{g}_{\mu\nu}\right) \tag{5.120}$$

We find the answers for Extrinsic curvature of the membrane seen from the inside and outside as

$$K_{\mu\nu} = \begin{cases} \frac{\eta_{\mu\nu}}{\mathbf{z}^2} + \mathbf{z}^{d-2} A_{\mu\nu}, & \text{for} \quad z \geq \mathbf{z} \\ \frac{\eta_{\mu\nu}}{\mathbf{z}^2} - \frac{d-2}{2}\mathbf{z}^{d-2} A_{\mu\nu}, & \text{for} \quad z \leq \mathbf{z} \end{cases} \tag{5.121}$$

Using (5.121) in (5.120) we get

$$A_{\mu\nu} = \frac{16\pi}{d}\frac{\mathbf{T}_{\mu\nu}}{\mathbf{z}^{d-2}} \tag{5.122}$$

It follows that the backreaction of the probe membrane modifies the metric of AdS spacetime to

$$\mathfrak{G}_{MN} dX^M dX^N = \begin{cases} \frac{1}{z^2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu + A_{\mu\nu}\mathbf{z}^d dx^\mu dx^\nu\right), & \text{for} \quad z \geq \mathbf{z} \\ \frac{1}{z^2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu + A_{\mu\nu} z^d dx^\mu dx^\nu\right), & \text{for} \quad z \leq \mathbf{z} \end{cases} \tag{5.123}$$

where $A_{\mu\nu}$ is given by (5.122). As we have explained above, the boundary Stress tensor is defined by the limit

$$\mathbb{T}^\mu_\nu = -\frac{1}{8\pi}\lim_{z\to 0}\frac{K^\mu_\nu - \delta^\mu_\nu}{z^d} \tag{5.124}$$

Using (5.121) in (5.124) we get the answer for boundary Stress tensor as

$$\mathbb{T}_{\mu\nu} = \frac{\mathbf{T}_{\mu\nu}}{\mathbf{z}^{d-2}} \tag{5.125}$$

(5.125) is the main result of this subsection. The analogous relationship between membrane and boundary metrics and energy currents takes the form

$$\begin{aligned} \mathbb{G}_{\mu\nu} &= \mathbf{z}^2 \mathbf{g}_{\mu\nu} \\ \mathbb{J}^\mu_E \equiv \mathbb{T}^\mu_\nu k^\nu &= \frac{\mathbf{T}^\mu_\nu k^\nu}{\mathbf{z}^d} = \frac{\mathbf{J}^\mu_E}{\mathbf{z}^d} \end{aligned} \tag{5.126}$$

Note that (5.126) are meaningful equations because we have used the 'same' x^μ coordinate on the membrane and on the boundary of spacetime. Note that (5.126) simply expresses the condition that the membrane velocity field and stress tensor are Weyl equivalent to the boundary velocity field and stress tensor in the case of stationary black branes. Note in particular that the boundary energy (charge carried by the current $\mathbb{J}^\mu_E$) contained in a part of a spacelike slice of the boundary is given by

$$\int \sqrt{\mathbb{G}_{ind}}\, \mathbb{J}^\mu_E \, r_\mu \tag{5.127}$$

where $\sqrt{\mathbb{G}_{ind}}$ is the boundary (fluid gravity) metric induced on the spatial slice, and r_μ is the unit normal to this slice. Noting that

$$\sqrt{\mathbb{G}_{ind}} = \mathbf{z}^{d-1} \sqrt{\mathbf{g}_{ind}}. \quad r_\mu = \mathbf{z}\, q_\mu$$

and using (5.126) it follows that (5.127) can be rewritten as

$$\int \sqrt{\mathbf{g}_{ind}}\, \mathbf{J}^\mu_E \, q_\mu \tag{5.128}$$

Identical comments apply to the entropy. In summary, the energy/entropy contained in any part of the boundary, computed using boundary currents, is identical to the energy/entropy of the 'same' region of the membrane, computed using membrane currents. In particular the formulae (5.86) in the planar black brane limit are easily reproduced directly from the membrane side.

Finally, to end this subsection we plug the explicit form of the membrane stress tensor T_{MN} into the formulae above and obtain explicit formulae for the linearized metric perturbation, the boundary stress tensor and the boundary entropy current dual to our flat membrane configuration. In order to obtain these quantities we note that the induced metric and extrinsic curvature of the membrane located at $z = \mathbf{z}$ are given by

$$ds^2_{ind} \equiv \mathbf{g}_{\mu\nu} dx^\mu dx^\nu = \frac{\eta_{\mu\nu}}{\mathbf{z}^2} dx^\mu dx^\nu = \frac{-dt^2 + dx^i dx_i}{\mathbf{z}^2}, \quad K_{\mu\nu} = \mathbf{g}_{\mu\nu} \tag{5.129}$$

It follows that the temperature of the flat membrane configuration is given by

$$T_m = \frac{\tilde{\mathcal{K}} \sqrt{-k.k}}{4\pi} = \frac{d}{4\pi \mathbf{z}} \tag{5.130}$$

and that the world volume membrane stress tensor is given by

$$16\pi \mathbf{T}_{\mu\nu} = \frac{\eta_{\mu\nu}}{\mathbf{z}^2} + d\, u_\mu u_\nu \tag{5.131}$$

Putting (5.131) in (5.125) it follows that the boundary stress tensor induced by the flat membrane of this subsubsection agrees exactly - at finite D - with the boundary stress tensor of the exact black brane solution (5.110) provided we make the identification (see (5.111), (5.130))

$$T_{bb} = T_m = \frac{d}{4\pi\ \mathbf{z}}, \quad v^\mu = \frac{u^\mu}{\mathbf{z}} \tag{5.132}$$

With these identifications the relations between the membrane and boundary entropy currents is given by

$$\mathbb{J}_S^\mu = \frac{\mathbf{J}_S^\mu}{\mathbf{z}^d} \tag{5.133}$$

We can get the explicit value of the total linearized metric outside the membrane using explicit value of the Stress tensor (5.131). Setting $v^\mu = (1, 0, 0...0)$ we find

$$h_{tt} = \left(\frac{d-1}{d}\right)\left(\frac{4\pi T_m}{d}\right)^d z^{d-2}, \quad h_{ij} = \left(\frac{1}{d}\right)\left(\frac{4\pi T_m}{d}\right)^d z^{d-2}\ \delta_{ij} \tag{5.134}$$

Note that (5.134) matches exactly with the expansion at linear order in $\frac{z^d}{\mathbf{z}^d}$ of the Fefferman Graham form of exact black brane metric i.e. (5.108). It follows immediately that the boundary stress tensor dual to our flat membrane configuration exactly matches the boundary stress tensor of a black brane once we use the identifications (5.132).

5.5.2 The boundary stress tensor in the derivative expansion

In the previous subsection we computed the linearized metric fluctuation (and thereby the boundary stress tensor) sourced by a membrane like stress tensor (5.114) that was localized at a constant value of z ($z = \mathbf{z}$) and was also uniform in space. Our final result was presented in (5.123). In this subsection will generalize the computation of the previous subsection in the following manner. We compute the linearized metric fluctuation sourced by the stress tensor

$$\begin{aligned} \mathcal{T}^{MN} &= z\sqrt{1+\partial_\mu \mathbf{z}\partial^\mu \mathbf{z}}\ \delta(z-\mathbf{z}(x))\ T^{MN} \\ T^{zz} &= \partial_\mu \mathbf{z}\ \partial_\nu \mathbf{z}\ \mathbf{T}^{\mu\nu}, \quad T^{\nu z} = \partial_\mu \mathbf{z}\ \mathbf{T}^{\mu\nu}, \quad T^{M=\mu, N=\nu} = \mathbf{T}^{\mu\nu} \end{aligned} \tag{5.135}$$

where $\mathbf{z}$ and $\mathbf{T}^{\mu\nu}$ are no longer constants but are slowly varying functions of x^μ. [36]. In what follows we view $\mathbf{T}^{\mu\nu}$ as a tensor valued field on the membrane world volume, and use the induced metric on the membrane to raise and lower its indices.

We will take advantage of the slowly varying nature of the functions $\mathbf{z}$ and $\mathbf{T}_{\mu\nu}$ to perform our computation to first nontrivial order (which turns out to be the second order) in an expansion of the derivatives of these fields.

At leading (zero) order in the derivative expansion the metric fluctuation sourced by (5.135) is simply given by the local form of (5.123), i.e.

$$\mathfrak{G}_{MN} dX^M dX^N = \frac{dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu}{z^2} + h_{\mu\nu} dx^\mu dx^\nu \tag{5.136}$$

[36]In the equation (5.135) μ indices have been raised using the induced metric on membrane $\mathbf{g}^{\mu\nu}$.

where

$$h_{\mu\nu} = \begin{cases} A_{\mu\nu}(x)\mathbf{z}(x)^d z^{-2}, & \text{for} \quad z \geq \mathbf{z}(x) \\ A_{\mu\nu}(x) z^{d-2}, & \text{for} \quad z \leq \mathbf{z}(x) \end{cases} \tag{5.137}$$

with

$$A_{\mu\nu}(x) = \frac{16\pi}{d} \frac{\mathbf{T}_{\mu\nu}(x)}{\mathbf{z}(x)^{d-2}} \tag{5.138}$$

Of course the metric (5.137) does not exactly solve Einstein's equations linearized about AdS space; when we plug (5.137) into Einstein's equations (5.4) with $\lambda = 1$, the LHS of these equations evaluates to an expression that is not zero, but turns out to be of second order in 'field theory' (i.e. x^μ) derivatives. In order to find a solution to the linearized Einstein equations valid to second order in derivatives we replace $h_{\mu\nu}$ in (5.137) by

$$h_{\mu\nu} = \begin{cases} A_{\mu\nu}(x)\mathbf{z}(x)^d z^{-2} + \delta h_{\mu\nu}(z,x), & \text{for} \quad z \geq \mathbf{z}(x) \\ A_{\mu\nu}(x) z^{d-2} + \delta h_{\mu\nu}(z,x), & \text{for} \quad z \leq \mathbf{z}(x) \end{cases} \tag{5.139}$$

where $\delta h_{\mu\nu}$ is an as yet unknown correction. We then plug (5.139) into the Einstein equation (5.4). We assume that $\delta h_{\mu\nu}$ is of second order in derivatives, and work consistently to this order (i.e. we ignore all terms in the equation that are of third or higher order). The LHS of (5.4) now has terms of two sorts. First we have the 'source' terms, independent of $\delta h_{\mu\nu}$ that we have already encountered earlier in this paragraph. In addition we have new terms proportional to $\delta h_{\mu\nu}$. Setting the sum of these terms to zero in the dynamical Einstein equations (dynamical w.r.t. evolution in z) yields an equations of the schematic form

$$H \delta h_{\mu\nu} = s_{\mu\nu} \tag{5.140}$$

where $s_{\mu\nu}$ are source terms and H is a differential operator of second order in z derivatives. Note that the differential operator has no derivatives in the x^μ directions - x^μ derivatives on $\delta h_{\mu\nu}$ result in expressions that are of third or higher order in derivatives and so are ignored at the order at which we work.

In order to obtain a unique solution to the equations (5.140) we impose the following boundary conditions. First we demand that the 'outside' solution is normalizable. Second we demand that the 'inside' solution does not blow up at $z = \infty$. Third we require that $\delta h_{\mu\nu}$ is continuous across the membrane located at $z = \mathbf{z}(x)$. Fourth we require the solution to obey the appropriate junction matching condition across the membrane (see below). These four conditions allow us to determine the four integration constants (two for the outside solution and two for the inside solution) that appear in the most general solution of (5.140) and thereby obtain a unique solution for $\delta h_{\mu\nu}$. The algebra involved in our work out is straightforward and we simply present our final results.

$$\delta h_{\mu\nu}(z,x) = \begin{cases} C^{(in)}_{\mu\nu} z^{-2} + B^{(in)}_{\mu\nu}, & \text{for} \quad z \geq \mathbf{z}(x) \\ C^{(out)}_{\mu\nu} z^{d-2} + B^{(out)}_{\mu\nu} z^d, & \text{for} \quad z \leq \mathbf{z}(x) \end{cases} \tag{5.141}$$

where

$$
\begin{aligned}
B_{\mu\nu}^{(out)} &= -\frac{\partial^2 A_{\mu\nu}^{(out)}}{2(d+2)} \\
B_{\mu\nu}^{(in)} &= -\frac{1}{2(d-2)}\left(\partial_\nu\partial^\alpha A_{\alpha\mu}^{(in)} + \partial_\mu\partial^\alpha A_{\alpha\nu}^{(in)} - \partial^2 A_{\mu\nu}^{(in)} - \frac{\partial^\alpha\partial^\beta A_{\alpha\beta}^{(in)}}{d-1}\eta_{\mu\nu}\right) \\
C_{\mu\nu}^{(out)} &= \frac{\mathbf{z}^2}{2d}\partial^2 A_{\mu\nu}^{(out)} - \frac{1}{d(d-2)}\left(\partial_\nu\partial^\alpha A_{\alpha\mu}^{(in)} + \partial_\mu\partial^\alpha A_{\alpha\nu}^{(in)} - \partial^2 A_{\mu\nu}^{(in)}\right) \\
&\quad - \frac{\partial^\alpha\mathbf{z}\partial_\alpha\mathbf{z}}{2}A_{\mu\nu}^{(out)} + \left(\frac{\partial^\alpha\partial^\beta A_{\alpha\beta}^{(in)}}{d(d-2)\mathbf{z}^{d-2}} - \partial_\alpha\mathbf{z}\partial_\beta\mathbf{z}\ A_{\alpha\beta}^{(out)}\right)\eta_{\mu\nu} \\
C_{\mu\nu}^{(in)} &= C_{\mu\nu}^{(out)}\mathbf{z}^d + B_{\mu\nu}^{(out)}\mathbf{z}^{d+2} - \mathbf{z}^2 B_{\mu\nu}^{(in)} \\
A_{\mu\nu}^{(in)} &= \mathbf{z}^d A_{\mu\nu}^{(out)} \equiv \mathbf{z}^d A_{\mu\nu}
\end{aligned}
\tag{5.142}
$$

In order to obtain the results listed above we have used the fact that the extrinsic curvature of the slice $z = \mathbf{z}(x)$ is given up to linear order in $h_{\mu\nu}$ and second order in field theory derivatives by

$$
\begin{aligned}
K_{\mu\nu} = \Bigg[&\frac{\eta_{\mu\nu}}{z^2} - \frac{z}{2}\partial_z h_{\mu\nu} + \frac{\partial_\mu\partial_\nu\mathbf{z}}{z} + \frac{\partial_\mu\mathbf{z}\partial_\nu\mathbf{z}}{z^2} + \frac{z}{4}\partial_z h_{\mu\nu}\partial^\alpha\mathbf{z}\partial_\alpha\mathbf{z} \\
&- \frac{z}{2}\partial^\alpha\mathbf{z}\left(\partial_\mu h_{\alpha\nu} + \partial_\nu h_{\alpha\mu} - \partial_\alpha h_{\mu\nu}\right) + \frac{1}{2z^2}\left(-\partial^\alpha\mathbf{z}\partial_\alpha\mathbf{z} + \mathbf{z}^2 h^{\alpha\beta}\partial_\alpha\mathbf{z}\partial_\beta\mathbf{z}\right)\eta_{\mu\nu} \\
&- \left(\frac{z}{2}\partial_z + 1\right)\left(h_{\mu\alpha}\partial^\alpha\mathbf{z}\partial_\nu\mathbf{z} + h_{\nu\alpha}\partial^\alpha\mathbf{z}\partial_\mu\mathbf{z}\right)\Bigg]_{z\to\mathbf{z}}
\end{aligned}
\tag{5.143}
$$

Note that the expression (5.143) depends on $\partial_z h_{\mu\nu}$. As this quantity jumps across the membrane, the extrinsic curvature 'above' the membrane is discontinuously different from the same quantity 'below' the membrane. The difference between these two quantities is governed by the 'junction condition' mentioned above

$$
\mathbf{T}_{\mu\nu} = -\frac{1}{8\pi}\left(\left[K_{\mu\nu}^{(out)} - K_{\mu\nu}^{(in)}\right] - \left[\mathcal{K}^{(out)} - \mathcal{K}^{(in)}\right]\mathbf{g}_{\mu\nu}\right) \tag{5.144}
$$

It is not difficult to evaluate the boundary stress tensor dual to the solution presented above using the definition (5.124); we find

$$
\begin{aligned}
\mathbb{T}_{\mu\nu} = &\frac{\mathbf{T}_{\mu\nu}}{\mathbf{z}^{d-2}} + \frac{\mathbf{z}^2}{2d}\ \partial^2\left(\frac{\mathbf{T}_{\mu\nu}}{\mathbf{z}^{d-2}}\right) - \frac{\partial^\alpha\mathbf{z}\partial_\alpha\mathbf{z}}{2\ \mathbf{z}^{d-2}}\mathbf{T}_{\mu\nu} + \frac{\partial^\alpha\partial^\beta\left(\mathbf{z}^2\mathbf{T}_{\alpha\beta}\right)}{d(d-2)\mathbf{z}^{d-2}}\eta_{\mu\nu} - \frac{\partial^\alpha\mathbf{z}\partial^\beta\mathbf{z}\mathbf{T}_{\alpha\beta}}{\mathbf{z}^{d-2}}\eta_{\mu\nu} \\
&- \frac{1}{d(d-2)\mathbf{z}^{d-2}}\left[\partial_\nu\partial^\alpha\left(\mathbf{z}^2\mathbf{T}_{\alpha\mu}\right) + \partial_\mu\partial^\alpha\left(\mathbf{z}^2\mathbf{T}_{\alpha\nu}\right) - \partial^2\left(\mathbf{z}^2\mathbf{T}_{\mu\nu}\right)\right]
\end{aligned}
\tag{5.145}
$$

The results (5.141), (5.142) were obtained by solving the dynamical Einstein equations. The Einstein constraint equations (for evolution along the z direction) remain to be solved. The situation with these equations is closely analogous to that encountered in section 4.3 of [14] in a distinct but related context.

Let us first recall the following general property of Einstein's equations: provided the dynamical equations are solved everywhere, the constraint equations are automatically solved everywhere if they are solved on a single slice. As we have already dealt with the dynamical equations, it remains only to solve the constraint equations on any one slice on the outside and on any other slice on the 'inside'. It is convenient to choose these slices to be the membrane world volume, approached either from the outside or from the inside.

Let us recall that the constraint equations are of two sorts; the momentum constraint equations and the 'Hamiltonian' constraint equations. Let us first deal with the momentum constraint equations. These equations are simply the statement that the Brown York tensor of the full metric (background plus fluctuation) is conserved on our slice. Now as in section 4.3 of [14], it turns out that this condition is automatic for the inside solution (this is suggested by the general argument of section 4.3 of [14] and we have explicitly algebraically verified that it is the case for the explicit solution presented above). On the other hand the Brown York stress tensor is not identically conserved just outside the membrane. However it follows from (5.144) that the difference between the conservation of the BY tensor outside and the BY tensor inside the membrane is simply the condition that the membrane stress tensor is conserved on its world volume. As the inside BY tensor is identically conserved, it follows that the outside BY tensor is conserved - and hence the outside Einstein constraint equation obeyed - if and only if the membrane stress tensor is conserved on its world volume.

We have already mentioned above, once the membrane stress tensor is conserved on the membrane world volume, this automatically ensures that the momentum constraint equations are solved everywhere. The momentum constraint equations are particularly interesting when evaluated on the boundary of AdS, where they assert the conservation of the boundary stress tensor (5.145). It follows, in other words, that conservation of the membrane stress tensor and the boundary stress tensor must be algebraically equivalent statements: one must imply the other. It is easy to directly verify that this is the case. In particular we have algebraically verified, using (5.145), that (5.2) is algebraically equivalent to the condition

$$\nabla^{\mu}\mathbb{T}_{\mu\nu} = 0 \tag{5.146}$$

(where ∇^{μ} in (5.146) is the boundary field theory covariant derivative -i.e. the raised partial derivative in flat space. [37]

In a similar manner, the Hamilton constraint equations are automatically (identically) obeyed for the inside solution. The condition that they are also obeyed on the outside solution follows provided that (5.3) holds (see around 4.25 of [14] for a proof). At the boundary of AdS, on the other hand, this constraint equation simply reduces to the condition that the boundary stress tensor is traceless. It follows, in other words, that the tracelessness of the boundary stress tensor

$$\mathbb{T}_{\mu\nu}\eta^{\mu\nu} = 0 \tag{5.147}$$

must be algebraically identical to the condition (5.3) for the membrane stress tensor. Using the explicit result (5.145) we have directly verified that this is the case.

[37] Note that, as in discussions of the fluid gravity correspondence, the equation (5.146) has an explicit derivative. It follows that the constraint equation (5.146) at $(n+1)^{th}$ order is completely determined by knowledge of the stress tensor at n^{th} order.

In summary, the solution (5.141), (5.142) solves all Einstein momentum constraint equations in addition to the Einstein dynamical equations if and only if the membrane stress tensor is conserved on its world volume and also obeys the equation (5.3). The resultant boundary stress tensor (5.145) is then automatically conserved and traceless.

5.5.3 Boundary stress tensor in terms of fluid variables

Plugging the explicit form of the membrane stress tensor, (5.7) into the general formula (5.145), we find that the boundary stress tensor dual to our membrane -accurate to second order in derivatives - is given by

$$\begin{aligned}
\mathbb{T}_{\mu\nu} &= t^{(0)}_{\mu\nu} + t^{(1)}_{\mu\nu} + t^{(2)}_{\mu\nu} \\
t^{(0)}_{\mu\nu} &= \frac{1}{\mathbf{z}^d}\left(\eta_{\mu\nu} + d\ v_\mu v_\nu\right) \\
t^{(1)}_{\mu\nu} &= -\frac{2}{\mathbf{z}^{d-1}}\sigma_{\mu\nu} \\
t^{(2)}_{\mu\nu} &= \frac{1}{\mathbf{z}^{d-2}}\Bigg[\left[-\left(\frac{d}{2}\right)\frac{\partial^\alpha\mathbf{z}\partial_\alpha\mathbf{z}}{\mathbf{z}^2} + \left(\frac{d-2}{d-1}\right)\frac{\partial^\alpha\partial_\alpha\mathbf{z}}{\mathbf{z}} - \left(\frac{d}{d-1}\right)\frac{v^\alpha v^\beta\partial_\alpha\partial_\beta\mathbf{z}}{\mathbf{z}}\right]P_{\mu\nu} \\
&+ \left[\left(\frac{d-1}{2}\right)\frac{\partial^\alpha\mathbf{z}\partial_\alpha\mathbf{z}}{\mathbf{z}^2} - \frac{\partial^\alpha\partial_\alpha\mathbf{z}}{\mathbf{z}}\right]\eta_{\mu\nu} + \left[\frac{\partial_\mu\mathbf{z}\partial_\nu\mathbf{z}}{\mathbf{z}^2} + \frac{\partial_\mu\partial_\nu\mathbf{z}}{\mathbf{z}}\right] - d\frac{\partial^\alpha\mathbf{z}}{\mathbf{z}}\left(v_\mu\partial_\alpha v_\nu + v_\nu\partial_\alpha v_\mu\right) \\
&+ \frac{1}{2}\left(\frac{d}{d-2}\right)\left(v_\mu\partial^2 v_\nu + v_\nu\partial^2 v_\mu + 2\partial^\alpha v_\mu\partial_\alpha v_\nu\right) - \frac{1}{d-2}\Big[\left(v_\mu\partial_\nu + v_\nu\partial_\mu\right)(\partial.v) \\
&+ (\partial.v)\left(\partial_\mu v_\nu + \partial_\nu v_\mu\right) + \left(\partial_\mu v^\alpha\partial_\alpha v_\nu + \partial_\nu v^\alpha\partial_\alpha v_\mu\right) + v.\partial\left(\partial_\mu v_\nu + \partial_\nu v_\mu\right)\Big] \\
&+ \left[d\frac{\partial^\alpha\mathbf{z}\partial_\alpha\mathbf{z}}{2\mathbf{z}^2} - \frac{\partial^\alpha\partial_\alpha\mathbf{z}}{2\mathbf{z}}\right]\left(\eta_{\mu\nu} + d v_\mu v_\nu\right) - \left[\frac{\partial^\alpha\mathbf{z}\partial_\alpha\mathbf{z}}{\mathbf{z}^2} + d\frac{(v.\partial\mathbf{z})^2}{\mathbf{z}^2}\right]\eta_{\mu\nu} \\
&+ \frac{1}{d-2}\left[(\partial.v)^2 + 2v.\partial(\partial.v) + \partial^\alpha v^\beta\partial_\beta v_\alpha\right]\eta_{\mu\nu} - d\frac{(v.\partial\mathbf{z})^2}{\mathbf{z}^2}v_\mu v_\nu\Bigg]
\end{aligned} \tag{5.148}$$

where,

$$v^\mu = \frac{u^\mu}{\mathbf{z}}, \quad P_{\mu\nu} = \eta_{\mu\nu} + v_\mu v_\nu, \quad \sigma_{\mu\nu} = \left(\frac{\partial_\alpha v_\beta + \partial_\beta v_\alpha}{2}\right)P^\alpha_\mu P^\beta_\nu - \left(\frac{\partial.v}{d-1}\right)P_{\mu\nu} \tag{5.149}$$

The expression for $t^{(2)}_{\mu\nu}$ above can be simplified by recalling that we are interested only in onshell configurations of our boundary fluid. At zero order in derivatives, the conservation of $t^{(0)}_{\mu\nu}$ yields

$$\partial^\mu t^{(0)}_{\mu\nu} = 0, \quad \partial_\alpha\partial^\mu t^{(0)}_{\mu\nu} = 0 \tag{5.150}$$

From (5.150) we get

$$\begin{aligned}
\frac{\partial_\mu \mathbf{z}}{\mathbf{z}} &= v.\partial v_\mu - \frac{\partial.v}{d-1} v_\mu \\
\frac{\partial_\mu \partial_\nu \mathbf{z}}{\mathbf{z}} &= \left(v.\partial v_\mu - \frac{\partial.v}{d-1} v_\mu\right)\left(v.\partial v_\nu - \frac{\partial.v}{d-1} v_\nu\right) - \frac{1}{d-1}\partial_\mu(\partial.v)v_\nu \\
&- \frac{\partial.v}{d-1}\partial_\mu v_\nu + \partial_\mu v^\lambda \partial_\lambda v_\nu + v.\partial(\partial_\mu v_\nu)
\end{aligned} \tag{5.151}$$

Of course the object that is really conserved is the full stress tensor rather than simply $t^{(0)}_{\mu\nu}$. This means that the RHS of (5.151) has corrections that are of higher order in derivatives. We will now use the equations (5.151) to simplify $t^{(2)}_{\mu\nu}$; the corrections to (5.151) yield terms of third or higher order in derivatives and so can be ignored. We thus proceed to simplify (5.148) by using (5.151) to replace occurrence of a term in $t^{(2)}_{\mu\nu}$ involving derivatives of $\mathbf{z}$ with the expressions on the RHS of (5.151). The resultant expression for $t^{(2)}_{\mu\nu}$ is a sum of two derivative terms with all derivatives acting on the velocity field v_μ. The final expression for the resulting expression is somewhat cumbersome and we do not explicitly list it here.

We will now perform a field redefinition from the natural membrane variables v_μ and $\mathbf{z}$ to more natural - and more standard - boundary variables. Let us define the Landau Frame boundary velocity field v_μ and and the boundary temperature T by the conditions

$$\mathbb{T}^\mu{}_\nu \mathsf{v}^\nu = -(d-1)\left(\frac{4\pi\mathsf{T}}{d}\right)^d \mathsf{v}^\mu \tag{5.152}$$

In other words v^μ is the unique timelike eigenvector of the boundary stress tensor (normalized to be a boundary velocity field) and T is simply defined in terms of its eigenvalue. It is not difficult to solve for v_μ and T in terms of v_μ and $\mathbf{z}$, order by order in the derivative expansion. At zero order in derivatives we work with the simple stress tensor $t^{(0)}_{\mu\nu}$; it is easily verified that

$$\mathsf{T} = T = \frac{d}{4\pi\ \mathbf{z}}, \quad \mathsf{v}^\mu = v^\mu = \frac{u^\mu}{\mathbf{z}} \tag{5.153}$$

Note that, at this order, (5.153) agrees with (5.132) as we might have anticipated on general grounds.

The relation

$$t^{(1)}_{\mu\nu} v^\mu = 0$$

immediately implies that the solution (5.153) continues to hold at first order in derivatives. The situation is more complicated at second order. At this order (5.153) is corrected to

$$\mathsf{T} = T(1+\delta T), \quad \mathsf{v}_\mu = v_\mu + \delta v_\mu \tag{5.154}$$

where,

$$\begin{aligned}
\delta T = \frac{1}{d(d-1)}\left(\frac{d}{4\pi T}\right)^2 \Bigg[&-\frac{1}{2}\left(\frac{d^2-7d+8}{d-2}\right)\sigma_{\alpha\beta}\sigma^{\alpha\beta} + \frac{1}{2}\left(\frac{d^2-3d+8}{d-2}\right)\omega_{\alpha\beta}\omega^{\alpha\beta} \\
&- \frac{(d-4)}{2} v.\partial(\partial.v) + \frac{(d-1)(d-2)}{2} v.\partial v_\lambda v.\partial v^\lambda - \frac{(d-1)(d-2)}{2}\left(\frac{\partial.v}{d-1}\right)^2 \Bigg]
\end{aligned} \tag{5.155}$$

and

$$\delta v_\mu = \frac{P^\lambda_\mu}{d}\left(\frac{d}{4\pi T}\right)^2\Bigg[-\frac{1}{2}\left(\frac{2d^2-5d+4}{(d-1)(d-2)}\right)(\partial.v)v.\partial v_\lambda + \frac{1}{2}\left(\frac{3d-4}{(d-1)(d-2)}\right)\partial_\lambda(\partial.v)$$
$$+\frac{(d-4)}{2(d-2)}v.\partial(v.\partial v_\lambda) - \frac{d}{2(d-2)}\partial^2 v_\lambda + (d)\ v.\partial v^\alpha\partial_\alpha v_\lambda - \frac{(d-4)}{2(d-2)}\partial_\lambda v_\alpha v.\partial v^\alpha\Bigg] \tag{5.156}$$

Plugging (5.155) and (5.156) into (5.148) we obtain our final expression for the boundary stress tensor expressed in terms of boundary Landau frame temperature and velocity fields

$$\begin{aligned}\mathbb{T}_{\mu\nu} &= p\,(\eta_{\mu\nu} + d\ \mathsf{v}_\mu\mathsf{v}_\nu) - 2\eta\sigma_{\mu\nu}\\ &+ 2\eta\left(\frac{d}{4\pi\mathsf{T}}\right)\Bigg[\left(\sigma^\lambda_\mu\sigma_{\lambda\nu} - \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P_{\mu\nu}\right) - \frac{2}{d-2}\left(\omega^\lambda_\mu\omega_{\lambda\nu} + \frac{\omega_{\alpha\beta}\omega^{\alpha\beta}}{d-1}P_{\mu\nu}\right)\\ &-\frac{1}{2}\left(\frac{d}{d-2}\right)\left(\omega^\lambda_\mu\sigma_{\lambda\nu} + \omega^\lambda_\nu\sigma_{\lambda\mu}\right) + \frac{1}{2}\left(\frac{d-4}{d-2}\right)(\mathsf{v}.D\sigma_{\mu\nu})\Bigg]\end{aligned} \tag{5.157}$$

Where we have

$$p = \frac{1}{16\pi}\left(\frac{4\pi\mathsf{T}}{d}\right)^d, \quad \eta = \frac{1}{16\pi}\left(\frac{4\pi\mathsf{T}}{d}\right)^{d-1}, \quad \mathsf{v}.D\sigma_{\mu\nu} = P^\alpha_\mu P^\beta_\nu\ \mathsf{v}.\partial\sigma_{\alpha\beta} + \frac{\partial.\mathsf{v}}{d-1}\sigma_{\mu\nu} \tag{5.158}$$

and the quantities $\sigma_{\mu\nu},\ \omega_{\mu\nu},\ P_{\mu\nu}$ are constructed from v. As a nontrivial check of the algebra leading up to (5.157) we note that the stress tensor (5.157) is Weyl covariant (see [16, 58]).

Let us now compare the second order hydrodynamical stress tensor (5.157) with the corresponding object obtained from the fluid gravity map listed in [38, 16, 58]. In the current chapter we have worked with a flat boundary metric, and so should set the boundary Weyl tensor in the fluid gravity papers listed above to zero. In this case the results of [38, 16, 58] are

$$\begin{aligned}\mathbb{T}^{(fg)}_{\mu\nu} &= p\,(\eta_{\mu\nu} + d\ \mathsf{v}_\mu\mathsf{v}_\nu) - 2\eta\sigma_{\mu\nu}\\ &- 2\eta\tau_\omega\left[\mathsf{v}.D\sigma_{\mu\nu} + \omega^\lambda_\mu\sigma_{\lambda\nu} + \omega^\lambda_\nu\sigma_{\lambda\mu}\right] + 2\eta\left(\frac{d}{4\pi\mathsf{T}}\right)\left[\mathsf{v}.D\sigma_{\mu\nu} + \sigma^\lambda_\mu\sigma_{\lambda\nu} - \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P_{\mu\nu}\right]\\ &\text{where,}\ \ \tau_\omega = \left(\frac{d}{4\pi\mathsf{T}}\right)\int_1^\infty \frac{y^{d-2}-1}{y(y^d-1)}dy = \left(\frac{d}{4\pi\mathsf{T}}\right)\left(\frac{1}{2} - \frac{\pi^2}{3d^2} + \mathcal{O}(\frac{1}{d^3})\right)\end{aligned} \tag{5.159}$$

The quantities p and η in (5.159) were listed in (5.158).

Clearly (5.157) agrees exactly (at finite d) with (5.159) at zero and first order in the derivative expansion. At second order the two stress tensors have the same tensor structures. The coefficients of individual tensor structures match perfectly at leading order in the large d limit, but deviate from each other at subleading orders in this expansion.

The 'flow' from membrane hydrodynamics to boundary hydrodynamics derived in this section has some similarities with the analysis of [59]. It might be interesting to explore this connection in greater detail in the future.

5.5.4 Quasinormal modes from membrane stress tensor about uniform planar membrane in AdS

In the previous subsection we demonstrated that the nonlinear equations that govern the motion of a membrane in planar AdS space reduce, in the derivative expansion, to the equations of boundary hydrodynamics. The boundary stress tensor is given in terms of the local boundary fluid velocity and temperature by a constitutive relation that agrees on the dot with the finite D fluid gravity constitutive relation at first order in the derivative expansion, but deviates (at finite D) from fluid gravity at second and higher orders in this expansion.

In this subsection we will explore related physics by performing a related but distinct computation - we use the membrane equations to compute the spectrum of small fluctuations about an exactly planar membrane in Poincare patch AdS space, and compare our results with the spectrum of quasinormal modes about the dual black brane in AdS space. Once again we find that the spectrum computed using our membrane equations perfectly reproduces black brane quasinormal mode spectrum to leading and first subleading order in k, but reproduces higher order corrections only in the large D limit.

We consider background spacetime AdS_D with $\lambda = \frac{1}{L^2} = 1$

$$ds^2 = -r^2 dt^2 + \frac{dr^2}{r^2} + r^2(dx^a dx_a) \tag{5.160}$$

Let the planar membrane be located at $r = r_0$. For convenience we choose $r_0 = 1$; it is easy to reinstate factors of r_0 in the final answer. In this section we closely follow the method used in [11]; we refer the reader interested in details to that paper and report only key results.

Consider the membrane configuration

$$\begin{aligned} r &= 1 + \epsilon\delta r(t, a) \\ u &= -(1 + \epsilon\delta r)dt + \epsilon\delta u_a(t, a)dx^a \end{aligned} \tag{5.161}$$

(the δr dependence in the velocity fluctuation is dictated by the requirement that $u^2 = -1$). The induced metric on membrane is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(1 + 2\epsilon\delta r)dt^2 + (1 + 2\epsilon\delta r)(dx^a dx_a) \tag{5.162}$$

The projector orthogonal to the fluid velocity is easily evaluated; we find

$$\mathcal{P}^a_b = \delta^a_b, \quad \mathcal{P}^t_t = 0, \quad \mathcal{P}^t_a = \epsilon\delta u_a, \quad \mathcal{P}^a_t = -\epsilon\delta u^a \tag{5.163}$$

We have the membrane equation

$$\begin{aligned} &\nabla . u = 0 \\ &16\pi\ \mathcal{P}^\nu_\alpha \nabla^\mu T_{\mu\nu} = \left(\tilde{\mathcal{K}}\ u.\nabla u_\nu + \nabla_\nu \tilde{\mathcal{K}} - 2\nabla^\mu \sigma_{\mu\nu}\right)\mathcal{P}^\nu_\alpha \equiv E_\nu \mathcal{P}^\nu_\alpha \end{aligned} \tag{5.164}$$

To linear order in fluctuations we find

$$\begin{aligned} &\sigma_{tt} = 0, \quad \sigma_{ta} = 0, \quad \sigma_{ab} = \epsilon\frac{\partial_a\delta u_b + \partial_b\delta u_a}{2} + \epsilon\partial_t\delta r\delta_{ab} \\ &\tilde{\mathcal{K}} = (D-1) + 2\epsilon\left(\partial_t^2\delta r - \partial^2\delta r\right) + \left(\frac{D-1}{D-2}\right)\epsilon\partial^2\delta r \\ &u.\nabla u_t = 0, \quad u.\nabla u_a = \epsilon\partial_t\delta u_a + \epsilon\partial_a\delta r \end{aligned} \tag{5.165}$$

Using these results the membrane equations (5.164) simplify to

$$\begin{aligned}
&\partial_a \delta u^a = -(D-2)\partial_t \delta r \\
&V_a \equiv (D-1)(\partial_t \delta u_a + \partial_a \delta r) + 2(\partial_a \partial_t^2 - \partial_a \partial^2 \delta r) + \left(\frac{D-1}{D-2}\right)\partial_a \partial^2 \delta r \\
&- \left(\partial^2 \delta u_a - (D-2)\partial_a \partial_t \delta r\right) - 2\partial_a \partial_t \delta r = 0
\end{aligned} \tag{5.166}$$

Inserting the plane wave expansion

$$\delta r = a e^{ik.x - i\omega t}, \quad \delta u_a = b_a e^{ik.x - i\omega t} \tag{5.167}$$

into (5.166), we find that our equations have solutions if and only if ω obeys either the sound wave dispersion relation (recall $d = D-1$ and $k = \sqrt{k.k}$)

$$\omega^s = \pm\left(\frac{k}{\sqrt{d-1}}\right)\left[\frac{\sqrt{d^2(d-1)^2 + 4(d-1)^2k^2 + 2(d-2)k^4}}{d(d-1)+2k^2}\right] - i\left[\frac{(d-2)k^2}{d(d-1)+2k^2}\right] \tag{5.168}$$

or the shear wave dispersion relation

$$\omega^v = -i\frac{k^2}{d} \tag{5.169}$$

Note, in particular, that (5.169) takes an incredibly simple purely imaginary form.

In order to compare with the spectrum of quasinormal modes about black branes, we expand these results in power series in k. We get

$$\omega^v = -i\frac{k^2}{d} + \mathcal{O}(k^3), \quad \omega^s = \pm\left(\frac{k}{\sqrt{d-1}}\right) - i\left[\frac{(d-2)k^2}{d(d-1)}\right] + \mathcal{O}(k^3) \tag{5.170}$$

The results (5.170) exactly (i.e. at arbitrary values of D and not merely at large D) match the spectrum of the lightest quasinormal modes expanded around a black brane to the respective orders reported in the derivative expansion [7] (see equation (6.1) and (6.2) in that paper); as might have been anticipated from the fact that our membrane exactly reproduces the fluid gravity stress tensor at zero and first order in derivatives even at finite D (see above). It is also, however, easily verified that (5.168) and (5.169) do not match the exact finite D gravitational results at higher orders in k. (however the match persists in the large D limit). This could also have been anticipated from the fact that our membrane accurately reproduces the second order terms in the hydrodynamical stress tensor only at large D (see above).

Note that the paper [7] directly computed the black brane quasi normal modes within gravity in an expansion in large D. They obtained results very similar to our (5.170); however the effective coefficient functions of the various terms in (5.170) were obtained in [7] order by order in an expansion in $\frac{1}{D}$ (upto a particular order see (4.23),(4.24),(4.25) of that paper). In contrast our membrane equations reproduces the reported coefficients exactly.

We find it very encouraging that the simple membrane equations of this chapter reproduce some gravitational results exactly as a function of D. It appears that the simple membrane equations presented in this chapter (whose form was dictated by physical consistency requirements) resum an infinite class of corrections of other approaches, and so do a particularly good job of reproducing gravitational results to higher accuracy than might have been reasonable to expect.

5.6 Discussion

In this chapter we have made four main points.

- At least at leading order, it is possible to 'improve' the large D perturbative expansion of black hole physics presented in earlier work. The improved leading order equations are chosen so that they agree with earlier derived results at leading order in the large D limit but also define consistent probe membrane dynamics at finite D . Even though our improved equations define consistent probe dynamics at finite D, they do not exactly reproduce black hole physics at finite D in generic situations, even though they appear to work surprisingly well in some equilibrium and near equilibrium configurations.

- The velocity field in stationary solutions of the improved membrane equations is always proportional to a killing vector of the background spacetime in which the membrane propagates. The membrane shape in such configurations obeys a differential equation that follows from extremizing a simple action for the membrane shape. Onshell this action reduces to the thermodynamical membrane partition function.

- The thermodynamics of static spherical membranes in flat space and global AdS space, obtained via this procedure, agrees exactly with that of their dual black holes even at finite D.

- The motion of a membrane in Poincare Patch AdS space sources linearized gravitational fluctuations and so a boundary stress tensor. In the long wavelength limit the resultant boundary stress tensor is a hydrodynamical stress tensor for a conformal boundary fluid. At zero and first order in the derivative expansion, this stress tensor exactly reproduces the results of the fluid gravity correspondence even at finite D. At second order in derivatives, the fluid dual to improved probe membrane agrees with the second order fluid gravity stress tensor at large D, but deviates from these exact results at finite D.

Each of the points listed above throws up several interesting questions and directions for future research. One immediate question is whether the improvement of the leading large D membrane equations, presented in this chapter can be systematically continued order by order, in large D perturbation theory. More precisely the question is the following. Given any positive integer n, can we always (in principle) find an improved n^{th} order membrane stress tensor with the following two properties. First, that the expansion of our improved stress tensor to n^{th} order in $\frac{1}{D}$ agrees with the 'naive' n^{th} order stress tensor obtained from the naive large D expansion (i.e. by following the algorithm presented in [10, 14]). Second, that our improved n^{th} order stress tensor autonomously defines consistent probe dynamics at finite D. [38] We suspect that the answer to this question is in the affirmative, and that the techniques developed in [60] and related subsequent work [61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76] will prove useful in demonstrating this issue. In particular, a device adopted in several of the papers [62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76] - namely the use

[38] As a first calculational check it would be useful to obtain explicit results for the improved large D expansion at first subleading order in $\frac{1}{D}$.

of 'diffeomorphisms' as the basic degrees of freedom to describe hydrodynamics - may have a very natural generalization to the context of this chapter, as a single bulk diffeomorphisms (starting from a prescribed membrane world volume) could generate both the most general membrane shape as well as the most general membrane velocity field. We hope to return to these questions in the future. [39]

There are also several interesting open questions relating to the action that governs equilibrium membrane configurations. First, as we have explained in the main text, we suspect that the very simple general structure of this action - namely that it is given by the sum of a Gibbons Hawking term and the action for a stationary fluid on the membrane - persists to every order in the $\frac{1}{D}$ expansion. It would be useful to explicitly verify this expectation, atleast at first subleading order in $\frac{1}{D}$. Second, it is natural to wonder whether this structure of the action - that it is the sum of a Gibbons Hawking like term plus a fluid action - generalizes to the study of an arbitrary higher derivative diffeomorphically invariant theory of gravity. Finally, it may be interesting to investigate whether there is a sense in which the offshell membrane action presented in this chapter can be obtained from an offshell gravitational action for an appropriate dual set of configurations.

In Appendix D.4 we have noted that the exact finite D agreement between spherical membranes and their dual Schwarzschild black holes appears not to carry over to rotating black holes. It may be possible to construct a further improved stress tensor (and correspondingly, improved membrane equations of motion and actions) whose rotating membrane solutions exactly reproduce the thermodynamics of arbitrary Myers Perry black holes at finite D. In this context it is encouraging to recall that, in the context of the fluid gravity correspondence, it was possible reproduce the exact thermodynamics of AdS Kerr black holes using only the second order corrected fluid stress tensor [16].

Finally, we find it absolutely fascinating that even the leading order large D membrane equations are equivalent to a set of equations of boundary hydrodynamics that reproduce the correct fluid constitutive relations at zero and first order in derivatives even at finite D, but also automatically resum a very particular infinite class of higher derivative corrections to the Navier Stokes equations - namely those that survive at large D. It would be interesting to compare this resummation with other partial resummations of the hydrodynamical derivative expansion investigated in the hydrodynamics literature (see e.g. [77, 78, 79, 80, 81]). We also note that some higher derivative corrections to the Navier Stokes equations - like the Israel Stewart correction - turn the parabolic Navier Stokes PDEs into hyperbolic PDEs. It would be interesting to investigate whether the corrections induced by our membrane also have this property (i.e. whether the membrane equations are hyperbolic PDEs).

We re-emphasize that our improved membrane equations define a generalization of the Navier Stokes equations that can be used to study the dynamics of thermal systems outside the validity of hydrodynamics (i.e. at length scales shorter than thermal length scales) atleast in the large D limit. We have already pointed out that the membrane picture suggests the possibility of qualitatively new phenomena - like membrane folds - that cannot be captured by the variables of hydrodynamics.

It would be useful to generalize the discussion of this chapter to the study of improved equations, the partition function and hydrodynamics of charged membranes (see [9, 14]).

[39] We thank M. Rangamani for discussions on this topic.

Apart from all these issues of principle, it would also be interesting to put the formulae presented in this chapter to practical use. It would be interesting to use the improved membrane equations presented in this chapter as the starting point for a 'rederivation' of the equations of black fold dynamics [40] and to compare our results with the exact gravitational results [82, 83, 84, 85]. Such a discussion could proceed along the lines of our 'rederivation' of boundary hydrodynamics from our improved membrane equations, presented earlier in this chapter.

It is already known that the black hole membranes have a 'Gregory-Laflamme like' instability at large D. At large D, however, this transition is of second order and ends up in a wiggly string. It would be interesting to re-investigate Gregory Laflamme physics using the improved membrane equations presented in this chapter. As our probe membranes define consistent dynamics even at finite D, it is meaningful to ask whether their Gregory Laflamme like transition switches from second to first order below a critical value of D (recall this is the case for actual black strings; the critical value of D is 13.5 [22]). Assuming this is the case as a related analysis suggests [22], it would be interesting to investigate whether the equations of membrane hydrodynamics presented in this chapter capture the fascinating dynamics of the 'self-similar cascade and pinch off' observed in [86]. It is far from clear that this will turn out to be the case [41]. Nonetheless we find the possibility tantalizing, as it holds out the promise of relating the mysterious process of horizon bifurcation to the more mundane process of hydrodynamical droplet formation in a semi quantitative manner.

Finally it is possible that the formalism developed in this chapter can be combined with that of [60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76] to establish a second law of thermodynamics for dynamical event horizons in higher derivative theories of gravity. We hope to return to this point in the future.

[40] We thank Mukund Rangamani for a discussion on this point.
[41] We thank R. Emparan for emphasizing this to us.

Appendix A

Appendices for Chapter 2

A.1 Method of calculation

In this Appendix we outline the method we have employed to obtain the results quoted in tables 2.2, 2.3, 2.4, 2.5, A.1, A.2.

As we have mentioned in the main text, our starting point is the metric listed in (2.12),(2.13),(2.14),(2.15). In order to obtain the equations of motion listed in table 2.2 (see also table 2.3) we simply plugged this metric into the vacuum Einstein equations. Assuming these equations are already obeyed at $n-1$ order we then obtained the form of the n^{th} order equations. As emphasized in table 2.2, each of these equations have a 'homogeneous' contribution and a 'source' contribution. The homogeneous contribution is linear in the (as yet unknown) n^{th} order fluctuation, and takes the same form at all orders. In order to evaluate the homogeneous contribution to all equations of motion, consequently, it is sufficient to work at first order.

While the first order computation is straightforward to perform analytically in principle, in practice the computations involved are rather lengthy [1]. In order to guard against error we employed Mathematica in our computations using the following device. Following [8, 9] we specialized to the particular case of metrics that preserve an $SO(D-p-2)$ isometry. Such special metrics effectively depend only on $p+3$ variables. For small values of p, therefore, all computations can be effectively performed on Mathematica (see [9] for a detailed explanation of how this is done). The first order computation performed in this manner yields the homogeneous part of the differential equations listed in tables 2.2 and 2.3 in a straightforward manner. Note that the homogeneous part of the equations are differential operators only in the variable R. They are 'ultra-local' on the membrane. Consequently, even though the assumption of isometry was used as a trick to facilitate the computation of the homogeneous part of the equation, the final result obtained for the structure of the equations listed in tables 2.2 and 2.3 is valid assuming only that all background quantities (e.g. $\mathcal{K}$) scale in the manner assumed in the text. In particular the homogeneous contribution to these equations are independent of p. By repeating all of our computations for $p=2$ and $p=3$ we have explicitly checked that this is the case.

Apart from the homogeneous pieces, the equations listed in tables 2.2 and 2.3 also have

[1] We have also done these computations analytically in the papers [11, 13]

contributions from sources. Source terms are different at different orders in the computation. We obtained our explicit results for the first order sources listed in tables 2.4, 2.5 and second order sources listed in tables A.1, A.2 as follows. Working separately in the scalar, vector and tensor channels we first explicitly listed all possible source structures that could appear in any given equation both at first and second order in perturbation theory. The source structures that appear in our classification are the analogues of the 'geometrical' quantities listed in the LHS of Table 4 in [9]. At any given order, it follows that the sources that appear in the equations of tables 2.2 and 2.3 are linear combinations of these structures with coefficients that are as yet unknown functions of R. We then worked out the analogue of the RHS of Table 4 of [9], i.e. we explicitly evaluated each of these basis source terms in terms of 'reduced source data' - the analogue of the expressions listed in table 1 of [9].

Using our explicit computations on Mathematica we read off the coefficients of all reduced sources in all of the equations listed in table 2.2 and 2.3. We then used our reduction formulae for 'geometrical sources in terms of reduced sources' (analogue of Table 4 in [9]) to determine the coefficients of all source terms in the original geometrical basis of possible source terms. The last step (determination of geometrical sources from the known coefficients of reduced sources) is unambiguous provided the map between geometrical and reduced sources in invertible, i.e. provided there does not exist a nontrivial linear combination of geometrical sources that maps to zero when re expressed in terms of reduced sources (i.e. vanishes under the the assumption of isometry). We have verified that this condition is met at first order provided $p \geq 2$ and at second order provided that $p \geq 3$. [2]. This is the reason we performed our computations at $p = 3$. [3]

A.2 Sources at second order

In this Appendix we present an explicit listing of all the sources that appear in the second order computation.By explicit computation we find that the sources listed in tables 2.1 and 2.2) ' are given at second order by the expressions we list in table A.1 below

[2]It is easy to understand the inequalities listed here. When $p = 1$, for instance, a potential source term proportional to the shear of the velocity field trivially vanishes just because fluids in one spatial dimension do not have a transverse direction in which to shear.

[3]We also performed all computations in $p = 2$ and verified that we obtained the same results for all sources from this computation - except in the case of a single second order source that vanished at $p = 2$ but not at $p = 3$. The coefficient of this term was left undetermined at $p = 2$ but we determined at $p = 3$.

Table A.1: Sources of R_{MN} equations at 2nd order

Scalar sector

$$\mathcal{S}^{S_1}(R) = e^{-R}(1-R)\left((u\cdot K - u\cdot\nabla u)\cdot\mathcal{P}\cdot(u\cdot K - u\cdot\nabla u)\right)$$

$$\mathcal{S}^{S_2}(R) = -\tfrac{1}{2}e^{-R}(R-2)\Big(K_{MN}K_{PQ}P^{NP}P^{MQ} - \tfrac{\mathcal{K}^2}{D-3}\Big) + \tfrac{1}{2}e^{-R}(R+2)\Big(\nabla_M u_N \nabla_P u_Q P^{NP}P^{MQ}\Big)$$
$$-\tfrac{e^{-R}}{2}\Big(\nabla_{[M}u_{N]}\nabla_{[P}u_{Q]}P^{NP}P^{MQ}\Big) - e^{-R}R\Big(\nabla_M u_N K_{PQ}P^{NP}P^{MQ}\Big)$$
$$+\tfrac{1}{\mathcal{K}}\tfrac{e^{-R}(R-2)R}{4}\nabla^A\left(\tfrac{D-3}{\mathcal{K}}\left(\tfrac{D-3}{\mathcal{K}^3}\left(\nabla\nabla^2\mathcal{K} - \nabla^2\nabla^2 u\right) + 8(u\cdot K - u\cdot\nabla u) + u\cdot K + \tfrac{\nabla^2 u}{\mathcal{K}}\right)_B \mathcal{P}^B_A\right)$$
$$-\tfrac{e^{-R}(R-2)R}{4}\tfrac{\nabla^2\nabla^2\mathcal{K}}{\mathcal{K}^3} + \tfrac{1}{4}e^{-2R}\left(e^R\left(R^2+2R-4\right) - 2(R-2)R\right)(u.\nabla u_M)(u.\nabla u_N)\mathcal{P}^{MN}$$
$$+\tfrac{1}{2}e^{-2R}\left(2e^R(R-1) - (R-2)R\right)(u^A K_{AM})(u^B K_{BN})\mathcal{P}^{MN} + e^{-2R}(R-2)R(u.\nabla u_M)(u^C K_{CN})P^{MN}$$
$$+\tfrac{1}{4}e^{-R}(R-2)R\left(\tfrac{\nabla^2 u_M}{\mathcal{K}}\right)\left(\tfrac{\nabla^2 u_N}{\mathcal{K}}\right)\ CP^{MN} - \tfrac{e^{-R}(R-2)R}{2}\left(\tfrac{\nabla^2 u_M}{\mathcal{K}}\right)(u.\nabla u_N)\mathcal{P}^{MN}$$
$$+\tfrac{1}{4}e^{-R}R\left(2R^2-3R-6\right)\tfrac{(u.\nabla\mathcal{K})^2}{\mathcal{K}^2} - \tfrac{e^{-R}\left(R^3-14R^2+20R+4\right)}{4}u.K.u\tfrac{\mathcal{K}}{(D-3)}$$
$$+\tfrac{e^{-R}\left(3R^3-38R^2+62R-4\right)}{4}\tfrac{\mathcal{K}}{(D-3)}\tfrac{u.\nabla\mathcal{K}}{\mathcal{K}} - \tfrac{1}{4}e^{-R}R\left(R^2-6\right)u.K.u\tfrac{u.\nabla\mathcal{K}}{\mathcal{K}} + e^{-R}(R-1)\tfrac{\mathcal{K}^2}{(D-3)^2}$$
$$-\tfrac{1}{4}e^{-R}\ \left(\nabla_{(A}u_{B)}\nabla_{(C}u_{D)}\mathcal{P}^{BC}\mathcal{P}^{AD}\right)$$

$$\mathcal{S}^{S_3}(R) = \mathcal{V}^{S_1}(R) - (1-e^{-R})\mathcal{S}^{S_2}(R)$$

$$\mathcal{S}^{S_4}(R) = (1-e^{-R})\mathcal{S}^{S_1}(R) - 2\mathcal{V}^{S_2}(R)$$

Vector sector

$$\mathcal{S}^{V_1}_M(R) = \tfrac{1}{(1-e^{-R})}\left(\mathcal{V}^V_L(R) - \mathcal{S}^{V_2}_L(R)\right)$$

$$\mathcal{S}^{V_2}_A(R) = \tfrac{\mathcal{K}^2}{2(D-3)^2}\Big[-e^{-2R}\left(e^R-1\right)\left(R^2-2\right)\tfrac{3}{2}\tfrac{D-3}{\mathcal{K}}\left(1 + 2\tfrac{u\cdot\nabla\mathcal{K}\ (D-3)}{\mathcal{K}^2} - \tfrac{u\cdot K\cdot u\ (D-3)}{\mathcal{K}}\right)(u\cdot\nabla u - u\cdot K)_B$$
$$-e^{-2R}\left(e^R-1\right)(R-1)\tfrac{(D-3)}{\mathcal{K}}\left(\tfrac{(D-3)}{\mathcal{K}^3}\left(\nabla\nabla^2\mathcal{K} - \nabla^2\nabla^2 u\right) + 8(u\cdot K - u\cdot\nabla u) + u\cdot K + \tfrac{\nabla^2 u}{\mathcal{K}}\right)_B$$
$$+Re^{-R}\left(-2\tfrac{(D-3)^2}{\mathcal{K}^2}\left(\tfrac{\nabla_M\mathcal{K}}{\mathcal{K}} - u^D K_{DM}\right)P^{MN}\left(\nabla_N u_B - K_{NB}\right) + \tfrac{(D-3)}{\mathcal{K}}\left(u^C K_{CB} - \tfrac{\nabla^2 u_B}{\mathcal{K}}\right)\right)\Big]\mathcal{P}^B_A$$
$$-\tfrac{e^{-R}}{2}\tfrac{\mathcal{K}}{(D-3)}\Big[-\mathcal{E}_M + D\tfrac{\nabla^2\nabla^2 u_M}{\mathcal{K}^3} - D\tfrac{\nabla_M(\nabla^2\mathcal{K})}{\mathcal{K}^3} + 3D\tfrac{(u\cdot K\cdot u)(u\cdot\nabla u_M)}{\mathcal{K}} - 3D\tfrac{(u\cdot K\cdot u)(u^A K_{AM})}{\mathcal{K}}$$
$$-6D\tfrac{(u\cdot\nabla\mathcal{K})(u\cdot\nabla u)}{\mathcal{K}^2} + 6D\tfrac{(u\cdot\nabla\mathcal{K})(u^A K_{AM})}{\mathcal{K}^2} + 3u\cdot\nabla u - 3u^A K_{AM}\Big]\mathcal{P}^M_L$$

Tensor sector

$$\mathcal{S}^T_{LP}(R) = \Big[e^{-R}\tfrac{\mathcal{K}}{(D-3)}\left(\left(K_{MN} - \nabla_{(M}u_{N)}\right) - \tfrac{\mathcal{P}_{MN}}{D}\left(K_{AB} - \nabla_{(A}u_{B)}\right)\mathcal{P}^{AB}\right)$$
$$-e^{-R}\left((K_{MC} - \nabla_C u_M)\mathcal{P}^{CD}(K_{DN} - \nabla_D u_N) - \tfrac{\mathcal{P}_{MN}}{D}(K_{AC} - \nabla_C u_A)\mathcal{P}^{CD}(K_{DB} - \nabla_D u_B)\mathcal{P}^{AB}\right)$$
$$-\tfrac{1}{2}e^{-2R}\left(R^2 - 4R + 2e^R(R-1) + 2\right)\Big((u_C K^C_M - u.\nabla u_M)(u_C K^C_N - u.\nabla u_N)$$
$$-\tfrac{\mathcal{P}_{MN}}{D}(u_C K^C_A - u.\nabla u_A)(u_C K^C_B - u.\nabla u_B)\mathcal{P}^{AB}\Big)\Big]\mathcal{P}^M_L\mathcal{P}^N_P$$

Table A.2: Sources to constraint equations at 2nd order

Vector constraint source
$\mathcal{V}^V_L(R) = \frac{1}{(D-3)}\nabla^P\Big[e^{-R}R\frac{D}{\mathcal{K}}\Big((K_{MC}-\nabla_C u_M)\mathcal{P}^{CD}(K_{DN}-\nabla_D u_N) - \frac{\mathcal{P}_{MN}}{(D-3)}(K_{AC}-\nabla_C u_A)\mathcal{P}^{CD}(K_{DB}-\nabla_D u_B)\mathcal{P}^{AB}\Big)\mathcal{P}^M_L\mathcal{P}^N_P - Re^{-R}\left((K_{MN}-\nabla_{(M}u_{N)}) - \frac{\mathcal{P}_{MN}}{(D-3)}(K_{AB}-\nabla_{(A}u_{B)})\mathcal{P}^{AB}\right)\mathcal{P}^M_L\mathcal{P}^N_P + (e^{-2R}\left(e^R-1\right)(R-2)R)\frac{(D-3)}{2\mathcal{K}}\Big((u_C K^C_M - u.\nabla u_M)(u_C K^C_N - u.\nabla u_N) - \frac{\mathcal{P}_{MN}}{(D-3)}(u_C K^C_A - u.\nabla u_A)(u_C K^C_B - u.\nabla u_B)\mathcal{P}^{AB}\Big)\mathcal{P}^M_L\mathcal{P}^N_P\Big] - \frac{e^{-R}}{2}\frac{\mathcal{K}}{(D-3)}\Big[-\mathcal{E}_M + D\frac{\nabla^2\nabla^2 u_M}{\mathcal{K}^3} - D\frac{\nabla_M(\nabla^2\mathcal{K})}{\mathcal{K}^3} + 3D\frac{(u\cdot K\cdot u)(u\cdot\nabla u_M)}{\mathcal{K}} - 3D\frac{(u\cdot K\cdot u)(u^A K_{AM})}{\mathcal{K}} - 6D\frac{(u\cdot\nabla\mathcal{K})(u\cdot\nabla u)}{\mathcal{K}^2} + 6D\frac{(u\cdot\nabla\mathcal{K})(u^A K_{AM})}{\mathcal{K}^2} + 3u\cdot\nabla u - 3u^A K_{AM}\Big]\mathcal{P}^M_L$
Scalar constraint 1 source
$\mathcal{V}^{S_1}(R) = \frac{\left(e^{-2R}R\left(e^R\left(R^2-6\right)+3(R+2)\right)\right)\mathcal{K}}{6(D-3)}\nabla^M\left(\frac{3}{2}\frac{(D-3)}{\mathcal{K}}\left(1+2\frac{u\cdot\nabla\mathcal{K}\ (D-3)}{\mathcal{K}^2} - \frac{u\cdot K\cdot u\ (D-3)}{\mathcal{K}}\right)(u\cdot\nabla u - u\cdot K)_B\,\mathcal{P}^B_M\right) + \frac{\left(e^{-2R}\left(e^R(R-2)+2\right)R\right)}{4\mathcal{K}}\nabla^M\left(\frac{(D-3)}{\mathcal{K}}\left(\frac{(D-3)}{\mathcal{K}^3}\left(\nabla\nabla^2\mathcal{K} - \nabla^2\nabla^2 u\right) + 8(u\cdot K - u\cdot\nabla u) + u\cdot K + \frac{\nabla^2 u}{\mathcal{K}}\right)_B\mathcal{P}^B_M\right) + \frac{\left(-e^{-R}R^2\right)}{4\mathcal{K}}\nabla^M\left(\left(-2\frac{(D-3)^2}{\mathcal{K}^2}\left(\frac{\nabla_M\mathcal{K}}{\mathcal{K}} - u^D K_{DM}\right)\mathcal{P}^{MN}\left(\nabla_N u_B - K_{NB}\right) + \frac{(D-3)}{\mathcal{K}}\left(u^C K_{CB} - \frac{\nabla^2 u_B}{\mathcal{K}}\right)\right)\mathcal{P}^B_M\right) - \frac{1}{4}e^{-R}\ \left(\nabla_{(A}u_{B)}\nabla_{(C}u_{D)}\mathcal{P}^{BC}\mathcal{P}^{AD}\right) + \frac{Re^{-R}}{2}\nabla^M\mathcal{E}_M$
Scalar constraint 2 source
$\mathcal{V}^{S_2}(R) = -\frac{1}{2}e^{-R}(R-1)\left(K_{MN}K_{PQ}P^{NP}P^{MQ} - \frac{\mathcal{K}^2}{D-3}\right) + \frac{1}{2}e^{-R}(3+R)\left(\nabla_M u_N\nabla_P u_Q P^{NP}P^{MQ}\right) + \frac{1}{2}\left(-e^{-R}\right)\left(\nabla_{[M}u_{N]}\nabla_{[P}u_{Q]}P^{NP}P^{MQ}\right) - e^{-R}(1+R)\left(\nabla_M u_N K_{PQ}P^{NP}P^{MQ}\right) + \frac{1}{\mathcal{K}}\frac{\left(e^{-R}(R+2)R\right)}{4}\nabla^M\left[\frac{(D-3)}{\mathcal{K}}\left(\frac{(D-3)}{\mathcal{K}^3}\left(\nabla\nabla^2\mathcal{K} - \nabla^2\nabla^2 u\right) + 8(u\cdot K - u\cdot\nabla u) + u\cdot K + \frac{\nabla^2 u}{\mathcal{K}}\right)_B\mathcal{P}^B_M\right] - \frac{\left(e^{-R}R^2\right)}{4}\frac{\nabla^2\nabla^2\mathcal{K}}{\mathcal{K}^3} + \frac{1}{4}\left(e^{-2R}R\left(2+R(e^R-1)\right)\right)(u.\nabla u_M)(u.\nabla u_N)\mathcal{P}^{MN} - \frac{1}{4}\left(e^{-2R}R(R-2)\right)(u^A K_{AM})(u^D K_{DN})\mathcal{P}^{MN} + \frac{1}{4}e^{-R}R^2\frac{\nabla^2 u_M}{\mathcal{K}}\frac{\nabla^2 u_N}{\mathcal{K}}\mathcal{P}^{MN} - \frac{\left(e^{-R}R^2\right)}{2}\frac{\nabla^2 u_M}{\mathcal{K}}u.\nabla u_N\mathcal{P}^{MN} + \frac{1}{2}\left(e^{-2R}R(-2+4e^R+R)\right)(u.\nabla u_M)(u^C K_{CN})\mathcal{P}^{MN} + \frac{\left(e^{-R}R\left(2R^2-R-12\right)\right)}{4}\frac{(u.\nabla\mathcal{K})^2}{\mathcal{K}^2} - \frac{\left(e^{-R}\left(R^3-14R^2-8R+2\right)\right)}{4}u.K.u\frac{\mathcal{K}}{(D-3)} + \frac{\left(e^{-R}R\left(3R^2-32R-2\right)\right)}{4}\frac{u.\nabla\mathcal{K}}{(D-3)} - \frac{\left(e^{-R}R\left(R^2-2R-18\right)\right)}{4}\frac{u.\nabla\mathcal{K}}{\mathcal{K}}u.K.u + e^{-R}R\frac{\mathcal{K}^2}{(D-3)^2} - \frac{1}{4}e^{-R}\ \left(\nabla_{(A}u_{B)}\nabla_{(C}u_{D)}\mathcal{P}^{BC}\mathcal{P}^{AD}\right)$

Appendix B

Appendices for Chapter 3

B.1 QNM for AdS/dS Schwarzschild Black hole

The Christoffel symbols (here we report only nonzero components) calculated for the metric (3.16) are given by (with $\bar{g}_{ab}$ and $\bar{\Gamma}^a_{bc}$ as metric and Christoffel symbol on unit sphere)

$$\begin{aligned}
&\Gamma^r_{ab} = -r\left(1-\frac{\sigma r^2}{L^2}\right)\bar{g}_{ab}, \quad \Gamma^a_{rb} = \frac{1}{r}\delta^a_b, \quad \Gamma^r_{tt} = -r\left(1-\frac{\sigma r^2}{L^2}\right)\frac{\sigma}{L^2} \\
&\Gamma^t_{rt} = -r\left(1-\frac{\sigma r^2}{L^2}\right)^{-1}\frac{\sigma}{L^2}, \quad \Gamma^r_{rr} = r\left(1-\frac{\sigma r^2}{L^2}\right)^{-1}\frac{\sigma}{L^2}, \quad \Gamma^a_{bc} = \bar{\Gamma}^a_{bc}
\end{aligned} \tag{B.1}$$

The normal vector to membrane surface is given by

$$n_r = \left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}}, \quad n_t = \left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}}(-\epsilon\partial_t\delta r), \quad n_a = \left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}}(-\epsilon\bar{\nabla}_a\delta r) \tag{B.2}$$

The answer for $\nabla_A n_B$ is given by

$$\begin{aligned}
&\nabla_r n_r = 0, \quad \nabla_r n_t = \left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{3}{2}}\frac{2\sigma r}{L^2}(-\epsilon\partial_t\delta r) \\
&\nabla_t n_r = \left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{3}{2}}\frac{\sigma r}{L^2}(-\epsilon\partial_t\delta r), \quad \nabla_a n_t = (-\epsilon\partial_t\bar{\nabla}_a\delta r)\left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}} \\
&\nabla_t n_t = \left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}}(-\epsilon\partial_t^2\delta r) + \left(1-\frac{\sigma r^2}{L^2}\right)^{\frac{1}{2}}\frac{\sigma r}{L^2} \\
&\nabla_r n_a = (-\epsilon\bar{\nabla}_a\delta r)\left[\frac{\sigma r}{L^2}\left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{3}{2}} - \frac{1}{r}\left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}}\right] \\
&\nabla_a n_r = (\epsilon\bar{\nabla}_a\delta r)\frac{1}{r}\left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}}, \quad \nabla_t n_a = (-\epsilon\partial_t\bar{\nabla}_a\delta r)\left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}} \\
&\nabla_a n_b = \left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}}(-\epsilon\bar{\nabla}_a\bar{\nabla}_b\delta r) + r\left(1-\frac{\sigma r^2}{L^2}\right)^{\frac{1}{2}}\bar{g}_{ab}
\end{aligned} \tag{B.3}$$

The answer for 'spacetime' projector $P^A_B = \delta^A_B - n^A n_B$ is

$$\begin{aligned} &P^r_r = 0, \quad P^t_t = 1, \quad P^a_b = \delta^a_b, \quad P^t_a = 0, \quad P^a_t = 0, \\ &P^r_t = \epsilon\partial_t\delta r, \quad P^t_r = \left(1 - \frac{\sigma r^2}{L^2}\right)^{-2}(-\epsilon\partial_t\delta r), \\ &P^r_a = \epsilon\bar{\nabla}_a\delta r, \quad P^a_r = \frac{1}{r^2}\left(1 - \frac{\sigma r^2}{L^2}\right)^{-1}(\epsilon\bar{\nabla}^a\delta r) \end{aligned} \tag{B.4}$$

The answer for the spacetime form of the Extrinsic curvature K_{MN} is

$$\begin{aligned} &K_{rr} = 0, \quad K_{rt} = \left(1 - \frac{\sigma r^2}{L^2}\right)^{-\frac{3}{2}}\frac{\sigma r}{L^2}(-\epsilon\partial_t\delta r), \quad K_{ra} = \frac{1}{r}\left(1 - \frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}}(\epsilon\bar{\nabla}_a\delta r) \\ &K_{ta} = \left(1 - \frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}}(-\epsilon\partial_t\bar{\nabla}_a\delta r), \quad K_{tt} = \left(1 - \frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}}(-\epsilon\partial_t^2\delta r) + \left(1 - \frac{\sigma r^2}{L^2}\right)^{\frac{1}{2}}\frac{\sigma r}{L^2} \\ &K_{ab} = \left(1 - \frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}}(-\epsilon\bar{\nabla}_a\bar{\nabla}_b\delta r) + r\left(1 - \frac{\sigma r^2}{L^2}\right)^{\frac{1}{2}}\bar{g}_{ab} \end{aligned} \tag{B.5}$$

Nonzero Christoffel symbol components for the metric (3.20) is given by

$$\begin{aligned} &\Gamma^t_{tt} = -\left(1 - \frac{\sigma}{L^2}\right)^{-1}\frac{\sigma}{L^2}(\epsilon\partial_t\delta r), \quad \Gamma^a_{tt} = -\frac{\sigma}{L^2}(\epsilon\bar{\nabla}^a\delta r) \\ &\Gamma^t_{at} = -\left(1 - \frac{\sigma}{L^2}\right)^{-1}\frac{\sigma}{L^2}(\epsilon\bar{\nabla}_a\delta r), \quad \Gamma^t_{ab} = \left(1 - \frac{\sigma}{L^2}\right)^{-1}(\epsilon\partial_t\delta r)\bar{g}_{ab} \\ &\Gamma^a_{tb} = (\epsilon\partial_t\delta r)\delta^a_b, \quad \Gamma^a_{bc} = \bar{\Gamma}^a_{bc} + \epsilon(\bar{\nabla}_b\delta r\delta^a_c + \bar{\nabla}_c\delta r\delta^a_b - \bar{\nabla}^a\delta r\bar{g}_{bc}) \end{aligned} \tag{B.6}$$

The answer for $\hat{\nabla}_\mu u_\nu$ is given by

$$\begin{aligned} &\hat{\nabla}_t u_t = 0, \quad \hat{\nabla}_t u_a = \epsilon\partial_t\delta u_a - \left(1 - \frac{\sigma}{L^2}\right)^{-\frac{1}{2}}\left(\frac{\sigma}{L^2}\right)(\epsilon\bar{\nabla}_a\delta r) \\ &\hat{\nabla}_a u_t = 0, \quad \hat{\nabla}_a u_b = \epsilon\bar{\nabla}_a\delta u_b + \left(1 - \frac{\sigma}{L^2}\right)^{-\frac{1}{2}}(\epsilon\partial_t\delta r)\bar{g}_{ab} \end{aligned} \tag{B.7}$$

B.2 Effective equations for AdS black brane from scaled membrane equations

B.2.1 Linearized fluctuation analysis

The nonzero Christoffel symbols for the metric (3.47) are

$$\Gamma^r_{rr} = \frac{-1}{r}, \quad \Gamma^r_{ab} = -r^3\delta_{ab}, \quad \Gamma^a_{rb} = \frac{1}{r}\delta^a_b, \quad \Gamma^r_{tt} = r^3, \quad \Gamma^t_{rt} = \frac{1}{r} \tag{B.8}$$

The normal to membrane surface is

$$n_r = \frac{1}{r}, \quad n_a = \frac{-\epsilon\partial_a\delta r}{r}, \quad n_t = \frac{-\epsilon\partial_t\delta r}{r} \tag{B.9}$$

The answer for $\nabla_M n_N$ is given by

$$\begin{aligned}
&\nabla_r n_r = 0, \quad \nabla_r n_t = \frac{2\epsilon\partial_t\delta r}{r^2}, \quad \nabla_t n_r = \frac{\epsilon\partial_t\delta r}{r^2}, \quad \nabla_t n_t = -\frac{\epsilon\partial_t^2\delta r}{r} - r^2, \\
&\nabla_r n_a = \frac{2\epsilon\partial_a\delta r}{r^2}, \quad \nabla_a n_r = \frac{\epsilon\partial_a\delta r}{r^2}, \quad \nabla_t n_a = \frac{-\epsilon\partial_t\partial_a\delta r}{r}, \\
&\nabla_a n_t = \frac{-\epsilon\partial_t\partial_a\delta r}{r}, \quad \nabla_a n_b = \frac{-\epsilon\partial_a\partial_b\delta r}{r} + r^2\delta_{ab}
\end{aligned} \tag{B.10}$$

where the rest of the components are zero.
The answer for the projector $P_A^B = \delta_A^B - n_A n^B$ is given by

$$\begin{aligned}
&P_r^r = 0, \quad P_t^t = 1, \quad P_b^a = \delta_b^a, \quad P_t^a = 0, \quad P_a^t = 0, \\
&P_t^r = \epsilon\partial_t\delta r, \quad P_r^t = \frac{-\epsilon\partial_t\delta r}{r^4}, \qquad P_a^r = \epsilon\partial_a\delta r, \qquad P_r^a = \frac{\epsilon\partial^a\delta r}{r^4}
\end{aligned} \tag{B.11}$$

The spacetime Extrinsic curvature K_{MN} is given by

$$\begin{aligned}
&K_{rr} = 0, \quad K_{rt} = \frac{\epsilon\partial_t\delta r}{r^2}, \quad K_{ra} = \frac{\epsilon\partial_a\delta r}{r^2} \\
&K_{tt} = \frac{-\epsilon\partial_t^2\delta r}{r} - r^2, \quad K_{ta} = \frac{-\epsilon\partial_t\partial_a\delta r}{r}, \quad K_{ab} = \frac{-\epsilon\partial_a\partial_b\delta r}{r} + r^2\delta_{ab}
\end{aligned} \tag{B.12}$$

where the rest of the components are zero.
The nonzero Christoffel symbols for the metric (3.49) are

$$\begin{aligned}
&\Gamma_{tt}^t = \epsilon\partial_t\delta r, \quad \Gamma_{tt}^a = \epsilon\partial^a\delta r, \quad \Gamma_{at}^t = \epsilon\partial_a\delta r, \quad \Gamma_{ab}^t = \epsilon\partial_t\delta r\delta_{ab} \\
&\Gamma_{tb}^a = \epsilon\partial_t\delta r\delta_b^a, \quad \Gamma_{bc}^a = \epsilon(\partial_b\delta r\delta_c^a + \partial_c\delta r\delta_b^a - \partial^a\delta r\delta_{bc})
\end{aligned} \tag{B.13}$$

The answer for $\hat{\nabla}_\mu u_\nu$ is given by

$$\hat{\nabla}_t u_t = 0, \quad \hat{\nabla}_t u_a = \epsilon\partial_t\delta u_a + \epsilon\partial_a\delta r, \quad \hat{\nabla}_a u_t = 0, \quad \hat{\nabla}_a u_b = \epsilon\partial_a\delta u_b + \epsilon\partial_t\delta r\delta_{ab} \tag{B.14}$$

where the rest of the components are zero.

B.2.2 Scaled nonlinear analysis

The nonzero Christoffel symbols for the metric (3.72) are

$$\begin{aligned}
&\Gamma_{\rho\rho}^\rho = -\frac{1}{D}\left(1+\frac{\rho}{D}\right)^{-1}, \quad \Gamma_{ab}^\rho = -\left(1+\frac{\rho}{D}\right)^3\delta_{ab}, \quad \Gamma_{ij}^\rho = -D\left(1+\frac{\rho}{D}\right)^3\delta_{ij}, \\
&\Gamma_{\rho b}^a = \frac{1}{D}\left(1+\frac{\rho}{D}\right)^{-1}\delta_b^a, \quad \Gamma_{\rho j}^i = \frac{1}{D}\left(1+\frac{\rho}{D}\right)^{-1}\delta_j^i, \\
&\Gamma_{tt}^\rho = D\left(1+\frac{\rho}{D}\right)^3, \quad \Gamma_{\rho t}^t = \frac{1}{D}\left(1+\frac{\rho}{D}\right)^{-1}
\end{aligned} \tag{B.15}$$

The normal to the membrane surface is given by

$$
\begin{aligned}
n_\rho &= \frac{1}{D}\left(1 - \frac{\rho}{D} - \frac{\partial^a Y \partial_a Y}{2D}\right) \\
n_t &= -\frac{\partial_t Y}{D}\left(1 - \frac{\rho}{D} - \frac{\partial^a Y \partial_a Y}{2D}\right) \\
n_a &= -\frac{\partial_a Y}{D}\left(1 - \frac{\rho}{D} - \frac{\partial^a Y \partial_a Y}{2D}\right)
\end{aligned} \tag{B.16}
$$

The projector $P^M_N = \delta^M_N - n^M n_N$ is given by

$$
\begin{aligned}
&P^\rho_\rho = \frac{\partial^a Y \partial_a Y}{D}, \quad P^t_t = 1, \quad P^i_j = \delta^i_j, \quad P^a_b = \delta^a_b, \quad P^t_a = \mathcal{O}(n^{-2}), \quad P^a_t = -\frac{\partial^a Y \partial_t Y}{D}, \\
&P^\rho_t = \partial_t Y, \quad P^t_\rho = \mathcal{O}(n^{-2}), \quad P^\rho_a = \partial_a Y, \quad P^a_\rho = \frac{\partial^a Y}{D}
\end{aligned} \tag{B.17}
$$

where the rest of the components are zero.
The answer for $\bar{\nabla}_M n_N$ is given by

$$
\begin{aligned}
&\bar{\nabla}_\rho n_\rho = \mathcal{O}(D^{-2}), \quad \bar{\nabla}_\rho n_t = \mathcal{O}(D^{-2}), \quad \bar{\nabla}_t n_\rho = \mathcal{O}(D^{-2}), \\
&\bar{\nabla}_t n_t = -\frac{\partial_t^2 Y}{D} - \left(1 + \frac{2\rho}{D} - \frac{\partial_a Y \partial^a Y}{2D}\right), \quad \bar{\nabla}_a n_\rho = \mathcal{O}(D^{-2}), \quad \bar{\nabla}_\rho n_a = \mathcal{O}(D^{-2}), \\
&\bar{\nabla}_t n_a = -\frac{\partial_t \partial_a Y}{D}, \quad \bar{\nabla}_a n_t = -\frac{\partial_t \partial_a Y}{D}, \quad \bar{\nabla}_a n_b = -\frac{\partial_a \partial_b Y}{D} + \frac{\delta_{ab}}{D}, \\
&\bar{\nabla}_i n_j = \delta_{ij}\left(1 + \frac{2\rho}{D} - \frac{\partial_a Y \partial^a Y}{2D}\right),
\end{aligned} \tag{B.18}
$$

where the rest of the components are zero.
The answer for spacetime form of the Extrinsic curvature K_{MN} is

$$
\begin{aligned}
&K_{\rho\rho} = \mathcal{O}(D^{-2}), \quad K_{\rho t} = \mathcal{O}(D^{-2}), \quad K_{\rho a} = \mathcal{O}(D^{-2}), \quad K_{\rho i} = \mathcal{O}(D^{-2}), \\
&K_{tt} = -\frac{\partial_t^2 Y}{D} - \left(1 + \frac{2\rho}{D} - \frac{\partial_a Y \partial^a Y}{2D}\right), \quad K_{ta} = -\frac{\partial_a \partial_t Y}{D}, \quad K_{ti} = \mathcal{O}(D^{-2}), \\
&K_{ab} = -\frac{\partial_a \partial_b Y}{D} + \frac{\delta_{ab}}{D}, \quad K_{ij} = \delta_{ij}\left(1 + \frac{2\rho}{D} - \frac{\partial_a Y \partial^a Y}{2D}\right)
\end{aligned} \tag{B.19}
$$

where the rest of the components are zero.
Nonzero Christoffel symbols components for the induced metric (3.74) are given by

$$
\begin{aligned}
&\Gamma^t_{tt} = \frac{\partial_t Y}{D}, \quad \Gamma^a_{tt} = \partial^a Y, \quad \Gamma^t_{at} = \frac{\partial_a Y}{D}, \quad \Gamma^t_{ab} = \frac{\partial_t Y}{D^2}\delta_{ab}, \quad \Gamma^t_{ij} = \frac{\partial_t Y}{D^2}\delta_{ij}, \\
&\Gamma^a_{tb} = \frac{\partial_t Y}{D}\delta^a_b, \quad \Gamma^i_{tj} = \frac{\partial_t Y}{D}\delta^i_j, \quad \Gamma^a_{bc} = \frac{1}{D}(\partial_b Y \delta^a_c + \partial_c Y \delta^a_b - \partial^a Y \delta_{bc}), \\
&\Gamma^a_{jk} = -\partial^a Y \delta_{jk}, \quad \Gamma^i_{ja} = \frac{\partial_a Y}{D}\delta^i_j
\end{aligned} \tag{B.20}
$$

The answer for $\tilde{\nabla}_\mu u_\nu$ is given by

$$\begin{aligned}
&\tilde{\nabla}_t u_t = -\frac{\partial_t(U_a U^a)}{2D}, \quad \tilde{\nabla}_t u_a = \frac{\partial_t U_a}{D} + \frac{\partial_a Y}{D}, \quad \tilde{\nabla}_a u_t = -\frac{\partial_a(U_b U^b)}{2D}, \\
&\tilde{\nabla}_a u_b = \frac{\partial_a U_b}{D}, \quad \tilde{\nabla}_i u_j = \frac{\partial_t Y}{D}\delta_{ij} + \frac{U^a \partial_a Y}{D}\delta_{ij}
\end{aligned} \tag{B.21}$$

where the rest of the components are zero.

Appendix C

Appendices for Chapter 4

C.1 QNM for AdS/dS Schwarzschild Black hole: Details of the calculation

In this section we shall present several computational details. We shall follow [9] and [11]. Steps are tedious but a straightforward extension of what has been done in [11].

The answers for non-zero components of Christoffel symbols for metric (4.3) are (denoting the metric on unit sphere by $\bar{g}_{ab}$, its Christoffel symbols by $\bar{\Gamma}^a_{bc}$ and the covariant derivatives with respect to $\bar{g}_{ab}$ by $\bar{\nabla}_a$)

$$\begin{aligned}
&\Gamma^r_{ab} = -r\left(1-\frac{\sigma r^2}{L^2}\right)\bar{g}_{ab}, \quad \Gamma^a_{rb} = \frac{1}{r}\delta^a_b, \quad \Gamma^r_{tt} = -r\left(1-\frac{\sigma r^2}{L^2}\right)\frac{\sigma}{L^2} \\
&\Gamma^t_{rt} = -r\left(1-\frac{\sigma r^2}{L^2}\right)^{-1}\frac{\sigma}{L^2}, \quad \Gamma^r_{rr} = r\left(1-\frac{\sigma r^2}{L^2}\right)^{-1}\frac{\sigma}{L^2}, \quad \Gamma^a_{bc} = \bar{\Gamma}^a_{bc}
\end{aligned} \tag{C.1}$$

The normal to membrane evaluates to

$$n_r = \left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}}, \quad n_t = \left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}}(-\epsilon\partial_t\delta r), \quad n_a = \left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}}(-\epsilon\bar{\nabla}_a\delta r) \tag{C.2}$$

$\nabla_A n_B$ evaluates to

$$
\begin{aligned}
&\nabla_r n_r = 0, \quad \nabla_r n_t = \left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{3}{2}}\frac{2\sigma r}{L^2}(-\epsilon\partial_t\delta r)\\
&\nabla_t n_r = \left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{3}{2}}\frac{\sigma r}{L^2}(-\epsilon\partial_t\delta r), \quad \nabla_a n_t = (-\epsilon\partial_t\bar{\nabla}_a\delta r)\left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}}\\
&\nabla_t n_t = \left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}}(-\epsilon\partial_t^2\delta r) + \left(1-\frac{\sigma r^2}{L^2}\right)^{\frac{1}{2}}\frac{\sigma r}{L^2}\\
&\nabla_r n_a = (-\epsilon\bar{\nabla}_a\delta r)\left[\frac{\sigma r}{L^2}\left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{3}{2}} - \frac{1}{r}\left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}}\right]\\
&\nabla_a n_r = (\epsilon\bar{\nabla}_a\delta r)\frac{1}{r}\left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}}, \quad \nabla_t n_a = (-\epsilon\partial_t\bar{\nabla}_a\delta r)\left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}}\\
&\nabla_a n_b = \left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}}(-\epsilon\bar{\nabla}_a\bar{\nabla}_b\delta r) + r\left(1-\frac{\sigma r^2}{L^2}\right)^{\frac{1}{2}}\bar{g}_{ab}
\end{aligned}
\tag{C.3}
$$

The projector $P_A^B = \delta_B^A - n^A n_B$ evaluates to

$$
\begin{aligned}
&P_r^r = 0, \quad P_t^t = 1, \quad P_b^a = \delta_b^a, \quad P_a^t = 0, \quad P_t^a = 0,\\
&P_t^r = \epsilon\partial_t\delta r, \quad P_r^t = \left(1-\frac{\sigma r^2}{L^2}\right)^{-2}(-\epsilon\partial_t\delta r),\\
&P_a^r = \epsilon\bar{\nabla}_a\delta r, \quad P_r^a = \frac{1}{r^2}\left(1-\frac{\sigma r^2}{L^2}\right)^{-1}(\epsilon\bar{\nabla}^a\delta r)
\end{aligned}
\tag{C.4}
$$

The spacetime form of Extrinsic curvature $K_{AB} = \Pi_A^C\nabla_C n_B$ evaluates to

$$
\begin{aligned}
&K_{rr} = 0, \quad K_{rt} = \left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{3}{2}}\frac{\sigma r}{L^2}(-\epsilon\partial_t\delta r), \quad K_{ra} = \frac{1}{r}\left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}}(\epsilon\bar{\nabla}_a\delta r)\\
&K_{ta} = \left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}}(-\epsilon\partial_t\bar{\nabla}_a\delta r), \quad K_{tt} = \left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}}(-\epsilon\partial_t^2\delta r) + \left(1-\frac{\sigma r^2}{L^2}\right)^{\frac{1}{2}}\frac{\sigma r}{L^2}\\
&K_{ab} = \left(1-\frac{\sigma r^2}{L^2}\right)^{-\frac{1}{2}}(-\epsilon\bar{\nabla}_a\bar{\nabla}_b\delta r) + r\left(1-\frac{\sigma r^2}{L^2}\right)^{\frac{1}{2}}\bar{g}_{ab}
\end{aligned}
\tag{C.5}
$$

Answers for the nonzero components of Christoffel symbols for metric (4.8) are

$$
\begin{aligned}
&\Gamma_{tt}^t = -\left(1-\frac{\sigma}{L^2}\right)^{-1}\frac{\sigma}{L^2}(\epsilon\partial_t\delta r), \quad \Gamma_{tt}^a = -\frac{\sigma}{L^2}(\epsilon\bar{\nabla}^a\delta r)\\
&\Gamma_{at}^t = -\left(1-\frac{\sigma}{L^2}\right)^{-1}\frac{\sigma}{L^2}(\epsilon\bar{\nabla}_a\delta r), \quad \Gamma_{ab}^t = \left(1-\frac{\sigma}{L^2}\right)^{-1}(\epsilon\partial_t\delta r)\bar{g}_{ab}\\
&\Gamma_{tb}^a = (\epsilon\partial_t\delta r)\delta_b^a, \quad \Gamma_{bc}^a = \bar{\Gamma}_{bc}^a + \epsilon(\bar{\nabla}_b\delta r\delta_c^a + \bar{\nabla}_c\delta r\delta_b^a - \bar{\nabla}^a\delta r\bar{g}_{bc})
\end{aligned}
\tag{C.6}
$$

$\hat{\nabla}_\mu u_\nu$ evaluates to

$$\begin{aligned}
&\hat{\nabla}_t u_t = 0, \quad \hat{\nabla}_t u_a = \epsilon \partial_t \delta u_a - \left(1 - \frac{\sigma}{L^2}\right)^{-\frac{1}{2}} \left(\frac{\sigma}{L^2}\right) (\epsilon \bar{\nabla}_a \delta r) \\
&\hat{\nabla}_a u_t = 0, \quad \hat{\nabla}_a u_b = \epsilon \bar{\nabla}_a \delta u_b + \left(1 - \frac{\sigma}{L^2}\right)^{-\frac{1}{2}} (\epsilon \partial_t \delta r) \bar{g}_{ab}
\end{aligned} \tag{C.7}$$

The projector $\mathcal{P}^\mu_\nu \equiv \delta^\mu_\nu + u^\mu u_\nu$ evaluates to

$$\mathcal{P}^t_t = 0, \quad \mathcal{P}^a_t = -\left(1 - \frac{\sigma}{L^2}\right)^{\frac{1}{2}} (\epsilon \delta u_a), \quad \mathcal{P}^t_a = \left(1 - \frac{\sigma}{L^2}\right)^{-\frac{1}{2}} (\epsilon \delta u_a), \quad \mathcal{P}^a_b = \delta^a_b \tag{C.8}$$

C.1.1 Computation of $\mathcal{K}_{\mu\nu}$

We define $\mathcal{K}_{\mu\nu}$ as the pullback of Extrinsic curvature K_{MN} (which is a spacetime tensor) on the membrane surface

$$\mathcal{K}_{\mu\nu} = \left(\frac{\partial X^M}{\partial y^\mu}\right) \left(\frac{\partial X^N}{\partial y^\nu}\right) K_{MN}|_{r=1+\epsilon\delta r} \tag{C.9}$$

where we denote the coordinates in spacetime (r, t, θ^a) by X^M and the coordinates on the membrane worldvolume (t, θ^a) by y^μ. The extrinsic curvature K_{AB} is defined as

$$K_{AB} = \Pi^C_A \nabla_C n_B, \quad \text{where} \quad \Pi_{AC} = g_{AC} - n_A n_C \tag{C.10}$$

Now equation (C.9) evaluated upto linear order for the QNM calculation implies that

$$\mathcal{K}_{\mu\nu} = \epsilon(\partial_\mu \delta r) K_{r\nu} + \epsilon(\partial_\nu \delta r) K_{r\mu} + K_{\mu\nu} + \mathcal{O}(\epsilon^2) \tag{C.11}$$

From (C.5) we see that $K_{rN} = \mathcal{O}(\epsilon)$. Using this fact along with (C.11) gives us

$$\begin{aligned}
\mathcal{K}_{tt} &= \left(1 - \frac{\sigma}{L^2}\right)^{-\frac{1}{2}} (-\epsilon \partial_t^2 \delta r) + \left(1 - \frac{\sigma}{L^2}\right)^{\frac{1}{2}} \left(\frac{\sigma}{L^2}\right) \left(1 + \epsilon \delta r - \frac{\sigma \epsilon \delta r}{L^2 - \sigma}\right) \\
\mathcal{K}_{ta} &= \left(1 - \frac{\sigma}{L^2}\right)^{-\frac{1}{2}} (-\epsilon \partial_t \bar{\nabla}_a \delta r) \\
\mathcal{K}_{ab} &= \left(1 - \frac{\sigma}{L^2}\right)^{-\frac{1}{2}} (-\epsilon \bar{\nabla}_a \bar{\nabla}_b \delta r) + \left(1 - \frac{\sigma}{L^2}\right)^{\frac{1}{2}} \left(1 + \epsilon \delta r - \frac{\sigma \epsilon \delta r}{L^2 - \sigma}\right) \hat{g}_{ab}
\end{aligned} \tag{C.12}$$

Trace of Extrinsic curvature (C.12) evaluates to

$$\begin{aligned}
\mathcal{K} = &\left(1 - \frac{\sigma}{L^2}\right)^{-\frac{3}{2}} (\epsilon \partial_t^2 \delta r) - \left(1 - \frac{\sigma}{L^2}\right)^{-\frac{1}{2}} \left(\frac{\sigma}{L^2}\right) \left(1 + \frac{\epsilon L^2 \delta r}{L^2 - \sigma}\right) \\
&+ \left(1 - \frac{\sigma}{L^2}\right)^{-\frac{1}{2}} (-\epsilon \bar{\nabla}^2 \delta r) + \left(1 - \frac{\sigma}{L^2}\right)^{\frac{1}{2}} \left(1 - \frac{\epsilon L^2 \delta r}{L^2 - \sigma}\right) (D - 2)
\end{aligned} \tag{C.13}$$

C.1.2 Computation of the terms relevant for the membrane equation

Here, we report the relevant terms needed to evaluate the membrane equation upto linear order. The relevant terms at leading order evaluate to

$$
\begin{aligned}
u^\nu \mathcal{K}_{\nu t} &= \frac{\sigma}{L^2} + \mathcal{O}(\epsilon) \\
u^\nu \mathcal{K}_{\nu a} &= \left(1 - \frac{\sigma}{L^2}\right)^{-1} (-\epsilon \partial_t \bar{\nabla}_a \delta r) + \left(1 - \frac{\sigma}{L^2}\right)^{\frac{1}{2}} (\epsilon \delta u_a) \\
u^\nu \hat{\nabla}_\nu u_t &= 0 \\
u^\nu \hat{\nabla}_\nu u_a &= \left(1 - \frac{\sigma}{L^2}\right)^{-\frac{1}{2}} (\epsilon \partial_t \delta u_a) - \left(1 - \frac{\sigma}{L^2}\right)^{-1} \frac{\sigma}{L^2} (\epsilon \bar{\nabla}_a \delta r) \\
\hat{\nabla}_t \mathcal{K} &= \mathcal{O}(\epsilon) \\
\hat{\nabla}_a \mathcal{K} &= \left(1 - \frac{\sigma}{L^2}\right)^{-\frac{3}{2}} (\epsilon \partial_t^2 \bar{\nabla}_a \delta r) - \left(1 - \frac{\sigma}{L^2}\right)^{-\frac{3}{2}} \frac{\sigma}{L^2} (\epsilon \bar{\nabla}_a \delta r) \\
&\quad + \left(1 - \frac{\sigma}{L^2}\right)^{-\frac{1}{2}} (-\epsilon \bar{\nabla}_a \bar{\nabla}^2 \delta r) - (D-2) \left(1 - \frac{\sigma}{L^2}\right)^{-\frac{1}{2}} (\epsilon \bar{\nabla}_a \delta r) \\
\hat{\nabla}^2 u_t &= \mathcal{O}(\epsilon) \\
\hat{\nabla}^2 u_a &= -\left(1 - \frac{\sigma}{L^2}\right)^{-1} (\epsilon \partial_t^2 \delta u_a) + \left(1 - \frac{\sigma}{L^2}\right)^{-\frac{3}{2}} \frac{\sigma}{L^2} (\epsilon \partial_t \bar{\nabla}_a \delta r) \\
&\quad + \epsilon \bar{\nabla}^2 \delta u_a + \left(1 - \frac{\sigma}{L^2}\right)^{-\frac{1}{2}} (\epsilon \partial_t \bar{\nabla}_a \delta r)
\end{aligned}
\tag{C.14}
$$

The relevant terms at subleading order evaluate to

$$\begin{aligned}
u^\nu \mathcal{K}_{\nu\mu}\mathcal{K}^\mu_t &= -\left(\frac{\sigma}{L^2}\right)^2\left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}\\
u^\nu \mathcal{K}_{\nu\mu}\mathcal{K}^\mu_a &= \left(1-\frac{\sigma}{L^2}\right)^{-\frac{3}{2}}\frac{\sigma}{L^2}\epsilon\partial_t\bar{\nabla}_a\delta r-\left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}\epsilon\partial_t\bar{\nabla}_a\delta r+\left(1-\frac{\sigma}{L^2}\right)\epsilon\delta u_a\\
\hat{\nabla}^2\hat{\nabla}^2 u_t &= \mathcal{O}(\epsilon)\\
\hat{\nabla}^2\hat{\nabla}^2 u_a &= \bar{\nabla}^2\bar{\nabla}^2\delta u_a\\
u.\hat{\nabla}\mathcal{K} &= \mathcal{O}(\epsilon)\\
\hat{\nabla}^\nu\mathcal{K}\hat{\nabla}_\nu u_t &= \mathcal{O}(\epsilon)\\
\hat{\nabla}^\nu\mathcal{K}\hat{\nabla}_\nu u_a &= \mathcal{O}(\epsilon)^2\\
\mathcal{K}^{\mu\nu}\hat{\nabla}_\mu\hat{\nabla}_\nu u_t &= \mathcal{O}(\epsilon)\\
\mathcal{K}^{\mu\nu}\hat{\nabla}_\mu\hat{\nabla}_\nu u_a &= \left(1-\frac{\sigma}{L^2}\right)^{\frac{1}{2}}\epsilon\hat{\nabla}^2\delta u_a\\
\hat{\nabla}_t\hat{\nabla}^2\mathcal{K} &= \mathcal{O}(\epsilon)\\
\hat{\nabla}_a\hat{\nabla}^2\mathcal{K} &= -\left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}\hat{\nabla}_a\hat{\nabla}^2\hat{\nabla}^2\delta r-(D-2)\left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}\hat{\nabla}_a\hat{\nabla}^2\delta r\\
\hat{\nabla}_t(\mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu}\mathcal{K}) &= \mathcal{O}(\epsilon)\\
\hat{\nabla}_a(\mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu}\mathcal{K}) &= -3(D-2)\left(1-\frac{\sigma}{L^2}\right)^{\frac{1}{2}}\epsilon\left(\hat{\nabla}_a\hat{\nabla}^2\delta r+(D-2)\hat{\nabla}_a\delta r\right)\\
(u\cdot\mathcal{K}\cdot u)u^\nu\hat{\nabla}_\nu u_t &= \mathcal{O}(\epsilon)\\
(u\cdot\mathcal{K}\cdot u)u^\nu\hat{\nabla}_\nu u_a &= \left(1-\frac{\sigma}{L^2}\right)^{-1}\frac{\sigma}{L^2}\epsilon\partial_t\delta u_a-\left(1-\frac{\sigma}{L^2}\right)^{-\frac{3}{2}}\left(\frac{\sigma}{L^2}\right)^2\epsilon\hat{\nabla}_a\delta r\\
(u\cdot\mathcal{K}\cdot u)u^\mu\mathcal{K}_{\mu t} &= \left(1-\frac{\sigma}{L^2}\right)^{-\frac{1}{2}}\left(\frac{\sigma}{L^2}\right)^2\\
(u\cdot\mathcal{K}\cdot u)u^\mu\mathcal{K}_{\mu a} &= -\left(1-\frac{\sigma}{L^2}\right)^{-\frac{3}{2}}\left(\frac{\sigma}{L^2}\right)\epsilon\partial_t\hat{\nabla}_a\delta r+\frac{\sigma}{L^2}\epsilon\delta u_a\\
(u\cdot\hat{\nabla}\mathcal{K})u^\mu\mathcal{K}_{\mu t} &= \mathcal{O}(\epsilon)\\
(u\cdot\hat{\nabla}\mathcal{K})u^\mu\mathcal{K}_{\mu a} &= \mathcal{O}(\epsilon)^2\\
(u\cdot\hat{\nabla}\mathcal{K})u^\nu\hat{\nabla}_\nu u_t &= \mathcal{O}(\epsilon)\\
(u\cdot\hat{\nabla}\mathcal{K})u^\nu\hat{\nabla}_\nu u_a &= \mathcal{O}(\epsilon)^2
\end{aligned} \tag{C.15}$$

C.1.3 Arguments leading to (4.12)

Firstly, for convenience, rewrite the membrane equation (4.11) as

$$E^{tot}_\mu \equiv \mathcal{P}^\nu_\mu E_\nu, \quad \text{where} \quad E_\mu \equiv \frac{\hat{\nabla}^2 u_\mu}{\mathcal{K}}-\frac{\hat{\nabla}_\mu\mathcal{K}}{\mathcal{K}}+u^\nu\mathcal{K}_{\nu\mu}-u^\nu\hat{\nabla}_\nu u_\mu+\dots$$

So, we get

$$
\begin{aligned}
E_t^{tot} &= E_t \mathcal{P}_t^t + E_b \mathcal{P}_t^b \\
E_a^{tot} &= E_t \mathcal{P}_a^t + E_b \mathcal{P}_a^b
\end{aligned}
\tag{C.16}
$$

We can see for a uniform membrane configuration with spherical symmetry that E_a would be zero and hence we have $E_a \sim \mathcal{O}(\epsilon)$ in case of fluctuations. Also we see that $\mathcal{P}_t^t = 0$ and $\mathcal{P}_t^a \sim \mathcal{O}(\epsilon)$. Hence we see from (C.16) that E_t^{tot} is identically zero at the linear order. Similarly because $\mathcal{P}_a^t = \mathcal{O}(\epsilon)$, only $\mathcal{O}(\epsilon^0)$ pieces of E_t are relevant for evaluating E_a^{tot} at linear order. Hence in subsection C.1.2 we evaluated only those terms in E_μ that are relevant for the linearized analysis.

Substituting the expressions derived in subsection (C.1.2) in the linearized vector membrane equation in the angular directions we finally get (4.12).

C.2 QNM for AdS Schwarzschild black brane: Details of the calculation

Just like previous section, here we shall provide the details of the computation required to determine the QNM frequencies for AdS Schwarzschild black brane.

The answers for nonzero components of Christoffel symbols for the background metric (4.27) are

$$
\Gamma^r_{rr} = \frac{-1}{r}, \quad \Gamma^r_{ab} = -r^3\delta_{ab}, \quad \Gamma^a_{rb} = \frac{1}{r}\delta^a_b, \quad \Gamma^r_{tt} = r^3, \quad \Gamma^t_{rt} = \frac{1}{r} \tag{C.17}
$$

Normal to the membrane evaluates to

$$
n_r = \frac{1}{r}, \quad n_a = \frac{-\epsilon\partial_a\delta r}{r}, \quad n_t = \frac{-\epsilon\partial_t\delta r}{r} \tag{C.18}
$$

Non zero components of $\nabla_M n_N$ evaluate to

$$
\begin{aligned}
&\nabla_r n_r = 0, \quad \nabla_r n_t = \frac{2\epsilon\partial_t\delta r}{r^2}, \quad \nabla_t n_r = \frac{\epsilon\partial_t\delta r}{r^2}, \quad \nabla_t n_t = -\frac{\epsilon\partial_t^2\delta r}{r} - r^2, \\
&\nabla_r n_a = \frac{2\epsilon\partial_a\delta r}{r^2}, \quad \nabla_a n_r = \frac{\epsilon\partial_a\delta r}{r^2}, \quad \nabla_t n_a = \frac{-\epsilon\partial_t\partial_a\delta r}{r}, \\
&\nabla_a n_t = \frac{-\epsilon\partial_t\partial_a\delta r}{r}, \quad \nabla_a n_b = \frac{-\epsilon\partial_a\partial_b\delta r}{r} + r^2\delta_{ab}
\end{aligned}
\tag{C.19}
$$

The projector $P_A^B = \delta_A^B - n_A n^B$ evaluates to

$$
\begin{aligned}
&P^r_r = 0, \quad P^t_t = 1, \quad P^a_b = \delta^a_b, \quad P^a_t = 0, \quad P^t_a = 0, \\
&P^r_t = \epsilon\partial_t\delta r, \quad P^t_r = \frac{-\epsilon\partial_t\delta r}{r^4}, \qquad P^r_a = \epsilon\partial_a\delta r, \qquad P^a_r = \frac{\epsilon\partial^a\delta r}{r^4}
\end{aligned}
\tag{C.20}
$$

Nonzero components of the spacetime form of Extrinsic curvature K_{MN} evaluate to

$$\begin{aligned} &K_{rr} = 0, \quad K_{rt} = \frac{\epsilon \partial_t \delta r}{r^2}, \quad K_{ra} = \frac{\epsilon \partial_a \delta r}{r^2} \\ &K_{tt} = \frac{-\epsilon \partial_t^2 \delta r}{r} - r^2, \quad K_{ta} = \frac{-\epsilon \partial_t \partial_a \delta r}{r}, \quad K_{ab} = \frac{-\epsilon \partial_a \partial_b \delta r}{r} + r^2 \delta_{ab} \end{aligned} \tag{C.21}$$

Nonzero components of Christoffel symbols for the induced metric (4.29) evaluate to

$$\begin{aligned} &\Gamma^t_{tt} = \epsilon \partial_t \delta r, \quad \Gamma^a_{tt} = \epsilon \partial^a \delta r, \quad \Gamma^t_{at} = \epsilon \partial_a \delta r, \quad \Gamma^t_{ab} = \epsilon \partial_t \delta r \delta_{ab} \\ &\Gamma^a_{tb} = \epsilon \partial_t \delta r \delta^a_b, \quad \Gamma^a_{bc} = \epsilon(\partial_b \delta r \delta^a_c + \partial_c \delta r \delta^a_b - \partial^a \delta r \delta_{bc}) \end{aligned} \tag{C.22}$$

The projector $\mathcal{P}^\mu_\nu = \delta^\mu_\nu + u^\mu u_\nu$ evaluates to

$$\mathcal{P}^a_b = \delta^a_b, \quad \mathcal{P}^t_t = 0, \quad \mathcal{P}^t_a = \epsilon \delta u_a, \quad \mathcal{P}^a_t = -\epsilon \delta u_a, \tag{C.23}$$

Nonzero components of $\hat{\nabla}_\mu u_\nu$ evaluate to

$$\begin{aligned} &\hat{\nabla}_t u_t = 0, \quad \hat{\nabla}_t u_a = \epsilon \partial_t \delta u_a + \epsilon \partial_a \delta r, \quad \hat{\nabla}_a u_t = 0, \\ &\hat{\nabla}_a u_b = \epsilon \partial_a \delta u_b + \epsilon \partial_t \delta r \delta_{ab} \end{aligned} \tag{C.24}$$

C.2.1 Computation of $\mathcal{K}_{\mu\nu}$

As done previously, $\mathcal{K}_{\mu\nu}$ is defined as the pullback of spacetime form of extrinsic curvature K_{MN} on the membrane worldvolume. Doing this procedure we find that the nonzero components of $\mathcal{K}_{\mu\nu}$ evaluate to

$$\mathcal{K}_{tt} = -\epsilon \partial_t^2 \delta r - (1 + 2\epsilon \delta r), \quad \mathcal{K}_{ta} = -\epsilon \partial_t \partial_a \delta r, \quad \mathcal{K}_{ab} = -\epsilon \partial_a \partial_b \delta r + (1 + 2\epsilon \delta r)\delta_{ab} \tag{C.25}$$

Trace of Extrinsic curvature $\mathcal{K}_{\mu\nu}$ evaluates to

$$\mathcal{K} = n + \epsilon \partial_t^2 \delta r - \epsilon \partial_a \partial^a \delta r \tag{C.26}$$

where we raised the index a in (C.26) with δ^{ab}.

C.2.2 Computation of the terms relevant for membrane equation

At leading order the relevant terms evaluate to

$$\begin{aligned} u^\nu \mathcal{K}_{\nu t} &= -1 + \mathcal{O}(\epsilon) \\ u^\nu \mathcal{K}_{\nu a} &= -\epsilon \partial_t \partial_a \delta r + \epsilon \delta u_a \\ u^\nu \hat{\nabla}_\nu u_t &= \mathcal{O}(\epsilon) \\ u^\nu \hat{\nabla}_\nu u_a &= \epsilon \partial_t \delta u_a + \epsilon \partial_a \delta r \\ \hat{\nabla}_t \mathcal{K} &= \mathcal{O}(\epsilon) \\ \hat{\nabla}_a \mathcal{K} &= \epsilon \partial_a \partial_t^2 \delta r - \epsilon \partial_a \partial^2 \delta r \\ \hat{\nabla}^2 u_t &= \mathcal{O}(\epsilon) \\ \hat{\nabla}^2 u_a &= -\epsilon \partial_t^2 \delta u_a + \epsilon \partial^2 \delta u_a \end{aligned} \tag{C.27}$$

While at subleading order the relevant terms evaluate to

$$\begin{aligned}
u^\nu K_{\nu\mu}K^\mu_t &= -1+\mathcal{O}(\epsilon)\\
u^\nu K_{\nu\mu}K^\mu_a &= -2\epsilon\partial_t\partial_a\delta r+\epsilon\delta u_a\\
\hat{\nabla}^2\hat{\nabla}^2 u_t &= \mathcal{O}(\epsilon)\\
\hat{\nabla}^2\hat{\nabla}^2 u_a &= \epsilon\partial_t^4\delta u_a-2\epsilon\partial_t^2\partial^2\delta u_a+\epsilon\partial^4\delta u_a\\
u.\hat{\nabla}\mathcal{K} &= \mathcal{O}(\epsilon)\\
\hat{\nabla}^\nu\mathcal{K}\hat{\nabla}_\nu u_t &= \mathcal{O}(\epsilon)\\
\hat{\nabla}^\nu\mathcal{K}\hat{\nabla}_\nu u_a &= \mathcal{O}(\epsilon)^2\\
K^{\mu\nu}\hat{\nabla}_\mu\hat{\nabla}_\nu u_t &= \mathcal{O}(\epsilon)\\
K^{\mu\nu}\hat{\nabla}_\mu\hat{\nabla}_\nu u_a &= -\epsilon\partial_t^2\delta u_a+\epsilon\partial^2\delta u_a\\
\hat{\nabla}_t\hat{\nabla}^2\mathcal{K} &= \mathcal{O}(\epsilon)\\
\hat{\nabla}_a\hat{\nabla}^2\mathcal{K} &= -\epsilon\partial_a\partial_t^4\delta r+2\epsilon\partial_a\partial_t^2\partial^2\delta r-\epsilon\partial_a\partial^2\partial^2\delta r\\
\hat{\nabla}_t(K_{\mu\nu}K^{\mu\nu}\mathcal{K}) &= \mathcal{O}(\epsilon)\\
\hat{\nabla}_a(K_{\mu\nu}K^{\mu\nu}\mathcal{K}) &= 3\epsilon(\partial_a\partial_t^2\delta r-\partial_a\partial^2\delta r)\\
(u.K.u)u^\nu\hat{\nabla}_\nu u_t &= \mathcal{O}(\epsilon)\\
(u.K.u)u^\nu\hat{\nabla}_\nu u_a &= -(\epsilon\partial_t\delta u_a+\epsilon\partial_a\delta r)\\
(u.K.u)u^\mu K_{\mu t} &= 1+\mathcal{O}(\epsilon)\\
(u.K.u)u^\mu K_{\mu a} &= \epsilon\partial_t\partial_a\delta r-\epsilon\delta u_a\\
(u.\hat{\nabla}\mathcal{K})u^\mu K_{\mu t} &= \mathcal{O}(\epsilon)\\
(u.\hat{\nabla}\mathcal{K})u^\mu K_{\mu a} &= \mathcal{O}(\epsilon)^2\\
(u.\hat{\nabla}\mathcal{K})u^\nu\hat{\nabla}_\nu u_t &= \mathcal{O}(\epsilon)\\
(u.\hat{\nabla}\mathcal{K})u^\nu\hat{\nabla}_\nu u_a &= \mathcal{O}(\epsilon)^2
\end{aligned}\tag{C.28}$$

C.2.3 Arguments leading to (4.31)

Following the same trick as previously done, we denote the vector membrane equation as

$$E^{tot}_\mu\equiv\mathcal{P}^\nu_\mu E_\nu,\quad\text{where}\quad E_\mu\equiv\left[\frac{\hat{\nabla}^2u_\alpha}{\mathcal{K}}-\frac{\hat{\nabla}_\alpha\mathcal{K}}{\mathcal{K}}+u^\beta K_{\beta\alpha}-u\cdot\hat{\nabla}u_\alpha\right]+\dots\tag{C.29}$$

Hence we have

$$\begin{aligned}
E^{tot}_t &= E_t\mathcal{P}^t_t+E_b\mathcal{P}^b_t\\
E^{tot}_a &= E_t\mathcal{P}^t_a+E_b\mathcal{P}^b_a
\end{aligned}\tag{C.30}$$

For the uniform planar membrane we have translational symmetry along the x_a directions, so we have $E_b\sim\mathcal{O}(\epsilon)$ in the case of fluctuations. Note that $\mathcal{P}^t_t=0$, $\mathcal{P}^a_t\sim\mathcal{O}(\epsilon)$ and also $E_b\sim\mathcal{O}(\epsilon)$, hence E^{tot}_t vanishes upto linear order. Note that $\mathcal{P}^t_a\sim\mathcal{O}(\epsilon)$, hence only $\mathcal{O}(\epsilon^0)$

pieces of E_t contribute when we evaluate E_a^{tot} upto linear order. Keeping these facts in mind we calculated only those terms that are relevant in subsection C.2.2.

Substituting the expressions derived in subsection (C.2.2) in the linearized vector membrane equation in the angular directions we finally get (4.31).

Appendix D

Appendices for Chapter 5

D.1 Shape variations

In this appendix, we demonstrate the results (5.66). That is, we calculate variations of various membrane quantities with respect to change in shape of membrane. We find it useful to use the Gaussian normal coordinates for this purpose. The form of the spacetime metric in Gaussian normal coordinates is

$$ds^2 = G_{MN}dx^M dx^N = dz^2 + g_{\mu\nu}(z, x^\mu)dx^\mu dx^\nu \tag{D.1}$$

and we take the membrane surface at $z = 0$. The induced metric on the membrane world-volume is $g_{\mu\nu}(0, x^\mu)$. We use overhead bar for the quantities defined in spacetime metric. Unbarred quantities are defined in induced metric on membrane. The normal to membrane is $n = dz$. The Christoffel symbols for the spacetime metric (D.1) are

$$\bar{\Gamma}^M_{zz} = 0, \quad \bar{\Gamma}^z_{zM} = 0, \quad \bar{\Gamma}^z_{\mu\nu} = -\frac{1}{2}\partial_z g_{\mu\nu}, \quad \bar{\Gamma}^\mu_{z\nu} = \frac{1}{2}g^{\mu\alpha}\partial_z g_{\alpha\nu}, \quad \bar{\Gamma}^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} \tag{D.2}$$

$\bar{\nabla}_M n_N$ and $\bar{P}^M_N \equiv \delta^M_N - n^M n_N$ evaluate to

$$\begin{aligned} &\bar{\nabla}_z n_z = 0, \quad \bar{\nabla}_z n_\mu = 0, \quad \bar{\nabla}_\mu n_z = 0, \quad \bar{\nabla}_\mu n_\nu = \frac{1}{2}\partial_z g_{\mu\nu} \\ &\bar{P}^z_z = \bar{P}^z_\mu = \bar{P}^\mu_z = 0, \quad \bar{P}^\mu_\nu = \delta^\mu_\nu \end{aligned} \tag{D.3}$$

Thus the Extrinsic curvature of the membrane evaluates to

$$K_{\mu\nu} = \frac{1}{2}\partial_z g_{\mu\nu}|_{z=0}, \quad K^{\mu\nu} = -\frac{1}{2}\partial_z g^{\mu\nu}|_{z=0} \tag{D.4}$$

Now we consider a new membrane surface $z = \delta z(x^i)$ (Note that δz is not a function of t, so x^i are rest of the spacial coordinates). We work in the linear order in shape perturbations. Using (D.4), the change in the induced metric on the membrane can be found to be

$$\begin{aligned} g_{\mu\nu}(z = \delta z, x^\mu) &= g_{\mu\nu}(z = 0, x^\mu) + \partial_z g_{\mu\nu}(z, x^\mu)|_{z=0}\ \delta z \\ \therefore \delta g_{\mu\nu} &= 2K_{\mu\nu}\delta z \end{aligned} \tag{D.5}$$

and for the inverse metric it is

$$\delta g^{\mu\nu} = -2K^{\mu\nu}\delta z \tag{D.6}$$

Using (D.5) we get the variation

$$\delta\sqrt{-g} = \sqrt{-g}\ \mathcal{K}\ \delta z \tag{D.7}$$

The normal to new surface is $n = dz - \partial_\mu \delta z(x^i)dx^\mu$. For the new surface, $\bar{\nabla}_M n_N$ and $\bar{P}^M_N \equiv \delta^M_N - n^M n_N$ evaluate to (with ∇_μ denotes the covariant derivative on the membrane worldvolume)

$$\begin{aligned} &\bar{\nabla}_z n_z = 0, \quad \bar{\nabla}_z n_\mu = \bar{\nabla}_\mu n_z = \frac{1}{2}\nabla^\rho \delta z\ \partial_z g_{\mu\rho}, \quad \bar{\nabla}_\mu n_\nu = \frac{1}{2}\partial_z g_{\mu\nu} - \nabla_\mu\nabla_\nu \delta z \\ &\bar{P}^z_z = 0, \quad \bar{P}^z_\mu = \nabla_\mu \delta z, \quad \bar{P}^\mu_z = \nabla^\mu \delta z, \quad \bar{P}^\mu_\nu = \delta^\mu_\nu \end{aligned} \tag{D.8}$$

Using (D.4) and (D.8), the Extrinsic curvature for the new surface is found to be

$$K_{\mu\nu}|_{z=\delta z} = \frac{1}{2}\partial_z g_{\mu\nu}|_{z=\delta z} - \nabla_\mu\nabla_\nu\delta z = -\nabla_\mu\nabla_\nu\delta z + K_{\mu\nu}|_{z=0} + \frac{1}{2}\partial_z^2 g_{\mu\nu}|_{z=0}\ \delta z \tag{D.9}$$

Hence we get

$$\delta K_{\mu\nu} = -\nabla_\mu\nabla_\nu\delta z + \frac{1}{2}\partial_z^2 g_{\mu\nu}\delta z \tag{D.10}$$

Ricci tensor $\bar{R}_{MN}$ in spacetime evaluates to

$$\begin{aligned} \bar{R}_{zz} &= -\frac{1}{2}g^{\mu\nu}\partial_z^2 g_{\mu\nu} - \frac{1}{4}\partial_z g_{\mu\nu}\partial_z g^{\mu\nu} \\ \bar{R}_{z\mu} &= \nabla_\nu \bar{\Gamma}^\nu_{z\mu} - \nabla_\mu \bar{\Gamma}^\nu_{\nu z} \\ \bar{R}_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\partial_z^2 g_{\mu\nu} + \frac{1}{2}\partial_z g_{\mu\alpha} g^{\alpha\beta}\partial_z g_{\beta\nu} - \frac{1}{4}\partial_z g_{\mu\nu}(g^{\alpha\beta}\partial_z g_{\alpha\beta}) \\ &= R_{\mu\nu} - \frac{1}{2}\partial_z^2 g_{\mu\nu} + 2K_{\mu\alpha}K^\alpha_\nu - \mathcal{K}K_{\mu\nu} \end{aligned} \tag{D.11}$$

Because the spacetime metric solves Einstein equations,

$$\bar{R} = -D(D-1)\lambda, \quad \bar{R}_{MN} = -(D-1)\lambda G_{MN} \tag{D.12}$$

Thus using (D.12) and (D.11) in (D.10) we get

$$\delta K_{\mu\nu} = (R_{\mu\nu} + (D-1)\lambda G_{\mu\nu} + 2K_{\mu\alpha}K^\alpha_\nu - \mathcal{K}K_{\mu\nu})\,\delta z - \nabla_\mu\nabla_\nu\delta z \tag{D.13}$$

Using (D.6), (D.13) and Gauss's identity, $R = \mathcal{K}^2 - K_{\mu\nu}K^{\mu\nu} - (D-1)(D-2)\lambda$ we get

$$\delta\mathcal{K} = \delta K_{\mu\nu}g^{\mu\nu} + K_{\mu\nu}\delta g^{\mu\nu} = (-K_{\mu\nu}K^{\mu\nu} + (D-1)\lambda)\,\delta z - \nabla^2\delta z \tag{D.14}$$

Using (D.7) and (D.14) we get

$$\delta(\sqrt{-g}\mathcal{K}) = \sqrt{-g}\left(\mathcal{K}^2 - K_{\mu\nu}K^{\mu\nu} + (D-1)\lambda - \nabla^2\right)\delta z \tag{D.15}$$

Notice that the term $\sqrt{-g}\nabla^2\delta z$ in (D.15) is total derivative.

The variation of the volume term can be seen to be

$$\delta \left[-(D-1)\lambda \int_V \sqrt{-G} \right] = -(D-1)\lambda \int_M \sqrt{-g}\ \delta z \tag{D.16}$$

The variation of γ becomes [1] (Recall $\gamma = \frac{1}{\sqrt{-k.k}}$)

$$\delta\gamma = (\partial_z \gamma)\ \delta z = \gamma(u.K.u)\delta z \tag{D.18}$$

Using (D.18) and (D.7) we get

$$\delta(\sqrt{-g}\ \gamma) = \sqrt{-g}\ \gamma(\mathcal{K} + u.K.u)\ \delta z \tag{D.19}$$

This completes the demonstration of (5.66).

D.2 QNM for spherical membrane in flat spacetime

In this section, we find the quasinormal mode spectrum for linearized fluctuations about a spherical membrane in arbitrary D dimensional flat spacetime background. Since the calculation is very similar to done e.g. in [9, 11] we present only key steps. For details, [9, 11] can be referred. We consider the background spacetime metric

$$ds^2_{ST} = -dt^2 + dr^2 + r^2 d\Omega^2_{D-2} \tag{D.20}$$

We consider the shape and velocity fluctuations about a uniform spherical membrane, so we consider the shape and the velocity field of the following form

$$r = 1 + \epsilon\delta r(t, \theta^a), \quad u = -dt + \epsilon\delta u_a(t, \theta^a) d\theta^a \tag{D.21}$$

We will always work in linear order in ϵ. Putting (D.21) in (D.20) we get the induced metric on the membrane

$$ds^2 = -dt^2 + (1 + 2\epsilon\delta r) d\Omega^2_{D-2} \tag{D.22}$$

We have the membrane equations

$$\begin{aligned} &\nabla.u = 0 \\ &16\pi\ \mathcal{P}^\nu_\alpha \nabla^\mu T_{\mu\nu} = \left(\tilde{\mathcal{K}}\ u.\nabla u_\nu + \nabla_\nu \tilde{\mathcal{K}} - 2\nabla^\mu \sigma_{\mu\nu} \right) \mathcal{P}^\nu_\alpha \equiv E_\nu \mathcal{P}^\nu_\alpha \end{aligned} \tag{D.23}$$

[1] This can be seen from the following manipulations

$$\partial_z \gamma = \frac{1}{2}\gamma^3 n.\nabla(k^M k_M) = \gamma^3 k^M n.\nabla k_M = -\gamma^3 k^M n^N \nabla_M k_N = \gamma^3 k^M k^N \nabla_M n_N = \gamma(u.K.u) \tag{D.17}$$

Where we have used the fact that there is a Killing vector k^M in spacetime whose pullback on the membrane is k^μ (see section 5.1.2). In the third step, we use the Killing equation. In the fourth and last step we use the fact that $k^M n_M = 0$ on the membrane.

We use the notation that Ω_{ab} denotes the metric on the unit sphere, ∇_a denotes the covariant derivative on the unit sphere, and $\nabla^2 \equiv \nabla^a \nabla_a$. To linear order in fluctuations we calculate the quantities present in (D.23)

$$\begin{aligned}
&\mathcal{P}^t_t = 0, \quad \mathcal{P}^a_t = -\epsilon \delta u^a, \quad \mathcal{P}^t_a = \epsilon \delta u_a, \quad \mathcal{P}^a_b = \delta^a_b \\
&\sigma_{tt} = 0, \quad \sigma_{ta} = 0, \quad \sigma_{ab} = \frac{\epsilon}{2}\Big(\nabla_a \delta u_b + \nabla_b \delta u_a\Big) + \epsilon \partial_t \delta r \Omega_{ab} \\
&\tilde{\mathcal{K}} = (D-3) - (D-3)\epsilon \delta r + 2\epsilon \partial_t^2 \delta r - \epsilon \left(\frac{D-3}{D-2}\right) \nabla^2 \delta r \\
&u.\nabla u_t = 0, \quad u.\nabla u_a = \epsilon \partial_t \delta u_a
\end{aligned} \tag{D.24}$$

Hence the membrane equations (D.23) simplify to

$$\begin{aligned}
&\nabla^a \delta u_a + (D-2)\partial_t \delta r = 0 \\
&V_a \equiv -(D-3)\nabla_a \delta r + 2\partial_t^2 \nabla_a \delta r - \left(\frac{D-3}{D-2}\right)\nabla_a \nabla^2 \delta r + (D-3)\delta_t \delta u_a \\
&- \nabla^b \nabla_a \delta u_b - \nabla^2 \delta u_a - 2\partial_t \nabla_a \delta r = 0
\end{aligned} \tag{D.25}$$

We write the velocity field as

$$\delta u_a = \delta v_a + \nabla_a \Phi, \quad \text{with} \quad \nabla^a \delta v_a = 0 \tag{D.26}$$

Putting (D.26) into the first equation in (D.25) we get

$$\nabla^2 \Phi = -(D-2)\partial_t \delta r \tag{D.27}$$

We expand the fluctuations in the Spherical Harmonic basis as

$$\delta r = \sum_{l,m} a_{l,m} Y_{l,m} e^{-i\omega^s_l t}, \quad \delta v_a = \sum_{l,m} b_{l,m} Y^{l,m}_a e^{-i\omega^v_l t} \tag{D.28}$$

Recall that for the Spherical Harmonics

$$\nabla^2 Y_{l,m} = -l(D-3+l)Y_{l,m}, \quad \nabla^2 Y^{l,m}_a = -\left(-l(D-3+l) - 1\right) Y^{l,m}_a \tag{D.29}$$

We take the divergence of the second equation in (D.25) i.e. $\nabla^a V_a$. We then eliminate the terms containing Φ using (D.27). We put the basis (D.28) and use (D.29), to get the scalar QNM frequencies, which are found to be

$$\omega^s_l = \pm \frac{\sqrt{-b^2 - 4ac}}{2a} - i\frac{b}{2a} \tag{D.30}$$

where,

$$\begin{aligned}
a &= l\big(l + D - 3\big) + \frac{\big(D-3\big)\big(D-2\big)}{2} \\
b &= \big(D-3\big)\Big[l\big(l+D-3\big) - \big(D-2\big)\Big] \\
c &= l\big(l+D-3\big)\left(\frac{D-3}{2}\right)\left[1 - \frac{l\big(l+D-3\big)}{D-2}\right]
\end{aligned} \tag{D.31}$$

Using the fact that δr solves the equation $\nabla^a V_a = 0$ the second equation in (D.25) reduces to the equation only for the variable δv_a. Putting (D.28) into this equation and using (D.29) we find the vector QNM frequencies

$$\omega_l^{(v)} = -i\left[\frac{l\big(l+D-3\big)-1}{D-3} - 1\right] \tag{D.32}$$

Expanding the answers (D.30) and (D.32) in a power series in $1/D$, we get

$$\begin{aligned} \omega_l^s &= \pm\sqrt{l-1} - i(l-1) \pm \frac{l\sqrt{l-1}(2l-3)}{2D} - i\frac{l(l-1)}{D} + \mathcal{O}(D^{-2}) \\ \omega_l^v &= -i(l-1) - \frac{i(l^2-1)}{D} + \mathcal{O}(D^{-2}) \end{aligned} \tag{D.33}$$

Whereas the actual answers found from gravity analysis in [6] and from Membrane paradigm approach in [10] are

$$\begin{aligned} \omega_l^s &= \pm\sqrt{l-1} - i(l-1) \pm \frac{\sqrt{l-1}(3l-4)}{2D} - i\frac{(l-1)(l-2)}{D} + \mathcal{O}(D^{-2}) \\ \omega_l^v &= -i(l-1) - \frac{i(l-1)^2}{D} + \mathcal{O}(D^{-2}) \end{aligned} \tag{D.34}$$

Note that the answers of (D.33) and (D.34) match at leading order but differ at the subleading orders in $1/D$.

D.3 Membrane Energy and Bulk Hamiltonian

In the main text we have demonstrated that the first two terms in the action (5.71) have a simple bulk interpretation - they are equal to half the action of the bulk region enclosed by the membrane. We will now present an alternative - but equivalent - reinterpretation of the same two terms in (5.71) in terms of the Hamiltonian of the region of spacetime enclosed by the membrane.

In order to do this we first rewrite the stationary spacetime (5.47) in the standard ADM form

$$ds^2_{ST} = G_{MN}dX^M dX^N = -N^2 dt^2 + q_{ab}(dX^a + N^a dt)(dX^b + N^b dt) \tag{D.35}$$

where, the various metric coefficients are related to (5.47) by the relations

$$q_{ab}N^b = -e^{2\Sigma}A_a, \quad -N^2 + q_{ab}N^a N^b = -e^{2\Sigma}, \quad q_{ab} = -e^{2\Sigma}A_a A_b + W_{ab} \tag{D.36}$$

Notice that

$$k^M = (\partial_t)^M, \quad q_M = -N(dt)_M, \quad k^M = Nq^M + N^a e^M_a \quad \text{with} \quad e^M_a = \left(\frac{\partial X^M}{\partial X^a}\right)_t \tag{D.37}$$

where k^M is the killing vector field as usual, q_M is the unit normalized normal vector orthogonal to slices of constant time t. As is well known, the offshell action of a region of spacetime can be rewritten in terms of the Hamiltonian of general relativity (see e.g. section 4.2 of [87])

$$\begin{aligned}\mathcal{S}_{\mathcal{G}} = & \frac{1}{16\pi}\Bigg[\int_V \sqrt{-G}\ (\bar{R}-2\Lambda)\ d^DX + 2\int_M \sqrt{-g}\ \mathcal{K}\ d^{D-1}x \\ & + 2\int_{\Sigma_{t_1}} \sqrt{q}\ C_{ab}\ d^{D-1}X - 2\int_{\Sigma_{t_2}} \sqrt{q}\ C_{ab}\ d^{D-1}X\Bigg] \\ = & \int dt\left(\int_{\Sigma_t} p^{ab}\dot{q}_{ab} - H_G\right) \\ \text{where}\quad & p^{ab} \equiv \frac{\sqrt{q}}{16\pi}(C^{ab}-Cq^{ab}),\quad \dot{q}_{ab}\equiv \mathcal{L}_t q_{ab} = (\mathcal{L}_k G_{MN})e^M_a e^N_b\end{aligned} \tag{D.38}$$

In (D.38), Σ_t is the spacelike slice of spacetime at time t. C_{ab}, C are the extrinsic curvature and its trace of the spacelike slice of the spacetime as embedded in the spacetime. $\Sigma_{t_2},\ \Sigma_{t_1}$ are respectively the initial and final spacelike slices. Focusing on the special case of the stationary solutions of interest to us we have

$$\dot{q}_{ab} = 0$$

Moreover, onshell, the Hamiltonian of spacetime is given by the ADM formula (see Equation (4.80) of [87])

$$H_G = -\frac{1}{8\pi}\int_{S_t}\sqrt{-g}\left(\mathcal{K} + q^M\nabla_M n_N q^N - \frac{N^a}{N}(C_{ab}-Cq_{ab})n^b\right) \tag{D.39}$$

In the special case at hand (D.39) can be further simplified.

$$\begin{aligned}N^a q_{ab} n^b &= \left(k^M - Nq^M\right) G_{MN} n^N = 0 \\ N^a C_{ab} n^b &= \left(k^M - Nq^M\right)\nabla_M q_N n^N = -k^M K_M{}^N q_N + Nq^M\nabla_M n_N q^N \\ &= NK^t_t + Nq^M\nabla_M n_N q^N\end{aligned} \tag{D.40}$$

Hence, we get

$$H_G = -\frac{1}{8\pi}\int_{S_t}\sqrt{-g}\ (\mathcal{K} - K^t_t) \tag{D.41}$$

Where S_t is the boundary of the Σ_t, that is the timeslice of the membrane worldvolume at time t.

It follows that the first two terms in the action (5.71) are equal both to the 'length of time' (equal to β in Euclidean space) times

1. Half of the General Relativistic Hamiltonian (i.e. ADM energy) of the region of spacetime enclosed by the membrane

2. The actual energy E of the membrane

The discussion in this Appendix provides an alternate derivation of the equation (5.27).

D.4 Rotating membranes in 4 dimensions

It would be interesting to find the exact solutions corresponding to rotating membrane solutions at all values of D. The problem we need to solve is the following. Specializing to the case of even D, consider flat space in the coordinates

$$ds^2 = -dt^2 + dz^2 + \sum_{i=1}^{[D/2]} dr_i^2 + r_i^2 d\phi_i^2 \tag{D.42}$$

Consider the killing vector

$$k = \partial_t + \sum_i \omega_i \partial_{\phi_i} \tag{D.43}$$

With this choice of k we need to find the membrane shape that obeys the equation (5.20). [2]

We postpone the general consideration of this problem to future work. For the present, we focus our attention on a simple special example, namely $D = 4$. In this case the most general velocity field is characterized by a single rotational velocity ω, and the construction of the membrane shape - dual to the Kerr black hole - turns out to be particularly easy. The trick turns out to be a good choice of coordinates; in this case the zero mass Boyer-Lindquist coordinates.

As our starting point consider the flat space metric in Minkowski coordinates

$$ds^2 = -dt^2 + dz^2 + dx^2 + dy^2 \tag{D.44}$$

Then perform the coordinate change to the zero mass Boyer-Lindquist coordinates

$$z = r\cos\theta, \quad x = \sqrt{r^2 + a^2}\sin\theta\cos\phi, \quad y = \sqrt{r^2 + a^2}\sin\theta\sin\phi \tag{D.45}$$

Under which (D.44) becomes

$$ds^2 = G_{MN}dx^M dx^N = -dt^2 + \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}dr^2 + \left(r^2 + a^2\cos^2\theta\right)d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2 \tag{D.46}$$

Under this coordinate change the killing vector (D.43) retains its form

$$k = \partial_t + \omega\partial_\phi \tag{D.47}$$

We will find it useful to define a new constant a of dimension length by the equation

$$\omega = \frac{a}{r_H^2 + a^2} \tag{D.48}$$

Working in the coordinate system (D.45) we will now demonstrate that the surface

$$r = r_H \tag{D.49}$$

(together with the choice of k listed in (D.47) and (D.48)) solve (5.20) with

$$4\pi T_0 = \frac{r_H}{a^2 + r_H^2} \tag{D.50}$$

[2]For other studies of reliable fluid descriptions of localized black holes see e.g. [88, 89, 53, 55].

In order to see this we note that the velocity field corresponding to the killing vector (D.48), (D.47) is given by

$$u^M = \gamma k^M, \quad \gamma = \frac{1}{\sqrt{-k^M G_{MN} k^N}} = \left(1 - \frac{a^2 \sin^2\theta}{r_H^2 + a^2}\right)^{-1/2} \tag{D.51}$$

For the surface (D.49) we find

$$\begin{aligned}
\mathcal{K} &= \frac{r_H}{\sqrt{r_H^2 + a^2}} \frac{2r_H^2 + a^2(1 + \cos^2\theta)}{(r_H^2 + a^2\cos^2\theta)^{3/2}} \\
K_{MN}K^{MN} &= \frac{r_H^2}{r_H^2 + a^2} \frac{(r_H^2 + a^2)^2 + (r_H^2 + a^2\cos^2\theta)^2}{(r_H^2 + a^2\cos^2\theta)^3} \\
u.K.u &= \frac{r_H}{\sqrt{r_H^2 + a^2}} \frac{a^2 \sin^2\theta}{(r_H^2 + a^2\cos^2\theta)^{3/2}}
\end{aligned} \tag{D.52}$$

It follows that

$$\frac{\tilde{\mathcal{K}}}{\gamma} = \frac{\mathcal{K}^2 - K_{MN}K^{MN}}{\gamma(\mathcal{K} + u.K.u)} = \frac{r_H}{r_H^2 + a^2} = 4\pi T_0 \tag{D.53}$$

demonstrating that the surface $r = r_H$ solves the equations (5.20) with T_0 given in (D.50).

Let us emphasize that the quantities γ, $\mathcal{K}$, K_{MN}, $u.K.u$ - which went into the LHS of (D.53) - all depend on θ in a nontrivial manner. Interestingly however, the θ dependences of the combination of these quantities that appears in $\tilde{\mathcal{K}}$ cancel out, allowing the configuration $r = r_H$ to solve (5.20).

Inverting (D.50) to solve for the parameter r_H in terms of a and T_0 we find

$$r_H = m \pm \sqrt{m^2 - a^2}, \quad \text{where} \ \ m \equiv \frac{1}{8\pi T_0} \tag{D.54}$$

It is not difficult to determine the thermodynamical charges of our solution. The entropy is given by

$$S_{ent} = \int_{sM} \sqrt{h}\ q_\mu J^\mu_S = \frac{1}{4}\int_{sM} \sqrt{-g}\ \gamma = \pi(r_H^2 + a^2) \tag{D.55}$$

Where, sM denotes integration over the spacelike slice of the membrane. h is the determinant of the metric on this slice. g is the determinant of the metric on the membrane worldvolume.

The mass of the membrane is given by

$$\begin{aligned}
M &= -\int_{sM} \sqrt{h}\ q^\mu T_{\mu\nu}(\partial_t)^\nu = \frac{-1}{16\pi}\int_{sM} \sqrt{h}\ q^\mu(\tilde{\mathcal{K}}P_{\mu\nu} + K_{\mu\nu} - \mathcal{K}g_{\mu\nu})(\partial_t)^\nu \\
&= \frac{r_H^2 + a^2}{2a} \tan^{-1}\left(\frac{a}{r_H}\right)
\end{aligned} \tag{D.56}$$

and the angular momentum

$$\begin{aligned}
J &= \int_{sM} \sqrt{h}\ q^\mu T_{\mu\nu}(\partial_\phi)^\nu = \frac{1}{16\pi}\int_{sM} \sqrt{h}\ q^\mu(\tilde{\mathcal{K}}P_{\mu\nu} + K_{\mu\nu} - \mathcal{K}g_{\mu\nu})(\partial_\phi)^\nu \\
&= \frac{r_H^2 + a^2}{4a}\left[-r_H + \frac{r_H^2 + a^2}{a}\tan^{-1}\left(\frac{a}{r_H}\right)\right]
\end{aligned} \tag{D.57}$$

It is easily verified that our results obey the first law of thermodynamics

$$dM = T_0 dS_{ent} + \omega dJ \tag{D.58}$$

The 'energy' of the membrane - i.e. conserved charge $E = M - \omega J$ of membrane dual to the killing vector k is given by

$$\begin{aligned} E &= -\int_{sM} \sqrt{h}\ q^{\mu} T_{\mu\nu} k^{\nu} = \frac{-1}{16\pi} \int_{sM} \sqrt{h}\ q^{\mu} (K_{\mu\nu} - \mathcal{K} g_{\mu\nu}) k^{\nu} \\ &= \frac{r_H}{4} \left[1 + \left(\frac{a}{r_H} + \frac{r_H}{a} \right) \tan^{-1} \left(\frac{a}{r_H} \right) \right] \end{aligned} \tag{D.59}$$

Provided we restrict attention to those variations that keep ω fixed we have (from (D.58))

$$dE = T_0 dS_{ent} \tag{D.60}$$

in agreement with the general analysis presented earlier in this chapter (recall that it was assumed - for the purpose of that analysis - that the killing vector k_μ - and hence ω of this subsection - is kept constant while taking all variations).

The Partition function for the rotating membrane in 4D flat spacetime, written in terms of chemical potentials becomes

$$\ln Z = \frac{-1}{4 T_0 \omega} \tan^{-1} \left(\frac{\omega}{4\pi T_0} \right) \tag{D.61}$$

Whereas, the partition function for actual Kerr black hole (see [90]) is (with M as the mass of black hole)

$$\ln Z = -\frac{M}{2T_0} = -\frac{1}{8\pi T_0^2 + 4T_0 \sqrt{4\pi^2 T_0^2 + \omega^2}} \tag{D.62}$$

Note that for $\omega \to 0$ we have both the partition functions reduce to $-\frac{1}{16\pi T_0^2}$ in agreement with (5.94) and (5.93) at $D = 4$. It is easy to check that the partition functions (D.61) and (D.62) satisfy the thermodynamic relations

$$J = T_0 \frac{\partial \ln Z}{\partial \omega}, \quad -T_0^2 \frac{\partial \ln Z}{\partial T_0} = -M + \omega J, \quad S_{ent} = \ln Z + T_0 \frac{\partial \ln Z}{\partial T_0} \tag{D.63}$$

It is also easy to check that the thermodynamical charges that we have computed for our 4D rotating membrane above obey the 'Smarr relation'

$$M = 2\omega J + 2 T_0 S_{ent} \tag{D.64}$$

(of course the exact thermodynamical charges for the Kerr black hole - see below - also obey (D.64)).

It is natural to interpret the rotating membrane solution presented in this chapter as the dual to the Kerr black hole solution given, for instance, pages (221, 222) of [87] with electromagnetic charge Q of [87] set to zero and the parameter r_+ of [87] identified with r_H of this subsection and a and M of [87] identified with a and M of this subsection. With these identifications, the entropy of the Kerr black hole agrees exactly with the (D.55). However the mass of the Kerr black hole does *not* agree exactly with (D.56); indeed we find the correct gravitational results for the Kerr black hole mass only once we make the replacement

$$\frac{1}{a}\tan^{-1}\frac{a}{r_H} \to \frac{1}{r_H}$$

. This replacement is exact in the limit $a \to 0$, and so at $\omega = 0$. However the two expressions above differ already at $\mathcal{O}(\omega^2)$.

The match between our membrane's angular momentum and the angular momentum of the Kerr black hole is even worse. The equation (D.57) reduces to the formula for *half of* the angular momentum of the Kerr black hole under the replacement

$$\frac{1}{a}\tan^{-1}\frac{a}{r_H} \to \frac{1}{r_H}$$

The surprise here is the additional factor of half which means that the membrane description fails to quantitatively reproduce the even the leading order - order ω. [3]

Of course the discrepancies of this subsection all occur at $D = 4$ - which is as far from the large D limit as we can be. Consequently the thermodynamical mismatches described above do not contradict any clearly established expectation. Nonetheless - given the fact that our membrane worked so remarkably well for static black holes, we find them disappointing. Given the fact that the second order fluid gravity correspondence was able to exactly reproduce the thermodynamics of Kerr-AdS black holes, it seems likely to us that the membrane stress tensor (5.7), will turn out to admit an additional improvement term that is irrelevant at large D and in static situations, but will allow us to reproduce the thermodynamics of rotating black hole solutions exactly at finite D. We postpone the study of this possibility to future work.

[3] Recall that the energy and angular momentum enter thermodynamical relations in the combination $E - \omega J$. The mismatch of J at order ω is, therefore, connected to the mismatch of E at order ω^2 noted above.

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