Scattering in Chern-Simons Theories
and
Black Holes in Large D

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by
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DECLARATION

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

The work was done under the guidance of Professor [guide's name], at the Tata Institute of Fundamental Research, Mumbai.

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[candidate's name and signature]

In my capacity as supervisor of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

Sminwalla

SHIRAZ MINWALLA

[guide's name and signature]

Date: 19th June 2018

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List of Publications

Papers relevant to the thesis work:

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  arXiv:1404.6373 [hep-th], JHEP 1504 (2015) 129.

- Y. Dandekar, M. Mandlik and S. Minwalla, "Poles in the S-Matrix of Relativistic Chern-Simons Matter theories from Quantum Mechanics,"
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- S. Bhattacharyya, M. Mandlik, S. Minwalla and S. Thakur,"A Charged Membrane Paradigm at Large D,"
  arXiv:1511.03432 [hep-th], JHEP 04 (2016) 128.

- S. Bhattacharyya, A. Mandal, M. Mandlik, U. Mehta, S. Minwalla, U. Sharma and S. Thakur. "Currents and Radiation from the large D Black Hole Membrane,"
  arXiv: 1611.09310 [hep-th], JHEP 05 (2017) 098.

Contents

1 Chapter 1: Unitarity, crossing symmetry and duality of the S-matrix

1.1 Introduction

It has recently been conjectured that $U(N)$ Chern-Simons theories coupled to a multiplet of fundamental Wilson-Fisher bosons at level k are dual to $U(|k| - N)$ Chern-Simons theories coupled to fundamental fermions at level $-k$. [1] The evidence for this conjecture is threefold. First the spectrum of 'single trace' operators and the three point functions of these operators have also been computed exactly in the 't Hooft limit, and have been found to match[2–7]. Second the thermal partition functions of these theories have also been computed in the 't Hooft large N limit and have been shown to match [2, 3, 6, 8–11]. Finally the duality described above has been demonstrated to follow from an extreme deformation of the known Giveon-Kutasov type duality [12, 13] between supersymmetric theories [14].

Assuming the duality described above does indeed hold, it is interesting to better understand the map that transforms bosons into fermions. Morally, we would like an explicit construction of the fundamental fermionic field $\psi_a(x)$ as a function of the fundamental bosonic fields [2]; such a formula cannot, however, be given precise meaning in the current context as $\psi_a(x)$ is not gauge invariant and its offshell correlators are ill-defined.

The on shell limit of correlators of the elementary bosonic and fermionic fields, however, are physical as they compute the S-matrix for the scattering of bosonic or fermionic quanta. As Chern-Simons theory has no propagating gluonic states, the S-matrix is free of soft gluon infrared divergences when the fundamental fields (bosons and fermions) are taken to be massive. An identity relating well-defined bosonic and fermionic S-matrices appears to be the closest we can come to a precise bosonization map. Motivated by this observation, in this chapter we present a detailed study of $2 \to 2$ S-matrices in Chern-Simons theories with fundamental bosonic and fermionic fields.

Even independent of the Bose-Fermi duality, it is interesting that it is possible to

[1] Our notation is as follows. k is the coefficient of the Chern-Simons term in the bulk Lagrangian in the dimensional reduction scheme utilized throughout in this chapter. It is useful to define $\kappa = \mathrm{sgn}(k)(|k| - N)$. $|\kappa|$ is the level of the WZW theory dual to the pure Chern-Simons theory. Note that $|k| > N$. In terms of κ and N the duality map takes the level-rank form $N' = |\kappa|, \quad \kappa' = -\mathrm{sgn}(\kappa)N$.

[2] The template here is the formula $\psi = e^{i\phi}$ of two dimensional bosonization. In some respects the already well known map between the gauge invariant higher spin currents on the two sides of the duality is the $2 + 1$ dimensional analogues of the $1 + 1$ dimensional relation between global $U(1)$ symmetry currents $\partial\phi \sim \tilde{\psi}\psi$.

determine exact results for the S-matrix of these theories as a function of the 't Hooft coupling constant $\lambda = \frac{N}{k}$. Exact results for scattering amplitudes as a function of a gauge coupling constant are rare, and should be studied when available for qualitative lessons. As we will see below, the explicit formulae for S-matrices presented in this chapter turn out to possess several unfamiliar and unusual structural features. Some of these unusual features appear to have a simple physical interpretation; we anticipate that they are general properties of S-matrices in all matter Chern-Simons theories. [3]

As we have mentioned above, it is possible to determine (or conjecture) explicit results for the $2 \to 2$ scattering amplitudes for large N fundamental matter Chern-Simons theories. In this chapter we present explicit formulae for all these scattering amplitudes. In the rest of this introduction we will describe the most important qualitative features of our results. We first briefly review some kinematics in order to set terminology.

Consider the $2 \to 2$ scattering of particles in representations R_1 and R_2 of $U(N)$. Let the tensor product of these two representations decompose as

$$R_1 \times R_2 = \sum_m R_m. \tag{1}$$

It follows from $U(N)$ invariance that the S-matrix for the process takes the schematic form

$$S = \sum_m P_m S_m, \tag{2}$$

where P_m is the projector onto the m^{th} representation, and S_m is the scattering matrix in the 'm^{th}' channel.

In this chapter we study the $2 \to 2$ scattering matrices of the elementary quanta of theories with only fundamental matter. In this situation R_1 and R_2, are either both fundamentals, or one fundamental and one antifundamental. [4] In the case of fundamental - fundamental scattering, R_m is either the 'symmetric' representation with two boxes in the first row (and no boxes in any other row) of the Young Tableaux, or the 'antisymmetric' representation with two boxes in the first column and no boxes in any other column. In the case of fundamental - antifundamental scattering, R_m is either the singlet or the adjoint representation. In this chapter we will present computations or conjectures for the all

[3]These features include the presence of an non-analytic δ function piece in the S-matrix localized on forward scattering, and modified crossing symmetry relations as we describe below.

[4]The scattering of two antifundamentals is simply related to the scattering of two fundamentals, and will not be considered separately in this chapter.

orders S-matrices in all the four channels mentioned above (symmetric, antisymmetric, singlet and adjoint) in both the bosonic and the fermionic theories.

The scattering matrices of interest to us in this chapter are already well known in the non-relativistic limit (i.e. in the limit in which the masses of the scattering particles and the center of mass energy are both taken to infinity at fixed momentum transfer) as we now very briefly review. The Chern-Simons equation of motion ensures that each particle traps magnetic flux. The Aharonov-Bohm effect then ensures that the particle R_1 picks up the phase $2\pi\nu_m$ as it circles around[5] the particle R_2, where

$$2\pi\nu_m = \frac{4\pi T_1^a T_2^a}{k} = 2\pi\frac{C_2(R_m) - C_2(R_1) - C_2(R_2)}{k}, \tag{3}$$

(where $T^a_{1/2}$ are the representation matrices for the group generators in representations R_1 and R_2 and $C_2(A)$ is the quadratic Casimir in representation A). It follows as a consequence [15] that the non-relativistic scattering amplitude in the R_m exchange channel is given by the Aharonov-Bohm scattering amplitude of a $U(1)$ particle of unit charge of a point like magnetic flux of strength $2\pi\nu_m$.

It is easily verified that $\nu_m = \mathcal{O}(\frac{1}{N})$ or smaller in the symmetric, antisymmetric or adjoint channels. In the singlet channel, however, it turns out that to leading order in the large N limit $\nu_m = \frac{N}{k} = \lambda$. It follows that the rotation by π which interchanges the two scattering particles is accompanied by a phase $e^{-i\pi\lambda_B}$ in the bosonic theory and $(-1)e^{-i\pi\lambda_F} = e^{-i\pi(-\mathrm{sgn}(\lambda_F)+\lambda_F)}$ in the fermionic theory. [6] Note that these phases are identical when

$$\lambda_B = \lambda_F - \mathrm{sgn}(\lambda_F). \tag{4}$$

However (4) is precisely the map between λ_B and λ_F [6] induced by the level-rank duality transformation described at the beginning of this introduction. In the singlet channel, in other words the bosons and conjecturally dual fermions are both effectively anyonic, with the same anyonic phase. This observation provides a partial physical explanation for the duality map (4).

We note in passing that the anyonic phase $\pi\lambda_B$ is precisely twice the phase of the bulk interaction term in the conjectured Vasiliev duals to these theories [2, 16]. Indeed the first

[5]Readers familiar with the relationship between Chern-Simons theory and WZW theory may recognize this formula in another guise. $\frac{C_2(R)}{k}$ is the holomorphic scaling dimension of a primary operator in the integrable representation R, and $e^{2\pi i\nu_m}$ is the monodromy of the four point function $< R_1,\ R_2,\ \bar{R}_1,\ \bar{R}_2 >$ in the conformal block corresponding to the OPE $R_1R_2 \to R_m$.

[6] The additional -1 in the fermionic theory comes from Fermi statistics. We have used $-1 = e^{\pm i\pi} = e^{-i\pi\mathrm{sgn}(\lambda_F)}$.

speculation of the bosonization duality for matter Chern-Simons theories [2] was motivated by argument very similar to that presented in the previous paragraph but in the context of Vasiliev theories (deformations of the bosonic and fermionic theory that lead to the same interaction phase ought to be the same theory). It would certainly be very interesting to find a logical link between the phase of interactions in Vasiliev theory and the anyonic phase of the previous paragraph, but we will not peruse this thread in this chapter.

Moving away from the non-relativistic limit, in this chapter we have (following the lead of [6]) summed all planar graphs to determine the exact relativistic S-matrix for both the bosonic as well as the fermionic theories in the symmetric, antisymmetric and adjoint channels. Even though our completely explicit solutions are quite simple, they possess a rich analytic structure (see section 1.3 for a detailed listing of results). It is a simple matter to compare the explicit results for the S-matrices in the bosonic and fermionic theories that are conjecturally dual to each other. We find that the bosonic and fermionic S-matrices agree perfectly in the adjoint channel. On the other hand the bosonic S-matrix in the symmetric/antisymmetric channels matches the fermionic S-matrix in the antisymmetric/symmetric channels. Our results are all consistent with the following rule: the bosonic S-matrix in the exchange channel R_m is identical with the fermionic S-matrix in the exchange channel R_m^T, where R_m^T is the dual representation under level-rank duality. [7]

The match of S-matrices upto transposition appears to make perfect sense from several points of view. Let us focus attention on the particle - particle scattering and consider a multi-particle asymptotic state. As the Aharonov-Bohm phases ν_m vanish in the large N limit considered in this chapter, the multi-particle state in question is effectively a collection of non interacting bosonic particles, and so must obey Bose statistics. As an example, consider a multi-particle state that is completely antisymmetric under the interchange of its momenta. In order to meet the requirement of Bose statistics, this state must also be completely antisymmetric under the interchange of color indices. The corresponding dual asymptotic state in the fermionic theory is also completely antisymmetric under the interchange of momenta. In order to meet the requirement of fermionic statistics, this state must thus be completely *symmetric* under the interchange of color indices. In other words the map between bosonic and fermionic asymptotic states must involve a transposition

[7] In the large N and large k limit, the dual of a representation with a finite number of boxes plus a finite number of anti-boxes in the Young Tableaux is given by the following rule: we simply transpose the boxes and the anti-boxes in the Young Tableaux (i.e. exchange rows and columns independently for boxes and anti-boxes). According to this rule the fundamental, antifundamental, singlet and adjoint representations are self-dual, while the symmetric and antisymmetric representations map to each other.

of color representations; this transposition is part of the duality map between asymptotic states of the two theories, and is a reflection of the bose -fermi nature of the duality. [8] See section 1.8 for further discussion of the map between the multi-particle states of this theory induced by duality.

The transposition of exchange representations above might also have been anticipated from another point of view. In the pure gauge sector (i.e. upon decoupling the fundamental bosonic and fermionic fields by making them very massive), the conjectured duality between the bosonic and fermionic theories reduces to the level-rank duality between two distinct pure Chern-Simons theories. It is well known that, under level-rank duality, a Wilson line in representation R maps to a Wilson line in the representation R^T. As a Wilson line in representation R represents the trajectory of a particle in representation R, it seems very natural that the exchange channels in a dynamical scattering process also map to each other only after a transposition.

Before proceeding we pause to address an issue of possible confusion. We have asserted above that scalar and spinor S-matrices map to each other under duality. The reader whose intuition is built from the study of four dimensional scattering processes may find this confusing. Scalar and spinor S-matrices cannot be equated in four or higher dimensions as they are functions of different variables. Scalar S-matrices are labelled by the momenta of the participating particles. On the other hand spinor S-matrices are labelled by both the momentum and the 'polarization spinor' of the participating particles. In precisely three dimensions, however, the Dirac equation uniquely determines the polarization spinor of particles and antiparticles as a function of of their momenta [9]. It follows that three dimensional spinorial and scalar S-matrices are both functions only of the momenta of the scattering particles, so these S-matrices can be sensibly identified.

For a technical reason we explain below we are unable to directly compute the S-matrix in the singlet exchange channel by summing graphs; given this technical limitation we are constrained to simply conjecture a result for this S-matrix. The reader familiar

[8] It is not difficult to see how the transposition of S-matrices emerges out of the difference between Bose and Fermi statistics at the diagrammatic level. Scattering processes involving identical particles (both fundamentals or both antifundamentals) receive contributions both from 'direct' scattering processes as well as 'exchange' scattering process. The usual rules tell us that direct and exchange processes must be added together with a positive sign in the bosonic theory but with a negative sign in the fermionic theory. The difference in relative signs implies that S-matrix in the symmetric channel (the sum of the exchange and direct S-matrices) in the bosonic theory is interchanged with the antisymmetric S-matrix (the difference between exchange and direct processes) the fermionic theory.

[9] A related fact: the little group for massive particles in 2+1 dimensions is $SO(2)$, which admits non-trivial one dimensional representations.

with the usual lore on scattering matrices may think this is an easy task. According to traditional wisdom, the S-matrices in a relativistic quantum field theory enjoy crossing symmetry. Particle-antiparticle scattering in both channels should be determined from the results of particle-particle scattering; given the scattering amplitudes in the symmetric and antisymmetric exchange channels, we should be able to obtain the results of scattering in the singlet and adjoint exchange channels by analytic continuation. This principle yields a conjecture for the S-matrix in the singlet channel which, however, fails every consistency check: it has the wrong non relativistic limit and does not obey the constraints of unitarity. For this reason we propose that the usual rules of crossing symmetry are modified in the study of S-matrices in matter Chern Simons theories.

A hint that crossing symmetry might be complicated in these theories is present already in the non-relativistic limit as the Aharonov-Bohm scattering amplitude has an unusual δ function contribution localized about forward scattering [17]. This contribution to the S-matrix has a simple physical origin: a wave packet of one particle that passes through another is diluted by the factor $\cos(\pi\nu_m)$ compared to the usual expectations because of destructive interference from Aharonov-Bohm phases; as a consequence the S-matrix includes a term proportional to $(\cos(\pi\nu_m) - 1)I$ (I is the identity S matrix; see subsection 1.2.3 for more details). The non-analyticity of this term makes it difficult to imagine it can be obtained from a procedure involving analytic continuation.

In addition to the singular δ function piece, the scattering amplitude has an analytic part. In this chapter (and in the large N limit studied here) we conjecture that this analytic piece is given by the naive analytic continuation from the particle-particle sector, multiplied by the factor

$$f(\lambda) = \frac{\sin(\pi\lambda)}{\pi\lambda}.$$

This conjecture passes several consistency checks; it yields a result consistent with the expectations of unitarity, and has the right non-relativistic limit, and yields S-channel S-matrices that transform into each other under Bose-Fermi duality.

The factor $f(\lambda)$ is familiar in the study of pure Chern-Simons theory; N times this factor is the expectation value of a circular Wilson loop on S^3 in the large N limit. In section 1.7.4 below we present a tentative explanation for why one should have *expected* S-matrices in matter Chern-Simons theories to obey the modified analyticity relation with precisely the factor $f(\lambda)$. Our tentative explanation has its roots in the fact that the fully gauge invariant object that obeys crossing symmetry is the 'S-matrix' computed in this chapter dressed with external Wilson lines linking the scattering particles. The presence of the Wilson lines leads to an additional contribution (in addition to those considered in this chapter) that we argue to be channel dependent; in fact we argue that the ratio of the additional contributions in the two channels is precisely the given by the factor above, explaining why the 'bare' S-matrix computed in this chapter has 'renormalized' crossing symmetry properties. If our tentative explanation of this feature is along the right tracks, then it should be possible to find a refined argument that predicts the analytic structure and crossing properties of the S-matrix at finite values of N and k. We leave this exciting task for the future.

We note also that the factor $f(\lambda)$ appears also in the normalization of two point functions of, for instance, two stress tensors (see [6]). The appearance of this factor in the two point functions of gauge invariant operators seems tightly tied to the appearance of the same factor in scattering in the singlet channel, as the diagrams that contribute to these processes are very similar. It would be interesting to understand this relationship better.

this chapter is organized as follows. In section 1.2 below we describe the theories we study in this chapter, review the conjectured level-rank dualities between the bosonic and fermionic theories, set up the notation and conventions for the scattering process we study, review the constraints of unitarity on scattering and review the known non-relativistic limits of the scattering matrices. In section 1.3 below we briefly summarize the method we use to compute S-matrices, and provide a detailed listing of the principal results and conjectures. We then turn to a systematic presentation of our results. In section 1.4 we compute the S-matrices of the bosonic theories by solving the relevant Schwinger-Dyson equations. In section 1.5 we verify the results of section 1.4 at one loop by a direct diagrammatic evaluation of the S-matrix in the covariant Landau gauge. In section 1.6 we compute the S-matrix of the fermionic theories by solving a Schwinger-Dyson equation and verify the equivalence of our bosonic and fermionic results under duality. In section 1.7 we present our conjecture for the S-channel scattering amplitudes (in the bosonic and fermionic systems) of our theory, and provide a heuristic explanation for the unusual transformation properties under crossing symmetry obeyed by our conjecture. In section 1.8 we end with a discussion of our results and of promising future directions of research. Several appendices contain technical details of the computations presented in this chapter.

1.2 Statement of the problem and review of background material

This section is organized as follows. In subsection 1.2.1 we describe the theories we study. In subsection 1.2.2 we review the conjectured duality between the bosonic and fermionic theories. In subsection 1.2.3 we review relevant aspects of the kinematics of $2 \to 2$ scattering in 3 dimensions, with particular emphasis on the structure of the 'identity' scattering amplitude, which will turn out to be renormalized in matter Chern-Simons theories. In subsection 1.2.4 describe the precise scattering processes we study in this chapter. In subsection 1.2.6 we review the known non-relativistic limits of these scattering amplitudes. In subsection 1.2.7 we describe the constraints on these amplitudes from the requirement of unitarity.

1.2.1 Theories

As we have explained above, in this chapter we study two classes of large N Chern-Simons theories coupled to matter fields in the fundamental representation. The first family of theories we study involves a single complex bosonic field, in the fundamental representation of $U(N)$, minimally coupled to a Chern-Simons coupled gauge field. In the rest of this chapter we refer to this class of theories as 'bosonic theories'. The second family of theories we study involves a single complex fermionic field in the fundamental representation of $U(N)$, minimally coupled to a Chern-Simons coupled gauge field. In the rest of this chapter we refer to this class of theories as 'fermionic theories'.

The bosonic system we study is described by the Euclidean Lagrangian

$$S = \int d^3x \left[i\varepsilon^{\mu\nu\rho} \frac{k_B}{4\pi} \mathrm{Tr}(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho) + D_\mu \bar{\phi} D^\mu \phi + m_B^2 \bar{\phi}\phi + \frac{1}{2N_B} b_4 (\bar{\phi}\phi)^2 \right] \tag{5}$$

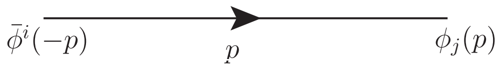
with $\lambda_B = \frac{N_B}{k_B}$. Throughout this chapter we employ the dimensional regularization scheme and light cone gauge employed in the original study of [2]. The theory (5) has been studied intensively in the recent literatures [3, 5–11, 14, 18]. It has in particular been demonstrated that in the regulation scheme and gauge employed in this chapter, the bosonic propagator is given, at all orders in λ_B, by the extremely simple form

$$\langle \phi_j(p) \bar{\phi}^i(-q) \rangle = \frac{(2\pi)^3 \delta^i_j \delta^3(-p+q)}{p^2 + c_B^2} \tag{6}$$

where the pole mass, c_B is a function of m_B, b_4 and λ_B, given by

$$c_B^2 = \frac{\lambda_B^2}{4} c_B^2 - \frac{b_4}{4\pi} |c_B| + m_B^2. \tag{7}$$

(see e.g. Eqn 1.5 of [14] setting $x_4 = 0$ setting temperature T to zero). In all the Feynman



(a)

Figure 1: Propagator of bosonic particles.

diagrams presented in this chapter, we adopt the following convention. The propagator (6) is denoted by a line with an arrow from $\bar{\phi}$ to ϕ, with moment p in the direction of the arrow (see Fig. 1).

The fermionic system we study is described by the Lagrangian

$$S = \int d^3x \left[i\varepsilon^{\mu\nu\rho}\frac{k_F}{4\pi}\mathrm{Tr}(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho) + \bar{\psi}\gamma^\mu D_\mu\psi + m_F\bar{\psi}\psi \right] \tag{8}$$

with $\lambda_F = \frac{N_F}{k_F}$. This theory has also been studied intensively in the recent literatures [2, 5, 7–11, 14, 18]. In particular it has been demonstrated that the fermionic propagator is given (in the light cone gauge and dimensional regulation scheme of this chapter), to all orders in λ_F, by [2, 7, 9, 14]

$$\left\langle \psi_j(p)\bar{\psi}^i(-q) \right\rangle = \frac{\delta^i_j(2\pi)^3\delta^3(-p+q)}{i\gamma^\mu p_\mu + \Sigma_F(p)}, \tag{9}$$

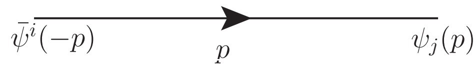
where

$$\begin{aligned}
\Sigma_F(p) =& i\gamma^\mu\Sigma_\mu(p) + \Sigma_I(p)I, \\
\Sigma_I(p) =& m_F + \lambda_F\sqrt{c_F^2 + p_s^2}, \\
\Sigma_\mu(p) =& \delta_{+\mu}\frac{p_+}{p_s^2}\left(c_F^2 - \Sigma_I^2(p)\right), \\
c_F^2 =& \left(\frac{m_F}{\mathrm{sgn}(m_F) - \lambda_F}\right)^2.
\end{aligned} \tag{10}$$

Here γ^μ compose the Euclidean Clifford algebra,

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}, \quad [\gamma^\mu, \gamma^\nu] = 2i\epsilon^{\mu\nu\rho}\gamma_\rho.$$

The fermionic propagator presented above has a pole at $p^2 = c_F^2$; so the quantity c_F is the pole mass - or true mass - of the fermionic quanta . In all the Feynman diagrams presented in this chapter, we adopt the following convention. The propagator (9) is denoted by a line



(a)

Figure 2: Propagator of fermionic particles.

with an arrow from $\bar{\psi}$ to ψ, with momentum p in the direction of the arrow (see Fig. 2).

1.2.2 Conjectured Bose-Fermi duality

The bosonic theory (5) may be rewritten as

$$\begin{aligned} S = \int d^3x \Big[& i\varepsilon^{\mu\nu\rho}\frac{k_B}{4\pi}\mathrm{Tr}(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho) + D_\mu\bar{\phi}D^\mu\phi + m_B^2\bar{\phi}\phi + \frac{1}{2N_B}b_4(\bar{\phi}\phi)^2 \\ & - \frac{N_B}{2b_4}\left(\sigma - \frac{b_4}{N_B}\bar{\phi}\phi - m_B^2\right)^2\Big]. \end{aligned} \tag{11}$$

We have introduced a new field σ in (11); upon integrating σ out (11) trivially reduces to (5). Expanding out the last bracket in (11) and ignoring the constant term, we find that (11) may be rewritten as

$$S = \int d^3x \left[i\varepsilon^{\mu\nu\rho}\frac{k_B}{4\pi}\mathrm{Tr}(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho) + D_\mu\bar{\phi}D^\mu\phi + \sigma\bar{\phi}\phi + N_B\frac{m_B^2}{b_4}\sigma - N_B\frac{\sigma^2}{2b_4}\right]. \tag{12}$$

The so called Wilson-Fisher limit of the bosonic theory is obtained by taking the limit

$$b_4 \to \infty, \quad m_B \to \infty, \quad \frac{4\pi m_B^2}{b_4} = m_B^{\mathrm{cri}} = \text{fixed}. \tag{13}$$

In this limit the last term in (12) may be omitted; moreover it follows from (7) that in this limit

$$|c_B| = m_B^{\mathrm{cri}}.$$

Note, of course, that this equation has no solution for negative m_B^{cri}. As was explained in [6, 14] this is plausibly a reflection of the fact that (7) is the saddle point equation for an uncondensed solution, whereas the scalar in the theory wants to condense when $m_B^{\mathrm{cri}} < 0$. The determination of the condensed saddle point is a fascinating but unsolved problem, and in this chapter we restrict our attention to the case $m_B^{\mathrm{cri}} > 0$.

As we have mentioned in the introduction, it has been has been conjectured that the scalar theory in the Wilson-Fisher limit described above is dual to the theory (8), [10] once we identify parameters according to

$$\begin{aligned} k_F &= -k_B, \\ N_F &= |k_B| - N_B, \\ \lambda_B &= \lambda_F - \mathrm{sgn}(\lambda_F), \\ m_F &= -m_B^{\mathrm{cri}} \lambda_B. \end{aligned} \tag{14}$$

As we have explained above, we will restrict our attention to bosonic theories with $m_B^{\mathrm{cri}} > 0$. It follows from (14) that, for the purpose of studying the bose-fermi duality, [11] we should restrict attention to fermionic theories that obey the inequality

$$\lambda_F m_F > 0. \tag{15}$$

It is easily verified that (14) implies that

$$|c_F| = |c_B|. \tag{16}$$

In other words the bosonic and fermionic fields have equal pole masses under duality. This observation already makes it seem likely that the duality map should involve some sort of identification of elementary bosonic and fermionic quanta. [12] The relationship between bosonic and fermionic S-matrices, proposed in this chapter, helps to flesh this identification out.

1.2.3 Scattering kinematics

In this chapter we study $2 \to 2$ particle scattering; for this purpose we work in Minkowski space. Let the 3 momenta of the initial particles be denoted by p_1 and p_2 and let the momenta of the final particles be denoted by $-p_3$ and $-p_4$. Momentum conservation

[10] A preliminary suggestion for this duality may be found in [2] . The conjecture was first clearly stated, for the massless theories in [9], making heavy use of the results of [4, 5]. The conjecture was generalized to the massive theories in [6] and further generalized in [14]. Additional evidence for this conjecture is presented in [7, 10, 11].

[11] We emphasize that all results obtained directly in the fermionic theory are valid irrespective of whether or not (15) is obeyed. However we do not have a corresponding bosonic results to compare with when this inequality is not obeyed.

[12] Note that this is very different from sine-Gordon-Thirring duality, in which elementary fermionic quanta are identified with solitons in the bosonic theory.

ensures $p_1 + p_2 + p_3 + p_4 = 0$. We use the mostly positive sign convention, and define the Lorentz invariants s, t, u in the usual manner

$$s = -(p_1 + p_2)^2, \quad t = -(p_1 + p_3)^2, \quad u = -(p_1 + p_4)^2, \quad s + t + u = 4c_B^2 \tag{17}$$

where c_B is the pole mass of the scattering particles (the scattering particles have equal mass).

The S matrix for the scattering processes is given by (see below for slight modifications to deal with bosonic or fermionic statistics)

$$\begin{aligned}\mathbf{S}(p_1, p_2, -p_3, -p_4) =& (2E_{\vec{p}_1})(2\pi)^2\delta^2(\vec{p}_1 + \vec{p}_3)(2E_{\vec{p}_2})(2\pi)^2\delta^2(\vec{p}_2 + \vec{p}_4) \\ &+ i(2\pi)^3\delta^3(p_1 + p_2 + p_3 + p_4)T(s, t, u, E(p_1, p_2, p_3)), \\ E_{\vec{p}} =& \sqrt{c_B^2 + \vec{p}^2}, \\ E(p_1, p_2, p_3) =& \pm 1 = \mathrm{sgn}\left(\epsilon_{\mu\nu\rho} p_1^\mu p_2^\nu p_3^\rho\right), \quad \epsilon_{012} = -\epsilon^{012} = 1\end{aligned} \tag{18}$$

The fact that $2 \to 2$ scattering can depend on the Z_2 valued variable $E(p_1, p_2, p_3)$ rather than just s, t, u is a kinematical peculiarity of 3-dimenensions. Note that $E(p_1, p_2, p_3)$ measures the 'handedness' of the triad of vectors p_1, p_2, p_3. The symbol $\vec{p}$ that appears in (18) denotes the spatial part of the 3-vector p. It might seem to be strange that $\vec{p}$ makes any appearance in the formula for a Lorentz covariant S-matrix. Note, however, that the various 3-vectors we deal with are always on-shell, so the knowledge of $\vec{p}$ is sufficient to permit the reconstruction of the full 3-vector p. Using the on-shell condition it is not difficult to verify that $(2E_{\vec{p}})(2\pi)^2\delta^2(\vec{p} + \vec{r})$ is Lorentz invariant, even though this is not completely manifest.

The manifestly Lorentz invariant rule for the multiplication of two S-matrices is

$$\begin{aligned}&[\mathbf{S_1 S_2}](p_1, p_2, -p_3, -p_4) \\ &= \int \frac{d^3r_1(2\pi)\theta(r_1^0)\delta(r_1^2 + c_B^2)}{(2\pi)^3} \frac{d^3r_2(2\pi)\theta(r_2^0)\delta(r_2^2 + c_B^2)}{(2\pi)^3} \\ &\quad \times \mathbf{S_1}(p_1, p_2, -r_1, -r_2)\mathbf{S_2}(r_1, r_2, -p_3, -p_4) \\ &= \int \frac{d^2\vec{r}_1}{2E_{\vec{r}_1}(2\pi)^2} \frac{d^2\vec{r}_2}{2E_{\vec{r}_2}(2\pi)^2} \mathbf{S_1}(p_1, p_2, -r_1, -r_2)\mathbf{S_2}(r_1, r_2, -p_3, -p_4).\end{aligned} \tag{19}$$

The quantity

$$I(p_1, p_2, -p_3, -p_4) = (2E_{\vec{p}_1})(2\pi)^2\delta^2(\vec{p}_1 + \vec{p}_3)(2E_{\vec{p}_2})(2\pi)^2\delta^2(\vec{p}_2 + \vec{p}_4) \tag{20}$$

that appears in the first line of (18) is clearly the identity matrix for this multiplication rule.

The identity matrix may be rewritten in a manifestly Lorentz invariant form (see Appendix 1.9.1)

$$I(p_1, p_2, -p_3, -p_4) = \lim_{\epsilon \to 0} 4\pi\sqrt{s}\delta\left(\sqrt{\frac{4t}{t+u}} - \epsilon\right)(2\pi)^3\delta^3(p_1 + p_2 + p_3 + p_4). \tag{21}$$

It is sometimes convenient to study $2 \to 2$ scattering in the center of mass frame. In this frame the scattering momenta may be taken to be

$$\begin{aligned} &p_1 = (\sqrt{k^2 + c_B^2}, k, 0), \quad p_2 = (\sqrt{k^2 + c_B^2}, -k, 0) \\ &p_3 = (-\sqrt{k^2 + c_B^2}, -k\cos(\theta), -k\sin(\theta)), \quad p_4 = (-\sqrt{k^2 + c_B^2}, k\cos(\theta), k\sin(\theta)). \end{aligned} \tag{22}$$

The kinematical invariants are given by

$$s = 4(c_B^2 + k^2), \quad t = -2k^2\left(1 - \cos(\theta)\right), \quad u = -2k^2\left(1 + \cos(\theta)\right), \tag{23}$$

and the S-matrix takes the form

$$\mathbf{S} = (2\pi)^3\delta(p_1 + p_2 + p_3 + p_4)S(\sqrt{s}, \theta), \tag{24}$$

where θ is the scattering angle - the angle between $-\vec{p}_3$ and $\vec{p}_1$. More precisely, let $\vec{p}_1$ point along the positive x axis so that $\vec{p}_2$ points along the negative x axis. $\theta \in (-\pi, \pi)$ is defined as the rotation in the clockwise direction (here clockwise is defined w.r.t. the orientation of the usual x, y axis system) that is needed to rotate $\vec{p}_1$ into $-\vec{p}_3$. Note that parity transformations, that take θ to $-\theta$, are generically not symmetries of our theory. In the center of mass system $E(p_1, p_2, p_3)$ defined in (18) is given by

$$E(p_1, p_2, p_3) = \text{sgn}(\theta). \tag{25}$$

For later use we note the following center of mass reduction formulae

$$
\begin{aligned}
E(p_1,p_2,p_3)\sqrt{\frac{su}{t}} &\to \sqrt{s}\cot\left(\frac{\theta}{2}\right), \\
E(p_1,p_2,p_3)\sqrt{\frac{st}{u}} &\to \sqrt{s}\tan\left(\frac{\theta}{2}\right), \\
E(p_1,p_2,p_3)\sqrt{\frac{tu}{s}} &\to \frac{2k^2}{\sqrt{s}}\sin(\theta).
\end{aligned} \tag{26}
$$

The rule (19) induces the following multiplication rule for the functions $S(\sqrt{s},\theta)$:

$$
[S_1S_2](\sqrt{s},\theta) = \int \frac{d\alpha}{8\pi\sqrt{s}} S_1(\sqrt{s},\alpha)S_2(\sqrt{s},\theta-\alpha), \tag{27}
$$

The identity matrix for this multiplication rule is clearly given by

$$
S_I(\sqrt{s},\theta) = 8\pi\sqrt{s}\delta(\theta) = \lim_{\epsilon\to 0} 4\pi\sqrt{s}\left[\delta(\theta+\epsilon)+\delta(\theta-\epsilon)\right], \tag{28}
$$

in agreement with (21) recast in center of mass coordinates.

The Hermitian conjugate of an S-matrix functions for $S^\dagger$ are given by

$$
\begin{aligned}
&[\mathbf{S}^\dagger](p_1,p_2,-p_3,-p_4) = \mathbf{S}^*(p_3,p_4,-p_1,-p_2), \\
&[S^\dagger](\sqrt{s},\theta) = S^*(\sqrt{s},-\theta).
\end{aligned} \tag{29}
$$

The S-matrix must be unitary, i.e. must obey the equation $S^\dagger S = 1$. This implies

$$
-i(T-T^\dagger) = T^\dagger T. \tag{30}
$$

Written out as an explicit equation for the T functions this boils down to

$$
\begin{aligned}
&-i\left(T(p_1,p_2,-p_3,-p_4) - T^*(p_3,p_4,-p_1,-p_2)\right)\delta^3(p_1+p_2+p_3+p_4) \\
&= \int \frac{d^3l}{(2\pi)^3}\frac{d^3r}{(2\pi)^3}\Big[\theta(-l_0)\theta(-r_0)\delta^3(p_1+p_2+p_3+p_4)\delta^3(p_1+p_2+l+r) \\
&\qquad \times (2\pi)\delta(r^2+c_B^2)(2\pi)\delta(l^2+c_B^2)T^*(-p_1,-p_2,l,r)T(-p_3,-p_4,l,r)\Big] + \dots
\end{aligned} \tag{31}
$$

where the ... denotes the contribution of intermediate states with more than two particles. We will return to this formula below

1.2.4 Channels of scattering

A theory of a fundamental field has two kinds of elementary quanta: those that transform in the fundamental of $U(N)$ and those that transform in the antifundamental of that gauge group. In this chapter we refer to quanta in the fundamental of $U(N)$ as particles; we refer to quanta in the antifundamental of $U(N)$ as antiparticle. We use the symbol $P_i(p)$ to denote a particle with color index i and three momentum p, while $A^i(p)$ denotes an antiparticle with color index i and three momentum p. We employ this notation for both the bosonic and the fermionic theories described in the previous subsection.

In this chapter we study $2 \to 2$ scattering. There are essentially two distinct $2 \to 2$ scattering process; particle-particle scattering and Particle-antiparticle scattering [13]

Particle - antiparticle scattering The tensor product of a fundamental and an antifundamental consists of the adjoint and the singlet representations. It follows that Particle-antiparticle scattering is characterized by two scattering functions. We adopt the following terminology: we refer to scattering in the singlet channel as scattering in the S-channel. Scattering in the adjoint channel is referred to as scattering in the T-channel.

It follows from $U(N)$ invariance that the S-matrix for the process

$$P_i(p_1) + A^j(p_2) \to P_m(-p_3) + A^n(-p_4) \tag{32}$$

is given by

$$\mathbf{S} = \delta_i^m \delta_n^j I(p_1, p_2, -p_3, -p_4) + iT_{in}^{jm}(p_1, p_2, -p_3, -p_4)(2\pi)^3 \delta^3(p_1 + p_2 + p_3 + p_4). \tag{33}$$

(see the previous subsection for the definition of I). The S-matrix may be decomposed into adjoint and singlet scattering matrices

$$\mathbf{S} = \left(\delta_i^m \delta_n^j - \frac{\delta_i^j \delta_n^m}{N} \right) S_T + \frac{\delta_i^j \delta_n^m}{N} S_S \tag{34}$$

where

$$\begin{aligned} \mathbf{S}_T &= I(p_1, p_2, -p_3, -p_4) + iT_T(p_1, p_2, -p_3, -p_4)(2\pi)^3 \delta^3(p_1 + p_2 + p_3 + p_4) \\ \mathbf{S}_S &= I(p_1, p_2, -p_3, -p_4) + iT_S(p_1, p_2, -p_3, -p_4)(2\pi)^3 \delta^3(p_1 + p_2 + p_3 + p_4) \end{aligned} \tag{35}$$

[13] The case of antiparticle-antiparticle scattering is related to that of particle-particle scattering by CPT, and so needn't be considered separately.

and

$$T_{in}^{jm}(p_1,p_2,-p_3,-p_4) = \left(\delta_i^m \delta_n^j - \frac{\delta_i^j \delta_n^m}{N}\right) T_T(p_1,p_2,-p_3,-p_4) + \frac{\delta_i^j \delta_n^m}{N} T_S(p_1,p_2,-p_3,-p_4) \tag{36}$$

Particle - particle scattering The tensor product of two fundamentals consists of the representation with two boxes in the first row of the Young Tableaux, and another representation with two boxes in the first column of the Young Tableaux. We refer to these two representations as the symmetric U-channel and the antisymmetric U-channel respectively. It follows that particle- particle scattering is characterized by the scattering functions in these two channels.

More quantitatively, the S-matrix for the process

$$P_i(p_1) + P_j(p_2) \to P_m(-p_3) + P_n(-p_4) \tag{37}$$

takes the form

$$\begin{aligned} \mathbf{S} = &\pm \delta_i^m \delta_j^n I(p_1,p_2,p_3,p_4) + \delta_i^n \delta_j^m I(p_1,p_2,p_4,p_3) \\ &+ iT_{ij}^{mn}(p_1,p_2,p_3,p_4)(2\pi)^3 \delta^3(p_1+p_2+p_3+p_4) \end{aligned} \tag{38}$$

where the $\pm$ in the first line is for bosons/fermions. The S-matrix may be decomposed into the symmetric and antisymmetric channels

$$\mathbf{S} = \frac{\delta_i^n \delta_j^m + \delta_i^m \delta_j^n}{2} \mathbf{S}_{U_s} + \frac{\delta_i^n \delta_j^m - \delta_i^m \delta_j^n}{2} \mathbf{S}_{U_a} \tag{39}$$

where

$$\begin{aligned} \mathbf{S}_{U_s} &= \pm I(p_1,p_2,p_3,p_4) + I(p_1,p_2,p_4,p_3) \\ &\quad + iT_{U_s}(p_1,p_2,p_3,p_4)(2\pi)^3 \delta(p_1+p_2+p_3+p_4) \\ \mathbf{S}_{U_a} &= -(\pm) I(p_1,p_2,p_3,p_4) + I(p_1,p_2,p_4,p_3) \\ &\quad + iT_{U_a}(p_1,p_2,p_3,p_4)(2\pi)^3 \delta(p_1+p_2+p_3+p_4). \end{aligned} \tag{40}$$

We will sometimes need to work with the direct and exchange scattering amplitudes ($\mathbf{S}_{U_d}$ and $\mathbf{S}_{U_e}$) by

$$\mathbf{S} = \delta_i^m \delta_j^n \mathbf{S}_{U_d} + \delta_i^n \delta_j^m \mathbf{S}_{U_e} \tag{41}$$

where

$$
\begin{aligned}
\mathbf{S}_{U_d} &= \pm I(p_1,p_2,p_3,p_4) + iT_{U_d}(p_1,p_2,p_3,p_4)(2\pi)^3\delta(p_1+p_2+p_3+p_4) \\
\mathbf{S}_{U_e} &= I(p_1,p_2,p_4,p_3) + iT_{U_a}(p_1,p_2,p_3,p_4)(2\pi)^3\delta(p_1+p_2+p_3+p_4)
\end{aligned}
\tag{42}
$$

where

$$
\mathbf{S}_{U_s} = \mathbf{S}_{U_d} + \mathbf{S}_{U_e}, \quad \mathbf{S}_{U_a} = \mathbf{S}_{U_e} - \mathbf{S}_{U_d}, \quad T_{U_s} = T_{U_d} + T_{U_e}, \quad T_{U_a} = T_{U_e} - T_{U_d}. \tag{43}
$$

And

$$
T_{ij}^{mn}(p_1,p_2,p_3,p_4) = \delta_i^m \delta_j^n T_{U_d}(p_1,p_2,p_3,p_4) + \delta_i^n \delta_j^m T_{U_e}(p_1,p_2,p_3,p_4). \tag{44}
$$

We refer to S_{U_d} as the 'direct S-matrix' in the U-channel. S_{U_e}, on the other hand is the 'exchange S-matrix in the U-channel.

In this chapter we study scattering in both the bosonic as well as fermionic theories described in the previous subsection. We use the superscript B/F to denote the corresponding functions in the bosonic/fermionic theories. For example S_T^B is the T-channel scattering matrix for bosons, while S_S^F denotes the S-channel scattering matrix for fermions.

1.2.5 Tree level scattering amplitudes in the bosonic and fermionic theories

The evaluation of full S-matrix of the bosonic and fermionic theories of subsection 1.2.1 is the main subject of this chapter. The evaluation of the all loop amplitudes will require summing all planar diagrams in lightcone gauge, together with some educated guesswork. However the tree level scattering amplitudes in these theories are, of course, easily evaluaed in a covariante Landau gauge. In this section we simply present the results for these tree level scattering amplitudes, in all scattering channels, in both the bosonic and the fermionic theories. In every case we present the results for the full S matrix (rather than the T matrix) to emphasize the relative sign between the identity piece and the scattering terms. In the scalar theories we work for simplicity at $b_4 = 0$. Our results in the fermionic theory are presented upto a physically irrelevant overall phase. The results presented in this subsection are all derived in Appendix 1.9.2.

At tree level we find

$$
\begin{aligned}
\mathbf{S}_{B,U_d} &= I(p_1,p_2,p_3,p_4) - \frac{4\pi}{k_B}\frac{\epsilon_{\mu\nu\rho}p_1^\mu p_2^\nu p_3^\rho}{(p_2+p_3)^2}(2\pi)^3\delta(p_1+p_2+p_3+p_4) \\
\mathbf{S}_{B,U_e} &= I(p_1,p_2,p_4,p_3) + \frac{4\pi}{k_B}\frac{\epsilon_{\mu\nu\rho}p_1^\mu p_2^\nu p_3^\rho}{(p_2+p_4)^2}(2\pi)^3\delta(p_1+p_2+p_3+p_4) \\
\mathbf{S}_{B,T} &= I(p_1,p_2,p_3,p_4) + \frac{4\pi}{k_B}\frac{\epsilon_{\mu\nu\rho}p_1^\mu p_2^\nu p_3^\rho}{(p_4+p_3)^2}(2\pi)^3\delta(p_1+p_2+p_3+p_4) \\
\mathbf{S}_{B,S} &= I(p_1,p_2,p_3,p_4) - 4\pi\lambda_B\frac{\epsilon_{\mu\nu\rho}p_1^\mu p_2^\nu p_3^\rho}{(p_2+p_4)^2}(2\pi)^3\delta(p_1+p_2+p_3+p_4) \\
\mathbf{S}_{F,U_d} &= I(p_1,p_2,p_3,p_4) + \frac{4\pi}{k_F}\left(\frac{\epsilon_{\mu\nu\rho}p_1^\mu p_2^\nu p_3^\rho}{(p_2+p_3)^2} - 2im_F\right)(2\pi)^3\delta(p_1+p_2+p_3+p_4) \\
\mathbf{S}_{F,U_e} &= I(p_1,p_2,p_4,p_3) - \frac{4\pi}{k_F}\left(\frac{\epsilon_{\mu\nu\rho}p_1^\mu p_2^\nu p_3^\rho}{(p_2+p_4)^2} + 2im_F\right)(2\pi)^3\delta(p_1+p_2+p_3+p_4) \\
\mathbf{S}_{F,T} &= I(p_1,p_2,p_3,p_4) - \frac{4\pi}{k_F}\left(\frac{\epsilon_{\mu\nu\rho}p_1^\mu p_2^\nu p_3^\rho}{(p_4+p_3)^2} + 2im_F\right)(2\pi)^3\delta(p_1+p_2+p_3+p_4) \\
\mathbf{S}_{F,S} &= -I(p_1,p_2,p_3,p_4) + 4\pi\lambda_F\left(\frac{\epsilon_{\mu\nu\rho}p_1^\mu p_2^\nu p_3^\rho}{(p_2+p_4)^2} - 2im_F\right)(2\pi)^3\delta(p_1+p_2+p_3+p_4).
\end{aligned} \tag{45}
$$

1.2.6 The non-relativistic limit and Aharonov-Bohm scattering

As we have explained above, in this chapter we wish to compute the $2 \to 2$ scattering matrix of fundamental matter coupled to Chern-Simons theory. The result of this computation is already well known in the non-relativistic limit, i.e. the limit in which

$$\frac{s-4c_B^2}{4c_B^2} \to 0. \tag{46}$$

[14] In this limit the S-matrix is obtained from the scattering of two non-relativistic particles interacting with a Chern-Simons gauge field. The quantum description of this system may be obtained by first eliminating the non dynamical gauge field in a suitable gauge and then writing down the effective two particle Schrodinger equation see e.g. [15]). Moving to center of mass and relative coordinates further simplifies the problem to the study of the quantum mechanics of a single particle interacting with a point like flux tube located at the origin. The S-matrix may then be read off from the scattering solution of Aharonov-Bohm [19] with one interesting twist; the effective value of the flux depends on the scattering channel.

[14] In the limit (46) $t/4m^2$ and $u/4m^2$ also tend to zero, as is most easily seen in the center of mass frame.

Let the scattering particles transform in the representations R_1 and R_2 of $U(N)$. As we have reviewed in the introduction, if

$$R_1 \times R_2 = \sum_m R_m \tag{47}$$

then

$$S = \sum_m S_m P_m$$

where P_m is the projector onto the representation R_m. It turns out that the scattering matrix in the m^{th} channel S_m is simply the Aharonov-Bohm scattering amplitude of a unit charge $U(1)$ particle scattering off a thin flux tube with integrated flux $2\pi\nu_m$ where

$$\nu_m = \frac{C_2(R_m) - C_2(R_1) - C_2(R_2)}{k}. \tag{48}$$

Let F denote the fundamental representation, A the antifundamental representation, S the 'symmetric' representation (with two boxes in the first row of the Young Tableaux, and no boxes in any other row), AS the antisymmetric representation (with two boxes in the first column of the Young Tableaux, and no boxes in any other column), Adj the adjoint representation and I the and the singlet. The Casimirs of these representations are

$$\begin{aligned} &C_2(F) = C_2(A) = \frac{N^2-1}{2N}, \quad C_2(S) = \frac{N^2+N-2}{N}, \quad C_2(AS) = \frac{N^2-N-2}{N} \\ &C_2(Adj) = N, \quad C_2(I) = 0. \end{aligned} \tag{49}$$

In the symmetric and antisymmetric exchange channels respectively (for particle-particle scattering)

$$\nu_S = \frac{1}{k} - \frac{1}{Nk}, \quad \nu_{AS} = -\frac{1}{k} - \frac{1}{Nk}. \tag{50}$$

In the singlet and adjoint exchange channels respectively (for particle - antiparticle scattering)

$$\nu_I = -\lambda_B + \frac{1}{Nk}, \quad \nu_{AdJ} = \frac{1}{Nk}. \tag{51}$$

Note that in the large N limit, ν_I is of order unity, ν_S and ν_{AS} are both of order $\mathcal{O}(1/N)$ and ν_{Adj} is of order $\mathcal{O}(1/N^2)$.

In the rest of this subsection we specialize to scattering in the scalar theory. As we have reviewed in great detail in Appendix 1.9.3, the quantum mechanics of a non-relativistic scalar scattering of a point like flux tube with integrated flux $2\pi\nu$ admits a 'scattering' solution (the Aharonov Bohm solution), whose large radius asymptotics is given by

$$\psi(r) = e^{ikx} + e^{-i\frac{\pi}{4}} h(\theta) e^{ikr} \sqrt{\frac{2\pi}{kr}} \tag{52}$$

where

$$h(\theta) = 2\pi \left(\cos(\pi\nu) - 1\right) \delta(\theta) + \sin(\pi\nu) \left(\mathrm{Pv} \cot\left(\frac{\theta}{2}\right) - i\mathrm{sgn}(\nu) \right) \tag{53}$$

where Pv denotes the principal value. In the non-relativistic limit and in the center of mass frame the scattering amplitude T is proportional to $h(\theta)$; more precisely

$$T(s,\theta) = -4ih(\theta)\sqrt{s}. \tag{54}$$

Using (23) (54) and (53) together imply the covariant prediction

$$\begin{aligned} T_m^{NR}(p_1,p_2,p_3,p_4,\lambda_B,b_4) = &-4i\sqrt{s}\sin(\pi\nu_m)\left(E(p_1,p_2,p_3)\sqrt{\frac{t}{u}} - i\mathrm{sgn}(\nu_m) \right) \\ &- i(\cos(\pi\nu_m)-1) I(p_1,p_2,p_3,p_4) \end{aligned} \tag{55}$$

(see (20) (21), (28) for a definition of I) where T_m^{NR} is the non-relativistic limit of scattering in the m^{th} channel, ν_m is the corresponding value of ν as described above.

(55) applies when the scattering particles are distinguishable (as in the case of particle - antiparticle scattering in the situation of interest here). When the scattering particles are identical - as in the case of particle - particle scattering here, $R_1 = R_2 = R$ and we have to add the contribution of exchange scattering. (55) is modified to

$$\begin{aligned} T_m^{NR}(p_1,p_2,p_3,p_4,\lambda_B,b_4) = &-4i\sqrt{s}\sin(\pi\nu_m)\left(E(p_1,p_2,p_3)\sqrt{\frac{t}{u}} - i\mathrm{sgn}(\nu_m) \right) \\ &- i(\cos(\pi\nu_m)-1) I(p_1,p_2,p_3,p_4) \\ &+ a\Bigg[-4i\sqrt{s}\sin(\pi\nu_m)\left(-E(p_1,p_2,p_3)\sqrt{\frac{u}{t}} - i\mathrm{sgn}(\nu_m) \right) \\ &\quad - i(\cos(\pi\nu_m)-1) I(p_2,p_1,p_3,p_4) \Bigg] \end{aligned} \tag{56}$$

where the sign $a = 1$ if the R_3 is symmetric in the Rs while $a = -1$ if R_3 is antisymmetric product of 2 Rs (in the case that the scattering particles are fermionic, a has an additional overall -1). In writing (56) we have used the fact that $E(p_2,p_1,p_3) = -E(p_1,p_2,p_3)$.

Non-relativistic limit of S-channel scattering In the S-channel $\nu_m = \lambda_B$ in the large N limit so the S-channel S-matrix must reduce, in the limit (46), to

$$\begin{aligned}(T_B^S)^{NR}(p_1,p_2,p_3,p_4,\lambda_B,b_4) =&4i\sqrt{s}\sin(\pi\lambda_B)\left(E(p_1,p_2,p_3)\sqrt{\frac{t}{u}}+i\mathrm{sgn}(\lambda_B)\right)\\ &-i(\cos(\pi\lambda_B)-1)I(p_1,p_2,p_3,p_4).\end{aligned} \tag{57}$$

This prediction for the non-relativistic limit of the S-matrix in the S-channel has several striking features.

- T_S^B is not an analytic function of kinematic variables. The term proportional to the δ function in that expression is singular, and is infact proportional to the identity scattering matrix (see subsection 1.2.3).
- T_S^B is not an analytic function of λ_B at $\lambda_B = 0$ (because of the term proportional to $\mathrm{sgn}(\lambda_B)$.)
- T_S^B is universal, in the sense that it is independent of b_4 in this limit.

As we will see below, the last two features are artifacts of the non-relativistic limit. On the other hand we will now argue that the last the term in (53) $\propto \delta(\theta)$ is an exact feature of the S-matrix at all energy scales.

The term proportional to $\delta(\theta)$ in (53) was infact missed in the original analysis by Aharonov and Bohm. The presence of this term was discovered much later by Ruijsenaars [17] (see also the later papers [15, 20–22] for further elaboration) where it was also pointed out that this contact term is necessary to unitarize Aharonov-Bohm scattering (see the next subsection for a review of this fact). In the rest of this subsection we will present a simple physical interpretation for this part of the Aharonov-Bohm S-matrix.

As we have reviewed extensively in (1.2.3), the scattering matrix is postulated the form $S = I + iT$ where the factor I accounts for the unscattered part of the wave packet. In the context of Aharonov-Bohm scattering, however, half of this unscattered wave packet passes above the scatterer and so picks up the phase $e^{i\pi\nu_m}$ while the other half passes below and so picks up the phase $e^{-i\pi\nu_m}$. The symmetry between up and down ensures that the part of the unscattered part of the S-matrix is modulated by a factor $\cos(\pi\nu_m)$ as it passes by the scatterer. In the current context, consequently, we should expect

$$S = \cos(\pi\nu_m) + iT'$$

where T' is an analytic function of momentum. If we insist nonetheless on using the usual split $S = I + iT$ then we will find

$$T = -i(\cos(\pi\nu_m) - 1)I + T'$$

(where T' is an analytic function of the scattering angle) in perfect agreement with (55). As our physical explanation of the last term on the RHS of (55) makes no reference to the non-relativistic limit, we expect this term to be an exact feature of the S-matrix in every channel, even away from the non-relativistic limit.

All our comments about the term proportional to I in the S-matrix hold also for the T and the U-channels; the last term in (55) is expected to be exact in these channels as well. As we have noted above, however, in these channels $\nu_m \leq \mathcal{O}(\frac{1}{N})$ so that $\cos(\pi\nu_m) - 1 \leq \mathcal{O}(\frac{1}{N^2})$. It follows that the $\mathcal{O}(\frac{1}{N})$ computations of these scattering matrices presented in this chapter will be insensitive to these terms.

Non-relativistic limit of scattering in the other channels As we have seen above, ν_m is of order $\frac{1}{N}$ or smaller in the other three scattering channels. All the calculations in this chapter are done to leading order in the $\frac{1}{N}$, and so capture the first term in the Taylor expansion in ν_m of the scattering amplitude. In this subsubsection we merely emphasize the simple but confusing fact that the non-relativistic limit of this term need not agree with the first term in the Taylor expansion of the non-relativistic limit (55) (this is an order of limits issue).

Let us consider a simple example for how this might work. Define $y = \frac{\nu_m(4m^2)}{s-4m^2}$, and consider the function

$$f = \frac{e^y - e^{-y}}{e^y + e^{-y}}.$$

Taylor expanding this function to first order in ν_m, we find

$$f = 2y + \mathcal{O}(\nu_m^2).$$

The non-relativistic limit the first term in this expansion diverges like y. On the other hand if we first take the non-relativistic limit (46)

$$f = \mathrm{sgn}(\nu_m).$$

Conservatively, therefore, we should conclude that the results of this subsection make no sharp prediction for the non-relativistic limit of the scattering amplitudes in the U and T-channels. This is certainly the case for the term independent of θ in (53); as in the toy example above, this term is non-analytic in ν_m, and so cannot be Taylor expanded in ν_m, and so makes no prediction for the non-relativistic limit of the Taylor expansion.

On the other hand the term in (53) proportional to $\cot\left(\frac{\theta}{2}\right)$ and $\delta(\theta)$ are both analytic in θ, and one might optimistically hope that the Taylor expansion of these terms in ν_m will accurately capture the non-relativistic limits of the scattering amplitudes in the U and the T-channels. Below we will see that this is indeed the case, though it works in a rather trivial way.

1.2.7 Constraints from unitarity

As we have already remarked above, the S matrix in any quantum theory obeys the equation $S^\dagger S = 1$. In subsection 1.2.3 we expanded this equation out in terms of the T-matrix to obtain (31).

In a general quantum field theory (31) does not constitute a closed equation for $2 \to 2$ scattering because of the terms indicated with the $\ldots$ - the contributions from $2 \times n$ scattering - in the RHS of (31). It is easily verified, however, that at leading order in the large N limit in the theories under consideration the contribution of $2 \to n$ processes to the RHS of (30) is suppressed, compared to the LHS, by a factor of $\frac{1}{N^{\frac{n-2}{2}}}$. In the large N limit of interest to this chapter, it follows that we can drop the $\ldots$ on the RHS of (31), which then turns into a powerful nonlinear closed constraint on $2 \to 2$ scattering matrix elements.

Constraints from unitarity in the various channels Let us work out the specific form of this constraint in the special case of particle - antiparticle scattering. Using (35), we find

$$
\begin{aligned}
&- i\Bigg[(T_T(p_1,p_2,-p_3,-p_4) - T_T^*(p_3,p_4,p_1,p_2)) \\
&\qquad \times (2\pi)^3 \left(\delta_{im}\delta_{jn} - \frac{1}{N}\delta_{ij}\delta_{mn}\right)(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\Bigg] \\
&- i\left[(T_S(p_1,p_2,-p_3,-p_4) - T_S^*(p_3,p_4,-p_1,-p_2))\,\delta^3(p_1+p_2-p_3-p_4)\frac{1}{N}\delta_{ij}\delta_{mn}\right] \\
&= \int \frac{d^3l}{(2\pi)^3}\frac{d^3r}{(2\pi)^3}\Bigg[(2\pi)^2\theta(l_0)\theta(r_0)\delta(r^2+c_B^2)\delta(l^2+c_B^2) \\
&\qquad\qquad \times (2\pi)^6\delta^3(p_1+p_2-p_3-p_4)\delta^3(p_1+p_2-l-r) \\
&\qquad\qquad \times \Bigg(\left(\delta_{im}\delta_{jn} - \frac{1}{N}\delta_{ij}\delta_{mn}\right) T_T(p_1,p_2,-l,-r)T_T^*(p_3,p_4,-l-,r) \\
&\qquad\qquad\qquad + \frac{1}{N}T_S(p_1,p_2,-l,-r)T_S^*(p_3,p_4,-l,-r)\delta_{ij}\delta_{mn}\Bigg)\Bigg].
\end{aligned}
\tag{58}
$$

Equating the coefficients of the different index structures on the LHS and RHS we conclude that

$$
\begin{aligned}
&- i\,(T_T(p_1,p_2,-p_3,-p_4) - T_T^*(p_3,p_4,-p_1,-p_2))\,\delta^3(p_1+p_2-p_3-p_4) \\
&= \int \frac{d^3l}{(2\pi)^3}\frac{d^3r}{(2\pi)^3}\Bigg[(2\pi i)^2\theta(l_0)\theta(r_0)\delta(r^2+c_B^2)\delta(l^2+c_B^2) \\
&\qquad\qquad \times \delta^3(p_1+p_2-p_3-p_4)(2\pi)^3\delta^3(p_1+p_2-l-r) \\
&\qquad\qquad \times T_T(p_1,p_2,-l,-r)T_T^*(p_3,p_4,-l,-r)\Bigg],
\end{aligned}
\tag{59}
$$

and that

$$
\begin{aligned}
&- i\,(T_S(p_1,p_2,-p_3,-p_4) - T_S^*(p_3,p_4,-p_1,-p_2))\,\delta^3(p_1+p_2-p_3-p_4) \\
&= \int \frac{d^3l}{(2\pi)^3}\frac{d^3r}{(2\pi)^3}\Bigg[(2\pi i)^2\theta(l_0)\theta(r_0)\delta(r^2+c_B^2)\delta(l^2+c_B^2) \\
&\qquad\qquad \times \delta^3(p_1+p_2-p_3-p_4)(2\pi)^3\delta^3(p_1+p_2-l-r) \\
&\qquad\qquad \times T_S(p_1,p_2,-l,-r)T_S^*(p_3,p_4,-l,-r)\Bigg].
\end{aligned}
\tag{60}
$$

Now recall that the scattering matrix T_T is $\mathcal{O}(\frac{1}{N})$. It follows that the RHS of (59) is subleading in $\frac{1}{N}$ compared to the LHS. In the large N limit, consequently, (59) may be rewritten as

$$(T_T(p_1,p_2,-p_3,-p_4) - T_T^*(p_3,p_4,-p_1,-p_2)) = 0. \tag{61}$$

Applying the same reasoning to particle-particle scattering, we reach the identical conclusion for U-channel scattering. It is easily verified that the slightly trivial, linear equations (61) (and the analogous equation for U-channel scattering) are infact obeyed by the exact solutions for T_T and T_U presented below[15]

On the other hand the S-channel scattering matrix T_S^B is $\mathcal{O}(1)$ in the large N limit. Consequently, the nonlinear equation (60) is a rather nontrivial constraint on S-channel scattering.

S-channel unitarity constraints in the center of mass frame The constraint on the S-channel S-matrix is most conveniently worked out in the center of mass frame. We choose the scattering momenta to take the form

$$\begin{aligned}
p_1 &= \left(\sqrt{p^2+c_B^2}, p, 0\right), \quad p_2 = \left(\sqrt{p^2+c_B^2}, -p, 0\right), \\
p_3 &= \left(-\sqrt{p^2+c_B^2}, -p\cos(\alpha), -p\sin(\alpha)\right), \quad p_4 = \left(-\sqrt{p^2+c_B^2}, p\cos(\alpha), p\sin(\alpha)\right),
\end{aligned} \tag{62}$$

In this frame $T_S = T_S(p,\alpha)$ or $T = T(s,\alpha)$ (recall $s = 4(p^2+c_B^2)$) and the constraint from unitarity is simply a constraint on this function of two variables.

In order to work out the precise form of this constraint we first process the delta functions inside the integrals.

$$\begin{aligned}
&\int \frac{d^3l}{(2\pi)^3}\frac{d^3r}{(2\pi)^3}(2\pi)^2\theta(l_0)\theta(r_0)\delta(r^2+c_B^2)\delta(l^2+c_B^2)(2\pi)^3\delta^3(p_1+p_2-l-r) \\
=&\int \frac{d^3l}{(2\pi)^3}(2\pi)^2\theta(l_0)\theta(-l_0+(p_1)_0+(p_2)_0)\delta(l^2+c_B^2)\delta((p_1+p_2)^2-2(p_1+p_2)\cdot l) \\
=&\frac{1}{8\pi\sqrt{s}}\int d\theta dl_0 d\ell_s \delta(l_0-\sqrt{p^2+c_B^2})\delta(\ell_s-p^2) \\
=&\frac{1}{8\pi\sqrt{s}}\int d\theta
\end{aligned} \tag{63}$$

where $E_p = \sqrt{p^2+m^2} = \frac{\sqrt{s}}{2}$ and $\ell_s = l^2 + l_0^2$. It follows that the unitarity constraint is

[15]This is related to the fact that these scattering amplitudes have no branch cuts in the physical domain for T and U-channel scattering.

given by

$$-i\left(T_S(s,\alpha)-T_S^*(s,-\alpha)\right)=\frac{1}{8\pi\sqrt{s}}\int d\theta T_S(s,\theta)T_S^*(s,-(\alpha-\theta)) \tag{64}$$

(this is essentially identical to the manipulation that produced the product rule (27)).

Unitarity of the non-relativistic limit As an example for how this works, we will now demonstrate that the non-relativistic limit of the S-channel S-matrix, (55), obeys the constraints of unitarity. In the center of mass frame (55) takes the form

$$T_S(\sqrt{s},\alpha)=H(\sqrt{s})T(\alpha)+W_1(\sqrt{s})-iW_2(\sqrt{s})\delta(\alpha), \tag{65}$$

where

$$T(\alpha)=i\cot\left(\frac{\alpha}{2}\right),$$

and

$$\begin{aligned} H(\sqrt{s}) &= 4\sqrt{s}\sin(\pi\lambda_B), \\ W_1(\sqrt{s}) &= -4\sqrt{s}\sin(\pi\lambda_B)\mathrm{sgn}(\lambda_B), \\ W_2(\sqrt{s}) &= 8\pi\sqrt{s}\left(\cos(\pi\lambda_B)-1\right). \end{aligned} \tag{66}$$

With an eye to application later in the chapter, we will first work out the unitarity constraint for arbitrary $H(\sqrt{s})$, $W_1(\sqrt{s})$ and $W_2(\sqrt{s})$, specializing to the specific forms (66) only at the end.

Using the formula

$$\int d\theta \mathrm{Pv}\cot\left(\frac{\theta}{2}\right)\mathrm{Pv}\cot\left(\frac{\alpha-\theta}{2}\right)=2\pi-4\pi^2\delta(\alpha), \tag{67}$$

(see footnote[16] for a check of (67)), (64) reduces to

$$
\begin{aligned}
H - H^* &= \frac{1}{8\pi\sqrt{s}}\left(W_2 H^* - H W_2^*\right), \\
W_2 + W_2^* &= -\frac{1}{8\pi\sqrt{s}}\left(W_2 W_2^* + 4\pi^2 H H^*\right), \\
W_1 - W_1^* &= \frac{1}{8\pi\sqrt{s}}\left(W_2 W_1^* - W_2^* W_1\right) - \frac{i}{4\sqrt{s}}\left(H H^* - W_1 W_1^*\right).
\end{aligned}
\tag{70}
$$

It is easily verified that the specific assignments (66) obey the equation (70). The first equation in (70) is obeyed because H and W_2, in (70), are both real. The third equation in (70) is obeyed because W_1 is also real and $|H|^2 = |W_1|^2$. The second equation in (66) reduces to the true trigonometric identity

$$
2\left(1 - \cos(\pi\lambda_B)\right) = \left(1 - \cos(\pi\lambda_B)\right)^2 + \sin^2(\pi\lambda_B).
$$

We conclude that the Aharonov-Bohm scattering amplitude obeys the equations of unitarity, though in a slightly trivial fashion as the coefficient of $\delta(\theta)$ was the only part of the S-matrix that had an imaginary piece.

Unitarity constraints on general S-matrices of the form (65) As we have seen in the last subsubsection, the functions $\mathrm{Pv}\cot\left(\frac{\theta}{2}\right)$, 1 and $\delta(\theta)$ form a closed algebra under convolution (i.e the convolution of any two linear combinations of these functions is, once again, a linear combination of the same three functions). This nontrivial fact allowed us

[16] We can check the (67) by calculating the Fourier coefficients,

$$
\begin{aligned}
&\int \frac{d\alpha}{2\pi} e^{-in\alpha} \int d\theta \, \mathrm{Pv}\cot\left(\frac{\theta}{2}\right) \mathrm{Pv}\cot\left(\frac{\alpha-\theta}{2}\right) \\
&= \oint \frac{d\omega}{2\pi\omega}\omega^{-n} \oint \frac{dz}{z} \mathrm{Pv}\left(\frac{z+1}{z-1}\right) \mathrm{Pv}\left(\frac{z+\omega}{\omega-z}\right) \\
&= \begin{cases} -i \oint dz \, \mathrm{Pv}\left(\frac{z+1}{z-1}\right) z^{-n-1} = -2\pi & (n>0) \\ 0 & (n=0) \\ i \oint dz \, \mathrm{Pv}\left(\frac{z+1}{z-1}\right) z^{-n-1} = -2\pi & (n<0) \end{cases}
\end{aligned}
\tag{68}
$$

where $z = e^{i\theta}$ and $\omega = e^{i\alpha}$. By comparing (68) with Fourier coefficients of delta function,

$$
\delta(\alpha) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in\alpha},
\tag{69}
$$

we can immediately check (67).

in the last subsection to find a simple solution of the unitarity equation of the form (65) (this was simply the Aharonov-Bohm solution).

Given the closure of (65) under convolution, it is tempting to conjecture that the scattering matrix in the S-channel takes the form (65) even outside the non-relativistic limit (we will find independent evidence below that this is indeed the case). With this conjecture in mind, in this subsection we will inquire to what extent the requirement of unitarity (70) determines S-matrices of the form (65).

Let us first do some counting. The data in S-matrices of the form (65) is three complex or six real functions of s and λ_B. Unitarity provides 3 real equations. It follows that if we impose no more than the condition of unitarity, the general S-matrix is given in terms of three unknown real functions.

In order to make further progress we need more information. In the previous subsection we have already argued that, on physical grounds, we expect the form of W_2 in (66) to be exact even away from the non-relativistic limit. If we make this assumption, unitarity gives us 3 real equations for the remaining 4 unknown functions, and so the S-matrix is determined in terms of one unknown function. Let us see how this works in more detail. The first equation in (70) forces the function H to be real. The second equation in (70) then forces H to be given exactly by the expression in (66). We are left with a single unknown complex function W_1 subject to a single real equation; the third of (70).

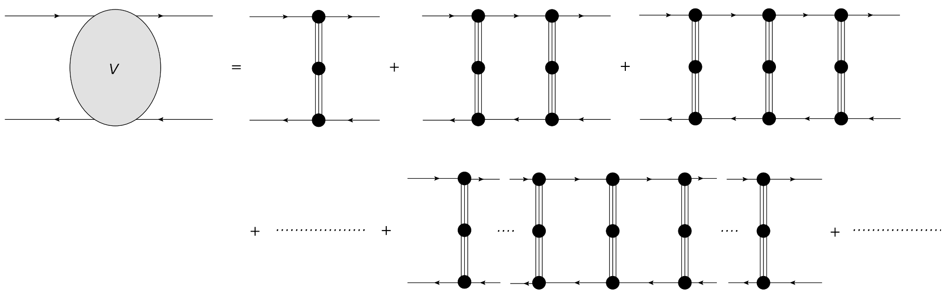
Let us summarize. If we assume that the S-matrix takes the form (65) and further assume that the expression for W_2 in (66) is exact, then unitarity also forces the expression for H in (66) to be exact, and constrains W_1 to obey the third of (66), which is one real equation for the unknown complex function W_1.

1.3 Summary: method, results and conjectures

In this section we summarize the method we use to compute S-matrices and list our principal results and conjectures.

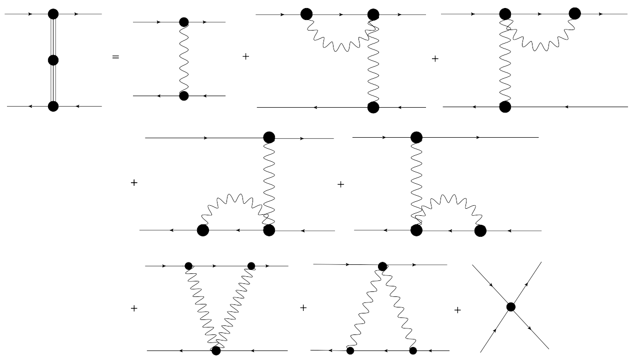
1.3.1 Method

In this chapter we compute the functions T_T, T_{U_d} and T_{U_e} for both the bosonic and the fermionic theories. We also present a conjecture for the functions S_S. We then study the transformation of our results under Bose-Fermi duality. The method we employ to compute the S-matrices is completely straightforward; we sum all the off shell planar graphs with



(a)

Figure 3: This diagram would contain a diagrammatic representation of the exact amplitude V as a sum over ladders, where the 'rungs' in the ladder are the triple line propagators.



(a)

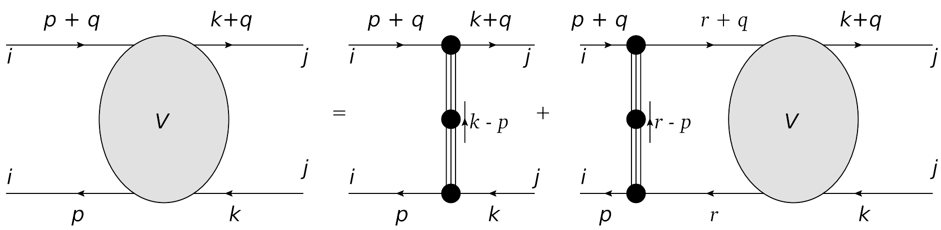
Figure 4: A diagrammatic representation of the effective single particle exchange four point amplitude for bosons. This amplitude is give by the sum of the tree level exchange of a gluon, dressed tree level exchanges of the gluon and the point interaction controlled by the parameter b_4

four external legs, and then obtain the S-matrices by taking the appropriate on shell limits.

Following [2] and several subsequent papers, we work in the lightcone gauge $A_- = 0$. [17] The off shell four point amplitude receives contributions from an infinite number of Feynman graphs. The graphs that contribute may be enumerated very simply; they are simply the sum of all ladder graphs Fig 3, where the triple line is the effective exchange interaction between fundamental particles. In the case of the bosonic theory, for instance, the triple line is given diagrammatically by Fig. 4. It is easy to convince oneself that the all orders amplitude depicted in Fig. 3 obeys the integral equation depicted in Fig 5 [2, 6].

According to the labeling of momenta in Fig. 5, q^μ is the three momentum that flows, from left to right in graphs of Fig. 3. q^μ is a 'constant of motion' in the sense that if a

[17] Our notation is as follows. x^+, x^- and x^3 are a set of coordinates on Minkowski space. x^+ and x^- are lightcone coordinates while x^3 is a spatial coordinate.



(a)

Figure 5: A diagrammatic depiction of the integral equation obeyed by offshell four point scattering amplitudes. The blob here represents the all orders scattering amplitude while the triple line represents the effective single particle exchange four point interaction between quanta. Here, and in every Feynman diagram in this chapter, all momenta flow in the direction of the arrows of the propagators.

given ladder diagram has a particular value of q^μ then every sub ladder within the original ladder also has the same value of q^μ (this is not true of the momenta p and k in Fig. 3). This implies that different values of q^μ do not 'mix' in the integral equation of Fig. 5. In other words Fig. 5 represents an infinite set of decoupled integral equations; one for every value of q^μ. It was pointed out in [6] that the integral equations in Fig. 5 simplifies dramatically when $q^\pm = 0$. The authors of [6] infact solved the relevant integral equations for the bosonic theory in massless limit. In this chapter to find exact formulae for the sum over planar graphs with four external lines with $q^\pm = 0$ by explicitly solving the integral equations relevant to that case. In the case of the bosonic theory our results are a generalization of those of [6] to nonzero mass[18] The integral equation turns out to be more complicated to solve in the case of the fermionic theory, but we are able to find the exact solution in this case as well.

With exact off shell results in hand, we proceed to evaluate the S-matrices for our problem by taking the appropriate on shell limits. The on shell condition determines the energy of each of the participating particles (in terms of their momenta) upto a sign. Energy and momentum conservation require that two of the external lines have positive energy while the other two have negative energy, leaving a total of six distinct cases. [19] Recalling

[18] [6] performed this summation in order to evaluate three point functions of gauge invariant operators in special kinematical configurations.

[19] We say an external line has positive energy if p_0 is positive (or p^0 is negative) going into the graph. An external line with an ingoing arrow and positive energy represents an initial particle. An external line with an outgoing arrow and positive energy into the graph (or negative energy in the direction of the arrow) is an ingoing antiparticle. An external line with an arrow going into the graph and negative energy going into the graph is an outgoing antiparticle. An external line whose arrow points out of the graph and whose energy is negative going into the graph (or positive in the direction of the arrow) is an outgoing particle.

that external lines with positive energy represent initial states while external lines with negative energy represent final states, it is not difficult to convince oneself that one of these six cases determines the function T_S, another determines T_T, two others determine T_{U_d}, T_{U_e} respectively, while the last two processes compute the CPT conjugates of scattering in the U-channel. In other words the four different scattering functions introduced, in the previous subsection, are all different limits of the single four point amplitudes determined by the integral equation of Fig. 5.

As we have emphasized above, we have been able to evaluate the off shell four point amplitude only in the special case $q^\pm = 0$. This technical limitation has different implications for our ability to compute the S matrices in the different channels.

q^μ turns out to be the center of mass 3 momentum for S-channel scattering. The condition $q^\pm = 0$ ensures that the center of mass energy is spacelike; this is impossible for an onshell scattering process. It follows that the technical limitations which restricted us to $q^\pm = 0$ forbid us from directly computing S-channel scattering, a fact that will force us to resort to conjecture in this channel.

In the T and U-channels, on the other hand, q represents the 3 momentum transfer between an initial and final particle. As all participating particles have the same mass, the 3 momentum transfer is always spacelike (this is most easily seen in the center of mass frame), there is no barrier to setting $q^\pm = 0$ in these processes. For an arbitrary T or U-channel process, it is always possible to find an inertial frame in which $q^\pm = 0$. In these channels, in other words, the restriction to $q^\pm = 0$ is simply a choice of frame. Assuming that the S-matrix for our process is Lorentz invariant, the on shell limits of our off shell four point amplitude completely fix the S-matrix in these channels. We are thus able to report definite results for the scattering matrices in these channels.

1.3.2 Results in the U and T channels

In this subsection we simply present our final results for U and T-channel scattering, separately for the bosonic and the fermionic theories. We first report our results for the bosonic theory. In the T-channel (adjoint exchange) we find

$$
\begin{aligned}
&T_T^B(p_1,p_2,p_3,p_4,k_B,\lambda_B,\tilde{b}_4,c_B)\\
=&E(p_1,p_2,p_3)\frac{4i\pi}{k_B}\sqrt{\frac{u\ t}{s}}\\
&-\frac{4\ i\pi}{k_B}\sqrt{-t}\ \ \frac{(\tilde{b}_4-4\pi i\lambda_B\sqrt{-t})e^{i\pi\lambda_B}+(\tilde{b}_4+4\pi i\lambda_B\sqrt{-t})e^{2i\lambda_B\tan^{-1}\left(\frac{2|c_B|}{\sqrt{-t}}\right)}}{-(\tilde{b}_4-4\pi i\lambda_B\sqrt{-t})e^{i\pi\lambda_B}+(\tilde{b}_4+4\pi i\lambda_B\sqrt{-t})e^{2i\lambda_B\tan^{-1}\left(\frac{2|c_B|}{\sqrt{-t}}\right)}}\\
=&E(p_1,p_2,p_3)\frac{4i\pi}{k_B}\sqrt{\frac{u\ t}{s}}\\
&-\frac{4\ i\pi}{k_B}\sqrt{-t}\ \ \frac{(\tilde{b}_4-4\pi i\lambda_B\sqrt{-t})+(\tilde{b}_4+4\pi i\lambda_B\sqrt{-t})e^{-2i\lambda_B\tan^{-1}\left(\frac{\sqrt{-t}}{2|c_B|}\right)}}{-(\tilde{b}_4-4\pi i\lambda_B\sqrt{-t})+(\tilde{b}_4+4\pi i\lambda_B\sqrt{-t})e^{-2i\lambda_B\tan^{-1}\left(\frac{\sqrt{-t}}{2|c_B|}\right)}}
\end{aligned}
\tag{71}
$$

where we have used

$$
\tan^{-1}(x)+\tan^{-1}(\frac{1}{x})=\frac{\pi}{2},\quad \text{for}\ \ \text{x}>0
$$

and $\tilde{b}_4=-b_4+2\pi\lambda_B^2|c_B|$. Here form of the $\tan^{-1}(x)$ is

$$
\tan^{-1}x=\frac{1}{2i}\ln\left(\frac{1+ix}{1-ix}\right)
\tag{72}
$$

and the domain and the branch cut structure of the function $\tan^{-1}(x)$ are depicted in Fig. 6.

In the special case $b_4\to\infty$, T_T reduces to

$$
\begin{aligned}
T_T^{B\infty}(p_1,p_2,p_3,p_4,k_B,\lambda_B,c_B)=&E(p_1,p_2,p_3)\frac{4i\pi}{k_B}\sqrt{\frac{u\ t}{s}}\\
&-\frac{4\ i\pi}{k_B}\sqrt{-t}\ \ \frac{1+e^{-2i\lambda_B\tan^{-1}\left(\frac{\sqrt{-t}}{2|c_B|}\right)}}{1-e^{-2i\lambda_B\tan^{-1}\left(\frac{\sqrt{-t}}{2|c_B|}\right)}}\ .
\end{aligned}
\tag{73}
$$

In the U-channel we find

$$
\begin{aligned}
&T_{U_d}^B(p_1,p_2,p_3,p_4,k_B,\lambda_B,\tilde{b}_4,c_B)\\
=&E(p_1,p_2,p_3)\frac{4i\pi}{k_B}\sqrt{\frac{s\ t}{u}}\\
&-\frac{4\ i\pi}{k_B}\sqrt{-t}\ \ \frac{(\tilde{b}_4-4\pi i\lambda_B\sqrt{-t})+(\tilde{b}_4+4\pi i\lambda_B\sqrt{-t})e^{-2i\lambda_B\tan^{-1}\left(\frac{\sqrt{-t}}{2|c_B|}\right)}}{-(\tilde{b}_4-4\pi i\lambda_B\sqrt{-t})+(\tilde{b}_4+4\pi i\lambda_B\sqrt{-t})e^{-2i\lambda_B\tan^{-1}\left(\frac{\sqrt{-t}}{2|c_B|}\right)}}.
\end{aligned}
\tag{74}
$$

In the limit $b_4 \to \infty$ we have

$$
\begin{aligned}
T_{U_d}^{B\infty}(p_1,p_2,p_3,p_4,k_B,\lambda_B,c_B) =& E(p_1,p_2,p_3)\frac{4i\pi}{k_B}\sqrt{\frac{s\ t}{u}} \\
& - \frac{4\ i\pi}{k_B}\sqrt{-t}\ \ \frac{1+e^{-2i\lambda_B \tan^{-1}\left(\frac{\sqrt{-t}}{2|c_B|}\right)}}{1-e^{-2i\lambda_B \tan^{-1}\left(\frac{\sqrt{-t}}{2|c_B|}\right)}}.
\end{aligned}
\tag{75}
$$

Finally, the amplitude $T_{U_e}^B$ is obtained from $T_{U_d}^B$ simply by interchanging the two initial momenta. The usual symmetry of bosonic amplitudes immediately implies

$$
T_{U_e}^B(p_1,p_2,p_3,p_4,k_B,\lambda_B,b_4,c_B) = T_{U_d}^B(p_2,p_1,p_3,p_4,k_B,\lambda_B,b_4,c_B) \tag{76}
$$

with a similar formula for $S_{U_e}^{\infty}(p_1,p_2,p_3,p_4,k_B,\lambda_B,c_B)$.

We now report our results for the fermionic theory. In this case S-matrix in the T-channel is given by

$$
\begin{aligned}
& T_T^F(p_1,p_2,p_3,p_4,k_F,\lambda_F,c_F) \\
=& -E(p_1,p_2,p_3)\frac{4i\pi}{k_F}\sqrt{\frac{u\ t}{s}} \\
& + \frac{4\ i\pi}{k_F}\sqrt{-t}\ \ \frac{e^{i\pi(\lambda_F-\mathrm{sgn}(m_F))}+e^{2i(\lambda_F-\mathrm{sgn}(m_F))\tan^{-1}\left(\frac{2|c_F|}{\sqrt{-t}}\right)}}{e^{i\pi(\lambda_F-\mathrm{sgn}(m_F))}-e^{2i(\lambda_F-\mathrm{sgn}(m_F))\tan^{-1}\left(\frac{2|c_F|}{\sqrt{-t}}\right)}} \\
=& -E(p_1,p_2,p_3)\frac{4i\pi}{k_F}\sqrt{\frac{u\ t}{s}} + \frac{4\ i\pi}{k_F}\sqrt{-t}\ \ \frac{1+e^{-2i(\lambda_F-\mathrm{sgn}(m_F))\tan^{-1}\left(\frac{\sqrt{-t}}{2|c_F|}\right)}}{1-e^{-2i(\lambda_F-\mathrm{sgn}(m_F))\tan^{-1}\left(\frac{\sqrt{-t}}{2|c_F|}\right)}}.
\end{aligned}
\tag{77}
$$

In the U-channel we find

$$
\begin{aligned}
& T_{U_d}^F(p_1,p_2,p_3,p_4,k_F,\lambda_F,c_F) \\
=& -\Bigg(-E(p_1,p_2,p_3)\frac{4i\pi}{k_F}\sqrt{\frac{s\ t}{u}} \\
& + \frac{4\ i\pi}{k_F}\sqrt{-t}\ \ \frac{e^{i\pi(\lambda_F-\mathrm{sgn}(m_F))}+e^{2i(\lambda_F-\mathrm{sgn}(m_F))\tan^{-1}\left(\frac{2|c_F|}{\sqrt{-t}}\right)}}{e^{i\pi(\lambda_F-\mathrm{sgn}(m_F))}-e^{2i(\lambda_F-\mathrm{sgn}(m_F))\tan^{-1}\left(\frac{2|c_F|}{\sqrt{-t}}\right)}}\Bigg) \\
=& -\Bigg(-E(p_1,p_2,p_3)\frac{4i\pi}{k_F}\sqrt{\frac{s\ t}{u}} + \frac{4\ i\pi}{k_F}\sqrt{-t}\ \ \frac{1+e^{-2i(\lambda_F-\mathrm{sgn}(m_F))\tan^{-1}\left(\frac{\sqrt{-t}}{2|c_F|}\right)}}{1-e^{-2i(\lambda_F-\mathrm{sgn}(m_F))\tan^{-1}\left(\frac{\sqrt{-t}}{2|c_F|}\right)}}\Bigg).
\end{aligned}
\tag{78}
$$

Finally, the usual symmetry for fermionic amplitudes immediately implies that

$$T^F_{U_e}(p_1, p_2, p_3, p_4, k_F, \lambda_F, c_F) = -T^F_{U_d}(p_2, p_1, p_3, p_4, k_F, \lambda_F, c_F). \tag{79}$$

As we have mentioned earlier in this introduction, in the limit $b_4 \to \infty$, the bosonic theory studied in this chapter has been conjectured to be dual to the fermionic theory, when the parameters of the two theories are related by (14). Our results for the scattering amplitudes reported above are in perfect agreement with this conjecture. In particular it may be verified that, provided the inequality (15) is obeyed, the bosonic and fermionic S-matrices (including the identity pieces, see subsections 1.2.3 and 1.2.4)

$$\begin{aligned}
\mathbf{S}^{B\infty}_T(p_1, p_2, p_3, p_4, -k_F, \lambda_F - \mathrm{sgn}(\lambda_F), c_F) &= \mathbf{S}^F_T(p_1, p_2, p_3, p_4, k_F, \lambda_F, c_F), \\
\mathbf{S}^{B\infty}_{U_d}(p_1, p_2, p_3, p_4, -k_F, \lambda_F - \mathrm{sgn}(\lambda_F), c_F) &= -\mathbf{S}^F_{U_d}(p_1, p_2, p_3, p_4, k_F, \lambda_F, c_F), \\
\mathbf{S}^{B\infty}_{U_e}(p_1, p_2, p_3, p_4, -k_F, \lambda_F - \mathrm{sgn}(\lambda_F), c_F) &= \mathbf{S}^F_{U_e}(p_1, p_2, p_3, p_4, k_F, \lambda_F, c_F), \\
\mathbf{S}^{B\infty}_{U_s}(p_1, p_2, p_3, p_4, -k_F, \lambda_F - \mathrm{sgn}(\lambda_F), c_F) &= \mathbf{S}^F_{U_a}(p_1, p_2, p_3, p_4, k_F, \lambda_F, c_F), \\
\mathbf{S}^{B\infty}_{U_a}(p_1, p_2, p_3, p_4, -k_F, \lambda_F - \mathrm{sgn}(\lambda_F), c_F) &= \mathbf{S}^F_{U_s}(p_1, p_2, p_3, p_4, k_F, \lambda_F, c_F).
\end{aligned} \tag{80}$$

1.3.3 A conjecture for identity exchange and modified crossing symmetry

In the case of the bosonic theory we conjecture that S matrix in the S-channel is given by

$$\mathbf{S}^B_S = \cos(\pi\lambda_B) I(p_1, p_2, p_3, p_4) + i\frac{\sin(\pi\lambda_B)}{\pi\lambda_B} T^{trial}_S \tag{81}$$

where T^{trial}_S is the S-channel S-matrix obtained from analytic continuation of the T or U-channel results using the usual rules of 'naive' crossing symmetry, and is given by

$$\begin{aligned}
T^{trial}_S =& (\pi\lambda_B)\, 4\, i\sqrt{s} E(p_1, p_2, p_3)\sqrt{\frac{u}{t}} \\
&+ (\pi\lambda_B)\, 4\sqrt{s} \left(\frac{\left(4\pi\lambda_B\sqrt{s} + \widetilde{b}_4\right) + e^{i\pi\lambda_B}\left(-4\pi\lambda_B\sqrt{s} + \widetilde{b}_4\right)\left(\frac{\frac{1}{2}+\frac{c_B}{\sqrt{s}}}{\frac{1}{2}-\frac{c_B}{\sqrt{s}}}\right)^{\lambda_B}}{\left(4\pi\lambda_B\sqrt{s} + \widetilde{b}_4\right) - e^{i\pi\lambda_B}\left(-4\pi\lambda_B\sqrt{s} + \widetilde{b}_4\right)\left(\frac{\frac{1}{2}+\frac{c_B}{\sqrt{s}}}{\frac{1}{2}-\frac{c_B}{\sqrt{s}}}\right)^{\lambda_B}} \right).
\end{aligned} \tag{82}$$

In the limit $b_4 \to \infty$, T_S^{trial} simplifies to

$$T_S^{trial} = (\pi\lambda_B)\, 4\, i\sqrt{s} \left(E(p_1,p_2,p_3)\sqrt{\frac{u}{t}} + \left(\frac{1+e^{i\pi\lambda_B}\left(\frac{\frac{1}{2}+\frac{c_B}{\sqrt{s}}}{\frac{1}{2}-\frac{c_B}{\sqrt{s}}}\right)^{\lambda_B}}{1-e^{i\pi\lambda_B}\left(\frac{\frac{1}{2}+\frac{c_B}{\sqrt{s}}}{\frac{1}{2}-\frac{c_B}{\sqrt{s}}}\right)^{\lambda_B}} \right) \right). \tag{83}$$

In a similar manner we expect that the fermionic S-matrix is given by

$$\begin{aligned} S_S^F &= \cos(\pi\lambda_F) I(p_1,p_2,p_3,p_4) + i\frac{\sin(\pi\lambda_F)}{\pi\lambda_F} T_F^{trial} \\ &= \sin(\pi\lambda_F)\left(4E(p_1,p_2,p_3)\sqrt{\frac{s\ t}{u}} + 4\sqrt{s}\ \ \frac{1+e^{-2i(\lambda_F-\mathrm{sgn}(m_F))\tan^{-1}\left(\frac{\sqrt{s}}{2|c_F|}\right)}}{1-e^{-2i(\lambda_F-\mathrm{sgn}(m_F))\tan^{-1}\left(\frac{\sqrt{s}}{2|c_F|}\right)}} \right) \\ &+ \cos(\pi\lambda_F) I(p_1,p_2,p_3,p_4). \end{aligned} \tag{84}$$

It follows from (81), (84) and the results of the previous subsection the fermionic and bosonic S-channel S matrices map to each other under duality upto an overall minus sign (recall that overall phases in an S-matrix are unobservable and so unimportant).

1.4 Scattering in the scalar theory

In this section we compute the four point scattering amplitude in the theory of fundamental bosons coupled to Chern-Simons theory. Very briefly we integrate out the gauge boson to obtain an offshell effective four boson term in the quantum effective action for our theory, given by

$$\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3} V(p,k,q)\phi_i(p+q)\bar{\phi}^j(-(k+q))\bar{\phi}^i(-p)\phi_j(k). \tag{85}$$

We then take an appropriate on shell limit to evaluate the S-matrix.

1.4.1 Integral equation for off shell four point amplitude

As explained in the previous section, $V(p,k,q)$ obeys the integral equation depicted in Fig 5. In formulas

$$\begin{aligned} V(p,k,q) &= V_0(p,k,q) - i\int \frac{d^3r}{(2\pi)^3} V(p,r,q_3)\frac{NV_0(r,k,q_3)}{(r^2+c_B^2-i\epsilon)\left((r+q)^2+c_B^2-i\epsilon\right)}, \\ V(p,k,q) &= V_0(p,k,q) - i\int \frac{d^3r}{(2\pi)^3} V_0(p,r,q_3)\frac{NV(r,k,q_3)}{(r^2+c_B^2-i\epsilon)\left((r+q)^2+c_B^2-i\epsilon\right)}, \end{aligned} \tag{86}$$

where the 'one particle' amplitude V_0 is given by the sum of graphs in Fig. 4. Summing these graphs (see Appendix 1.9.4 for details) we find[20]

$$\begin{aligned} NV_0(p,k,q_3) &= -4\pi i\lambda_B q_3 \frac{(k+p)_-}{(k-p)_-} + \widetilde{b}_4, \\ \widetilde{b}_4 &= 2\pi\lambda_B^2 c_B - b_4. \end{aligned} \tag{87}$$

Here

$$d^3r = dr^0 dr^1 dr^3, \quad k_\pm = \frac{\pm k_0 + k_1}{\sqrt{2}}. \tag{88}$$

[21] (87) is actually ambiguous as stated. The first term on the RHS of (87) is proportional to $\frac{1}{(k-p)_-}$: the gauge boson propagator in lightcone gauge. This term is ill defined when $k_- = p_-$, a point that lies on the integration contour on the RHS of (34).

The reason that the gauge boson has a codimension two singularity in momentum space is that the choice of lightcone gauge, $A_- = 0$, leaves unfixed the residual gauge transformations that depend only on x^+ and x^3. In this chapter we resolve this ambiguity of the propagator at $p_- = 0$ with the 'Feynman' prescription

$$\frac{1}{p_-} \to \frac{p_+}{p_+ p_- - i\epsilon}. \tag{89}$$

We adopt this prescription for several reasons.

- 1. It is the only resolution of the singularity of the gauge propagator that permits

[20] If we include other multi-trace terms such as $\frac{\lambda_p}{N^{p-1}}(\bar{\phi}\phi)^p$ in the action (5), this effect only reflects a shift of $\tilde{b}_4$ by a linear term of $c_B^{p-2}\lambda_p$ with a suitable coefficient. The rest of calcuation of $2\to 2$ scattering is the same as presented in this chapter.

[21] Note in that our definition of k_- is the negative of the definition usually adopted in studies of Minkowskian physics. We adopt this definition because it will prove convenient once we continue to Euclidean space.

continuation to Euclidean space. It therefore appears to be the only resolution of the singularity that can make contact with all the beautiful Euclidean results of [2–6, 8–11].

- 2. Its use leads to sensible results with no unphysical divergences. [22]

- 3. In special cases, results obtained by use of this prescription turn out to agree with results in the covariant Landau gauge (see subsection 1.5 below).

Of course the pragmatic reasons spelt out above are ultimately unsatisfactory; we would like eventually to have a justification of this prescription on physical grounds (such a justification would presumably involve a careful accounting for the unfixed gauge symmetry of the problem). However we leave this potentially subtle exercise to future work.

1.4.2 Euclidean continuation

In order to solve the integral equation (86) we will find it convenient to use a standard maneuver to 'continue this equation to Euclidean space'. Operationally, the procedure is to define a Euclidean amplitude via $V^E(p^0, k^0) = V(ip^0, ik^0)$. [23] Once the amplitude V^E has been solved for, the amplitude of real physical interest, V, is obtained by the inverse relation

$$V(p^0, k^0) = V^E(-ip^0, -ik^0).$$

Even though the method of Euclidean continuation is standard in the study of scattering amplitudes, for completeness we recall the justification of this method, in the context of our problem, in Appendix 1.9.4. We emphasize that this procedure is valid only when the singularities of all propagators in the Lorentzian problem are resolved by the Feynman $i\epsilon$ prescription. This is one of the main reasons we adopted the $i\epsilon$ prescription of (89) above.

[22] Other potential resolutions of this singularity appear to lead to pathological results. For instance the replacement of $\frac{1}{p_-}$ by its principal value leads to unacceptable divergences in propagators.

[23] In this paragraph we are interested only in the dependence of all quantities on p^0 and k^0 so we suppress the dependence of V on other components of the momenta.

The Euclidean continuation of the scattering amplitude obeys the integral equation

$$\begin{aligned}
V^E(p,k,q) &= V_0^E(p,k,q) + \int \frac{d^3r}{(2\pi)^3} V_0^E(p,r,q_3) \frac{N V^E(r,k,q_3)}{(r^2+c_B^2)\left((r+q)^2+c_B^2\right)} \\
V^E(p,k,q) &= V_0^E(p,k,q) + \int \frac{d^3r}{(2\pi)^3} V^E(p,r,q_3) \frac{N V_0^E(r,k,q_3)}{(r^2+c_B^2)\left((r+q)^2+c_B^2\right)} \\
N V_0^E(p,k,q_3) &= -4\pi i \lambda_B q_3 \frac{(k+p)_-}{(k-p)_-} + \widetilde{b}_4
\end{aligned} \tag{90}$$

where

$$d^3r = dr^0 dr^1 dr^3, \quad k_\pm = \frac{k_1 \pm i k_0}{\sqrt{2}}. \tag{91}$$

Note, in particular, that $k_\pm$ are now complex conjugates of each other. Below we will sometimes use the notation

$$k_s^2 = 2k_+k_- = k_1^2 + k_0^2. \tag{92}$$

1.4.3 Solution of the Euclidean integral equation

The integral equation (90) may be solved in a completely systematic manner. We have presented a detailed derivation of our solution of this equation in Appendix 1.9.4. In this subsection we simply quote our final results.

Our solution takes the form

$$NV = e^{-2i\lambda_B\left(\tan^{-1}\left(\frac{2(a(k))}{q_3}\right) - \tan^{-1}\left(\frac{2(a(p))}{q_3}\right)\right)} \left(4\pi i \lambda_B q_3 \frac{p_- + k_-}{p_- - k_-} + j(q_3,\lambda_B)\right) \tag{93}$$

where

$$a(p) = \sqrt{2p_+p_- + c_B^2} \tag{94}$$

and

$$j(q_3,\lambda_B) = 4\pi i \lambda_B q_3 \left(\frac{\left(4\pi i\lambda_B q_3 + \widetilde{b}_4\right) e^{2i\lambda_B \tan^{-1}\left(\frac{2c_B}{q_3}\right)} + \left(-4\pi i \lambda_B q_3 + \widetilde{b}_4\right) e^{\pi i \lambda_B \mathrm{sgn}(q_3)}}{\left(4\pi i\lambda_B q_3 + \widetilde{b}_4\right) e^{2i\lambda_B \tan^{-1}\left(\frac{2c_B}{q_3}\right)} - \left(-4\pi i \lambda_B q_3 + \widetilde{b}_4\right) e^{\pi i \lambda_B \mathrm{sgn}(q_3)}} \right). \tag{95}$$

It is not difficult to verify that

$$j(q_3,\lambda_B) = j(-q_3,\lambda_B) = j(q_3,-\lambda_B) = j(-q_3,-\lambda_B) = j(|q_3|,|\lambda_B|). \tag{96}$$

In other words, j is an even function of q_3 and λ_B separately. It follows in particular that

$$j(q,\lambda_B) = 4\pi i\lambda_B|q| \left(\frac{\left(4\pi i\lambda_B|q| + \widetilde{b}_4\right) e^{2i\lambda_B \tan^{-1}\left(\frac{2c_B}{|q|}\right)} + \left(-4\pi i\lambda_B|q| + \widetilde{b}_4\right) e^{\pi i\lambda_B}}{\left(4\pi i\lambda_B|q| + \widetilde{b}_4\right) e^{2i\lambda_B \tan^{-1}\left(\frac{2c_B}{|q_3|}\right)} - \left(-4\pi i\lambda_B|q| + \widetilde{b}_4\right) e^{\pi i\lambda_B}} \right). \quad (97)$$

This formula may be rewritten as follows. Let us define

$$\begin{aligned} H(q) &= \int \frac{d^3r}{(2\pi)^3} \frac{1}{(r^2+c_B^2)\left((r+q)^2+c_B^2\right)} = \left(-\frac{\tan^{-1}\left(\frac{2c_B}{|q_3|}\right)}{4\pi|q_3|} + \frac{1}{8|q_3|} \right) \\ &= \frac{\tan^{-1}\left(\frac{|q_3|}{2c_B}\right)}{4\pi|q_3|} \\ &= \frac{1}{8\pi i|q|} \ln\left(\frac{\frac{1}{2}+\frac{c_B}{iq}}{-\frac{1}{2}+\frac{c_B}{iq}} \right). \end{aligned} \quad (98)$$

Here to get the last line, we have used the formula (72). $H(q)$ is simply the one loop four boson scattering amplitude in ϕ^4 theory. In terms of this function we have

$$j(q) = 4\pi i\lambda_B|q| \left(\frac{\left(4\pi i\lambda_B|q| + \widetilde{b}_4\right) + \left(-4\pi i\lambda_B|q| + \widetilde{b}_4\right) e^{8i\pi\lambda_B|q|H(q)}}{\left(4\pi i\lambda_B|q| + \widetilde{b}_4\right) - \left(-4\pi i\lambda_B|q| + \widetilde{b}_4\right) e^{8i\pi\lambda_B|q|H(q)}} \right). \quad (99)$$

Using the last line in (98) $j(q)$ may also be rewritten as

$$j(q) = 4\pi iq\lambda_B \left(\frac{\left(4\pi iq\lambda_B + \widetilde{b}_4\right) + \left(-4\pi iq\lambda_B + \widetilde{b}_4\right) \left(\frac{\frac{1}{2}+\frac{c_B}{iq}}{-\frac{1}{2}+\frac{c_B}{iq}}\right)^{\lambda_B}}{\left(4\pi iq\lambda_B + \widetilde{b}_4\right) - \left(-4\pi iq\lambda_B + \widetilde{b}_4\right) \left(\frac{\frac{1}{2}+\frac{c_B}{iq}}{-\frac{1}{2}+\frac{c_B}{iq}}\right)^{\lambda_B}} \right). \quad (100)$$

Transformation under parity While parity transformations are not a symmetry of the bosonic theory, the simultaneous action of a parity transformation and the flip in the sign of k_B (or λ_B) is symmetry of this theory. Every physical quantity in this theory must, therefore, transform in a suitably 'nice' way under the combined action of these two transformations.

The off shell Greens function computed in the previous subsection is not physical as it is not gauge invariant, and so need not transform 'nicely' under parity operations. Indeed it is easily verified by inspection that the amplitude V is left invariant by a reflection in the

3 direction accompanied by a flip in the sign of λ_B. However the combined operation of a flip in the sign of λ_B and a reflection in either the 0 or 1 directions is not an invariance of this amplitude. The reason for this asymmetry is that reflections in the 3 direction are the only parity transformations that commute with the choice of light cone gauge (for instance a reflection in the 1 direction changes the gauge $A_- = 0$ to $A_+ = 0$.).

As we will see below, the physical S matrix indeed enjoys the full parity symmetry expected of this theory.

1.4.4 Analytic continuation of $j(q)$

In our study of S-channel scattering later in this chapter we will need to continue the function $j(q)$ to $q^2 = -s$. This analytic continuation is achieved by setting $q = \lim_{\alpha \to \frac{\pi}{2}} e^{-i\alpha\frac{\pi}{2}}\sqrt{s}$ or equivalently by setting

$$q \to -i\left(\sqrt{s} + i\epsilon\right)$$

The precise analytic continuation we will use is the following. We will take the function $j(q)$ to be defined by (99), where $H(q)$ is defined by (98). The function $\tan^{-1}(x)$ that appears in some versions of the definition of $H(q)$ is taken to have the analytic structure depicted in Fig. 6 [24]

The function $H(q)$ (see (98)) analytically continues to $H^M(\sqrt{s})$

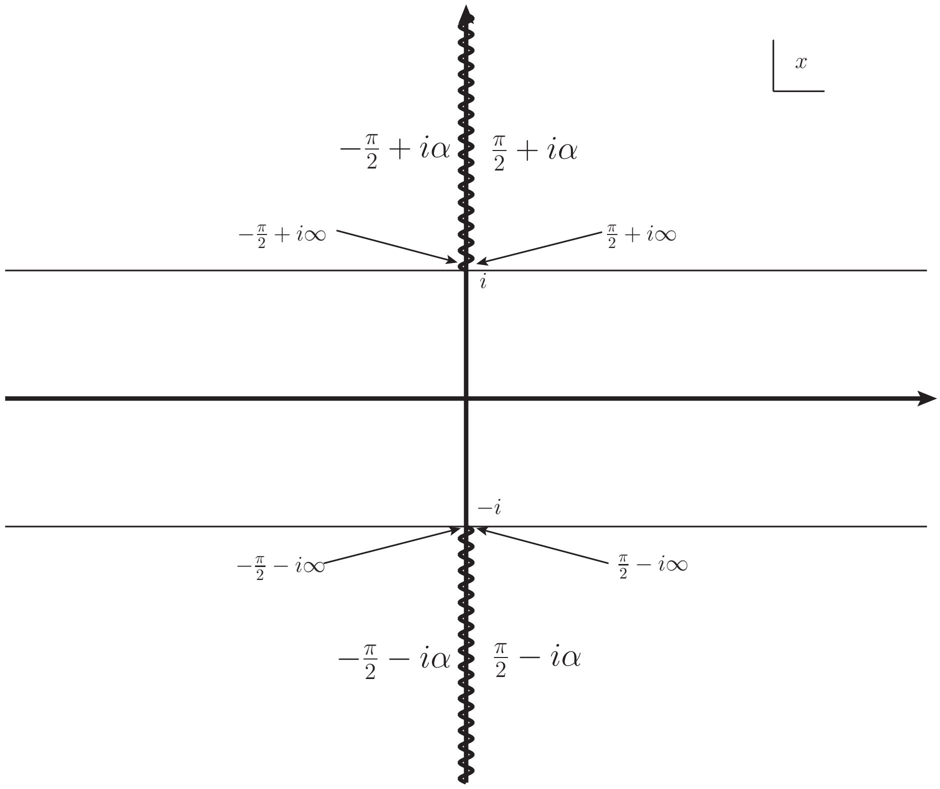
$$\begin{aligned} H^M(\sqrt{s}) =& -i\int \frac{d^3r}{(2\pi)^3}\frac{1}{(r^2+c_B^2-i\epsilon)\left((r+q)^2+c_B^2-i\epsilon\right)} \\ =&\frac{1}{8\pi\sqrt{s}}\ln\left(\frac{\frac{1}{2}+\frac{c_B}{\sqrt{s+i\epsilon}}}{-\frac{1}{2}+\frac{c_B}{\sqrt{s+i\epsilon}}}\right). \end{aligned} \tag{102}$$

For $\sqrt{s} < 2c_B$, the factors of $i\epsilon$ make no difference in the formula (102) and may simply be dropped. When $\sqrt{s} > 2c_B$, the factors of $i\epsilon$ choose out the branch of logarithmic function

[24] This analytic structure follows from the formula

$$\tan^{-1} x = \frac{1}{2i}\ln\left(\frac{1+ix}{1-ix}\right). \tag{101}$$

if we define the logarithmic to be the usual log for positive real values, but to have a branch cut along the negative real axis.



(a)

Figure 6: Branch cut structure of the function $\tan^{-1} x$. α is a real function of x along the branch cut which vanishes at infinities and becomes ∞ at $|Im(x)| = 1$.

and we have

$$H^M(\sqrt{s}) = \begin{cases} \frac{1}{8\pi\sqrt{s}} \ln\left(\frac{\frac{1}{2}+\frac{c_B}{\sqrt{s}}}{-\frac{1}{2}+\frac{c_B}{\sqrt{s}}}\right) & (\sqrt{s} < 2c_B) \\ \frac{1}{8\pi\sqrt{s}} \left(\ln\left(\frac{\frac{1}{2}+\frac{c_B}{\sqrt{s}}}{\frac{1}{2}-\frac{c_B}{\sqrt{s}}}\right) + i\pi\right) & (\sqrt{s} > 2c_B) \end{cases} . \tag{103}$$

It follows, in particular, that

$$-i\left(H(\sqrt{s} - H^*(\sqrt{s})\right) = \frac{\theta(\sqrt{s} - 2c_B)}{4\pi\sqrt{s}}. \tag{104}$$

Let j^M denote the analytic continuation of $j(q)$. It follows that

$$j^M(\sqrt{s}) =(\pi\lambda_B)(4\sqrt{s}) \left(\frac{\left(4\pi\lambda_B\sqrt{s} + \widetilde{b}_4\right) + \left(-4\pi\lambda_B\sqrt{s} + \widetilde{b}_4\right) e^{8\pi\lambda_B\sqrt{s}H^M(\sqrt{s})}}{\left(4\pi\lambda_B\sqrt{s} + \widetilde{b}_4\right) - \left(-4\pi\lambda_B\sqrt{s} + \widetilde{b}_4\right) e^{8\pi\lambda_B\sqrt{s}H^M(\sqrt{s})}}\right),$$

$$j^M(\sqrt{s}) = \begin{cases} (\pi\lambda_B)(4\sqrt{s}) \left(\frac{\left(4\pi\lambda_B\sqrt{s}+\widetilde{b}_4\right)+\left(-4\pi\lambda_B\sqrt{s}+\widetilde{b}_4\right)\left(\frac{\frac{1}{2}+\frac{c_B}{\sqrt{s}}}{-\frac{1}{2}+\frac{c_B}{\sqrt{s}}}\right)^{\lambda_B}}{\left(4\pi\lambda_B\sqrt{s}+\widetilde{b}_4\right)-\left(-4\pi\lambda_B\sqrt{s}+\widetilde{b}_4\right)\left(\frac{\frac{1}{2}+\frac{c_B}{\sqrt{s}}}{-\frac{1}{2}+\frac{c_B}{\sqrt{s}}}\right)^{\lambda_B}}\right), & (\sqrt{s} < 2c_B) \\ (\pi\lambda_B)(4\sqrt{s}) \left(\frac{\left(4\pi\lambda_B\sqrt{s}+\widetilde{b}_4\right)+e^{i\pi\lambda_B}\left(-4\pi\lambda_B\sqrt{s}+\widetilde{b}_4\right)\left(\frac{\frac{1}{2}+\frac{c_B}{\sqrt{s}}}{\frac{1}{2}-\frac{c_B}{\sqrt{s}}}\right)^{\lambda_B}}{\left(4\pi\lambda_B\sqrt{s}+\widetilde{b}_4\right)-e^{i\pi\lambda_B}\left(-4\pi\lambda_B\sqrt{s}+\widetilde{b}_4\right)\left(\frac{\frac{1}{2}+\frac{c_B}{\sqrt{s}}}{\frac{1}{2}-\frac{c_B}{\sqrt{s}}}\right)^{\lambda_B}}\right), & (\sqrt{s} > 2c_B) \end{cases} \tag{105}$$

1.4.5 Poles of the functions $j(q)$ and $j^M(\sqrt{s})$

In this subsection we will analyze the conditions under which the functions $j(q)$ and $j^M(\sqrt{s})$ have poles for real values of their arguments. The conditions are most conveniently presented in terms of inequalities on b_4 for fixed values of all other parameters.

Substituting $\widetilde{b}_4 = 2\pi\lambda_B^2 c_B - b_4$ in the formulas (105) and (100) we can see that for $b_4 > -2\pi\lambda_B c_B(4 - \lambda_B)$ neither of the functions above has a pole at real values of its argument. When $-2\pi\lambda_B c_B(4-\lambda_B) \geq b_4 \geq -2\pi c_B(4-\lambda_B^2)$ the function j^M has a pole, but j has no pole. At the upper end of this interval the pole occurs at $\sqrt{s} = 2c_B$. At the lower end of this interval the pole value is $\sqrt{s} = 0$. For $b_4 \leq -2\pi c_B(4 - \lambda_B^2)$, $j^M(\sqrt{s})$ has no real poles, but the function $j(q)$ develops a pole. This pole starts out at $q = 0$ and migrates to $q = \infty$ as $b_4 \to -\infty$.

A pole in the function $j^M(\sqrt{s})$ at $s = s_B$ signals the presence of a particle - antiparticle bound state in the singlet channel. As we have seen above, bound states exist only for b_4

less than a certain minimum value. We will now explain how this result fits with physical intuition; let us first focus on the special case $\lambda_B = 0$. In this case poles exist for $b_4 \leq 0$. In the non-relativistic limit a term $+\int b_4 \frac{(\bar{\phi}\phi)^2}{2N}$ in the Minkowskian action represents a negative (attractive) delta function interaction between particles and antiparticles when $b_4 > 0$. It seems plausible that such an attractive potential could support a bound state, as appears to be the case. Clearly the binding energy of this system is proportional to b_4, and so goes to zero in the limit $b_4 \to 0$. In other words we should expect the mass of the bound state to be given precisely by $2c_B$ at $b_4 = 0$, exactly as we find. As b_4 decreases we should expect the binding energy to increase, i.e. for the bound state energy to decrease, exactly as we find. Above a critical value of c_B we find above that the binding energy is so large that the bound state energy vanishes. At even lower values of b_4 the vacuum is unstable as it is energetically favorable for particle - antiparticle pairs to spontaneously bubble out of the vacuum. This instability is, presumably, signalled by the appearance of the tachyonic pole in b_4. The instability of the vacuum also seems reasonable from the viewpoint of quantum field theory; a large negative value of b_4 the classical scalar potential is unbounded from below; plausibly the same is true of the exact potential in the quantum effective action in this regime.

The pattern is very similar at nonzero λ_B; though the precise values of the critical values for b_4 shift around. Apparently the anyonic interaction in the singlet channel renormalizes the effective interaction of the theory.

Note that bound states do not exist in the limit $b_4 \to \infty$, the limit in which the bosonic theory is dual to the fermionic theory.

It would be interesting to flesh out the qualitative discussion presented in this subsection. Near the threshold of bound state formation the interacting particles are approximately non-relativistic, so it may be possible to reproduce the pole mass in this regime by solving a Schrodinger equation. We leave this to future work.

1.4.6 Various limits of the function $j(q)$.

The explicit form of the function $j(q)$ (here $q = \sqrt{|q_3|^2}$) is one of the principal computational results of this section. $j(q)$ has the dimensions of mass. It is a function of one dimensionless variable λ_B, and three quantities of mass dimension 1; q, c_B and b_4. It follows that j takes the form $j = qh(x, y, \lambda_B)$ where

$$x = \frac{q}{2c_B}, \quad y = \frac{q}{b_4}. \tag{106}$$

In this subsection we study the behavior of the function j at extreme values of its three dimensionless arguments.

Large b_4 limit When $|b|_4 \gg \lambda_B q$ (i.e. when $\lambda_B y \ll 1$) the function $j(q)$ simplifies to

$$j(q_3, \lambda_B) = 4\pi i \lambda_B |q_3| \left(\frac{1 + e^{8i\pi\lambda_B |q_3| H(q)}}{1 - e^{8i\pi\lambda_B |q_3| H(q)}} \right) = -4\pi i \lambda_B |q_3| \left(\frac{1 + e^{-2i\lambda_B \tan^{-1}\left(\frac{|q_3|}{2c_B}\right)}}{1 - e^{-2i\lambda_B \tan^{-1}\left(\frac{|q_3|}{2c_B}\right)}} \right). \quad (107)$$

Small λ_B The function j may be expanded in a Taylor series in λ_B at fixed values of x and y. We find

$$j = \frac{-b_4}{1 + b_4 H(q)} - \frac{16\pi^2 \lambda_B^2 q_3^2}{3b_4} \left(\frac{1}{(b_4 H(q) + 1)^{\,2}} - b_4 H(q) - 1 \right) + \mathcal{O}(\lambda_B^4). \quad (108)$$

The limits $\lambda_B \to 0$ and $b_4 \to \infty$ (i.e $\lambda_B \to 0$ and $y \to 0$ at fixed x) commute, so one may obtain the small λ_B expansion of (107) by simply setting $b_4 \to \infty$ in (108).

Note that, in the strict $\lambda_B \to 0$ limit,

$$\lim_{\lambda_B \to 0} NV(p, k, q^3) = \lim_{\lambda_B \to 0} j(q_3) = \frac{-b_4}{1 + b_4 \frac{\tan^{-1}\left(\frac{|q_3|}{2c_B}\right)}{4\pi |q_3|}} = \frac{-b_4}{1 + H(q_3) b_4}. \quad (109)$$

(109) is the well known result for the off shell amplitude in large N ϕ^4 theory. It is easily verified by directly solving the integral equation (90) at $\lambda_B = 0$.

The limit $|\lambda_B| \to 1$ The expression for $j(q)$ simplifies somewhat in the limit $\lambda_B \to 1$. The simplification is especially dramatic if we also take the limit $b_4 \to \infty$. In the combined limit $y \to 0$ and $\lambda_B \to 1$ (the order of limits does not matter) we have

$$\begin{aligned} j(q) =& 4\pi i q \frac{e^{i \tan^{-1}\left(\frac{2c_B}{q}\right)} - e^{-i \tan^{-1}\left(\frac{2c_B}{q}\right)}}{e^{i \tan^{-1}\left(\frac{2c_B}{q}\right)} + e^{-i \tan^{-1}\left(\frac{2c_B}{q}\right)}} \\ =& 4\pi i q (i) \tan\left(\tan^{-1}\left(\frac{2c_B}{q} \right) \right) \\ =& -8\pi c_B. \end{aligned} \quad (110)$$

The ultra-relativistic limit If c_B and b_4 are held fixed while $\sqrt{-t}$ is taken to infinity (this is the case, for instance, in fixed angle high energy scattering in the U and T-channels, see below) , we take x and y to infinity at fixed λ_B and j simplifies to

$$j(q) = 4\pi q \lambda_B \tan \frac{\pi \lambda_B}{2}. \tag{111}$$

The ultra relativistic limit does not commute with the limit $b_4 \to \infty$. If b_4 is taken to ∞ first and $q \to \infty$ next then we work with $y \to 0$, $x \to \infty$ at fixed λ_B and find

$$j(q) = -4\pi \lambda_B q \cot \left(\frac{\pi \lambda_B}{2} \right). \tag{112}$$

The ultra relativistic limit also does not commute with the limit $\lambda_B \to 0$. At $\lambda_B = 0$ the function $j(q)$ tends to a constant proportional to b_4. Physically this is we have a dimensionless coupling constant at nonzero λ_B, but only a dimensionful coupling constant at any finite λ_B; at zero lambda the theory is very weakly coupled at high energies, and receives contributions only from tree level graphs.

The massless limit If c_B is taken to zero at fixed b_4, λ_B and q (i.e. if x is taken to infinity at fixed y and λ_B) then j simplifies to the rational function

$$j(q) = 4\pi \lambda_B q \left(\frac{4\pi \lambda_B \sin\left(\frac{\pi \lambda_B}{2}\right) q + \tilde{b}_4 \cos\left(\frac{\pi \lambda_B}{2}\right)}{4\pi \lambda_B \cos\left(\frac{\pi \lambda_B}{2}\right) q - \tilde{b}_4 \sin\left(\frac{\pi \lambda_B}{2}\right)} \right). \tag{113}$$

The massless limit commutes with the limit $b_4 \to \infty$. In this limit (113) reduces to (112).

The non-relativistic limit in the U and T-channels As we will see below, the non-relativistic limit in the U and T-channels is obtained by taking c_B to infinity at fixed q. In other words, this limit is obtained by taking x to zero at fixed λ_B and y. In this limit $2i\lambda_B \tan^{-1}\left(\frac{2c_B}{q_3}\right)$ in (97) reduces to $\pi i \lambda_B$ and we have

$$j(\sqrt{-t}) = \tilde{b}_4. \tag{114}$$

In this limit, in other words, the function j receives contributions only from tree level scattering with the effective four point coupling $\tilde{b}_4$ in this limit. No genuine loop diagrams contribute to T and U-channel scattering in this limit.

If we first take $b_4 \to \infty$ and then take the non-relativistic limit we find

$$j(q) = -8\pi c_B \tag{115}$$

As (114) and (115) both tend to infinity in the combined non-relativistic and $b_4 \to \infty$ limit, the reader may find herself tempted to conclude that the non-relativistic and $b_4 \to \infty$ commute. This conclusion is, infact, slightly misplaced. As we have emphasized in section 1.2.6, the true dynamical information in the non-relativistic limit lies in the function

$$h = -\frac{j}{8ic_B}$$

which is derived from (54). The correct interpretation of the results of this subsection are that the function h vanishes in the non-relativistic limit at fixed b_4, but reduces to a λ_B independent numerical constant if b_4 is first taken to infinity.

1.4.7 The non-relativistic limit in the S-channel

As we will see below, the function relevant for scattering in the S-channel is the analytically continued function $j^M(\sqrt{s})$, see (105). The non-relativistic limit of S-channel scattering is obtained in the limit $\sqrt{s} \to 2c_B$ where the limit is taken from above with all other parameters held fixed. It is easily seen from (105) that in this limit

$$j^M(\sqrt{s}) = -(\pi\lambda_B)(4\sqrt{s})\mathrm{sgn}(\lambda_B). \tag{116}$$

Note that $j^M(\sqrt{s})$ is a non-analytic function of λ_B as $\lambda_B \to 0$ in this limit. The non-analyticity is precisely of the form expected from the non-relativistic limit; infact, in this limit

$$W_1(\sqrt{s}) = \frac{\sin(\pi\lambda_B)}{\pi\lambda_B} j^M(\sqrt{s}). \tag{117}$$

We will suggest an interpretation of this fact in section 1.7 below.

1.4.8 The onshell limit

In order to compute the physical S-matrix we analytically continue the amplitude V to Minkowski space. It follows from (85) that the onshell value of this analytically continued V may directly be identified with the scattering amplitude T (see subsection 1.2.3) once all momenta are taken onshell.

As the 3 vectors p and $p+q$ are simultaneously onshell, it follows that $p_3 = -\frac{q_3}{2}$. Similarly $k_3 = -\frac{q_3}{2}$. As p and k are themselves onshell it follows that [25]

$$a(p)^2 = -\frac{q_3^2}{4}, \quad a(k)^2 = -\frac{q_3^2}{4}, \quad a(p) = a(k) = -i\frac{q_3}{2}.$$

An infrared 'ambiguity' and its resolution The offshell amplitude (241) takes the form

$$\begin{aligned} &NV = PT, \\ &T = \left(4\pi i\lambda_B q_3 \frac{p_- + k_-}{p_- - k_-} + j(q_3)\right), \\ &P = e^{-2i\lambda_B\left(\tan^{-1}\left(\frac{2(a(k)}{q_3}\right) - \tan^{-1}\left(\frac{2(a(p)}{q_3}\right)\right)}. \end{aligned} \tag{118}$$

The expression T defined above has a perfectly smooth on shell limit that we will study below. The onshell limit of P is more singular,

$$P = e^{-2i\lambda_B\left(\tan^{-1}(-i) - \tan^{-1}(-i)\right)}, \tag{119}$$

recall that $\tan^{-1}(i)$ diverges, P thus takes the schematic form

$$P = e^{i\lambda_B(\infty-\infty)}$$

and is ambiguous.

The ambiguity in the expression for P has its origins in ladder graphs in which the scalars interact via the exchange of a very soft gauge boson. The integration over very small gauge boson momenta is divergent; however we encounter two classes of divergences which could potentially cancel, leading to the ambiguous result for P.

In a theory with physical gluonic states, the IR divergence obtained upon integrating out soft gluons is a real effect in scattering amplitudes (even though it cancels out in physical IR safe observables). However Chern-Simons theory has no physical gluons. On physical grounds, therefore, we do not expect the scattering amplitude to be divergent or ambiguous in any way. We will now explain that the correct on shell value for P is infact unity.

[25] The sign in the last two equations follows from the fact that $a(p)$ is defined with a square root with a branch cut on the negative real axis coupled with the fact that the rotation from Euclidean to Minkowski space proceeds in the clockwise direction.

We first note that the λ_B dependence of the ambiguity is extremely simple; it follows that if we can accurately establish the on shell value of P at one loop, we know its correct value at all loops. In order to determine P at one loop, in Appendix 1.9.4 we have performed a careful computation of the one loop amplitude directly in Minkowski space. Offshell our result agrees perfectly with the analytic continuation of (93), as we would expect. On being careful about all factors of $i\epsilon$ however, we find that the on shell result is unambiguous, and we find that the two terms in (119) actually cancel. It follows that the correct on shell continuation of P above is simply unity. In the next subsection we present a completely independent verification of this result from a rather different point of view.

In this subsubsection we have already encountered an unusual phenomenon: the analytic continuation of the Euclidean answer is ambiguous or incomplete due to potential IR on shell singularities, and this ambiguity is resolved by performing a computation directly in Minkowski space. In the case at hand the ambiguity had a relatively simple and straightforward resolution. A similar issue will come back to haunt us in a more virulent form in our study of S-channel scattering below.

Covariantization of the amplitude We now turn to the onshell limit of T in (118). In this limit the expression for T may equally well be written in the manifestly covariant form

$$T = 4\pi i\lambda_B \epsilon_{\mu\nu\rho} \frac{q^\mu (p-k)^\nu (p+k)^\rho}{(p-k)^2} + j(\sqrt{q^2}). \tag{120}$$

[26] The manifestly covariant expression (120) also enjoys invariance under the simultaneous operation of an arbitrary parity flip together with a flip in the sign of λ_B. The first term in (120) is odd under parity flips as well as under a flip in the sign of λ_B. The second term in (120) is even under both operations.

As we will explain in more detail below, the magnitude of the expression $\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2}$ can be written in terms of the standard kinematical invariants s, t, u. However the sign of this expression is not a function of these invariants. This is a peculiar kinematical feature of 2-2 scattering in $2+1$ dimensions. The most general amplitude in this dimension is a

[26] The equivalence between (120) and (118) follows from the observation that, in onshell,

$$q \cdot (p-k) = 0 \Rightarrow p_3 - k_3 = 0 \Rightarrow (p_- - k_-)(p_+ - k_+) = \frac{1}{2}(p-k)^2$$

and the observation (see the previous subsection) that $j(q_3) = j(-q_3)$.

function of s,t and the Z_2 valued variable

$$E(q,p-k,p+k) = \text{sgn}\left(\epsilon_{\mu\nu\rho}q^{\mu}(p-k)^{\nu}(p+k)^{\rho}\right).$$

The quantity $E(a,b,c)$ measures the 'handedness' of the triad of three vectors a,b,c. Note that it is odd under parity as well as under the interchange of any two vectors.

In order to obtain the onshell amplitude from the offshell one, one can utilize LSZ formula. By making different choices for the signs of the energies of the four external particles, the single master expression (120) determines the T-matrix for particle-particle scattering in both channels, as well as the T-matrix for particle antiparticle scattering in the adjoint channel; this observation also makes clear that these three T-matrices are related as usual by crossing symmetry. In the rest of this section we explicitly evaluate the T-matrix in each of these channels and comment on our results.

1.4.9 The S-matrix in the adjoint channel

In order to determine the scattering function T_T^B (particle - antiparticle scattering in the adjoint channel) we study the scattering process

$$P_i(p_1) + A^j(p_2) \to P_i(p_3) + A^j(p_4) \tag{121}$$

for $i \neq j$. It follows from the definitions (36) that the scattering amplitude for this process is precisely the function T_T^B.

The S-matrix for the scattering process (121) is evaluated by the exact onshell amplitude (120), once we make the identifications

$$p_1 = p+q, \quad p_2 = -(k+q), \quad p_3 = -p, \quad p_4 = k.$$

It follows that

$$s = -(p-k)^2, \quad t = -q^2, \quad u = -(p+q+k)^2$$

which implies

$$\begin{aligned} p_1^2 = p_2^2 = p_3^2 = -c_B^2, \quad p_1 \cdot p_2 = \frac{-s+2c_B^2}{2}, \\ p_1 \cdot p_3 = \frac{-t+2c_B^2}{2}, \quad p_2 \cdot p_3 = \frac{-u+2c_B^2}{2}. \end{aligned}$$

Note also that [27]

$$
\begin{aligned}
&|\epsilon_{\mu\nu\rho}q^\mu(p-k)^\nu(p+k)^\rho|^2 = 4|\epsilon_{\mu\nu\rho}(p+q)^\mu(k+q)^\nu p^\rho|^2 = 4|\epsilon_{\mu\nu\rho}p_1^\mu p_2^\nu p_3^\rho|^2 \\
=&-4\biggl(p_1^2p_2^2p_3^2 + 2(p_1\cdot p_2)(p_2\cdot p_3)(p_3\cdot p_1) \\
&\qquad - p_3^2(p_1\cdot p_2)^2 - p_2^2(p_1\cdot p_3)^2 - p_1^2(p_3\cdot p_2)^2\biggr) \\
=&-\left(16c_B^6 - 8c_B^4(s+t+u) + c_B^2(s+t+u)^2 - s\ t\ u\right) \\
=&s\ t\ u.
\end{aligned}
\tag{122}
$$

It follows that

$$
T_T^B(p_1,p_2,p_3,p_4,\lambda_B,b_4,c_B) = \frac{4i\pi}{k_B}E(p_1,p_2,p_3)\sqrt{\frac{tu}{s}} + \frac{1}{N}j(\sqrt{-t}), \tag{123}
$$

where the field renormalization factor is trivial in the leading order in $1/N$ expansion. In the center of mass frame, this S-matrix is given by

$$
T_T^B(s,\theta,\lambda_B,b_4,c_B) = \frac{4i\pi}{k_B}\frac{s-4c_B^2}{2\sqrt{s}}\sin(\theta) + \frac{1}{N}j\left(\sqrt{s-4c_B^2}\left|\sin\left(\frac{\theta}{2}\right)\right|\right). \tag{124}
$$

Notice that the scattering amplitude is completely regular at $\theta = 0$; in particular In the non-relativistic limit we find that the scattering function $h(\theta)$ is given by

$$
h_T^B(\theta) = 0 \tag{125}
$$

at finite b_4. If b_4 is taken to infinity first, on the other hand, in the non-relativistic limit we find

$$
h_T^B(\theta) = -i\pi. \tag{126}
$$

Notice that in neither case does $h(\theta)$ have a term proportional either to $\cot\left(\frac{\theta}{2}\right)$ or to $\delta(\theta)$) as anticipated in our discussion of the non-relativistic limit in subsection 1.2.6.

1.4.10 The S-matrix for particle- particle scattering

In order to determine the scattering function $T_{U_d}^B$ we study the scattering process

$$
P_i(p_1) + P_j(p_2) \to P_i(p_3) + P_j(p_4). \tag{127}
$$

[27] In our notation $\epsilon_{012} = -\epsilon^{012} = 1$.

It follows from the definitions (44) that the scattering amplitude for this process is precisely the function $T^B_{U_d}$, provided $i \neq j$.

The S-matrix for the scattering process (121) is evaluated by the exact onshell amplitude (120), once we make the identifications

$$p_1 = p + q, \quad p_2 = k, \quad p_3 = -p, \quad p_4 = -(k + q).$$

It follows that

$$s = -(p + q + k)^2, \quad t = -q^2, \quad u = -(p - k)^2$$

$$T^B_{U_d}(p_1, p_2, p_3, p_4, \lambda_B, b_4, c_B) = \frac{4i\pi}{k_B} E(p_1, p_2, p_3)\sqrt{\frac{ts}{u}} + \frac{1}{N} j(\sqrt{-t}) \tag{128}$$

where $E(p_1, p_2, p_3)$ was defined in (18). Notice that, upto the issues involving the sign E, $T^B_{U_d}$ is obtained from T^B_T by the interchange $s \leftrightarrow u$.

In the bosonic theory under study, $T^B_{U_e}$ is obtained from $T^B_{U_d}$ by the interchange $p_1 \leftrightarrow p_2$. This interchange flips the sign of E and also interchanges u and t, so we find

$$T^B_{U_e}(p_1, p_2, p_3, p_4, \lambda_B, b_4, c_B) = -\frac{4i\pi}{k_B} E(p_1, p_2, p_3)\sqrt{\frac{us}{t}} + \frac{1}{N} j(\sqrt{-u}). \tag{129}$$

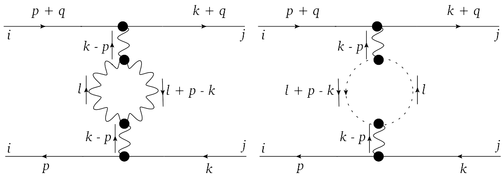
If the non-relativistic limit is taken at nonzero b_4 we have using (54)

$$\begin{aligned} h^B_{U_d}(\theta) &= -\frac{\pi}{k_B} \tan\left(\frac{\theta}{2}\right), \\ h^B_{U_e}(\theta) &= \frac{\pi}{k_B} \cot\left(\frac{\theta}{2}\right). \end{aligned} \tag{130}$$

If b_4 is first taken to infinity, on the other hand, we have

$$\begin{aligned} h^B_{U_d}(\theta) &= -\frac{\pi}{k_B} \tan\left(\frac{\theta}{2}\right) - i\pi, \\ h^B_{U_e}(\theta) &= \frac{\pi}{k_B} \cot\left(\frac{\theta}{2}\right) - i\pi, \end{aligned} \tag{131}$$

in good agreement with the predictions of subsection 1.2.6.



(a)

Figure 7: Gauge loop in gauge field propagator is cancelled by the ghost loop.

1.5 The onshell one loop amplitude in Landau Gauge

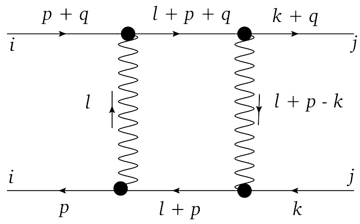
In this section we present a consistency check of (120) and (95), the main results of the previous section. Our check proceeds by independently evaluating the onshell 4 point function at one loop in the covariant Landau gauge. As we describe below, the results of our computation are in perfect agreement with the expansion of (120) and (95) to $\mathcal{O}(\lambda_B^2)$.

We believe that the check performed in this subsection has value for several reasons. First, the lightcone gauge employed in this chapter is nonstandard in several respects. It is not manifestly covariant. It leads to a gauge boson propagator that is singular when $p_- = 0$: as we have emphasized above, in order to make progress in our computation we were forced to simply postulate an $i\epsilon$ prescription that resolves this singularity in an appealing manner. And finally the offshell result of this computation appears, at first sight, to be ambiguous when continued onshell.

The computation we describe in this subsection, on the other hand, suffers from none of these deficiencies. It is manifestly covariant; it is an entirely standard computation, following rules that have been developed and repeatedly utilized over several decades, and it will turn out to have no confusing IR ambiguities. [28] For this reason, the match between our results of the previous subsection and those that we report in this subsection may be regarded as rather nontrivial evidence that we have correctly dealt with all the tricky aspects of the computation in the lightcone gauge.

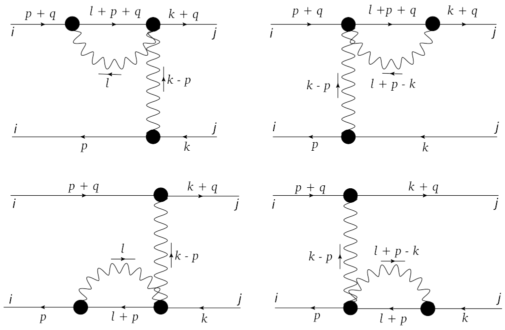
We now turn to a brief description of the Landau gauge computation, relegating most details to Appendix 1.9.5. For simplicity we work with the scalar theory in special case $b_4 = 0$. In the Landau Gauge, the gauge boson propagator receives two corrections at one loop: from a gauge boson loop and from a ghost loop. It is easily verified that these two diagrams cancel each other (see Fig 7). It is also easily seen that the ghosts make

[28]Of course the weakness of the Landau gauge is that, unlike in the lightcone gauge, it is very difficult to perform explicit computations in this gauge beyond low loop order, as the gauge condition does not remove all gauge boson self interactions.



(a)

Figure 8: The box diagram in Landau Gauge

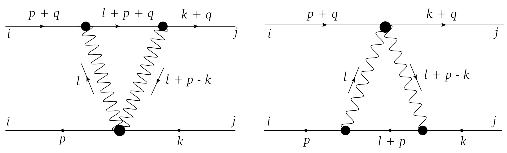


(a)

Figure 9: H diagrams in the lightcone gauge.

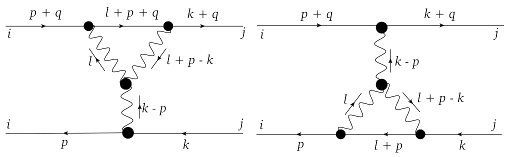
no appearance in any other diagram that contributes to one loop scattering of four gauge bosons. It follows that, at the one loop level, we may ignore both renormalizations of the gauge boson propagator as well as the ghosts: These two complications cancel each other out.

With this understanding it is easily verified that the one loop scattering amplitude of four scalar bosons receives contributions from six classes of diagrams, (see six figures, Figs. 8∼13). These are the box diagrams of Fig. 8, the h diagrams of Fig. 9, the V diagrams of Fig. 10, the Y diagrams of Fig. 11, the Eye diagram of Fig. 12, and the Lollipop diagram of Fig. 13. In order to evaluate the one loop contribution to four scalar scattering, we need to evaluate the sum of these six classes of diagrams. It is well known,



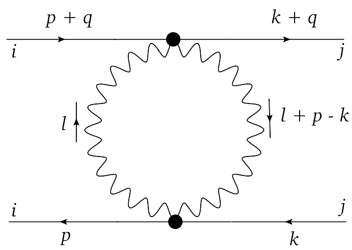
(a)

Figure 10: V diagrams in the Landau Gauge.



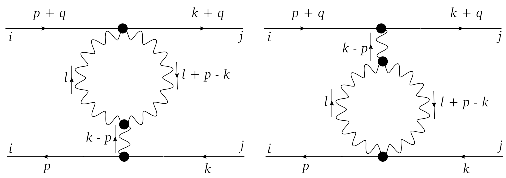
(a)

Figure 11: Y diagram in the Landau gauge.



(a)

Figure 12: Eye diagram in the Landau gauge.



(a)

Figure 13: Lollipop diagram in the Landau Gauge.

however, that in the study of planar diagrams there is a canonical way to sum the integrands of these diagrams before performing the integral. We choose a uniform definition of the loop momentum across all the six sets of graphs; the loop momentum l is the momentum that flows clockwise between the external line with momentum p and the external line with momentum $p+q$ (see Fig. 8). Adopting this definition, we then evaluate the integrand for each class of diagrams, and sum the integrands.

It turns out that the process of summing integrands leads to several cancellations and simplifications. In order to see the cancellations between integrands, it is important that each integrand be expressed in a canonical form. There is, of course, a standard way to achieve this. It is a well known result that an arbitrary one loop integrand in d dimensions may be reduced, under the integral sign [29] to a linear sum over scalar integrals [30] with at most d propagators. The coefficients in this decomposition are rational functions of the external momenta. There also exists a rather simple algorithmic procedure for decomposing an arbitrary integrand into this canonical form. Finally the scalar integrals are not all independent. The canonical form of the integrand is obtained by decomposing the integrand into a linear combination of linearly independent scalar integrands.

Implementing this procedure (see Appendix 1.9.5 for several details) we find that the full one loop integrand for 4 scalar boson scattering turns out to be given by the remarkably simple expression

$$\begin{aligned} I_{full} = 4\pi^2\lambda_B^2\Bigg(&-\frac{2}{c_B^2+(l+p)^2} - \frac{2}{(l+p-k)^2} \\ &- \frac{8k\cdot q}{\left(c_B^2+(l+p)^2\right)\left(c_B^2+(p+q+l)^2\right)}\Bigg). \end{aligned} \tag{132}$$

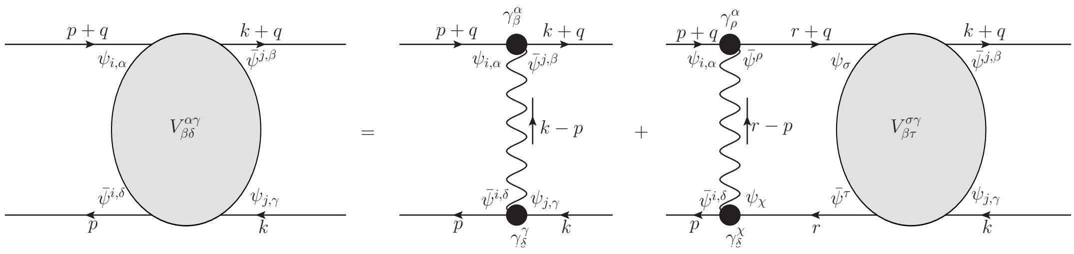
In the dimensional regulation scheme that we employ, the integral of the first term in (132) is $4\pi^2\lambda_B^2 \times \frac{c_B}{2\pi}$. The integral of the second term simply vanishes. The integral of the third term is $32\pi^2(k\cdot q)\lambda_B^2 H(q)$ where $H(q)$, the one loop amplitude for four boson scattering, was defined in (109). It follows that the full one loop onshell scattering amplitude is given by

$$\int \frac{d^3l}{(2\pi)^3} I_{full} = V_{\text{one loop}} = 2\pi c_B\lambda_B^2 + 32\pi^2(k\cdot q)\lambda_B^2 H(q) \tag{133}$$

in perfect agreement with (108) at $b_4 = 0$.

[29]i.e. upto terms that integrate to zero.

[30]A scalar integral, by definition, is the loop integral over a product of propagators in the loop, but with numerator unity.



(a)

Figure 14: A diagrammatic depiction of the integral equation obeyed by offshell four point scattering amplitudes in the fermionic theory. The blob here represents the all orders scattering amplitude.

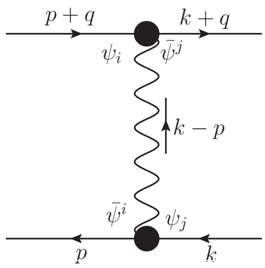
We end this brief subsection with two further comments. We first note that the one loop amplitude in the Landau gauge was manifestly infrared safe. While integrands that would have given rise to infrared divergences (associated with the exchange of arbitrarily soft gluons in loop) appear at intermediate stages in the computation, they all cancel already at the level of the integrand (i.e. before performing any integrals). This is the analogue of the slightly more subtle cancellation of IR divergences in lightcone gauge mentioned above and described in more detail in Appendix 1.9.4.

The second comment is that the derivation integrand reported in (132) uses a reduction formula that is valid only at generic values of external momenta. Our derivation of this formula fails, for instance, when two of the external momenta are collinear. In more familiar quantum field theories this caveat would be of little consequence; the analyticity of the amplitude as a function of external momenta would guarantee that the result applied at all values of the momenta. As we will see below, however, this amplitudes in Chern-Simons theories sometimes appear to have non analytic singularities, so the caveat spelt out in this paragraph may turn out to be more than a pedantic technicality.

1.6 Scattering in the fermionic theory

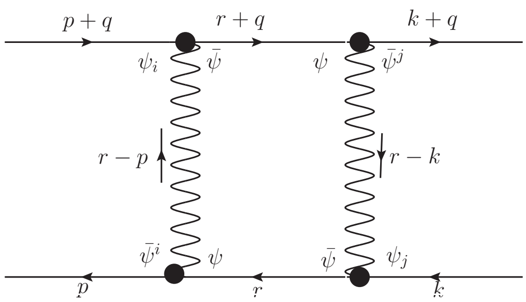
In this section we compute the four point scattering amplitude in the theory of fundamental fermions coupled to Chern-Simons theory. As in the bosonic theory, we integrate out the gauge boson to obtain an offshell effective four fermi term in the quantum effective action for our theory, given by

$$\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}V^{\alpha\gamma}_{\beta\delta}(p,k,q)\psi_{i,\alpha}(p+q)\bar{\psi}^{j,\beta}(-(k+q))\bar{\psi}^{i,\delta}(-p)\psi_{j,\gamma}(k). \tag{134}$$



(a)

Figure 15: Fermionic tree level diagram



(a)

Figure 16: Fermionic 1 loop diagram

We then take an appropriate onshell limit to evaluate the S-matrix.

1.6.1 The offshell four point amplitude

As in the case of the bosonic theory, the offshell four point amplitude $V^{\alpha\gamma}_{\beta\delta}(p,k,q)$ obeys a closed Schwinger-Dyson equation. As for the bosonic theory, we work with the special case $q^{\pm}=0$. As above we first set up this Schwinger-Dyson equation for the Lorentzian theory, but find it more convenient, technically, to work with the Euclidean rotated amplitude. The Euclidean rotated amplitude is defined in a manner very similar to the bosonic theory (see below for a few more details), and may be shown to obey the Schwinger-Dyson equation

$$\begin{aligned} V^{\alpha\gamma}_{\beta\delta}(p,k,q) &= \frac{1}{2}(\gamma^{\mu})^{\alpha}_{\beta}G_{\mu\nu}(p-k)(\gamma^{\nu})^{\gamma}_{\delta} \\ &+ \frac{1}{2}\int \frac{d^3r}{(2\pi^3)}[\gamma^{\mu}G(r+q)]^{\alpha}_{\sigma}V^{\sigma\gamma}_{\beta\tau}(r,k,q)[G(r)\gamma^{\nu}]^{\tau}_{\delta}G_{\mu\nu}(p-r). \end{aligned} \tag{135}$$

Here $G(p)_{\alpha\sigma}$ is the exact fermionic propagators determined in (9) (see also [2]), while $G_{\mu\nu}$ is the gauge boson propagator defined by

$$\langle A^a_{\mu}(-p)A^b_{\nu}(q) \rightarrow= (2\pi)^3\delta^3(p-q)G_{\mu\nu}(q) \tag{136}$$

where $A_{\mu}=A^aT^a$ and we work with generators normalized so that

$$\sum_a (T^a)^i_j(T^a)^k_l = \frac{1}{2}\delta^i_l\delta^k_j. \tag{137}$$

And here γ^{μ} compose the Euclidean Clifford algebra,

$$\{\gamma^{\mu},\gamma^{\nu}\}=2\delta^{\mu\nu}, \quad [\gamma^{\mu},\gamma^{\nu}]=2i\epsilon^{\mu\nu\rho}\gamma_{\rho}, \qquad (\epsilon^{103}=\epsilon_{103}=1).$$

In the lightcone gauge in which we work the only nonzero components of $G_{\mu\nu}$ are

$$G_{+3}(p)=-G_{3+}(p)=\frac{4\pi\ i}{\kappa\ p^{+}}. \tag{138}$$

Now noting the fact that only non zero component is $G_{+3}(p)=-G_{3+}(p)$ and using rearrangement with $\gamma^{+}=\frac{i\gamma^0+\gamma^1}{\sqrt{2}}$,

$$(\gamma^{+})^{\alpha}_{\beta}(\gamma^{3})^{\gamma}_{\delta}-(\gamma^{3})^{\alpha}_{\beta}(\gamma^{+})^{\gamma}_{\delta} = -\left(\delta^{\gamma}_{\beta}(\gamma^{+})^{\alpha}_{\delta}-(\gamma^{+})^{\gamma}_{\beta}\delta^{\alpha}_{\delta}\right), \tag{139}$$

as well as

$$\gamma^+ X \gamma^3 - \gamma^3 X \gamma^+ = -2(X_I \gamma^+ - X_- I), \tag{140}$$

(here $X = X_i\gamma^i + X_I I$ is an arbitrary $2 \to 2$ matrix), we conclude that in the α, δ indices of R.H.S of the Eq.(135) - and therefore the LHS , and so V takes the form

$$V^{\alpha\gamma}_{\beta\delta}(p,k,q) = g(p,k,q)\delta^\alpha_\delta \delta^\gamma_\beta + f(p,k,q)(\gamma^+)^\alpha_\delta \delta^\gamma_\beta + g_1(p,k,q)\delta^\alpha_\delta (\gamma^+)^\gamma_\beta + f_1(p,k,q)(\gamma^+)^\alpha_\delta (\gamma^+)^\gamma_\beta. \tag{141}$$

Plugging this form V into (135) yields a set of four integral equations for the four component functions in (141). We have succeeded in finding the exact solution to these equations. We present the derivation of our solution in Appendix 1.9.6. The final result for this offshell amplitude is extremely complicated. The result, which takes multiple pages to write, is given in (300), (307) and (308) of the Appendix. We see no benefit in reproducing this extremely complicated final result in the main text.

1.6.2 The onshell limit

As we have seen above, the offshell four point function defined in (134) is quite a complicated object. In this section we will argue that the onshell S-matrix is, however, rather simple.

In order to study the S-matrix it is first convenient to continue our result for V in (134) to Minkowski space. This is achieved by making the substitution

$$p^0 \to -ip^0, \quad k^0 \to -ik^0, \quad \gamma^0 \to -i\gamma^0,$$

on the Euclidean result of the previous subsection. This substitution yields the four fermi term in the effective action (85) in Lorentzian space.

In order to convert this four point vertex to a scattering amplitude, we must now go onshell. We now pause to carefully explain how this is achieved.

In free field theory (i.e. in the absence of the four point function interaction) the fermion field operators may be expanded in creation and annihilation modes in the standard fashion

$$
\begin{aligned}
\psi(x) &= \int \frac{d^3p}{(2\pi)^3}\psi(p)e^{ip\cdot x} \\
&= \int \frac{d^2p}{(2\pi)^2}\frac{1}{\sqrt{2\ E_p}}\left(u(\vec{p})a_{\vec{p}}e^{ip\cdot x} + v(\vec{p})b^\dagger_{\vec{p}}e^{-ip\cdot x}\right), \\
\bar{\psi}(x) &= \int \frac{d^3p}{(2\pi)^3}\bar{\psi}(p)e^{ip\cdot x} \\
&= \int \frac{d^2p}{(2\pi)^2}\frac{1}{\sqrt{2\ E_p}}\left(\bar{u}(\vec{p})a^\dagger_{\vec{p}}e^{-ip\cdot x} + \bar{v}(\vec{p})b_{\vec{p}}e^{ip\cdot x}\right),
\end{aligned}
\tag{142}
$$

where $p^0 = \omega = \sqrt{c_F^2 + p_1^2 + p_3^2}$. As always we use the mostly positive convention, so $e^{ip.x}$ has negative 'frequency' in time, while $e^{-ip.x}$ has positive frequency in time. As is usual, the coefficients of negative frequency wave functions are annihilation operators, while the coefficients of positive frequency wave functions are creation operators. We refer to a and $a^\dagger$ as particle destruction and creation operators, while b and $b^\dagger$ are antiparticle destruction and creation operators. The wave functions $u(p)e^{ip.x}$ and $v(p)e^{-ip.x}$ are solutions to the Dirac equation

$$(i(p_\mu + \Sigma_\mu)\gamma^\mu + \Sigma_I)\,\psi(p) = 0$$

and, as usual, $\bar{\psi} = i\psi^\dagger\gamma^0$, where Σ is defined in (10). For later convenience, we introduce the following notation,

$$\Sigma_I(p_s) = f(p_s)p_s, \quad \Sigma_+ = g(p_s)p_s.$$

The Dirac equation uniquely determines $u(p)$ and $v(p)$ upto multiplicative constants. We fix the normalization ambiguity by demanding

$$\bar{u}(\vec{p})u(\vec{p}) = 2f(p_s)p_s, \quad \bar{v}(\vec{p})v(\vec{p}) = -2f(p_s)p_s. \tag{143}$$

[31] These requirements leave the phase of the functions $u(p)$ and $v(p)$ undetermined: we will make an arbitrary choice for this phase below.

We will find it useful to have explicit expressions for u and v. In order to obtain these expressions, it is useful to fix a particular convention for γ matrices. In Euclidean space we

[31] This normalization convention may be justified by performing a double analytic continuation, so that x^0 becomes a spatial direction and x^3 a temporal direction. Once this is done, the free Lagrangian is of first order in time, and so may be canonically quantized in the usual manner. The normalization described above are chosen to ensure that the usual anticommutation relations for the field operators ψ translate to standard anticommutation relations for the creation and annihilation operators $a, a^\dagger, b, b^\dagger$.

make the choice $\gamma^+ = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}$, $\gamma^- = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}$ and $\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This choice determines the Lorentzian γ matrices to be

$$\begin{aligned} \gamma^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^+ = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \\ \gamma^- &= \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}, \qquad \gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{144}$$

The quadratic Dirac Lagrangian consequently takes the explicit form

$$\int \frac{d^3p}{(2\pi)^3} \bar{\psi}(-p) \begin{pmatrix} ip_2 + f(p_s)p_s & i\sqrt{2}p_+(1+g(p_s)) \\ i\sqrt{2}p_- & -ip_2 + f(p_s)p_s \end{pmatrix} \psi(p). \tag{145}$$

The equations of motion for u and $\bar{u}$ are

$$\begin{aligned} &\begin{pmatrix} ip_2 + f(p_s)p_s & -i(E_{\vec{p}} - p_1)(1+g(p_s)) \\ i(E_{\vec{p}} + p_1) & -ip_2 + f(p_s)p_s \end{pmatrix} u(\vec{p}) = 0, \\ &\bar{u}(\vec{p}) \begin{pmatrix} ip_2 + f(p_s)p_s & -i(E_{\vec{p}} - p_1)(1+g(p_s)) \\ i(E_{\vec{p}} + p_1) & -ip_2 + f(p_s)p_s \end{pmatrix} = 0, \end{aligned} \tag{146}$$

while those for v and $\bar{v}$ are

$$\begin{aligned} &\begin{pmatrix} ip_2 - f(p_s)p_s & -i(E_{\vec{p}} - p_1)(1+g(p_s)) \\ i(E_{\vec{p}} + p_1) & -ip_2 - f(p_s)p_s \end{pmatrix} v(\vec{p}) = 0, \\ &\bar{v}(\vec{p}) \begin{pmatrix} ip_2 - f(p_s)p_s & -i(E_{\vec{p}} - p_1)(1+g(p_s)) \\ i(E_{\vec{p}} + p_1) & -ip_2 - f(p_s)p_s \end{pmatrix} = 0. \end{aligned} \tag{147}$$

Note that, (146) and (147) admits solution only when, determinant of the matrix appearing in those equations are zero. This gives onshell condition $p^2 + c_F^2 = 0$, equivalently

$$p_2^2 + f(p_s)^2 p_s^2 - \left(E_{\vec{p}}^2 - p_1^2\right)(1 + g(p_s)) = 0.$$

Solving these equations subject to the normalization conventions described above (plus an arbitrary choice of phase) we find

$$\begin{aligned}
u(\vec{p}) &= \frac{1}{\sqrt{E_{\vec{p}} + p_1}} \begin{pmatrix} ip_2 - f(p_s)p_s \\ i(E_{\vec{p}} + p_1) \end{pmatrix}, \\
\bar{u}(\vec{p}) &= \frac{1}{\sqrt{E_{\vec{p}} + p_1}} \begin{pmatrix} -(E_{\vec{p}} + p_1) & p_2 - if(p_s)p_s \end{pmatrix},
\end{aligned} \tag{148}$$

and

$$\begin{aligned}
v(\vec{p}) &= \frac{1}{\sqrt{E_{\vec{p}} + p_1}} \begin{pmatrix} ip_2 + f(p_s)p_s \\ i(E_{\vec{p}} + p_1) \end{pmatrix}, \\
\bar{v}(\vec{p}) &= \frac{1}{\sqrt{E_{\vec{p}} + p_1}} \begin{pmatrix} -(E_{\vec{p}} + p_1) & p_2 + if(p_s)p_s \end{pmatrix}.
\end{aligned} \tag{149}$$

1.6.3 S-matrices

With explicit expressions for $u(\vec{p})$ and $v(\vec{p})$ in hand, it might seem like an easy task to take the onshell limit of the ofshell 4 Fermi correlators. Infact that is not the case. As in the bosonic theory the onshell limit of these correlators is apparently ambiguous, and must be taken very carefully. The reader will recall that we discussed this issue at great detail in the bosonic theory, came to the conclusion that the correct final prescription is simply to first set $|\vec{k}|$ to $|\vec{p}|$ before taking either of these momenta individuallyonshell. We adopt a similar prescription for the bosonic theories. We first replace the quantities E_p and E_k that appear in our solutions for $u(\vec{p})$ and $v(\vec{p})$ with $\pm p^0$ and $\pm k^0$ respectively. We then evaluate the offshell amplitude with $|\vec{k}| = |\vec{p}|$ and only then take the momenta to individually be onshell. This process yields unambiguous answers which we present below. As in the bosonic case, it should be possible to justify this order of limits with a careful evaluation of the amplitude directly in Minkoski space keeping careful track of the factors of $i\epsilon$ but we have not persued this thought.

S-matrix for adjoint exchange in particle - antiparticle scattering As we have explained above, the offshell four fermion scattering amplitude is extremely complicated. Quite remarkably, however, the onshell limit displays remarkable simplifications. In the

T-channel the onshell S-matrix is given by

$$
\begin{aligned}
T_T^F &= V^{\alpha\gamma}_{\beta\delta}(p,k,q) u_{i,\alpha}(p+q)\bar{v}^{j,\beta}(-(k+q))\bar{u}^{i,\delta}(p)v_{j,\gamma}(-k) \\
&= -\frac{4\pi i}{k_F} q_3 \frac{p_- + k_-}{p_- - k_-} \\
&- \frac{4i\pi}{k_F} q_3 \ \frac{(q_3 - 2i\ \mathrm{sgn}(m_F)|c_F|)\, e^{2i\lambda_F \tan^{-1}\left(\frac{2|c_F|}{q_3}\right)} - e^{i\pi \mathrm{sgn}(q_3)\lambda_F}\,(q_3 + 2i\ \mathrm{sgn}(m_F)|c_F|)}{e^{i\pi \mathrm{sgn}(q_3)\lambda_F}\,(q_3 + 2i\ \mathrm{sgn}(m_F)|c_F|) + (q_3 - 2i\ \mathrm{sgn}(m_F)|c_F|)\, e^{2i\lambda_F \tan^{-1}\left(\frac{2|c_F|}{q_3}\right)}} \\
&= -E(p_1,p_2,p_3)\frac{4i\pi}{k_F}\sqrt{\frac{u\ t}{s}} \\
&- \frac{4i\pi}{k_F}\sqrt{-t}\ \frac{\left(\sqrt{-t} - 2i\ \mathrm{sgn}(m_F)|c_F|\right) e^{2i\lambda_F \tan^{-1}\left(\frac{2|c_F|}{\sqrt{-t}}\right)} - e^{i\pi\lambda_F}\left(\sqrt{-t} + 2i\ \mathrm{sgn}(m_F)|c_F|\right)}{e^{i\pi\lambda_F}\left(\sqrt{-t} + 2i\ \mathrm{sgn}(m_F)|c_F|\right) + \left(\sqrt{-t} - 2i\ \mathrm{sgn}(m_F)|c_F|\right) e^{2i\lambda_F \tan^{-1}\left(\frac{2|c_F|}{\sqrt{-t}}\right)}} \\
&= -E(p_1,p_2,p_3)\frac{4i\pi}{k_F}\sqrt{\frac{u\ t}{s}} \\
&- \frac{4i\pi}{k_F}\sqrt{-t}\ \frac{\left(\sqrt{-t} - 2i\ \mathrm{sgn}(m_F)|c_F|\right) e^{2i\lambda_F \tan^{-1}\left(\frac{2|c_F|}{\sqrt{-t}}\right)} - e^{i\pi\lambda_F}\left(\sqrt{-t} + 2i\ \mathrm{sgn}(m_F)|c_F|\right)}{e^{i\pi\lambda_F}\left(\sqrt{-t} + 2i\ \mathrm{sgn}(m_F)|c_F|\right) + \left(\sqrt{-t} - 2i\ \mathrm{sgn}(m_F)|c_F|\right) e^{2i\lambda_F \tan^{-1}\left(\frac{2|c_F|}{\sqrt{-t}}\right)}} \\
&= -E(p_1,p_2,p_3)\frac{4i\pi}{k_F}\sqrt{\frac{u\ t}{s}} + \frac{4\ i\pi}{k_F}\sqrt{-t}\ \frac{e^{i\pi(\lambda_F - \mathrm{sgn}(m_F))} + e^{2i(\lambda_F - \mathrm{sgn}(m_F))\tan^{-1}\left(\frac{2|c_F|}{\sqrt{-t}}\right)}}{e^{i\pi(\lambda_F - \mathrm{sgn}(m_F))} - e^{2i(\lambda_F - \mathrm{sgn}(m_F))\tan^{-1}\left(\frac{2|c_F|}{\sqrt{-t}}\right)}} \\
&= -E(p_1,p_2,p_3)\frac{4i\pi}{k_F}\sqrt{\frac{u\ t}{s}} + \frac{4\ i\pi}{k_F}\sqrt{-t}\ \frac{1 + e^{-2i(\lambda_F - \mathrm{sgn}(m_F))\tan^{-1}\left(\frac{\sqrt{-t}}{2|c_F|}\right)}}{1 - e^{-2i(\lambda_F - \mathrm{sgn}(m_F))\tan^{-1}\left(\frac{\sqrt{-t}}{2|c_F|}\right)}}.
\end{aligned}
\tag{150}
$$

As we have emphasized above, we have obtained this result only after taking the onshell limit in a particular manner. In particular, in the solution in (148),(149) we treated E_p as a free symbol to start with; we set $p_s = k_s$ first and then set $E_p^2 = \overrightarrow{p}^2 + c_F^2$.

S-matrix for particle - particle scattering In the U-channel

$$
\begin{aligned}
T^F_{U_d} &= V^{\alpha\gamma}_{\beta\delta}(p,k,q)u_{i,\alpha}(p+q)\bar{u}^{j,\beta}(k+q)\bar{u}^{i,\delta}(p)u_{j,\gamma}(k) \\
&= \frac{4\pi i}{k_F}q_3\frac{p_- + k_-}{p_- - k_-} \\
&+ \frac{4i\pi}{k_F}q_3 \; \frac{(q_3 - 2i\ \mathrm{sgn}(m_F)|c_F|)\, e^{2i\lambda_F \tan^{-1}\left(\frac{2|c_F|}{q_3}\right)} - e^{i\pi \mathrm{sgn}(q_3)\lambda_F}\,(q_3 + 2i\ \mathrm{sgn}(m_F)|c_F|)}{e^{i\pi \mathrm{sgn}(q_3)\lambda_F}\,(q_3 + 2i\ \mathrm{sgn}(m_F)|c_F|) + (q_3 - 2i\ \mathrm{sgn}(m_F)|c_F|)\, e^{2i\lambda_F \tan^{-1}\left(\frac{2|c_F|}{q_3}\right)}} \\
&= -\Bigg(E(p_1,p_2,p_3)\frac{4i\pi}{k_B}\sqrt{\frac{s\ t}{u}} \\
&- \frac{4i\pi}{k_F}\sqrt{-t}\; \frac{\left(\sqrt{-t} - 2i\ \mathrm{sgn}(m_F)|c_F|\right) e^{2i\lambda_F \tan^{-1}\left(\frac{2|c_F|}{\sqrt{-t}}\right)} - e^{i\pi\lambda_F}\left(\sqrt{-t} + 2i\ \mathrm{sgn}(m_F)|c_F|\right)}{e^{i\pi\lambda_F}\left(\sqrt{-t} + 2i\ \mathrm{sgn}(m_F)|c_F|\right) + \left(\sqrt{-t} - 2i\ \mathrm{sgn}(m_F)|c_F|\right) e^{2i\lambda_F \tan^{-1}\left(\frac{2|c_F|}{\sqrt{-t}}\right)}}\Bigg) \\
&= -\left(- E(p_1,p_2,p_3)\frac{4i\pi}{k_F}\sqrt{\frac{s\ t}{u}} + \frac{4\ i\pi}{k_F}\sqrt{-t}\; \frac{e^{i\pi(\lambda_F - \mathrm{sgn}(m_F))} + e^{2i(\lambda_F - \mathrm{sgn}(m_F))\tan^{-1}\left(\frac{2|c_F|}{\sqrt{-t}}\right)}}{e^{i\pi(\lambda_F - \mathrm{sgn}(m_F))} - e^{2i(\lambda_F - \mathrm{sgn}(m_F))\tan^{-1}\left(\frac{2|c_F|}{\sqrt{-t}}\right)}} \right) \\
&= -\left(- E(p_1,p_2,p_3)\frac{4i\pi}{k_F}\sqrt{\frac{s\ t}{u}} + \frac{4\ i\pi}{k_F}\sqrt{-t}\; \frac{1 + e^{-2i(\lambda_F - \mathrm{sgn}(m_F))\tan^{-1}\left(\frac{\sqrt{-t}}{2|c_F|}\right)}}{1 - e^{-2i(\lambda_F - \mathrm{sgn}(m_F))\tan^{-1}\left(\frac{\sqrt{-t}}{2|c_F|}\right)}} \right).
\end{aligned}
\tag{151}
$$

As in the previous subsubsection, we have obtained this resultafter taking the onshell limit in a particular manner. In particular, in the solution in (148),(149) we treated E_p as a free symbol to start with; we set $p_s = k_s$ first and then set $E_p^2 = \overrightarrow{p}^2 + c_F^2$.

1.7 Scattering in the identity channel and crossing symmetry

1.7.1 Crossing symmetry

It is sometimes asserted that the S-matrix for particle - antiparticle scattering, in any quantum field theory, may be obtained from the S-matrix for particle - particle scattering. This claim goes by the name of crossing symmetry. In the context of the $2 \to 2$ scattering studied in this chapter, the formulae asserted with the claim are (we work with the bosonic theory for definiteness)

$$
T_S(s,t,u) = N T_{U_d}(t,u,s), \qquad T_T(s,t,u) = T_{U_d}(u,t,s). \tag{152}
$$

These equations assert that the formulae for particle - antiparticle scattering may be read off from the analytic continuation of the physical particle - particle scattering amplitude [32] .

In the case of an ungauged field theory - or in the case of the scattering of gauge invariant particles in a gauge theory, there is a rather straightforward intuitive argument for crossing symmetry of amplitudes. The LSZ formula relates S-matrices to onshell limits of well-defined offshell correlators. The offshell correlators are expected to be analytic functions of their insertion positions. The on shell limit of these correlators is the 'master function' referred to in the footnote above which plausibly inherits analytic properties from those of the underlying correlators.

This intuitive argument does not work for the scattering of non gauge singlet particles in a gauge theory, as the relevant scattering amplitudes cannot be obtained from the onshell limit of an offshell correlator (the putative offshell correlators are not gauge invariant and so are ill defined).

While the argument for crossing symmetry presented in this subsection does not apply to, for instance, the scattering of gluons in $\mathcal{N} = 4$ Yang Mills theory, the final result (i.e. that scattering amplitudes obey crossing symmetry) is widely expected to hold true for these amlitudes, at least with a suitable definition of the scattering amplitudes (a definition is needed to deal with IR ambiguities having to do with soft gluons and other soft particles). In this context we expect that the failure of the argument outlined in this subsection is just a technicality; other arguments (perhaps based on diagrammatics) guarantee the final result.

As in the previous paragraph, we are also interested in the scattering of non singlet excitations. Unlike the case of gluonic scattering in $\mathcal{N} = 4$ Yang Mills, however, we will argue below that the failure of the argument for crossing symmetry is more than a technicality. The crossing relations are *actually* modified in our theories. We suspect that the underlying reason for the modification is that the Chern-Simons action, which controls the dynamics of our gauge fields, effectively turns our scattering particles into anyons. Apparently, the usual crossing relations are true for the scattering of bosons and fermions, but are modified in the scattering of anyons.

[32] Analytic continuation is needed because physical scattering processes in the different channels utilize non overlapping domains of the (allegedly) single analytic 'master' scattering formula. Consider, for instance, the first of (152). Physical particle- particle scattering process are captured by the function $T_{U_d}(x, y, z)$ for $y, z < 0$; given that $x + y + z = 4m^2$, this implies $x > 4m^2$. On the other hand on the RHS of the first of (152) we need the same function at $x, y < 0$ and so $z > 4m^2$. It is clear that there is no overlap between these different domains.

1.7.2 A conjecture for the S-matrix in the singlet channel

As we have explained above, a naive application of crossing symmetry predicts that, the S-channel scattering amplitude is given by $T_S^B(s,t,u) = NT_{U_d}^B(t,u,s)$. We have performed the analytic continuations needed to make sense of this formula in subsection 1.4.4. Utilizing the results of that subsection, the naive prediction of crossing symmetry is

$$\begin{aligned}
T_S^{trial} &= (\pi\lambda_B)\, 4i\sqrt{s}E(p_1,p_2,p_3)\sqrt{\frac{u}{t}} + j^M(\sqrt{s}) \\
&= (\pi\lambda_B)\, 4\sqrt{s}\left(i\ E(p_1,p_2,p_3)\sqrt{\frac{u}{t}} + \left(\frac{\left(4\pi\lambda_B\sqrt{s}+\widetilde{b}_4\right) + \left(-4\pi\lambda_B\sqrt{s}+\widetilde{b}_4\right)e^{8\pi\lambda_B\sqrt{s}H^M(\sqrt{s})}}{\left(4\pi\lambda_B\sqrt{s}+\widetilde{b}_4\right) - \left(-4\pi\lambda_B\sqrt{s}+\widetilde{b}_4\right)e^{8\pi\lambda_B\sqrt{s}H^M(\sqrt{s})}}\right)\right) \\
&= (\pi\lambda_B)\, 4\sqrt{s}\left(i\ E(p_1,p_2,p_3)\sqrt{\frac{u}{t}} + \left(\frac{\left(4\pi\lambda_B\sqrt{s}+\widetilde{b}_4\right) + e^{i\pi\lambda_B}\left(-4\pi\lambda_B\sqrt{s}+\widetilde{b}_4\right)\left(\frac{\frac{1}{2}+\frac{c_B}{\sqrt{s}}}{\frac{1}{2}-\frac{c_B}{\sqrt{s}}}\right)^{\lambda_B}}{\left(4\pi\lambda_B\sqrt{s}+\widetilde{b}_4\right) - e^{i\pi\lambda_B}\left(-4\pi\lambda_B\sqrt{s}+\widetilde{b}_4\right)\left(\frac{\frac{1}{2}+\frac{c_B}{\sqrt{s}}}{\frac{1}{2}-\frac{c_B}{\sqrt{s}}}\right)^{\lambda_B}}\right)\right),
\end{aligned} \tag{153}$$

(in the last line we have specialized to the physical domain $s \geq 4c_B^2$).

The function T_S^{trial} cannot be the true scattering matrix in the S-channel for three related reasons.

- T_S^{trial} does not include the last term in (53); a term delta function localized on forward scattering with a coefficient proportional to $(\cos(\pi\lambda_B) - 1)$. This term is certainly present in the scattering amplitude at least in the non-relativistic limit.
- Even ignoring the term localized at forward scattering, the non-relativistic limit of T_S^{trial} does not agree with (53).
- T_S^{trial} does not obey the unitarity relation (64).

In the rest of this subsection we will demonstrate that all these problems are simultaneously cured if we conjecture that the scattering matrix in the S-channel is given by a rescaled T_S^{trial} plus a contact term added by hand. We conjecture that the bosonic scattering matrix in the S-channel is given by

$$T_S^B = \frac{\sin(\pi\lambda_B)}{\pi\lambda_B}T_S^{trial} - i(\cos(\pi\lambda_B) - 1)I(p_1,p_2,p_3,p_4) \tag{154}$$

(see subsection 1.2.3 for a definition of the Identity matrix). In subsection 1.7.4 we will present a tentative justification for the modification of the usual rules of crossing symmetry implicit in

(81). In the rest of this subsection we will demonstrate that the conjectured scattering amplitude T_S^B passes various consistency checks.

In the center of mass frame our conjectured scattering amplitude (154) takes the form (65) with

$$
\begin{aligned}
&H(\sqrt{s}) = 4\sqrt{s}\sin(\pi\lambda_B),\\
&W_1(\sqrt{s}) = 4\sqrt{s}\sin(\pi\lambda_B)G,\\
&W_2(\sqrt{s}) = 8\pi\sqrt{s}\left(\cos(\pi\lambda_B)-1\right),\\
&G = \left(\frac{\left(4\pi\lambda_B\sqrt{s}+\widetilde{b}_4\right)+e^{i\pi\lambda_B}\left(-4\pi\lambda_B\sqrt{s}+\widetilde{b}_4\right)\left(\frac{\frac{1}{2}+\frac{c_B}{\sqrt{s}}}{\frac{1}{2}-\frac{c_B}{\sqrt{s}}}\right)^{\lambda_B}}{\left(4\pi\lambda_B\sqrt{s}+\widetilde{b}_4\right)-e^{i\pi\lambda_B}\left(-4\pi\lambda_B\sqrt{s}+\widetilde{b}_4\right)\left(\frac{\frac{1}{2}+\frac{c_B}{\sqrt{s}}}{\frac{1}{2}-\frac{c_B}{\sqrt{s}}}\right)^{\lambda_B}}\right).
\end{aligned}
\tag{155}
$$

Let us first demonstrate that our conjectured expressions (155) have the correct non-relativistic limit. The functions H and W_2 in (66) are independent of the energy s and already agree perfectly with the same functions in (66). Moreover

$$
\lim_{\sqrt{s}\to 2c_B} G = -\mathrm{sgn}(\lambda_B) \tag{156}
$$

it follows that

$$
\lim_{\sqrt{s}\to 2c_B} W_1(\sqrt{s}) = -4\sqrt{s}|\sin(\pi\lambda_B)| \tag{157}
$$

in agreement with (66). We conclude that our conjectured scattering amplitude (155) reduces precisely to the expected Aharonov-Bohm scattering amplitude in the non-relativistic limit.

We next demonstrate that our conjecture for the S-channel S-matrix obeys the constraints of unitarity, i.e. that (155) obeys the equations (70). As we have explained in subsection 1.2.7, the fact that H and W_2 in (155) agree with the corresponding functions in (66) immediately implies that the first two equations in (70) are obeyed. We will now demonstrate that the functions in (155) also obey the third equation in (70). [33]

[33] A point here requires explanation. In our study of unitarity in section 1.2.7, the function H multiplies an S-matrix proportional to $\mathrm{Pv}\cot\frac{\theta}{2}$. Feynmam diagrams produce a scattering amplitude in which the function H multiplies $\frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}-i\epsilon}$. These two expressions clearly coincide at nonzero θ; interestingly enough they also coincide at $\theta=0$. Indeed it is not difficult to demonstrate that

$$
\mathrm{Pv}\frac{1}{\theta} = \frac{\theta}{\theta^2 - i\epsilon}.
$$

The key point here is that the second expression above has two poles; one of these lies above the real θ axis while the second one lies below it. The residue of each of these two poles is precisely half what it would have been for the simple pole $\frac{1}{\theta}$, demonstrating that the expression on the RHS is identical to the principal value.

The third equation in (70) may be rewritten, in terms of the function G, as

$$G - G^* = (1 - \cos(\pi\lambda_B))(G - G^*) - i\sin(\pi\lambda_B)(1 - GG^*)$$

This equation is holds if

$$G - G^* = -i\tan(\pi\lambda_B)(1 - GG^*). \tag{158}$$

Now

$$G = \frac{1 + e^{i\pi\lambda_B}y}{1 - e^{i\pi\lambda_B}y},$$

where

$$y = \frac{\left(-4\pi\lambda_B\sqrt{s} + \widetilde{b}_4\right)}{\left(4\pi\lambda_B\sqrt{s} + \widetilde{b}_4\right)}\left(\frac{\frac{1}{2} + \frac{c_B}{\sqrt{s}}}{\frac{1}{2} - \frac{c_B}{\sqrt{s}}}\right)^{\lambda_B}.$$

Note in particular that y is real (its detailed form is irrelevant for what follows). It follows that

$$G - G^* = \frac{4iy\sin(\pi\lambda_B)}{|1 - e^{i\pi\lambda_B}y|^2}, \quad (1 - GG^*) = \frac{-4y\cos(\pi\lambda_B)}{|1 - e^{i\pi\lambda_B}y|^2}.$$

It follows that (158) is satisfied so that our proposal (81) defines a unitary S-matrix.

Finally, in the limit $\lambda_B \to 0$, our conjecture reduces to (see the second line of (82))

$$T_S^B = \frac{-b_4}{1 + b_4 H^M(\sqrt{s})}.$$

It is easily independently verified that this is the correct formula for the scattering amplitude of the large N ϕ^4 theory that (5) reduces to in the small λ_B limit. In other words our conjectured scattering amplitude has the correct small λ_B limit.

1.7.3 Bose-Fermi duality in the S-channel

We have conjectured above that, in the S-channel, the bosonic S-matrix is given by

$$T_S^B(s,t,u,\lambda_B) = \frac{k_B\sin(\pi\lambda_B)}{\pi}T_{U_d}^B(t,u,s,\lambda_B) - i\left(\cos(\pi\lambda_B) - 1\right)I(p_1,p_2,p_3,p_4), \tag{159}$$

This implies that the S-matrix in the S-channel is given by

$$S_S^B(s,t,u,\lambda_B) = i\frac{k_B\sin(\pi\lambda_B)}{\pi}T_{U_d}^B(t,u,s,\lambda_B) + \cos(\pi\lambda_B)I(p_1,p_2,p_3,p_4), \tag{160}$$

where I is the identity S-matrix, see subsection 1.2.3.

In this section we have, so far, presented our conjecture for the S-channel S-matrix in the bosonic theory. It is natural to conjecture a similar formula in the fermionic theory. In analogy with our conjecture for the bosonic theory we conjecture that

$$T_S^F(s,t,u,\lambda_F) = \frac{k_F \sin(\pi\lambda_F)}{\pi} T_{U_d}^F(t,u,s,\lambda_F) - i\left(\cos(\pi\lambda_F) - 1\right) I(p_1,p_2,p_3,p_4) \tag{161}$$

so that

$$S_S^F(s,t,u,\lambda_F) = i\frac{k_F \sin(\pi\lambda_F)}{\pi} T_{U_d}^F(t,u,s,\lambda_F) + \cos(\pi\lambda_F) I(p_1,p_2,p_3,p_4). \tag{162}$$

We will now demonstrate that these two conjectures map to each other under duality.

$$\begin{aligned}
&\frac{k_B \sin(\pi\lambda_B)}{\pi} = \frac{k_F \sin(\pi\lambda_F)}{\pi},\\
&T_{U_d}^B(t,u,s,\lambda_B) = -T_{U_d}^F(t,u,s,\lambda_F),\\
&\cos(\pi\lambda_B) = -\cos(\pi\lambda_F),
\end{aligned} \tag{163}$$

(through this subsection we specialize to the limit $b_4 \to \infty$ in the bosonic theory). it follows that

$$S_S^B(s,t,u,\lambda_B) = -S_S^F(s,t,u,\lambda_F), \tag{164}$$

which implies that

$$\begin{aligned}
S_S^F(s,t,u,\lambda_F) = &\sin(\pi\lambda_F)\left(4E(p_1,p_2,p_3)\sqrt{\frac{s\ t}{u}} + 4\sqrt{s}\ \ \frac{1 + e^{-2i(\lambda_F - \mathrm{sgn}(m_F))\tan^{-1}\left(\frac{\sqrt{s}}{2|c_F|}\right)}}{1 - e^{-2i(\lambda_F - \mathrm{sgn}(m_F))\tan^{-1}\left(\frac{\sqrt{s}}{2|c_F|}\right)}}\right)\\
&+ \cos(\pi\lambda_F) I(p_1,p_2,p_3,p_4).
\end{aligned} \tag{165}$$

Note that, $S_S^F(s,t,u,\lambda_F)$ reduces to correct tree level S-matrix presented in section 1.2.5. The overall minus sign on the RHS of (164) has no physical significance, as the sign of fermionic scattering amplitudes is largely a matter of convention. [34] (164) demonstrates the unitarity singlet fermionic S-matrix obtained from the conjecture (162), as we have already checked the unitarity of the bosonic S-matrix.

In summary, our conjecture for the S-channel S-matrices is consistent with Bose-Fermi duality. This observation may be taken as one more piece of evidence in support of our conjecture. [35]

[34] Indeed there does not even exist a particularly natural convention for the sign of a fermionic S-matrix. A fermionic transition amplitude could be defined either by $< a_4 a_3 | a_2^\dagger a_1^\dagger >$ or by the amplitude $< a_3 a_4 | a_2^\dagger a_1^\dagger >$; both conventions are equally natural and yield S-matrices that differ by a minus sign. Note that the sign of all components of the S-matrix, including the identity term is flipped by this maneuver, just as in (164).

[35] The function T_S^{trial}, and its fermionic counterpart clearly map to each other under duality. In order to account for the nature of anyonic scattering, unitarity and the non-relativistic limit, we were forced to modify T_B^{trial} and its fermionic counterpart by multiplicative and additive shifts. It is nontrivial that these shift functions, which were determined purely by consistency requirements in each theory, also turn out to

1.7.4 A heuristic explanation for modified crossing symmetry

In this section we have conjectured that the naive crossing symmetry (152) are modified in fundamental matter Chern-Simons theory; in the large N limit of interest to this chapter, we have proposed that the second of (152) continues to apply, while the first of (152) is replaced by (81). The arguments presented so far for this replacement have been entirely pragmatic; we guessed the modified crossing relation in order that the S-matrix in the S-channel obey various consistency conditions.

In this subsection we will attempt to sketch a logical explanation for this modified crossing relation (82). Our explanation is heuristic in several respects, but we hope that its defects will be remedied by more careful studies in the future.

The starting point of our analysis is the argument for crossing symmetry in the bosonic theory in the limit $\lambda_B \to 0$, briefly alluded to in subsection 1.7.1. When λ_B is set to zero, the bosonic theory effectively reduces to a theory of scalars with global $U(N)$ symmetry . In this theory the offshell correlator

$$C = \langle \phi_i(x_1)\bar{\phi}^j(x_2)\bar{\phi}^k(x_3)\phi_m(x_4)\rangle \tag{166}$$

is a well-defined meromorphic function of its arguments. By $U(N)$ invariance this correlator is given by

$$C^{jk}_{im}(x_1,x_2,x_3,x_4) = A(x_1,x_2,x_3,x_4)\delta^j_i\delta^k_m + B(x_1,x_2,x_3,x_4)\delta^k_i\delta^j_m \tag{167}$$

where the coefficient functions A and B are functions of the insertion points $x^1 \ldots x^4$. crossing symmetry follows from the observation that distinct scattering amplitudes are simply distinct onshell limits of the same correlators.

This statement is usually made precise in momentum space, but we will find it more convenient to work in position space. Consider an S^2 of size R, inscribed around the origin in Euclidean R^3 (we will eventually be interested in the limit $R \to \infty$). The S-matrices S_{U_d} and S_S may both be obtained from the correlator A as follows. Consider free incoming particles of momentum p_i and p_m starting out at very early times and focussed so that their worldlines will both intersect the origin of R^3. These two world lines intersect the S^2 described above at easily determined locations x_1 and x_4 respectively. Similarly the coordinates x_2 and x_3 are chosen to be the intercepts of the world lines of particles with index j and k, starting out from the origin of R^3 and proceeding to the future along world lines of momentum p_2 and p_3 respectively. Having now chosen the insertion points of all operators as definite functions of momenta, the correlator $A(x_1,x_2,x_3,x_4)$ is now a function only of the relevant particle - particle scattering data; the particle-particle S-matrix may infact be read off from this correlator in the limit $R \to \infty$ after we strip off factors pertaining to free propagation of our particles from the surface of the S^2 to the origin of R^3. Particle-

transform into each other under duality.

antiparticle scattering may be obtained in an identical manner, by choosing x_1 and x_2 to lie along the trajectory of incoming particles or antiparticles of momentum p_1 and p_2 respectively, while x_3 and x_4 lie along particle trajectories of outgoing particles and antiparticles of momentum p_3 and p_4 respectively. Intuitively we expect that crossing symmetry - the first of (152) - follows from the analyticity of the correlator A as a function of x_1, x_2, x_3 and x_4 on the large S^2.

In the large N limit A may be obtained from the correlator C_{im}^{jk} in (167) from the identity

$$A = \frac{1}{N^2} C_{im}^{jk} \delta_j^i \delta_k^m \tag{168}$$

At nonzero λ_B the correlator C_{im}^{jk} no longer makes sense as it is not gauge invariant. In order to construct an appropriate gauge invariant quantity let W_{12} denote an open Wilson line, in the fundamental representation, starting at x_1, ending at x_2 and running entirely outside the S^2 one which the operators are inserted. In a similar manner let W_{43} denote an open Wilson starting at x_4 and ending at x_3, once again traversing a path that lies entirely outside the S^2 on which operators are inserted. Then the quantity

$$A' = C_{im}^{jk} (W_{12})_j^i (W_{43})_k^m \tag{169}$$

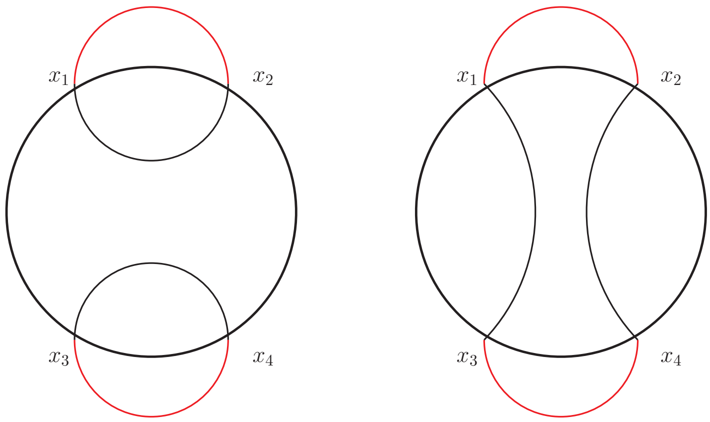
is a rough analogue of A in the gauged theory. The precise relationship is that A' reduces to A in the limit $\lambda_B \to 0$ in which gauge dynamics decouples from matter dynamics. A' is clearly gauge invariant at all λ_B; moreover there seems no reason to doubt that A' is an analytic function of $x_1 \ldots x_4$.

We can now evaluate A' in the same two onshell limits discussed in the paragraph above; as in the paragraph above this yields two functions of onshell momenta that are analytic continuations of each other. In the limit $\lambda_B \to 0$ these two functions are simply the direct channel and singlet channel S-matrices. We will now address the following question: what is the interpretation of these two functions, obtained out of A', at finite λ_B?

The path integral that evaluates the quantity A' may conceptually be split up into three parts. The path integral inside the S^2 may be thought of as defining a ket $|\psi_1>$ of the field theory that lives on S^2. The path integral outside the S^2 defines a bra of the field theory on S^2, lets call it $<\psi_2|$. And, finally, the path integral on S^2 evaluates $<\psi_2|\psi_1>$.

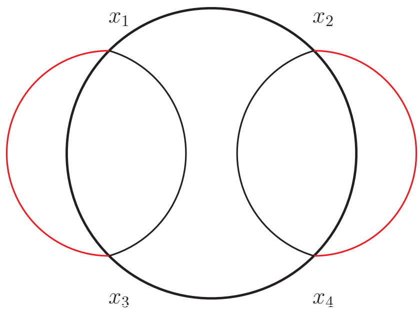
The key observation here is that the inner product occurs in the direct product of the matter Hilbert space, and the pure gauge Hilbert Space. The pure gauge Hilbert space is the two dimensional Hilbert Space of conformal blocks of pure Chern-Simons theory on S^2 with two fundamental and two antifundamental Wilson line insertions.

The inner product in the gauge sector depends only on the topology of the paths of matter particles inside the S^2. The distinct topological sectors are distinguished by a relative wind-



(a)

Figure 17: The full effective Wilson lines for S and U_d channels



(a)

Figure 18: The full effective Wilson lines for T-channel

ing number of the two scattering particles around each other. In the large N limit where the probability for reconnections in the Skein relations (see Eq. 4.22 of [23]) vanishes, the gauge theroy inner product in a sector of winding number w diffters from the inner product in a sector of winding number zero merely by the relevant Aharonov-Bohm phase. This relative weighting is, of course, a very important part of the scattering amplitude of the theory, producing all the nontrivial behaviour. However the gauge theory inner product is nontrivial even at $w = 0$. The details of this extra factor depend on the apparently unphysical external Wilson lines. This extra factor is not present in the 'S-matrix' computed in this chapter (as we had no external Wilson lines connecting the various particles). In order to compare with the S-matrices presented in this chapter, we must remove this overall inner product factor.

The gauge inner product $< \psi_2^G | \psi_1^G >$ corresponding to identity matter scattering (i.e. the geodesic paths of the matter particles from prduction to annihilations) depends on the scattering channel. Let us first study scattering in the identity channel. The initial particle created at x_1

connects up to the final particle at x_3, while the particle created at x_2 connects up with the final particle at x_4. Combining with the external lines, the full effective Wilson line is topologically a circle, see the second of Fig. 17. On the other hand, in the case of particle-particle scattering, the dominant dynamical trajectories are from the initial insertion at x_1 to the final insertion at x_2 and from the initial insertion at x_4 to the final insertion at x_3. Including the external lines, the net effective Wilson line has the topology of two circles, see the first of Fig 17.

As the topology of the effective Wilson loops in the first and second of Fig. 17 differs, it follows that the gauge theory inner product (even at zero winding) is different in the two sectors. It was demonstrated by Witten in [23] that the ratio of the path integral with two circular Wilson lines to the path integral with a single circular Wilson line is infact given by

$$\frac{k\sin(\pi\lambda_B)}{\pi} = N\frac{\sin(\pi\lambda_B)}{\pi\lambda_B}$$

in the large N limit. It follows that we should expect that

$$T_S = \frac{k\sin(\pi\lambda_B)}{\pi}T_{U_d} \tag{170}$$

in perfect agreement with (82) (the δ function piece in (82) is presumably related to a contact term in the correlators described in this subsection).

A similar argument relates T_{U_e} to T_T without any relative factor, as in this case the closed Wilson lines described above has the topology of two circles in both cases.

1.7.5 Direct evaluation of the S-matrix in the identity channel

The fact that we were able to solve the integral equation that determines four particle scattering only for $q^{\pm} = 0$ prevented us from evaluating the S-matrix in the identity channel by direct computation. For this reason we have been forced, in this section, to resort to guesswork and indirect arguments to conjecture a result for the S-matrix in the channel with identity exchange. It would, of course, be very satisfying to be able to verify our conjecture by direct computation. Unfortunately we have not succeeded in doing this. In this subsection we briefly report two potentially promising ideas for a direct evaluation.

Double analytic continuation As we have already explained above, the planar graphs that evaluate $2 \to 2$ scattering may be summed by an integral equation. As a technical trick to solve the integral equation, earlier in this chapter we found it convenient to analytically continue momenta to Euclidean space according to the formula $p^0 = ip^0_E$. We then proceeded to solve the integral equation in Euclidean space. In order to evaluate T and U-channel scattering we then analytically continued the final result back to Lorentzian space by setting $p^0_E = ip^0$.

There is, however, a natural, inequivalent analytic continuation of the Euclidean space integral equation to Lorentzian space: the continuation

$$p^3 = -ip_L^3$$

Under this continuation x^3 turns into a time like coordinate, while $x^\pm$ are complex coordinates $x^+ \sim z$, $x^- \sim \bar{z}$ that parameterize the spatial R^2. This at first strange sounding analytic continuation has been employed with great apparent success in several studies of the thermal partition function of large N Chern-Simons theories [2–6, 8–11], a fact that suggests this analytic continuation should be taken seriously.

Under this analytic continuation a center of mass momentum with $q^\pm = 0$ is timelike; indeed the condition $q^\pm = 0$ is simply the assertion that the center of mass momentum points entirely in the time direction, so that in the S-channel we are studying scattering in the center of mass frame. [36]

In summary, it seems plausible that the double analytic continuation of the integral equation (86) at $q^\pm = 0$ provides a direct computational handle on the S-matrix in the identity channel.

The discussion of this subsection may seem, at first, to directly contradict (82); surely the solution of an analytically continued integral equation is simply the analytic continuation of the solution of the original equation without any factors or additional singular terms? Infact this is not the case. It turns out that the integral equation after double analytic continuation has new singularities in the integral. These singularities - which are absent in the original equation - spoil naive analytic continuation. We illustrate this complicated set of affairs in Appendix 1.9.7.

If the central idea of this subsection is correct, then it should be possible to obtain the scattering cross section with identity exchange by solving the double analytic continued integral equation taking the new singular contributions into account. This appears to be a delicate task that we have not managed to implement.

As a warm up to the exercise suggested in this section it would be useful to rederive the ordinary non-relativistic Aharonov-Bohm equation by solving the Lippmann Schwinger equation, order by order in perturbation theory, in momentum space, perhaps at the value of the self adjoint extension parameter $w = 1$ (see [22]) at which point the Aharonov-Bohm amplitude is an analytic function of ν so perturbation theory is well-defined. We suspect that this exercise will encounter all the subtle singularities discussed in this section, and it would be useful to learn how to carefully deal with these singularities in a context where the answer is known without doubt. We postpone further study of these ideas to future work.

[36] Recall that the 3 momentum q^μ had the interpretation of momentum transfer in the T and the U-channels. As momentum transfer is necessarily spacelike for an onshell process, it follows that the U and T channel scattering processes are never onshell with this choice of Lorentzian continuation.

Schrodinger equation in lightfront quantization? It is striking that in the non-relativistic limit, the exact S-matrix was obtained rather easily by solving a Schrodinger equation in position space. One might wonder if the full non-relativistic S-matrix may similarly be obtained by solving an appropriate Schrodinger equation.

An observation that supports this hope is the fact that genuine 'particle creation' never occurs in the large N limit. A Feynman diagram that describes virtual fundamental particles being created and destroyed during a scattering process has additional index loops and is suppressed in the large N limit. It thus seems plausible that the scattering matrices of interest to us in this chapter may be obtained by solving the relevant quantum mechanical problem.

Although we will not present the details here, we have succeeded in reproducing the effective scattering amplitude of the ungauged large N ϕ^4 theory by solving a two particle Schrodinger equation. The Schrodinger equation in question is obtained from a lightcone quantization of the quantum field theory. It may well prove possible to extend this analysis to the gauged theory, and thereby extract the S-matrix from an effective Schrodinger equation; however we have not yet succeeded in implementing this idea. We leave further study of this idea to future work.

1.8 Discussion

In this chapter we have presented computations and conjectures for the formulas for $2 \to 2$ scattering in large N matter Chern-Simons theories at all orders in the 't Hooft coupling. All the computations presented in this chapter were performed in the light cone gauge together with an assumption of involving the precise definition of the gauge propagator in this gauge. It would be useful to have checks of our results using different methods - perhaps working in a covariant gauge. It might be possible (and would be very interesting) to generalize the covariant computation of section 1.5 to two loops. It would also be very interesting to study how (and whether) the unusual structural features predicted here manifest themselves in a covariant computation.

Obvious extensions of this chapter include the generalization of the computations presented here to the simplest $\mathcal{N} = 1$ and 2 supersymmetric matter Chern-Simons theories, and also to the large class of single boson-fermion theories studied in [7]. The authors of [24] study the most general renormalizable $\mathcal{N} = 1$ $U(N)$ Chern-Simons gauge theory coupled to a single (generically massive) fundamental matter multiplet. Their S-matrices are in perfect agreement with the self duality of this class of theories. And excitingly, The consistency of their results with unitarity requires a modification of the usual rules of crossing symmetry in precisely the manner anticipated in lending substantial support to our conjecture. They also find that in a certain range of coupling constants S-matrices have a pole whose mass vanishes on a self dual codimension one surface in the space of couplings.

In [25] the finite b_4 results of the bosonic computations in this chapter are matched with a generalized fermionic computation in which we include a $(\bar{\psi}\psi)^2$ and $(\bar{\psi}\psi)^3$ terms in the fermionic Lagrangian.

Perhaps the most interesting formula presented in this chapter is the formula for the scattering matrix in the S-channel. (see (81), (82)). This formula is manifestly unitary: it includes an unusual rescaling of the identity piece in the S-matrix; it agrees with the formula for Aharonov-Bohm scattering in the non-relativistic limit, and the formula for large N ϕ^4 scattering in the small λ_B limit. It is also tightly related to scattering in the other channels via rescaled relations of crossing symmetry. In the case of the scalar theory, this S-matrix also has poles signalling the existence of a stable singlet bound state of two particles in the singlet channel over a range of values of b_4. Unfortunately the formula for S-channel scattering presented in this chapter has not been derived but has simply been conjectured. A very important problem for the future is to honestly derive the formula for S-channel scattering, perhaps along the lines sketched in subsection 1.7.5.

Another reason to understand scattering after the double analytic continuation described in subsection 1.7.5 is to better understand the detailed connection between the Lorentzian results of this chapter and the Euclidean results of earlier computations [2, 3, 6, 8–11].

The S-matrices derived here have all been obtained for the scattering of massive particles. There is no barrier to taking the high energy (or equivalently zero mass) limit of our scattering amplitudes. Interestingly, the scattering amplitudes develop no new infrared singlularities in this limit. This fact is probably an artifact of the large N limit that supresses the pair creation of fundamental particles; it seems likely that $\frac{1}{N}$ corrections to the results presented in this chapter will have new infrared singularities in the zero mass limit.

As we have explained, the formulas (and conjectures) presented in this chapter imply that the usual rules of crossing symmetry are modified in matter Chern-Simons theories. In this chapter we have presented a conjecture for the nature of that modification in the 't Hooft large N limit. It would be interesting to prove this rule analogue of crossing symmetry (perhaps using a refinement of the arguments in subsection 1.7.4).

A simplifying feature of the 't Hooft large N is that scattering was truly anyonic (i.e. was characterized by a nonzero anionic phase) only in the singlet channel of Particle-antiparticle scattering. In particular the anyonic phase vanishes in particle-particle scattering (see subsection 1.2.6) so that we were never forced in this chapter to address issues having to do with the generalization of Bose or Fermi statistics. At finite N and k this situation will change, presumably leading to nontrivial phases between particle-particle scattering in the direct and exchange channels. These considerations suggest that the crossing symmetry structure of scattering amplitudes will be very rich at finite N and k; it would be fascinating to have even a well motivated conjecture for this structure. It is conceivable that the S-matrix presented in this chapter and its generalization to the finite N and k case may have useful applications in the condensed matter problems and also in the area of the topological quantum computation [26]. [37]

If the unusual structural properties conjectured in this chapter withstand further scrutiny, then they are likely to be general features of all matter Chern-Simons theories. We should, in particular, be able to probe these features in the scattering of maximally supersymmetric Chern-Simons theories (ABJ theories). In this connection it is interesting to note that there is an unresolved paradox in the study of scattering amplitudes in ABJM and ABJ theory. In this theory the $2 \to 2$ scattering amplitude has been argued to vanish at one loop [27–29], but to be non vanishing at two loops [29–31]. The paradox arises because although the proposed two loop formula for four particle scattering in ABJM theory has cuts [30], there do not seem to exist any candidate intermediate processes to saturate these cuts. [38] While scattering amplitudes in ABJ theory are more confusing than those considered in this chapter because they receive infrared divergences from intermediate massless scalar and fermion propagation, it is at least conceivable that the resolution to this apparent unitarity paradox lies along the lines sketched in this chapter. The results of this chapter should generalize, in the most straightforward fashion, to scattering in $U(M) \times U(N)$ theory when $\frac{M}{N} << 1$ (in this limit the ABJ theory begins to closely resemble a theory with a single gauge group and only fundamental matter, like the theories studied in this chapter). The analysis of this chapter suggests that the $2 \to 2$ scattering amplitude does not completely vanish at one loop: it should at least have a δ function localized singular piece. The contribution of this piece in a one loop sub diagram to two loop graphs could then, additionally, modify the scattering amplitudes as well as the usual rules of crossing symmetry. It would be fascinating to verify these expectations via a direct analysis of scattering amplitudes in the supersymmetric theories[39].

A significant check of all the computations and conjectures presented in this chapter is that they are all consistent with the recently conjectured level-rank duality between bosonic and fermionic Chern-Simons theories. This works in a rather remarkable way. The bosonic S-matrices have nontrivial analytic structure (e.g. two particle cuts) at all values of λ_B (including $\lambda_B = 0$ where the cuts come from the four boson contact interaction) provided $|\lambda_B| \neq 1$. Precisely at $\lambda_B = 1$, however, the bosonic S-matrix collapses into precisely the analytically trivial constant that one predicts from fermionic tree level scattering. Indeed the agreement between bosonic and

[37] Perhaps there is a sense in which the finite N and k result is 'quantum', and results in the 't Hooft limit are obtained from the 'classical limit' of the corresponding 'quantum structure'.

[38] There appear to be only two candidates for the processes that could produce these cuts. The first is by sewing together two $2 \to 3$ tree level amplitudes, but there are no such amplitudes in ABJM theory. The second is by sewing together a tree level $2 \to 2$ amplitude with a one loop $2 \to 2$ amplitude, but as we have remarked, the latter have been argued to vanish.

[39] It is interesting to note that, in [32] it was argued that in the case of ABJM, three loop amplitude is non zero. However, again they missed the existance of delta function. It would be interesting to see whether higher point functions also shows some nontrivial analytical structure. For a discussion of higher point function in ABJM theory, we refer reader to [33].

fermionic S-matrices works at all values of λ_B, not just at extreme ends.

Indeed the results of this chapter shed some additional light on the working of this duality. The first point, as we have already emphasized in the introduction, is that our S-matrix is effectively anyonic in the singlet channel. The effective anyonic phase can be estimated very simply in the non-relativistic limit, and the duality map from λ_B to λ_F can simply be deduced by demanding that the dual theories have equal anyonic phases.

In the U-channel, on the other hand, the anyonic phase is trivial. Bosonic and fermionic S matrices map to each other only after we transpose the exchange representations. As we have explained in more detail in the introduction, this suggests that, for scattering purposes, there exists a map between asymptotic multi bosonic states that transform in representation R of $U(N_B)$ and multi- fermionic states that transform in representation R^T of $U(N_F)$.

There is an obvious puzzle about the identification suggested above; namely the number of states on the two sides do not match (this is true even if we restrict to the simplest representation, namely the fundamental, simply because $N_B \neq N_F$). It seems possible that the duality between the bosonic and fermionic theories really works only on compact manifolds (and so, effectively, only in the singlet sector on R^2). If this turns out to be the correct eventual statement of the duality, then the perfect match under duality of the scattering amplidues in non singlet sectors may eventually find its explaination in the match of factorized subsectors in higher point scattering in the singlet channel. For instance one could consider the scattering of two particles, and simultaneously the scatering of two antiparticles very far away, with colour indices chosen so that the full four particle initial state is a singlet and so duality invariant. Presumably the scattering amplitudes factor into the scattering amplitude for particle - particle scattering and the scatterng amplitude for antiparticle-antiparticle scattering, implying the duality invariance of these more basic 2 particle scattering amplitudes, even though they do not occur in a gauge singlet sector, explaining the results obtained in this chapter. It would certainly be nice to understand this better.

In summary, the results and conjectures presented in this chapter have several unexpected features, have intriguing implications, and throw up several puzzles. If our results stand up to further scrutiny they suggest several fascinating new directions of investigation.

1.9 Appendices for Chapter 1

1.9.1 The identity S-matrix as a function of s, t, u

As explained in subsection 1.2.3, the identity S-matrix has a simple form in the center of mass frame; it is given by

$$(2\pi)^3\delta^3(p_1 + p_2 - p_3 - p_4)8\pi\sqrt{s}\delta(\theta)$$

As we will see below, the expression $\delta(\theta)$ is slightly singular when recast in terms of invariants, so we will find it convenient to regulate this expression as

$$(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)4\pi\sqrt{s}\lim_{\epsilon\to 0}\left(\delta(\theta-\epsilon)+\delta(\theta+\epsilon)\right).$$

Using (23), this expression may be recast in invariant form

$$\delta\left(\sqrt{\frac{4t}{t+u}}-\epsilon\right)(2\pi)^3\delta^3(p_1+p_2-p_3-p_4) \tag{171}$$

as we have already noted in (21).

In this Appendix we present a cumbersome but direct algebraic check that I as defined in (171) coincides with I defined in (20). Our strategy is as follows. We start with the expression (171), and express the arguments of the delta functions in (171) entirely in terms of the 8 variables $p_1^x, p_1^y, p_2^x, p_2^y, p_3^x, p_3^y, p_4^x, p_4^y$ (the energies of the ingoing and outgoing particles are solved for using the on shell condition). We choose to view the resultant expression as follows. We think of $p_1^x, p_1^y, p_2^x, p_2^y$ as fixed initial data and the remaining quantities $p_3^x, p_3^y, p_4^x, p_4^y$ as variable scattering data. The four delta functions in (171) thus determine $p_3^x, p_3^y, p_4^x, p_4^y$ as functions of $p_1^x, p_1^y, p_2^x, p_2^y$. At leading order in ϵ is not difficult to explicitly determine the values for $p_3^x, p_3^y, p_4^x, p_4^y$ obtained in this manner. We find

$$p_{3,x}=p_{1,x}\pm\epsilon a_{3,x},\ \ p_{4,x}=p_{2,x}\pm\epsilon a_{4,x},\ \ p_{3,y}=p_{1,y}\pm\epsilon a_{3,y},\ \ p_{4,y}=p_{2,y}\pm\epsilon a_{4,y}. \tag{172}$$

where the four a variables are obtained by solving four linear equations (the $\pm$ above corresponds to the two possibilities $\theta=\epsilon$ or $\theta=-\epsilon$ in the centre- of-mass frame). In what follows below we will not need the explicit form of the solutions for the a variables, but will only need certain identities obeyed by these solutions. These identities turn out, in fact, to be three of the four equations that the a variables obey. The relevant three equations are

$$\begin{gathered}a_{4,x}=-a_{3,x},\quad a_{4,y}=-a_{3,y},\quad a_{3,x}=Ba_{3,y},\\ \text{where}\quad B=\frac{p_{2,y}\sqrt{m^2+p_{1,x}^2+p_{1,y}^2}-p_{1,y}\sqrt{m^2+p_{2,x}^2+p_{2,y}^2}}{p_{1,x}\sqrt{m^2+p_{2,x}^2+p_{2,y}^2}-p_{2,x}\sqrt{m^2+p_{1,x}^2+p_{1,y}^2}}.\end{gathered} \tag{173}$$

Let us now return to our task of rewriting the delta function in (171) in terms of delta functions linear in $p_3^x, p_3^y, p_4^x, p_4^y$. It follows from the usual rules for manipulating delta functions that

$$\begin{aligned}
&\delta\left(\sqrt{\frac{4t}{t+u}}-\epsilon\right)\delta^3(p_1+p_2-p_3-p_4) \\
&=J_1\delta^2(\overrightarrow{p}_3-\overrightarrow{p}_1+\epsilon a_3)\delta^2(\overrightarrow{p}_4-\overrightarrow{p}_2+\epsilon a_4)+J_2\delta^2(\overrightarrow{p}_3-\overrightarrow{p}_1-\epsilon a_3)\delta^2(\overrightarrow{p}_4-\overrightarrow{p}_2-\epsilon a_4)
\end{aligned} \tag{174}$$

where J_1 and J_2 are the relevant Jacobians. It remains to compute these Jacobians.

The reader might naively expect that the Jacobians are independent of a_3 and a_4 in the limit $\epsilon \to 0$, but that is not the case. It is not difficult verify that, in the $\epsilon \to 0$ limit the derivatives $\frac{\partial\sqrt{\frac{4t}{t+u}}}{\partial p_3^x}$ and $\frac{\partial\sqrt{\frac{4t}{t+u}}}{\partial p_3^y}$ (which enter the expression for the Jacobians) are of the form $\frac{A}{B}$ where A and B are both expressions of unit homogeneity in a_3 and a_4. The ratio $\frac{A}{B}$ does not depend on the overall scale of $a_{3,x}, a_{3,y}$ and $a_{4,x}, a_{4,y}$, but does depend on their relative magnitudes. It turns out that the equations (173) are sufficient to unambiguously determine the ratio $\frac{A}{B}$ (which turns out to be the same for the two solutions corresponding to the $\pm$ signs so that $J_1 = J_2 = J$) ; we find

$$J=\sqrt{s}\frac{1}{E_1E_2} \tag{175}$$

where $E_i = \sqrt{m^2 + p_{i,x}^2 + p_{i,y}^2}$ and

$$s=\sqrt{2}\sqrt{\sqrt{m^2+p_{1,x}^2+p_{1,y}^2}\sqrt{m^2+p_{2,x}^2+p_{2,y}^2}+m^2-p_{1,x}p_{2,x}-p_{1,y}p_{2,y}}. \tag{176}$$

Collecting factors, it follows that the RHS of (171) coincides with the RHS of (20) in the limit $\epsilon \to 0$.

1.9.2 Tree level S-matrix

The bosonic effective action is

$$T_B=\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}V(p,k,q)\phi_i(p+q)\bar{\phi}^j(-k-q)\bar{\phi}^i(-p)\phi_j(k), \tag{177}$$

where at tree level

$$V(p,k,q)=8\pi i\lambda\epsilon_{\mu\nu\rho}\frac{q^\mu p^\nu k^\rho}{(k-p)^2}. \tag{178}$$

And the fermionic effective action is

$$T_F=\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}V_{\beta\delta}^{\alpha\gamma}(p,k,q)\psi_{i,\alpha}(p+q)\bar{\psi}^{j,\beta}(-k-q)\bar{\psi}^{i,\delta}(-p)\psi_{j,\gamma}(k), \tag{179}$$

where at tree level

$$V^{\alpha\gamma}_{\beta\delta}(p,k,q) = 2i\pi\lambda\epsilon_{\mu\nu\rho}\frac{(\gamma^{\mu})^{\alpha}_{\beta}(\gamma^{\nu})^{\gamma}_{\delta}(k-p)^{\rho}}{(k-p)^{2}}. \tag{180}$$

The gauge field propagator that we work with in this section is

$$\langle A_{\mu}(p)A_{\nu}(-q)\rangle = (2\pi)^{3}\delta^{3}(p-q)\frac{4\pi}{p^{2}}\epsilon_{\mu\nu\rho}p^{\rho}. \tag{181}$$

Particle-particle scattering According to the momentum assignments in (37), The bosonic S-matrix is given by

$$\begin{aligned}
&S_B(p_1,p_2,p_3,p_4)\\
=&\langle out|1+iT_B|in\rangle\\
=&\langle 0|a_n(p_4)a_m(p_3)a^{b\dagger}(p_2)a^{a\dagger}(p_1)|0\rangle\\
&+\frac{i}{2}\int\frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\Bigg[V(p,k,q)\\
&\qquad\times\langle 0|a_n(p_4)a_m(p_3)\left(\phi_i(p+q)\bar{\phi}^j(-k-q)\bar{\phi}^i(-p)\phi_j(k)\right)a^{b\dagger}(p_2)a^{a\dagger}(p_1)|0\rangle\Bigg].
\end{aligned} \tag{182}$$

Using appropriate contractions and commutation relations, we find

$$\begin{aligned}
S_B(p_1,p_2,p_3,p_4) = {}&\delta^a_m\delta^b_n\left(I(p_1,p_2,p_3,p_4)+iV(-p_3,p_2,p_1+p_3)\right)\\
&+\delta^a_n\delta^b_m\left(I(p_1,p_2,p_4,p_3)+iV(-p_4,p_2,-p_3-p_2)\right)
\end{aligned} \tag{183}$$

The first term is for the U_d channel while the other is for the U_e channel.
Whereas the fermionic S-matrix is

$$\begin{aligned}
&S_F(p_1,p_2,p_3,p_4)\\
=&\langle out|1+iT_F|in\rangle\\
=&\langle 0|a_n(p_4)a_m(p_3)a^{b\dagger}(p_2)a^{a\dagger}(p_1)|0\rangle\\
&+\frac{i}{2}\int\frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\Bigg[V^{\alpha\gamma}_{\beta\delta}(p,k,q)\\
&\times\langle 0|a_n(p_4)a_m(p_3)\left(\psi_{i,\alpha}(p+q)\bar{\psi}^{j,\beta}(-(k+q))\bar{\psi}^{i,\delta}(-p)\psi_{j,\gamma}(k)\right)a^{b\dagger}(p_2)a^{a\dagger}(p_1)|0\rangle\Bigg]
\end{aligned} \tag{184}$$

Using appropriate contractions and anticommutation relations,

$$
\begin{aligned}
&S_F(p_1,p_2,p_3,p_4)\\
=&-\delta^a_m\delta^b_n I(p_1,p_2,p_3,p_4)\\
&-i\delta^a_m\delta^b_n V^{\alpha\gamma}_{\beta\delta}(-p_3,p_2,p_1+p_3)\bar{u}^\beta(-p_4)\bar{u}^\delta(-p_3)u_\alpha(p_1)u_\gamma(p_2)\\
&+\delta^a_n\delta^b_m I(p_1,p_2,p_4,p_3)\\
&+i\delta^a_n\delta^b_m V^{\alpha\gamma}_{\beta\delta}(-p_4,p_2,-p_3-p_2)\bar{u}^\beta(-p_3)\bar{u}^\delta(-p_4)u_\alpha(p_1)u_\gamma(p_2)
\end{aligned}
\tag{185}
$$

Again, the first term is for the U_d channel while the other is for the U_e channel.

Particle-antiparticle scattering According to the momentum assignments in (32), The bosonic S-matrix is given by

$$
\begin{aligned}
&S_B(p_1,p_2,p_3,p_4)\\
=&\langle out|1+iT_B|in\rangle\\
=&\langle 0|b^n(p_4)a_m(p_3)b^\dagger_b(p_2)a^{a\dagger}(p_1)|0\rangle\\
&+\frac{i}{2}\int\frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\Bigg[V(p,k,q)\\
&\quad\times\langle 0|b^n(p_4)a_m(p_3)\left(\phi_i(p+q)\bar{\phi}^j(-k-q)\bar{\phi}^i(-p)\phi_j(k)\right)b^\dagger_b(p_2)a^{a\dagger}(p_1)|0\rangle\Bigg]
\end{aligned}
\tag{186}
$$

Using appropriate contractions and commutation relations, we find

$$
\begin{aligned}
S_B(p_1,p_2,p_3,p_4)=&\left(\delta^a_m\delta^n_b-\frac{\delta^n_m\delta^a_b}{N}\right)\left(I(p_1,p_2,p_3,p_4)+iV(-p_3,p_4,p_1+p_3)\right)\\
&+\frac{\delta^n_m\delta^a_b}{N}\left(I(p_1,p_2,p_3,p_4)+iV(-p_2,p_4,p_1+p_2)\right)
\end{aligned}
\tag{187}
$$

The first term is for the T-channel while the other is for the S-channel.
Whereas the fermionic S-matrix is

$$
\begin{aligned}
&S_F(p_1,p_2,p_3,p_4)=\langle out|1+iT_F|in\rangle=\langle 0|b^n(p_4)a_m(p_3)b^\dagger_b(p_2)a^{a\dagger}(p_1)|0\rangle\\
&+\frac{i}{2}\int\frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}V^{\alpha\gamma}_{\beta\delta}(p,k,q)\langle 0|b^n(p_4)a_m(p_3)\\
&\left(\psi_{i,\alpha}(p+q)\bar{\psi}^{j,\beta}(-k-q)\bar{\psi}^{i,\delta}(-p)\psi_{j,\gamma}(k)\right)b^\dagger_b(p_2)a^{a\dagger}(p_1)|0\rangle
\end{aligned}
\tag{188}
$$

Using appropriate contractions and anticommutation relations, we find

$$\begin{aligned}
&S_F(p_1,p_2,p_3,p_4)\\
&= -\left(\delta^a_m\delta^n_b - \frac{\delta^n_m\delta^a_b}{N}\right)\Big(I(p_1,p_2,p_3,p_4) - iV^{\alpha\gamma}_{\beta\delta}(-p_3,p_4,p_1+p_3)\bar{u}^\beta(-p_3)\bar{v}^\delta(p_2)u_\alpha(p_1)v_\gamma(-p_4)\Big)\\
&-\frac{\delta^n_m\delta^a_b}{N}\Big(I(p_1,p_2,p_3,p_4) + iV^{\alpha\gamma}_{\beta\delta}(-p_2,p_4,p_1+p_2)\bar{v}^\beta(p_2)\bar{u}^\delta(-p_3)u_\alpha(p_1)v_\gamma(-p_4)\Big)
\end{aligned} \tag{189}$$

Again, the first term is for the T-channel while the other is for the S-channel.

Explicit tree level computation Now we substitute for the Vs for the respective channels in bosonic case, and obtain

While the fermionic expressions for S, T, U_d and U_e channels are (with respect to the identity) respectively,

$$\begin{aligned}
T_S &= \frac{2i\pi}{k_F(p_2+p_4)^2}\epsilon_{\mu\nu\rho}\left(\bar{u}(-p_3)\gamma^\mu u(p_1)\right)\left(\bar{v}(p_2)\gamma^\mu v(-p_4)\right)(p_2+p_4)^\rho\\
T_T &= -\frac{2i\pi}{k_F(p_3-p_4)^2}\epsilon_{\mu\nu\rho}\left(\bar{v}(p_2)\gamma^\mu u(p_1)\right)\left(\bar{u}(-p_3)\gamma^\mu v(-p_4)\right)(p_3+p_4)^\rho\\
T_{U_d} &= \frac{2i\pi}{k_F(p_2+p_3)^2}\epsilon_{\mu\nu\rho}\left(\bar{u}(-p_4)\gamma^\mu u(p_1)\right)\left(\bar{u}(-p_3)\gamma^\mu u(p_2)\right)(p_2+p_3)^\rho\\
T_{U_d} &= \frac{2i\pi\lambda_F}{(p_2+p_4)^2}\epsilon_{\mu\nu\rho}\left(\bar{u}(-p_3)\gamma^\mu u(p_1)\right)\left(\bar{u}(-p_4)\gamma^\mu u(p_2)\right)(p_2+p_4)^\rho
\end{aligned} \tag{190}$$

These expressions can be manipulated conveniently using the Gordon Identities which are derived below:

The Dirac equation satisfied by $u(p), \bar{u}(p), v(p), \bar{v}(p)$ are given by

$$\begin{aligned}
&(i\gamma^\mu p_\mu + m)\,u(p) = 0, \quad \bar{u}(p)\,(i\gamma^\mu p_\mu + m) = 0,\\
&(-i\gamma^\mu p_\mu + m)\,v(p) = 0, \quad \bar{v}(p)\,(-i\gamma^\mu p_\mu + m) = 0.
\end{aligned} \tag{191}$$

The gamma matrices are given by

$$\begin{aligned}\gamma^0 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ \gamma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \gamma^2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\end{aligned} \tag{192}$$

They satisfy

$$\gamma^\mu \gamma^\nu = g^{\mu\nu} - \epsilon^{\mu\nu\rho}\gamma_\rho. \tag{193}$$

Now, using Dirac equation (191), it is easy derive the Gordon identities

$$\begin{aligned}-\bar{u}(p_1)\gamma^\mu u(p_2) &= i\left(\bar{u}(p_1)\frac{(p_1+p_2)^\mu}{2m}u(p_2) - \epsilon^{\mu\nu\rho}\frac{(-p_1+p_2)_\nu}{2m}\bar{u}(p_1)\gamma_\rho u(p_2)\right) \\ -\bar{u}(p_1)\gamma^\mu v(p_2) &= i\left(\bar{u}(p_1)\frac{(p_1-p_2)^\mu}{2m}v(p_2) + \epsilon^{\mu\nu\rho}\frac{(p_1+p_2)_\nu}{2m}\bar{u}(p_1)\gamma_\rho v(p_2)\right) \\ -\bar{v}(p_1)\gamma^\mu u(p_2) &= i\left(\bar{v}(p_1)\frac{(-p_1+p_2)^\mu}{2m}u(p_2) - \epsilon^{\mu\nu\rho}\frac{(p_1+p_2)_\nu}{2m}\bar{v}(p_1)\gamma_\rho u(p_2)\right) \\ -\bar{v}(p_1)\gamma^\mu v(p_2) &= i\left(-\bar{v}(p_1)\frac{(p_1+p_2)^\mu}{2m}v(p_2) + \epsilon^{\mu\nu\rho}\frac{(p_1+p_2)_\nu}{2m}\bar{v}(p_1)\gamma_\rho v(p_2)\right)\end{aligned} \tag{194}$$

Using this, it is easy to show that

$$\begin{aligned}\bar{u}(p_1)\gamma^\mu u(p_2) &= \frac{1}{1+(\frac{p_1-p_2}{2m})^2}\left(-i\frac{(p_1+p_2)^\mu}{2m} - \frac{1}{2m^2}\epsilon^{\mu\nu\rho}(p_1)_\nu(p_2)_\rho\right)\bar{u}(p_1)u(p_2) \\ \bar{v}(p_1)\gamma^\mu u(p_2) &= \frac{1}{1+(\frac{p_1+p_2}{2m})^2}\left(-i\frac{(-p_1+p_2)^\mu}{2m} + \frac{1}{2m^2}\epsilon^{\mu\nu\rho}(p_1)_\nu(p_2)_\rho\right)\bar{v}(p_1)u(p_2) \\ \bar{u}(p_1)\gamma^\mu v(p_2) &= \frac{1}{1+(\frac{p_1+p_2}{2m})^2}\left(-i\frac{(p_1-p_2)^\mu}{2m} + \frac{1}{2m^2}\epsilon^{\mu\nu\rho}(p_1)_\nu(p_2)_\rho\right)\bar{u}(p_1)v(p_2) \\ \bar{v}(p_1)\gamma^\mu v(p_2) &= \frac{1}{1+(\frac{p_1-p_2}{2m})^2}\left(i\frac{(p_1+p_2)^\mu}{2m} - \frac{1}{2m^2}\epsilon^{\mu\nu\rho}(p_1)_\nu(p_2)_\rho\right)\bar{v}(p_1)v(p_2)\end{aligned} \tag{195}$$

The only thing that is remaining is to compute the quantities, $\bar{u}(p')u(p), \bar{v}(p')v(p), \bar{u}(p')v(p), \bar{v}(p')u(p)$. For this, we explicitly construct the solution for V and u starting boosting the rest frame results which are easily computable to a frame where the momenta is p. In the rest frame, equation satisfied by the u and v is given by,

$$\left(-i\gamma^0 + I\right)u(0) = 0, \quad \left(i\gamma^0 + I\right)v(0) = 0, \tag{196}$$

and for $\bar{u}$ and $\bar{v}$

$$\bar{u}(0)\left(-i\gamma^0 + I\right) = 0, \quad \bar{v}(0)\left(i\gamma^0 + I\right) = 0, \tag{197}$$

where I denotes, the 2×2 identity matrix. The solutions are

$$u(0) = \sqrt{m}\,(1, -i)\,, \quad v(0) = \sqrt{m}\,(1, i)\,, \quad \bar{u}(0) = \sqrt{m}\,(1, i)\,, \quad \bar{v}(0) = \sqrt{m}\,(-1, i)\,. \tag{198}$$

Suppose we are now interested in solution for u and v at momenta p, given by

$$p_\mu = (-m\cosh(\alpha), m\sinh(\alpha)\cos(\theta), m\sinh(\alpha)\sin(\theta))\,. \tag{199}$$

The solutions are given by

$$\begin{aligned}
u(p) &= \left(\cosh(\frac{\alpha}{2})I - \sinh(\frac{\alpha}{2})\left(\cos(\theta)\gamma^2 - \sin(\theta)\gamma^1\right)\right)u(0) \\
\bar{u}(p) &= \bar{u}(0)\left(\cosh(\frac{\alpha}{2})I + \sinh(\frac{\alpha}{2})\left(\cos(\theta)\gamma^2 - \sin(\theta)\gamma^1\right)\right) \\
v(p) &= \left(\cosh(\frac{\alpha}{2})I - \sinh(\frac{\alpha}{2})\left(\cos(\theta)\gamma^2 - \sin(\theta)\gamma^1\right)\right)v(0) \\
\bar{v}(p) &= \bar{v}(0)\left(\cosh(\frac{\alpha}{2})I + \sinh(\frac{\alpha}{2})\left(\cos(\theta)\gamma^2 - \sin(\theta)\gamma^1\right)\right).
\end{aligned} \tag{200}$$

It is now easy to compute $\bar{u}(p')u(p), \bar{v}(p')v(p), \bar{u}(p')v(p), \bar{v}(p')u(p)$. Results are given by

$$\begin{aligned}
\bar{u}(p_1)u(p_2) =& e^{i\tan^{-1}\frac{\sin(\theta_2-\theta_1)}{\cos(\theta_2-\theta_1)-\coth(\alpha_1)\coth(\alpha_2)}}\sqrt{(2m^2 - 2\;p_1\cdot p_2)}, \\
\bar{v}(p_1)v(p_2) =& e^{i\tan^{-1}\frac{\sin(\theta_1-\theta_2)}{\cos(\theta_1-\theta_2)-\coth(\alpha_1)\coth(\alpha_2)}}\sqrt{(2m^2 - 2\;p_1\cdot p_2)}, \\
\bar{v}(p_1)u(p_2) =& e^{i\tan^{-1}\frac{\sinh\left(\frac{\alpha_1}{2}\right)\cosh\left(\frac{\alpha_2}{2}\right)\sin(\theta_1)-\sinh\left(\frac{\alpha_2}{2}\right)\cosh\left(\frac{\alpha_1}{2}\right)\sin(\theta_2)}{\sinh\left(\frac{\alpha_1}{2}\right)\cosh\left(\frac{\alpha_2}{2}\right)\cos(\theta_1)-\sinh\left(\frac{\alpha_2}{2}\right)\cosh\left(\frac{\alpha_1}{2}\right)\cos(\theta_2)}} \\
&\times\sqrt{(-2m^2 - 2\;p_1\cdot p_2)}, \\
\bar{u}(p_1)v(p_2) =& e^{i\tan^{-1}\frac{\sinh\left(\frac{\alpha_2}{2}\right)\cosh\left(\frac{\alpha_1}{2}\right)\sin(\theta_2)-\sinh\left(\frac{\alpha_1}{2}\right)\cosh\left(\frac{\alpha_2}{2}\right)\sin(\theta_1)}{\sinh\left(\frac{\alpha_1}{2}\right)\cosh\left(\frac{\alpha_2}{2}\right)\cos(\theta_1)-\sinh\left(\frac{\alpha_2}{2}\right)\cosh\left(\frac{\alpha_1}{2}\right)\cos(\theta_2)}} \\
&\times\sqrt{(-2m^2 - 2\;p_1\cdot p_2)},
\end{aligned} \tag{201}$$

As a final ingredient to compute the tree level scattering is

$$
\begin{aligned}
&\epsilon_{\mu\nu\rho}\bar{u}(p_1)\gamma^\mu u(p_2)\bar{u}(p_3)\gamma^\nu u(p_4)p_5^\rho \\
=&\frac{\left(\bar{u}(p_1)u(p_2)\right)\left(\bar{u}(p_3)u(p_4)\right)}{\left(1+\frac{(p_1-p_2)^2}{4m^2}\right)\left(1+\frac{(p_3-p_4)^2}{4m^2}\right)} \\
&\times\Big[-\frac{1}{4m^2}\epsilon_{\mu\nu\rho}(p_1+p_2)^\mu(p_3+p_4)^\nu p_5^\rho \\
&\quad+\frac{1}{4m^4}\left((p_1\cdot p_5)\epsilon_{\mu\nu\rho}p_2^\mu p_3^\nu p_4^\rho-(p_2\cdot p_5)\epsilon_{\mu\nu\rho}p_1^\mu p_3^\nu p_4^\rho\right) \\
&\quad+\frac{i}{4m^3}\Big(\left(p_4\cdot p_5\right)\left(p_3\cdot(p_1+p_2)\right)-\left(p_3\cdot p_5\right)\left(p_4\cdot(p_1+p_2)\right)\Big) \\
&\quad+\frac{i}{4m^3}\Big(\left(p_1\cdot p_5\right)\left(p_2\cdot(p_3+p_4)\right)-\left(p_2\cdot p_5\right)\left(p_1\cdot(p_3+p_4)\right)\Big)\Big],
\end{aligned}
\tag{202}
$$

where $p\cdot p'=p_\mu p'^\mu$. Now just by few interchange of signs, as it follows from (195), one can compute tree level with any appropriate combinatios of $u's$ and $v's$ using (201). For example,

$$
\begin{aligned}
&\epsilon_{\mu\nu\rho}\bar{v}(p_1)\gamma^\mu v(p_2)\bar{u}(p_3)\gamma^\nu u(p_4)p_5^\rho \\
=&\frac{\left(\bar{v}(p_1)v(p_2)\right)\left(\bar{u}(p_3)u(p_4)\right)}{\left(1+\frac{(p_1-p_2)^2}{4m^2}\right)\left(1+\frac{(p_3-p_4)^2}{4m^2}\right)} \\
&\times\Big[\frac{1}{4m^2}\epsilon_{\mu\nu\rho}(p_1+p_2)^\mu(p_3+p_4)^\nu p_5^\rho \\
&\quad+\frac{1}{4m^4}\left((p_1\cdot p_5)\epsilon_{\mu\nu\rho}p_2^\mu p_3^\nu p_4^\rho-(p_2\cdot p_5)\epsilon_{\mu\nu\rho}p_1^\mu p_3^\nu p_4^\rho\right) \\
&\quad+\frac{i}{4m^3}\left(-\left(p_4\cdot p_5\right)\left(p_3\cdot(p_1+p_2)\right)+\left(p_3\cdot p_5\right)\left(p_4\cdot(p_1+p_2)\right)\right) \\
&\quad+\frac{i}{4m^3}\left(\left(p_1\cdot p_5\right)\left(p_2\cdot(p_3+p_4)\right)-\left(p_2\cdot p_5\right)\left(p_1\cdot(p_3+p_4)\right)\right)\Big].
\end{aligned}
\tag{203}
$$

Using formulas presented in (202), (203) we find

$$
\begin{aligned}
\mathbf{S}_{F,U_d}&=-I(p_1,p_2,p_3,p_4)-e^{i\alpha_1}\frac{8\pi}{k_F}\left(\frac{\epsilon_{\mu\nu\rho}p_1^\mu p_2^\nu p_3^\rho}{(p_2+p_3)^2}-2im_F\right)(2\pi)^3\delta(p_1+p_2+p_3+p_4),\\
\mathbf{S}_{F,U_e}&=I(p_1,p_2,p_4,p_3)-e^{i\alpha_2}\frac{8\pi}{k_F}\left(\frac{\epsilon_{\mu\nu\rho}p_1^\mu p_2^\nu p_3^\rho}{(p_2+p_4)^2}+2im_F\right)(2\pi)^3\delta(p_1+p_2+p_3+p_4),\\
\mathbf{S}_{F,T}&=-I(p_1,p_2,p_3,p_4)+e^{i\alpha_3}\frac{8\pi}{k_F}\left(\frac{\epsilon_{\mu\nu\rho}p_1^\mu p_2^\nu p_3^\rho}{(p_4+p_3)^2}+2im_F\right)(2\pi)^3\delta(p_1+p_2+p_3+p_4),\\
\mathbf{S}_{F,S}&=-I(p_1,p_2,p_3,p_4)+e^{i\alpha_4}8\pi\lambda_F\left(\frac{\epsilon_{\mu\nu\rho}p_1^\mu p_2^\nu p_3^\rho}{(p_2+p_4)^2}-2im_F\right)(2\pi)^3\delta(p_1+p_2+p_3+p_4),
\end{aligned}
\tag{204}
$$

where α_1 to α_4 are some complicated physically irelevant phase factors. They obey interchange symmetry and for equal momneta (for example, in (201), $p_1=p_2$) phase vanishes. In particular,

this implies that the phase factor in U_d and U_e channel are the same. Although, these phases has no physical relevance, we present the results in the C.M. frame. Let the incoming momenta be p_1, p_2 and out going momenta are $-p_3, -p_4$ and the angle between p_1 and $-p_3$ is given by θ then we find $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -\theta$. Note that, inparticular this has the property that, near identity, phase factors has no contribution, this is what we expect also from physical ground. So the answers obey the duality with the Bosonic answers in the respective channels.
For completeness, we also write answers for bosonic case.

$$
\begin{aligned}
\mathbf{S}_{B,U_d} &= I(p_1, p_2, p_3, p_4) - \frac{8\pi}{k_B} \frac{\epsilon_{\mu\nu\rho} p_1^\mu p_2^\nu p_3^\rho}{(p_2 + p_3)^2} (2\pi)^3 \delta(p_1 + p_2 + p_3 + p_4) \\
\mathbf{S}_{B,U_e} &= I(p_1, p_2, p_4, p_3) + \frac{8\pi}{k_B} \frac{\epsilon_{\mu\nu\rho} p_1^\mu p_2^\nu p_3^\rho}{(p_2 + p_4)^2} (2\pi)^3 \delta(p_1 + p_2 + p_3 + p_4) \\
\mathbf{S}_{B,T} &= I(p_1, p_2, p_3, p_4) + \frac{8\pi}{k_B} \frac{\epsilon_{\mu\nu\rho} p_1^\mu p_2^\nu p_3^\rho}{(p_4 + p_3)^2} (2\pi)^3 \delta(p_1 + p_2 + p_3 + p_4) \\
\mathbf{S}_{B,S} &= I(p_1, p_2, p_3, p_4) - 8\pi\lambda_B \frac{\epsilon_{\mu\nu\rho} p_1^\mu p_2^\nu p_3^\rho}{(p_2 + p_4)^2} (2\pi)^3 \delta(p_1 + p_2 + p_3 + p_4).
\end{aligned}
\tag{205}
$$

1.9.3 Aharonov-Bohm scattering

In this section we will review the classic computation, first performed by Aharonov and Bohm, of the scattering of a charged non-relativistic particle off a flux tube; see [15, 17, 19–22] for relevant references. We assume that the flux tube is oriented in the z direction, and sits at the origin of the transverse two dimensional space. We focus on states that also preserve translational invariance along the z direction, so our problem is effectively two (spatial) dimensional. We assume that the integrated flux of the flux tube equals $2\pi\nu$ so that the phase associated with the charge particle circling the flux tube is $2\pi i\nu$ (the particle is assumed to carry unit charge and mass m). Throughout this appendix we assume $|\nu| < 1$.

Derivation of the scattering wave function We will find scattering state solutions at energy $E = \frac{k^2}{2m}$ of the Schrodinger equation for this particle; intuitively k is the momentum of the particle incident on the flux.

The time independent Schrodinger equation that governs our system is

$$
\left(-\frac{1}{2m} \left(\nabla + 2\pi i\nu \mathbf{G} \right)^2 - \frac{k^2}{2m} \right) \psi = 0 \tag{206}
$$

where

$$
G_i = \frac{\epsilon_{ij}}{2\pi} \partial_j \ln r \tag{207}
$$

In polar coordinates the one form G is given by

$$G = \frac{d\phi}{2\pi}.$$

Following Aharonov and Bohm we adopt 'regular' boundary conditions at the origin of our space, i.e. we demand that the wave function at the origin remain finite. As we will see below this requirement forces the wave function to vanish at the origin like $r^{|\nu|}$ in the s wave channel. The appearance of $|\nu|$ in this boundary condition results in a scattering amplitude that is non-analytic as a function of ν and $\nu = 0$. [40]

The most general solution to the Schrodinger equation consistent with the boundary conditions described above is given by

$$\psi(r,\theta) = \sum_{n>0} a_n e^{in\theta} J_{n+\nu}(kr) + \sum_{n>0} a_{-n} e^{-in\theta} J_{n-\nu} + a_0 J_{|\nu|}(kr) \tag{208}$$

Recall the asymptotic expansion of Bessel functions at small and large values of the argument

$$J_\alpha(x) = \frac{\left(\frac{x}{2}\right)^\alpha}{\Gamma(\alpha+1)} + \ldots, \quad = \frac{1}{\sqrt{2\pi x}} \left(e^{ix - i\frac{\pi}{4} - i\frac{\alpha\pi}{2}} + e^{-ix + i\frac{\pi}{4} + i\frac{\alpha\pi}{2}} \right) \tag{209}$$

and the expansion of the plane wave in terms of Bessel functions

$$e^{ikx} = \sum_n i^n J_n(kr) e^{in\theta} \tag{210}$$

and the large r expansion of this plane wave (obtained by substituting (208) into (210))

$$\begin{aligned}
e^{ikx'} &= e^{ikr'\cos(\theta)} = \sum_n i^n e^{in\theta} J_n(kr) \\
\sum_n i^n e^{in\theta} J_n(kr) &\approx \frac{1}{\sqrt{2\pi kr}} \sum_n i^n e^{in\theta} \left((e^{ikr - \frac{i\pi n}{2} - \frac{i\pi}{4}} + e^{-ikr + \frac{i\pi n}{2} + \frac{i\pi}{4}} \right) \qquad (r \gg 1) \\
&= \frac{2\pi}{\sqrt{2\pi kr}} \left(e^{\frac{-i\pi}{4}} e^{ikr} \delta(\theta) + e^{\frac{i\pi}{4}} e^{-ikr} \delta(\theta - \pi) \right).
\end{aligned} \tag{211}$$

[41] It is easy to see that the unique solution of the form (208) whose ingoing part - i.e. part

[40] See [22] for a fascinating one parameter self adjoint relaxation of this boundary condition (which infact yields analytic S-matrices at $w = 1$) .

[41] This formula is very picturesque; it describes an incoming wave from the negative x axis (so at $\theta = -\pi$) and an outgoing wave along the positive x axis (so at $\theta = 0$). In particular, the outgoing part of the incident wave is equivalent to a contribution to the scattering amplitude proportional to $\delta(\theta)$.

proportional to e^{-ikr} - is identical to the plane wave (210) is given by

$$\psi(r,\theta)=\sum_{n=1}^{\infty} i^n e^{-i\frac{\pi\nu}{2}} J_{n+\nu}(kr)e^{in\theta}+\sum_{n=1}^{\infty} i^n e^{i\frac{\pi\nu}{2}} J_{n-\nu}(kr)e^{-in\theta}+e^{-i\frac{\pi|\nu|}{2}} J_{|\nu|}(kr) \tag{212}$$

The scattering amplitude At large r $\psi(r)$ reduces to

$$\frac{1}{\sqrt{2\pi kr}}\left(2\pi e^{i\frac{\pi}{4}}\delta(\theta-\pi)e^{-ikr}+H(\theta)e^{-i\frac{\pi}{4}}e^{ikr}\right) \tag{213}$$

where

$$H(\theta)=e^{-i\pi|\nu|}+\sum_{n=1}^{\infty}\left(e^{-i\pi\nu}e^{in\theta}+e^{i\pi\nu}e^{-in\theta}\right). \tag{214}$$

Decomposing $H(\theta)$ up into its even and odd parts and then further processing we find

$$\begin{aligned}
H(\theta) &= \left(\sum_{n=1}^{\infty} 2\cos(\pi\nu)\cos(n\theta)\right)+e^{-i|\nu|\pi}+\left(\sum_{n=1}^{\infty} 2\sin(\pi\nu)\sin(n\theta)\right)\\
&= \left(\cos(\nu\pi)+\sum_{n=1}^{\infty} 2\cos(\pi\nu)\cos(n\theta)\right)-i|\sin(\nu\pi)|+\left(\sum_{n=1}^{\infty} 2\sin(\pi\nu)\sin(n\theta)\right)\\
&= 2\pi\cos(\pi\nu)\delta(\theta)-i|\sin(\nu\pi)|+\left(\sum_{n=1}^{\infty} 2\sin(\pi\nu)\sin(n\theta)\right)\\
&= 2\pi\cos(\pi\nu)\delta(\theta)+\sin(\pi\nu)\mathrm{Pv}\left(\cot\left(\frac{\theta}{2}\right)\right)-i|\sin(\pi\nu)|\\
&= 2\pi\cos(\pi\nu)\delta(\theta)+\sin(\pi\nu)\mathrm{Pv}\left(\frac{e^{-i\frac{\theta\,\mathrm{sgn}[\nu]}{2}}}{\sin\left(\frac{\theta}{2}\right)}\right).
\end{aligned} \tag{215}$$

[42] It is conventional to write the wave function as a plane wave plus a scattered piece ; at large r

$$\psi(r) = e^{ikx} + \frac{h(\theta)e^{-i\frac{\pi}{4}}e^{ikr}}{\sqrt{2\pi kr}}. \tag{218}$$

Plugging (211) into (218) and comparing with (213) we conclude that

$$h(\theta) = H(\theta) - 2\pi\delta(\theta) \tag{219}$$

so that

$$h(\theta) = 2\pi\left(\cos(\pi\nu) - 1\right)\delta(\theta) + \sin(\pi\nu)\mathrm{Pv}\left(\frac{e^{-i\frac{\theta\mathrm{sgn}[\nu]}{2}}}{\sin\left(\frac{\theta}{2}\right)}\right). \tag{220}$$

Physical interpretation of the δ function at forward scattering It is intuitively clear that the amplitude for propagation (path integral) for a particle starting out a large distance away from the origin on the negative real axis, to a position nearer the scattering center has enough information to compute the scattering S-matrix. [43] The amplitude for a particle to propagate from far to the left of the origin to a point near the origin (lets say at angle $\theta \approx \pi$ for definiteness) receives contributions from path whose angular winding around the origin are approximately

---

[42] In going from the third to the fourth line above we have used the formula

$$\mathrm{Pv}\left(\cot\left(\frac{\theta}{2}\right)\right) = 2\sum_{m=1}^{\infty}\sin(m\theta) \tag{216}$$

This formula is equivalent to the assertion that

$$\int \frac{d\theta}{2\pi i}\mathrm{Pv}\cot\left(\frac{\theta}{2}\right)e^{im\theta} = \mathrm{sgn}(m) \tag{217}$$

(the integral on the RHS of (217) clearly vanishes when m=0 as $\mathrm{Pv}(\cot\left(\frac{\theta}{2}\right))$ is an odd function). The integral on the LHS of (217) can be converted into a contour integral about the unit circle on the complex plane via the substitution $z = e^{i\theta}$. The contour integral in question is simply

$$\oint \frac{dz}{2\pi i}\mathrm{Pv}\frac{z^{m-1}(z+1)}{z-1}$$

This integral is easily seen to evaluate to unity for $m \geq 1$ when it receives contributions only from the pole at unity. The substitution $z = \frac{1}{w}$ allows one to conclude as easily that the integral evaluates to -1 for $m \leq -1$, establishing (216).

[43] Let us explain how scattering data may be extracted in practice. Recall that the amplitude for a free particle to propagate from polar coordinates r, θ to polar coordinates r', θ' in time t is given by

$$A_F(r,\theta,r',\theta',t) = \frac{1}{2\pi it}e^{i\left(\frac{r^2+(r')^2-2rr'\cos(\theta-\theta')}{2t}\right)} \tag{221}$$

$$\phi_k^F(r',\theta') = 2\pi i\sqrt{t}e^{-i\frac{r^2}{2t}}A_F(r,\theta,r',\theta',t) \tag{222}$$

$...-3\pi, -\pi, \pi, 3\pi....$ Of these infinitely many paths those with winding approximately π and $-\pi$ are special. These sectors consist of paths that go below the origin, and paths that go above the origin, but do not otherwise wind the origin. It may be shown that these paths are entirely responsible for the terms in $H(\theta)$ (see the previous subsection) proportional to $\delta(\theta)$.

For a free plane wave $H(\theta) = 2\pi\delta(\theta)$. In a 'traditional' scattering problem $H(\theta) = 2\pi\delta(\theta) +$ nonsingular i.e. the incident wave goes through largely untouched, and in addition we have some scattering. In the problem with Aharonov-Bohm scattering, however, we have seen in the last subsection that $H(\theta) = 2\pi\cos(\pi\nu)\delta(\theta)$. This fact is easily interpreted. The contribution of paths with winding π and $-\pi$ in this problem is identical to the contribution of the same paths in the free theory except that the paths with winding π are weighted by an additional phase $e^{i\pi\nu}$ while the paths with winding $-\pi$ are weighted by the additional phase $e^{-i\pi\nu}$. The two sectors are flipped by reflection and so otherwise contribute equally. This explains the modulation of the $\delta(\theta)$ part of $H(\theta)$ by $\cos(\pi\nu)$, and the consequent appearance of the term $2\pi(\cos(\pi\nu) - 1)\delta(\theta)$ in $h(\theta)$.

1.9.4 Details of the computation of the scalar S-matrix

Computation of the effective one particle exchange interaction In this subsection we explicitly compute the summation over the effective 'one particle exchange' four point interactions depicted in Fig 4. We perform our computation in Euclidean space and analytically continue our final result back to Euclidean space. In Figure 19 we redraw the diagrams of Fig 4, this time including detailed momentum assignments for all legs.

---

is the wave function at time t of a particle, initially localized to a delta function located at r, θ. In the limit

$$r \to \infty, \quad t \to \infty \quad \frac{mr}{\bar{h}t} = k = \text{fixed}, \quad r', \theta' = \text{fixed} \tag{223}$$

we have

$$\phi_k^F = e^{ikx'} \tag{224}$$

i.e the wave function reduces to a plane wave. In the case of an interacting theory with interactions localized around the origin, let the amplitude for the particle to propagate from r, θ to polar coordinates r', θ' in time t be denoted by $A(r, \theta, r', \theta', t)$. It follows that the scattering wave function for our problem is given by

$$\phi_k(r', \theta') = 2\pi i\sqrt{t}e^{-i\frac{r^2}{2t}}A(r, \theta, r', \theta', t) \tag{225}$$

in the limit (223) as this path integral produces a wave function with an incoming piece that is indistinguishable from a plane wave near the origin. The scattering amplitude $h(\theta)$ is read off from the large r' expansion of $\phi_k(r', \theta)$ in the usual manner.

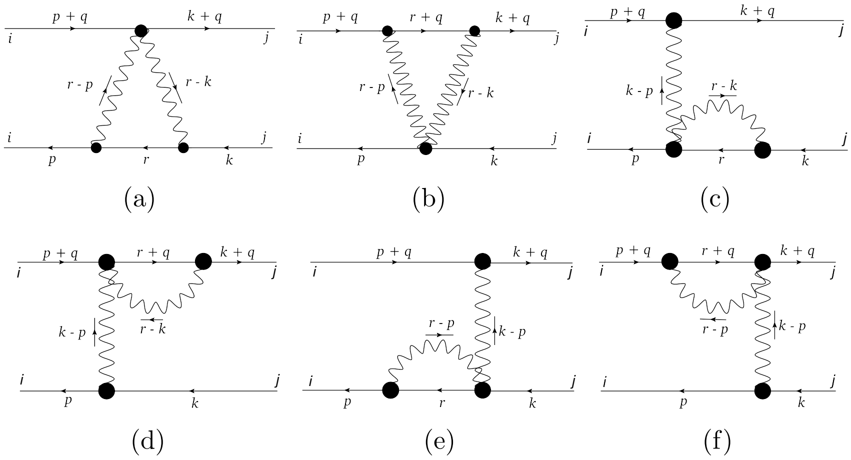


Figure 19: The one loop diagrams that contribute to the unit represented by the triple line excluding the tree level diagram. Note that box diagram is not included here as it is one of the contributions from two units sewn together.

[44].

The graph in Fig. 19(a) evaluates to

$$
\begin{aligned}
NA_1 &= (-4\pi^2\lambda^2)\int -\frac{(r+p)_-}{(r-p)_-}\frac{(r+k)_-}{(r-k)_-}\frac{1}{r^2+c_B^2}\frac{d^3r}{(2\pi)^3} \\
&= \int -\left(1+2\frac{(p+k)_-}{(p-k)_-}\left(\frac{p_-}{(r-p)_-}-\frac{k_-}{(r-k)_-}\right)\right)\frac{1}{r^2+c_B^2}\frac{d^3r}{(2\pi)^3}
\end{aligned}
$$

Let θ denote the phase of the complex number r_-. Since r^2 doesn't have a θ dependence, performing the θ integration first,

$$
NA_1 = (-4\pi^2\lambda^2)\int -\left(1-2\frac{(p+k)_-}{(p-k)_-}\left(\theta(p_s-r_s)-\theta(k_s-r_s)\right)\right)\frac{1}{r^2+c_B^2}\frac{dr_3 r_s dr_s}{(2\pi)^2} \tag{226}
$$

The graph in Fig. 19(b) evaluates to

$$
NA_2 = (-4\pi^2\lambda^2)\int -\frac{(r+p+2q)_-}{(r-p)_-}\frac{(r+k+2q)_-}{(r-k)_-}\frac{1}{(r+q)^2+c_B^2}\frac{d^3r}{(2\pi)^3}
$$

[44]All the graphs below have the common overall factor $-4\pi^2\lambda^2$, because they each have a single internal scalar propagator, two internal gauge propagators, and two $\phi\phi A$ 3. The scalar propagators contribute with no factors. The gauge propagators are each proportional to $2\pi i\lambda$. The triple vertices each contribute a factor of i. And finally we get an overall minus sign from the fact that we are computing the contribution to the Euclidean effective action which appears in the path integral as e^{-S_E}

we can change the integration variable$r \to r-q$ and define variables $p' = p+q$,$k' = k+q$.

$$
\begin{aligned}
NA_2 &= (-4\pi^2\lambda^2)\int -\frac{(r+p')_-}{(r-p')_-}\frac{(r+k')_-}{(r-k')_-}\frac{1}{r^2+c_B^2}\frac{d^3r}{(2\pi)^3} \\
&= \int -\left(1+2\frac{(p'+k')_-}{(p-k)_-}\left(\frac{p'_-}{(r-p')_-}-\frac{k'_-}{(r-k')_-}\right)\right)\frac{1}{r^2+c_B^2}\frac{d^3r}{(2\pi)^3}
\end{aligned}
$$

Again, performing the θ integration first,

$$
NA_2 = (4\pi^2\lambda^2)\int\left(1-2\frac{(p'+k')_-}{(p-k)_-}\left(\theta(p'_s-r_s)-\theta(k'_s-r_s)\right)\right)\frac{1}{r^2+c_B^2}\frac{dr_3 r_s dr_s}{(2\pi)^2} \tag{227}
$$

Fig. 19(c) evaluates to

$$
\begin{aligned}
NA_3 &= (-4\pi^2\lambda^2)\int -\frac{(p+k+2q)_-}{(p-k)_-}\frac{(r+k)_-}{(r-k)_-}\frac{1}{r^2+c_B^2}\frac{d^3r}{(2\pi)^3} \\
&= (-4\pi^2\lambda^2)\int -\frac{(p'+k')_-}{(p-k)_-}\left(1+2\frac{k_-}{(r-k)_-}\right)\frac{1}{r^2+c_B^2}\frac{d^3r}{(2\pi)^3} \\
&= (-4\pi^2\lambda^2)\int -\frac{(p'+k')_-}{(p-k)_-}\left(1-2\theta(k_s-r_s)\right)\frac{1}{r^2+c_B^2}\frac{dr_3 r_s dr_s}{(2\pi)^2}
\end{aligned} \tag{228}
$$

Fig. 19(d) evaluates to

$$
\begin{aligned}
NA_4 &= (-4\pi^2\lambda^2)\int -\frac{(p+k)_-}{(p-k)_-}\frac{(r+k+2q)_-}{(r-k)_-}\frac{1}{(r+q)^2+c_B^2}\frac{d^3r}{(2\pi)^3} \\
&= (-4\pi^2\lambda^2)\int -\frac{(p+k)_-}{(p-k)_-}\frac{(r+k')_-}{(r-k')_-}\frac{1}{r^2+c_B^2}\frac{d^3r}{(2\pi)^3} \\
&= (-4\pi^2\lambda^2)\int -\frac{(p+k)_-}{(p-k)_-}\left(1+2\frac{k'_-}{(r-k')_-}\right)\frac{1}{r^2+c_B^2}\frac{d^3r}{(2\pi)^3} \\
&= (-4\pi^2\lambda^2)\int -\frac{(p+k)_-}{(p-k)_-}\left(1-2\theta(k'_s-r_s)\right)\frac{1}{r^2+c_B^2}\frac{dr_3 r_s dr_s}{(2\pi)^2}
\end{aligned} \tag{229}
$$

Fig. 19(e) evaluates to

$$
\begin{aligned}
NA_5 &= (-4\pi^2\lambda^2)\int \frac{(p+k+2q)_-}{(p-k)_-}\frac{(r+p)_-}{(r-p)_-}\frac{1}{r^2+c_B^2}\frac{d^3r}{(2\pi)^3} \\
&= (-4\pi^2\lambda^2)\int \frac{(p'+k')_-}{(p-k)_-}\left(1+2\frac{p_-}{(r-p)_-}\right)\frac{1}{r^2+c_B^2}\frac{d^3r}{(2\pi)^3} \\
&= (-4\pi^2\lambda^2)\int \frac{(p'+k')_-}{(p-k)_-}\left(1-2\theta(p_s-r_s)\right)\frac{1}{r^2+c_B^2}\frac{dr_3 r_s dr_s}{(2\pi)^2}
\end{aligned} \tag{230}
$$

Fig. 19(f) evaluates to

$$\begin{aligned} NA_6 &= (-4\pi^2\lambda^2)\int \frac{(p+k)_-}{(p-k)_-}\frac{(r+p+2q)_-}{(r-p)_-}\frac{1}{(r+q)^2+c_B^2}\frac{d^3r}{(2\pi)^3} \\ &= (-4\pi^2\lambda^2)\int -\frac{(p+k)_-}{(p-k)_-}\frac{(r+p')_-}{(r-p')_-}\frac{1}{r^2+c_B^2}\frac{d^3r}{(2\pi)^3} \\ &= (-4\pi^2\lambda^2)\int \frac{(p+k)_-}{(p-k)_-}\left(1+2\frac{p'_-}{(r-p')_-}\right)\frac{1}{r^2+c_B^2}\frac{d^3r}{(2\pi)^3} \\ &= (-4\pi^2\lambda^2)\int \frac{(p+k)_-}{(p-k)_-}\left(1-2\theta(p'_s-r_s)\right)\frac{1}{r^2+c_B^2}\frac{dr_3 r_s dr_s}{(2\pi)^2} \end{aligned} \tag{231}$$

The total Amplitude is

$$NA_{tot} = \sum_{i=1}^{6} A_i \tag{232}$$

Which gives

$$\begin{aligned} NA_{tot} = &\int \frac{dr_3 r_s dr_s}{(2\pi)^2}\left[\frac{(-4\pi^2\lambda^2)}{r^2+c_B^2}\right. \\ &\times \left.\left(-2+\frac{4q_-}{(p-k)_-}\left[\theta(p'_s-r_s)-\theta(k'_s-r_s)+\theta(k_s-r_s)-\theta(p_s-r_s)\right]\right)\right] \end{aligned} \tag{233}$$

Where we recall that

$$p' = p+q, \quad k' = k+q.$$

We are interested in the special case $q^{\pm} = 0$. In this case the $p'_{\pm} = p_{\pm}$ and $k'_{\pm} = k_{\pm}$, and so $k's = k_s$ and $p'_s = p_s$. It follows that the θ functions in (233) cancel in pairs and

$$\begin{aligned} NA_{tot} &= (-2)(-4\pi^2\lambda^2)\int \frac{1}{r^2+c_B^2}\frac{dr_3 r_s dr_s}{(2\pi)^2} \\ &= 8\pi^2\lambda^2 \int \frac{1}{r^2+c_B^2}\frac{d^3r}{(2\pi)^3} \end{aligned} \tag{234}$$

We use dimensional regularization, which replaces the integral by $(\frac{m}{4\pi})$. So ultimately these diagrams give

$$NA_{loop} = 2\pi\lambda^2 c_B \tag{235}$$

Euclidean rotation The integral equation (90) may be used to solve for the function $V(p^0, \vec{p}, k^0, \vec{k}, q^3)$. In this subsection we will be interested only in the dependence of V on p^0 and k^0 and so use the notation $V = V(p^0, k^0)$.

As is often the case in the study of relativistic scattering amplitudes, in this chapter we will find it convenient determine V by first computing its 'Euclidean continuation'. In this brief subsection we pause to define the Euclidean continuation of V, and to determine the integral equation it obeys.

Given the amplitude $V(p^0, k^0)$ we define the one parameter set of amplitudes, $V_\alpha(p^0, k^0)$ for $0 \leq \alpha \leq \frac{\pi}{2}$ as follows. Let us assume that $V(p^0, k^0)$ admits an analytic continuation to the function $V(z, w)$ for $0 < Arg(z) < \frac{\pi}{2}$ and $0 < Arg(w) < \frac{\pi}{2}$. We also assume that this function can be defined to be free of singularities when $Arg(z) = Arg(w)$. In terms of this analytic function, we define a one parameter extension, V_α of V by

$$V_\alpha(p^0, k^0) = V(p^0 e^{i\alpha}, k^0 e^{i\alpha}).$$

It follows in particular that V_α is a smooth function of α.

The Euclidean continuation, V_E of V is defined by

$$V_E(p^0, q^0) = V_{\frac{\pi}{2}}(p^0, k^0).$$

Note in particular that

$$V_E(p^0, k^0) = V(ip^0, ik^0)$$

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In order to obtain the integral equation obeyed by $V_\alpha(p^0, q^0)$ one must, of course, make the replacement $p^0 \to e^{i\alpha} p^0$, $k^0 \to e^{i\alpha} k^0$ in (34). However this replacement must also be accompanied by a simultaneous change in the contour of integration of the variable r^0. If the r^0 contour is left unchanged then the pole

$$\frac{1}{(p-r)_+(p-r)_- - i\epsilon}$$

in the integrand in the first of (34) could cross the contour of integration at a particular value of α, leading to a non-analyticity in V_α as a function of α. In order to define V_α as a smooth function with no singularities, we adopt the following procedure. For any given p^0 and α we first deform the contour of integration over the variable r^0. This deformation is performed without crossing any singularities in the integrand, and so does not change the value of the integral. It is chosen in a manner that ensures that the rotation $p^0 \to p^0 e^{i\alpha}$ can be performed without the pole crossing the contour of integration; for any fixed p^0 and α such a deformation may always be found. After the rotation on p^0 is now performed, the integration contour for r^0 is further modified to suit convenience. It is convenient to choose the final contour for integration over r^0 to be the rotation of the initial contour counterclockwise by the angle α, together with two arcs of angle α at ∞. It is easily verified that the arcs at infinity do not contribute to the integral (because the integrand dies off fast enough at infinity) .

[45] This equation that is sometimes summarized by the mnemonic $p^0_E = -ip^0_L, \quad k^0_E = -ik^0_L$.

In summary, the integral equation obeyed by the function $V_\alpha(p^0,k^0)$ is given by making the replacements $p^0 \to e^{i\alpha}p^0,\ \ k^0 \to e^{i\alpha}k^0,\ \ r^0 \to e^{i\alpha}r^0$ in (34) and then continuing to integrate the new r^0 variable over its real axis. The integral equation for V_E is given by the special case $\alpha = \frac{\pi}{2}$.

Solution of the Euclidean integral equations In this subsection we determine the solution to the scalar Euclidean integral equation (90).

Differentiating the first equation in (90) w.r.t. p^3 we conclude that $\partial_{p^3}V^E = 0$. It follows that V is independent of k_3 and p_3. In a similar manner, from the second equation we conclude that $\partial_{k^3}V^E = 0$. The identity

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)((x+y)^2+a^2)} = \frac{2\pi}{|a|(y^2+4a^2)}$$

may now be used to perform the integral over r_3 on the RHS of the first two equations in (90). Defining

$$a(p) = \sqrt{c_B^2 + \vec{p}^2} = \sqrt{c_B^2 + 2p_+p_-}$$

where the square root on the RHS is positive by definition, we find

$$\begin{aligned}
V^E(p,k,q) &= V_0^E(p,k,q) + \int \frac{d^2r}{(2\pi)^2} V_0^E(p,r,q_3) \frac{N}{a(r)(q_3^2+4a^2(r))} V^E(r,k,q_3) \\
V^E(p,k,q) &= V_0^E(p,k,q) + \int \frac{d^2r}{(2\pi)^2} V^E(p,r,q_3) \frac{N}{a(r)(q_3^2+4a^2(r))} V_0^E(r,k,q_3) \\
NV_0^E(p,k,q_3) &= -4\pi i\lambda q_3 \frac{(k+p)_-}{(k-p)_-} + \widetilde{b}_4
\end{aligned} \tag{236}$$

Now if $z = \frac{x+iy}{\sqrt{2}}$ then

$$\partial_z = \frac{1}{2}\left(\partial_x - i\partial_y\right), \quad \nabla^2 = 2\partial_z\partial_{\bar{z}}, \quad \partial_{\bar{z}}\frac{1}{z} = \partial_z\partial_{\bar{z}}\ln(z\bar{z}) = \nabla^2 \ln r = 2\pi\delta^2(\vec{r}).$$

It follows from (236) that

$$\begin{aligned}
\partial_{p_+}\left(V - V_0\right) &= \frac{4i\lambda q_3 p_-}{a(p)(q_3^2+4a^2(p))}V, \\
\partial_{k_+}\left(V - V_0\right) &= -\frac{4i\lambda q_3 k_-}{a(k)(q_3^2+4a^2(k))}V.
\end{aligned} \tag{237}$$

The equations (237) may be regarded as first order ordinary differential equations in the variables p_+ and k_+ respectively. These equations are easily solved. Using the identities

$$\int \frac{dp_+p_-}{a(p)(q_3^2+4a(p)^2)} = \int \frac{da}{q_3^2+4a^2} = \frac{1}{2|q_3|}\tan^{-1}\left(\frac{2a}{|q_3|}\right)$$

If we agree to choose a definition of $\tan^{-1}$ that makes it an odd function we can drop the modulus signs in this formula. Of course we would also like the $\tan^{-1}$ function to be continuous; these requirements together fix the branch choice

$$-\frac{\pi}{2} < \tan^{-1}(x) < \frac{\pi}{2}$$

It follows that (237) may be recast as

$$\begin{aligned}\partial_{p_+}\left(e^{-2i\lambda\tan^{-1}\left(\frac{2a(p)}{q_3}\right)}V\right) &= \left(e^{-2i\lambda\tan^{-1}\left(\frac{2a(p)}{q_3}\right)}\right)\partial_{p_+}V_0,\\ \partial_{k_+}\left(e^{2i\lambda\tan^{-1}\left(\frac{2a(k)}{q_3}\right)}V\right) &= \left(e^{2i\lambda\tan^{-1}\left(\frac{2a(k)}{q_3}\right)}\right)\partial_{k_+}V_0.\end{aligned} \tag{238}$$

The equations (238) are now easily solved by integration. It might at first seem that the integral of the RHS of these equations is complicated by the fact that the term multiplying $\partial_{p_+}V_0$ in the first equation on the RHS of (238) is actually a function of p. Recall, however, that $\partial_{p_+}V_0$ is proportional to the δ function; using the formula $f(x)\delta(x-a) = f(a)\delta(x-a)$ we can replace the argument of this prefactor by the corresponding function of k_+. Similar remarks apply to the second of (238). Integrating these two equations it follows that

$$\begin{aligned}NV &= (4\pi i\lambda q_3)\frac{p_- + k_-}{p_- - k_-}e^{-2i\lambda\left(\tan^{-1}\left(\frac{2(a(k)}{q_3}\right)-\tan^{-1}\left(\frac{2(a(p)}{q_3}\right)\right)}\\ &\quad - e^{2i\lambda\tan^{-1}\left(\frac{2a(p)}{q_3}\right)}h(k,p_-,q_3)\\ &= (4\pi i\lambda q_3)\frac{p_- + k_-}{p_- - k_-}e^{-2i\lambda\left(\tan^{-1}\left(\frac{2(a(k)}{q_3}\right)-\tan^{-1}\left(\frac{2(a(p)}{q_3}\right)\right)}\\ &\quad - e^{-2i\lambda\tan^{-1}\left(\frac{2a(k)}{q_3}\right)}\tilde{h}(k_-,p,q_3)\end{aligned} \tag{239}$$

Comparing these two equations determines the k_+ dependence of h and the p_+ dependence of $\tilde{h}$, and we conclude

$$NV(p,k,q_3) = e^{-2i\lambda\left(\tan^{-1}\left(\frac{2(a(k)}{q_3}\right)-\tan^{-1}\left(\frac{2(a(p)}{q_3}\right)\right)}\left(4\pi i\lambda q_3\frac{p_- + k_-}{p_- - k_-} + j(k_-,p_-,q_3)\right) \tag{240}$$

Now the function $j(k_-,p_-)$ above must be a function of charge zero, and so must be a function of $\frac{k_-}{p_-}$. It must also be singularity free (i.e. its derivative w.r.t both p_+ and k_+ must vanish. This seems impossible unless the function j is a constant, so we conclude

$$NV = e^{-2i\lambda\left(\tan^{-1}\left(\frac{2(a(k)}{q_3}\right)-\tan^{-1}\left(\frac{2(a(p)}{q_3}\right)\right)}\left(4\pi i\lambda q_3\frac{p_- + k_-}{p_- - k_-} + j(q_3)\right) \tag{241}$$

In order to evaluate $j(q_3)$ we now plug the form (241) back into (236), explicitly perform the integral over $\vec{r}$ and compare both sides of the integral equation. The integral over $\vec{r}$ may be evaluated in polar coordinates by integrating over the modulus r and the angle θ. We will find it convenient to perform the angular integral by contour methods. Let us define $z = e^{i\theta}$. Then $\int d\theta = \int_C \frac{dz}{2\pi i z}$ where the contour C runs counterclockwise over the unit circle on the complex plane. The first of (236) turns into

$$\begin{aligned}
& e^{2i\lambda \tan^{-1}\left(\frac{2\sqrt{k^2+c_B^2}}{q_3}\right)} (NV(p,k,q) - NV_0(p,k,q)) \\
&= \int dr \frac{r e^{2i\lambda \tan^{-1}\left(\frac{2\sqrt{r^2+c_B^2}}{q_3}\right)}}{\sqrt{c_B^2 + r^2(q_3^2 + 4(c_B^2 + r^2))}} I(r) = \frac{1}{4i\lambda q_3} \int dr \partial_r \left(e^{2i\lambda \tan^{-1}\left(\frac{2\sqrt{r^2+c_B^2}}{q_3}\right)} \right) I(r) \\
I(r) &= \int_C \frac{dz}{(2\pi)^2 i z} \left(-4\pi i \lambda q_3 \frac{rz + p_-}{rz - p_-} + \widetilde{b}_4 \right) \left(-4\pi i \lambda q_3 \frac{rz + k_-}{-rz + k_-} + j(q_3) \right)
\end{aligned} \tag{242}$$

Where z in $I(r)$ is integrated over the unit circle. We now proceed to evaluate I(r) using Cauchy's theorem. We find

$$\begin{aligned}
2\pi I(r) &= (4\pi i \lambda q_3 + \widetilde{b}_4)(-4\pi i \lambda q_3 + h) \\
&- \theta(r-p) 8\pi i \lambda q_3 \left(-4\pi i \lambda q_3 \frac{k_- + p_-}{k_- - p_-} + j(q_3) \right) \\
&+ \theta(r-k) 8\pi i \lambda q_3 \left(-4\pi i \lambda q_3 \frac{k_- + p_-}{k_- - p_-} + \widetilde{b}_4 \right)
\end{aligned} \tag{243}$$

where the first line is the contribution from the pole at $z = 0$, the second line is the contribution from the pole at $z = \frac{p_-}{r}$ and the third line is the contribution of the pole at $z = \frac{k_-}{r}$. Let us define

$$F(r) = e^{2i\lambda \tan^{-1}\left(\frac{2\sqrt{r^2+c_B^2}}{q_3}\right)}$$

It follows from (243) and (242) that

$$\begin{aligned}
& (8\pi i \lambda q_3) e^{2i\lambda \tan^{-1}\left(\frac{2\sqrt{k^2+c_B^2}}{q_3}\right)} (NV(p,k,q) - NV_0(p,k,q)) \\
&= (4\pi i \lambda q_3 + \widetilde{b}_4)(-4\pi i q_3 + j(q_3))(F(\infty) - F(0)) \\
&- 8\pi i \lambda q_3 \left(-4\pi i \lambda q_3 \frac{k_- + p_-}{k_- - p_-} + j(q_3) \right) (F(\infty) - F(p)) \\
&+ 8\pi i \lambda q_3 \left(-4\pi i \lambda q_3 \frac{k_- + p_-}{k_- - p_-} + \widetilde{b}_4 \right) (F(\infty) - F(k))
\end{aligned} \tag{244}$$

Substituting in for V and V_0, the LHS of this equation may be rewritten as

$$(8\pi i\lambda q_3)\left(F(p)\left(4\pi i\lambda q_3\frac{p_-+k_-}{p_--k_-}+j(k_-,p_-,q_3)\right)-F(k)\left(4\pi i\lambda q_3\frac{p_-+k_-}{p_--k_-}+\widetilde{b}_4\right)\right)$$

It follows LHS exactly cancels the terms proportional to $F(k)$ and $F(p)$, and (244) may be rewritten as

$$(-4\pi i\lambda q_3+\widetilde{b}_4)(+4\pi i\lambda q_3+j(q_3))F(\infty)=(4\pi i\lambda q_3+\widetilde{b}_4)(-4\pi i\lambda q_3+j(q_3))F(0)$$

This is a linear equation for $j(q_3)$ whose solution is given by

$$j(q_3)=4\pi i\lambda q_3\left(\frac{\left(4\pi i\lambda q_3+\widetilde{b}_4\right)F(0)+\left(-4\pi i\lambda q_3+\widetilde{b}_4\right)F(\infty)}{\left(4\pi i\lambda q_3+\widetilde{b}_4\right)F(0)-\left(-4\pi i q_3+\widetilde{b}_4\right)F(\infty)}\right) \tag{245}$$

Using

$$F(\infty)=e^{\pi i\lambda \mathrm{sgn}(q_3)},\quad F(0)=e^{2i\lambda\tan^{-1}\left(\frac{2c_B}{q_3}\right)}$$

we have

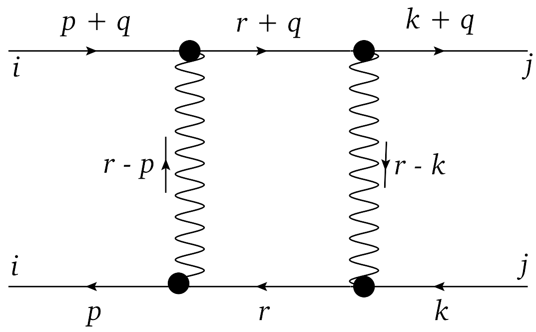
$$j(q_3)=4\pi i\lambda q_3\left(\frac{\left(4\pi i\lambda q_3+\widetilde{b}_4\right)e^{2i\lambda\tan^{-1}\left(\frac{2c_B}{q_3}\right)}+\left(-4\pi i\lambda q_3+\widetilde{b}_4\right)e^{\pi i\lambda\mathrm{sgn}(q_3)}}{\left(4\pi i\lambda q_3+\widetilde{b}_4\right)e^{2i\lambda\tan^{-1}\left(\frac{2c_B}{q_3}\right)}-\left(-4\pi i\lambda q_3+\widetilde{b}_4\right)e^{\pi i\lambda\mathrm{sgn}(q_3)}}\right) \tag{246}$$

In the limit $b_4\to\infty$ we have

$$\begin{aligned} j(q_3)&=-4\pi i\lambda q_3\left(\frac{e^{\pi i\lambda\mathrm{sgn}(q_3)}+e^{2i\lambda\tan^{-1}\left(\frac{2c_B}{q_3}\right)}}{e^{\pi i\lambda\mathrm{sgn}(q_3)}-e^{2i\lambda\tan^{-1}\left(\frac{2c_B}{q_3}\right)}}\right)\\ &=-4\pi i\lambda|q_3|\left(\frac{1+e^{-2i\lambda\tan^{-1}\left(\frac{|q_3|}{2c_B}\right)}}{1-e^{-2i\lambda\tan^{-1}\left(\frac{|q_3|}{2c_B}\right)}}\right)\end{aligned} \tag{247}$$

In summary, the off shell Euclidean sum of the diagrams depicted in Fig 3 is given by (241) with $j(q_3)$ given by (246).

The one loop box diagram computed directly in Minkowski space In this subsection, by the direct calculation of the one loop box diagram in the Minkowski space, we will show the cancellation of IR divergence of gauge propagator and that P in (119) becomes unity



(a)

Figure 20: Box diagram in the light cone gauge.

$P = 1$. In Minkowski space, the one loop box diagram (see Fig 20) evaluates to

$$\begin{aligned} I_{oneloop} = (4\pi\lambda q_3)^2 \int \frac{d^3r}{(2\pi)^3} \frac{(r+p)_-(p-r)_+}{(p-r)_+(p-r)_- - i\epsilon_1} \frac{(r+k)_-(k-r)_+}{(k-r)_+(k-r)_- - i\epsilon_1} \\ \frac{1}{2r_-r_+ + r_3^2 + c_B^2 - i\epsilon} \frac{1}{2(r+q)_-(r+q)_+ + (r+q)_3^2 + c_B^2 - i\epsilon}. \end{aligned} \tag{248}$$

Although we are interested in the value of this integral at $q_\pm = 0$, we have allowed $q_\pm \neq 0$ in the scalar propagators as a regulator; we will take the limit at the end of the computation. This manoeever allows us to evaluate the integral in a particularly simple manner.

Before embarking on the calculation, let us recall the issues involved. The term of $\mathcal{O}(\lambda^2)$ in the expansion of the offshell amplitude (93) (we set $b_4 = 0$ for simplicity) is

$$\begin{aligned} V_2 =& 8\pi\lambda^2 q_3 \left(\tan^{-1}\left(\frac{2a(k)}{q_3}\right) - \tan^{-1}\left(\frac{2a(p)}{q_3}\right)\right) \frac{p_- + k_-}{p_- - k-} \\ &+ 16\pi^2 q_3^2 \lambda^2 H(q_3) + 2\pi c_B \lambda^2. \end{aligned} \tag{249}$$

The last term in this equation is the contribution of the one loop diagrams in Fig. 4 to V. Offshell, consequently, we expect (248) to evaluate to

$$\begin{aligned} -iI_{oneloop} =& 8\pi\lambda^2 q_3 \left(\tan^{-1}\left(\frac{2a(k)}{q_3}\right) - \tan^{-1}\left(\frac{2a(p)}{q_3}\right)\right) \frac{p_- + k_-}{p_- - k-} \\ &+ 16\pi^2 q_3^2 \lambda^2 H(q_3). \end{aligned} \tag{250}$$

Here extra $-i$ factor comes from the analytic continuation as we can check by the relationship between (86) and (90). As mentioned at the beginning of this subsection, the reason we are undertaking this whole exercise is that the first term in (249) is naively ambiguous onshell, and we aim to discover its true value via a careful evaluation of (248).

In order to evaluate (248) we first evaluate the integral over r_+ integral using the methods of complex analysis. The integral over r_+ may be regarded as contour integral, where the contour runs from left to right along the real axis and then closes in a giant semi circle at infinity in the upper half plane. The integrand has four poles located at

$$\begin{aligned} r_+ &= p_+ + i\frac{\epsilon_1}{(r-p)_-}, \\ r_+ &= k_+ + i\frac{\epsilon_1}{(r-k)_-}, \\ r_+ &= -\frac{r_3^2 + c_B^2}{2r_-} + i\frac{\epsilon}{2r_-}, \\ r_+ &= -q_+ - \frac{(r_3+q_3)^2 + c_B^2}{2(r+q)_-} + i\frac{\epsilon}{2(r+q)_-}. \end{aligned} \tag{251}$$

Scalar poles From the point of view of IR divergences, the main point of interest in this section is the contribution from the first two poles in (251); the poles that have their origin in the gauge boson propagator. In order to be able to focus on the interesting part, however, it is useful to first get the 'boring' part of the answer out of the way. (Irrelevant part for the subtraction between $\tan^{-1}$ functions.) In this subsection we evaluate the contribution of the last two poles to the integral. In this subsection we assume for definiteness that the regulator $q_- < 0$ (it is not difficult to see that the final results do not depend on this assumption).

If $r_- < 0$ then neither of the third or fourth poles in (251) lie in the upper half plane, and so these poles do not contribute to the r_+ integral. On the other hand if $r_- > -q_- > 0$, both poles contribute to the integral, and it is not difficult to verify that the contribution of the two poles infact cancels. In other words the poles of interest contribute only in the range

$$0 < r_- < -q_-.$$

When r_- is in this window, we integrate over r_+ receives contributions only from the third pole in (251). Evaluating the residue of this pole redefining $r_- = -q_- x$, it is easily seen that

$$\begin{aligned} I^3_{oneloop} = \frac{i}{2}(4\pi\lambda q_3)^2 \int_0^1 \frac{dx}{2\pi} \int_{-\infty}^{\infty} \frac{dr_3}{2\pi} \Bigg[&\left(1 - 2\frac{(p+k)_-}{(p-k)_-}\left(\frac{p_-}{p_- + q_-\ x} - \frac{k_-}{k_- + q_-\ x}\right)\right) \\ &\times \frac{1}{r_3^2 + c_B^2 + q_3^2\ x + 2q_3 r_3 x - i\epsilon - 2q_- q_+(x^2 - x)}\Bigg]. \end{aligned} \tag{252}$$

In the limit $q_\pm \to 0$,

$$I^3_{oneloop} = \frac{i}{2}(4\pi\lambda q_3)^2 \int_0^1 \frac{dx}{2\pi} \int_{-\infty}^{\infty} \frac{dr_3}{2\pi} \frac{1}{r_3^2 + c_B^2 + q_3^2\ x + 2q_3 r_3 x - i\epsilon}, \tag{253}$$

(where $I^3_{oneloop}$ is the contribution of the third and fourth poles to the integral (248).)

We now evaluate the integral over r_3 by closing the contour in the upper half plane. The pole that contributes is at

$$r_3 = -q_3 x + i\sqrt{c_B^2 + q_3^2 x - q_3^2 x^2 - i\epsilon},$$

(note that $c_B^2 + q_3^2 x - q_3^2 x^2 > 0$). We find

$$\begin{aligned} I^3_{oneloop} &= \frac{i}{4}(4\pi\lambda q_3)^2 \int_0^1 \frac{dx}{2\pi} \frac{1}{\sqrt{c_B^2 + q_3^2 x - q_3^2 x^2}} \\ &= 2\pi\lambda^2 \sqrt{q_3^2}\left(\log(2m + i\sqrt{q_3^2}) - \log(2m - i\sqrt{q_3^2})\right) \\ &= i(4\pi\lambda q_3)^2 H(q), \end{aligned} \tag{254}$$

in precise agreement with the second term in (250).

As the sum of the third and fourth poles in (251) yields the second term in (250), the sum of the first two poles must give rise to the first term in (250). We will now verify that that is indeed the case.

Contributions of the gauge boson poles off shell The first two poles in (251) are a consequence of our resolution of the singularity of the gauge boson propagator. Offshell, the contribution of these poles to the integral (248) is very simple: we pause to explain this fact. Consider the integral

$$\int dl_+ dl_- \frac{l_+}{l_+ l_- - i\epsilon_1} f(l_+, l_-),$$

where f is any sufficiently smooth function. The integrand has a pole at

$$l_+ = \frac{i\epsilon_1}{l_-}.$$

If we evaluate the l_+ integral by closing the contour with a giant semicircle in the upper half plane, this pole contributes only if $l_- > 0$. The contribution of this pole to the integral is

$$2\pi i \int_0^\infty dl_- \frac{i\epsilon_1}{l_-^2} \theta(l_-) f(\frac{i\epsilon_1}{l_-}, l_-).$$

Provided $f(l_+, l_-)$ has no singularities if either of its arguments vanish, then in the limit $\epsilon_1 \to 0$ the integral over l_- receives contributions only from $l_- \sim \epsilon_1$, i.e. at finite values of the variable $y = \frac{\epsilon_1}{l_-}$. Changing integration variables to y we find that the contribution of this pole to the integral is given by

$$-2\pi \int_0^\infty dy f(iy, 0). \tag{255}$$

Provided all external momenta are offshell the analysis of the paragraph above applies, and allows us to easily evaluate the contribution of the first two poles to (248). Identifying the function f, applying the formula (255) and performing the integral over r_3 we find that the contribution of the first pole

$$\begin{aligned} I^1_{oneloop} &= -(4\pi\lambda q_3)^2 \frac{(k+p)_-}{(k-p)_-} \\ &\times \int_0^\infty \frac{d\widetilde{y}}{2\pi} \frac{1}{\sqrt{2p_-p_+ + c_B^2 - i\epsilon + i\widetilde{y}}} \frac{1}{4(2p_-p_+ + c_B^2 + i\widetilde{y} - i\epsilon) + q_3^2}, \end{aligned} \tag{256}$$

where

$$\tilde{y} = 2p_-y.$$

In (256), we took $\epsilon_1 \to 0$ limit already, and we also take into account that $p_- < 0$ inside the lightcone. Evaluating the integral we obtain

$$\begin{aligned} I^1_{oneloop} &= -(4\pi\lambda q_3)^2 \frac{(k+p)_-}{(k-p)_-} \\ &\times \frac{i}{2\pi\sqrt{q_3^2}} \left(-\frac{\pi}{2} + \tan^{-1}\left(2\sqrt{\frac{2p_-p_+ + c_B^2 - i\epsilon}{q_3^2}} \right) \right). \end{aligned} \tag{257}$$

Similarly the contribution of the second term is

$$\begin{aligned} I^2_{oneloop} &= (4\pi\lambda q_3)^2 \frac{(k+p)_-}{(k-p)_-} \\ &\times \frac{i}{2\pi\sqrt{q_3^2}} \left(-\frac{\pi}{2} + \tan^{-1}\left(2\sqrt{\frac{2k_-k_+ + c_B^2 - i\epsilon}{q_3^2}} \right) \right). \end{aligned} \tag{258}$$

Summing these two contributions we find perfect agreement with the first term in (250).

All we have seen so far is that the one loop four point function in Minkowski space is, indeed, the continuation of its Euclidean counterpart. Of course we knew this had to be true on general grounds, so the agreements obtained so far have simply been internal consistency checks. In order to get new information we will now investigate the contribution of the first two poles in (251) to the amplitude (248) when the external particles are all onshell. Recall that the continuation of the Euclidean answer - and the naive analysis of this subsection - yielded ambiguous answers for this quantity. Obtaining the correct result for this amplitude requires a more careful calculation which we now turn to .

The onshell contribution of the gauge boson poles In this subsubsection finally we will show that P in (119) is unity $P = 1$. In the previous two subsubsections, we have seen that gauge boson poles contribute to the first term of (250) while the scalar boson poles contributes to second term of the (250). Therefore our concerning factor

$$\left(\tan^{-1}\left(\frac{2a(k)}{q_3}\right) - \tan^{-1}\left(\frac{2a(p)}{q_3}\right)\right) \to \left(\tan^{-1}(-i) - \tan^{-1}(-i)\right) \tag{259}$$

in the first term of (250) is given by the contribution of the on-shell gauge boson poles.

When the momenta p and k are onshell, the analysis of Appendix 1.9.4 yields an ambiguous result. This is because the analysis presented above applies only when the function f of the previous section is sufficiently well behaved. This assumption is valid for generic values of p and k. When the two external momenta are onshell, however, it turns out that the function $f(l_-, l_+)$ of the previous subsection blows up at $l_- = 0$, invalidating the approximations used in the previous subsection. We will now present a more careful analysis of this special case. In this subsection we ignore the overall factor $(4\pi\lambda q_3)^2$ in (248); the factor is not important as the conclusion of this subsection is that the net contribution of the two gauge boson poles for the onshell 4 point function actually vanishes.

The contribution of the first gauge boson pole to the r_+ integral in (248) is given as

$$\begin{aligned}\int_0^\infty dy \int_{-\infty}^\infty dr_3 \Bigg[&-\frac{1}{2\pi}(1+\frac{\epsilon_1}{y})\frac{1}{X} \\ &\times \left(\frac{(y(k_- + p_-) + 2p_-\epsilon_1)(2p_-(k_+ - p_+) - iy)}{(y(k_- - p_-) - 2p_-\epsilon_1)(2p_-(k_+ - p_+) - iy) - i2p_- y\epsilon_1}\right)\Bigg]\end{aligned} \tag{260}$$

where we have made the variable redefinition

$$r_- = p_- + 2p_-\frac{\epsilon_1}{y} \tag{261}$$

and where

$$\begin{aligned}X = &\left(r_3^2 - p_3^2 - 2(p_3^2 + c_B^2)\frac{\epsilon_1}{y} + i(y - \epsilon + 2\epsilon_1)\right) \\ &\times \left((r_3 + q_3)^2 - p_3^2 - 2(p_3^2 + c_B^2)\frac{\epsilon_1}{y} + i(y - \epsilon + 2\epsilon_1)\right).\end{aligned} \tag{262}$$

We can obtain the second pole contribution by exchanging the momentum k and p.

By noting that

$$p_s^2 = k_s^2, \qquad p_3^2 = k_3^2,$$

when $p, k, (p+q), (k+q)$ are on-shell, we can see that the first line of (260) is symmetric under the exchange. We can see that $O(\epsilon_1^0)$ term of the second line of (260) is antisymmetric under the exchange of p and k because its form is

$$\frac{k_- + p_-}{k_- - p_-}.$$

Hence the sum of the contributions from first and second pole of (248) should be $O(\epsilon^1)$. In the integrand of (260), the variable r_3 appears only in factor X. Therefore the sum of the first and second pole contribution becomes following form

$$\int_y^\infty dy \int_{-\infty}^\infty dr_3 \left(\epsilon_1 \times \tilde{I}(y) \frac{1}{X(r_3, y)} \right). \tag{263}$$

Because of this explicit factor, in order to establish that (263) vanishes in the limit $\epsilon_1 \to 0$ it is sufficient to verify that the integral in (263) has no compensating singularity as $\epsilon_1 \to 0$. To investigate it, it is important to note that

$$(y(k_- - p_-) - 2p_-\epsilon_1)(2p_-(k_+ - p_+) - iy) - i2p_-y\epsilon_1 = 0 \Rightarrow y = \epsilon_1 = 0. \tag{264}$$

at the denominator of second line of (260) if $k_- \neq p_-$. [46] It is also useful to expand

$$\begin{aligned} -2\pi\tilde{I}(y) \sim & \frac{(k_- + p_-)}{(k_- - p_-)^2} \left(\frac{2ip_-}{2p_-(k_+ - p_+) - iy} + \frac{2ik_-}{2k_-(p_+ - k_+) - iy} \right) \\ & + \frac{8p_-k_-}{y(k_- - p_-)^2} + \mathcal{O}(\epsilon_1). \end{aligned} \tag{265}$$

The integral over r_3 in factor X is elementary, and may be explicitly performed; however the resultant expression is a slightly messy function of y and we do not present the explicit form here.

After performing the r_3 integral and further changing variables to $y_1 = \frac{y}{\epsilon_1}$, (263) reduces to an expression of the schematic form

$$I = \int_0^\infty dy_1 I(y_1). \tag{266}$$

Naively $I(y_1)$ is of order ϵ_1^2 (it picks up an additional factor of ϵ_1 from the change of variables $y = \epsilon_1 y_1$). Infact the singular behavior that results in the ill definition of the naive expression modifies this estimate for y_1 of order unity or smaller. Nonetheless it is possible to demonstrate

[46] In the case of $k_- - p_- = 0$ on-shell, LHS of (264) is always zero for any y, ϵ_1. This may intrigue to the delta function in the S-channel.

that $I(y_1) \leq O(\epsilon_1)$ throughout its integration domain. In pariticular [47]

$$I(y_1) \sim \begin{cases} \frac{\epsilon_1\sqrt{y_1}}{(c_B^2+p_3^2)^{\frac{3}{2}}} & (y_1 \ll 1) \\ \epsilon_1 & (y_1 \sim 1) \\ \epsilon_1^2 & (y_1 \sim \frac{p_3^2}{\epsilon_1}) \\ \frac{1}{\sqrt{\epsilon_1}}\left(\frac{1}{y_1}\right)^{\frac{5}{2}} & (y_1 \gg \frac{p_3^2}{\epsilon_1}) \end{cases} . \tag{271}$$

We can immediately see that (263) vanishes in the limit $\epsilon_1 \to 0$ in the first three cases in (271). Actually also in the case $y_1 \gg \frac{p_3^2}{\epsilon_1}$, we can check that it vanishes if we integrate over y_1

$$\left| \int_{\frac{p_3^2}{\epsilon_1}}^{\infty} dy_1 \frac{1}{\sqrt{\epsilon_1}} \left(\frac{1}{y_1}\right)^{\frac{5}{2}} \right| \sim \frac{1}{\sqrt{\epsilon_1}} (\epsilon_1)^{\frac{3}{2}} \sim \epsilon_1 \to 0. \tag{272}$$

So the net contribution of two gauge boson poles for one loop 4 point function, namely contribution for the first term of (248) vanishes. This results that the subtraction of $\tan^{-1}$ function vanishes as

$$\begin{aligned} 0 &= \left(\tan^{-1}\left(\frac{2a(k)}{q_3}\right) - \tan^{-1}\left(\frac{2a(p)}{q_3}\right)\right)\frac{p_- + k_-}{p_- - k-} \\ \Rightarrow 0 &= \left(\tan^{-1}\left(\frac{2a(k)}{q_3}\right) - \tan^{-1}\left(\frac{2a(p)}{q_3}\right)\right) \end{aligned} \tag{273}$$

in the on-shell p and k. Then finally we conclude that the P in (119) becomes unity $P = 1$.

---

[47]For instance at $y \ll \epsilon_1$ namely $y_1 \ll 1$ by performing the contour integral we get

$$\int dr_3 \frac{1}{X} \sim \int dr_3 \frac{1}{r_3^2 - 2(p_3^2 + c_B^2)\frac{\epsilon}{y} + i\tilde{\epsilon}} \frac{1}{(r_3 + q_3)^2 - 2(p_3^2 + c_B^2)\frac{\epsilon}{y} + i\tilde{\epsilon}} \sim \frac{\pi i y_1^{\frac{3}{2}}}{4\sqrt{2}(p_3^2 + c_B^2)^{\frac{3}{2}}}. \tag{267}$$

Then (260) behaves as

$$\int dy \left[\frac{\epsilon_1}{y} \int dr_3 \frac{1}{X} \left(\frac{2p_-\epsilon_1(2p_-(k_+ - p_+))}{-2p_-\epsilon_1(2p_-(k_+ - p_+))}\right)\right] \sim \int dy_1 \epsilon_1 \frac{\sqrt{y_1}}{(p_3^2 + c_B^2)^{\frac{3}{2}}}. \tag{268}$$

At $y \gg p_3^2$, namely $y_1 \gg \frac{p_3^2}{\epsilon_1}$, the integration over r_3 gives

$$\int dr_3 \frac{1}{X} \sim \int dr_3 \frac{1}{r_3^2 + iy} \frac{1}{(r_3 + q_3)^2 + iy} \sim \frac{-\pi e^{\frac{i\pi}{4}}}{2y^{\frac{3}{2}}}. \tag{269}$$

Then from (265) and $dy = \epsilon_1 dy_1$, we can see that (263) behaves as

$$\int dy \int_{-\infty}^{\infty} dr_3 \left(\epsilon_1 \times \tilde{I}(y) \frac{1}{X(r_3, y)}\right) \sim \int dy_1 \epsilon_1^2 \frac{1}{y_1^{\frac{5}{2}} \epsilon_1^{\frac{5}{2}}} \sim \int dy_1 \frac{1}{y_1^{\frac{5}{2}} \epsilon_1^{\frac{1}{2}}}. \tag{270}$$

1.9.5 Details of the one loop Landau gauge computation

In this subsection we provide some details for our evaluation of the one loop scattering amplitude in the covariant Landau gauge. As we have explained in the main text, the evaluation consists of determining the integrand for each graph, and then following standard manipulations that allow one to re-express the integrand in a standard basis. In order to illustrate how this works, we first present all steps in detail for the most complicated diagram (this is the box graph). For the remaining diagrams we content ourselves with a brief explanation or simply stating our results.

Simplification of the integrand of the box graph Straightforward use of the Feynman rules leads to an expression for the integrand of the box graph depicted in Fig. 8

$$\begin{aligned}\frac{1}{64\pi^2\lambda^2} I_{box} =& \frac{(\epsilon_{\nu_1\nu\beta} q_{\nu_1} p_\nu l_\beta \epsilon_{\mu_1\mu\beta_1} q_{\mu_1} (l+p)_\mu k_{\beta_1})}{l^2((l+p)^2+c_B^2)(l+p-k)^2((l+p+q)^2+c_B^2)} \\ =& \frac{k\cdot q\left[2\left((l\cdot k)\left(c_B^2 - l\cdot p\right) + (k\cdot p)(l\cdot(l+p))\right) - (l\cdot q)(l\cdot(k+p))\right]}{l^2((l+p)^2+c_B^2)(l+p-k)^2((l+p+q)^2+c_B^2)} \\ &+ \frac{(k\cdot q)(q\cdot l)\left(k\cdot p + c_B^2\right) + (k\cdot q)^2(l\cdot(-k+l+p)) + (k\cdot p)(q\cdot l)^2}{l^2((l+p)^2+c_B^2)(l+p-k)^2((l+p+q)^2+c_B^2)}\end{aligned} \tag{274}$$

The denominator of the expression above is the product $E_1E_2E_3E_4$ where

$$E_1 = c_B^2 + (l+p)^2,\ E_2 = c_B^2 + (p+q+l)^2,\ E_3 = l^2,\ E_4 = (l+p-k)^2. \tag{275}$$

The terms in the numerator RHS of (274) that involve the loop momentum l can be re-expressed as functions of the denominators plus terms independent of l. For example

$$\begin{aligned}&l\cdot l = E_3, \quad 2\ p\cdot l = E_1 - E_3, \\ &2\ q\cdot l = E_2 - E_1, \quad 2\ k\cdot l = E_1 - E_4 - 2\ c_B^2 - 2\ k\cdot p,\end{aligned}$$

where we have used onshell conditions

$$p^2 + c_B^2 = 0, \quad k^2 + c_B^2 = 0, \quad (p+q)^2 + c_B^2 = 0, \quad (k+q)^2 + c_B^2 = 0.$$

Judiciously using these and similar identities, it is easy to show that the integrand in (274) may be rewritten as

$$
\begin{aligned}
&-\frac{(k\cdot k-k\cdot p)(k\cdot q)(k\cdot q+2k\cdot k)}{E_1E_2E_3E_4}+\frac{k\cdot q\left(k\cdot q-2c_B^2\right)}{2E_1E_2E_3}+\frac{k\cdot q\left(k\cdot q-2c_B^2\right)}{2E_1E_2E_4}\\
&+\frac{k\cdot q\left(k\cdot p+c_B^2\right)}{E_2E_3E_4}+\frac{k\cdot q\left(k\cdot p+c_B^2\right)}{E_1E_3E_4}-\frac{(k\cdot p)(q\cdot l)}{2E_2E_3E_4}+\frac{(k\cdot p)(q\cdot l)}{2E_1E_3E_4}\\
&-\frac{k\cdot q}{2E_1E_2}+\frac{k\cdot q}{4E_1E_3}+\frac{k\cdot q}{4E_1E_4}+\frac{k\cdot q}{4E_2E_3}+\frac{k\cdot q}{4E_2E_4}-\frac{k\cdot q}{2E_3E_4}
\end{aligned}
\tag{276}
$$

The expression in (276) includes a term with four denominators. As we have mentioned in the main text, under the integral sign it is always possible to reduce any such expression into a linear combination of expressions with three or fewer denominators (recall we work in 3 spacetime dimensions). This reduction may be achieved by the systematic procedure spelt out in [34]. Implementing this procedure in the case at hand we find the replacement rule

$$
\begin{aligned}
-\frac{(k\cdot k-k\cdot p)k\cdot q(k\cdot q+2k\cdot k)}{E_1E_2E_3E_4}=&\frac{k\cdot q\left(2c_B^2-k\cdot q\right)}{2E_1E_2E_3}+\frac{k\cdot q\left(2c_B^2-k\cdot q\right)}{2E_1E_2E_4}\\
&-\frac{k\cdot q\left(k\cdot p+c_B^2\right)}{2E_1E_3E_4}-\frac{k\cdot q\left(k\cdot p+c_B^2\right)}{2E_2E_3E_4}
\end{aligned}
\tag{277}
$$

Using (277), the integrand for the box diagram reduces to

$$
\begin{aligned}
\frac{1}{64\pi^2\lambda^2}I_{box}=&\frac{k\cdot q\left(k\cdot p+c_B^2\right)}{2E_1E_3E_4}+\frac{k\cdot q\left(k\cdot p+c_B^2\right)}{2E_2E_3E_4}-\frac{(k\cdot p)(q\cdot l)}{2E_2E_3E_4}+\frac{(k\cdot p)(q\cdot l)}{2E_1E_3E_4}\\
&-\frac{k\cdot q}{2E_1E_2}+\frac{k\cdot q}{4E_1E_3}+\frac{k\cdot q}{4E_1E_4}+\frac{k\cdot q}{4E_2E_3}+\frac{k\cdot q}{4E_2E_4}-\frac{k\cdot q}{2E_3E_4}
\end{aligned}
\tag{278}
$$

We now turn to a discussion of the relations between distinct scalar (and other) integrands. Expressing the corresponding integrals in terms of Feynman parameters, it is not difficult to demonstrate that, under the integral sign

$$
\begin{aligned}
&\frac{1}{E_1E_2E_3}=\frac{1}{E_1E_2E_4}, \qquad \frac{1}{E_1E_3E_4}=\frac{1}{E_2E_3E_4}\\
&\frac{q.l}{E_1E_3E_4}=-\frac{q.l}{E_2E_3E_4}\\
&\frac{1}{E_1E_3}=\frac{1}{E_1E_4}=\frac{1}{E_2E_3}=\frac{1}{E_2E_4}.
\end{aligned}
\tag{279}
$$

For instance

$$
\begin{aligned}
\frac{1}{E_1 E_3} &= \int_0^1 \frac{dx}{\left(xE_3 + (1-x)E_1\right)^2} \\
&= \int_0^1 \frac{dx}{\left(l^2 + 2(1-x)l \cdot p\right)^2} \\
&= \int_0^1 \frac{dx}{\left(\tilde{l}^2 + (1-x)^2 c_B^2\right)^2}.
\end{aligned}
\tag{280}
$$

Similarly

$$
\begin{aligned}
\frac{1}{E_1 E_4} &= \int_0^1 \frac{dx}{\left(xE_4 + (1-x)E_1\right)^2} \\
&= \int_0^1 \frac{dx}{\left(l^2 + 2l \cdot p - 2\ p \cdot kx - 2\ l \cdot k\ x - 2c_B^2\ x\right)^2} \\
&= \int_0^1 \frac{dx}{\left(\tilde{l}^2 + (1-x)^2 c_B^2\right)^2}.
\end{aligned}
\tag{281}
$$

Using these relations we may rewrite the integrand for the box diagram as

$$
\frac{1}{64\pi^2\lambda^2} I_{box} = \frac{k \cdot q \left(k \cdot p + c_B^2\right)}{E_1 E_3 E_4} + \frac{(k \cdot p)(q \cdot l)}{E_1 E_3 E_4} - \frac{k \cdot q}{2E_1 E_2} + \frac{k \cdot q}{E_1 E_3} - \frac{k \cdot q}{2E_3 E_4}. \tag{282}
$$

In order to complete our simplification, we must now re-express the term

$$
\frac{(k \cdot p)(q \cdot l)}{E_1 E_3 E_4}
$$

in terms of scalar integrals. The procedure for doing this is once again standard [35, 36] and we find

$$
\begin{aligned}
\frac{(k \cdot p)(q \cdot l)}{E_1 E_3 E_4} \rightarrow & \frac{(k \cdot p)(k \cdot q)}{\left(c_B^2 - k \cdot p\right)} \frac{1}{E_1\ E_4} - \frac{(k \cdot p)(k \cdot q)}{c_B^2 - k \cdot p} \frac{1}{E_3\ E_4} \\
& + \frac{(k \cdot p)(k \cdot q)\left(k \cdot p + c_B^2\right)}{c_B^2 - k \cdot p} \frac{1}{E_1\ E_3\ E_4}.
\end{aligned}
$$

Using this replacement rule the integrand for the box diagram finally reduces to

$$
\begin{aligned}
I_{box} = 4\pi^2\lambda^2 \Bigg(& - \frac{8k \cdot q}{E_1\ E_2} - \frac{8(c_B^2 + k \cdot p)k \cdot q}{c_B^2 - k \cdot p} \frac{1}{E_3\ E_4} \\
& + \frac{16\ c_B^2 (k \cdot q)}{c_B^2 - k \cdot p} \frac{1}{E_1\ E_4} + \frac{16\ c_B^2 (c_B^2 + k \cdot p) k \cdot q}{c_B^2 - k \cdot p} \frac{1}{E_1\ E_3\ E_4} \Bigg).
\end{aligned}
\tag{283}
$$

Simplification of the remaining integrands We are left with the task of evaluating and simplifying the integrand of the remaining one loop scattering diagrams. These diagrams are listed in Fig. 9-7. The simplification of the integrand follows a procedure that similar to but much simpler than that adopted in the previous subsection. The diagrams of 9-7 are simpler than the box diagram considered in the previous subsection because none of them involves more than 3 propagators, so we never have to employ a replacement rule analogous to (277).

We briefly illustrate how things work in the specially simple case of the h graphs of Fig. 9. Since all of the four h diagrams are interrelated by linear momentum redefinitions, we can evaluate any one of them and multiply the result by 4. We consider first of these diagrams. Apart from come constant overall factor it gives

$$\begin{aligned}&\int \frac{d^3l}{(2\pi)^3}\frac{\epsilon^{\mu\nu\rho}(l+2p+2q)_\nu l_\rho g_{\mu\chi}\epsilon^{\chi\sigma\phi}(p+k)_\sigma(k-p)_\phi}{(k-p)^2l^2((l+p+q)^2+c_B^2)}\\ =&4\int \frac{d^3l}{(2\pi)^3}\frac{\epsilon^{\mu\nu\rho}(p+q)_\nu l_\rho g_{\mu\chi}\epsilon^{\chi\sigma\phi}p_\sigma k_\phi}{(k-p)^2l^2((l+p+q)^2+c_B^2)}.\end{aligned} \tag{284}$$

Introducing Feynman parameter x and eliminating cross-terms including l in the denominator by usual drill we get

$$4\int \frac{\epsilon^{\mu\nu\rho}(p+q)_\nu l_\rho g_{\mu\chi}\epsilon^{\chi\sigma\phi}p_\sigma k_\phi}{(k-p)^2(l^2+x(1-x)(p+q)^2+xc_B^2)^2}\frac{d^3l}{(2\pi)^3} \tag{285}$$

The integrand is odd in all components of l, hence the integration vanishes. It follows that

$$I_h = 0.$$

In a similar manner we find that the integrand for the sum of the two V diagrams (see Fig. 10) is

$$I_V = 4\pi^2\lambda^2\left(-\frac{2}{E_1}-\frac{8\ c_B^2}{E_1\ E_3}+\frac{6(c_B^2+k\cdot p)}{E_3\ E_4}-\frac{8\ c_B^2\ (c_B^2+k\cdot p)}{E_1\ E_3\ E_4}\right).$$

The integrand for the sum of the two Y diagrams (see Fig. 11) is

$$\begin{aligned}\frac{1}{4\pi^2\lambda^2}I_Y =&\frac{8c_B^2\left(k\cdot p+c_B^2\right)\left(-k\cdot p-2k\cdot q+c_B^2\right)}{(c_B^2-k\cdot p)\ E_1\ E_3\ E_4}\\ &+\frac{8c_B^2\left(-k\cdot p-2k\cdot q+c_B^2\right)}{(c_B^2-k\cdot p)\ E_1\ E_4}-\frac{4\left(k\cdot p+c_B^2\right)\left(-k\cdot p-2k\cdot q+c_B^2\right)}{(c_B^2-k\cdot p)\ E_3\ E_4}.\end{aligned} \tag{286}$$

The integrand for the sum of the eye diagrams (see Fig. 12) is

$$I_{Eye} = 4\pi^2\lambda^2\left(-\frac{2}{E_4}-\frac{2\left(k\cdot p+c_B^2\right)}{E_3\ E_4}\right).$$

Note that, contribution from lollipop diagrams (see Fig. 13) vanishes. Similarly, one can show that two diagrams in Fig. 7 each other. Summing all these contributions together, we find the remarkably simple integrand

$$I_{full} = 4\pi^2\lambda^2 \left(-\frac{2}{E_1} - \frac{2}{E_4} - \frac{8k \cdot q}{E_1\ E_2} \right). \tag{287}$$

It follows that (modulo possible subtleties at special values of external momenta, see the main text) the full one loop four boson scattering amplitude is given by

$$S_{\text{one loop}} = 2\pi m\lambda^2 + 32\pi^2(k \cdot p)\lambda^2 H(q). \tag{288}$$

Note, of course, that this result precisely matches the $\mathcal{O}(\lambda^2)$ term in the Taylor expansion of the function $j(q)$ at $b_4 = 0$.

Absence of IR divergences Notice that our scattering amplitude is finite without regulation; in particular the amplitude has no IR divergences. This is satisfying. IR divergences in theories like QED result from the fact that the asymptotic electron states of the theory are surrounded by a cloud of soft photons. The IR finiteness of our amplitude reflects the fact that Chern-Simons theories does not have massless gluonic states. Although the absence of IR divergences is physically very reasonable, at the technical level it appears to be a bit of a miracle, given the appearance of the massless gauge boson propagator at intermediate steps in the computation. Integrands of the form, for instance

$$\frac{1}{E_1E_3E_4}, \quad \frac{1}{E_1E_4}, \quad \frac{1}{E_1E_3} \tag{289}$$

that appear at intermediate steps in the computation, give rise to integrals that are IR divergent. The lack of IR divergences in our final result is a consequence of the cancellation of all these expressions in the final result for the integrand. For instance, the box diagram integrand Eq.282, first and second term are IR finite where as third and fourth are IR divergent. However, one can show log divergence arising from both the third and fourth integrands cancel each other[48]. Note

[48] One simple way to check this is using the following trick (refer to Bern's paper [37])

$$\begin{aligned} \int \frac{d^3p}{2\pi^3}\frac{1}{E_1E_3E_4} = &-\frac{1}{(k\cdot p + c_B^2)}(\int \frac{d^3p}{2\pi^3}\frac{1}{E_1E_4}) + \frac{1}{2c_B^2}\int \frac{d^3p}{2\pi^3}\frac{1}{E_3E_4} \\ &- N\left(\frac{2}{(p-k)^2} + \frac{1}{2c_B^2}\right)\int \frac{d^5p}{2\pi^5}\frac{1}{E_1E_3E_4}. \end{aligned} \tag{290}$$

where in the last line N is some number which is not important for our argument. Note that, third (last) term in the last line is IR convergent as this is in the higher dimension. The second term in the last line also IR convergent where first term is not, however, this IR divergence explicitly cancels the IR divergence that, the first line of box integral of Eq.274 has no IR divergence (near $l \sim 0$), so final should also have no IR divergence.

Absence of gauge boson cuts The imaginary part of any Feynman diagram may be determined using Cutkosky's rules. We pause to briefly review these rules (we follow a presentation due to 't Hooft and Veltman [38]). Given a graph one divides the vertices of the graph into two groups; circled and uncircled vertices. Associated with a particular distribution of circles for vertices , one defines a 'cut graph'. The expression for the cut graph is obtained from a sequence of modifications on the expression for the usual (uncut) Feynman graph as we now describe. The factor of i in each circled vertex is replaced by a factor of $-i$. Propagators between two circled vertices are replaced by their complex conjugates. Every factor of $\frac{1}{p^2+c_B^2-i\epsilon}$ in a cut propagator: i.e. a propagator that runs between a circled and uncircled vertex - is replaced by $\theta(p^0)\delta(p^2 + c_B^2 - i\epsilon)$ where p^0 is the energy running from the uncircled to the circled vertices. The sequence of modifications described above gives the expression for the 'cut graph' associated with a given distribution of circles for vertices.

Cutkowski's rules state that the imaginary part of any Feryman diagram is given by the sum of the expressions for cut graphs for all possible ways of distributing circles among the vertices of that graph subject to the restriction that at least one vertex in the graph is circled and at least one vertex is uncircled. Cutkowski's rules are the diagrammatic reflection of the unitarity of scattering amplitudes.

If we were to apply these rule to the one loop diagrams depicted in Fig. 8,9,10,11,1213, it would, at first appear that the imaginary part of the one loop graph would receive contributions from graphs in which two scalar propagators are cut and graphs in which two gauge boson propagators are cut. [49] Our extremely simple final answer (132) and (133) does have two scalar cuts, but has no cut contribution from two intermediate gauge boson lines. From a physical standpoint this is extremely satisfying; the Chern-Simons theory we study has no propagating gauge boson states, and so a two gauge boson cut would likely have signalled a contradiction with unitarity. From the purely technical point of view, however, the absence of two gauge boson cuts seems striking. Individual graphs in Fig. 8,9,10,11,1213 certainly have these cuts, which must, therefore cancel between graphs. In this subsubsection we verify that this is indeed the case.

Two gauge boson cuts naively occur in the T-channel. In this channel the two external scalar lines at the top of the graphs in Fig. 8,9,10,11,1213 represent initial states (one particle one antiparticle) while the two external lines at the bottom of the graph are final states. In order

coming from third term of (282).

[49] A graph in which a one gauge boson and propagator is cut will contribute zero to the imaginary part. All cut graphs may be regarded as the square of tree level processes. One of the tree process corresponding to such a cut would be decay of a single scalar to a scalar and a gauge boson: this is kinematically forbidden and so does not contribute.

to focus on this channel we must take $p_0 > 0$, $k_0 < 0$, $(p+q)_0 > 0$ and $(k+q)_0 < 0$. We find it useful to work in the 'center of mass frame' in which the two incoming quanta approach each other along the x axis. Let the final scattering angle be α. It follows that

$$\begin{aligned} &p = (p_0, p, 0), \quad k = (-p_0, p, 0), \\ &p + q = (p_0, p\cos(\alpha), p\sin(\alpha)), \quad k + q = (-p_0, p\cos(\alpha), p\sin(\alpha)). \end{aligned} \tag{291}$$

All two gauge boson cuts have a universal factor that comes from delta functions that puts the gauge bosons on shell. This factor is given by

$$\begin{aligned} &\int \frac{d^3l}{(2\pi)^3}(-2\pi i)^2\delta(-l_0^2 + l^2)\delta((l+p-k)^2)\theta(-l_0)\theta(l_0+2p_0) \\ &= \int \frac{d^3l}{(2\pi)^3}(-2\pi i)^2\frac{1}{2|l_0|}\delta(l_0+l)\delta(-4l_0p_0 - 4\ p_0^2)\theta(-l_0)\theta(l_0+2p_0) \\ &= \int \frac{1}{(2\pi)^3} l dl_0 dl d\theta(-2\pi i)^2\frac{1}{8\ p_0^2}\delta(l_0+l)\delta(l_0+p_0)\theta(-l_0)\theta(l_0+2p_0) \\ &= \int \frac{1}{(2\pi)^3} p_0 dl_0 dl d\theta(-2\pi i)^2\frac{1}{8\ p_0^2}\delta(-p_0+l)\delta(l_0+p_0)\theta(-l_0)\theta(l_0+2p_0) \\ &= -\frac{1}{16\pi p_0}\int dl_0 dl d\theta\delta(-p_0+l)\delta(l_0+p_0)) \end{aligned} \tag{292}$$

in the last line we have dropped the theta function because delta function clicks with in the theta function. [50]

In addition to the universal factor evaluated in (292) each diagram has its own particular factors that arise from the vertex factors, from propagators between circles or between crosses, and from the numerator of the cut gauge boson propagators that we have not yet included in our

[50] The two delta functions in the final line of (292) have a simple physical interpretation. As the two gauge fields are on shell, the cut graph proceeds via two intermediate (tree level) scattering processes, each of which take two scalar photons to two gauge bosons. The usual kinematical restrictions applied to these intermediate processes implies that the 3 momenta of the two intermediate gauge bosons - which, according to the labelling of 3 momenta in Fig 8 - is $-l$ and $l + p + k$ -

$$(p_0, \pm p_0\cos(\alpha), \pm p_0\sin(\alpha)).$$

The δ functions in the last line of δ enforce this.

analysis. For the various diagrams with two gauge bosons cuts, these factors are given by

$$\begin{aligned}
\text{Eye}_{\text{ diagram}} =& -4p_0^2\delta(-p_0+l)\delta(l_0+p_0).\\
\text{V}_{\text{ diagram}} =& 4\left(2p_0^2 - l\cdot k - \frac{2c_B^2\ p_0^2}{l\cdot p}\right)\delta(-p_0+l)\delta(l_0+p_0)\\
\frac{1}{16}\left(\text{Box}_{\text{ diagram}}\right) =& \Bigg[-\frac{k\cdot q}{2} + \frac{2k\cdot q\left(k\cdot p + c_B^2\right) + (k\cdot p)(l\cdot q)}{4l\cdot p}\\
&+\frac{2k\cdot q\left(k\cdot p + c_B^2\right) - (k\cdot p)(l\cdot q)}{4l\cdot(p+q)}\\
&-\frac{k\cdot q\left(k\cdot p + c_B^2\right)\left(2c_B^2 - k\cdot q\right)}{4(l\cdot p)\ \ (l\cdot(p+q))}\Bigg]\times\delta(l-p_0)\delta(l_0+p_0)\\
\text{Y}_{\text{ diagram}} =& \Bigg[8\left(-k\cdot p - k\cdot q - c_B^2 + l\cdot p + q\cdot l\right)\\
&+4\frac{c_B^2\left(k\cdot p + c_B^2 - 2q\cdot l\right)}{l\cdot p}\Bigg]\times\delta(l-p_0)\delta(l_0+p_0).
\end{aligned} \tag{293}$$

We must now sum these factors, multiply with the universal term in δ and then integrate the result over the 3 momentum l. The delta functions in (292) effectively turn this last integral into an integral over the angle of the spatial part of l. This angular integral is easily performed using

$$\begin{aligned}
&\int_c \frac{l\cdot q}{l\cdot p} = 2\pi(\cos(\alpha)-1) - 2\pi(\cos(\alpha)-1)\frac{p_0}{m},\\
&\int_c \frac{1}{l\cdot(p+q)\ l\cdot p} = \frac{4\pi}{p_0 m(2c_B^2 + p^2 - p^2\cos(\alpha))},\\
&\int_c 1 = 2\pi, \quad \int_c \frac{1}{l\cdot p} = \frac{2\pi}{m\ p_0}, \qquad \int_c l\cdot k = -\int_c l\cdot p = -2\pi p_0^2, \qquad \int_c l\cdot q = 0
\end{aligned} \tag{294}$$

where the notation $\int_c$ is the angle integral or more formally

$$\int_c = \int dl_0 dl d\theta\delta(-p_0+l)\delta(l_0+p_0). \tag{295}$$

We find that the cut due to the various diagrams is given by $-\frac{1}{16\pi\ p_0}$ times

$$\begin{aligned}
\text{Box}_{\text{ cut}} =& -\frac{1}{16\pi\ p_0}\times 2(-m+p_0)\sin^2\left(\frac{\alpha}{2}\right),\\
\text{Y}_{\text{ cut}} =& -\frac{1}{16\pi\ p_0}\times(p_0-m)\cos(\alpha),\\
\text{Eye}_{\text{ cut}} =& -\frac{1}{16\pi\ p_0}\frac{p_0}{2},\\
\text{V}_{\text{ cut}} =& -\frac{1}{16\pi\ p_0}(m - \frac{3p_0}{2}).
\end{aligned} \tag{296}$$

It follows that

$$(\text{Box}_{\text{ cut}}) + (\text{Y}_{\text{ cut}}) + (\text{Eye}_{\text{ cut}}) + (\text{V}_{\text{ cut}}) = 0. \tag{297}$$

Potential subtlety at special values of external momenta We now turn to the discussion of an important subtlety that we have, so far, glossed over. As we have emphasized above, our determination of the integrand for the box diagram made crucial use of the replacement rule (277). The derivation of this replacement rule works at generic values of the external momenta but turns out to fail when any two of the three independent external momenta are parallel (in this case the Gram-determinant vanishes) to each other. As an example, consider the situation when $p^\mu \parallel k^\mu$ as appearing in 8. In this case, in the centre-of-mass frame, the angle of scattering, $\theta = 0$ in S-channel. Of course if the amplitude was an analytic function of external momenta then we could simply ignore these exceptional momenta. The scattering amplitude at exceptional external momenta could be obtained by analytic continuation from the generic case. However we have seen that, the scattering amplitude is *not* an analytic function of external momenta (in the S-channel, in the centre-of-mass frame we have a piece $\delta(\theta)$ and this is precisely one of the points where the reduction that we discussed in (277) breaks down). The amplitude actually has singularities that are localized on the s, t plane. Moreover these singularities play an important role in the discussion of unitarity in these theories, as we have already emphasized.

1.9.6 Details of scattering in the fermionic theory

Off shell four point function We now restrict our attention to the special case $q^\pm = 0$. Plugging (141) into the Schwinger-Dyson equation (135), performing the integral over the 3 component of the momentum, and comparing coefficients of the different index structures on the two sides of this equation we find

$$\begin{aligned}
&f(p,k,q) = -\frac{\lambda}{2} G_{+3}(p-k) \\
&- 4\pi i \lambda \int \frac{d^2p'}{(2\pi)^2} \frac{\left(p'_- f(p',k,q)(q_3 - 2i\Sigma_I(p')p'_s) + 2g(p',k,q)((-1+\Sigma_I^2(p'))p'^2_s - c_F^2)\right) G_{+3}(p'+p)}{\sqrt{p'^2_s + c_F^2}\,\left(q_3^2 + 4(p'^2_s + c_F^2)\right)} \\
&g(p,k,q) = \\
&- 4\pi i \lambda \int \frac{d^2p'}{(2\pi)^2} \frac{p'_- G_{+3}(p'+p)}{\sqrt{p'^2_s + c_F^2}\,\left(q_3^2 + 4(p'^2_s + c_F^2)\right)} \left(2p'_- f(p',k,q) + g(p',k,q)(q_3 + 2i\Sigma_I(p')p'_s)\right)
\end{aligned} \tag{298}$$

$$
\begin{aligned}
&f_1(p,k,q) = \\
&-4\pi i\lambda \int \frac{d^2p'}{(2\pi)^2} \frac{\left(p'_- f_1(p',k,q)(q_3 - 2i\Sigma_I(p')p'_s) + 2g_1(p',k,q)((-1+\Sigma_I^2(p'))p'^2_s - c_F^2)\right) G_{+3}(p'+p)}{\sqrt{p'^2_s + c_F^2}\left(q_3^2 + 4(p'^2_s + c_F^2)\right)}
\end{aligned}
$$

$$
\begin{aligned}
&g_1(p,k,q) = \frac{\lambda}{2} G_{+3}(p-k) \\
&-4\pi i\lambda \int \frac{d^2p'}{(2\pi)^2} \frac{p'_- G_{+3}(p'+p)}{\sqrt{p'^2_s + c_F^2}\left(q_3^2 + 4(p'^2_s + c_F^2)\right)} \left(2p'_- f_1(p',k,q) + g_1(p',k,q)(q_3 + 2i\Sigma_I(p')p'_s)\right)
\end{aligned}
\tag{299}
$$

We have played around with these equations and discovered that they admit a solution of the following structure

$$
\begin{aligned}
g(p,k,q) &= \frac{-p_-}{2(p-k)_-} W_0(y,x,q_3) + \frac{1}{2} W_1(y,x,q_3) \\
f(p,k,q) &= \frac{1}{2(p-k)_-} W_3(y,x,q_3) + \frac{-p_+}{q_s^2} W_2(y,x,q_3) \\
g_1(p,k,q) &= \frac{k_+ p_-}{2(p-k)_-} B_2(y,x,q_3) + \frac{1}{2(p-k)_-} B_3(y,x,q_3) \\
f_1(p,k,q) &= \frac{-p_+}{p_s^2(p-k)_-} B_0(y,x,q_3) + \frac{-k_+}{2(p-k)_-} B_1(y,x,q_3)
\end{aligned}
\tag{300}
$$

where we use $y = \frac{2}{q_3}\sqrt{k_s^2 + c_F^2}$ and $x = \frac{2}{q_3}\sqrt{p_s^2 + c_F^2}$. Our ansatz completely specifies the dependence of V on the argument of the complex variables p^+ and k^+, leaving undetermined the dependence of V on the modulus of these variables. Plugging the above ansatz, it is possible to perform all angular integrals in Eq. 298,299 using the formulae

$$\begin{aligned}
\int_0^{2\pi}\frac{d\theta}{2\pi}(p'_-)^2\frac{1}{p_- - p'_-} &= -p_-\theta(p'_s - p_s)\\
\int_0^{2\pi}\frac{d\theta}{2\pi}p'_-\frac{1}{p_- - p'_-} &= -\theta(p'_s - p_s)\\
\int_0^{2\pi}\frac{d\theta}{2\pi}\frac{1}{p'_- - p_-} &= -2\frac{p_+}{p_s^2}\theta(p_s - p'_s)\\
\int_0^{2\pi}\frac{d\theta}{2\pi}\frac{1}{(p'_- - p_-)(k-p')_-} &= \frac{2}{(k-p)_-}\left(\frac{k_+}{k_s^2}\theta(k_s - p'_s) - \frac{p_+}{p_s^2}\theta(p_s - p'_s)\right)\\
\int_0^{2\pi}\frac{d\theta}{2\pi}\frac{p'_- k_+}{(p'_- - p_-)(k-p')_-} &= \frac{k_+}{(k-p)_-}\big(\theta(k_s - p'_s) - \theta(p_s - p'_s)\big)\\
\int_0^{2\pi}\frac{d\theta}{2\pi}\frac{(p'_-)^2}{(p'_- - p_-)(k-p')_-} &= -\frac{1}{(k-p)_-}\big(k_-\theta(-k_s + p'_s) - p_-\theta(-p_s + p'_s)\big)
\end{aligned} \tag{301}$$

Equating the coefficients of the different functions of the arguments of k^+ and p^+ we obtain the following equations for the coefficient functions $W_1 \dots W_4$ and $B_1 \dots B_4$.

$$
\begin{aligned}
W_1(y,x,q_3) &= \frac{i\lambda}{q_3}\int_y^\infty dx' \frac{XW_0(y,x',q_3)+2W_3(y,x',q_3)}{(1+x'^2)} - \frac{i\lambda}{q_3}\int_x^\infty dx' \frac{XW_1(y,x',q_3)+2W_2(y,x',q_3)}{(1+x'^2)} \\
W_0(y,x,q_3) &= \frac{i\lambda}{q_3}\int_y^x dx' \frac{XW_0(y,x',q_3)+2W_3(y,x',q_3)}{(1+x'^2)} \\
W_3(y,x,q_3) &= -\frac{i\lambda}{q_3}\int_x^y dx' \frac{Y_1W_0(y,x',q_3)+YW_3(y,x',q_3)}{(1+x'^2)} - 4\pi i\lambda \\
W_2(y,x,q_3) &= \frac{i\lambda}{q_3}\int_{\frac{2|c_F|}{q_3}}^x dx' \frac{Y_1W_1(y,x',q_3)+YW_2(y,x',q_3)}{(1+x'^2)} \\
B1(y,x,q_3) &= \\
-\frac{i\lambda}{q_3}&\left(\frac{2}{(-c_F^2+q_3^2\frac{y^2}{4})}\int_{\frac{2|c_F|}{q_3}}^y \frac{YB_0(y,x',q_3)+Y_1B_3(y,x',q_3)}{x'^2+1}dx' + \int_x^y \frac{YB_1(y,x',q_3)+Y_1B_2(y,x',q_3)}{x'^2+1}dx'\right) \\
B_0(y,x,q_3) &= \frac{i\lambda}{q_3}\int_{\frac{2|c_F|}{q_3}}^x \frac{YB_0(y,x',q_3)+Y_1B_3(y,x',q_3)}{x'^2+1}dx' \\
B_2(y,x,q_3) &= -\frac{i\lambda}{q_3}\int_x^\infty \frac{2B_1(y,x',q_3)+XB_2(y,x',q_3)}{x'^2+1}dx' \\
B_3(y,x,q_3) &= 4\pi i\ \lambda + \\
\frac{i\lambda}{q_3}&\left(\frac{(-c_F^2+q_3^2\frac{y^2}{4})}{2}\int_y^\infty \frac{2B_1(y,x',q_3)+XB_2(y,x',q_3)}{x'^2+1}dx' - \int_x^y \frac{2B_0(y,x',q_3)+XB_3(y,x',q_3)}{x'^2+1}dx'\right)
\end{aligned}
\tag{302}
$$

where

$$
\begin{aligned}
&a = 2\frac{|c_F|}{q_3}, \quad X = q_3\left(1+i\left(\frac{2m_f}{q_3}+\lambda x\right)\right), \\
&Y = q_3\left(1-i\left(\frac{2m_f}{q_3}+\lambda x\right)\right), \quad Y_1 = \frac{1}{2}q_3^2\left(\left(\frac{2m_f}{q_3}+\lambda x\right)^2 - x^2\right) \\
&x = \frac{2}{q_3}\sqrt{p_s^2+c_F^2}, \quad y = \frac{2}{q_3}\sqrt{k_s^2+c_F^2}
\end{aligned}
\tag{303}
$$

All of the equations above may be converted into differential equations by differentiating w.r.t. x. Notice that the first four equations in (302) (equations for the W variables) are decoupled from the last four variables (equations for the B variables). Furthermore the second and third of the equations above involve only the functions W_0 and W_3. These two equations are a set of linear first order differential equations for W_0 and W_3. These equations are given by

$$\begin{aligned}
\partial_x W_0(y,x,q_3) &= I\frac{\lambda}{q_3}\frac{1}{1+x^2}\left(W_0(y,x,q_3)\ X(x) + 2W_3(y,x,q_3)\right) \\
\partial_x W_3(y,x,q_3) &= I\frac{\lambda}{q_3}\frac{1}{1+x^2}\left(W_0(y,x,q_3)\ Y_1(x) + Y(x)\ W_3(y,x,q_3)\right)
\end{aligned} \tag{304}$$

It is not difficult to simultaneously solve these equations, using the observation that

$$\partial_x\left(W_3(y,x,q_3) - \frac{Y(x)}{2}W_0(y,x,q_3)\right) = 0. \tag{305}$$

With this solution in hand, the first of (302) may then be used to solve for W_1 (we merely have to solve a linear first order differential equation) and the fourth of (302) may be solved for W_2. A very similar process may be employed to solve for B_1, B_2, B_3, and B_4. Of course the solution to the differential equations so obtained have four integration 'constants' (in the Ws) and four integration 'constants' in the Bs. These integration 'constants' are really arbitrary functions of y. However their y dependence may be determined either from the requirement of symmetry - or equivalently by setting up the analogue of the (135) 'from the right' (this process yields a solutions to Ws and Bs upto unknown functions of x. Implementing these steps we find that our functions are given by

$$\begin{aligned}
W_0(y,x,q_3) &= \frac{C_1(y) + C_2(y)e^{2i\lambda\tan^{-1}(x)}}{q_3} \\
W_3(y,x,q_3) &= -C_1(y) + \frac{\left(C_1(y) + C_2(y)e^{2i\lambda\tan^{-1}(x)}\right)(-2im_f - i\lambda q_3 x + q_3)}{2q_3} \\
W_1(y,x,q_3) &= \frac{D_1(y) + D_2(y)e^{2i\lambda\tan^{-1}(x)}}{q_3} \\
W_2(y,x,q_3) &= -D_1(y) + \frac{\left(D_1(y) + D_2(y)e^{2i\lambda\tan^{-1}(x)}\right)(-2im_f - i\lambda q_3 x + q_3)}{2q_3} \\
B_2(y,x,q_3) &= \frac{h_1(y) + h_2(y)e^{2i\lambda\tan^{-1}(x)}}{q_3} \\
B_1(y,x,q_3) &= -h_1(y) + \frac{(-2im_f - i\lambda q_3 x + q_3)\left(h_1(y) + h_2(y)e^{2i\lambda\tan^{-1}(x)}\right)}{2q_3} \\
B_3(y,x,q_3) &= \frac{h_3(y) + h_4(y)e^{2i\lambda\tan^{-1}(x)}}{q_3} \\
B_0(y,x,q_3) &= -h_3(y) + \frac{(-2im_f - i\lambda q_3 x + q_3)\left(h_3(y) + h_4(y)e^{2i\lambda\tan^{-1}(x)}\right)}{2q_3}.
\end{aligned} \tag{306}$$

The 8 undetermined constants in our solution are an artifact of the fact that we solved a set of integral equations by converting them into differential equations. In order to determine the

8 integration constants, we plug our solution back directly into the integral equations (302). It turns out that all integrals on the RHS of the equations (302) may be explicitly performed. The undetermined constants are then easily obtained by comparing the LHS and RHS of (302). Implementing this procedure we obtain the final solution

$$\begin{aligned}
W_0(y,x,q_3) &= -\frac{4i\pi\lambda\left(-1+e^{2i\lambda\left(\tan^{-1}(x)-\tan^{-1}(y)\right)}\right)}{q_3}\\
W_1(y,x,q_3) &= \frac{4i\pi\lambda\left(-1+e^{i\lambda\left(\pi-2\tan^{-1}(y)\right)}\right)\left(e^{2i\lambda\tan^{-1}(a)}(a\lambda+m_{f1}+i)-(a\lambda+m_{f1}-i)e^{2i\lambda\tan^{-1}(x)}\right)}{q_3\left(e^{i\pi\lambda}(a\lambda+m_{f1}-i)-e^{2i\lambda\tan^{-1}(a)}(a\lambda+m_{f1}+i)\right)}\\
W_2(y,x,q_3) &= \frac{2\pi\lambda\left(-1+e^{i\lambda\left(\pi-2\tan^{-1}(y)\right)}\right)}{e^{i\pi\lambda}(a\lambda+m_{f1}-i)-e^{2i\lambda\tan^{-1}(a)}(a\lambda+m_{f1}+i)}\\
&\quad\left(e^{2i\lambda\tan^{-1}(a)}(a\lambda+m_{f1}+i)(m_{f1}+\lambda x-i)-(a\lambda+m_{f1}-i)(m_{f1}+\lambda x+i)e^{2i\lambda\tan^{-1}(x)}\right)\\
W_3(y,x,q_3) &= 2\pi\lambda\left(-(m_{f1}+\lambda x+i)e^{2i\lambda\left(\tan^{-1}(x)-\tan^{-1}(y)\right)}+m_{f1}+\lambda x-i\right).
\end{aligned} \tag{307}$$

where

$$m_{f1} = 2\frac{m_f}{q_3}.$$

The other components are

$$
\begin{aligned}
&B_0(y,x,q_3)\\
&=\frac{\pi\lambda q_3 e^{-2i\lambda\tan^{-1}(y)}\left(e^{i\pi\lambda}(m_{f1}+\lambda y-i)-(m_{f1}+\lambda y+i)e^{2i\lambda\tan^{-1}(y)}\right)}{e^{i\pi\lambda}(a\lambda+m_{f1}-i)-e^{2i\lambda\tan^{-1}(a)}(a\lambda+m_{f1}+i)}\\
&\left((ia\lambda+im_{f1}+1)(m_{f1}+\lambda x+i)e^{2i\lambda\tan^{-1}(x)}-ie^{2i\lambda\tan^{-1}(a)}(a\lambda+m_{f1}+i)(m_{f1}+\lambda x-i)\right)\\
&B_1(y,x,q_3)\\
&=\frac{2\pi\lambda q_3 e^{-2i\lambda\tan^{-1}(y)}\left(e^{i\pi\lambda}(m_{f1}+\lambda x-i)-(m_{f1}+\lambda x+i)e^{2i\lambda\tan^{-1}(x)}\right)}{\left(\frac{y^2q_3^2}{4}-c_F^2\right)\left(e^{i\pi\lambda}(a\lambda+m_{f1}-i)-e^{2i\lambda\tan^{-1}(a)}(a\lambda+m_{f1}+i)\right)}\\
&\left(e^{2i\lambda\tan^{-1}(a)}(a\lambda+m_{f1}+i)(im_{f1}+i\lambda y+1)-i(a\lambda+m_{f1}-i)(m_{f1}+\lambda y+i)e^{2i\lambda\tan^{-1}(y)}\right)\\
&B_2(y,x,q_3)=\\
&\frac{4\pi\lambda\left(e^{i\pi\lambda}-e^{2i\lambda\tan^{-1}(x)}\right)e^{-2i\lambda\tan^{-1}(y)}}{\left(\frac{y^2q_3^2}{4}-c_F^2\right)\left(e^{i\pi\lambda}(a\lambda+m_{f1}-i)-e^{2i\lambda\tan^{-1}(a)}(a\lambda+m_{f1}+i)\right)}\\
&\left((a\lambda+m_{f1}-i)(m_{f1}+\lambda y+i)e^{2i\lambda\tan^{-1}(y)}-e^{2i\lambda\tan^{-1}(a)}(a\lambda+m_{f1}+i)(m_{f1}+\lambda y-i)\right)\\
&B_3(y,x,q_3)=\\
&-\frac{2\pi\lambda e^{-2i\lambda\tan^{-1}(y)}\left(e^{2i\lambda\tan^{-1}(a)}(a\lambda+m_{f1}+i)-(a\lambda+m_{f1}-i)e^{2i\lambda\tan^{-1}(x)}\right)}{e^{i\pi\lambda}(a\lambda+m_{f1}-i)-e^{2i\lambda\tan^{-1}(a)}(a\lambda+m_{f1}+i)}\\
&\left((m_{f1}+\lambda y+i)e^{2i\lambda\tan^{-1}(y)}-e^{i\pi\lambda}(m_{f1}+\lambda y-i)\right)
\end{aligned}
\tag{308}
$$

In summary, the offshell four point amplitude, defined in (85), takes the form (141), with the functions in this equation given by (300) with the W and B functions given in (307) and (308) respectively.

1.9.7 Preliminary analysis of the double analytic continuation

Analysis of the scalar integral equation after double analytic continuation In this appendix we initiate a very preliminary discussion of the bosonic integral equation after double analytic continuation discussed in subsection 1.7.5 above. In subsection 1.9.7 below we evaluate the one loop contribution to four boson scattering after double analytic continuation, and demonstrate that the computation includes a singular contribution, absent from the naive analytic continuation of the U and T-channel results to the S-channel. Under certain assumptions this singular piece precisely reproduces the $\mathcal{O}(\lambda^2)$ term in the contact δ function part of the S-

channel scattering amplitudes. In subsection 1.9.7 below we take a non-relativistic limit of the double analytic continued integral equation and demonstrate that it reduces to the non-relativistic Aharonov-Bohm equation with $\nu = \lambda$.

The oneloop box diagram after double analytic continuation Appendix 1.9.4 was devoted to a detailed study of the one loop diagram Fig. 20 at $q^{\pm} = 0$ directly in usual Minkowski space. The conclusions of Appendix 1.9.4 may be summarized as follows. In the case that the momenta p and k both lie offshell, the Minkowskian one loop diagram agrees with the unambiguous analytic continuation of the Euclidean answer. In the case that the momenta p and k were both onshell, the continuation from Euclidean space was ambiguous, but the Minkowskian computation resolved the ambiguity.

In this Appendix we revisit the one loop diagram of Fig. 20 after performing the double analytic continuation described in subsection 1.7.5. We recompute the diagram, this time in the double analytically continued Minkowski space - the space in which the 3 direction is taken to be time. We address the following question: how does the answer of this computation compare with analytic continuation from usual Minkowski space (and the analytic continuation from Euclidean space, when this analytic continuation is unambiguous).

Although we will not present the detailed computation here we have indeed verified that when p and k are both offshell, the computation performed directly after the double analytic continuation agrees with the appropriate analytic continuations from usual Minkowski space as well as from Euclidean space.

The situation is more delicate when p and k are both onshell. In this case though the Euclidean answer is ambiguous, the 'usual' Minkowskian answer is not. We outline the computation of the double analytically continued result in this Appendix. In particular we show that the analytic continuation of this 'usual' answer does *not* agree with the answer of the computation performed directly in double analytically continued Euclidean space. The details of the difference between these answer depends in a very unusual way on the relative smallness of the $i\epsilon$ in scalar propagators and $i\epsilon$ in the gauge propagators. In a natural limit (the one in which these two have the same degree of smallness), the difference between the two results agrees precisely with the difference between T_S^{trial} and T_S^B (see (154)) lending some support to the conjecture (154).

Setting up the computation Let $T(\alpha)$ denote the double analytic continuation of the one loop contribution to the T matrix. $T(\alpha)$ is given by (see (248))

$$\begin{aligned}\frac{iT(\alpha)}{(4\pi\lambda q_3)^2} = -\int \frac{d^3r}{(2\pi)^3}\Bigg[&\frac{2(r+p)_-(r-p)_+}{2(r-p)_-(r-p)_+ - i\epsilon}\frac{2(r+k)_-(r-k)_+}{2(r-k)_-(r-k)_+ - i\epsilon}\\ &\times\frac{1}{r_s^2 - r_3^2 + c_B^2 - i\epsilon_1}\frac{1}{r_s^2 - (r_3+q_3)^2 + c_B^2 - i\epsilon_1}\Bigg]\end{aligned} \tag{309}$$

Note that after double analytic continuation v_+ is a complex number and v_- is its complex conjugate for all $v_\pm$ (this was true also in Euclidean space). As in Euclidean space, we will find it convenient to work with the magnitude and phase of these complex numbers. Choosing axes so that p_+ is a real number we have

$$p_\pm = \frac{p_s}{\sqrt{2}}, \quad k_\pm = \frac{k_s}{\sqrt{2}} e^{\pm i\alpha}, \quad r_\pm = \frac{r_s}{\sqrt{2}} e^{\pm i\theta}. \tag{310}$$

As we focus on the case of onshell scattering (and as $q^\pm = 0$) we have

$$p_s = k_s, \quad q_3 = -2p_3 = -2k_3 = 2\sqrt{p_s^2 + c_B^2} = \sqrt{s} \tag{311}$$

Plugging (310) and (311) into (309) and using the fact that the scalar propagators are independent of θ and α, while the gauge boson propagators are independent of r_3 we find

$$\frac{iT(\alpha)}{(4\pi\lambda q_3)^2} = \int_0^\infty \frac{r_s dr_s}{2\pi} \mathcal{I}_1(r_s,\alpha)\mathcal{I}_2(r_s) \tag{312}$$

where $\mathcal{I}_2(r_s)$ is the integral of the product of the scalar propagators over the timelike coordinate r_3

$$\mathcal{I}_2(r_s) = \int_{-\infty}^{\infty} \frac{dr_3}{2\pi} \frac{1}{r_s^2 - r_3^2 + c_B^2 - i\epsilon_1} \frac{1}{r_s^2 - (r_3+q_3)^2 + c_B^2 - i\epsilon_1} \tag{313}$$

and $\mathcal{I}_1(r_s,\alpha)$ is the integral of the product of the gauge boson propagators over the angle θ

$$\mathcal{I}_1(r_s,\alpha) = -\int_0^{2\pi} \frac{d\theta}{2\pi} \frac{(r_s e^{i\theta} - p_s)(r_s e^{-i\theta} + p_s)}{(r_s e^{i\theta} - p_s)(r_s e^{-i\theta} - p_s) - i\epsilon} \frac{(r_s e^{i\theta} - p_s e^{i\alpha})(r_s e^{-i\theta} + p_s e^{-i\alpha})}{(r_s e^{i\theta} - p_s e^{i\alpha})(r_s e^{-i\theta} - p_s e^{-i\alpha}) - i\epsilon} \tag{314}$$

The integral over r_3 in (313) is easily evaluated by contour methods and we find

$$\begin{aligned}\mathcal{I}_2(r_s) &= \frac{-i}{\sqrt{r_s^2 + c_B^2}(q_3^2 - 4r_s^2 - 4c_B^2 + 4i\epsilon_1)} \\ &= \frac{-i}{4\sqrt{r_s^2 + c_B^2}(p_s^2 - r_s^2 + i\epsilon_1)}\end{aligned} \tag{315}$$

The integral over θ in (314) may also be evaluated by contour techniques. Let

$$z = e^{i\theta}, \quad w = e^{i\alpha} \tag{316}$$

so that

$$\mathcal{I}_1(r_s,\alpha) = -\oint_{|z|=1} \frac{dz}{2\pi i z} \frac{(z + \frac{r_s}{p_s})(z - \frac{p_s}{r_s})}{(z - \frac{r_s}{p_s})(z - \frac{p_s}{r_s}) + \frac{i\epsilon z}{r_s p_s}} \frac{(z + w\frac{r_s}{p_s})(z - w\frac{p_s}{r_s})}{(z - w\frac{r_s}{p_s})(z - w\frac{p_s}{r_s}) + \frac{i\epsilon z w}{r_s p_s}} \tag{317}$$

where the integration contour in (317) runs over the unit circle.

The contribution of the pole at zero The integrand in (317) is a meromorphic function of z with 5 poles. The simplest of these poles is at $z = 0$. The contribution of this pole to $\mathcal{I}_1(r_s, \alpha)$ is simply -1 ; plugging this together with (315) into (312) we find that the contribution of the pole at zero to iT is given by

$$iT = i(4\pi\lambda q_3)^2 H(q) \tag{318}$$

in perfect agreement with the analytic continuation of (254). As the contribution of the pole at zero has already reproduced the analytic continuation of the 'real' Minkowski scattering amplitude, It follow that the contribution of the remaining 4 poles in (317) is simply the difference between this analytic continuation, and the result directly computed after double analytic continuation

The contribution of the remaining four poles Let us retreat from the onshell limit for a moment, i.e. allow p_s and k_s to be different. A naive evaluation of the contribution of the remaining four poles in (317) in the limit of vanishing ϵ_1 yields and answer proportional to

$$\theta(p_s - r_s) - \theta(k_s - r_s)$$

This quantity vanishes when $p_s = k_s$ suggesting that the contribution of the remaining four poles to the angle integral should vanish in the onshell limit. [51] However this reasoning is a bit too quick for the following reason. Suppose $p_s - k_s = a$ where a is a very small number and k_s is the onshell value of spatial momentum. Then r_s is indeed constrained vary over a very small range. However this is not sufficient to guarantee that the integral over r_s will vanish. The reason for this is that this small interval is concentrated around precisely the value of r_s at which (315) is singular, and a singular integrand may well integrate to a finite quantity over a vanishing small integration domain. Cautioned by these considerations we now turn to a careful and honest evaluation of the contribution of the remaining 4 poles in (317) to $\mathcal{I}_1(r_s, \alpha)$

The remaining four poles in (317) are located at $z_\pm$ and $wz_\pm$ where

$$z_\pm = \frac{1}{2}\left(\frac{r_s}{p_s} + \frac{p_s}{r_s} - i\epsilon \pm \sqrt{(\frac{r_s}{p_s} + \frac{p_s}{r_s} - i\epsilon)^2 - 4}\right), \quad w = e^{i\alpha} \tag{319}$$

where the square root function is defined to have a branch cut along the negative real axis. It is

[51] This is indeed how things worked in our derivation of the Euclidean integral equation for V.

easily verified that

$$z_+z_- = 1, \quad z_+ + z_- = \frac{r_s}{p_s} + \frac{p_s}{r_s}, \quad z_+ - z_- = \sqrt{\left(\frac{r_s}{p_s} - \frac{p_s}{r_s}\right)^2 - 2i\epsilon\left(\frac{r_s}{p_s} + \frac{p_s}{r_s}\right)} \tag{320}$$

It may also be verified that $|z_+| > 1$, so $|z_-| < 1$. The two poles enclosed by the unit contour in (317) are located at z_- and wz_- (the remaining two poles lie outside the contour and do not contribute to the integral). The contribution of these two poles to (317) is given by

$$\mathcal{I}_1(r_s, \alpha) = -\frac{(z_- + \frac{r_s}{p_s})(z_- - \frac{p_s}{r_s})}{z_-^2(w-1)(z_+ - z_-)} \left(\frac{(z_- + w\frac{r_s}{p_s})(z_- - w\frac{p_s}{r_s})}{z_- - wz_+} - \frac{(wz_- + \frac{r_s}{p_s})(wz_- - \frac{p_s}{r_s})}{wz_- - z_+} \right) \tag{321}$$

Using (320) several times, (321) may be simplified to

$$\begin{aligned} \mathcal{I}_1(r_s, \alpha) &= \frac{-w(z_- + \frac{r_s}{p_s})(z_- - \frac{p_s}{r_s})(z_- - z_+ - \frac{r_s}{p_s} + \frac{p_s}{r_s})}{z_-(wz_+ - z_-)(wz_- - z_+)} \frac{z_+ + z_-}{z_+ - z_-} \\ &= \frac{-w(z_- - z_+ + \frac{r_s}{p_s} - \frac{p_s}{r_s})(z_- - z_+ - \frac{r_s}{p_s} + \frac{p_s}{r_s})}{(wz_+ - z_-)(wz_- - z_+)} \frac{z_+ + z_-}{z_+ - z_-} \\ &= \frac{2i\epsilon w(r_s^2 + p_s^2)^2}{r_s^3 p_s^3 (z_+ - z_-)(wz_+ - z_-)(wz_- - z_+)} \end{aligned} \tag{322}$$

Note that (322) ϵ in apparent vindication of the intuition that suggests that these poles contribute vanishingly to the integral. Let us anyway proceed to complete our careful evaluation: we conclude that the contribution of these poles to (312) is given by

$$iT(\alpha) = \frac{4\epsilon\pi\lambda^2 q_3^2}{p_s^3} \int_0^\infty \frac{dr_s w(r_s^2 + p_s^2)^2}{r_s^2(z_+ - z_-)(w - \frac{z_-}{z_+})(w - \frac{z_+}{z_-})} \frac{1}{\sqrt{r_s^2 + c_B^2(p_s^2 - r_s^2 + i\epsilon_1)}} \tag{323}$$

In the limit $\epsilon \to 0$, the RHS in (323) vanishes unless the integral in that equation develops a singularity. The integrand in (323) does have a singularity that approaches the integration contour at $r_s = p_s$. If $w \neq 1$, however, no other singularity in the integrand approaches the integration contour $r_s = (0, \infty)$. A single singularity approaching an integration contour does not give rise to a singular contribution to the integral (because the integration contour can always be deformed to avoid the singularity). Provided $w \neq 1$ it follows that the integral on the RHS of (323) is nonsingular, and so the RHS of (323) vanishes in the limit $\epsilon \to 0$.

The situation is different, however, if w tends to unity. In this case the singularities caused by the factors $(w - \frac{z_-}{z_+})$, $(w - \frac{z_+}{z_-})$ and $(p_s^2 - r_s^2 + i\epsilon_1)$ all approach the same contour point, namely $r_s = p_s$ as $w \to 1$ and $\epsilon_1, \epsilon \to 0$. In this case the integral on the RHS conceivably develops a pinch singularity, and the RHS of (323) does not necessarily vanish in this case.

In summary we have concluded that $iT(\alpha)$ vanishes for nonzero α, but not necessarily at

$\alpha = 0$. In order to better understand the behaviour of $iT(\alpha)$ near $\alpha = 0$ we now evaluate the integral of this quantity over α. This integral may be affected by contour techniques and we find

$$\begin{aligned}
&\int_0^{2\pi} d\alpha \ iT(\alpha) \\
&= \oint_{|w|=1} \frac{dw}{iw} \frac{4\epsilon\pi\lambda^2 q_3^2}{p_s^3} \int_0^\infty \frac{dr_s w(r_s^2+p_s^2)^2}{r_s^2(z_+-z_-)(w-\frac{z_-}{z_+})(w-\frac{z_+}{z_-})} \frac{1}{\sqrt{r_s^2+c_B^2}(p_s^2-r_s^2+i\epsilon_1)}
\end{aligned} \tag{324}$$

The integral runs counterclockwise over the unit circle in the w plane. This contour encloses a single pole, at $w = \frac{z_-}{z_+}$. Evaluating the residue of this pole we find

$$\int_0^{2\pi} d\alpha \ iT(\alpha) = -\frac{8\pi^2\epsilon\lambda^2 q_3^2}{p_s^2} \int_0^\infty \frac{dr_s(r_s^2+p_s^2)}{r_s(z_+-z_-)^2} \frac{1}{\sqrt{r_s^2+c_B^2}(p_s^2-r_s^2+i\epsilon_1)} \tag{325}$$

Because of the overall factor of ϵ, it is clear that (325) receives contributions - if at all - only from r_s in the neighborhood of p_s. It is not too difficult to convince oneself that the dominant contribution is from $r_s \sim \sqrt{\epsilon}$. In order to see this we make the variable change $r_s = \sqrt{\epsilon}x$. To leading order in $\sqrt{\epsilon}$ we find

$$\int_0^{2\pi} d\alpha \ iI(\alpha) = -\frac{16\pi^2\lambda^2 p_s q_3^2}{\sqrt{p_s^2+c_B^2}} \int_{-\infty}^{\infty} \frac{\sqrt{\epsilon}dx}{(i\epsilon_1 - 2xp_s\sqrt{\epsilon})(x^2-i)} \tag{326}$$

(to obtain (326) we have used here that $(z_+ - z_-)^2 = 2\epsilon(x^2 - i)$ at leading order in ϵ)

Let us now assume that $\epsilon_1 \ll \sqrt{\epsilon}$ (this would in particular have been the case if $\epsilon_1 = \epsilon$). In this case (326) simplifies to

$$\int_0^{2\pi} d\alpha \ iI(\alpha) = 4\pi^2\lambda^2\sqrt{s} \int_{-\infty}^{\infty} \frac{dx}{(x-ib)(x^2-i)} \tag{327}$$

Where b is a positive infinitesimal. The integral on the RHS of (327) evaluates (by a straightforward application of contour techniques) to $-\pi$. We conclude that

$$iT = -4\pi^3\lambda^2\sqrt{s}\delta(\alpha) \tag{328}$$

This is in perfect agreement with the expectation

$$T = -8\pi i\sqrt{s}\left(\cos(\pi\lambda) - 1\right)\delta(\alpha) = 4i\pi^3\lambda^2\sqrt{s}\delta(\alpha) + \mathcal{O}(\lambda^4)$$

Solutions of the Dirac equation at $q^{\pm} = 0$ after double analytic continuation.
In order to compute S matrices in he Fermionic theory after double analytic continuation we need solutions to the relevant Dirac equations. We present the relevant solutions in this Appendix.

After a double analytic continuation $k_0 = ik_3$ and the gamma matrix convention is $\gamma^0 = -i\gamma^3$. The Dirac equation is give by

$$\bar{\psi}(-p)\big(i\left(p_0\gamma^0 + p_-\gamma^- + p_+\gamma^+\left(1 + g(p_s)\right)\right) + f(p_s)p_s\big)\psi(p) = 0. \tag{329}$$

Where

$$p_s^2 = p_1^2 + p_2^2. \tag{330}$$

Our gamma matrix convention is

$$\gamma^0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{331}$$

$$\gamma^+ = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \tag{332}$$

$$\gamma^- = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix} \tag{333}$$

So now the Dirac equation is

$$\bar{\psi}(-p)\begin{pmatrix} p_0 + f(p_s)p_s & i\sqrt{2}p_+(1 + g(p_s)) \\ i\sqrt{2}p_- & -p_0 + f(p_s)p_s \end{pmatrix}\psi(p) = 0 \tag{334}$$

Now we use the on-shell condition

$$p_0 = \pm E_{\vec{p}} \tag{335}$$

Where

$$E_{\vec{p}} = \sqrt{p_1^2 + p_2^2 + C_f^2} \tag{336}$$

C_f is the fermion pole mass.
The solution with $p_0 = -E_{\vec{p}}$ is particle solution $u(\vec{p})$ while the solution with $p_0 = E_{\vec{p}}$ is the antiparticle solution $v(-\vec{p})$.

Now we need to solve

$$\bar{u}(\vec{p})\begin{pmatrix} -E_{\vec{p}} + f(p_s)p_s & i\sqrt{2}p_+(1 + g(p_s)) \\ i\sqrt{2}p_- & E_{\vec{p}} + f(p_s)p_s \end{pmatrix}u(\vec{p}) = 0 \tag{337}$$

Which on solving on right and on left gives respectively,

$$u(\vec{p}) = \frac{1}{\sqrt{E_{\vec{p}} + f(p_s)p_s}} \begin{pmatrix} E_{\vec{p}} + f(p_s)p_s \\ -i\sqrt{2}p_- \end{pmatrix} \tag{338}$$

$$\bar{u}(\vec{p}) = \frac{1}{\sqrt{E_{\vec{p}} + f(p_s)p_s}} \begin{pmatrix} E_{\vec{p}} + f(p_s)p_s & -i\sqrt{2}p_+(1+g(p_s)) \end{pmatrix} \tag{339}$$

Where normalization is set to be $\bar{u}(\vec{p})u(\vec{p}) = 2f(p_s)p_s$.

We also need to solve

$$\bar{v}(\vec{p}) \begin{pmatrix} -E_{\vec{p}} - f(p_s)p_s & i\sqrt{2}p_+(1+g(p_s)) \\ i\sqrt{2}p_- & E_{\vec{p}} - f(p_s)p_s \end{pmatrix} v(\vec{p}) = 0 \tag{340}$$

Which on solving on right and on left gives respectively,

$$v(\vec{p}) = \frac{1}{\sqrt{E_{\vec{p}} - f(p_s)p_s}} \begin{pmatrix} E_{\vec{p}} - f(p_s)p_s \\ -i\sqrt{2}p_- \end{pmatrix} \tag{341}$$

$$\bar{v}(\vec{p}) = \frac{1}{\sqrt{E_{\vec{p}} - f(p_s)p_s}} \begin{pmatrix} E_{\vec{p}} - f(p_s)p_s & -i\sqrt{2}p_+(1+g(p_s)) \end{pmatrix} \tag{342}$$

Where normalization is set to be $\bar{v}(\vec{p})v(\vec{p}) = -2f(p_s)p_s$.

Aharonov-Bohm in the non-relativistic limit After double analytic continuation, the four boson four point function satisfies the integral equation

$$V(\vec{p},\vec{k}) = V_0(\vec{p},\vec{k}) + \int \frac{(i)^2 V_0(\vec{p},\vec{l})V(\vec{l},\vec{k})\frac{d^3l}{(2\pi)^3}}{\left(-l_0^2 + l_s^2 + c_B^2 - i\epsilon\right)\left(-(l_0+q_0)^2 + l_s^2 + c_B^2 - i\epsilon\right)} \tag{343}$$

where

$$V_0(\vec{p},\vec{k}) = 4\pi i\lambda q_0 \frac{(k+p)_-}{(k-p)_-} - 2i\pi\lambda^2 c_B \tag{344}$$

Since both V_0 and V depend only on the spatial components of momenta, we can perform l_0 integral in (343) to get

$$V(\vec{p},\vec{k}) = V_0(\vec{p},\vec{k}) + i\int \frac{V_0(\vec{p},\vec{l})V(\vec{l},\vec{k})}{\sqrt{l_s^2 + c_B^2}\left(q_0^2 - 4l_s^2 - 4c_B^2 + i\epsilon\right)} \frac{d^2l}{(2\pi)^2} \tag{345}$$

Let us focus on the special case in which k and $k+q$ are taken to be onshell, i.e. $q_0 = -2k_0 = -2\sqrt{k_s^2 + c_B^2}$ while p and $p+q$ are generically offshell. Let us define

$$\psi(\vec{p}) = (2\pi)^2 \delta^2(\vec{p} - \vec{k}) + i \frac{V(\vec{p}, \vec{k})}{4\sqrt{p_s^2 + c_B^2}(k_s^2 - p_s^2 + i\epsilon)} \tag{346}$$

Where k is onshell. Then (345) can be written as

$$-4i\sqrt{p_s^2 + c_B^2}\left(k_s^2 - p_s^2\right)\psi(\vec{p}) = \int V_0(\vec{p}, \vec{l})\psi(\vec{l})\frac{d^2l}{(2\pi)^2} \tag{347}$$

In the non-relativistic limit

$$\sqrt{p_s^2 + c_B^2} = c_B$$
$$q_0 = -2c_B$$

and so (347) becomes

$$\left(k_s^2 - p_s^2\right)\psi(\vec{p}) = \int \left(2\pi\lambda \frac{(l+p)_-}{(l-p)_-} + \frac{\pi\lambda^2}{2}\right)\psi(\vec{l})\frac{d^2l}{(2\pi)^2} \tag{348}$$

(348) takes the form of a non-relativistic Schrodinger equation of a particle propagating in a potential whose nature we will soon identify. (346) is the assertion that the wave function $\psi(r)$ that obeys this Schrodinger equation takes the Lippmann Schwinger scattering form, with a scattering function (roughly $h(\theta)$) proportional to $V(k,p)$ once p is set onshell. Restated, the non relativistic limit of the integral equation (343) is simply the Lippmann Schwinger equation for the scattering matrix of a non-relativistic quantum mechanical problem, whose precise nature we now investigate.

In order to better understand the Schrodinger equation (348) we transform it to position space. Multiplying (348) by $\frac{e^{ipx}}{(2\pi)^2}$ and integrating over p we find

$$\int \left(k_s^2 - p_s^2\right)\psi(\vec{p})e^{ip.x}\frac{d^2p}{(2\pi)^2} = \int \left(2\pi\lambda \frac{(l+p)_-}{(l-p)_-} + \frac{\pi\lambda^2}{2}\right)\psi(\vec{l})\frac{d^2l}{(2\pi)^2}e^{ip.x}\frac{d^2p}{(2\pi)^2} \tag{349}$$

Let us define the position space wave function

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}e^{ip.x}\psi(p)$$

Changing the integration variable on the RHS of (348) as $p \to p+l$, and recalling $z = x^+ = \frac{x^1+ix^2}{\sqrt{2}}$

and $\bar{z} = x^- = \frac{x^1 - ix^2}{\sqrt{2}}$, (348) may be rewritten as

$$\left(2\partial_z\partial_{\bar{z}} + k_s^2\right)\psi(z,\bar{z}) = \int \left(-4\pi\lambda\frac{l_-}{p_-} + \frac{\pi\lambda^2}{2} - 2\pi\lambda\right)\psi(\vec{l})\frac{d^2l}{(2\pi)^2}e^{ip.x}e^{il.x}\frac{d^2p}{(2\pi)^2} \tag{350}$$

The first term on RHS of (349) is

$$-4\pi\lambda\int\frac{e^{ip.x}}{p_-}\frac{d^2p}{(2\pi)^2}\int l_-\psi(\vec{l})e^{il.x}\frac{d^2l}{(2\pi)^2} = -4\pi\lambda\left(\frac{i}{2\pi z}\right)\left(-i\partial_{\bar{z}}\psi(z,\bar{z})\right) \tag{351}$$

$$= \frac{-2\lambda}{z}\psi(z,\bar{z}) \tag{352}$$

While the rest of the RHS of (349) is

$$\left(\frac{\pi\lambda^2}{2} - 2\pi\lambda\right)\int\psi(\vec{l})e^{il.x}\frac{d^2l}{(2\pi)^2}\int e^{ip.x}\frac{d^2p}{(2\pi)^2} = \left(\frac{\pi\lambda^2}{2} - 2\pi\lambda\right)\psi(z,\bar{z})\delta^2(z) \tag{353}$$

It follows that (349) may be recast as

$$\left(\partial_z\partial_{\bar{z}} + \frac{k_s^2}{2}\right)\psi(z,\bar{z}) = \frac{-\lambda}{z}\partial_{\bar{z}}\psi(z,\bar{z}) + \left(\frac{\pi\lambda^2}{4} - \pi\lambda\right)\psi(z,\bar{z})\delta^2(z) \tag{354}$$

Let us now define a gauge covariant derivative as

$$\begin{aligned} D_z &= \partial_z + iA_z \\ A_z &= \frac{-i\lambda}{z} \\ D_{\bar{z}} &= \partial_{\bar{z}} \end{aligned} \tag{355}$$

in terms of which (349) reduces to

$$\left(D_zD_{\bar{z}} + \frac{k_s^2}{2}\right)\psi(z,\bar{z}) = -\left(\frac{\pi\lambda^2}{4} + \pi\lambda\right)\psi(z,\bar{z})\delta^2(z) \tag{356}$$

How is the gauge potential A_z in (355) to be interpreted? Firstly, clearly this potential is pure gauge away from $z = 0$, as the antiholomorphic derivative of A_z vanishes away from $z = 0$. In other words A_z is the gauge potential of a localized point flux. The magnitude of this flux is given by the contour integral $\int A_z dz$ over the unit circle and so is $2\pi^2\lambda$. In other words (356) is the Schrodinger equation for the Aharonov-Bohm problem with $\nu = \lambda$ (plus delta function contact interaction), in an unusual complex gauge. The contact interaction plausibly makes do difference to scattering computations if the Schrodinger equation is studied with boundary conditions (like those adopted by Aharonov and Bohm) that force $\psi(r)$ to vanish at the origin.

2 Chapter 2: Poles in the S-Matrix

2.1 Introduction, analysis and conclusions

In the previous chapter based on [39] has initiated the study of the S-matrix in fundamental matter Chern-Simons theories to all orders in the 't Hooft coupling. In particular it presented a detailed study of $2 \to 2$ scattering in the most general renormalizable theory of a single fundamental scalar interacting with a $U(N_B)$ Chern-Simons gauge field

$$S = \int d^3x \left[i\varepsilon^{\mu\nu\rho} \frac{\kappa_B}{4\pi} \mathrm{Tr}(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho) + D_\mu \bar{\phi} D^\mu \phi + m_B^2 \bar{\phi}\phi + \frac{1}{2N_B} b_4 (\bar{\phi}\phi)^2 \right], \qquad (357)$$

to all orders in $\lambda_B = \frac{N_B}{\kappa_B}$.

The theory (357) has elementary quanta that transform in either the fundamental or the antifundamental representations of $U(N_B)$. We refer to quanta in the fundamental representation as particles, and quanta in the antifundamental representation as antiparticles. It was possible to explicitly compute the particle - particle scattering matrix together with the particle - antiparticle scattering matrix in the channel corresponding to adjoint exchange. They also presented the following conjectured formula for the particle - antiparticle S-matrix in the channel corresponding to singlet exchange:

$$\begin{aligned} T_S(\sqrt{s}, \theta) &= 8\pi i \sqrt{s}(1 - \cos(\pi\lambda_B))\delta(\theta) + 4i\sqrt{s}\sin(\pi\lambda_B)\mathrm{Pv}\left(\cot\left(\frac{\theta}{2}\right)\right) \\ &+ 4\sqrt{s}\sin(\pi|\lambda_B|)\left(\frac{\left(4\pi|\lambda_B|\sqrt{s} + \widetilde{b}_4\right) + e^{i\pi|\lambda_B|}\left(-4\pi|\lambda_B|\sqrt{s} + \widetilde{b}_4\right)\left(\frac{\frac{1}{2}+\frac{c_B}{\sqrt{s}}}{\frac{1}{2}-\frac{c_B}{\sqrt{s}}}\right)^{|\lambda_B|}}{\left(4\pi|\lambda_B|\sqrt{s} + \widetilde{b}_4\right) - e^{i\pi|\lambda_B|}\left(-4\pi|\lambda_B|\sqrt{s} + \widetilde{b}_4\right)\left(\frac{\frac{1}{2}+\frac{c_B}{\sqrt{s}}}{\frac{1}{2}-\frac{c_B}{\sqrt{s}}}\right)^{|\lambda_B|}} \right) \end{aligned} \qquad (358)$$

where

$$\begin{aligned} c_B &= \text{pole mass of the single scalar excitation}, \\ \sqrt{s} &= \text{centre of mass energy}, \\ \theta &= \text{angle of scattering}, \\ \widetilde{b}_4 &= 2\pi\lambda_B^2 c_B - b_4. \end{aligned} \qquad (359)$$

As explained in chapter 1, the S-matrix (358) does *not* agree with the simple analytic continuation of the particle - particle S-matrix. Instead, the nonsingular part of (358) is given by the analytic continuation of the particle - particle S-matrix rescaled by the factor $\frac{\sin(\pi\lambda_B)}{\pi\lambda_B}$. In other words the correctness of the conjectured S-matrix (358) requires an intriguing modification of the usual text book rules of crossing symmetry in the case of matter Chern-Simons theories. As with any conjecture that challenges accepted wisdom, the formula (358) should be subjected to stringent checks. In this note we confront the conjecture of chapter 1 with a nontrivial consistency check and find that it passes the test, as we now describe.

The S-matrix (358) has a pole for $\widetilde{b}_4 \geq \widetilde{b}_4^{crit} = 8\pi c_B|\lambda_B|$ indicating the existence of a particle - antiparticle bound state in the singlet channel at these values of parameters. [52] As $\widetilde{b}_4$ approaches $\widetilde{b}_4^{crit}$ from above, the mass of the bound state approaches $2c_B$. In other words, if we set $\widetilde{b}_4 = \widetilde{b}_4^{crit} + \delta b_4$, the binding energy E_B is small at small δb_4 (it turns out $E_B \sim (\delta b_4)^{1/|\lambda_B|}$) [53] and vanishes when $\delta b_4 = 0$.

Motivated by this observation, in this note we focus on the field theory (357) in a sector containing a singlet particle - antiparticle pair in a particular scaling limit we call the 'near threshold limit'. This limit is defined by scaling δb_4 to zero while simultaneously scaling $\sqrt{s} - 2c_B$ to zero like $(\delta b_4)^{1/|\lambda_B|}$. In this limit the particles are non-relativistic and we may set $\sqrt{s} - 2c_B = \frac{k^2}{c_B}$.[54] In our scaling limit

$$\frac{\delta b_4}{c_B} \to 0, \quad \frac{k}{c_B} \to 0, \quad \frac{k}{c_B}\left(\frac{c_B}{\delta b_4}\right)^{\frac{1}{2|\lambda_B|}} = \text{ fixed.} \tag{361}$$

Like any non-relativistic limit, our limit focuses attention on a sector of the theory in which kinetic energies of the particle and antiparticle are small compared to rest masses. In this limit our system must admit an effective description in terms of the non-relativistic quantum mechanics of two particles interacting via Chern-Simons gauge boson exchange, plus a contact interaction. We will now describe this quantum mechanical system in more detail, following Amelino-Camelia and Bak [22].

It is well known (see, for instance, [15, 21]) that the entire effect of the Chern-Simons interactions between non-relativistic particles is to implement anyonic statistics for the particles. This happens because the Chern-Simons equation of motion forces each particle to trap a unit of flux; the other particle picks up a phase when circumnavigating this flux. The magnitude of the phase depends on the coupling colour factors: when the colour factors of the two particles (which

[52] b_4 is always negative when bound states exist, so it possible that (357) is non perturbatively unstable in this range of parameters. While the study of the nonperturbative stability of (357) is an interesting question (one that can presumably be settled by the evaluation of the all orders effective action for ϕ), it is irrelevant for the perturbative considerations of this note, and will not be studied in this chapter.

[53] More precisely, at lowest nontrivial order in δb_4

$$\frac{E_B}{4c_B} = \left(\frac{\delta b_4}{16\pi|\lambda_B|c_B}\right)^{\frac{1}{|\lambda_B|}}. \tag{360}$$

[54] Note that our definition of the near threshold limit does not constrain $\sqrt{s} - 2c_B$ to take a particular sign. This quantity is negative in the study of bound states, and positive in the study of scattering.

transform in representations R_1 and R_2 respectively) Clebsch-Gordon couple into representation R_m it turns out that the magnitude of the phase is given by [15]

$$\nu_m = \frac{c_2(R_m) - c_2(R_1) - c_2(R_2)}{\kappa}, \tag{362}$$

where $c_2(R)$ is the quadratic Casimir of the representation R.

The effect of this phase is most simply described when we change variables to work with the centre of mass and relative degrees of freedom of the particle - antiparticle system. The centre of mass motion is free, and is ignored in what follows. In terms of relative coordinates, in the gauge singlet sector (i.e. $c_2(R_m) = 0$), the entire effect of the Chern-Simons coupled gauge field is implemented by inserting a point like solenoid of integrated flux $-2\pi\lambda_B$ at the origin of the two dimensional plane. The quantum mechanical description of this system is given by a non-relativistic Schroedinger equation (363) below for a particle of effective mass $\frac{c_B}{2}$ and of effective $U(1)$ charge unity, minimally coupled to a $U(1)$ gauge field corresponding to this point like solenoid In other words, the time independent Schroedinger equation for our system at energy $E = \sqrt{s} - 2c_B = \frac{k^2}{c_B}$ is given by

$$\begin{aligned} -D_i D^i \psi &= k^2 \psi, \\ D_i &= \nabla_i + i A_i, \\ A_i &= \nu \frac{\epsilon_{ij} x^j}{x^2}, \end{aligned} \tag{363}$$

where, in the singlet sector, (as in Chapter 1)

$$\nu = -\lambda_B. \tag{364}$$

It turns out that the point like interaction between the particle and the antiparticle imposes modified boundary conditions for this effective Schroedinger wave function at origin [22, 40](see the Appendix 2.2 for an intuitive explanation). As explained in [22, 40] there exists a one parameter set of consistent and self-adjoint boundary conditions for the wave function at the origin. These boundary conditions are specified as follows. Let

$$\psi(\vec{r}) = \sum_m e^{im\theta} \psi_m(r). \tag{365}$$

The functions $\psi_m(r)$ for $m \neq 0$ are required, as usual to vanish at $r = 0$. For $m = 0$, on the other hand, we require that

$$\psi_0(r) \propto \left(r^{|\lambda_B|} + \frac{w R^{2|\lambda_B|}}{r^{|\lambda_B|}} \right), \tag{366}$$

where R is a reference length scale and w is the self-adjoint extension parameter as introduced in [22].

In other words ψ_0 is not forced to vanish at the origin but has a component that blows up. We refer to (366) as the Amelino-Camelia-Bak boundary conditions.

The modified boundary conditions (366) are labeled by the single dimensionful parameter $wR^{2|\lambda_B|}$. It follows from dimensional analysis that the effect of this parameter on any process with characteristic momentum scale k (like the scattering of particles with momentum k) is proportional to $w(Rk)^{2|\lambda_B|}$. As $w(Rk)^{2|\lambda_B|} \to 0$ the boundary conditions above effectively reduce to the 'usual' Aharonov-Bohm boundary conditions; the boundary conditions that force ψ_0 to vanish at the origin.

In summary, the low energy effective description of the particle - antiparticle system in the near threshold limit is given by the quantum mechanics of a single non-relativistic particle propagating in two dimensions. The wave function of this particle obeys the Schroedinger equation (363) and the boundary conditions (366). The boundary condition parameter $wR^{2|\lambda_B|}$ in (366) is an as yet unknown function of δb_4.

It follows from the discussion above that the S-matrix (358) must reduce in the near threshold limit, to the S-matrix computed by solving (363) subject to the Amelino-Camelia- Bak boundary conditions. This expectation is a nontrivial consistency check of the conjecture (358), which we now proceed to verify.

The near threshold limit of the S-matrix (358) is easily determined. As above we set

$$\sqrt{s} = 2c_B + \frac{k^2}{c_B}. \tag{367}$$

In the limit (361), the second line of (358) reduces to

$$8c_B|\sin(\pi\lambda_B)|\frac{1 + e^{i\pi|\lambda_B|}\left[\frac{\delta b_4\left(\frac{2c_B}{k}\right)^{2|\lambda_B|}}{16\pi|\lambda_B|c_B}\right]}{1 - e^{i\pi|\lambda_B|}\left[\frac{\delta b_4\left(\frac{2c_B}{k}\right)^{2|\lambda_B|}}{16\pi|\lambda_B|c_B}\right]},$$

so that the S-matrix (358) reduces to

$$\begin{aligned}
T_S(\sqrt{s},\theta) &= -16\pi i c_B(\cos(\pi\lambda_B)-1)\delta(\theta) + 8ic_B\sin(\pi\lambda_B)\mathrm{Pv}\left(\cot\left(\frac{\theta}{2}\right)\right) \\
&+ 8c_B|\sin(\pi\lambda_B)|\frac{1+e^{i\pi|\lambda_B|}\frac{A_R}{k^{2|\lambda_B|}}}{1-e^{i\pi|\lambda_B|}\frac{A_R}{k^{2|\lambda_B|}}}, \\
A_R &= \left[\frac{\delta b_4\,(2c_B)^{2|\lambda_B|}}{16\pi|\lambda_B|c_B}\right].
\end{aligned} \tag{368}$$

On the other hand the S-matrix obtained by solving the Schroedinger equation (363) subject to the boundary conditions (366) has already been determined in [22] and we rederive it in the Appendix 2.2. [55] It turns out that

$$\begin{aligned}
T_{NR} &= -16\pi i c_B\left(\cos\left(\pi\lambda_B\right)-1\right)\delta(\theta) + 8ic_B\sin(\pi\lambda_B)\mathrm{Pv}\left(\cot\frac{\theta}{2}\right) \\
&+ 8c_B|\sin\pi\lambda_B|\frac{1+e^{i\pi|\lambda_B|}\frac{A_{NR}}{k^{2|\lambda_B|}}}{1-e^{i\pi|\lambda_B|}\frac{A_{NR}}{k^{2|\lambda_B|}}}, \\
A_{NR} &= \frac{-1}{w}\left(\frac{2}{R}\right)^{2|\lambda_B|}\frac{\Gamma(1+|\lambda_B|)}{\Gamma(1-|\lambda_B|)}.
\end{aligned} \tag{371}$$

The S-matrices (368) and (371) are identical in structure. They agree in all details provided we identify

$$-w\,(c_B R)^{2|\lambda_B|} = \frac{c_B}{\delta b_4}\left(16\pi|\lambda_B|\frac{\Gamma(1+|\lambda_B|)}{\Gamma(1-|\lambda_B|)}\right). \tag{372}$$

(372) determines the hitherto unknown dependence of the boundary condition parameter $wR^{2|\lambda_B|}$ as a function of δb_4.

[55] More precisely in the Appendix 2.2 we show that the Schroedinger equation described above has a scattering solution that takes the form

$$\begin{aligned}
\psi(\vec{r}) &= e^{ikx} + \zeta(\vec{r}), \\
\zeta(\vec{r}) &= \frac{e^{-\frac{i\pi}{4}}e^{ikr}h(\theta)}{\sqrt{2\pi kr}} + \mathcal{O}\left(\frac{1}{r^{\frac{3}{2}}}\right), \\
h(\theta) &= 2\pi\left(\cos\left(\pi\lambda_B\right)-1\right)\delta(\theta) - \sin(\pi\lambda_B)\mathrm{Pv}\left(\cot\frac{\theta}{2}\right) + i|\sin\left(\pi\lambda_B\right)|\frac{1+e^{i\pi|\lambda_B|}\left[\frac{-1}{w}\left(\frac{2}{kR}\right)^{2|\lambda_B|}\frac{\Gamma(1+|\lambda_B|)}{\Gamma(1-|\lambda_B|)}\right]}{1-e^{i\pi|\lambda_B|}\left[\frac{-1}{w}\left(\frac{2}{kR}\right)^{2|\lambda_B|}\frac{\Gamma(1+|\lambda_B|)}{\Gamma(1-|\lambda_B|)}\right]}.
\end{aligned} \tag{369}$$

The non-relativistic limit of the usual invariant scattering amplitude is given by

$$T_{NR} = -8ic_B h(\theta). \tag{370}$$

In summary, in the near threshold limit, the S-matrix (358) agrees perfectly with the S-matrix computed from the Schroedinger equation (363) subject to the boundary conditions

$$\psi_0(r) \propto \left(r^{|\lambda_B|} - \frac{\frac{c_B}{\delta b_4}\left(16\pi|\lambda_B|\frac{\Gamma(1+|\lambda_B|)}{\Gamma(1-|\lambda_B|)}\right)}{\left(rc_B^2\right)^{|\lambda_B|}} \right). \tag{373}$$

As we have emphasized above, however, the effect of the modified boundary conditions on a process at momentum scale k is measured by $w(Rk)^{2|\lambda_B|}$. It follows from (372) that in the current situation, the effect of the modified boundary conditions on a process at momentum scale k is measured by

$$M = \frac{c_B}{\delta b_4}\left(\frac{k}{c_B}\right)^{2|\lambda_B|}\left(16\pi|\lambda_B|\frac{\Gamma(1+|\lambda_B|)}{\Gamma(1-|\lambda_B|)}\right). \tag{374}$$

Note that M is held fixed in the near threshold scaling limit (361). The modified boundary condition can be ignored when $M \to 0$. M tends to zero in, for instance, the usual non-relativistic limit (where k is scaled to zero with all other parameters like δb_4 held fixed). Consequently $wR^{2|\lambda_B|}$ is effectively zero in the quantum mechanical description of the usual non-relativistic limit, explaining why (358) reduces to the $w = 0$ Aharonov-Bohm-Ruijsenaars [17, 19] S matrix in this limit, as noted in chapter 1.

The agreement of the S-matrix (358) (and in particular of its poles) with (371) in the near threshold limit immediately demonstrates that the spectrum of near threshold bound states of the singlet particle - antiparticle sector of (357) agrees with the spectrum of bound states of the Schroedinger equation (363) subject to the boundary conditions (373).

The scattering matrices T_S and T_{NR} are quite involved functions of k and λ_B; for this reason we view the matching of these two functions in the appropriate limit as a rather nontrivial test of the conjectured S matrix (358). Note that T_S would not have matched with T_{NR} without the the additional factor $\frac{\sin(\pi\lambda_B)}{\pi\lambda_B}$ invoked in chapter 1. As a consequence the results of this note provide indirect support to the modified crossing symmetry properties for the S matrix of matter Chern-Simons theories conjectured in chapter 1.

In this chapter we have argued that the S matrix (358) may be derived from a Schroedinger equation in a particular scaling limit. Perhaps it is possible to derive the full relativistic formula (358) from the solution to an appropriate Schroedinger equation in lightcone slicing; we leave the further investigations of this issue to future work.

2.2 Appendix: The quantum mechanics of anyons with point like interactions

In the main text we have followed [22, 40] to assert that point like interactions between anyons effectively impose modified local boundary conditions on the Schroedinger equation in the relative coordinates. This assertion may appear unfamiliar as contact interactions usually lead to delta function potentials for relative coordinates. In fact these viewpoints are equivalent. In the subsections 2.2 and 2.2 below we demonstrate that the correct treatment of the two dimensional delta function at $\lambda_B = 0$ does, in fact, effectively modify the boundary conditions at the origin and has no other effect. Moreover the boundary conditions so obtained agree with $\lambda_B \to 0$ limit of the boundary conditions (366).

In subsection 2.2 we proceed to rederive the scattering amplitude for the Schroedinger equation (363) subject to the boundary conditions (366); our results agree with those of [22].

Quantum mechanics with a two dimensional delta function

Renormalization of the coupling constant In this section we review the dynamics of the quantum mechanical system governed by the two dimensional Schroedinger equation

$$-\frac{\nabla^2}{2m}\psi(\vec{x}) + V(\vec{x})\psi(\vec{x}) = \frac{k^2}{2m}\psi(\vec{x}), \tag{375}$$

where $V(\vec{x})$ is taken to be proportional to a suitably renormalized version of the attractive two dimensional δ function. This system has been studied in great detail in several papers (see e.g. [41]); we review the principal results.

Let

$$\psi(\vec{x}) = \int \frac{d^2k}{(2\pi)^2} e^{i\vec{k}.\vec{x}} \widetilde{\psi}(\vec{k}). \tag{376}$$

The time independent solution of (375) that describes the scattering of an incoming particle with momentum $\vec{k}$ off an arbitrary potential $V(x)$ is given by the solution to the Lippmann-Schwinger equation

$$\widetilde{\psi}(\vec{p}) = (2\pi)^2\delta^2(\vec{p}-\vec{k}) + 2m\int \frac{d^2q}{(2\pi)^2}\frac{\widetilde{V}(\vec{q})\widetilde{\psi}(\vec{p}-\vec{q})}{k^2-p^2+i\epsilon}. \tag{377}$$

Let $V(x) = -g\delta^2(\vec{x})$ so that its Fourier transform is given by $\widetilde{V}(\vec{k}) = -g$. Plugging into (377) we find

$$\widetilde{\psi}(\vec{p}) = (2\pi)^2\delta^2(\vec{p}-\vec{k}) - \frac{2mgA(\vec{k})}{k^2-p^2+i\epsilon}, \tag{378}$$

where

$$A(\vec{k}) = \frac{1}{1 - 2mg \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 - k^2 - i\epsilon}}. \tag{379}$$

The integral in (379) diverges logarithmically. Evaluating the integral with a cut off Λ we have

$$A(\vec{k}) = \frac{1}{1 - \frac{mg}{2\pi} \ln\left(\frac{\Lambda^2}{-k^2}\right)}. \tag{380}$$

The function $A(\vec{k})$ is proportional to the scattering amplitude of our quantum mechanical system. In order to define a sensible scattering problem we must regulate and renormalize (380) by choosing the coupling constant g to scale to zero logarithmically with the cut off Λ. We choose $g(\Lambda)$ so that

$$\frac{1}{g(\Lambda)} = \frac{1}{g_R(\mu)} + \frac{m}{2\pi} \ln\left(\frac{\Lambda^2}{\mu^2}\right), \tag{381}$$

where the renormalized coupling $g_R(\mu)$ is held fixed as Λ is taken to infinity. $g_R(\mu)$ is, of course, a function of the renormalization scale μ. (380) now takes the form [56]

$$A(\vec{k}) = \frac{1}{1 - \frac{mg_R}{2\pi} \ln\left(\frac{\mu^2}{-k^2}\right)}. \tag{384}$$

Description in terms of modified boundary conditions We will now find an alternative effective description of the renormalized two dimensional delta function in terms of modified boundary conditions at $r = 0$. For this purpose it will prove convenient to work in position rather than momentum space. For this reason we regulate the δ function potential as the 'circular square well'

$$\begin{aligned} V(r) &= -\frac{g}{\pi r_0^2} \quad ; \ r < r_0, \\ &= 0 \qquad\quad ; \ r > r_0. \end{aligned} \tag{385}$$

Let us now study rotationally invariant solutions of the two dimensional Schroedinger equation with the potential (385). [57] Clearly the most general regular (at $r = 0$) solution to the

[56] As an application notice that the scattering amplitude (384) has a pole at

$$k^2 = -\mu^2 e^{-\frac{2\pi}{mg_R}}, \tag{382}$$

implying that our renormalized δ function potential quantum mechanics has a single bound state with binding energy

$$E = -\frac{\mu^2}{2m} e^{-\frac{2\pi}{mg_R}}. \tag{383}$$

[57] Only rotationally invariant solutions are affected by the potential (385) in the limit $r_0 \to 0$, as the

Schroedinger equation takes the form

$$\begin{aligned} aJ_0(lr)\ ;\ r < r_0, \\ cJ_0(kr) + dY_0(kr)\ ;\ r > r_0, \end{aligned} \tag{386}$$

where,

$$l^2 = 2m\left(\frac{g}{\pi r_0^2} + E\right) \quad , \quad k^2 = 2mE. \tag{387}$$

The requirement of continuity of the wave function and its first derivative across $r = r_0$ determines d and c in terms of a. In the small r_0 limit it is easily verified that

$$\frac{d}{c} = \frac{-1}{\frac{2}{mg} + \frac{2}{\pi}\left[\gamma + \ln\left(\frac{kr_0}{2}\right)\right]}, \tag{388}$$

where, γ is Euler-Mascheroni constant.

As in the previous section (388) does not have a well defined $r_0 \to 0$ limit. In order that the LHS of (388) is well defined as $r_0 \to 0$ we must choose g to be a function of r_0 and take g to zero as r_0 is scaled to zero, keeping g_R fixed where

$$\frac{1}{g_R(\mu)} = \frac{1}{g(r_0)} + \frac{m}{\pi}\left[\ln\left(\frac{r_0\mu}{2}\right) + \gamma\right]. \tag{389}$$

Note that(389) agrees exactly with (381) under the replacement $\frac{\mu r_0 e^{\gamma}}{2} \to \frac{\mu}{\Lambda}$.

Implementing this limit we find

$$\frac{d}{c} = \frac{-1}{\frac{2}{mg_R} + \frac{2}{\pi}\ln\left(\frac{k}{\mu}\right)}. \tag{390}$$

It follows that the Schroedinger problem with a delta function potential with renormalized strength g_R is equivalent to the free Schroedinger equation subject to the $r \to 0$ boundary condition

$$\psi_0(r) \propto \left[\left(-\frac{2}{mg_R} - \frac{2}{\pi}\ln\frac{k}{\mu}\right) J_0\left(kr\right) + Y_0(kr)\right]. \tag{391}$$

Using the small argument expansions

$$J_0(kr) = 1 + \mathcal{O}\left((kr)^2\right), \quad Y_0(kr) = \frac{2}{\pi}\ln\left(\frac{kr}{2}\right) + 2\frac{\gamma}{\pi} + \mathcal{O}\left((kr)^2\ln(kr)\right), \tag{392}$$

we see that the k dependence cancels from (391) and the boundary condition on $\psi(r)$ takes the

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wave function at nonzero angular momentum dies rapidly at small r due to the angular momentum barrier.

local form

$$\psi_0(r) \propto \left[\left(-\frac{2}{mg_R} + 2\frac{\gamma}{\pi}\right) + \frac{2}{\pi}\ln\left(\frac{\mu r}{2}\right) + \mathcal{O}\left(r^2 \ln r\right)\right]. \tag{393}$$

In summary, the Schroedinger equation in the presence of a renormalized δ function potential is exactly equivalent to the free Schroedinger equation subject to the local boundary conditions (393) at the origin.

It is easily verified that the boundary conditions (393) are obtained as a limit of the Amelino-Camelia-Bak boundary conditions (366) if we set

$$w = -1 + |\lambda_B|\left(-\frac{2\pi}{mg_R} + 2\gamma + 2\ln\left(\frac{\mu R}{2}\right)\right),$$

and take the limit $|\lambda_B| \to 0$. In other words the usual (i.e. δ function) description of contact interactions is indeed equivalent to the appropriate $|\nu| \to 0$ limit of the Schroedinger equation (363) subject to the boundary conditions (366). This suggests that the boundary conditions (366) do indeed capture the effect of contact interactions at general λ_B. This has been argued to be true in [22, 40].

Derivation of the scattering amplitude In this section we will derive the scattering amplitude for the Schroedinger equation (363) subject to the boundary conditions (366). We assume $|\nu| < 1$.

We wish to find scattering state solutions at energy $E = \frac{k^2}{2m}$ of the Schroedinger equation for this particle; i.e. k is the magnitude of the momentum of the particle incident on the solenoid. The most general solution of the Schroedinger equation that meets the boundary conditions for $\psi_m(r)$ at the origin ($m \neq 0$) is

$$\psi(\vec{r}) = \sum_{n>0} a_n e^{in\theta} J_{n+\nu}(kr) + \sum_{n>0} a_{-n} e^{-in\theta} J_{n-\nu}(kr) + a_0 J_{|\nu|}(kr) + b_0 J_{-|\nu|}(kr). \tag{394}$$

The scattering solution we wish to find obeys the boundary condition (366); moreover at large r its ingoing piece (part proportional to e^{-ikr}) must reduce to that of the incoming wave e^{ikx}. It is not difficult to see that the unique solution that meets our boundary conditions is given by (see 1.9.3 for the detailed derivation for the special case $w = 0$)

$$\begin{aligned}\psi(\vec{r}) =& \sum_{n=1}^{\infty} i^n e^{-i\frac{\pi\nu}{2}} J_{n+\nu}(kr) e^{in\theta} + \sum_{n=1}^{\infty} i^n e^{i\frac{\pi\nu}{2}} J_{n-\nu}(kr) e^{-in\theta} \\ &+ \frac{\Gamma(|\nu|+1)\left(\frac{2}{k}\right)^{|\nu|} J_{|\nu|}(kr) + wR^{2|\nu|}\Gamma(1-|\nu|)\left(\frac{k}{2}\right)^{|\nu|} J_{-|\nu|}(kr)}{\Gamma(|\nu|+1)\left(\frac{2}{k}\right)^{|\nu|} e^{i\frac{\pi|\nu|}{2}} + wR^{2|\nu|}\Gamma(1-|\nu|)\left(\frac{k}{2}\right)^{|\nu|} e^{-i\frac{\pi|\nu|}{2}}}.\end{aligned} \tag{395}$$

At large r, $\psi(\vec{r})$ reduces to

$$\frac{1}{\sqrt{2\pi kr}}\left(e^{i\frac{\pi}{4}}\delta(\theta-\pi)e^{-ikr}+H(\theta)e^{-i\frac{\pi}{4}}e^{ikr}\right),$$

where,

$$\begin{aligned}H(\theta) =& \sum_{n=1}^{\infty}\left(e^{-i\pi\nu}e^{in\theta}+e^{i\pi\nu}e^{-in\theta}\right)\\ &+\frac{\Gamma(|\nu|+1)\left(\frac{2}{k}\right)^{|\nu|}e^{-i\frac{\pi|\nu|}{2}}+wR^{2|\nu|}\Gamma(1-|\nu|)\left(\frac{k}{2}\right)^{|\nu|}e^{i\frac{\pi|\nu|}{2}}}{\Gamma(|\nu|+1)\left(\frac{2}{k}\right)^{|\nu|}e^{i\frac{\pi|\nu|}{2}}+wR^{2|\nu|}\Gamma(1-|\nu|)\left(\frac{k}{2}\right)^{|\nu|}e^{-i\frac{\pi|\nu|}{2}}}.\end{aligned} \tag{396}$$

Now we can write

$$\begin{aligned}\sum_{n=1}^{\infty}\left(e^{-i\pi\nu}e^{in\theta}+e^{i\pi\nu}e^{-in\theta}\right) &= \left(\sum_{n=1}^{\infty}2\cos(\pi\nu)\cos(n\theta)\right)+\left(\sum_{n=1}^{\infty}2\sin(\pi\nu)\sin(n\theta)\right)\\ &= \left(\cos(\pi\nu)+\sum_{n=1}^{\infty}2\cos(\pi\nu)\cos(n\theta)\right)-\cos(\pi\nu)\\ &+\left(\sum_{n=1}^{\infty}2\sin(\pi\nu)\sin(n\theta)\right)\\ &= 2\pi\cos(\pi\nu)\delta(\theta)-\cos(\pi\nu)+\left(\sum_{n=1}^{\infty}2\sin(\pi\nu)\sin(n\theta)\right)\\ &= 2\pi\cos(\pi\nu)\delta(\theta)+\sin(\pi\nu)\mathrm{Pv}\left(\cot\left(\frac{\theta}{2}\right)\right)-\cos(\pi\nu).\end{aligned} \tag{397}$$

Substituting in (396)

$$\begin{aligned}H(\theta) &= 2\pi\cos(\pi\nu)\delta(\theta)+\sin(\pi\nu)\mathrm{Pv}\left(\cot\left(\frac{\theta}{2}\right)\right)\\ &+\frac{\Gamma(|\nu|+1)\left(\frac{2}{k}\right)^{|\nu|}e^{-i\frac{\pi|\nu|}{2}}+wR^{2|\nu|}\Gamma(1-|\nu|)\left(\frac{k}{2}\right)^{|\nu|}e^{i\frac{\pi|\nu|}{2}}}{\Gamma(|\nu|+1)\left(\frac{2}{k}\right)^{|\nu|}e^{i\frac{\pi|\nu|}{2}}+wR^{2|\nu|}\Gamma(1-|\nu|)\left(\frac{k}{2}\right)^{|\nu|}e^{-i\frac{\pi|\nu|}{2}}}-\cos(\pi\nu)\\ &= 2\pi\cos(\pi\nu)\delta(\theta)+\sin(\pi\nu)\mathrm{Pv}\left(\cot\left(\frac{\theta}{2}\right)\right)\\ &-i\sin(\pi|\nu|)\frac{\Gamma(|\nu|+1)\left(\frac{2}{k}\right)^{|\nu|}e^{i\pi|\nu|}-wR^{2|\nu|}\Gamma(1-|\nu|)\left(\frac{k}{2}\right)^{|\nu|}}{\Gamma(|\nu|+1)\left(\frac{2}{k}\right)^{|\nu|}e^{i\pi|\nu|}+wR^{2|\nu|}\Gamma(1-|\nu|)\left(\frac{k}{2}\right)^{|\nu|}}.\end{aligned} \tag{398}$$

In order to compute the scattering amplitude, we must rewrite the wave function as a plane wave plus a scattered piece; at large r

$$\psi(r) = e^{ikx} + \frac{h(\theta)e^{-i\frac{\pi}{4}}e^{ikr}}{\sqrt{2\pi kr}}. \tag{399}$$

We find

$$h(\theta) = H(\theta) - 2\pi\delta(\theta), \tag{400}$$

so that

$$\begin{aligned} h(\theta) &= 2\pi\left(\cos(\pi\nu) - 1\right)\delta(\theta) + \sin(\pi\nu)\mathrm{Pv}\left(\cot\left(\frac{\theta}{2}\right)\right) \\ &\quad - i\sin(\pi|\nu|)\frac{\Gamma(|\nu|+1)\left(\frac{2}{k}\right)^{|\nu|}e^{i\pi|\nu|} - wR^{2|\nu|}\Gamma(1-|\nu|)\left(\frac{k}{2}\right)^{|\nu|}}{\Gamma(|\nu|+1)\left(\frac{2}{k}\right)^{|\nu|}e^{i\pi|\nu|} + wR^{2|\nu|}\Gamma(1-|\nu|)\left(\frac{k}{2}\right)^{|\nu|}}. \end{aligned} \tag{401}$$

This yields (369).

3 Chapter 3: A Charged Membrane Paradigm at Large D

3.1 Introduction

Emparan, Suzuki, Tanabe (EST) and collaborators have recently noted [42–48] that the classical dynamics of black holes simplifies at large D (D is the dimensionality of space time). Schwarzschild black holes in a large number of dimensions are characterized by two widely separated length scales. The first of these is the Schwarzschild radius r_0, while the second is the distance δr away from Schwarzschild radius after which spacetime ceases to be warped by the black hole. In other words δr is defined so that spacetime is effectively flat for $r > r_0 + \delta r$. At large D the membrane thickness, δr, is easily estimated; it turns out that $\delta r \sim r_0/D \ll r_0$. Similar observations apply to static charged black holes at large D.

The separation of scales between the membrane thickness and the black hole radius results in the simplification of black hole dynamics at large D. The first hint of this fact appeared in the results for the large D spectrum of quasinormal modes of Schwarzschild black holes obtained by EST and collaborators [45, 47, 48]. It turns out that most of the quasinormal modes are heavy with frequencies $\sim 1/\delta r$. The remaining modes are anomalously light; their frequencies are of order $1/r_0$. [58] As we will see below, the spectrum of quasinormal modes about Reissner-Nordstrom black holes is qualitatively similar.

The pattern of the quasinormal mode frequencies described above may be understood intuitively as follows. A quasinormal mode is a linearized solution of Einstein's equations about the black hole background, subject to the condition that it is ingoing at the horizon and outgoing in the asymptotically flat exterior region. As the second boundary condition is effectively imposed at the outer edge of the membrane region, the quasinormal problem is analogous to the analysis of the harmonics of the wave equation in a hollow, leaky spherical shell. The radius of this shell is r_0 and its thickness is δr. Clearly modes with nonzero 'harmonic number' in the radial direction all have frequencies of order $1/\delta r$; these are EST's generic heavy modes. Modes of zero radial harmonic number, if present, have frequencies of order $1/r_0$; these are EST's anomalously light modes.

The imaginary part of all heavy quasinormal mode frequencies are of order $1/\delta r$; it follows that these modes all decay away after a time scale of order δr. On the other hand the light quasinormal modes have lifetimes of order r_0. Consider a violent dynamical process like a black hole collision. For a time of order δr after the event, dynamics is complicated and involves all quasinormal modes. For times $t \gg \delta r$, however, the heavy quasinormal modes have all decayed

[58] More precisely, all but a finite number of quasinormal modes at every angular momentum are heavy. A finite number of modes at every angular momentum are light.

away and the subsequent dynamics is governed by a nonlinear interacting theory of only the light quasinormal modes, the principal focus of this chapter. [59]

Light quasinormal modes may roughly be thought of as 'Goldstone bosons' for the symmetries of flat space that are broken by the black hole. Non rotating black holes appear in family of solutions labeled by a set of parameters α^i; the black hole location, radius, boost velocity and charge. By infinitesimally varying each of these parameters we obtain a set of time independent linearized solutions of the Einstein-Maxwell equations about any of these black holes. Now consider configurations that locally resemble these zero modes but with $\delta\alpha^i = \delta\alpha^i(\theta)$, i.e. with the infinitesimal parametric variations chosen to be functions of the black hole angular coordinates with spherical harmonic numbers of order unity. It follows that in any patch of size of order δr (i.e. of angular extent of order $\delta r/r_0$) the $\delta\alpha^i$ are approximately constant. In any such patch the fluctuation closely approximates a zero mode, and so is static on the time scale δr. However the variation of $\delta\alpha^i$ on length scales of order r_0 cause such configurations evolve over times of the same order. It follows that quasinormal modes built out of such configurations have frequencies of order $1/r_0$, and may be identified with charged generalizations of the light modes of EST.

The identification of light quasinormal modes with 'Goldstone bosons' immediately suggests the possibility of using the collective coordinate method to derive the *nonlinear* 'chiral Lagrangian' of these light modes. [60]. On general grounds one expects that the effective nonlinear equations of motion for the light modes will admit a power series expansion in the ratio of the energy scales of the light and heavy modes, i.e. in a power series in $\delta r/r_0 \sim 1/D$. In other words the collective coordinate equations for light modes dynamics are a reformulation of black hole dynamics that is exact at large D.

At leading nontrivial order in $1/D$, the equations that govern the collective coordinate dynamics of uncharged black holes were derived in the recent paper [1] (see [49–53] for closely related work) [61]. In this chapter we build on the work of [1] in two different ways. First we improve the construction of [1] in several respects. We use collective coordinate variables with a direct physical significance and present our final equations and spacetimes in an explicitly 'geometrical' form. Second - using the same improvements - we generalize the work of [1] to obtain the nonlinear collective coordinate dynamics of *charged* black holes in a large number of dimensions.

[59]At time scales large compared to r_0 the light quasinormal modes also decay away and the black holes settle down into their equilibrium state. The approach to equilibrium is governed by the linearized theory of quasinormal modes.

[60]As the resulting system turns out to be dissipative, however, it is easier to deal with the effective equations of motion than an effective action.

[61]The papers [49] and [50] worked out the effective collective coordinate expansions for the special case of uncharged stationary configurations. When restricted to flat space and lowest order in D the results of these papers are special cases of [1] and this chapter. The papers [51–53] analyze dynamics at length and time scales of order $r_0/\sqrt{D}$ (this turns out to be the relevant length scale for the Gregory-Laflamme phenomenon at large D), as opposed to this chapter where we focus on length scales of order unity.

In the rest of this introduction we will provide a more detailed description of the collective coordinate construction presented in this chapter and present our main results.

A more detailed introduction and summary

In the technical heart of this chapter we follow [1] to simply write down a class of leading order collective coordinate spacetimes (see (408) below). We then carefully verify that our spacetimes and gauge fields (written down by physically guided guesswork following [1]) obey the Einstein-Maxwell equations of motion at leading order in $1/D$ [62] and so constitute a good starting point for the construction of true solutions to the Einstein-Maxwell equations in an expansion in $1/D$. Our collective coordinate spacetimes are built sewing together patches of Reissner-Nordstrom black holes with different radii, charges and boost velocities into a single smooth spacetime. These spacetimes are in one to one correspondence with the configurations of a non gravitational codimension one membrane propagating in flat D dimensional space. The dynamical degrees of freedom of the membrane are

- 1. The embedding of its timelike world volume in flat D dimensional spacetime, i.e. the shape of the membrane. Through this chapter we use the symbols n_A and K_{AB} to denote the normal and extrinsic curvature of the membrane surface in D dimensional Minkowski space. We also use the symbol $\mathcal{K} = \eta^{AB} K_{AB}$ to denote the trace of the extrinsic curvature.
- 2. A velocity vector field u^A in the membrane world volume (so that $u \cdot n = 0$) whose world volume divergence vanishes (i.e. $\nabla \cdot u = 0$ where ∇ is the covariant derivative on the membrane world volume). The velocity field is normalized in the usual manner $u \cdot u = -1$.
- 3. A scalar charge density field Q [63] that lives on the membrane (this field is absent in the neutral case).

To reiterate, the starting point of the technical analysis presented in this chapter is a class of 'collective coordinate spacetimes' - that are simply guessed. We have one such spacetime for every distinct membrane configuration. Our collective coordinate spacetimes turn out to solve the Einstein-Maxwell equations at leading order in $1/D$ everywhere outside their event horizons.

The strategy adopted in the rest of this chapter is to use these spacetimes as the first term in the perturbative construction of true solutions of the Einstein-Maxwell equations in a power series

[62] More precisely the equations of motion are obeyed everywhere outside the even horizons of these configurations. This is sufficient, as regions inside the event horizon are causally disconnected - and invisible - from those outside, and so may be ignored for the purposes of predicting observations outside the event horizon.

[63] More precisely the field Q utilized in this chapter is a variable proportional to the actual conserved charge density field on the membrane.

expansion in $1/D$. In this chapter we explicitly implement this expansion to first subleading order in $1/D$. In other words we correct the leading order collective coordinate spacetimes described above to ensure that they obey the Einstein-Maxwell equations not just at the leading order in $1/D$ but also at first subleading order in this expansion. We discover that it is possible to accomplish this task with only nonsingular corrections if and only if the membrane shape, charge density and velocity fields obey the following local equations of motion

$$\begin{aligned}
&\left(\frac{\nabla^2 u}{\mathcal{K}} - (1-Q^2)\frac{\nabla\mathcal{K}}{\mathcal{K}} + u\cdot K - (1+Q^2)(u\cdot\nabla)u\right)\cdot\mathcal{P} = 0,\\
&\frac{\nabla^2 Q}{\mathcal{K}} - u\cdot\nabla Q - Q\left(\frac{u\cdot\nabla\mathcal{K}}{\mathcal{K}} - u\cdot K\cdot u\right) = 0,
\end{aligned} \tag{402}$$

$$\begin{aligned}
&\text{where } \nabla = \text{the covariant derivative on the membrane world volume,}\\
&\text{and } \mathcal{P}_{AB} = \eta_{AB} - n_A n_B + u_A u_B.
\end{aligned}$$

64 65

Corresponding to every solution of the equations (402) we are able to improve (408). The improvements are computed to ensure that the corrected configurations (see (473),(474), (475), (476), (477)) solve the Einstein-Maxwell equations at leading *and first subleading* order in $1/D$. We expect the construction presented in this chapter to constitute the first couple of terms in a systematic expansion of solutions to the Einstein-Maxwell equations order by order in $1/D$.

As we have explained above, membrane spacetimes are parameterized by the shape of the membrane (one function), the charge density field (one function) and a unit normalized divergence free velocity field on the membrane ($D-3$ functions) and so by $D-1$ functions in total. The membrane equations (402) are also $D-1$ in number (the first equation in (402) is a vector projected orthogonal to n and u and so has $D-2$ components, while the second is a scalar and so has one component). It follows that we have as many equations as variables and so (402) define an initial value problem for membrane motion. (402) are simply the large D collective coordinate equations of black hole motion.

Following [1], in this chapter we have derived the membrane equations (402) *under the assumption* that our spacetimes preserve an $SO(D-p-2)$ isometry subgroup for p held fixed as $D\to\infty$. [66] Even though have made this assumption in our derivation, the final membrane

[64]The expression in the first bracket in the first of (402) is a vector in the membrane world volume and so is orthogonal to n. When acting on such a vector the projector $\mathcal{P}_{AB} = g^{(WV)}_{AB} + u_A u_B$ where $g^{(WV)}_{AB}$ is the induced metric on the membrane world volume.

[65]In the uncharged limit, the equation (402) are easily demonstrated to reduce to the membrane equation of motion presented in [1] once we account for the fact that the velocity field of this chapter differs from the velocity field employed in [1] (see subsection 3.3.11 for relevant details.).

[66]This requirement guarantees that there are no unaccounted for factors of D in, for instance, derivatives of the metric and gauge field.

equations (402) (and the spacetimes dual to solutions of these membrane equations) make no explicit reference to the isometry group. Our final equations are entirely covariant; they treat the isometry directions and other directions democratically. We refer to equations with this property as geometrical.

Given the geometrical nature of our membrane equations and spacetimes, it is natural to wonder whether our equations apply more generally than their derivation. Could it be that (402) captures the dynamics of black hole motions on time scales of order unity, even in the absence of a large isometry symmetry? While an appropriate version of such a conjecture might well be true, we would like to emphasize a subtlety. There are several pairs of independent geometrical expressions that reduce to each other at leading order in the large D limit under the assumption of an SO$(D-p-2)$ isometry but differ from each other more generally [67]. For this reason it turns out that there are different geometrical ways of presenting the equations of motion (402), all of which are identical at leading order in $1/D$ when evaluated on any membrane configuration that preserves an SO$(D-p-2)$ isometry but which differ on more general configurations. As the results of this chapter are all obtained assuming an SO$(D-p-2)$ isometry, they cannot distinguish between these different geometrical presentations of the membrane equations. For example, the divergence of the first equation in (402) turns out to coincide, at leading order in large D, with the equation

$$(1-Q^2)\left[\frac{\nabla^2\mathcal{K}}{\mathcal{K}^2}-\frac{u\cdot\nabla\mathcal{K}}{\mathcal{K}}\right]-(1+Q^2)\left(\frac{u\cdot\nabla\mathcal{K}}{\mathcal{K}}-u\cdot K\cdot u\right)=0, \tag{403}$$
$$\text{where } \nabla = \text{the covariant derivative on the membrane world volume,}$$

under the assumption of SO$(D-p-2)$ symmetry. It follows that the computations presented in this chapter cannot resolve the question of which of these is the 'correct' leading order membrane equation in the absence of an isometry .[68]

The membrane equations (402) are nonlinear and rather complicated. In future work we will demonstrate that these equations admit simple classes of solutions in which the membrane velocity field u^μ is that of rigid rotations and the charge density field is proportional to u^0 (the time component of the velocity vector). The membrane shape is constrained to obey a single nonlinear partial differential equation. Solutions obtained in this manner include the duals to charged rotating black hole solutions at large D. For the special case of uncharged black holes

[67]For example, the independent geometrical quantities $u\cdot\nabla\mathcal{K}/D$ and $\nabla_\mu(u\cdot K)^\mu$ may be shown to agree with each other at leading order in $1/D$ for any membrane configuration that preserves an SO$(D-p-2)$ invariance. On the other hand the same two expressions could differ at leading order when evaluated on configurations that do not enjoy any symmetry.

[68]Even staying within the class of isometric spacetimes, the iteration of the computations of this chapter to one higher order could help to resolve this question. We hope to report on the results of a higher order computation in the not too distant future.

this nonlinear partial differential equation turns out to exactly match the constraint on the shape of stationary membranes derived in a different way in [49, 50], establishing that the results of [49, 50] (at leading order and in flat space) are a special case of the more general results of [1] and this chapter.

The simplest solution of the sort described in the previous paragraph is obtained upon switching off all angular velocities; the membrane solution is a static spherical 'soap bubble' with a uniform charge density. In section 3.4.4 below we have verified that the metric and gauge field dual to this solution agree perfectly with the exactly known static Reissner-Nordstrom black hole solution expanded to first subleading order in $1/D$.

The membrane equations (402) capture all of the complexities of black hole horizon dynamics at large D, at time scales of order unity [69]. The detailed study of (402) should teach us a great deal about black hole horizon dynamics. As a first small step in this program, in section (3.5) we linearize the membrane equations (402) about the exact spherical solution dual to the Reissner-Nordstrom black hole, and determine the spectrum of small fluctuations about this background (see section 3.4.4) for details. This spectrum of linearized fluctuations may be regarded as a prediction for the spectrum of light quasinormal modes about charged black holes at large D.

In the course of obtaining the quasinormal mode spectrum described in the previous paragraph, we reduce the manifestly geometrical but slightly abstract equations (402) to explicit linear differential equations for two scalar fields and a divergence free vector field on S^{D-2} times time (this reduction is valid for linearized fluctuations about the spherical membrane surface). This explicit form of the equations helps us verify that the equations (402) do indeed constitute a well posed initial value problem for the membrane shape, charge density and velocity fields at least for these linearized configurations, as we had anticipated above on intuitive grounds. Our explicit results for the quasinormal modes also reveals that the membrane equations (402) are highly dissipative. As an independent test of the equations (402) it would be useful to verify our prediction for the large D quasinormal spectrum by direct analysis of the Einstein-Maxwell equations about the Reissner-Nordstrom black hole background. While we make some remarks about this, we leave a detailed verification to future work.

[69]We believe this to be true at least for spacetimes that preserve an $\mathrm{SO}(D-p-2)$ isometry for any p that is held fixed as D is taken to infinity.

3.2 The collective coordinate ansatz

3.2.1 Boosted charged black holes in Kerr-Schild coordinates

The Reissner-Nordstrom black hole in the 'Kerr-Schild' coordinate system [70] is given by

$$
\begin{aligned}
ds^2 &= -dt^2 + dr^2 + r^2 d\Omega_{D-2}^2 + \left((1+Q^2 c_D) \left(\frac{r_0}{r}\right)^{D-3} - c_D Q^2 \left(\frac{r_0}{r}\right)^{2(D-3)} \right) (dt+dr)^2, \\
&= ds_{flat}^2 + \left((1+Q^2 c_D) \left(\frac{r_0}{r}\right)^{D-3} - c_D Q^2 \left(\frac{r_0}{r}\right)^{2(D-3)} \right) (dt+dr)^2, \\
A &= \sqrt{2} Q \left(\frac{r_0}{r}\right)^{D-3} (dt+dr).
\end{aligned}
\tag{404}
$$

(404) describes a black hole at rest, i.e. a black hole moving with velocity $u = -dt$. The solution for a black hole moving at an arbitrary constant velocity u may be obtained by boosting (404) and is given by

$$
\begin{aligned}
& g_{MN} = \eta_{MN} + \left((1+Q^2 c_D) \frac{1}{\rho^{D-3}} - c_D Q^2 \frac{1}{\rho^{2(D-3)}} \right) O_M O_N, \\
& A_M = \frac{\sqrt{2} Q O_M}{\rho^{D-3}}, \\
& O = n - u, \quad u = \text{const}, \quad u \cdot u = -1, \quad \rho = \frac{r}{r_0}\,, \\
& r^2 = P_{MN} x^M x^N, \quad P_{MN} = \eta_{MN} + u_M u_N, \quad n = r_0 d\rho, \quad \text{note } \; u \cdot n = 0\,.
\end{aligned}
\tag{405}
$$

Note that the function ρ in (405) obeys the identity

$$\rho \nabla^2 \rho = (D-2) d\rho \cdot d\rho\,. \tag{406}$$

Here and through most of this chapter we view ρ as a function that lives in flat D dimensional space. In particular ∇ in (406) is the covariant derivative in flat space rather than in the metric (405).

Through this chapter we will use the term membrane to refer to the surface $\rho = 1$ viewed as a submanifold of flat Minkowski space. Note also that u^μ may be thought of a vector field that lives on the membrane. It is obvious that

$$\nabla \cdot u = 0, \tag{407}$$

where ∇ is the covariant derivative on the membrane.

[70] See Appendix 3.7.1 for a lightning introduction to this coordinate system and its advantages.

3.2.2 Collective coordinate spacetimes from boosted black holes

Consider the spacetime given by

$$
\begin{aligned}
g_{MN} &= \eta_{MN} + \left[\left(1+Q^2\right) \frac{1}{\rho^{D-3}} - \frac{Q^2}{\rho^{2(D-3)}} \right] O_M O_N, \\
A_M &= \frac{\sqrt{2} Q O_M}{\rho^{D-3}}, \\
O &= n - u, \quad u \cdot u = -1, \quad n = \frac{d\rho}{\sqrt{d\rho . d\rho}}, \quad u \cdot n = 0,
\end{aligned}
\tag{408}
$$

where ρ, Q and u are *arbitrary* smooth functions and vector fields in flat D dimensional Minkowski spacetime subject only to the requirement that the function ρ obeys (406) on the membrane surface and that the velocity field restricted to the membrane obeys (407).

The codimension one membrane worldvolume will play a special role in this chapter. We assume that the function ρ is chosen to ensure that the membrane surface is a smooth timelike submanifold of flat Minkowski space. [71] The membrane separates regions of spacetime where with $\rho < 1$ (inside the membrane) from regions with $\rho > 1$ (outside the membrane). The function ρ is chosen to ensure that the outside region is a connected spacetime and that includes all of spacelike infinity as well as $\mathcal{I}^+$ and $\mathcal{I}^-$. The membrane worldvolume itself is not necessarily connected.

The spacetimes (408) have the following properties.

- 1. Upto corrections of order $1/D$, the static black holes (405) are special cases of (408) with the ρ, Q and u functions given as in (405). In these special cases $\rho = 1$ is the black hole event horizon.

- 2. It is easily verified the membrane surface $\rho = 1$ is a null submanifold of the metric (408) for a general spacetime of this form. At least when (408) settles down to a stationary black hole at late times (as we will assume throughout this chapter) this submanifold may be identified with the spacetime event horizon. [72]

- 3. Consider a point x_0^μ on the membrane ($\rho = 1$) of the spacetime (408). Let u_0^μ, Q_0 and $\mathcal{K}_0$ denote the velocity, charge density field and trace of membrane extrinsic curvature at that point. Comparing with (405), we will see in subsection 3.3.6 below that a patch of size of order $\frac{1}{D}$ centered about x_0^μ is identical, at leading order in D, to the metric and gauge

[71] We will see below that the same surface - $\rho = 1$ - is a null when viewed as a submanifold of the metric (408).

[72] The dissipative nature of the membrane equations of motion we derive below suggests that all solutions reduce to stationary solutions at late times.

field of a patch centered about the membrane of a Reissner-Nordstrom black hole of radius $(D-2)/\mathcal{K}$, Q parameter Q_0 and boost velocity u_0^μ.

- 4. It seems plausible from point (3) above that every patch centered about the membrane of the configuration (408) obeys the Einstein-Maxwell equations at leading order in $1/D$. In subsection 3.3.6 below we demonstrate that this is the case provided the spacetime (408) enjoys an SO$(D-p-2)$ isometry for any p that is held fixed as D is taken to infinity.

- 5. The gauge field in (408) and the deviation of the metric from ds^2_{flat} scales like $e^{-D(\rho-1)}$. It follows (408) approaches flat space exponentially rapidly for $\rho - 1 \gg 1/D$.

- 6. Combining (4) and (5) above it follows that (408) also obeys the Einstein-Maxwell equations at leading order in $1/D$ (or better) everywhere outside its event horizon.

- 7. The equations of motion are not well solved when $1-\rho \gg 1$. However points that lie inside the event horizon of (408) are causally disconnected from dynamics on and outside the membrane and will be ignored in the rest of this chapter.

In summary, the metric (408) is built by stitching together bits of the event horizon of Reissner-Nordstrom black holes of varying radii, charge densities and boost velocities. The spacetime (408) obeys the Einstein-Maxwell equations at leading order in large D everywhere outside its horizon at least provided it preserves an SO$(D-p-2)$ isometry. It follows that metrics of the form (408) are useful starting points for a perturbative construction of the solutions of the Einstein-Maxwell equation in an expansion in $\frac{1}{D}$.

3.2.3 Subsidiary constraints on ρ, u and Q

The spacetimes (408) are parameterized by the functions ρ and Q and u^μ. These functions are defined on all of D dimensional Minkowski space. However we have already noted that (408) rapidly tends to flat space when $\rho - 1 \gg \frac{1}{D}$. Consequently two spacetimes whose ρ and Q and u^μ functions agree on the surface $\rho = 1$ but deviate at larger values of ρ actually describe spacetimes that agree at leading order in $1/D$ on and outside their event horizons. [73]

In this chapter we use spacetimes of the form (408) as the starting point for a perturbative expansion of true solutions of the Einstein-Maxwell system in a power series in $1/D$. Any two

[73] In and around subsection 3.3.6 we show that for this statement to be true it is also necessary the gradients $\nabla\rho$ of the two ρ functions coincide on the membrane $\rho = 1$ at leading order in the large D limit. However this is automatic, given the conditions we have imposed on our construction. Upto a position dependent normalization, $\nabla\rho$ is proportional to the normal vector of the surface $\rho = 1$. It follows that the two $\nabla\rho$ functions agree with each other upto normalization at $\rho = 1$. The condition that both ρ functions obey (406) at $\rho = 1$ guarantees that the normalizations also agree at leading order in the large D limit (see (430)).

configurations of the form (408) that differ from each other only at subleading orders in $1/D$ constitute equivalent starting points for perturbation theory. In order to restrict attention only to inequivalent configurations it is convenient to invent a set of rules that determine the functions ρ, u and Q everywhere in spacetime, in terms of the shape of the membrane and the values of the velocity and charge density fields *on the membrane*. We refer to these arbitrary rules as subsidiary constraints on the functions ρ, Q and u.

There is a great deal of freedom in the choice of subsidiary constraints. Two different choices of these conditions lead to the same solution at any given order in perturbation theory. The differences between the starting points in perturbation theory are compensated for by the differences in the results of the perturbative expansion.

While all choices of subsidiary constraints are on equal footing in principle, in practice some choices (those that most accurately approximate the true eventual solutions) lead to simpler results in perturbation theory than others. After experimenting with a few options we have chosen, in this chapter, to impose the following subsidiary constraints on ρ, u and Q:

$$\begin{aligned}
&\rho\nabla^2\rho = (D-2)d\rho\cdot d\rho,\\
&u\cdot u = -1, \quad n\cdot u = 0, \quad \mathcal{P}^{MN}\left[(n\cdot\nabla)\,u_M + (u\cdot\nabla)\,n_M\right] = 0,\\
&n\cdot\nabla Q = 0,\\
\text{where}\ \ &n = \frac{d\rho}{\sqrt{d\rho\cdot d\rho}}, \quad \mathcal{P}^{MN} = \eta^{MN} - n^M n^N + u^M u^N.\\
&\text{and}\ \nabla = \text{the covariant derivative in the embedding flat space.}
\end{aligned} \tag{409}$$

Let us pause to comment on our choice of subsidiary constraints. Recall that it is an important element of our construction that (409) is obeyed *on the surface* $\rho = 1$ (see (406)). This is a physical requirement, independent of arbitrary choices of subsidiary conditions. Our first subsidiary condition (409) simply asserts that (406) continues hold everywhere; even away from the membrane. This condition is sufficient to determine the function everywhere in terms of the shape of the membrane (i.e. solutions to the equation $\rho - 1 = 0$).

The third condition in (409) asserts that Q is defined off the membrane surface by parallel transporting it along integral curves of the normal vector $n \propto d\rho$. The second condition (409) determines u in terms of its value on the membrane by specifying its evolution under parallel transport under the same integral curves. [74]

[74]The subsidiary constraints adopted in this chapter are chosen to permit simple comparison with exact uncharged rotating black hole solutions, see [54] for details. These conditions imposed in this differ from the rather elegant geometrical subsidiary constraints imposed in [1].

3.2.4 Fixing coordinate and gauge invariance

In the next section we will describe the perturbative procedure we will employ to correct the spacetime (408) in order to obtain a spacetime that solves the Einstein-Maxwell equations upto first subleading order in $1/D$. In order to find an unambiguous solution to this problem we need to fix coordinate redefinition and Maxwell gauge ambiguities. In this subsection we describe our choice of coordinates and gauge.

Let the spacetime metric in the solutions described by this chapter take the form

$$g_{MN} = \eta_{MN} + h_{MN}, \tag{410}$$

where h_{MN} is given, at leading order, by (408). We fix coordinate redefinition ambiguity by imposing the condition

$$O^M h_{MN} = 0, \tag{411}$$

where

$$O = n - u, \tag{412}$$

and all indices in (411) are raised and lowered using the flat metric η_{MN}. Using the fact that $O \cdot O = 0$, it is easily verified that the leading order metric (408) does indeed obey (411).

In a similar manner we fix the Maxwell gauge ambiguity by imposing the condition

$$O^M A_M = 0. \tag{413}$$

Note that (413) is obeyed at leading order (see (408)).

Note that our choice of gauge depends on O, and so on n and u, which, in turn, depend on the membrane shape and velocity field in the particular solution under study. Our choice of gauge is somewhat analogous to a background field gauge in the study of gauge theories, or, more closely, to the gauges adopted in the study of the fluid gravity correspondence (see e.g. [55–60]).

Note also that the coordinate choice adopted in this chapter differs in detail from that of [1]. As is clear from the discussion of this section, the gauge adopted here is completely geometrical. This is not true of the gauge adopted in [1], which singles out the isometry direction as special.

3.2.5 Perturbation theory

In the next section we will implement a perturbative procedure that can be used to correct (408) at first subleading order in $1/D$. Roughly speaking we search for a metric and gauge field of the form

$$
\begin{aligned}
g_{MN} &= \eta_{MN} + h_{MN}, \\
h_{MN} &= \sum_{n=0}^{\infty} \frac{h^{(n)}_{MN}}{D^n}, \\
A_M &= \sum_{n=0}^{\infty} \frac{A^{(n)}_{M}}{D^n}, \\
h^{(0)}_{MN} &= O_M O_N \left[(1+Q^2)\rho^{D-3} - Q^2 \rho^{-2(D-3)} \right], \\
A^{(0)}_{M} &= \frac{\sqrt{2}Q}{\rho^{D-3}},
\end{aligned}
\tag{414}
$$

and attempt to find the correction fields $h^{(1)}_{MN}$ and $A^{(1)}_{M}$ that ensure that the Einstein-Maxwell equations are satisfied not just at leading order but also at first subleading order in $1/D$. In order to technically implement this idea, it turns out to be very helpful to assume our solutions preserve a large isometry group, as we describe in detail in the next section

3.3 Perturbation theory assuming SO$(D-p-2)$ invariance

3.3.1 Careful definition of the large D limit

In the computational part of this chapter we follow [1] to take the limit $D \to \infty$ while preserving an SO$(D-p-2)$ symmetry with p held fixed. We take the large D limit while maintaining a large isometry subgroup so that we can reliably estimate the scaling with D of all terms in the equations we encounter.

The requirement that our solutions preserve an isometry group is less restrictive than it first appears for two reasons. First, several spacetimes of physical interest (e.g. those that describe classes of black hole collisions) indeed preserve large isometry groups. Secondly, although the derivation of the membrane equations that we present below assumes an SO$(D-p-2)$ isometry, we will see that all our final equations are entirely geometrical on the membrane world volume; the isometry directions are not special in any way. In particular our final equations are independent of p.

While none of our final results will depend on p, all intermediate computations are performed within a framework that explicitly preserves SO$(D-p-2)$ invariance. In order to perform computations we assume that our metric and gauge field take the form

$$\begin{aligned} ds^2 &= g_{\mu\nu}(x^\mu)dx^\mu dx^\nu + e^{\phi(x^\mu)}d\Omega_d^2 \ , \\ A &= A_\mu(x^\mu)dx^\mu, \\ d &= D-p-3, \quad \mu = 1\ldots p+3, \end{aligned} \tag{415}$$

where $g_{\mu\nu}$, ϕ and A_μ are all arbitrary functions of the coordinates x^μ but are independent of the angular coordinates on the S^d in (415). [75] Under this assumption the D dimensional Einstein-Maxwell equations effectively reduce to a $p+3$ dimensional Einstein-Maxwell system coupled to the effective scalar field ϕ.

3.3.2 The Einstein-Maxwell equations in the SO$(D-p-2)$ invariant sector

In this chapter we study solutions of the Einstein-Maxwell equations governed by the Lagrangian

$$\mathcal{S} = \frac{1}{16\pi G_D}\int \sqrt{-\tilde{g}}\ d^Dx\ \left(\tilde{R} - \frac{F_{MN}F^{MN}}{4}\right), \tag{417}$$

where

$$\begin{aligned} F_{MN} &= \partial_M A_N - \partial_N A_M, \\ \tilde{R} &= \text{Ricci scalar in full D dimensional spacetime}, \\ \tilde{g} &= \text{Determinant of the metric in full D dimensional spacetime}. \end{aligned} \tag{418}$$

[76] We wish to focus attention on metrics and gauge fields of the form (415). In this section we will work out the effective dynamical equations for such configurations.

Substituting (415) into (417) we find the effective Lagrangian [77]

$$\begin{aligned} \mathcal{S} &= \frac{\Omega_d}{16\pi G_D}\int \sqrt{-g}\ d^{p+3}x\ e^{\frac{d\phi}{2}}\left(R + d(d-1)e^{-\phi} + \frac{d(d-1)}{4}(\partial\phi)^2 - \frac{F_{\mu\nu}F^{\mu\nu}}{4}\right), \\ (\partial\phi)^2 &= g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi). \end{aligned} \tag{419}$$

[75] In the special case of flat space

$$ds^2 = \eta_{\alpha\beta}dw^\alpha dw^\beta + dS^2 + S^2 d\Omega_d^2 = \eta_{\alpha\beta}dw^\alpha dw^\beta + dz_M dz^M, \tag{416}$$

where z^M are the $d+1$ Euclidean coordinates built out of angular coordinates on S^d and the radial coordinate S.

[76] In (418) the gauge field A_μ and the metric $g_{\mu\nu}$ are both taken to be dimensionless while Newton's constant G_D has length dimension $D-2$.

[77] Due to the presence of SO$(D-p-2)$ symmetry all the quantities depend only on w^α, S coordinates, while all the vectors (in particular, A) have components only in dw^α, dS directions. Hence when we go to the $p+3$ dimensional space, the M, N indices are replaced by μ, ν.

Varying this Lagrangian we obtain the equations of motion

$$
\begin{aligned}
&(d-1)e^{-\phi} - \frac{d}{4}(\partial\phi)^2 - \frac{1}{2}\nabla^2\phi + \frac{1}{4(d+p+1)}F_{\mu\nu}F^{\mu\nu} = 0, \\
&R_{\mu\nu} - \frac{d}{4}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{d}{2}\nabla_\mu\nabla_\nu\phi - \frac{1}{2}F_{\mu\rho}F_\nu{}^\rho + \frac{1}{4(d+p+1)}F_{\rho\sigma}F^{\rho\sigma}g_{\mu\nu} = 0, \\
&\nabla_\mu F^{\mu\nu} + \frac{d}{2}(\partial_\mu\phi)F^{\mu\nu} = 0, \\
&\text{where } \ d = D-p-3, \\
&\text{and } \nabla = \text{covariant derivative taken w.r.t. the metric } g_{\mu\nu}.
\end{aligned}
\tag{420}
$$

3.3.3 Setting up the perturbative computation

Convenient coordinates for flat space The metric (408) is completely determined once we specify the two scalar fields ρ and Q and the vector field u^μ. These fields live in *flat space* and are constrained to obey the equation (409).

The following coordinates for flat space

$$
ds^2_{flat} = \eta_{\alpha\beta}dw^\alpha dw^\beta + dS^2 + S^2 d\Omega_d^2 \ , \qquad i = \{0, 1, \cdots, p+1\}, \quad d = D-p-3. \tag{421}
$$

are particularly useful for studying SO($D-p-2$) invariant configurations. In these coordinates the requirement of SO($D-p-2$) isometry implies that ρ, Q and u are functions of ($\{w^\alpha, S\} \equiv \{x^\mu\}$) only. Moreover $u_{\theta_i} = 0$ in every angular direction θ_i on the S^d.

The perturbative expansion of SO($D-p-2$) invariant solutions Metrics and gauge fields that preserve an SO($D-p-2$) isometry can be parameterized in the form

$$
\begin{aligned}
ds^2 &= g_{\mu\nu}(S, w^\alpha)dx^\mu dx^\nu + S^2 e^{\delta\phi(S,w^\alpha)}d\Omega_d^2 \ , \\
A_M dX^M &= A_\mu(S, w^\alpha)dx^\mu.
\end{aligned}
\tag{422}
$$

Note that

$$
\phi = \phi^0 + \delta\phi, \quad \phi^0 = 2\ln(S).
$$

(ϕ^0 is simply value of ϕ in flat space).

As explained around (410), in this chapter we will expand the metric and gauge field in a power series expansion in $1/D$. [78] The schematic expansion (410) takes the precise form

$$
g_{\mu\nu} = \sum_{k=0}^{\infty}\left(\frac{1}{D}\right)^k g_{\mu\nu}^{(k)}, \quad A_\mu = \sum_{k=0}^{\infty}\left(\frac{1}{D}\right)^k A_\mu^{(k)}, \quad \delta\phi = \sum_{k=1}^{\infty}\left(\frac{1}{D}\right)^k \delta\phi^{(k)}. \tag{423}
$$

[78] The central advantage of the assumption of SO($D-p-2$) isometry is that the variables of the perturbation expansion are independent of D.

From (408) we read off the leading values of $g_{\mu\nu}$ and A_μ

$$\begin{aligned} g^{(0)}_{\mu\nu} dx^\mu dx^\nu &= \eta_{\alpha\beta} dw^\alpha dw^\beta + dS^2 + \left[(1+Q^2)\rho^{-(D-3} - Q^2 \rho^{-2(D-3)}\right] (O_\mu dx^\mu)^2, \\ A^{(0)}_\mu &= \sqrt{2} Q \rho^{-(D-3)} O_\mu . \end{aligned} \tag{424}$$

More detailed parameterization of the first order corrections to the metric and gauge field After imposing the gauge conditions (411) and (413), the metric correction $g^{(1)}_{\mu\nu}$ and gauge field correction $A^{(1)}_\mu$ can can be parameterized in terms of 6 unknown scalar, three unknown vector and one unknown tensor functions [79]as

$$\begin{aligned} g^{(1)}_{\mu\nu} &= S_{(VV)} O_\mu O_\nu + 2 S_{(Vz)} O_{(\mu} Z_{\nu)} + S_{(zz)} Z_\mu Z_\nu + S_{(Tr)} P_{\mu\nu} \\ &+ 2 V^{(V)}{}_{(\mu} O_{\nu)} + 2 V^{(z)}{}_{(\mu} Z_{\nu)} + T_{\mu\nu}, \\ A^{(1)}_\mu &= S_{(AV)} O_\mu + S_{(Az)} Z_\mu + V^{(A)}_\mu, \end{aligned} \tag{425}$$

where

$$\begin{aligned} &O = n - u, \quad Z = \frac{dS}{S} - \left(\frac{n \cdot dS}{S}\right) n, \\ &P_{\mu\nu} = \text{projector perpendicular to } u, n \text{ and } Z, \quad P^{\mu\nu} T_{\mu\nu} = 0. \end{aligned}$$

The vectors ($V^{(V)}_\mu$, $V^{(Z)}_\mu$, $V^{(A)}_\mu$) and tensor ($T_{\mu\nu}$) above are all projected orthogonal to O, n and Z (the tensor $T_{\mu\nu}$ is also assumed to be traceless).

Let us now consider the corrections of the 'dilaton' function $\delta\phi$. We see from (433) and (434) that $\chi = D(d\phi)$ appears in the equations of motion. Were ϕ to have an $\mathcal{O}\left(\frac{1}{D}\right)$ fluctuation $\delta\phi^{(1)}$, this term would contribute to the equations of motion at leading order, invalidating the fact that the starting metric (408) solves the Einstein-Maxwell equations at leading order. For the same reason ϕ at $\mathcal{O}\left(\frac{1}{D}\right)^2$, contributes to the Einstein-Maxwell equations at $\mathcal{O}\left(\frac{1}{D}\right)$. It follows that $\delta\phi^{(2)}$ is an unknown function that contributes to the first order perturbative equations at the same order as the 6 scalars that appear in (425), and so will have to be determined together with these six functions in the computation of the first corrections to (408).

Auxiliary embedding space The coordinate system (421) describes flat R^D as the 'fibration' of an S^d over a $p+3$ dimensional base space with metric

$$ds^2_{flat} = \eta_{\alpha\beta} dw^\alpha dw^\beta + dS^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \qquad x^\mu = \{w^\alpha, S\}. \tag{426}$$

The radius of the fibred S^d is given by the coordinate S.

[79]The terms scalar, vector and tensor refer to the transformation properties of the fields under those rotations in the tangent space that leave n, u and dS fixed. See below for more details.

Under the assumption of SO($D-p-2$) symmetry, the membrane world volume can be thought of as a codimension one ($p+2$ dimensional) surface in the base space together with the d dimensional spheres fibred over each of the base points on this surface. More generally all the ingredients - the functions ρ, u^μ and Q - that go into the construction of the seed metric (408) can all be regarded as functions and vector fields on the base space - which then determine SO($d+1$) invariant functions and vector fields on all of R^D in the obvious manner. This is the viewpoint we will adopt while doing the computations described in this section. This viewpoint is convenient because the auxiliary space (426) makes no reference to D. Once we formulate our perturbation theory in terms of fields propagating on the auxiliary space (426), all factors of D in the equations are completely manifest, allowing for a clean formulation of large D perturbation theory.

The end result of the first stage of our computation (e.g. the results presented in (3.3.13)) are all presented in terms of covariant derivatives of the field ϕ, u and Q viewed as scalar and one-form fields that live in the base or auxiliary space (426).

It is important to note general expressions built out of covariant derivatives of SO($D-p-2$) invariant fields in the auxiliary space (426) do *not* agree with the corresponding expressions built out of covariant derivatives of the same fields in the metric (421) of the embedding space [80]. In Appendix 3.7.4 we have explored the dictionary between covariant expressions in the full flat D dimensional space and the auxiliary space. Using these translation formulae, we are then able to rewrite our final results for the first order corrected metric and gauge fields in terms of full spacetime covariant derivatives of ρ, u and Q. Our final results, presented in the next section, are given in this language, and turn out to be geometrical, in a sense we describe in detail below.

Constraints and Subsidiary conditions recast in auxiliary space As we have explained in the previous section our construction (408) works provided the functions ρ and u obey the conditions (406) and (407). The ∇^2 in (406) is a Laplacian in the full flat space (421), while the ∇ operator in (407) is the covariant derivative on the membrane, viewed as a submanifold of the full flat space (421). In order to use these conditions in our computations below, we need to rewrite them in terms of covariant derivatives on (426) and on the membrane world volume viewed as a submanifold of (426). [81]

Depending on context, we will use the symbol $\tilde{\nabla}$ to denote the covariant derivative either in the base space (426) or on the membrane viewed as a submanifold of (426). As we have explained

[80] Roughly speaking the difference comes about in terms involving expressions like Γ^M_{SM} with M summed over. This expression receives contributions from M ranging over the angular directions of Ω_d in the case of (421) but not in the case of (426).

[81] All computations in the paper [1] were performed in the auxiliary space (426). The final results of [1] were presented in this auxiliary space, without being reconverted to the full space. Note also that in the auxiliary space, because of our choice of coordinates, all Christoffel symbols vanish and the covariant derivatives are same as partial derivatives.

in Appendix 3.7.4,

$$\nabla \cdot u = (D-p-2) Z \cdot u + \tilde{\nabla} \cdot u, \tag{427}$$

(in this equation ∇ is the covariant derivative of the membrane viewed as a submanifold of R^D while $\tilde{\nabla}$ is the covariant derivative on the membrane viewed as a submanifold of (426)). Here

$$Z = \frac{dS}{S} - \left(\frac{n \cdot dS}{S}\right) n. \tag{428}$$

Using the fact that (407) is assumed to hold for our ansatz metrics it follows from (428) that

$$Z \cdot u = -\frac{\tilde{\nabla} \cdot u}{D-p-2}. \tag{429}$$

In a similar manner the fact that (406) is assumed to hold on the membrane of (408) implies that

$$\begin{aligned} &(D-p-2)\frac{dS \cdot \tilde{\nabla}\rho}{S} + \tilde{\nabla}^2 \rho = (D-2) d\rho \cdot d\rho. \\ &\text{where } \tilde{\nabla} = \text{the covariant derivative on the space (489).} \end{aligned} \tag{430}$$

In an entirely analogous manner, the subsidiary condition (409) can be recast in terms of covariant derivatives in the auxiliary space (426).

$$\begin{aligned} &(D-p-2)\frac{\rho}{S} dS \cdot \tilde{\nabla}\rho + \rho \tilde{\nabla}^2 \rho = (D-2) d\rho \cdot d\rho, \\ &u_\mu u^\mu = -1, \quad n_\mu u^\mu = 0, \\ &(\eta^{\mu\nu} + u^\mu u^\nu - n^\mu n^\nu) \left[\left(n^\alpha \tilde{\nabla}_\alpha\right) u_\mu + \left(u^\alpha \tilde{\nabla}_\alpha\right) n_\mu\right] = 0, \\ &n^\mu \tilde{\nabla}_\mu Q = 0, \\ \text{where } \ &n_\mu = \frac{\tilde{\nabla}_\mu \rho}{\sqrt{(\tilde{\nabla}_\nu \rho)(\tilde{\nabla}^\nu \rho)}}, \\ &\text{and } \tilde{\nabla} = \text{the covariant derivative on (426).} \end{aligned} \tag{431}$$

3.3.4 Zooming in on patches

In this subsection we will identify a scaling limit of distance scales that admits an interesting large D limit. For this purpose we turn back to the Einstein-Maxwell equations specialized to the case of SO$(D-p-2)$ invariant configurations and note that derivatives of the scalar field ϕ appear in (420) with additional factors of D as compared to terms with an equal number of derivatives of $g_{\mu\nu}$ or A_μ. This observation (see [1]) suggests that we will obtain one class of nontrivial solutions to these equations if we assume that $g_{\mu\nu}$ and A_μ vary on length scale $1/D$, i.e. the length scale of

δr (see the introduction) while ϕ varies at the length scale unity (at least upto corrections that are subleading in $1/D$). Under this assumption the solutions we study are characterized by two widely separated length scales, exactly like the black holes described in the introduction. [82]

In order to describe the large D limit of solutions characterized by two different length scales ($1/D$ and unity) we adopt the following procedure. We view our manifold as a union of patches, each of size $1/D$. Each patch is centered around a particular coordinate x_0^μ. In each such patch we work with the scaled coordinates, metric, connections and gauge fields

$$\begin{aligned} x^\mu &= x_0^\mu + \frac{\alpha^\mu(q^a)}{D}, \\ G_{ab} &= D^2 \times (\partial_a \alpha^\mu)\ (\partial_b \alpha^\nu) g_{\mu\nu}, \\ \mathcal{A}_a &= D \times (\partial_a \alpha^\mu) A_\mu, \end{aligned} \tag{432}$$

where α^μ are any convenient (D independent) functions of the coordinates q^a. Note that G_{ab} differs from $g_{\mu\nu}$ transformed to q^a coordinates by the scale factor D^2. In the same way the gauge field $\mathcal{A}_a$ differs from A_μ transformed to the coordinates q^a by a scale factor D. The scale factors are chosen to scale up distances and holonomies on the patch to order unity. We also find it convenient to define the one-form field

$$\chi_a \equiv D\ \partial_a \phi = \alpha_a^\mu \partial_\mu \phi. \tag{433}$$

Note that χ_μ is of order unity and constant (to leading order in $1/D$) in scaled patch coordinates (see [1] for more discussion). The equations of motion may be rewritten in terms of scaled quantities as

$$\begin{aligned} \mathcal{E}_\phi \equiv &\ \left(\frac{d}{D}\right) \nabla_a \chi^a + \frac{\chi^2}{2} - \frac{2(d-1)}{d} e^{-\phi} - \left[\frac{D^2}{2d(D-2)}\right] F_{cd} F^{cd} = 0, \\ \mathcal{E}_{ab} \equiv &\ R_{ab} - \left(\frac{d}{D}\right) \left(\frac{\nabla_a \chi_b + \nabla_b \chi_a}{2}\right) - \left(\frac{d}{4D^2}\right) \chi_a \chi_b - \frac{1}{2} F_{ac} F_b{}^c + g_{ab} \left[\frac{F_{cd}F^{cd}}{4(D-2)}\right] = 0, \\ \mathcal{E}_a \equiv &\ \nabla_a F^{ab} + \frac{d}{2D} \chi_a F^{ab} = 0, \\ &\ \text{where } \nabla = \text{the covariant derivative w.r.t. metric } g_{\mu\nu}. \end{aligned} \tag{434}$$

All quantities (curvatures, Christoffel symbols, field strengths) in (434) are constructed out of the scaled metric G_{ab} and scaled gauge field $\mathcal{A}_a$.

The variables in these equations are all assumed to be of order unity. All factors of D in these equations are explicit, and so the equations (434) are easily expanded in a power series in $1/D$.

[82] See [1] for a more detailed discussion of the rational behind choosing this scaling limit.

At leading order, in particular, the equations reduce to

$$\begin{aligned}
\mathcal{E}_\phi|_{\text{leading}} \equiv&\ \nabla_a\chi^a + \frac{\chi^2}{2} - 2e^{-\phi} - \left[\frac{F_{cd}F^{cd}}{2}\right] = 0,\\
\mathcal{E}_{ab}|_{\text{leading}} \equiv&\ R_{ab} - \frac{\nabla_a\chi_b + \nabla_b\chi_a}{2} - \frac{1}{2}F_{ac}F_b{}^c = 0,\\
\mathcal{E}_a|_{\text{leading}} \equiv&\ \nabla_a F^{ab} + \frac{1}{2}\chi_a F^{ab} = 0.
\end{aligned} \tag{435}$$

In this chapter we search for solutions of these equations in each patch of the manifold. We require that solutions in neighbouring patches agree with each other where they overlap. We will find solutions of our equations order by order in an expansion in $\frac{1}{D}$.

3.3.5 Choice of 'patch coordinates'

In this chapter we will follow [1] to implement perturbation in $1/D$ in a patch of size $\sim \mathcal{O}\left(\frac{1}{D}\right)$ centered around an arbitrary point x_0^μ on the membrane ($\rho = 1$ surface). We will then sew together the results from each patch to obtain a global correction to the metric and gauge field in (408).

In order to set up the computation in any given patch, we need an explicit choice of local coordinates in each patch, i.e. an explicit choice of the coordinates $\sim \{y^a\}$ as defined in equation (432).

Having imposed $\mathrm{SO}(D-p-2)$ invariance we have three distinguished one-form fields in each patch. These one-forms are $n(x_0^\mu)$, $u(x_0^\mu)$ and $Z(x_0^\mu)$. Note that from (429) it follows that

$$Z\cdot n = 0, \quad Z\cdot O = -Z\cdot u = \mathcal{O}\left(\frac{1}{D}\right),$$

where '$\cdot$' denotes contraction with respect to flat metric.
Let Y^i denote a set of p one-form fields chosen so that

$$Y^i\cdot Z = Y^i\cdot n = Y^i\cdot O = 0, \quad Y^i\cdot Y^j = \delta^{ij}.$$

There is, of course, a great deal of ambiguity in the precise details of the Y^i fields that will play no role in what follows.

Let $\{x_0^\mu\} = \{w_0^\alpha, S_0\}$ represent a point on the membrane in the metric (408). We wish to focus on the patch of size of order $\frac{1}{D}$ around x_0^μ. We set up a local coordinate system for this patch as follows.

$$
\begin{aligned}
R &= D(\rho - 1), \\
V &= D(x^\mu - x_0^\mu) O_\mu(x_0), \\
\frac{z}{S_0} &= D(x^\mu - x_0^\mu) Z_\mu(x_0), \\
y^i &= D(x^\mu - x_0^\mu) Y_\mu^i(x_0).
\end{aligned}
\tag{436}
$$

3.3.6 The perturbative metric in a patch

In these coordinates and at leading order in the $\frac{1}{D}$ expansion, the rescaled metric and gauge field (432) take the form

$$
\begin{aligned}
ds^2 &= 2\left(\frac{S_0}{n_S^0}\right) dR\, dV - \left[1 - (1 + Q_0^2) e^{-R} + Q_0^2 e^{-2R}\right] dV^2 \\
&\quad + \left[\frac{1}{1 - (n_S^0)^2}\right] dz^2 + \sum_{i=1}^{p} dy^i dy^i + \mathcal{O}\left(\frac{1}{d}\right), \\
e^\phi &= S_0^2, \\
A &= \sqrt{2}\, Q_0 e^{-R} dV + \mathcal{O}\left(\frac{1}{d}\right),
\end{aligned}
\tag{437}
$$

where $Q_0 = Q(x_0^\mu), \quad n_S^0 = (n \cdot dS)|_{x^\mu = x_0^\mu}$.
(437) describes a configuration that is translationally invariant in the coordinates V z and y^i (but not in R). We refer to (437) as the black brane metric. Notice that black brane metrics are parameterized by S_0, n_S^0 and the charge $Q = Q_0$. Recall $r_0 = S_0/n_S^0$ is the radius of the static black hole whose patch, when blown up about a membrane point with $S = S_0$, yields the black brane metric (437).

It is easily directly verified that the black brane configuration (437) solves the leading large D equations of motion (435).

After appropriate scaling the metric and gauge field fluctuation at first order in $\left(\frac{1}{D}\right)$ (see(425)) takes the following form in the 'patch coordinates'

$$
\begin{aligned}
G^{(1)}_{ab} dq^a dq^b = & S_{(VV)} dV^2 + 2 \left[\frac{S_{(Vz)}}{S_0} \right] dV dz + \left[\frac{S_{(zz)}}{S_0^2} \right] dz^2 + S_{(Tr)} dy^i dy^i \\
& + 2V_i^{(V)} dy^i dV + 2 \left[\frac{V_i^{(z)}}{S_0} \right] dy^i dz + T_{ij} dy^i dy^j ,
\end{aligned}
\tag{438}
$$

$$
\mathcal{A}^{(1)}_a dq^a = S_{(AV)} dV + \left[\frac{S_{(Az)}}{S_0} \right] dz + V_i^{(A)} dy^i .
$$

3.3.7 The structure of perturbative equations at first order

Let us begin the process of determining the correction to our metric and gauge field in a patch (centered about an arbitrary point on the membrane). Upon plugging first order corrected metric and gauge field into the Einstein-Maxwell equations, we find that each of these equations takes the schematic form

$$
H v^{(1)} = s^{(1)} . \tag{439}
$$

The term $v^{(1)}$ in (439) is a schematic for the collection of unknown functions in (425). The 'source' terms $s^{(1)}$ have their origin in the fact that a blown up patch of (408) fails to solve the Einstein-Maxwell equations at first subleading order in $1/D$. This failure has its roots in the following facts:

- 1 A patch of (408) differs from the black brane metric at first subleading order in $1/D$. This difference is visible upon Taylor expanding the fields n, u and Q to first order about the special point x_0^μ and results in source terms proportional to the first derivative of n u and Q.
- 2. The black brane itself fails to solve the Einstein- Maxwell equations at first subleading order in $1/D$. This shows up in the fact that the equations (434) themselves have corrections in the $1/D$ expansion. This gives rise to derivative free source terms.

Note that all source terms are entirely determined by the data (membrane shape, velocity field, charge field) that go into defining the ansatz metric and gauge field. (408)

All source terms are fast varying functions of the coordinate R but slow varying functions of all other coordinates. This implies that

$$
v^{(1)} = v^{(1)}(R, \frac{V}{D}, \frac{z}{D}, \frac{y^i}{D}),
$$

where R and the other scaled coordinates are are defined in (436). As $v^{(1)}$ is already a fluctuation variable at order $1/D$, derivatives of $v^{(1)}$ in all directions other than R contribute to the Einstein-Maxwell equations only at order $1/D^2$. It follows that the homogeneous operator H is a differential operator only in the variable R. In other words the equations (439) are linear ordinary differential equations.

Even though the RHS of (439) has its origin partly in the Taylor expansion of (408) about the special point x_o^μ, the source functions $s^{(1)}$ in the patch about x_0^μ do not explicitly depend on the expansion coordinates V, z, y^i. The reason for this is simple. The locality of the Einstein-Maxwell equations ensures that $s^{(1)}$ is a $\frac{1}{D}$ times a local functions of the fields ρ^{p+d}, n^μ, u^μ, Q and their derivatives. Dependence on the coordinates V, z and y^i dependence could only arise from Taylor expanding the fields n^μ, u^μ and Q about the point x_0^μ. The terms proportional to V, z, y^i in this Taylor expansion are all manifestly of order $1/D^2$ or smaller. [83]

Let us also reiterate that source $s^{(1)}$ contains at most one derivative of n^μ, u^μ and Q. This follows immediately from the observation that ρ, u and Q are functions of $\frac{V}{D}$, $\frac{z}{D}$ and $\frac{y^i}{D}$ in the patch, and every derivative of these functions is weighted by a factor of $\frac{1}{D}$.

Let us summarize. (439) is a collection of an infinite number of *linear ordinary differential equations* in the variable R; one such equation at each point on the membrane world volume. At each membrane point the source functions are explicit function of R, with coefficients that depend on the values and (at most) one derivatives of the ρ, u and Q fields at that point. In to find $G^{(1)}_{ab}$, $\mathcal{A}^{(1)}_a$ and $\delta\phi^{(2)}$we need to solve these linear differential equations at each membrane point and then sew these solutions together into a global correction to (408). At the technical level, the procedure for perturbation theory is strongly reminiscent of the procedure adopted in studies of the fluid gravity correspondence, see e.g. [55, 58–60]

3.3.8 Equations in the three symmetry channels

As we have explained above, the variables in $G^{(1)}_{ab}$, $\mathcal{A}^{(1)}_a$ and $\delta\phi(1)$ consist of 7 scalar functions, 3 vector functions and one tensor function (where 'scalar', 'vector' and 'tensor' refer to the transformation property of the modes under SO(p) rotations in part of x^μ tangent space that is orthogonal to X, n and u). The black brane background (437)), and so the operator H, preserves SO(p) symmetry. It follows that the equations (439) do not mix the scalar vector and tensor modes; the equations in these three sectors decouple from each other.

Tensor Sector:

[83]On the other hand source functions have nontrivial dependence on R at leading order in $1/D$; this is a consequence of the fact that ρ^{p+d} evaluates to e^R at leading order in the large d expansion, and so powers and derivatives of this function naturally appear in sources.

In the tensor sector the differential equations (439) reduce to a single ordinary second order differential equation for a single unknown, $T_{ij}(R)$; this equation is easily solved for an arbitrary source function. We present our explicit results below.

Vector Sector:

In the vector sector we have four coupled equations for three unknown functions. The four equations in question are

$$\begin{aligned} &\mathcal{E}_{Ri} = 0, \quad \mathcal{E}_{Vi} = 0, \\ &\mathcal{E}_{zi} = 0, \quad \mathcal{E}_{i} = 0, \end{aligned} \tag{440}$$

(see (434) for definitions of the equations). The directions i are the Y^i directions. They are assumed to be orthogonal to O, u and dS.

At first order it turns out that the following linear combination of equations vanishes identically.

$$\begin{aligned} &\partial_R\left[\left(\frac{S_0}{n_S^0}\right)\mathcal{E}_{Vi} + f_0(R)\mathcal{E}_{Ri}\right] + \left[\left(\frac{S_0}{n_S^0}\right)\mathcal{E}_{Vi} + f_0(R)\mathcal{E}_{Ri}\right] + \left[\frac{1-(n_S^0)^2}{S_0}\right]\mathcal{E}_{zi} = 0, \\ &\text{where} \;\; f_0(R) = 1 - (1+Q_0^2)e^{-R} + Q_0^2 e^{-2R}. \end{aligned} \tag{441}$$

We thus have only three independent vector equations for our three vector unknowns. It turns out that the the remaining three equations are easily solved for arbitrary source terms that obey (441), and in particular for the source terms that actually appear in the first order computation (see below for more details).

Scalar Sector:

In the scalar sector we have 11 equations for 7 variables. The 11 equations are

$$\begin{aligned} &\mathcal{E}_{RR} = 0, \quad \mathcal{E}_{RV} = 0, \quad \mathcal{E}_{Rz} = 0, \\ &\mathcal{E}_{VV} = 0, \quad \mathcal{E}_{Vz} = 0, \quad \mathcal{E}_{zz} = 0, \\ &\mathcal{E}_{R} = 0, \quad \mathcal{E}_{V} = 0, \quad \mathcal{E}_{z} = 0, \\ &\sum_{i=1}^{p}\mathcal{E}_{ii} = 0, \quad \mathcal{E}_{\phi} = 0, \end{aligned} \tag{442}$$

(see (434) for the definition of these equations). At first order it turns out that the following four linear combination of equations automatically vanish.

$$
\begin{aligned}
\text{Combination-1: } & \partial_R \mathcal{E}_R + \mathcal{E}_R + \frac{\mathcal{E}_z}{S_0} = 0, \\
\text{Combination-2: } & \partial_R \left[\mathcal{E}_{VV} + \left(\frac{n_S^0}{S_0} \right) f_0(R) \mathcal{E}_{RV} \right] + \left[\mathcal{E}_{VV} + \left(\frac{n_S^0}{S_0} \right) f_0(R) \mathcal{E}_{RV} \right] \\
& + \left(n_S^0 - \frac{1}{n_S^0} \right) \left[\mathcal{E}_{Vz} + \frac{Q_0 S_0 e^{-R}}{\sqrt{2}\left[1-(n_S^0)^2\right]} \mathcal{E}_R \right] = 0, \\
\text{Combination-3: } & \partial_R \left[\left(\frac{n_S^0}{S_0} \right) f_0(R) \mathcal{E}_{Rz} + \mathcal{E}_{Vz} \right] + \left[\left(\frac{n_S^0}{S_0} \right) f_0(R) \mathcal{E}_{Rz} + \mathcal{E}_{Vz} \right] \\
& - \left(n_S^0 - \frac{1}{n_S^0} \right) \left(\frac{n_S^0}{S_0} \right) \mathcal{E}_{zz} = 0, \\
\text{Combination-4: } & \partial_R \left[\mathcal{E}_\phi + 2 \left(\frac{n_S^0}{S_0} \right)^2 f_0(R)\ \mathcal{E}_{RR} - 2[1-(n_S^0)^2]\ \mathcal{E}_{zz} - \mathcal{E}_{ii} \right] \\
& + 2 \left(\frac{n_S^0}{S_0} \right)^2 \left[\partial_R f_0(R) + 2 f_0(R) \right]\ \mathcal{E}_{RR} + 4 \left(\frac{n_S^0}{S_0} \right) \mathcal{E}_{RV} \\
& + 4 \left[\frac{1-(n_S^0)^2}{S_0} \right] \mathcal{E}_{Rz} - 2\sqrt{2} Q_0 e^{-R} \mathcal{E}_V = 0.
\end{aligned}
\tag{443}
$$

We thus have exactly seven independent equations to solve for the seven unknowns in the scalar sector. It turns out that the remaining seven equations are easily solved for arbitrary sources that obey (443), and in particular for the source terms that actually appear in the first order computation (see below for more details).

3.3.9 Basis for Source Functions

Let us now turn to a description of the sources that appear on the RHS of (439). In the scalar sector there are two kinds of sources. The first kind of source has its origin in the fact that the black brane metric (437) solves the Einstein-Maxwell equations only at large D and not at first subleading order in $\frac{1}{D}$. This fact gives rise to sources (RHS of (439)) that are simply functions of R. We also have sources from the first term in the Taylor expansion of the functions n_μ, u^μ and Q expanded about x_0^μ. Let $\mathfrak{s}^{(a)}$ $(a = 1 \dots N_S)$ denote the set of scalar first derivatives of the functions n, u and Q . Let $\mathfrak{s}_0 = 1$ (this allows us to deal with the first kind of source mentioned above). On general grounds, the source $\mathcal{S}_m$ terms in the mth scalar equations E_m^S takes the form

$$
\mathcal{S}_m = \sum_{a=0}^{N_S} \mathcal{S}_m^a(R)\ \mathfrak{s}^{(a)}. \tag{444}
$$

In a similar manner we let $\mathfrak{v}^{(a)}$ $(a = 1 \dots N_V)$ denote the set of scalar first derivatives of the functions n, u and Q. The source terms $\mathcal{V}_{mi}$ in the mth vector equation E^V_{mi} take the form

$$\mathcal{V}_{mi} = \sum_{a=1}^{N_V} \mathcal{V}^a_m(R)\mathfrak{v}_i^{(a)}. \tag{445}$$

Finally if $\mathfrak{t}_{ij}^{(a)}$ $(a = 1 \dots N_T)$ denote the set of tensor first derivatives of the functions n, u and Q, then the source terms $\mathcal{T}_{ij}$ in the unique tensor equation must take the form

$$\mathcal{T}_{ij} = \sum_{a=1}^{N_T} \mathcal{T}^a(R)\mathfrak{t}_{ij}^{(a)}. \tag{446}$$

It turns out at first order $(N_S = 6, \quad N_V = 5, \quad N_T = 2)$. In table (1) we have listed and explicit basis for independent scalar vector and tensor data at first order. Here $P_{\mu\nu}$ is the projector perpendicular to u_μ, n_μ and Z_μ.

Scalars (6)	Vectors (5)	Tensors (2)
$\mathfrak{s}^{(1)} = u^\mu u^\nu K_{\mu\nu}$	$\mathfrak{v}_\mu^{(1)} = u^\nu P^\alpha_\mu K_{\nu\alpha}$	$\mathfrak{t}_{\mu\nu}^{(1)} = P^\alpha_\mu P^\beta_\nu \left[\frac{K_{\alpha\beta}}{2} - \left(\frac{\mathfrak{s}_3}{p}\right)\eta_{\alpha\beta}\right]$
$\mathfrak{s}^{(2)} = u^\mu Z^\nu K_{\mu\nu}$	$\mathfrak{v}_\mu^{(2)} = u^\nu P^\alpha_\mu \partial_\nu u_\alpha$	$\mathfrak{t}_{\mu\nu}^{(2)} = P^\alpha_\mu P^\beta_\nu \left[\frac{\partial_\alpha u_\beta + \partial_\beta u_\alpha}{2} - \left(\frac{\mathfrak{s}_4}{p}\right)\eta_{\alpha\beta}\right]$
$\mathfrak{s}^{(3)} = P^{\mu\nu} K_{\mu\nu}$	$\mathfrak{v}_\mu^{(3)} = P^\alpha_\mu (Z \cdot \partial) u_\alpha$	
$\mathfrak{s}^{(4)} = P^{\mu\nu} \partial_\mu u_\nu$	$\mathfrak{v}_\mu^{(4)} = P^\alpha_\mu \partial_\alpha Q$	
$\mathfrak{s}^{(5)} = u^\mu \partial_\mu Q$	$\mathfrak{v}_\mu^{(5)} = Z^\nu P^\alpha_\mu K_{\nu\alpha}$	
$\mathfrak{s}^{(6)} = Z^\mu Z^\nu K_{\mu\nu}$		

Table 1: Data at 1st order in $\frac{1}{D}$ expansion

3.3.10 Equations of motion from regularity at the horizon

We are interested in solutions to the equations of perturbation theory that are everywhere regular (away from the black hole singularity that will turn out to be shielded by an event horizon). Even though all our source functions are regular, this condition is not automatic at $R = 0$ (i.e. $\rho = 1$). This perhaps surprising fact plays a key role in this chapter. This subsection is devoted to a more detailed exposition of this fact.

Let E^{MN} denote the Einstein equation obtained by varying the Einstein-Maxwell Lagrangian w.r.t g_{MN}, and let M_N denote the Maxwell equation obtained by varying the Einstein-Maxwell Lagrangian w.r.t A_M. As we have explained above, the perturbative procedure of this chapter is geared to determining the ρ dependence of unknown metric and gauge field components. For our purposes it is thus natural to view the ρ direction as a Euclidean 'time' direction in which we wish to understand 'dynamics'. From this point of view the equations

$$C^M_{Ein} = E^{MN}(d\rho)_M = E^{M\rho}, \quad C_{Max} = M^N(d\rho)_N = M^\rho, \tag{447}$$

are, respectively, the Einstein and Maxwell 'constraint' equations.

The dot product of the Einstein scalar equation C^M_{Ein} with n and u (or n and O) appears to play no role in the discussions of this subsection. For that reason in the rest of this section we will deal with C^M_{Einp}, the constraint Einstein equations that are projected orthogonal to n and O. From the 'geometrical' viewpoint (see below for much more discussion) C^M_{Einp} is a vector equation while C_{Max} is a scalar equation. However perturbative procedure described so far is not geometrical: it treats the isometry directions as special. From our current point of view C^M_{Einp} may be decomposed into a single SO(p) scalar $C_{Einp} \cdot Z$ and an SO(p) vector (C^M_{Einp} projected orthogonal to Z).

In the scalar sector it is easily verified that

$$\begin{split}(C_{Einp} \cdot Z) \propto & \left[\left(\frac{S_0}{n^0_S} \right) \mathcal{E}_{Vz} + f_0(R)\mathcal{E}_{Rz} \right], \\ & \propto f_0(R)^2 \frac{d}{dR} \left[\frac{S_{(Vz)}(R)}{f_0(R)} \right] + \Sigma_{(Vz)}(R) = 0,\end{split} \tag{448}$$

$$\begin{split}C_{Max} \propto & \ \mathcal{E}_R \\ \propto & \ f_0(R) \left(\frac{d}{dR} S_{(Az)}(R) \right) + \sqrt{2} Q_0 e^{-R} S_{(Vz)}(R) + \Sigma_{(Az)}(R) = 0.\end{split} \tag{449}$$

Here $\Sigma_{(Vz)}(R)$ the full source term for the combination of equations $\left[\left(\frac{S_0}{n^0_S} \right) \mathcal{E}_{Vz} + f_0(R)\mathcal{E}_{Rz} \right]$ while $\Sigma_{(Az)}(R)$ is the source term in $\mathcal{E}_R$. [84]

An inspection of (448) reveals that this equation admits nonsingular solutions at $R = 0$ if and only if the linear term, in the Taylor expansion of $\Sigma_{(Vz)}(R)$ about $R = 0$, vanishes. Provided this condition is met the solution to (448) is nonsingular. Once this condition is met it follows from (448) that

$$S_{(Vz)}(R=0) = \frac{\Sigma_{(Vz)}(R=0)}{f'(R=0)}. \tag{450}$$

Turning to the equation (449), it is easily seen that the solution to this equation is nonsingular if and only if $\left[\sqrt{2}Q_0 e^{-R} S_{(Vz)}(R) + \Sigma_{(Az)}(R)\right]$ vanishes at $R = 0$. Using (450), this condition is

[84] Clearly, each of $\Sigma_{(VZ)}(R)$ and $\Sigma_{(Az)}(R)$ are linear combinations of the previously defined quantities $\mathcal{S}^a_i(R)$.

equivalent to the requirement that $\left[\sqrt{2}Q_0\frac{\Sigma_{(Vz)}(R=0)}{f'(R=0)} + \Sigma_{(Az)}(R=0)\right]$ vanish. Plugging in the explicit expressions for the source functions $\Sigma_{(Az)}(R)$ and $\Sigma_{(Az)}(R)$ we find that we have nonsingular solutions if and only if

$$
\begin{aligned}
&(X-u)\cdot K\cdot(X-u) - \left[\frac{2Q^2}{(1-Q^2)}\right]\left[(X-u)\cdot K\cdot u\right] = \left(\frac{1-n_S^2}{S\ n_S}\right),\\
&(X-u)\cdot\partial Q = Q\left[(X-u)\cdot K\cdot u\right],\\
&\text{where } X = \frac{dS}{n_S} - n = \left(\frac{n_S}{S}\right)Z.
\end{aligned}
\tag{451}
$$

In the vector sector, the projection of C_{Einp} may be shown to be proportional to

$$
\left[\left(\frac{S^0}{n_S^0}\right)\mathcal{E}_{Vi} + f_0(R)\mathcal{E}_{Ri}\right] \propto f_0(R)\frac{d}{dR}\left[V_i^{(z)}(R)\right] + \mathcal{V}_i^{(Z)}(R) = 0. \tag{452}
$$

Here $\mathcal{V}_i^{(Z)}(R)$ is the combination of source terms in the first line of (452) - and so an appropriate linear combination of $\mathcal{V}_{mi}^a(R)$ This equation has regular solutions if $\mathcal{V}_i^{(Z)}(R)$ vanishes at $R=0$ i.e. if

$$
\begin{aligned}
&P_j^i\left[(X-u)\cdot\partial(u-n)_i + Q^2\left(X\cdot\partial n_i - u\cdot\partial u_i\right)\right] = 0,\\
&\text{where } X = \frac{dS}{n_S} - n = \left(\frac{n_S}{S}\right)Z.
\end{aligned}
\tag{453}
$$

It may be verified that (451) and (453) exhaust the constraints of regularity; once these equations hold the solution for the first order correction to the black brane metric and gauge field can always be chosen (by choosing appropriate integration constants in the solutions of the differential equation) to be regular at $R=0$ (and everywhere else within the patch).

In summary, the perturbative procedure described in this subsection yields regular solutions if and only if the equations of motion (451) and (453) are obeyed.

3.3.11 Equivalence to the equations of [1] in the uncharged limit

Note that the same null one-form O_μ has been parametrized in a different way in [1].

$$
O = A(dS - u^{there}) = n - u^{here}, \tag{454}
$$

where u^{there} is the velocity field used in [1] and in this subsection, u^{here} will denote the velocity field we used in this chapter . Recall that u^{there} was chosen to obey $u^{there}.dS = 0$. Dotting (454) with dS we find $A = n_S = n \cdot dS$ from which it follows that

$$u^{there} = \frac{u^{here} - n}{n_S} + dS. \tag{455}$$

This is the reason the equations of motion for the uncharged membrane as reported in equation 1.7 of [1] apparently do not match with the $Q \to 0$ limit of the equations of motion we derived in (451) and (453). However, we shall see that once we take into account this difference in the definition of u, the uncharged limit of our equations of motion exactly matches with that of [1].

The equations of motion for the uncharged membrane were reported in equation 1.7 of [1] as

$$\begin{aligned}
&U_\perp \cdot K \cdot U_\perp + n_S(n_S^2 - 1)/S = 0,\\
&\Big((U_\perp \cdot \nabla)\, u^{there}\Big) \cdot P^{\mu\nu}_{there} = 0,\\
&U_\perp = U - (U \cdot n)n, \quad U = dS + n_S^2(dS - u^{there}_\mu dx^\mu).
\end{aligned} \tag{456}$$

The projector $P^{\mu\nu}_{there}$ projects orthogonal to the subspace spanned by u^{there}, n and dS. But u_{there} is a linear combination of u_{here} and n. Therefore it follows that the projector $P^{\mu\nu}_{there}$ employed in (456) agree with the projector $P^{\mu\nu}$ in (453). The covariant derivative '∇' is a derivative defined in the auxiliary space. In our choice of coordinates, this could be replaces by '∂'.

Using (455) we could express the vector $U_\perp$ in (456) in terms of the velocity u^{here}

$$\begin{aligned}
&U_\perp = U - 2n_S n = dS - n_S(u^{here} + n) = n_S(X - u^{here}),\\
&\text{where} \quad X = \frac{dS}{n_S} - n = \left(\frac{n_S}{S}\right) Z.
\end{aligned} \tag{457}$$

Substituting equation(455) and (457) in (456), we find [85]

$$\begin{aligned}
&(X - u^{here}) \cdot K \cdot (X - u^{here}) + \frac{n_S^2 - 1}{S n_S} = 0,\\
&\Big[\Big(\Big(X - u^{here}\Big) \cdot \partial\Big)\,(n - u^{here})\Big] \cdot P = 0.
\end{aligned} \tag{458}$$

Equations (458) exactly match with the $(Q \to 0)$ limit of equations (451) and (453).

3.3.12 Conditions to fix the integration constants

As we have explained above, the first order corrections to (408) are obtained by solving a collection of linear ordinary differential equations at each point on the membrane. As mentioned above these equations turn out to be explicitly solvable and yield regular solutions provided the equa-

[85]note that the projected derivative of $\frac{u_{here} - n}{n_S}$ equals $\frac{1}{n_S}$ times the projected derivative of $u - n$ as the term with n_S differentiated vanishes under projection. Where u is the new velocity.

tions of motion of Subsection 3.3.10 are obeyed. The solutions to these equations are, however, not yet unique. as they depend on as yet undetermined integration constants at each membrane point. As we have mentioned in the previous subsection, some of these constants are determined by the requirement of regularity at $R = 0$. This condition however leaves several integration constants undetermined. [86] In order to obtain a unique solution to our equations we will impose additional physically motivated constraints that will uniquely determine these integration constants.

Asymptotic flatness:

An obvious requirement that we impose is that the correction metric and gauge field $g^{(1)}_{\mu\nu}$ and $A^{(1)}_{\mu}$ vanish exponentially rapidly as $R \to \infty$. This condition ensures that the full spacetime metric rapidly approaches the metric of flat space upon moving a large distance (in units of $\frac{1}{D}$) away from the membrane. This condition sets the value of several integration constants.

Normalization Conditions:

Even after imposing the condition of asymptotic flatness, it turns out that we still have two undetermined integration constants in the scalar sector and one in the vector sector. This is precisely as should be expected on physical grounds. Our starting spacetime (408) was parameterized by two scalar functions (the shape of the membrane and its charge density field) plus one vector function (the velocity field). A redefinition of these fields (e.g. $Q \to Q + \mathcal{O}(1/D)$ leaves (408) unchanged at leading order, but modifies it at first subleading order. Such a redefinition will modifies the first order correction to the metric by a compensating amount. For this reason we should expect the first order correction to have a two parameter ambiguity in the scalar sector and a one parameter ambiguity in the vector sector, precisely as we find. [87]

The ambiguity described above is a result of the fact that we have not yet supplied a precise all orders definition of the shape, velocity and charge density fields that enter into the leading order solution (408). Such a definition may be supplied by specifying an additional constraint on all higher order corrections to (408) that would fix the field redefinition ambiguity described in the previous paragraph. In this chapter we choose to do this by requiring that $S_{(VV)}$, $V^{(V)}_{\mu}$ and $S_{(AV)}$ all vanish at $R = 0$. More invariantly we impose the condition that

$$H_{MN} n^N = A_M n^M = 0 \quad \text{when} \quad \rho = 1.$$

[86] As the integration 'constants' can, in general, be unconstrained functions of the membrane world volume (they are constants only in that they do not depend on R) they are in fact undetermined integration functions on the membrane world volume.

[87] A very similar issue arose in the study of the fluid gravity correspondence, and was dealt with in a manner similar to that described below. See e.g. [55, 58–60].

We refer to these additional conditions - that effectively define the shape, velocity and charge density fields - as 'normalization' conditions.

It may be checked that the normalization conditions we have chosen ensure, in particular that the surface $\rho = 1$ is a null surface which we will later identify with the event horizon of the spacetime.

The conditions of asymptotic flatness together with the normalization conditions are sufficient to fix all integration constants, and yield unique expressions for the first order correction the the metric and gauge field (408).

3.3.13 Results for the first order correction on the patch

In this subsection we present the explicit solution for the metric and the gauge field corrections at first order in $\mathcal{O}\left(\frac{1}{D}\right)$. Our explicit results are presented for $(p = 2)$, but will be generalized to all p in the next section. As mentioned above, our solution takes the form (438). In the rest of this subsection we present our explicit results for the functions that appear in (438)[88]

The functions appearing in the gauge field

$$\begin{aligned} V_i^A(R) = &- \sqrt{2}\, Q \left(\frac{S}{n_S}\right)^2 \left[(1-Q^2)\mathfrak{v}_i^{(5)} + (1+Q^2)\left(\frac{n_S}{S}\right)\mathfrak{v}_i^{(2)}\right] R e^{-R} \\ &+ \sqrt{2} Q^3 \left(\frac{S}{n_S}\right)^2 \left(\mathfrak{v}_i^{(5)} - \left(\frac{n_S}{S}\right)\mathfrak{v}_i^{(2)}\right)\left[1 + \log(1 - Q^2 e^{-R})\right] e^{-R} \end{aligned} \tag{459}$$

$$\begin{aligned} S_{(Az)}(R) = &- \left[\frac{2\sqrt{2}\, S^2 Q^3 e^{-R}}{(1-n_S^2)(1-Q^2)}\right]\left[1 + \log(1 - Q^2 e^{-R})\right]\mathfrak{s}_1 \\ &+ \left[\frac{2\sqrt{2}\, S^3 Q\, e^{-R}}{n_S(1-n_S^2)(1-Q^2)}\right]\left[(Q^2 - R + Q^2 R) + Q^2 \log(1 - Q^2 e^{-R})\right]\mathfrak{s}_2. \end{aligned} \tag{460}$$

$$\begin{aligned} S_{(AV)}(R) = &\sqrt{2}\, Q\, R e^{-R}\left(\frac{S}{n_S}\right)\left(\frac{\mathfrak{s}^{(5)}}{Q} - \mathfrak{s}^{(1)} + \frac{S}{n_S}\mathfrak{s}^{(2)}\right) \\ &+ 2\sqrt{2}\left(\frac{Q^3}{1-Q^2}\right) e^{-R}\, \Upsilon_A(R)\, \left(\frac{S}{n_S}\right)\left(\mathfrak{s}^{(1)} - \frac{S}{n_S}\mathfrak{s}^{(2)}\right), \end{aligned} \tag{461}$$

[88]The solution presented in this subsection depends on three functions Q, S and n_S. Strictly speaking they should be written as Q_0, S_0 and n_S^0, the values of these functions at $x^\mu = x_0^\mu$. But we did not write it that way firstly because of notational simplicity and secondly because we know that the difference is always suppressed by terms of order $\mathcal{O}\left(\frac{1}{D}\right)$.

where

$$\Upsilon_A(R) = \int_0^R dx\ \log(1 - Q^2 e^{-x}). \tag{462}$$

The functions appearing in the metric:

$$T_{ij}(R) = \left(\frac{2S}{n_S}\right)\left(\mathfrak{t}^{(1)}_{ij} - \mathfrak{t}^{(2)}_{ij}\right)\log(1 - Q^2 e^{-R}). \tag{463}$$

$$V^{(z)}_i(R) = \left[\frac{S^2(1+Q^2)}{n_S(1-n_S^2)}\right]\left(\mathfrak{v}^{(5)}_i - \left(\frac{n_S}{S}\right)\mathfrak{v}^{(2)}_i\right)\log(1 - Q^2 e^{-R}) \tag{464}$$

$$\begin{aligned} V^{(V)}_i(R) = &\left(\frac{QS}{n_S}\right)^2 \left[1 - e^{-R} - f_0(R)\left(1 + \log[1 - Q^2 e^{-R}]\right)\right]\left(\mathfrak{v}^{(5)}_i - \left(\frac{n_S}{S}\right)\mathfrak{v}^{(2)}_i\right) \\ &- R\left[1 - f_0(R)\right]\left(\frac{S}{n_S}\right)^2 \left[(1-Q^2)\mathfrak{v}^{(5)}_i + (1+Q^2)\left(\frac{n_S}{S}\right)\mathfrak{v}^{(2)}_i\right] \end{aligned} \tag{465}$$

$$\begin{aligned} S_{(Vz)}(R) =& S^{(1)}_{(Vz)}(R)\ \mathfrak{s}^{(1)} + S^{(2)}_{(Vz)}(R)\ \mathfrak{s}^{(2)}, \\ S^{(1)}_{(Vz)}(R) =& -\left[\frac{2Q^2S^2}{(1-n_S^2)(1-Q^2)}\right]\left[Q^2\left(e^{-R} - e^{-2R}\right) - f_0(R)\ \log(1 - Q^2 e^{-R})\right], \\ S^{(2)}_{(Vz)}(R) =& \left[\frac{2Q^2S^3}{n_S(1-n_S^2)(1-Q^2)}\right]\Bigg[(e^{-R} - e^{-2R})(Q^2 - R + Q^2 R) \\ &- f_0(R)\ \log(1 - Q^2 e^{-R})\Bigg] - \frac{2S^3 R e^{-R}}{n_S(1-n_S^2)}. \end{aligned} \tag{466}$$

$$S_{(zz)}(R) = \left[\mathfrak{s}^{(2)} - \left(\frac{n_S}{S}\right)\ \mathfrak{s}^{(1)}\right]\left[\frac{2S^4(1+Q^2)}{(1-n_S^2)^2(1-Q^2)}\right]\log(1 - Q^2 e^{-R}). \tag{467}$$

$$S_{(Tr)}(R) = \left[-2 + \left(\frac{S}{n_S}\right)(\mathfrak{s}^{(3)} - \mathfrak{s}^{(4)})\right]\log(1 - Q^2 e^{-R}). \tag{468}$$

$$
\begin{aligned}
S_{(VV)}(R) = & -\sqrt{2}Q\ e^{-R}S_{AV}(R) + Q^2\left[e^{-2R} - e^{-R}\right] \\
& + 2e^{-R}\left[Q^2\ R\ \left(\frac{\mathfrak{s}^{(5)}}{Q} - \mathfrak{s}^{(1)} + \frac{S}{n_S}\mathfrak{s}^{(2)}\right) + \Upsilon_H(R)\left(\mathfrak{s}^{(1)} - \frac{S}{n_S}\mathfrak{s}^{(2)}\right)\right],
\end{aligned}
\tag{469}
$$

where

$$
\begin{aligned}
f_0(R) &= 1 - \left[(1+Q^2)e^{-R} - Q^2 e^{-2R}\right], \\
\Upsilon_H(R) &= \left[(e^{-R} - Q^2)\log(1 - Q^2 e^{-R}) - (1-Q^2)\log(1-Q^2) + \left(\frac{Q^2(1+Q^2)}{1-Q^2}\right)\Upsilon_A(R)\right].
\end{aligned}
\tag{470}
$$

Correction (2nd order in $\frac{1}{D}$) to the scalar field ϕ

$$
\begin{aligned}
\delta\phi &= \sum_{k=1}\left(\frac{1}{D}\right)^k \delta\phi^{(k)}, \\
\delta\phi^{(1)} &= 0, \\
\delta\phi^{(2)}(R) &= -2S_{(Tr)}(R) - \left(\frac{1-n_S^2}{S^2}\right)S_{(zz)}(R).
\end{aligned}
\tag{471}
$$

The $Q \to 0$ limit If we set Q to zero in equation (459) to (471), most of the functions vanish except $V_i^{(V)}$ and $S_{(Vz)}$. In the uncharged limit, the metric takes the following simple form,

$$
G^{(1)}_{ab}dq^a dq^b|_{\text{uncharged}} = -2Re^{-R}\left[\left(\frac{S}{n_S}\right)^2\left(\mathfrak{v}_i^{(5)} + \frac{n_S}{S}\mathfrak{v}_i^{(1)}\right)dy^i + \frac{S^3\ \mathfrak{s}^{(2)}}{n_S(1-n_S^2)}dz\right]dV. \tag{472}
$$

3.3.14 The global first order metric

With the first order corrected patch metric in hand (see the previous subsection), it is straightforward to find the global form of the metric and gauge field which, when expanded in any patch around a membrane point, will reproduce the results of Appendix 3.3.13. In order to obtain this global form we simply make the replacements

$$
e^R \to \rho^{-D}, \quad R \to D\times(\rho - 1), \quad dV \to O_M dx^M, \quad dR \to D\times(d\rho),
$$

in the results of subsection 3.3.13. The final metric obtained in this manner is already reasonably compact. There is, however, a physically motivated rewriting of this result in a form that is both more elegant and also makes manifest the 'geometrical' nature of our final result, as we explain in more detail in the next section.

3.4 Geometrical Form of the first order corrected metric

3.4.1 Redistribution invariance and the Geometrical form

The membrane equations (451) and (453) make make special reference to e^{ϕ}, n_S and the one-form field Z_μ. The same is true of our explicit results for the first order correction to (408), presented in subsection 3.3.13. Expressions involving S, n_S and Z of course are only well defined for configurations that preserve an SO$(D-p-2)$ symmetry. Moreover the definition of, e.g. S depends on the details of the isometry.

Unconstrained dependence on S and n_S is unacceptable for the following reason. A solution that preserves an SO$(D-p-2)$ isometry also preserves an SO$(D-p'-2)$ isometry for all $p' > p$. It follows that any solution of the equations for a particular choice of p must also be a solution of the same equation for all larger p. We refer to this requirement as the requirement of redistribution invariance.

The requirement of redistribution invariance is most simply met if the equation of motion and the metric and gauge field can both be written in an explicitly geometrical form that makes no reference to the particular isometry group of the solution. The membrane equation and first order metric and gauge field obtained in this section do indeed turn out to have this property.

The reader may, at first, wonder how it is possible for expressions with explicit appearances of S and n_S to also be geometrical. This is, infarct, possible in the large D limit, as we now explain with an example. Consider the manifestly geometrical expression $\nabla^2\rho$ where ∇ refers to the covariant derivative on the full flat D dimensional embedding spacetime. Let us now evaluate this expression in the large D limit restricting attention to membrane configurations that preserve an SO$(D-p-2)$ isometry. The computation is most conveniently performed using the following coordinates

$$ds^2 = \eta_{\alpha\beta}dw^\alpha dw^\beta + dS^2 + S^2 d\Omega_d^2,$$

in the embedding flat space. Using these coordinates

$$\nabla^2\rho = \frac{1}{S^d}\partial_\mu\left(S^d\partial^\mu\rho\right).$$

At leading order in the large D limit this expression reduces to $D\frac{dS\cdot d\rho}{S}$ It follows that $\frac{dS\cdot d\rho}{S} = \frac{\nabla^2\rho}{D}$ at leading order in the large D limit. Consequently any appearance of $\frac{dS\cdot d\rho}{S}$ in any equation may be explicitly geometrized.

Similar manipulations allow us to geometrize several other expressions involving S, n_S and Z. Of course not every expression involving these quantities can be geometrized (expressions that are not redistribution invariant certainly cannot). However it turns out that all terms in the equations of motion (451) and (453) and all terms in our explicit expression for the metric and gauge field in subsection 3.3.13 can be geometrized. The final geometrical expressions for equations of motion and the the first order corrected metric and gauge field are more compact than the unprocessed expressions. In the next section we present our final results for the first order corrections to the metric and gauge field in explicitly geometrical form. In the subsequent subsection we do the same for the equations of motion.

3.4.2 Metric and Gauge field in Geometric Form

While we expect the first order correction to the metric and gauge field to be geometrizable on physical grounds, this requirement is nontrivial at the algebraic level. The vector Z_μ - which is treated as a special in the computation described above and in subsection 3.3.13- has no intrinsic geometrical significance [89]. If the first order correction to the metric and gauge field is completely geometrical, it should be possible to rewrite it in a manner that makes no reference to Z_μ. In fact it should be possible to rewrite the metric and gauge field in the form

$$\begin{aligned}
&h_{MN} = F(\rho) O_M O_N + H^{(T)}_{MN} + 2 O_{(M} H^{(V)}_{N)} + H^{(S)} O_M O_N + H^{(Tr)} \mathcal{P}_{MN},\\
&A_M = \sqrt{2} Q\ \rho^{-(D-3)}\ O_M + \left(A^{(S)} O_M + A^{(V)}_M \right),\\
&\text{where}\\
&F(\rho) = \left[(1+Q^2)\rho^{-(D-3)} - Q^2 \rho^{-2(D-3)} \right],\\
&\mathcal{P}_{MN} = \eta_{MN} - O_M n_N - O_N n_M + O_M O_N,\\
&\mathcal{P}^{MN} H^{(V)}_N = \mathcal{P}^{MN} A^{(V)}_N = 0, \quad \mathcal{P}^{MN} H^{(T)}_{MQ} = 0, \quad \mathcal{P}^{MN} H^{(T)}_{MN} = 0,
\end{aligned} \tag{473}$$

(473) should reproduce the expressions for $g_{\mu\nu}$, A_μ (see (425)) as well as the scalar ϕ (recall that ϕ is part of the full D dimensional metric).

The general metric and gauge field presented in (473) are parameterized by three unknown scalar functions (rather than the seven scalar functions in (425) and in the expansion of the scalar ϕ) and by two vector functions (rather than three vector functions in (425)). It follows that the explicit results of subsection 3.3.13 can be recast into the form (473) only if the seven scalar functions determined in subsection 3.3.13 obey four constraints, and the three vector functions

[89] On the other hand the vectors n^μ and u^μ are intrinsically geometrical, as they describe the membrane shape and velocity field in D dimensional spacetime.

determined in the same Appendix obey a single constraint equation.

We have verified that our explicit results do infarct obey all constraints. We view this fact as an impressive consistency check of the complicated algebra that went into obtaining the explicit results of subsection 3.3.13.

Scalars	$\mathfrak{S}_{(1)} = \left(\frac{D}{\mathcal{K}}\right)\left[\frac{u\cdot\partial Q}{Q} - u\cdot K\cdot u + \frac{(u\cdot\partial)\mathcal{K}}{\mathcal{K}}\right]$ $\mathfrak{S}_{(2)} = \left(\frac{D}{\mathcal{K}}\right)\left[u\cdot K\cdot u - \frac{(u\cdot\partial)\mathcal{K}}{\mathcal{K}}\right]$
Vectors	$\mathfrak{V}^M_{(1)} = \left(\frac{D}{\mathcal{K}}\right)\left[\frac{\nabla_N\mathcal{K}}{\mathcal{K}} + (u\cdot\nabla)u_N\right]\ \mathcal{P}^{NM}$ $\mathfrak{V}^M_{(2)} = \left(\frac{D}{\mathcal{K}}\right)\left[\frac{\nabla_N\mathcal{K}}{\mathcal{K}} - (u\cdot\nabla)u_N\right]\ \mathcal{P}^{NM}$
Tensor	$\mathfrak{T}^{MN} = \mathcal{P}^{MQ_1}\ \left(\frac{D}{\mathcal{K}}\right)\left[\frac{\nabla_{Q_1}O_{Q_2}+\nabla_{Q_2}O_{Q_1}}{2} - \eta_{Q_1Q_2}\left(\frac{\nabla\cdot O}{D-2}\right)\right]\ \mathcal{P}^{Q_2N}$

Table 2: We list the data that enters into our explicit results for the first order correction to the metric and the gauge field. All data is presented in explicitly geometrical form. ρ, Q and u^μ should be thought of as two functions and a vector field in flat D dimensional space. All derivatives that appear in this table are covariant derivatives w.r.t flat D dimensional space.

As our explicit results obey all consistency conditions, it is possible to rewrite our final results in the explicitly geometric form (473). We find that the various free functions in (410) are given by

$$\begin{aligned}A^{(V)}_M =& -\left(\frac{\sqrt{2}}{D}\right)Q\rho^{-D}\Big[D(\rho-1)(\mathfrak{V}_{(1)} - Q^2\ \mathfrak{V}_{(2)}) - Q^2[1+\log(1-\rho^{-D}Q^2)]\mathfrak{V}_{(2)}\Big]_M\\ &+\mathcal{O}\left(\frac{1}{D}\right)^2,\\ A^{(S)} =& \left(\frac{1}{D}\right)\left[\sqrt{2}\ Q\ D(\rho-1)\ \rho^{-D}\mathfrak{S}_{(1)} + 2\sqrt{2}\left(\frac{Q^3}{1-Q^2}\right)\rho^{-D}\ \Upsilon_A(\rho)\ \mathfrak{S}_{(2)}\right] + \mathcal{O}\left(\frac{1}{D}\right)^2.\end{aligned} \tag{474}$$

$$\begin{aligned}H^{(T)}_{MN} =& \left(\frac{2}{D}\right)\log(1-Q^2\rho^{-D})\ \mathfrak{T}_{MN} + \mathcal{O}\left(\frac{1}{D}\right)^2,\\ H^{(V)}_M =& \left(\frac{1}{D}\right)\Big\{Q^2\left[(F(\rho)-\rho^{-(D-3)}) + (F(\rho)-1)\log(1-Q^2\rho^{-D})\right]\mathfrak{V}_{(2)M}\\ &-D(\rho-1)F(\rho)\ [\mathfrak{V}_{(1)} - Q^2\ \mathfrak{V}_{(2)}]_M\Big\} + \mathcal{O}\left(\frac{1}{D}\right)^2.\end{aligned} \tag{475}$$

$$
\begin{aligned}
H^{(S)} =& -\sqrt{2}Q\ \rho^{-D}A^{(S)} + \left(\frac{1}{D}\right)\left[\rho^{-(D-3)} - F(\rho)\right] \\
&+ \left(\frac{2}{D}\right)\rho^{-D}\left[Q^2\ D(\rho-1)\ \mathfrak{S}_{(1)} + \Upsilon_H(\rho)\mathfrak{S}_{(2)}\right] + \mathcal{O}\left(\frac{1}{D}\right)^2,
\end{aligned}
\tag{476}
$$

$$
H^{(Tr)} = \mathcal{O}\left(\frac{1}{D}\right)^3,
$$

where

$$
\begin{aligned}
F(\rho) &= \left[(1+Q^2)\rho^{-(D-3)} - Q^2\rho^{-2(D-3)}\right], \\
\Upsilon_A(\rho) &= \int_0^{D(\rho-1)} dx\ \log(1-Q^2e^{-x}), \\
\Upsilon_H(\rho) &= \left[(\rho^D - Q^2)\log(1-Q^2\rho^{-D}) - (1-Q^2)\log(1-Q^2) + Q^2\left(\frac{1+Q^2}{1-Q^2}\right)\Upsilon_A(\rho)\right].
\end{aligned}
\tag{477}
$$

The limit $Q \to 0$ The results of the previous subsection simplifies drastically in the limit $Q \to 0$. In this limit the gauge field simply vanishes, and the full first order corrected metric is given by the remarkably simple expression

$$
\begin{aligned}
ds^2_{uncharged} =& ds^2_{\text{flat}} + \rho^{-(D-3)}(O_M dx^M)^2 \\
&- 2(\rho-1)\rho^{-(D-3)}[\mathfrak{V}_{(1)}]_M O_N dx^M dx^N + \mathcal{O}\left(\frac{1}{D}\right)^2,
\end{aligned}
\tag{478}
$$

where $\mathfrak{V}_1$ is defined in table 2.

In Appendix 3.7.2 we have shown how this geometric form of the metric and gauge field reduce to the solution presented in subsection 3.3.13, once we impose the constraint of SO$(D-p-2)$ invariance on all geometric data.

3.4.3 Geometrizable form of the membrane equations of motion

The membrane equations of motion (451) and (453) may be recast into a simpler looking form. We have a combined equation capturing both vector equation and one of the scalar equations.

$$
\left[(u-X)\cdot\tilde{\nabla}O - Q^2(u\cdot\tilde{\nabla})u + Q^2(X\cdot K)\right]\cdot\mathcal{P} + \left(\frac{n_S}{S}\right)(1-Q^2)X = 0, \tag{479}
$$

$$(X-u)\cdot\tilde{\nabla}Q+Q\left[\left(\frac{S}{n_S}\right)(u\cdot\tilde{\nabla})\left(\frac{n_S}{S}\right)-(u\cdot K\cdot u)\right]=0, \tag{480}$$

where

$$\mathcal{P}_{\mu\nu} = \text{Projector perpendicular to } u_\mu \text{ and } n_\mu,$$
$$X=\frac{dS}{n_S}-n=\left(\frac{S}{n_S}\right)Z, \qquad O=n-u.$$

In this equation $\tilde{\nabla}$ above is the partial derivative on the membrane world volume viewed as a submanifold of (426).

The projection of equation (479) perpendicular to Z_μ directly reduces to the vector equation of motion as given in equation (453). In appendix 3.7.3 we have shown that equation (480) is equal to second equation of (451). Moreover the dot product (479) with Z_μ equals the first equation in (451) upto correction of $\mathcal{O}\left(\frac{1}{D}\right)$.

We re emphasize that the projector employed in (479) projects orthogonal to n and u but not to Z_μ. In other words (479) unifies a SO(p) scalar and SO(p) vector equation into a single 'geometrical' vector equation. This fact may lead the reader to suspect that the equations (479) and (480) are geometrizable (i.e. can be written without any explicit reference to the isometry direction. This is indeed the case. It is not too difficult to demonstrate that the geometric form of the equations of motion, (402) (see the introduction) reduce immediately to (479) and (480) upon using the dictionary of translation as presented in appendix 3.7.4.

Constraint equations and the membrane equations of motion In the previous section we explained that the Einstein and Maxwell constraint equations play a special role in our construction. We obtained the membrane equations of motion from the requirement that these bulk equations admit nonsingular solutions. Once the membrane equations were imposed, it was possible to utilize the constraint equations to solve for some unknown bulk scalar and vector fields in terms of others in a nonsingular manner. We have already mentioned in subsection 3.3.10 that the geometric nature of the membrane equations is a direct consequence of the geometric nature of Einstein's constraint equations.

In this subsection we wish to focus on the fact the constraint equations played two roles in the perturbative program of the previous section.

- 1 They yielded the membrane equations of motion.

- 2 They allowed us to solve for some unknowns bulk fields in terms of others.

Interestingly enough, the relations that we obtain from item (2) above are all *automatic* in the expression (473). In other words the relations of item (2) above are simply a subset of the relations between various unknown bulk vectors fields and various unknown bulk scalar scalars fields that are forced on us once we assume that the first order metric and gauge field correction take on a geometric form.

Had we used hindsight to set up our perturbative expansion in a manifestly geometric manner by simply assuming that our first order correction takes the form (473) then the constraint equations of subsection 3.3.10 would simply have *reduced* to the membrane equations (402), exactly as in the studies of the fluid gravity correspondence (see e.g. [55–60]).

Recall that the Einstein constraint equations assert the conservation of the Brown York Stress tensor, while the Maxwell constraint equation asserts the conservation of a 'charge current' $F^{\rho M}$. These observations suggest that it may be possible to recast our membrane equations (402) as conservation equations for a manifestly geometric membrane stress tensor and charge current, as was the case in the study of fluid gravity. We will not pursue this point further in this chapter but hope to return to it in the near future.

3.4.4 Comparison with the Reissner-Nordstrom solution

As an elementary check of the results reported in Subsection 3.4.2, consider the following membrane configuration. Let $u_M dx^M = -dt$, $Q = const$ and let the membrane surface be given by $x^M(\eta_{MN} + u_M u_N)x^N = r_0^2$. It is easily verified that this configuration solves the membrane equations (402); clearly this static soap bubble solution is dual to the Reissner-Nordstrom black hole (405).

We will now use the formalism developed in this chapter to determine the spacetime metric and gauge field dual to this membrane solution, to first order in $1/D$.

Let us first start with the leading order solution (408) dual to this solution. We need to find a function ρ that obeys the first of equation (409) and s.t. $\rho = 1$ on the membrane surface listed above. The unique solution to this mathematical problem is given by

$$\rho = \sqrt{\frac{x^M(\eta_{MN} + u_M u_N)x^N}{r_0^2}}.$$

Next we must determine the spacetime fields u and Q fields that reduce to $-dt$ and q on the membrane and obey (409) everywhere in the bulk of D dimensional flat space. The unique solution to this problem is given by $u = -dt$ and Q in all of flat space. The leading order solution with this data is given by (408) with these choices for ρ, Q and u.

Let us now turn to the first order correction. It is easily verified that relevant geometric data as given in table 2 vanish on this particular profile of ρ and u_M and Q. It follows that the first order correction to the gauge field vanishes. The first order correction to the metric also almost vanishes. Of all the quantities listed in (474), (475) and (476) vanish except for H^S which evaluates to $\left(\frac{1}{D}\right)\left[\rho^{-(D-3)} - F(\rho)\right]$ with $F(\rho)$ listed in (477).

Plugging these values of ρ, u_M and Q into (408) and adding the correction terms (473) , it follows that the metric and gauge field dual to our simple solution of the membrane equations is given, to first order in $\frac{1}{D}$, by

$$
\begin{split}
g_{MN} = \eta_{MN} + \left[(1+Q^2)\rho^{-(D-3)} - Q^2\rho^{-2(D-3)}\right] O_M O_N \\
+ \frac{Q^2}{D}\left(\rho^{-2(D-3)} - \rho^{-(D-3)}\right) O_M O_N + \mathcal{O}\left(\frac{1}{D}\right)^2,
\end{split}
\tag{481}
$$

(481) is easily seen to agree with the exact RN black hole solution (405) expanded to leading nontrivial order in $1/D$. The function ρ of (405) agrees exactly with the function ρ reported above. The only appearance of D in the solution (405) is in the factor c_D. Upon plugging the expansion

$$
c_D = 1 - \frac{1}{D} + \mathcal{O}\left(\frac{1}{D}\right)^2,
$$

into (405) we immediately recover (481).

The matching performed in this subsection was almost trivial. In an unpublished work we use the same method to match the metric dual to a rigidly rotating solution of the membrane equations to the much more complicated exact solution of an uncharged rotating Myers Perry black hole [61]. Once again we find a perfect match between the two expansions.

3.5 Light quasinormal spectrum or the RN black hole

Our membrane equations (402) should describe all SO$(D-p-2)$ invariant black hole dynamics (over times scales much larger than $1/D$) at large D. As a first application of these equations, in this section we will use them to obtain a prediction for the spectrum of light quasinormal modes (those with frequencies of order unity rather than of order D) about charged Reissner-Nordstrom black holes in the large D limit.

For the purposes of this section we work in polar coordinates in D spacetime dimensions. Our coordinate system for flat space is given by

$$
ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{D-2}^2 \quad . \tag{482}
$$

The exact solution of (402) dual to RN black holes was presented in subsection 3.4.4. In the coordinates (482) this solution takes the particularly simple form

$$r = 1, \quad Q = Q_0 = \text{const}, \quad u = -dt, \tag{483}$$

where we have chosen units that set the size of the membrane to unity. [90]

The most general linearized perturbation around (483) takes the form

$$\begin{aligned} r &= 1 + \epsilon\ \delta r(t,\theta), \\ Q &= Q_0 + \epsilon\ \delta Q(t,\theta), \\ u &= -dt + \epsilon\ \delta u_\mu(t,\theta) dx^\mu. \end{aligned} \tag{484}$$

We now adopt the following strategy. We simply insert the expansions (484) into (402), work to linear order in ϵ and obtain the effective linear equations for the fluctuation fields δr, δQ and δu_μ defined in (484). These fields live on the membrane world volume. A useful set of coordinates on this world volume are the angular coordinates θ^a on Ω_{D-2} and time. The metric on the membrane world volume in these coordinates is obtained by inserting the first of (484) into (482), and is given in terms of the function $\delta r(t,\theta)$. To linear order in ϵ the metric on the membrane surface is given by

$$ds^2 = -dt^2 + (1 + 2\epsilon\delta r)\, d\Omega_{D-2}^2 \quad . \tag{485}$$

It is useful to have a dictionary to go between forms and vectors that live on the membrane and those that live in spacetime. Consider a vector field A^μ that lives on the membrane. This vector field may be uplifted to spacetime. The spacetime components $A^M_{(ST)}$ of this vector field are given in terms of the world volume components A^a by the formulae

$$A^a_{(ST)} = A^a, \quad A^t_{(ST)} = A^t, \quad A^r_{(ST)} = \epsilon \left(A^t \partial_t \delta r + A^a \partial_a \delta r \right). \tag{486}$$

In a similar manner, a one-form field in spacetime $B_a^{(ST)}$ is easily pulled back to a one-form field

[90] We do not loose generality by making this choice. The classical Einstein Maxwell equations studied in this chapter enjoy invariance under the following 'scaling' symmetry:

$$\tilde{g}_{MN} = \alpha^2 g_{MN}, \quad \tilde{F}_{MN} = \alpha F_{MN}.$$

This scale transformation together with the coordinate change $\tilde{x}^M = \alpha x^M$ transforms a Reissner Nordstrom black hole with Schwarzschild radius r_0 and charge parameter Q_0 into a Reissner Nordstrom black hole with Schwarzschild radius αr_0 and charge parameter Q_0. It follows that the quasinormal mode frequencies of the black hole parameterized by (r_0, Q_0) are simply $\frac{1}{r_0}$ times those for the black hole parameterized by $(1, Q_0)$. For this reason we will perform all computations in this section with black holes of radius unity, and simply reinsert factors of r_0 in the final answer.

B_a on the membrane. In formulae

$$B_a = B_a^{(ST)} + \epsilon B_r^{(ST)} \partial_a \delta r, \quad B_t = B_t^{(ST)} + \epsilon B_r^{(ST)} \partial_t \delta r. \tag{487}$$

As a simple consistency check on these formulae, it is easily verified that $A^\mu B_\mu = A^M_{(ST)} B_M^{(ST)}$. Below we will treat the field u_μ in (484) as a one-form field on the membrane. Recall that u_μ is constrained by the requirement $\nabla \cdot u = 0$, i.e. that the velocity field is divergence free. Here ∇ is the covariant derivative taken in the metric (485).

In order to evaluate the terms in (402) we need to compute the, normal one-form and extrinsic curvature of the membrane as well as a few derivatives of the velocity field. The computations involved are straightforward: to linear order in ϵ we find

$$\begin{aligned}
n_r &= 1, \\
n_\mu &= -\epsilon \partial_\mu \delta r, \\
K_{tt} &= -\epsilon \partial_t^2 \delta r, \\
K_{ta} &= -\epsilon \partial_t \nabla_a \delta r, \\
K_{ab} &= -\epsilon \nabla_a \nabla_b \delta r + (1 + \epsilon \delta r) g_{ab}, \\
\delta u_t &= 0, \qquad\qquad (u \cdot u = -1) \\
(u \cdot K)_t &= K_{tt} = -\epsilon \partial_t^2 \delta r, \\
(u \cdot K)_a &= -\epsilon \partial_t \nabla_a \delta r + \epsilon \delta u_a, \\
\mathcal{K} = K_A{}^A &= D \left(1 - \epsilon \left(1 + \frac{\nabla^2}{D} \right) \delta r \right),
\end{aligned} \tag{488}$$

where a, b are angular directions on Ω_{D-2}, the symbol μ runs over these angular coordinates and time (i.e $\mu = (t, a)$) and g_{ab} is the round metric on S^{D-2}. All indices in (488) are indices on the spherical metric world volume, i.e. on a space with metric

$$ds^2 = -dt^2 + d\Omega_{D-2}^2 \ , \tag{489}$$

and all covariant derivatives in (488) are taken in the background spacetime (489).

Using (488), the first equation in (402) may be shown to reduce, at linearized order in ϵ, to

$$\left(1 + \frac{\nabla^2}{D} \right) \delta u_a + (1 - Q_0^2) \nabla_a \left(1 + \frac{\nabla^2}{D} \right) \delta r - \partial_t \nabla_a \delta r - (1 + Q_0^2) \partial_t \delta u_a = 0. \tag{490}$$

(All covariant derivatives are once again evaluated on the metric (489)).

As we have noted above, the fluctuation velocity field δu_a is constrained by the condition $\nabla \cdot u = 0$. The divergence in this equation is evaluated in the full membrane metric. Rewriting this constraint in terms of fields that are taken to propagate on the fixed metric (489) (to linear order in ϵ and leading order in D) we find

$$\nabla_\mu \delta u^\mu = -D\partial_t \delta r, \tag{491}$$

with the covariant derivatives now evaluated on the metric (489). From now on until the end of this section our fluctuation fields will all be taken to propagate on the fixed background (489) and all covariant derivatives will refer to this metric unless explicitly otherwise declared.

As δu^t vanishes (this follows upon linearizing the equation $u \cdot u = -1$), (491) may be rewritten as

$$\nabla_a \delta u^a = -D\partial_t \delta r. \tag{492}$$

In order to solve this equation it is useful to define

$$\delta u_a = \nabla_a \Phi + \delta v_a, \tag{493}$$

where

$$\nabla \cdot \delta v = 0. \tag{494}$$

It follows from (492) that

$$\nabla^2 \Phi = -D\partial_t \delta r. \tag{495}$$

Below we will use this equation to eliminate Φ in favour of δr. Note that (495) admits a solution if and only if its RHS has no overlap with the kernel of the operator ∇^2. As the kernel of ∇^2 consists of functions that are constant on the sphere, it follows that (495) is consistent if and only if the spatially constant (i.e. $l = 0$ mode) of δr is time independent. When this condition is obeyed, Φ may be solved for in terms of δr, as we will do below.

Plugging the expansion (493) into the equation (490) we find

$$\begin{aligned} \left(1 + \frac{\nabla^2}{D} - (1+Q_0^2)\partial_t\right)\delta v_a = &- \left((1-Q_0^2)\nabla_a\left(1+\frac{\nabla^2}{D}\right) - \partial_t \nabla_a\right)\delta r \\ &- \left(1 + \frac{\nabla^2}{D} - (1+Q_0^2)\partial_t\right)\nabla_a \Phi. \end{aligned} \tag{496}$$

3.5.1 The spectrum of shape fluctuations

Taking the divergence of (496) and using (494) and (495) we obtain the following decoupled scalar equation for the fluctuation field δr

$$D(1+Q_0^2)\partial_t^2 \delta r - 2D\left(1+\frac{\nabla^2}{D}\right)\partial_t \delta r + (1-Q_0^2)\left(1+\frac{\nabla^2}{D}\right)\nabla^2 \delta r = 0. \tag{497}$$

91

The most general linearized fluctuation δr can be expanded as

$$\delta r = \sum_{l,m} a_{lm} Y_{lm} e^{-i\omega_l^r t} \quad . \tag{499}$$

Here Y_{lm} are spherical harmonics on S^{D-2}, l labels the spherical harmonic representation, m is a collective label for all the internal quantum numbers within a given spherical harmonic representation.

Let us pause to give a more complete description of scalar spherical harmonics in arbitrary dimensions, and in particular to compute the eigenvalue under ∇^2 acting on the l^{th} spherical harmonic. The l^{th} spherical harmonic, Y_{lm}, are composed of the collection of functions on S^{D-2} obtained by restricting homogeneous degree l polynomials in R^{D-1} to the unit sphere. The polynomials in questions are linear combinations of monomials of the form $a_{\mu_1\mu_2\mu_3\ldots\mu_l} x^{\mu_1} x^{\mu_2} \ldots x^{\mu_l}$ where $a_{\mu_1\mu_2\mu_3\ldots\mu_l}$ are symmetric and traceless tensors. It is easily shown that

$$-\nabla^2_{S^{D-2}} Y_{lm} = l(D+l-3) Y_{lm}. \tag{500}$$

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Plugging the expansion (499) into (497) and using (500) we find (at leading order in large D)

$$\omega_l^r = \frac{-i(l-1) \pm \sqrt{(l-1)(1-lQ_0^4)}}{1+Q_0^2}. \tag{502}$$

Re inserting factors of r_0 (see the discussion in the introduction to this section) we find (500) we

[91]In order to obtain (497) we have used and

$$\begin{aligned} \nabla^a \nabla^2 \delta u_a &= \nabla^2 \nabla_a \delta u^a + R^{ab} \nabla_a \delta u_b, \\ &= \nabla^2 \nabla_a \delta u^a + D\, g^{ab} \nabla_a \delta u_b, \\ &= D \left(1 + \frac{\nabla^2}{D}\right) \nabla_a \delta u^a = -D^2 \left(1 + \frac{\nabla^2}{D}\right) \partial_t \delta r. \end{aligned} \tag{498}$$

[92]This may be demonstrated as follows. The condition of tracelessness ensures that the degree l polynomials described above obey the equation $\nabla^2 \Phi = 0$, where ∇^2 is evaluated in R^{D-1}. But

$$0 = \nabla^2_{R^{D-1}} \Phi = \frac{1}{r^{D-2}} \partial_r \left(r^{D-2} \partial_r r^l\right) + \frac{\nabla^2_{S^{D-2}} \Phi}{r^2}. \tag{501}$$

(the RHS of this equation is ∇^2 of the function in R^{D-1} evaluated in polar coordinates). Here $\nabla^2_{S^{D-2}}$ is the Laplacian evaluated on the unit sphere. (500) follows from (501).

find (at leading order in large D)

$$r_0\omega_l^r = \frac{-i(l-1) \pm \sqrt{(l-1)(1-lQ_0^4)}}{1+Q_0^2}. \tag{503}$$

93

(503), our final result for the light quasinormal mode frequencies associated with shape fluctuations, is correct as stated for $l > 1$, but requires clarification for in special cases $l = 0$ and $l = 1$ for the reasons we now describe.

Let us first consider $l = 0$. In this case (503) predicts the existence of quasinormal modes with frequencies $\omega r_0 = 0$ and $\omega r_0 = \frac{2i}{1+Q_0^2}$. As noted under (495), however, modes at $l = 0$ are physical only if they are time independent. It follows that we have only one mode at $l = 0$: this mode has $\omega = 0$. [94] This zero mode has a simple physical interpretation; it corresponds to an infinitesimal uniform rescaling of the black hole radius.

Let us now turn to $l = 1$. In this case we have a degeneracy of quasinormal mode frequencies; both modes have $\omega = 0$. The formula (503) was obtained by assuming harmonic dependence in time and solving for the harmonic frequencies, and it is well known that this procedure requires modification in the case that the frequencies are degenerate. In order to see how this works, we note that the specialization of (497) to modes with $l = 1$ yields the very simple equation

$$\partial_t^2 \delta r = 0. \tag{504}$$

It follows that the two solutions to this equation are $\delta r = Y_1^m(a_m + b_m t)$ where a_m and b_m are arbitrary constants. These two zero modes also have a simple physical interpretation. The mode multiplying a_m is an infinitesimal translation of the black hole, while b_m parameterizes an infinitesimal boost of the black hole. Note that the m labels for $l = 1$ scalar spherical harmonics are precisely the labels for a vector in $(D-1)$ dimensions, as appropriate for translations and boosts.

As we have mentioned above, (503) is correct as stated for $l \geq 2$. It is easily verified [95] that all quasinormal modes with $l \geq 2$ have negative imaginary components (and so represent decaying fluctuations).

[93] K. Tanabe has informed us that he is also studying the dynamics of charged black holes at large D and has independently obtained the result (503).

[94] Had the mode with $\omega r_0 = \frac{2i}{1+Q_0^2}$ been physical, it would have represented an instability, contradicting the well known stability of Reissner Nordstrom black holes (atleast of sufficiently small charge) in arbitrary dimensions.

[95] Note that a Reissner Nordstrom black hole with $Q_0 = 1$ is extremal at large D. All regular black holes have $|Q_0| < 1$.

3.5.2 The spectrum of velocity fluctuations

The fact that the shape fluctuation δr obeys the equation of the previous subsection ensures that the RHS of (496) vanishes. The velocity fluctuations, δv_a, are thus effectively constrained to obey (496) with its RHS set to zero.

The fluctuation field δv may be expanded in vector spherical harmonics

$$\delta v_a = \sum_{l,m} b_{lm} Y_a^{lm} e^{-i\omega_l^v t} \tag{505}$$

Let us pause to describe vector spherical harmonics in arbitrary dimension in more detail. The l^{th} vector spherical harmonic may be obtained as a restriction of a vector field on R^{D-1} to the unit sphere. The vector field in question is made up as a linear sum of vector valued monomials of the form $V_{\mu\mu_1\mu_2\ldots\mu_l} x^{\mu_1} x^{\mu_2} \ldots x^{\mu_l}$ where $V_{\mu\mu_1\mu_2\ldots\mu_l}$ is traceless, symmetric in all of its indices except the first one, and it is zero when it's first index is symmetrized with any of the others. In particular, tracing the first index of b with any of the others gives zero.

It follows that each of the vector valued monomials listed above obeys the equations

$$\nabla.V = 0, \qquad \nabla^2 V = 0 \tag{506}$$

where the covariant derivatives are taken in the flat space R^{D-1}. The restriction of each of these vector valued monomials to the unit sphere yields a vector field tangent to the unit sphere (this is because the r component of these vector fields - proportional to the monomial with first index dotted with x^μ - vanishes). Let this vector field be denoted by V. It is easily verified that $\nabla.V = 0$ (where the covariant derivative is now taken on the unit sphere). We demonstrate in Appendix 3.7.5 that

$$\nabla^2 V = -[(D + l - 3)l - 1]V \tag{507}$$

where, in this equation, V is viewed as a vector field on the unit sphere and ∇ is the covariant derivative on the unit sphere.

Plugging the expansion (505) into (496) and setting the coefficient of every independent vector spherical harmonic to zero we obtain, at leading order in large D

$$\omega_l^v = \frac{-i(l-1)}{1+Q_0^2}. \tag{508}$$

This formula agrees with the formula for the spectrum of vector quasinormal modes presented in [1] in the limit $Q_0 \to 0$. Reinstating factors of r_0 we have

$$r_0 \omega_l^v = \frac{-i(l-1)}{1+Q_0^2} \quad (l = 1, 2, 3 \ldots.) \tag{509}$$

Note that all the velocity quasinormal modes are pure (negative) imaginary, and so represent a ring down that decays without oscillations. Vector harmonics with $l = 1$ are zero modes. These modes transform in the representation $(1, 1, 0, 0, \ldots 0)$ - i.e. the adjoint representation - of $SO(D-1)$ and have a simple physical interpretation. These zero modes turn on an infinitesimal spin on the for the black holes, i.e. begin to take one along the branch of the large D version of Kerr Newman black holes.

3.5.3 The spectrum of charge fluctuations

The spectrum of charge fluctuations is governed by the second of (402), which we repeat here for convenience

$$\frac{\nabla^2 Q}{\mathcal{K}} - u \cdot \nabla Q = Q \left(\frac{u \cdot \nabla \mathcal{K}}{\mathcal{K}} - u \cdot K \cdot u \right). \tag{510}$$

Plugging (484) into this equation we obtain the linearized equation

$$\left(\frac{\nabla^2}{D} - \partial_t \right) \delta Q = Q_0 \left(\partial_t^2 - \partial_t \left(\frac{\nabla^2}{D} + 1 \right) \right) \delta r. \tag{511}$$

Plugging in the expansion

$$\delta Q = \sum_{l,m} q_{lm} Y_{lm}(\theta) e^{-it\omega_l^Q} \quad , \tag{512}$$

and focusing on the coefficient of Y_{lm} for a particular value of l, the RHS of (511) is a source term which drives δQ at the frequency ω_l^r given by (522). A source of the form

$$\delta r = \sum_{l,m} a_{lm} Y_{lm}(\theta) e^{-i\omega_l^r t} \tag{513}$$

induces the response

$$\delta Q_f = \sum_{l,m} a_{lm} \frac{i\omega_l^r Q_0 (l - 1 - i\omega_l^r)}{l - i\omega_l^r} Y_{lm}(\theta) e^{-i\omega_l^r t} \quad . \tag{514}$$

The most general solution of (511) is given by a linear sum of the particular solution (514) and the most general solution to the homogeneous equation, i.e. to the equation (511) with the RHS set to zero. In order to determine the quasinormal frequencies we associated with Q oscillations we solve for the frequencies of these homogeneous modes. This is easily accomplished. Using (500) we find, at leading order in large D,

$$-l + i\omega_l^Q = 0, \tag{515}$$

which gives the QN frequency for the charge perturbations

$$\omega_l^Q = -il. \tag{516}$$

Reinstating factors of r_0 we have

$$r_0\omega_l^Q = -il. \tag{517}$$

As in the case of velocity fluctuations, the charge fluctuation quasinormal modes are pure negative imaginary, and so represent diffusive decay without oscillation. ω vanishes when $l = 0$. The corresponding zero mode is simply an infinitesimal uniform rescaling of Q_0.

3.5.4 A consistency check for shape fluctuations

The spectrum of shape fluctuations can be rederived starting from the equation (403), i.e.

$$(1-Q^2)\left[\frac{\nabla^2\mathcal{K}}{\mathcal{K}^2} - \frac{u\cdot\nabla\mathcal{K}}{\mathcal{K}}\right] = (1+Q^2)\left(\frac{u\cdot\nabla\mathcal{K}}{\mathcal{K}} - u\cdot K\cdot u\right). \tag{518}$$

The linearized equation is given by

$$(1-Q_0^2)\left(\partial_t - \frac{\nabla^2}{D}\right)\left(1+\frac{\nabla^2}{D}\right)\delta r = (1+Q_0^2)\left(\partial_t^2 - \partial_t\left(1+\frac{\nabla^2}{D}\right)\right)\delta r. \tag{519}$$

Now let's consider the perturbation in membrane shape function to be a particular mode, namely

$$f(t,\theta) = \sum_{l,m} a_{lm}Y_{lm}(\theta)e^{-i\omega_l^r t} \quad . \tag{520}$$

This turns (519) into an algebraic equation for a given mode

$$(\omega_l^r)^2 + \frac{2i\omega_l^r(l-1)}{1+Q_0^2} - \frac{1-Q_0^2}{1+Q_0^2}l(l-1) = 0, \tag{521}$$

which has roots

$$\omega_l^r = \frac{-i(l-1)\pm\sqrt{(l-1)(1-lQ_0^4)}}{1+Q_0^2}. \tag{522}$$

They exactly match with (502).

Recall that we have argued above that the divergence of (402) agrees with (403) for all configurations that preserve an SO$(D-p-2)$ isometry. In this subsection we have shown, however, the spectrum of shape fluctuations computed from the divergence of (402) agrees with the spectrum computed from (403) even though arbitrary spherical harmonics do not, in general, preserve a large isometry subgroup. The reason this had to work is as follows. In any spherical harmonic representation there exist special spherical harmonics that preserve a large isometry group. It follows from our general arguments above the two equations considered in this subsection must give the same spectrum of shape fluctuations for these special modes. However the equations analyzed in this subsection are geometrical, and in particular respect the full $\mathrm{SO}(D-1)$ rotational symmetry group of the background solution, and so generate the same spectrum of oscillations for all spherical harmonics in a given representation.

In summary the two equations had to give the same spectrum for *some* particular elements of the spherical harmonic representation. Rotational invariance then forces them to give the same spectrum for all spherical harmonics in the same representation, as we actually find.

3.6 Discussion

In this chapter we have presented a construction of a large class of solutions of the Einstein-Maxwell equations. Our solutions are in one to one correspondence with the solutions of the equations of a charged, nongravitational membrane propagating in flat space according to the dynamical equations (402).

We have used our membrane equations to generate a prediction for the spectrum of light quasinormal modes about Reissner-Nordstrom black holes in Einstein-Maxwell gravity. As a check on our results it would be useful to independently compute these quasinormal mode frequencies, perhaps using the gauge invariant formulation of [62].

All of the computations presented in this chapter have been performed at first nontrivial order in the expansion in $1/D$. It is of great interest to generalize the computations presented herein to the next order in this expansion. Apart from determining second order corrections to the membrane equations presented in this chapter such a computation would allow us to distinguish between different geometrical presentations of the first order equations (e.g the equation (403) and the divergence of (402)) (see the introduction for a discussion).

In this chapter we have derived equations of membrane dynamics assuming that our configuration preserves an $SO(D-p-2)$ isometry. As we have explained above, however, our final results are geometrical (in that they make no reference to the isometry algebra and treat all dimensions democratically). It is possible that the final geometrical equations are valid in more general situations, i.e. for configurations that preserve no isometry but perhaps obey some other weaker conditions [96]. It would be interesting to investigate this further.

In order to gain intuition for the membrane equations derived in this chapter, it would be useful to determine and study the properties of a class of simple solutions of these equations. In future work we will present a detailed study of stationary solutions to the membrane equations (402). As we have mentioned in the introduction, this allows us to make contact between the

[96] We thank A. Strominger for a question about this possibility.

membrane equations presented in this chapter and the membrane analysis of static and stationary black holes at large D presented in [49, 50].

It would also be interesting to follow the lead of [51–53] and attempt to investigate Gregory-Laflamme type instabilities using an appropriate extension of the framework presented in this chapter.

The solutions presented in this chapter rapidly approach empty flat space away from their event horizons. At every order in the expansion in $1/D$ the gauge field and metric simply vanishes far away from the membrane. Non perturbatively in $1/D$ (most likely at order $1/D^D$) we expect our membrane motions to excite a Maxwell and gravitational radiation field. As this radiation field is the means by which an external observer can actually observe the black hole dynamics described in this chapter, it is of great interest to find the formula that determines this field. We hope to return to this question in the near future.

On a related note, any extended object that consistently sources gravity and Maxwell radiation should possess a conserved charge current and stress tensor. It would be interesting to find all orders formulae (within the $1/D$ expansion) for the charge current and stress tensor of the membrane.

Once all these issues have been settled satisfactorily, it would of course be interesting to simulate complicated dynamical processes (e.g. black hole collisions) using our membrane equations, and compare our results with numerical simulations in $D = 4$. Such a comparison would throw light on the question of whether the beautiful structures that emerge in black hole dynamics at large D are also a useful starting point for a perturbative expansion for the dynamics of astrophysical black holes.

3.7 Appendices for Chapter 3

3.7.1 Reissner-Nordstrom solution in Kerr-Schild coordinates

The static Reissner-Nordstrom black hole solution is very familiar. This solution is most usually presented in Schwarzschild like coordinates. In these coordinates the spacetime manifestly Minkowskian at infinity. However the coordinates are singular at the black hole horizon. Let $\tilde{t}$ be the Schwarzschild time coordinate. The coordinate change

$$d\tilde{t} = dv - \frac{dr}{f(r)}, \tag{523}$$

recasts the solution as

$$
\begin{aligned}
ds^2 &= 2dvdr - f(r)dv^2 + r^2 d\Omega_{D-2}^2\,, \\
A &= \sqrt{2}Q\left(\frac{r_0}{r}\right)^{D-3} dv\,, \\
f(r) &= 1 - (1+Q^2 c_D)\left(\frac{r_0}{r}\right)^{D-3} + c_D Q^2\left(\frac{r_0}{r}\right)^{2(D-3)}\,, \\
c_D &= \frac{D-3}{D-2}.
\end{aligned}
\tag{524}
$$

In these so called Eddington-Finkelstein coordinates the advantages and disadvantages of the Schwarzschild coordinate system are reversed. The black hole metric is now smooth at the future event horizon. However in the limit $r \to \infty$ the spacetime metric $ds^2 = 2dvdr - dv^2 + r^2 d\Omega_{D-2}^2$ is not manifestly Minkowskian.

The further coordinate change to the 'Kerr-Schild' time coordinate t is specified by

$$dv = dt + dr. \tag{525}$$

It is easy to see that the Kerr-Schild time t agrees with the Schwarzschild time coordinate at large r, but effectively reduces to the Eddington-Finkelstein time coordinate at the first zero of $f(r)$ (when approached from large r), i.e. at the outer event horizon. For this reason one might anticipate that the black hole solution in Kerr-Schild coordinates is both manifestly Minkowskian at large r as well as manifestly smooth at the outer future event horizon. A glance at the explicit black hole solution, (404) is sufficient to convince oneself that this is indeed the case.

3.7.2 Relating the geometric form of the metric and gauge field with the answer found in explicit computation

In this appendix we shall present how the different structures and functions appearing in section 3.4 are related to the functions and data appearing in subsection 3.3.13 (the explicit computation with SO$(d+1)$ invariance).
As explained in subsection 3.3.3, for explicit computation we assumed the following metric for the flat space-time.

$$ds^2_{flat} = \eta_{MN} dx^M dx^N = \eta_{\mu\nu} dx^\mu dx^\nu + S^2 \Omega_{ij} d\theta^i d\theta^j\,,$$

where $\eta_{\mu\nu} dx^\mu dx^\nu = dw_a dw^a + dS^2$ is the metric in the auxiliary space of $\{x^\mu\}$ (see subsection 3.3.3) and Ω_{ij} is the metric of a d dimensional unit sphere ($\{\theta^i\}$ are the angular coordinates along the isometry directions). Now because of the isometry the geometric forms will have the following properties.

1. For any geometric vector $\mathfrak{V}_M$, the components along the $\{\theta^i\}$ directions will be zero (i.e., $\mathfrak{V}_{\theta^i} = 0$).

2. Similarly for any geometric tensor $\mathfrak{H}_{MN}$

$$\mathfrak{H}_{\mu\theta^i} = 0 \ \text{ and } \ \mathfrak{H}_{\theta^i\theta^j} \propto \Omega_{ij} \, .$$

3. As explained before, apart from n_μ and u_μ there is one more special vector in the auxiliary space: $Z_M dx^M = Z_\mu dx^\mu = \frac{dS}{S} - \left(\frac{n_S}{S}\right) n_\mu dx^\mu$. Using these Z one-form we can further decompose any geometric vector and tensor scalar, vector and tensor of the SO(p) isometry group in the $(p+3)$ dimensional auxiliary space.

Using these properties we can translate the results in the geometric form to the language of 'auxiliary space'. For most of the functions, the translation is straightforward and we present the dictionary in Table 3 and Table 4. In Table 3 we present how the three scalar, two vector and one tensor function appearing in the geometric form of the metric and gauge field ((473)) decompose into seven scalar, three vector and one tensor function appearing in equation ((438)). Then in table 4 we decompose the geometric data in terms of the non-geometric ones (with SO$(d+1)$ invariance) used for explicit computation.

However for for $\delta\phi$, i.e. the fluctuation in the radius of the d dimensional sphere, the translation rule becomes a bit more subtle to be presented in a table. For convenience we shall explain it separately in subsection 3.7.2.

Using the tables 3 and 4 and the argument presented in subsection 3.7.2 we could easily see that if we specialize the metric and the gauge field as presented in equations (473), (474), (475) and (476) to SO$(d+1)$ isometry (where $d = D - p - 3$) , they indeed reduce to the explicit solution presented in subsection 3.3.13 upto correction of $\mathcal{O}\left(\frac{1}{D}\right)^2$.

Relating $\delta\phi$ to the geometric forms Note that any geometric tensor will have some nonzero components along the isometry directions and also because of symmetry the components must be proportional to the metric of the d dimensional unit sphere. We can explicitly compute this proportionality factor which will be directly related to $\delta\phi$.
Consider the traceless tensor $H^{(T)}_{MN}$ appearing in equation (473) and suppose $H^{(T)}_{\theta^i\theta^j} = S^2 \mathfrak{H} \, \Omega_{ij}$

Explicit Computation	Geometric Form
$\left(\frac{1}{D}\right) S_{(VV)}$	$H^{(S)} + \mathcal{O}\left(\frac{1}{D}\right)^2$
$\left(\frac{1}{D}\right) S_{(AV)}$	$A^{(S)} + \mathcal{O}\left(\frac{1}{D}\right)^2$
$\left(\frac{1}{D}\right) S_{(Vz)}$	$Z^\mu H^{(V)}_\mu + \mathcal{O}\left(\frac{1}{D}\right)^2$
$\left(\frac{1}{D}\right) S_{(zz)}$	$Z^\mu Z^\nu H^{(T)}_{\mu\nu} + \mathcal{O}\left(\frac{1}{D}\right)^2$
$\left(\frac{1}{D}\right) S_{(Az)}$	$Z^\mu A^{(V)}_\mu + \mathcal{O}\left(\frac{1}{D}\right)^2$
$\left(\frac{1}{D}\right) S_{(Tr)}$	$\left(\frac{1}{p}\right) P^{\mu\nu} H^{(T)}_{\mu\nu} + H^{(Tr)} + \mathcal{O}\left(\frac{1}{D}\right)^2$
$\left(\frac{1}{D}\right) V^{(V)}_\mu$	$P^\nu_\mu H^{(V)}_\nu + \mathcal{O}\left(\frac{1}{D}\right)^2$
$\left(\frac{1}{D}\right) V^{(z)}_\mu$	$P^\alpha_\mu Z^\beta H^{(T)}_{\alpha\beta} + \mathcal{O}\left(\frac{1}{D}\right)^2$
$\left(\frac{1}{D}\right) V^{(A)}_\mu$	$P^\nu_\mu A^{(V)}_\nu + \mathcal{O}\left(\frac{1}{D}\right)^2$
$\left(\frac{1}{D}\right) T^{\mu\nu}$	$P^{\mu\alpha} P^{\nu\beta}\left[H^{(T)}_{\alpha\beta} - \left(\frac{\mathfrak{g}_{\alpha\beta}}{p}\right) P^{\nu_1\nu_2} H^{(T)}_{\nu_1\nu_2}\right] + \mathcal{O}\left(\frac{1}{D}\right)^2$

Table 3: Here we relate how the functions appearing in equation (438) are related to the geometric form of the metric and the gauge field as in equation (473)

where $\mathfrak{H}$ is some scalar function. Then it follows that

$$\begin{aligned} 0 = \eta^{MN} H^{(T)}_{MN} &= \eta^{\mu\nu} H^{(T)}_{\mu\nu} + \frac{\Omega^{ij}}{S^2} H^{(T)}_{\theta^i\theta^j} = \eta^{\mu\nu} H^{(T)}_{\mu\nu} + d \times \mathfrak{H} \\ \Rightarrow \ \mathfrak{H} &= -\frac{\eta^{\mu\nu} H^{(T)}_{\mu\nu}}{D-p-3}. \end{aligned} \tag{526}$$

Similarly consider the tensor $H^{(Tr)}\mathcal{P}_{MN}$ appearing in (473)). Since we know that $n_{\theta^i} = u_{\theta^i} = 0$, the nonzero components of this tensor along the isometry directions are simply given by

$$H^{(Tr)}\mathcal{P}_{\theta^i\theta^j} = H^{(Tr)} S^2 \Omega_{ij}\,. \tag{527}$$

From equation (526) and (527) and the definition of $\delta\phi$ (recall that the fluctuation in the radius of the d dimensional sphere $= S^2\delta\phi$), it follows that

$$\delta\phi = H^{(Tr)} + \mathfrak{H} = H^{(Tr)} - \left(\frac{1}{D-p-3}\right) \eta^{\mu\nu} H^{(T)}_{\mu\nu}\,. \tag{528}$$

The second term in the RHS of equation (528) is of $\mathcal{O}\left(\frac{1}{D}\right)^2$ since by construction $H^{(T)}_{\mu\nu}$ starts at $\mathcal{O}\left(\frac{1}{D}\right)$. Now from explicit computation we know that $\delta\phi$ is of $\mathcal{O}\left(\frac{1}{D}\right)^2$ (see equation (471)) . Then it immediately follows that $H^{(Tr)}$ also must start from terms of $\mathcal{O}\left(\frac{1}{D}\right)^2$.

We could explicitly compute the second term in RHS of (528) in terms of the functions

Geometric data	Data used in computation
$\mathfrak{S}_{(1)}$	$\left(\frac{S}{n_S}\right)\left[\frac{\mathfrak{s}^{(5)}}{Q} - \mathfrak{s}^{(1)} + \left(\frac{S}{n_S}\right)\mathfrak{s}^{(2)}\right]$
$\mathfrak{S}_{(2)}$	$\left(\frac{S}{n_S}\right)\left[\mathfrak{s}^{(1)} - \left(\frac{S}{n_S}\right)\mathfrak{s}^{(2)}\right]$
$Z_\mu \mathfrak{V}^\mu_{(1)}$	$\left(\frac{S}{n_S}\right)^2 \mathfrak{s}^{(6)} + \mathfrak{s}^{(1)} - \left(\frac{1-n_S^2}{S\times n_S}\right)^2$
$Z_\mu \mathfrak{V}^\mu_{(2)}$	$\left(\frac{S}{n_S}\right)^2 \mathfrak{s}^{(6)} - \mathfrak{s}^{(1)} - \left(\frac{1-n_S^2}{S\times n_S}\right)^2$
$Z_\mu Z_\nu \mathfrak{T}^{\mu\nu}$	$\mathfrak{s}^{(6)} - \left(\frac{n_S}{S}\right)\mathfrak{s}^{(2)}$
$P_{\mu\nu}\mathfrak{T}^{\mu\nu}$	$\mathfrak{s}^{(3)} - \mathfrak{s}^{(4)}$
$P_{\mu\nu}\mathfrak{V}^\nu_{(1)}$	$\left(\frac{S}{n_S}\right)^2\left[\mathfrak{v}^{(5)}_\mu + \left(\frac{n_S}{S}\right)\mathfrak{v}^{(2)}_\mu\right]$
$P_{\mu\nu}\mathfrak{V}^\nu_{(2)}$	$\left(\frac{S}{n_S}\right)^2\left[\mathfrak{v}^{(5)}_\mu - \left(\frac{n_S}{S}\right)\mathfrak{v}^{(2)}_\mu\right]$
$Z_\nu P_{\alpha\mu}\mathfrak{T}^{\alpha\nu}$	$\mathfrak{v}^{(5)}_\mu - \mathfrak{v}^{(3)}_\mu - \left(\frac{n_S}{S}\right)\mathfrak{v}^{(1)}_\mu$

Table 4: Decomposition of geometric data in the special case of SO$(d+1)$ symmetry in terms of the data in auxiliary space used for explicit computation

appearing in equation (438). Note that $\eta^{\mu\nu}$ could be expanded as

$$\eta^{\mu\nu} = n^\mu n^\nu - u^\mu u^\nu + \frac{S^2}{1-n_S^2} Z^\mu Z^\nu + P^{\mu\nu} + \mathcal{O}\left(\frac{1}{D}\right).$$

Using this expansion of $\eta^{\mu\nu}$ and the translation rules as given in table 3 we find

$$\left(\frac{1}{D-p-3}\right)\eta^{\mu\nu}H^{(T)}_{\mu\nu} = \frac{1}{D^2}\left[\left(\frac{1-n_S^2}{S^2}\right)S_{zz} + p\times S_{(Tr)}\right] + \mathcal{O}\left(\frac{1}{D}\right)^3. \tag{529}$$

In equation (529) we have used the fact that $H^{(Tr)}$ is of $\mathcal{O}\left(\frac{1}{D}\right)^2$ and by construction $u^\mu H^{(T)}_{\mu\nu} = n^\mu H^{(T)}_{\mu\nu} = 0$. Substituting equation (529) in equation (528) we found that

$$H^{(Tr)} = \frac{1}{D^2}\left[\delta\phi^{(2)} + \left(\frac{1-n_S^2}{S^2}\right)S_{zz} + p\times S_{(Tr)}\right] + \mathcal{O}\left(\frac{1}{D}\right)^3.$$

Now from equation (471) it directly follows that

$$H^{(Tr)} = \mathcal{O}\left(\frac{1}{D}\right)^3.$$

3.7.3 Relating equations of motion expressed in different forms

In this section we shall first state a set of algebraic identities that are true upto corrections of $\mathcal{O}\left(\frac{1}{D}\right)$. Using these identities we could easily show that the equations of motion as derived in subsection 3.3.10 ((451) and (453)) are equivalent to equations (479) and (480).

1. Identity-1:

$$\begin{aligned}
\tilde{V}_\perp \cdot K \cdot u = u \cdot K \cdot \tilde{V}_\perp &= [(u\cdot\partial)n]\cdot\tilde{V}_\perp \\
&= -[(u\cdot\partial)\tilde{V}_\perp]\cdot n \qquad \text{Since } n\cdot\tilde{V}_\perp = 0 \\
&= \left(\frac{n_S}{S}\right)[(u\cdot\partial)u]\cdot n + (u\cdot\partial)\left(\frac{n_S}{S}\right) + \mathcal{O}\left(\frac{1}{D}\right) \qquad \text{Since } u\cdot dS = \mathcal{O}\left(\frac{1}{D}\right) \\
&= -\left(\frac{n_S}{S}\right)(u\cdot K\cdot u) + (u\cdot\partial)\left(\frac{n_S}{S}\right) + \mathcal{O}\left(\frac{1}{D}\right) \qquad \text{Since } u\cdot n = 0.
\end{aligned} \tag{530}$$

Here $\tilde{V}_\perp = Z - \left(\frac{n_S}{S}\right)u = \frac{dS}{S} - \left(\frac{n_S}{S}\right)(n+u) = \left(\frac{S}{n_S}\right)(X-u)$.

2. Identity-2:

$$\begin{aligned}
\tilde{V}_\perp\cdot\partial u\cdot Z &= -\,\tilde{V}_\perp\cdot\partial Z\cdot u + \mathcal{O}\left(\frac{1}{D}\right) \qquad \text{since } u\cdot Z = \mathcal{O}\left(\frac{1}{D}\right) \\
&= \left(\frac{n_S}{S}\right)\tilde{V}_\perp\cdot K\cdot u + \mathcal{O}\left(\frac{1}{D}\right) \qquad \text{since } u\cdot dS = \mathcal{O}\left(\frac{1}{D}\right), \quad u\cdot n = 0.
\end{aligned} \tag{531}$$

3. Identity-3:

$$\begin{aligned}
u\cdot\partial u\cdot Z &= -\,u\cdot\partial Z\cdot u + \mathcal{O}\left(\frac{1}{D}\right) \qquad \text{since } u\cdot Z = \mathcal{O}\left(\frac{1}{D}\right) \\
&= \left(\frac{n_S}{S}\right)u\cdot K\cdot u + \mathcal{O}\left(\frac{1}{D}\right) \qquad \text{since } u\cdot dS = \mathcal{O}\left(\frac{1}{D}\right), \quad u\cdot n = 0.
\end{aligned} \tag{532}$$

4. Identity-4:

$$\begin{aligned}
&\left(\frac{S}{n_S}\right)\tilde{V}_\perp\cdot\partial(u-n)\cdot Z \\
=\;& \tilde{V}_\perp\cdot K\cdot u - \left(\frac{S}{n_S}\right)\tilde{V}_\perp\cdot K\cdot Z + \mathcal{O}\left(\frac{1}{D}\right) \qquad \text{Using (531)} \\
=\;& -\left(\frac{S}{n_S}\right)\tilde{V}_\perp\cdot K\cdot\tilde{V}_\perp + \mathcal{O}\left(\frac{1}{D}\right) \qquad \text{Since } Z = \tilde{V}_\perp + \frac{n_S}{S}u.
\end{aligned} \tag{533}$$

5. Identity-5:

$$\begin{aligned}
& \left[u\cdot\partial u - \left(\frac{S}{n_S}\right) Z\cdot\partial n\right]\cdot Z \\
= & \left(\frac{n_S}{S}\right) u\cdot K\cdot u - \left(\frac{S}{n_S}\right) Z\cdot K\cdot Z + \mathcal{O}\left(\frac{1}{D}\right) \qquad \text{Using (532)} \\
= & -\left(\frac{S}{n_S}\right)\tilde{V}_\perp\cdot K\cdot\tilde{V}_\perp - 2(Z\cdot K\cdot u) + 2\left(\frac{n_S}{S}\right) u\cdot K\cdot u + \mathcal{O}\left(\frac{1}{D}\right) \\
= & -\left(\frac{S}{n_S}\right)\tilde{V}_\perp\cdot K\cdot\tilde{V}_\perp - 2(\tilde{V}_\perp\cdot K\cdot u) + \mathcal{O}\left(\frac{1}{D}\right).
\end{aligned} \tag{534}$$

Using (533)and (534) we could very easily compute the projection of (479) along Z direction. It turns out to be the following,

$$\begin{aligned}
0 = & \left[(u-X)\cdot\partial O - Q^2(u\cdot\partial)u - Q^2(X\cdot K)\right]\cdot Z + \left(\frac{n_S}{S}\right)(1-Q^2)(X\cdot Z) \\
= & -(1-Q^2)\left[\left(\frac{S}{n_S}\right)\tilde{V}_\perp\cdot K\cdot\tilde{V}_\perp - Z\cdot Z\right] + 2Q^2(\tilde{V}_\perp\cdot K\cdot u) + \mathcal{O}\left(\frac{1}{D}\right) \\
= & -\left(\frac{S}{n_S}\right)(1-Q^2)\left[\tilde{V}_\perp\cdot K\cdot\tilde{V}_\perp - \left(\frac{n_S(1-n_S^2)}{S^3}\right)\right] + 2Q^2(\tilde{V}_\perp\cdot K\cdot u) + \mathcal{O}\left(\frac{1}{D}\right).
\end{aligned} \tag{535}$$

Equation (535) is simply equal to $\left[-\frac{S}{n_S}(1-Q^2)\right]$ times the first equation in (451). Second equation of (451) follows once we substitute (530) in equation (480).

3.7.4 Notation and translation

Through this chapter we have had occasion to work with functions (like ρ and Q) and one-form or vector fields (like u and $n = \frac{d\rho}{|d\rho|}$) that live in flat D dimensional space. We also often deal with functions and one-forms that live on the the membrane world volume. Through the chapter we use the dummy indices $M, N \ldots$ to denote coordinates in the embedding flat D dimensional spacetime, and the indices $A, B \ldots$ to denote coordinates on the membrane world volume. M, N indices run over D values, while A, B indices run over $D-1$ values.

In the computational part of this chapter we have assumed that our spacetimes and membrane world volumes both preserve an SO$(D-p-2)$ isometry group. It follows that the spacetime metric takes the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + e^\phi d\Omega_d^2, \tag{536}$$

where μ, ν run over $p+3$ values and $g_{\mu\nu}$ and ϕ are functions only of x^μ. We will often use the notation

$$e^{\phi} = S^2.$$

In a similar manner the metric on the membrane world volume takes the form

$$ds^2 = g_{ab}dx^a dx^b + e^{\phi} d\Omega_d^2, \tag{537}$$

where a, b run over $p+2$ values. As all spacetime vector and scalar fields also preserve SO$(d+1)$, for computational purposes it is sometimes useful to view these fields as living on the reduced $p+3$ dimensional manifold

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu},$$

(in the case of bulk fields) and

$$ds^2 = g_{ab}dx^a dx^b,$$

(in the case of fields that live on the membrane world volume). We use the symbols ∇_M and ∇_A to denote covariant derivatives on all of spacetime (or all of the membrane world volume) and $\tilde{\nabla}_{\mu}$ and $\tilde{\nabla}_a$ to denote fields on reduced spacetime (or the reduced membrane world volume).

Consider a vector field v^M defined on all of flat space. If we assume that v^M preserves SO(d) invariance, it is easily verified that

$$\nabla_M v^M = d\frac{v.\tilde{\nabla}S}{S} + \tilde{\nabla}_{\mu}v^{\mu}. \tag{538}$$

In a similar manner, if v^A is a vector field on the membrane then

$$\nabla_A v^A = d\frac{v.\tilde{\nabla}S}{S} + \tilde{\nabla}_a v^a. \tag{539}$$

If ψ is a scalar field in spacetime or on the membrane then it is easily verified that

$$\nabla^2\psi = d\frac{\tilde{\nabla}S.\tilde{\nabla}\psi}{S} + \tilde{\nabla}^2\psi, \tag{540}$$

(where dS is regarded as a one-form in either spacetime or on the membrane depending on the space on which ∇^2 is evaluated). Note that dS on the membrane world volume is simply dS in spacetime, projected onto the membrane world volume.

Finally if v is a vector field in either spacetime or the membrane world volume then

$$\nabla^2 v = d\left(\frac{\tilde{\nabla}S.\tilde{\nabla}v}{S} - dS\ v.\tilde{\nabla}S\right) + \tilde{\nabla}^2 v. \tag{541}$$

To end this section we note that the extrinsic curvature tensor for the membrane world volume takes the form

$$K_{AB} = (K_{\mu\nu}, \frac{n^S}{S}\Omega_{ij}),$$

where θ^i are angles on the unit d sphere and Ω_{ij} is a metric on this space.

3.7.5 Eigenvalues of the Laplacian for Vector Spherical Harmonics

In this Appendix we evaluate the eigenvalue of the Laplacian acting on the l^{th} vector spherical harmonic. This spherical harmonic was defined in terms of the restriction of a collection of vector valued monomials to the unit sphere in subsection 3.5.2.

Our strategy is to evaluate the Laplacian of V - viewed as a vector valued monomial in R^{D-1} - in spherical polar coordinates, and use the fact that this Laplacian vanishes (see subsection 3.5.2) to evaluate the Laplacian of the same vector field restricted to the unit sphere.

Consider any divergenceless vector field on R^{D-1} with vanishing radial component, i.e. $V_r = 0$. Using explicit expressions for the Christoffel symbols for flat space in polar coordinates we find

$$\begin{aligned}
\nabla_r V_r &= 0, \\
\nabla_r V_a &= \partial_r V_a - \frac{V_a}{r}, \\
\nabla_a V_r &= \frac{V_a}{r}, \\
\nabla_a V_b &= \hat{\nabla}_a V_b,
\end{aligned} \tag{542}$$

where $\hat{\nabla}$ denotes the covariant derivative taken on a unit sphere.
We will now use these results to evaluate $\nabla^2 V$ on R^{D-1} in spherical polar coordinates. The result of this computation depends on the free index in this equation. Let us first consider the case with the free index equal to r. In this case

$$\begin{aligned}
\nabla^2 V_r &= \nabla_r(\nabla_r V_r) + \frac{1}{r^2} g^{ab}\nabla_a\nabla_b V_r, \\
&= \frac{1}{r^2}\hat{\nabla}_a\hat{\nabla}^a V_r - \frac{1}{r^2}\hat{\nabla}_a V^a, \\
&= 0.
\end{aligned} \tag{543}$$

In other words the vanishing of the r component of $\nabla^2 V$ is just a triviality - it follows as an identity upon assuming $V_r = 0$ and $\nabla . V = 0$.

Let us now turn to the more interesting case of the free index being an angular direction on the unit sphere. In this case

$$
\begin{aligned}
\nabla^2 V_c =&\nabla_r(\nabla_r V_c) + \frac{1}{r^2} g^{ab}\nabla_a\nabla_b V_c, \\
=&\partial_r\left(\partial_r V_c - \frac{V_c}{r}\right) - \Gamma^a_{rc}\left(\partial_r V_a - \frac{V_a}{r}\right) + \frac{1}{r^2}\hat{\nabla}_a\hat{\nabla}^a V_c \\
&+ \Gamma^a_{ar}\left(\partial_r V_c - \frac{V_c}{r}\right) + \frac{1}{r^2}\Gamma^r_{ac}\frac{V^a}{r}, \\
=&\partial_r\left(\partial_r V_c - \frac{V_c}{r}\right) - \frac{1}{r}\left(\partial_r V_c - \frac{V_c}{r}\right) + \frac{1}{r^2}\hat{\nabla}_a\hat{\nabla}^a V_c \\
&+ \frac{D-2}{r}\left(\partial_r V_c - \frac{V_c}{r}\right) - \frac{V_c}{r^2} \quad .
\end{aligned} \tag{544}
$$

Let us now specialize to V_c is the vector field corresponding to the l^{th} vector spherical harmonic. In this case $V_c \propto r^{l+1}$. Using this fact and $\nabla^2 V_c = 0$ we get

$$
-\hat{\nabla}^2 V_c = (l(l+1) - l - l + (D-2)l - 1)V_c = [(D+l-3)l - 1]V_c. \tag{545}
$$

4 Chapter 4: Currents and Radiation from the large D Black Hole Membrane

4.1 Introduction

4.1.1 Review of Black hole - Membrane duality

The classical dynamics of black holes in asymptotically Minkowski spacetimes has recently been shown to simplify in a large number of dimensions D. Consider a violent dynamical process such as a collision between two black holes. The dynamics of this situation is complicated when the black holes first 'collide' . After a time of order $1/D$ after the 'merger' however, it turns out that the spacetime metric settles down into a configuration whose near horizon geometry is a union of overlapping patches, each of size $1/D$. The geometry of each patch closely resembles that of a Schwarzschild or Reissner Nordstrom black hole. The effective radius, boost velocity and charge of these patches varies on the event horizon over time and length scales of order unity. The subsequent evolution of the spacetime is governed by an effective dynamical system whose variables are the effective shape of the event horizon (one function) together with its local boost velocity field ($D-2$ functions) and charge density field (one function), a total of D functions of $D-1$ variables. The dynamical evolution of these variables is governed by a set of local membrane equations of motion. The underlying Einstein-Maxwell equations that govern the dynamics of this system uniquely determine the membrane equations in a power series expansion in $1/D$. At leading order in $1/D$ the membrane equations of motion take the form

$$
\begin{aligned}
\hat{\nabla} \cdot u &= 0 \\
p^{\nu}_{\mu} \left(u \cdot \hat{\nabla}\right) u_{\nu} &= p^{\nu}_{\mu} \left(\frac{\hat{\nabla}^2 u_{\nu} - (1-Q^2)\hat{\nabla}_{\nu} K + K \ (u^{\alpha} K_{\alpha\nu})}{K(1+Q^2)} \right), \\
u^{\nu} \hat{\nabla}_{\nu} (KQ) &= \hat{\nabla}^2 Q - KQ \ (u^{\alpha} K_{\alpha\beta} u^{\beta}),
\end{aligned}
\tag{546}
$$

(546)[97] [98] are a set of D equations for as many variables. It follows that (546) defines a well

[97] The equations (546) were first obtained in the papers [1, 63] building on the earlier work [42–48]. See also [49, 50, 64, 65] for the independent derivation of membrane equations in for the special case of stationary solutions. (546) had been generalized in [66] to include first correction in $1/D$ for the special case of uncharged black hole membranes. [51–53, 67, 68] have also independently derived the equations of membrane dynamics in the so called 'black brane' limit. At least for the case of uncharged black holes, the equations of [51–53, 67, 68] were demonstrated in [69] to be a special case (a special scaling limit) of the equation (546). See [70–76] for recent related work.

[98] The notation used in this equation goes as follows. Here we view the membrane as embedded in flat

posed initial value problem for membrane dynamics.

We have presented the membrane equations (546) at leading order in the expansion in $\frac{1}{D}$; as a consequence all terms in each of the equations (546) are of the same order in D, where orders of D are counted according to the rules spelt out in chapter 3. According to the rules of that chapter in particular, all divergences and Laplacians are of order D, while contractions of indices of the form $A_M B^M$ are of order unity. As an example of an application of this rule, $\nabla^2 u^M$ and $K = \nabla_A n^A$ are both taken to be of order $\mathcal{O}(D)$ while $(u^A K_{AB} u^B)$ is assigned order $\mathcal{O}(1)$ This rule applies irrespective of whether we are dealing with space-time indices or worldvolume indices. See chapter 3 for an explanation of the rational behind this rule.

Using the rule spelt out in the previous paragraph, it follows that the LHS of the first equation in (546) is of order D. Every term in the third equation in (546) is also of order D. However each term in the second equation of (546) is of order unity.

The membrane whose dynamics is described by (546) may be thought of in the following picturesque terms. The membrane consists of a bunch of 'particles' of density $u^0 = \gamma$ whose velocity is given by $\frac{u^i}{u^0}$. u^M is the 'density current' of these 'particles' and the first equation in (546) is a statement of the conservation of this density current. With this interpretation, the conservation of this density current is simply the statement that our fictional particles flow from one point to another but are never created or destroyed [99] The second equation in (546) may be regarded as a statement of Newton's laws for the constituent particles of the membrane. This equation asserts that the acceleration of any given membrane particle is governed by 'forces' (the RHS of the second equation in (546)) which depend on the trajectories of neighbouring particles. [100] The terms on the RHS of the second of (546) are reminiscent of the force terms that act on a regular fluid. The first term on the RHS of (546) captures the force of shear viscosity while the second term is analogous to a pressure force, with the role of the pressure played by K the

Minkowski space. Small Greek indices denotes the intrinsic coordinates along the membrane worldvolume. $\hat{\nabla}_\mu$ denotes the covariant derivative with respect to the intrinsic metric of the membrane, $g^{(ind,f)}_{\mu\nu}$. All raising and lowering of indices are also done using this intrinsic metric. $K_{\mu\nu}$ is the extrinsic curvature of the membrane , $K = K^\mu_\mu$ is the trace of the extrinsic curvature, $p_{\mu\nu}$ is the projector orthogonal to the velocity field

$$p_{\mu\nu} = g^{(ind,f)}_{\mu\nu} + u_\mu u_\nu,$$

u_μ is the velocity.

[99] As we will see below, the 'particles' in question will turn out to be the basic carriers of entropy of the membrane, and the 'particle density current' mentioned here is closely related to the membrane's entropy current. The conservation of entropy density holds only at first order; we will show below that the divergence of the entropy current is generically nonzero (but positive) at second order in the expansion in $1/D$. This means that the fictional 'particles' mentioned in the text above are created in dynamical flows at second and higher order in $1/D$.

[100] We have a $D-2$ parameter set of particles which execute a $D-2$ parameter set of particle flows. The $D-1$ dimensional membrane world volume is simply the congruence of these flow lines. Note that the extrinsic curvature of the membrane at any given point is completely determined by the shape of particle flow lines in the neighbourhood of that point.

trace of the extrinsic curvature of the membrane. This term drives flows that reduces gradients of K and works to iron out wrinkles in the membrane world volume that might otherwise have developed over the course of a dynamical flow. In some sense this term is responsible for stitching the independent particle world lines (or, more visually, world threads) into a smooth membrane surface.

The last equation in (546) asserts that our particles carry a separate independent 'charge' - with density proportional to KQ. This charge is carried along by our particles as they move. In addition it 'diffuses' between particles in the manner specified by the RHS of the third equation in (546). This charge density is, of course, closely related to the electromagnetic charge current of the membrane, a statement we will make precise in this chapter.

Let us re-emphasize the main point. If we wait for a time large compared to $1/D$ after a cataclysmal event, the equations that govern black hole dynamics reduce to the equations that govern the motion of a relativistic membrane that propagates in flat space. At first nontrivial order, the membrane may usefully be thought of as generated by the flow lines of a collection of 'particles' which interact with each other locally as they flow. The membrane equations (546) - which define a good initial value problem for the membrane shape and velocity field - are simply a rewriting of Einstein's equations for black hole dynamics at leading order in $1/D$ and in the appropriate regime.

4.1.2 Membrane coupling to radiation: qualitative discussion

In this chapter we refer to all degrees of freedom that vary on time and length scales of order unity (rather than, say, $1/D$) as slow. The collective coordinate membrane motions described above are one set of slow degrees of freedom in black hole spacetimes. A second simpler set of slow degrees of freedom are gravitons and photons that live far away from the black hole and have wavelengths of order unity or larger. It is natural to wonder how these two distinct classes of slow modes interact with each other. In this chapter we present a detailed analysis of the coupling of these two classes of slow modes. We demonstrate, in particular, that the coupling between membrane modes and light gravitons is of order $\frac{1}{D^{\frac{D}{2}}}$, and so is nonperturbatively small in the $1/D$ expansion.

As we explain in section 4.2 below the smallness of this coupling at large D may be understood as follows. The slow modes that describe the collective coordinate motions of membranes are localized to a region very near the the black hole horizon by a large potential barrier. The barrier is kinematical in origin and schematically takes the form of a repulsive potential $V(r) = \frac{D^2}{4r^2}$ in an effective one dimensional Schrodinger problem. In order to escape as radiation, a membrane mode which lives at the edge of the black hole of radius r_0 has to to tunnel through this barrier all the way out to $r \approx \frac{D}{2\omega}$ before it can start to propagate. The amplitude for this tunneling process is suppressed by the area under the potential curve, and is of order $e^{-\frac{D}{2}\ln\frac{D}{2\omega r_0}} \sim \left(\frac{2\omega r_o}{D}\right)^{\frac{D}{2}}$. When $r_0\omega$ is of order unity, this amplitude is nonperturbatively small in the $1/D$ expansion. It follows that membrane motions on time scale of order $1/r_0$ do not source radiation at any finite order in the $1/D$ expansion.

The discussion of the previous paragraph is reminiscent of Maldacena's argument for the decoupling of the near horizon geometry of a D3 brane from the external bulk in the context of the AdS/CFT correspondence [77]. Indeed at energies of order unity, the limit $D \to \infty$ is effectively a decoupling limit for the near horizon region of the Schwarzschild and Reissner Nordstrom black holes, analogous in many respects to the Maldacena decoupling limit in which energies are held fixed as α' is taken to zero.

We would like to emphasize that the decoupling between membrane degrees of freedom and asymptotic infinity is accurate only for the classical theory of gravity and appears to fail quantum mechanically, even semiclassically. The reason for this is simply that near horizon modes with $\omega \sim \frac{D}{r_0}$ do not decouple from infinity. As we will review below, however, the Hawking temperature of a black hole of radius r_0 scales like $\frac{D}{r_0}$ at large D. It follows that the Hawking radiation emitted by a black hole at large D does not decouple from infinity. This observation suggests that it is misguided to hope that there exists a quantum microscopic theory of the large D membrane described in this chapter. Such a theory - which might have been hoped to stand in the same relation to the membrane equations (546) as $\mathcal{N} = 4$ Yang Mills theory does to the hydrodynamics of [55, 58] - appears never to decouple from asymptotic infinity. In other words the analysis of this chapter should be viewed purely in terms of the classical equations of gravity and not as the first step in a programme to quantize gravity at large D.

4.1.3 Membrane coupling to radiation: quantitative discussion

Although membrane degrees of freedom couple very weakly to external gravitons and photons at large D, they do couple to these modes at any finite D no matter how large. In other words membrane motions source gravitational and electromagnetic radiation. One of the principle accomplishments of this chapter is the derivation of a formula for the radiation sourced by any given membrane motion.

In order to obtain this formula we first note that the explicit $\frac{1}{D}$ expansion of spacetime solutions dual to membrane motions (see [1, 63, 66]) is valid only at points whose distance from the event horizon, S, obeys the inequality $S \ll r_0$ (here r_0 is the local black hole radius). [101] When, on the other hand, $S \gg \frac{r_0}{D}$ the solution reduces to a small fluctuation about flat space.

[101]More precisely $r_0 = \frac{D}{K}$ where K is the trace of the extrinsic curvature of the membrane surface. We use the notation of the previous chapter through this chapter. Recall that K is of order D so r_0 is of order unity.

In this region the solution is well approximated by a solution of the Einstein Maxwell equations linearized about flat space. Notice that the domains of validity of these two approximations overlap: the $1/D$ expansion of [1, 63, 66] and linearization are both valid approximations in the overlap regime[102]

$$\frac{r_0}{D} \ll S \ll r_0. \tag{547}$$

In the previous subsection we have explained that the radiation field first begins to propagate at distances S of order $\frac{D}{\omega}$ away from the membrane. These distances lie well outside the regime of the $1/D$ expansion of [1, 63, 66]. However the radiation fields are extremely small, and so are well described by the linearized Einstein Maxwell equations. In order to obtain the radiation field due to a given membrane motion, all we need to do is to identify the effective linearized solution that the spacetimes of [1, 63, 66] reduce to in the overlap region (547) and then continue this linearized solution to infinity.

The implementation of this programme is, however, complicated by an important detail. In order to explain this point we first pause to provide a qualitative description of space of linearized solutions to the Einstein Maxwell equations away from the membrane , i.e. at distances $S \gg \frac{r_0}{D}$ to the exterior of the membrane. The linearized solutions in this region turn out to be a superposition of two classes of modes; modes whose integrated flux decays towards infinity (we call these the decaying modes) and modes whose integrated flux grow towards infinity (we call these the growing modes). These can be understood as the decaying and growing modes of the effective Schrodinger problem under the potential barrier $V(r) = \frac{D^2}{4r^2}$ mentioned in the previous subsection. As we show in section 4.2 below, decaying modes of the effective Schrodinger problem start out at order unity very near the membrane and decay rapidly upon progressing outwards. On the other hand growing modes start out at order $1/D^D$ near the membrane but grow equally rapidly away from the membrane. The growing modes catch up in magnitude with the decaying mode at a distances of order $\frac{D}{2\omega}$ away from the membrane. This is also precisely the point beyond which both the modes emerge out from under the effective potential barrier. At larger distances the modes cease to grow or decay but oscillate, propagating in form of radiation fields. The integrated flux of both modes stays constant as r is further increased.

As mentioned above, the $\frac{1}{D}$ expansion of [1, 63, 66] is valid simultaneously with the linearized approximation only in the region (547). The decaying solution is sizeable in this region and is perfectly captured by the $\frac{1}{D}$ expansion. On the other hand the growing mode is of order $\frac{1}{D^D}$ in this region. It is thus nonperturbatively small and so is completely invisible to the $\frac{1}{D}$ expansion of [1, 63]. In other words the solutions of [1, 63] capture only half of the information of the linearized solution in the overlap region (547). In order to complete our specification of the

[102]We explicitly verify below that the metric and gauge field presented in chapter 3 is a solution of the linearized Einstein Maxwell equations in this regime.

linearized solution and to extend it into the radiation region we need more information. The extra data comes from the physical expectation that radiation from the membrane motion is necessarily outgoing at infinity. The absence of ingoing radiation at infinity provides the second piece of data needed to continue the linearized solutions to large S.

We now explain how the membrane solutions may actually be continued to infinity in a practically useful manner. In this chapter we demonstrate that the decaying part of a linearized solution of the Einstein- Maxwell equations uniquely defines a stress tensor and a charge current on the membrane at large D. [103]. The sources thus defined may be thought of as giving rise to (the decaying part of) the linearized solution we started with. More precisely the convolution of a Greens function against this source produces a response whose decaying part agrees with the solution we started out with. [104] The absence of ingoing radiation at infinity dictates that we use the retarded Greens function. This convolution produces the correct solution in and outside the overlap region (547). In the overlap region the convolution produces the nonperturbatively small growing part of the solution in addition to the decaying piece obtained from the solutions of [1, 63]. In the region $r \gg \frac{D}{\omega}$ the convolution produces the radiation field that we wished to calculate.

In sections 4.4 and 4.5 below - the technical heart of this chapter - we explain in detail how the map between decaying solutions of the Einstein-Maxwell system and a stress tensor and charge current on the membrane is constructed. Though the derivation takes a lot of work the final prescription is very simple. The charge current J_B is given by

$$J_B = J_B^{(out)} - J_B^{(in)}.$$

Here

$$J_B^{(out)} = n^A F_{AB}^{out}, \tag{548}$$

where F_{AB}^{out} is the field strength of the decaying part of external solution that was given to us, evaluated on the membrane, and n^A is the outward pointing unit normal to the membrane. Note that $n^B J_B^{(out)} = 0$. It follows that this current may also be viewed as the current $J_\mu^{(out)}$ that lives on the world volume of the membrane. [105] In a similar manner the current $J_B^{(in)}$ turns out to obey $n^B J_B^{(in)} = 0$ and can also be thought of as a current $J_\mu^{(in)}$ that lives on the membrane world volume. It turns out

$$J_\mu^{(in)} = -\frac{\delta}{\delta A_\mu} S_{ctrA}, \tag{549}$$

[103]The existence of such a map is plausible from a counting perspective; both sides of the map depend on a single piece of data on a slice (think the membrane) of spacetime.

[104]This convolution procedure also produces a growing mode. The detailed magnitude of that growing mode - which is always of order $1/D^D$ - depends on the Greens function we use.

[105]See section (4.3) for the precise relationship between $J_B^{(out)}$ and $J_\mu^{(out)}$.

where, to first order in the expansion in $1/D$,

$$S_{ctrA} = -\frac{1}{4}\int \frac{F_{\mu\nu}F^{\mu\nu}}{\sqrt{\mathcal{R}}}, \tag{550}$$

where $\mathcal{R}$ is the Ricci scalar on the world volume of the membrane and $F_{\mu\nu}$ is the field strength of the linearized external solution restricted to the membrane. [106]

In a similar manner the stress tensor T_{AB} on the membrane is given by

$$T_{AB} = T^{(out)}_{AB} - T^{(in)}_{AB}. \tag{551}$$

Here

$$8\pi T^{(out)}_{AB} = \mathcal{K}^{(out)}_{AB} - \mathcal{K}^{(out)}\mathfrak{p}^{(out)}_{AB}\ , \tag{552}$$

is the Brown York stress tensor of the external solution evaluated on the membrane surface. Here $\mathcal{K}^{(out)}_{AB}$ and $\mathfrak{p}^{(out)}_{AB}$ are the extrinsic curvature and the projector on the membrane world volume viewed as a submanifold of the bulk whose metric is that of Minkowski space perturbed by the decaying external solution. As above, $T^{(out)}_{AB}$ and $T^{(in)}_{AB}$ are both tangential to the membrane world volume and so can equally well be regarded as stress tensors, $T^{(out)}_{\mu\nu}$ and $T^{(in)}_{\mu\nu}$ that live on the membrane world volume. [107] It turns out that

$$\sqrt{-g^{(ind)}}T^{(in)}_{\mu\nu} = -\frac{\delta}{\delta g^{\mu\nu}_{(ind)}}S_{(in)}, \tag{553}$$

where

$$S_{(in)} = -\frac{1}{8\pi}\int \sqrt{-g^{(ind)}}\left[\sqrt{\mathcal{R}} + \frac{1}{2}\left(\frac{\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}}{\mathcal{R}^{\frac{3}{2}}_{(in)}}\right) + \mathcal{O}\left(\frac{1}{D}\right)\right], \tag{554}$$

$\mathcal{R}$, $\mathcal{R}_{\mu\nu}$ and $g^{(ind)}_{\mu\nu}$ are respectively the intrinsic Ricci scalar , intrinsic Ricci tensor and the intrinsic metric of the membrane.

The stress tensors (553) and (552) are both evaluated on the membrane world volume using the prescribed external solution. Recall the external solution is flat space plus the decaying linearized solution of Einstein's equations, which we assume is given to us. In the particular case of interest to this chapter, this decaying linearized solution is given by matching with the metric presented in [1, 63].

[106] At leading order in the large D expansion the Gauss Codazzi equations may be used to show that $\sqrt{\mathcal{R}} = K$.

[107] Once again see section 4.3 for the precise relationship between the spacetime and world volume stress tensors.

4.1.4 Explicit formula for the Membrane Stress Tensor and Charge Current

It is not difficult to implement the procedure described in the previous subsection on the solutions of [1, 63] and so obtain a formula for the membrane stress tensor and charge current. We find

$$
\begin{aligned}
T_{\mu\nu} = &\left(\frac{1}{8\pi}\right)\Bigg[\left(\frac{K}{2}\right)(1+Q^2)u_\mu u_\nu + \left(\frac{1-Q^2}{2}\right)K_{\mu\nu} - \left(\frac{\hat{\nabla}_\mu u_\nu + \hat{\nabla}_\nu u_\mu}{2}\right) \\
& - \left(\frac{KQ^2}{2D} + \frac{2Q\hat{\nabla}^2 Q}{K} + Q^2 u^\alpha u^\beta K_{\alpha\beta}\right)u_\mu u_\nu - (u_\mu \mathcal{V}_\nu + u_\nu \mathcal{V}_\mu) \\
& - \left[\left(\frac{1+Q^2}{2}\right)\left(u^\alpha u^\beta K_{\alpha\beta}\right) + \left(\frac{1-Q^2}{2}\right)\left(\frac{K}{D}\right)\right] g^{(ind,f)}_{\mu\nu}\Bigg] \\
& + \mathcal{O}\left(\frac{1}{D}\right),
\end{aligned}
\tag{555}
$$

$$
\begin{aligned}
J^\mu = &\left(\frac{Q}{2\sqrt{2\pi}}\right)\left[Ku^\mu - \left(\frac{p^{\nu\mu}\hat{\nabla}_\nu Q}{Q}\right) - (u\cdot\hat{\nabla})u^\mu - \left(\frac{\hat{\nabla}^2 u^\mu}{K}\right) + K^{\alpha\mu}u_\alpha\right] \\
& + \mathcal{Q}\; u^\mu + \mathcal{O}\left(\frac{1}{D}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{V}_\mu = &\; Q\;\hat{\nabla}_\mu Q + Q^2(u^\alpha K_{\alpha\mu}) + \left(\frac{2Q^4 - Q^2 - 1}{2}\right)\left(\frac{\hat{\nabla}_\mu K}{K}\right) \\
& - \left(\frac{Q^2 + 2Q^4}{2}\right)(u\cdot\hat{\nabla})u_\mu + \left(\frac{1+Q^2}{K}\right)\hat{\nabla}^2 u_\mu, \\
\mathcal{Q} = &\left(\frac{Q}{2\sqrt{2\pi}}\right)\left[\frac{\hat{\nabla}^2 K}{K^2} - \frac{2K}{D} - \frac{(u\cdot\hat{\nabla})K}{K} - \left(\frac{2\hat{\nabla}^2 Q + K(u\cdot\hat{\nabla})Q}{Q\;K}\right) + \left(u^\alpha u^\beta K_{\alpha\beta}\right)\right].
\end{aligned}
\tag{556}
$$

Here $g^{(ind,f)}_{\mu\nu}$ denotes the induced metric on the membrane as embedded in flat space and $\hat{\nabla}_\mu$ denotes the covariant derivative with respect to $g^{(ind,f)}_{\mu\nu}$. Extrinsic curvature of the membrane is denoted by $K_{\mu\nu}$ and K is the trace of the extrinsic curvature.

According to the rules for D counting explained earlier in this introduction, the first term on the RHS for the expressions for stress tensor and charge currents presented in (555) are each of order D. All other terms in both expressions are of order unity. We emphasize, in particular, that the membrane stress tensor and charge current are not parametrically small in the large D limit. The radiation sourced by these currents is nonetheless nonperturbatively small in the appropriate regimes, for the kinematical reasons - the heavily damped grey body factor - described earlier in this introduction.

Several terms in the stress tensor and charge current above have familiar hydrodynamical interpretations. In particular, relativistic fluids propagating on fixed background manifolds always have a contribution to their stress tensor proportional to $-\eta\sigma_{MN}$ where σ_{MN} is the symmetrized derivative of the velocity field projected orthogonal to the velocity and η is called the shear viscosity of the fluid. An inspection of the first line of (555) reveals that our membrane stress tensor also has such a contribution with effective value of $\eta = \frac{1}{16\pi}$. Below we will see that the entropy density of the membrane is given, to leading order, by $\frac{1}{4}$. It follows that the ratio of shear viscosity to entropy density for our membrane equals $\frac{1}{4\pi}$, in agreement with [78].

Keeping only the leading terms (i.e the terms that scale like D) in (555) we find the much simplified expressions

$$\begin{aligned} T_{\mu\nu} =& \left(\frac{K}{16\pi}\right)(1+Q^2)u_\mu u_\nu, \\ J_\mu =& \frac{1}{2\sqrt{2\pi}}\left(QKu_\mu\right). \end{aligned} \tag{557}$$

Note that the leading order stress tensor and charge current is simply that of a collection of pressure free 'dust' particles. Note, in particular, that the leading order stress tensor lacks a surface tension term (a term proportional to $\Pi_{\mu\nu}$). In this respect the stress tensor of the large D black hole membrane differs significantly from more familiar membranes like soap bubbles or $D2$ branes.

4.1.5 Equations of motion from conservation

As the fractional loss of energy to radiation is non perturbative in the large D limit, it follows that membrane energy, momentum and charge are conserved at each order in the $\frac{1}{D}$ expansion. In fact a stronger result must hold; in order for the formula for gravitational and electromagnetic radiation from the membrane to be gauge invariant, the membrane stress tensor and charge current must be conserved currents. Indeed the conservation of the membrane stress tensor and charge current turn out to be an alternate - and conceptually very satisfying - way of restating the membrane equations of motion (546). The fact that the membrane equations (546) are simply statements of conservation of an appropriate membrane stress tensor and charge current emphasizes that our membrane equations are hydrodynamical in nature.

We have explained above that the expressions for the stress tensor and charge current (555) each have one term of order D and several terms of order unity. The reader may at first suppose that only the leading order terms (those of order D) are needed to obtain the leading order membrane equations of motion via conservation. This is indeed the case for the first equation (546). The divergence of the leading order stress tensor a term of order D^2. This term is proportional to $Ku^\mu \nabla.u$. It follows that the term in $\partial_\nu T^{\nu\mu}$ proportional to u^μ indeed receives its leading contribution from the order D part of the stress tensor; the condition that this term vanish is simply the first equation of (546)

Let us turn our attention, however, to the projection of $\partial_\nu T^{\nu\mu}$ orthogonal to u_μ. According to the rules of large D counting summarized earlier in this introduction, this projected expression is of order D rather than of order D^2. At leading order (order D) this expression receives contributions both from the order D as well as the order unity contributions to the stress tensor (recall that the divergence of a tensor or vector of order unity is generically of order D). The order D piece of $T^{\mu\nu}$, (557), yields the LHS of the second equation in (546); the RHS of that equation is obtained from the divergence of the order unity parts of the stress tensor (555). A similar statement is true of the relationship between the conservation of the charge current and the third equation in (546).

In summary, in order to obtain the first equation in (546) we needed to know only the leading order stress tensor (557). In order to obtain the second and third equations of (546), however, we need to know the subleading terms in (555) as well.

4.1.6 Entropy Current

We have, so far, focused our attention on the conserved currents that live on the membrane. A key fact about black holes, however, is that that they carry entropy in addition to charge and energy. While charge and energy obey the first law of thermodynamics, and so are conserved, entropy obeys the second law and so is a non decreasing function of time.

The entropy carried by a black hole is mirrored in the fact that the membrane carries an entropy current. In this chapter we define this current and demonstrate that it obeys a local version of the second law of thermodynamics, i.e.

$$\hat{\nabla}_\mu J^\mu_S \geq 0.$$

Our construction of the membrane entropy current proceeds in a manner analogous to the construction of [79]. The current is constructed by pulling the area form on the event horizon back onto the membrane. A local form of the Hawking area increase theorem then ensures that the divergence of this entropy current is point wise non negative for every membrane motion. At first leading and subleading order in the $1/D$ expansion we find the extremely simple result

$$J^M_S = \frac{u^M}{4}, \tag{558}$$

(see (771) for the correction to this equation at second subleading order in the special case of uncharged fluids). By explicit use of equation 1.5 of [66] at leading nontrivial order in $\frac{1}{D}$ we find

$$\partial_M J^M_S = \frac{1}{8K} \sigma_{AB} \sigma^{AB}, \tag{559}$$

where

$$\sigma_{AB} = \left(\partial_M u_N + \partial_N u_M \right) P^M_A P^N_B. \tag{560}$$

Note in particular that entropy production vanishes at leading order if and only if the fluid velocity flow is shear free. As the flow is always also divergence free, it follows that every time independent (i.e. stationary) velocity vector field is proportional to a killing vector on the membrane world volume [80]. This observation may be used as the first step in a systematic classification of stationary solutions of the membrane equations, a topic we hope to return to in the near future.

4.1.7 Radiation from small fluctuations

In the section 4.8 to this chapter we develop the general theory of radiation for the Maxwell and Einstein equations (877) coupled to sources after linearization. In that section we work in a particular Lorentz frame, expand all modes in spherical harmonics and present very explicit radiation formulae. As an application of these formulae, in the main text we evaluate the radiation that results from a general linearized fluctuation about a spherical membrane. It follows from the formulae of that section that energy lost to radiation per unit time is smaller by a factor of $1/D^D$ when compared to the membrane energy stored in the fluctuation, providing a clear demonstration of the smallness of radiation.

4.1.8 Organization

This chapter is organized as follows. In section 4.2 we review the properties of retarded Greens functions in arbitrary dimensions with a special emphasis on the large D limit. In section 4.3 we review the structure of currents and stress tensors localized on a codimension one membrane. Sections 4.4 and 4.5 are the technical heart of this chapter. In these sections we construct a membrane charge current and stress tensor dual to any decaying linearized solution of the Einstein Maxwell equations in the exterior neighbourhood of the membrane world volume. In section 4.6 we apply the general formalism of the previous two sections to the special case of the membrane spacetimes of chapter 3, and find the stress tensor and charge current that lives on the membrane dual to large D black holes at leading order in $\frac{1}{D}$. In section 4.7 we define an entropy current on the membrane and demonstrate that its divergence is point wise non negative. In section 4.8 we proceed to review and develop the general theory of linearized radiation from localized sources for the Einstein Maxwell equations in an arbitrary number of dimensions. We then proceed, in section 4.9, to use these formulae to determine the radiation sourced by small fluctuations about the spherical membrane solution. Finally in section 4.10 we present a discussion of our results. This chapter also includes several appendices in which we present details of algebraically intensive computations.

4.2 Review of background material: Greens functions in general dimensions

In this section we review elementary background material on Greens functions in arbitrary dimensions, with a focus on the large D limit. In the rest of this chapter we will use the results of this subsection for qualitative as well as quantitative purposes. The key qualitative results from this subsection that will be of importance to us below are

- In the large D limit distinct Greens functions (e.g. retarded and Feynman Greens functions) differ from each other only at order $1/D^{\frac{D}{2}}$ at spatial distances and time frequencies of order unity (see subsection 4.2.2 below).
- The fractional energy loss per unit time into gravitational radiation, from a stress tensor that varies over distance and time scales of order unity, is of order $1/D^D$.

At the quantitative level, in section 4.8 we use the results of this section to derive detailed formulae for the electromagnetic and linearized gravitational radiation from arbitrary sources in general dimensions, once again with a focus on the large D limit.

4.2.1 Greens function in frequency space

Consider the retarded Greens function $G(x_\mu, x'_\mu)$ defined by the equation

$$-\Box\, G(x - x') = \delta^D(x - x'), \tag{561}$$

together with the boundary condition that G vanishes if x lies outside the future lightcone of x'. In (561) the d'Alembertian [108] is taken is taken w.r.t the coordinate x. G may be thought of as the causal response at the point x to a unit normalized delta function source at x'.

Although the equation (561) is Lorentz invariant, our Greens function cannot be thought of as a function only of x^2 (this is a consequence of retarded boundary conditions). In order to solve for the Greens function (and to understand its properties) we found it most convenient to sacrifice manifest Lorentz invariance. We choose a particular rest frame and so a particular time coordinate. In this section we further locate the source point x' of our Greens function at the

[108]Throughout this chapter we employ the mostly positive sign convention.

origin of spatial coordinates and Fourier transform w.r.t. time

$$G(\omega, \vec{r}) = \int G(t, \vec{r}) e^{i\omega t} dt. \tag{562}$$

It follows from (561) that $G(\omega, \vec{r})$ obeys the equation

$$-\left(\omega^2 + \vec{\nabla}^2\right) G(\omega, \vec{r}) = \delta^{D-1}(\vec{r}). \tag{563}$$

As $G(\omega, \vec{r})$ is spherically symmetric it is convenient to work in polar coordinates, i.e. in coordinates in which the Minkowskian metric is given by

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{D-2}^2,$$

(563) simplifies to

$$\omega^2 G(\omega, r) + \frac{1}{r^{D-2}} \partial_r \left(r^{D-2} \partial_r G(\omega, r)\right) = -\delta^{D-1}(\vec{r}). \tag{564}$$

The boundary conditions on $G(r, t)$ require $G(\omega, r)$ to be purely outgoing (i.e. $\propto e^{i\omega r}$) at infinity. The unique solution to (564) subject to these boundary conditions is

$$G(\omega, r) = \frac{i}{4} \left(\frac{\omega}{2\pi r}\right)^{\frac{D-3}{2}} H^{(1)}_{\frac{D-3}{2}}(\omega r). \tag{565}$$

Here $H^{(1)}_n(x)$ is the n^{th} Hankel function of first kind, whose small and large argument asymptotics are given by

$$\begin{aligned} H^{(1)}_n(x) &\approx \quad -i\left(\frac{2}{x}\right)^n \frac{1}{\pi}\Gamma(n)\left(1 + \frac{x^2}{4(n-1)} + \mathcal{O}\left(x^4/n^2\right)\right) \quad \text{for} \quad x^2 \ll n, \\ H^{(1)}_n(x) &\approx \quad (1-i)e^{-\frac{in\pi}{2}} e^{ix} \frac{1}{\sqrt{\pi x}} \left(1 + i\frac{4n^2-1}{8x} + \mathcal{O}\left(n^4/x^2\right)\right) \quad \text{for} \quad x \gg n^2. \end{aligned} \tag{566}$$

Using (566) it follows that our Greens function is given by

$$\begin{aligned} &G(\omega, r) \approx \frac{1}{(D-3)\Omega_{D-2}} \frac{1}{r^{D-3}} \left(1 + \frac{\omega^2 r^2}{2(D-5)} + \mathcal{O}(\omega^4 r^4/D^2)\right) \qquad \text{for } r^2\omega^2 \ll D, \\ &G(\omega, r) \approx -(2i)^{-\frac{D}{2}} \left(\frac{\omega}{\pi r}\right)^{\frac{D-2}{2}} \frac{e^{i\omega r}}{\omega} \left(1 + i\frac{(D-2)(D-4)}{8\omega r} + \mathcal{O}(D^4/r^2\omega^2)\right) \\ &\text{for } r\omega \gg D^2. \end{aligned} \tag{567}$$

Lightcone structure of the retarded Greens function In the previous subsubsection we presented an exact result for the retarded Greens function as a function of ω and r. In

Appendix 4.11.5 we evaluate the Fourier transform of the expressions of the previous subsection and obtain a formula for the retarded Greens function directly in position space. In this brief subsection we simply report the final results of Appendix 4.11.5.

When D is even we find

$$G(x,x') = \frac{\theta(X^0)}{2}\left(\frac{1}{\pi}\right)^{\frac{D-2}{2}} \delta^{\left(\frac{D-4}{2}\right)}\left(-X_M X^M\right), \tag{568}$$

where

$$X^M = x^M - (x')^M, \quad \delta^n(X) = \partial_X^n \delta(X).$$

When D is odd, on the other hand we find

$$G(r,t) = \frac{\Omega_{D-3}}{(2\pi)^{D-4}}(\partial_M \partial^M)^{\frac{D-3}{2}}\left(\frac{\theta(t-r)}{\sqrt{-x_M x^M}}\right), \tag{569}$$

where Ω_n is the volume of the unit n sphere

$$\Omega_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}. \tag{570}$$

In either case the Greens function is given by linear sums of finite numbers of derivatives acting on expressions that vanish outside the future lightcone; it follows that these Greens functions never propagate signals faster than light. [109]

Although the expressions (568) and (569) are exact, they are not particularly well suited for taking the large D and obscure various features of the Greens functions in this limit. In the rest of this chapter we will revert to working with the non manifestly Lorentz invariant but highly explicit representation Greens functions (565). We will now proceed to estimate the expression (565) in the large D limit; we find that the large D limit is smooth and can be taken without differentiating between odd and even D.

4.2.2 Large D expansion through WKB

In this section we will use the WKB approximation to determine the large D limit of the retarded Greens function. The main conclusions of this subsection are

- Upto a scaling (see (571)) the scalar Greens function is given by the solution to the one dimensional Schrodinger equation listed in (572)

[109]Note, however, that the number of derivatives that appears in the expression for the Greens functions increases without bound in the large D limit. This allows naive large D approximations of the Greens function to mimic apparently acausal behaviour in some situations. When used correctly, however, the Greens function is causal in every D.

- In the large D limit the potential in this Schrodinger equation exceeds the energy when $\omega r < 2D$ and is less than the energy when $\omega r > 2D$. The wave function that yields the Greens function describes a process of tunneling through a wide potential barrier. The exponential tunneling suppression ensures that the oscillating solution that emerges when $\omega r > 2D$ is very small. This explains the smallness of radiation at large D.
- All Greens functions (e.g. retarded, advanced, Feynman) are all essentially identical for $\omega r \ll 2D$. In particular when ωr is of order unity, the differences between different Greens functions are of order $1/D^D$.

In the rest of this subsection we will explain these points in some more detail relegating detailed derivations to appendices.

The transformation

$$G(\omega, r) = \frac{1}{r^{\frac{D-2}{2}}}\psi(\omega, r), \tag{571}$$

recasts the equation (564) into

$$-\partial_r^2\psi(\omega, r) + \frac{(D-2)(D-4)}{4r^2}\psi(\omega, r) = \omega^2\psi(\omega, r), \tag{572}$$

[110] i.e. a one dimensional Schrodinger equation with potential V and energy E given by

$$V(r) = \frac{D^{*2}}{4r^2}, \quad E = \omega^2 \quad \text{where } D^* = \sqrt{(D-4)(D-2)} \approx D - 3 + \mathcal{O}(1/D).$$

This potential divides the r axis into the classically allowed and disallowed regions

$$2\omega r > D^* : \text{ allowed;}$$

$$2\omega r < D^* : \text{ disallowed.}$$

In Appendix 4.11.5 we demonstrate that WKB approximation of the solutions to this equation are exact in the large D limit away from the turning points. [111]

Let us first consider the classically disallowed region. We define

$$\kappa(r) = \left(\frac{D^{*2}}{4r^2} - \omega^2\right)^{\frac{1}{2}}. \tag{573}$$

[110] For the purposes of this discussion we stay away from the point $r = 0$ and so ignore the term proportional to the δ function.

[111] Although we do not go beyond leading order in this chapter, higher order corrections to the WKB approximation generate a systematic expansion of the Greens function in a power series in $1/D$.

The WKB solution to $\psi(\omega, r)$ takes the form

$$\psi(\omega, r) = \frac{1}{\sqrt{\kappa(r)}} \left(A \left(\frac{e\omega}{D} \right)^{D-3} e^{\int \kappa(r) dr} + B e^{-\int \kappa(r) dr} \right), \tag{574}$$

(where e is Euler's number 2.7182...) for some constants A and B. In (574) we have chosen to multiply A by the constant factor $\left(\frac{e\omega}{D}\right)^{D-3}$ for future convenience. Note that this factor is of order $1/D^D$.

At small r and with an appropriate choice of integration constants we have

$$\int \kappa(r) dr \approx \frac{D^*}{2} \ln r - \frac{r^2\omega^2}{2D^*} + \mathcal{O}(r^4\omega^4/(D^*)^3),$$

so that

$$e^{\int \kappa(r) dr} \approx r^{D^*/2} \left(1 - \frac{r^2\omega^2}{2D^*} + \dots \right).$$

[112] It follows that at small r

$$G(\omega, r) = A \left(\frac{e\omega}{D} \right)^{D-3} + \frac{B}{r^{D-3}}, \tag{575}$$

where we have accounted for the proportionality factor between $G(\omega, r)$ and $\psi(\omega, r)$ (see (571)).[113]

Now the equation

$$\nabla^2 G(\omega, r) = -\delta(r);$$

leaves A undetermined but fixes the constant B to

$$B = \frac{1}{(D-3)\Omega_{D-2}}, \tag{576}$$

(Ω_n, the volume of the unit n sphere, is listed in (570)). The constant A is determined by matching with the solution in the classically allowed region as we explain below.

In the classically allowed region we have $k(r) = \sqrt{\omega^2 - \frac{D^{*2}}{4r^2}}$. The usual formulae of the WKB approximation yield

$$\psi(\omega, r) = \frac{1}{\sqrt{k(r)}} \left(C e^{i \int k(r) - \frac{iD\pi}{4}} + E e^{-i \int k(r) + \frac{iD\pi}{4}} \right) \approx \frac{1}{\sqrt{\omega}} \left(C e^{i\left(\omega r - \frac{D\pi}{4}\right)} + E e^{-i\left(\omega r - \frac{D\pi}{4}\right)} \right). \tag{577}$$

[112]Note, in particular, that the correction to the leading order small r behaviour in $e^{\int \kappa(r) dr} \approx r^{D^*/2}$ is negligible provided $\frac{r^2\omega^2}{2D^*} \ll 1$, in agreement with the estimate for the validity of the small argument expansion of the exact formula for the Greens function presented in (567).

[113]In fact we choose the integration constants in (574) to ensure that (575) is valid. The constants The combination of the equations (574) and (575) give a complete definition of the constants A and B.

The last expression in (577) holds in the limit $2\omega r \gg D^*$. [114]

For the special case of the retarded Greens function the wave function must be outgoing at infinity so that $E = 0$. The constants A and C are both determined by matching across the turning point; in Appendix 4.11.5 we use standard WKB matching formulae to find

$$\begin{aligned} C &= \frac{(1+i)}{\sqrt{2}} B \sqrt{\frac{D^*}{2}} \left(\frac{D^*}{\omega}\right)^{-\frac{D^*}{2}} e^{\frac{D^*}{2}} = \frac{(1+i)}{\sqrt{2}} (2)^{-\frac{D}{2}} \frac{\omega^{\frac{D-3}{2}}}{\pi^{\frac{D-2}{2}}}, \\ A &= \frac{iB}{2} = \frac{i}{2(D-3)\Omega_{D-2}}. \end{aligned} \tag{578}$$

The parametric dependences of these results may be understood as follows. At the turning point we expect the two terms in (574) to be of comparable magnitude. Using the WKB approximation to evolve the solution inwards to small r we obtain the following estimate. The ratio of the decaying to the growing solution at the point r should approximate $e^{2\int_r^{\frac{D^*}{2\omega}} \kappa(r) dr}$. At large D and when $r \ll \frac{D^*}{2\omega}$ we find

$$2\int_r^{\frac{D^*}{2\omega}} \kappa(r) dr \approx D^* \ln \frac{D^*}{e\omega r}.$$

Comparing with (575) it follows that

$$A \sim B, \tag{579}$$

in approximate agreement with the more precise formulae (578). Using similar logic we can use (577) to estimate the value of $\psi(\omega, r)$ when we approach the turning point from the large r limit. Matching this estimate with the value of the wave function when the turning point is approached from the small r limit we find

$$C \sim B \left(\frac{2\omega}{D^*}\right)^{\frac{D-2}{2}}, \tag{580}$$

an estimate that is once again in agreement with the precise result (578).

The utility of the rough approximations (579) and (580) is that they are equally valid for other Greens functions (e.g. the retarded Greens function or the Feynman Greens function). It follows that for all these Greens functions the term in (574) proportional to B dominates over the term proportional to A when $r\omega \ll D/2$. When $r\omega$ is of order unity, in particular, the term proportional to A (which is sensitive to the precise nature of the Greens function) is subdominant to the term proportional to B (which is universal) at relative order $1/D^D$. It follows that different reasonable Greens functions [115] differ from each other only at order $1/D^D$ when ωr is of order

[114] The integration constants in the integrals in the first expression in (577) are determined by the requirement that it reduce to the second expression in the same equation at large r.

[115] We call a Greens function 'reasonable' if the large r boundary condition that defines it ensures that the ratio of the decaying and growing solutions at the turning point is of order unity. The retarded, advanced and Feynman Greens functions are all reasonable by this criterion. It is possible to rig up Greens functions

unity.

The fact that C/B is of order $1/D^{D/2}$ captures the smallness of radiation in the large D limit.

Let us end this subsubsection with a brief discussion of a subtle point. In the limit that $r^2\omega^2 \ll D$ the Greens function $G(\omega, r)$ is effectively independent of ω. Upon Fourier transforming, this observation suggests that the Greens function in this limit is time independent but nonlocal in space (in fact the spatial dependence of the propagator is exactly that of the Euclidean propagator for ∇^2 in $D-1$ Euclidean dimensions). This suggests that the retarded propagator mediates instantaneous action at a distance and so is acausal. Of course the exact formulae of subsubsection 4.2.1 make it clear that this conclusion is erroneous. While we have not carefully tracked down the fallacy in the naive argument, we believe it has its roots in the following fact. In order to really argue for acausality one should turn on a source that is sharply localized in time and detect a response outside the lightcone of this source. Such a source is necessarily non analytic and so always has significant support at arbitrarily high ω. It follows that the approximations of the previous paragraph, which work for ω of order unity cannot really be consistently used to argue for acausality. It would be interesting to understand this point better but we leave it for future work.

4.3 Review of Background Material: the stress tensor and conserved currents on codimension one membranes

In this section we study conserved currents and stress tensors localized on codimension one surfaces in space time.

Consider the flat space $R^{D-1,1}$. Consider a function ρ defined on this spacetime, and consider a membrane whose world volume is given by the solutions to the equation $\rho - 1 = 0$. The normal to the membrane world volume is given by the equation

$$n_M = \frac{\partial_M \rho}{|\partial\rho|}, \quad |\partial\rho| = \sqrt{\partial_M \rho \partial^M \rho} \tag{581}$$

and is assumed to be everywhere spacelike.

4.3.1 Scalar sources localized on a membrane

As a warm up consider the minimally coupled scalar equation

$$-\Box\phi = \mathcal{S}. \tag{582}$$

whose boundary conditions are finely tuned (in a D dependent way) so as to violate the conclusions of this paragraph. Such Greens functions are unphysical for our purposes, and will be ignored through the rest of this chapter.

Consider a situation in which the source $\mathcal{S}$ of that equation is given by the distributional valued field $\mathcal{S}_{ST}$ localized on the membrane

$$\mathcal{S}_{ST} = |\partial\rho|\delta(\rho - 1)\mathcal{S}, \tag{583}$$

where $\mathcal{S}$ is a smooth function on the membrane. Integrating (583) over a pillbox whose faces are just above and just below the membrane we conclude that

$$\vec{n} \cdot \partial\phi_{out} - \vec{n} \cdot \partial\phi_{in} = -\mathcal{S}, \tag{584}$$

where $\vec{n}$ is the outward pointing unit normal to the membrane (i.e. from 'in' to 'out'), ϕ_{out} is the scalar field just outside the membrane and ϕ_{in} is the scalar field just inside the membrane.

The source $\mathcal{S}$ can also be given the following interpretation. Let ϕ_0 be the value of the field ϕ on the membrane world volume. Let $S_{out}[\phi_0]$ represent the action of the outer part of the solution as a functional of ϕ_0, the value of the field ϕ on the membrane [116]. Using

$$S_{out} = -\frac{1}{2}\int (\partial\phi)^2,$$

it follows that

$$\delta S_{out} = \int \delta\phi_{out}\partial^2\phi_{out} - \int \partial_M \left(\delta\phi_{out}\partial^M\phi_{out}\right) = \int \delta\phi_{out}(n\cdot\partial)\phi_{out}. \tag{585}$$

The first two integrals on the RHS of (585) are taken over the bulk spacetime to the exterior of the membrane. The last integral is taken over the membrane world volume. In the final step in (585) we have used the scalar equation of motion and Stokes theorem.

It follows from (585) that

$$(\vec{n}\cdot\partial)\phi_{out} = \frac{\delta S_{out}}{\delta\phi_0} \tag{586}$$

(this is simply the Hamilton Jacobi equation: the LHS is evaluated on the membrane approached from the outside). In a similar manner, making similar definitions we have

$$(\vec{n}\cdot\partial)\phi_{in} = -\frac{\delta S_{in}}{\delta\phi_0}. \tag{587}$$

The difference in sign between (587) and (586) stems from the fact that the normal n is outward pointing from the point of view of the inside, but inward pointing from the point of view of the

[116] If the external region of spacetime has an additional boundary, the action would also depend on the value of the field ϕ on this additional boundary. This dependence plays no role in what follows and is suppressed in the notation. Similar remarks hold for the internal solution.

outside. It follows that (584) can be rewritten as

$$\mathcal{S} = -\frac{\delta S_{in}}{\delta \phi_0} - \frac{\delta S_{out}}{\delta \phi_0}. \tag{588}$$

It is not difficult to present explicit expressions for the actions S_{out} and S_{in} in terms of integrals over the membrane of ϕ_0 and the normal derivatives of ϕ on the outer and inner solutions respectively on the membrane.

$$S_{in}[\phi_0] = -\frac{1}{2}\int (\partial\phi)^2 = \left(-\frac{1}{2}\int \partial_M \left(\phi \partial^M \phi\right) + \frac{1}{2}\int \phi \partial^2 \phi\right) = -\frac{1}{2}\int \phi_0 (n\cdot\partial)\phi_{in}. \tag{589}$$

The integral in the last expression in (589) is taken over the membrane world volume; all other integrals are taken over the region of bulk spacetime that lies to the interior of the membrane; in obtaining the last equality we have used the bulk equation of motion and Stokes theorem. In a similar manner

$$S_{out}[\phi_0] = \frac{1}{2}\int \phi_0 (n\cdot\partial)\phi_{out}. \tag{590}$$

4.3.2 Membrane Charge current

Let us now study the Maxwell equation. Consider the action for the bulk gauge field A_M coupled to a current $\mathcal{J}^M$

$$\text{Action} = -\int \left(\frac{F_{MN}F^{MN}}{4} + \mathcal{J}^M A_M\right), \tag{591}$$

where

$$F_{MN} = \partial_M A_N - \partial_N A_M. \tag{592}$$

The equation of motion that follows from this action

$$\partial^M F_{MN} = \mathcal{J}^N. \tag{593}$$

Let the charge current that is tangent to and localized on the membrane .

$$\mathcal{J}^M = |\partial\rho|\delta(\rho - 1)J^M, \tag{594}$$

where J^M is a smooth vector field tangent to the membrane (i.e. $J^M n_M = 0$). Integrating (798) over a pillbox that encloses the membrane we conclude that

$$n_M F^{MN}_{(out)} - n_M F^{MN}_{(in)} = J^N, \tag{595}$$

where n is the outward pointing normal to the membrane.

As in the previous subsection, (595) may be rewritten as

$$J^N = \frac{\delta S_{out}[(A_0)_N]}{\delta (A_0)_N} + \frac{\delta S_{in}[(A_0)_N]}{\delta (A_0)_N}, \tag{596}$$

$S_{out}[(A_0)_N]$ is the action of the outer part of the solution as a functional of the gauge field restricted to the membrane.

As in the previous subsection it is not difficult to present explicit expressions for the actions S_{out} and S_{in} in terms of integrals over the membrane of $(A_0)_M$ and the normal derivatives of the gauge field in the outer and inner solutions respectively.

$$\begin{aligned} S_{in}[A_0] &= -\frac{1}{2}\int (A_0)_N n_M F^{MN}_{(in)}, \\ S_{out}[A_0] &= \frac{1}{2}\int (A_0)_N n_M F^{MN}_{(out)}. \end{aligned} \tag{597}$$

We will now demonstrate that the divergence of $\mathcal{J}^M_{ST}$, viewed as a distributional current in spacetime, vanishes provided J^M is a conserved current on the membrane.

In order to see this we note that

$$\begin{aligned} \partial_M \mathcal{J}^M &= \delta(\rho-1)|\partial\rho| \left[\partial_M \left(\ln\left(\sqrt{\partial_M\rho\partial^M\rho}\right)\right) J^M + \partial_M J^M\right] \\ &= \delta(\rho-1)|\partial\rho| \left[J^N (n\cdot\partial) n_N + \partial_M J^M\right] \\ &= \delta(\rho-1)|\partial\rho| \left[\Pi^N_M \partial_N J^M\right]. \end{aligned} \tag{598}$$

Here $\Pi_{MN} = \eta_{MN} - n_M n_N$. In the first line of (598) have used $(\partial_M\rho)J^M = 0$. In order to obtain the second line of the equation we have used $\partial_M\partial_N\rho = \partial_N\partial_M\rho$ and $n_M J^M = 0$. In order to obtain the third line we have used $n_M J^M = 0$ to conclude that $n^N n^M \partial_M J^N = -J^N (n\cdot\partial) n_N$. As $\Pi^N_M \partial_N J^M$ is simply the divergence of J^N viewed as a vector field on the membrane, it follows from (598) that the $\mathcal{J}^M$ is conserved in spacetime if and only if the J^M is conserved on the membrane world volume.

4.3.3 Membrane localized stress tensor

Let us now turn to a study of the Einstein equation . the action for the bulk gauge field g_{MN} coupled to a current $\mathcal{T}^{MN}$

$$\text{Action} = \frac{1}{16\pi}\int \sqrt{-g}R - \left(\frac{1}{2}\right)\int h^{MN}\mathcal{T}_{MN}. \tag{599}$$

Consider a membrane localized stress tensor given by

$$\mathcal{T}^{MN} = |\partial\rho|\delta(\rho - 1)T^{MN}. \tag{600}$$

The equation of motion that follows from this action

$$R_{MN} - \left(\frac{R}{2}\right) g_{MN} = 8\pi\mathcal{T}_{MN}, \tag{601}$$

where T_{MN} [81]is a symmetric tensor that is tangent to and smooth on the membrane. By integrating Einstein's equations over a pill box that surrounds the membrane one can show that

$$\left(\mathcal{K}^{(out)}_{MN} - \mathcal{K}^{(out)}(g_0)_{MN}\right) - \left(\mathcal{K}^{(in)}_{MN} - \mathcal{K}^{(in)}(g_0)_{MN}\right) = -8\pi T_{MN}, \tag{602}$$

where $(g_0)_{MN}$ is the space-time metric restricted to the membrane. $\mathcal{K}^{(out)}_{MN}$ and $\mathcal{K}^{(in)}_{MN}$ are the extrinsic curvature computed from 'outside' and 'inside' the membrane respectively.
In other words the discontinuity of the Brown- York stress tensor across the membrane is proportional to T_{MN}.

As in the previous subsection, (602) may be rewritten as

$$T^{MN} = -\left[\frac{\delta S_{out}[(g_0)_{MN}]}{\delta((g_0)_{MN})} + \frac{\delta S_{in}[(g_0)_{MN}]}{\delta((g_0)_{MN})}\right], \tag{603}$$

$S_{out}[(g_0)_{MN}]$ is the action of the outer part of the solution as a functional $(g_0)_{MN}$, the space-time metric, restricted to the membrane.

As in the previous subsection it is not difficult to present explicit expressions for the actions S_{out} and S_{in} in terms of integrals over the membrane of $(g_0)_{MN}$ and the normal derivatives of the metric in the outer and inner solutions respectively. The action is given entirely by the Gibbons Hawking term and takes the form

$$S = S_{out} + S_{in} \tag{604}$$

where

$$\begin{aligned} S_{in} &= -\frac{1}{8\pi}\int \sqrt{-g_{(ind)}}\,\mathcal{K}_{in}, \\ S_{out} &= \frac{1}{8\pi}\int \sqrt{-g_{(ind)}}\,\mathcal{K}_{out}, \end{aligned} \tag{605}$$

where the integral is taken over the world volume of the membrane, viewed as a boundary of the internal and external solutions respectively. The difference in signs in the two equations above is because K is defined as the trace of the extrinsic curvature of the normal vector n which always runs from in to out.

We emphasize that T^{MN} is assumed tangent to the membrane, i.e. $T^{MN}n_M = 0$. We will now demonstrate that $\mathcal{T}^{MN}$ is conserved in spacetime if and only if

- T^{MN} is a conserved stress tensor on the membrane world volume
- $T^{MN}\mathcal{K}_{MN} = 0$, where $\mathcal{K}_{MN}$ is the extrinsic curvature on the membrane.

Unlike the equation for charge conservation, the equation for the conservation of the spacetime stress tensor has a free index. We get the first condition above when the free index in this equation is in the membrane world volume, and the second condition when the free index is chosen proportional to the membrane normal.

Let us first consider the equation for stress tensor conservation projected tangent to the membrane world volume:

$$\begin{aligned}
\mathfrak{p}^P_N \nabla_M \mathcal{T}^{MN} &= \delta(\rho-1)|\partial\rho|\mathfrak{p}^P_N \left[\nabla_M \left(\ln\left(\sqrt{\partial_M\rho\partial^M\rho}\right)\right) T^{MN} + \nabla_M T^{MN}\right] \\
&= \delta(\rho-1)|\partial\rho|\mathfrak{p}^P_N \left[(n\cdot\nabla)n_M T^{MN} + \nabla_M T^{MN}\right] \\
&= \delta(\rho-1)|\partial\rho| \left[\mathfrak{p}^P_N \mathfrak{p}^Q_M \nabla_Q T^{MN}\right].
\end{aligned} \tag{606}$$

The manipulations in (606) are essentially identical to those in (598). Note that $\mathfrak{p}^P_N \mathfrak{p}^Q_M \nabla_Q T^{MN}$ is the membrane world volume divergence of the membrane stress tensor T^{MN}.

On the other hand

$$n_N \nabla_M \mathcal{T}^{MN} = -\left(\nabla_M n_N\right)\mathcal{T}^{MN} = -\mathcal{K}_{MN}\mathcal{T}^{MN} = -\delta(\rho-1)|\partial\rho|\mathcal{K}_{MN}T^{MN}, \tag{607}$$

(in going from the first to the second expression in (607) we have used $\mathcal{T}^{MN}n_N = 0$). It follows that the normal component of the stress tensor conservation equation is satisfied if and only if $\mathcal{K}_{MN}T^{MN} = 0$.

4.3.4 The stress tensor for a Nambu-Goto membrane

In order to gain some intuition for membrane stress tensors is useful to consider a simple example. Consider a relativistic membrane whose only degree of freedom is its shape and whose dynamics is governed by the relativistic Nambu-Goto action

$$S = -\sigma \int \sqrt{-g_{(ind)}}, \tag{608}$$

where $g_{(ind)}$ is the determinant of the metric $g^{(ind)}_{\mu\nu}$ induced on the world volume of the membrane and σ is the tension of the membrane. It is easily verified that the equation of motion that follows from this action is simply

$$\mathcal{K} = 0, \tag{609}$$

where $\mathcal{K}$ is the trace of the extrinsic curvature of the membrane world volume. The spacetime stress tensor for this system may be obtained by varying the action w.r.t the spacetime metric. The stress tensor is easily verified to take the form (600) with

$$T^{MN} = -\sigma \ \mathfrak{p}^{MN}. \tag{610}$$

Note that T^{MN} is proportional to the world volume metric; it follows that T^{MN} - viewed as membrane world volume stress tensor - is trivially conserved. On the other hand the requirement that $T_{MN}\mathcal{K}^{MN} = \sigma\mathcal{K} = 0$ is nontrivial and yields the membrane equation of motion.

In the simple example reviewed above the conservation of the membrane stress tensor was trivial in the world volume directions as a consequence of diffeomorphism invariance in these directions. On the other hand the conservation of the stress tensor in the normal direction was nontrivial and yields the equations of motion - a relativistic version of Newton's laws in the normal direction. Below we will see that the large D gravitational membranes of interest to this chapter behave in an orthogonal fashion. In that case the equation of stress tensor conservation in the normal direction is obeyed in a relatively trivial manner, while the equation for world volume conservation of the stress tensor yields the membrane equations of motion.

4.4 Membrane Currents from Linearized solutions: Description of the Map

In this section and the next we study the minimally coupled scalar, Maxwell and linearized Einstein equation in the vicinity of the world volume of a codimension one membrane. We assume that our membrane is embedded in a flat D dimensional spacetime and work in the large D limit.

Let us suppose we are given a solution to the exterior of the membrane world volume that decays rapidly towards infinity. [117] We then search for a corresponding *regular* solution in the interior region of the membrane subject to the requirement that the scalar field, tangential components of field strengths and curvatures are continuous across the membrane while allowing for first derivatives of these quantities to be discontinuous across the membrane. Our continuity requirement effectively imposes a Dirichlet type boundary condition for the (as yet unknown) solution in the interior of the membrane. This boundary condition, together with the requirement of regularity, turns out to be sufficient to uniquely - and practically - determine the interior

[117] As we will see later, the true exterior solution also has small constant modes with coefficient of order $\frac{1}{D^D}$ (see (575) for an example). At distances of order unity from the membrane - where we work in this section - the constant modes (the mode proportional to A in (575)) are nonperturbatively smaller than the decaying piece, and so are invisible to the large D analysis of this section. However the details of this constant piece shape the nature of the radiation far away; see e.g. the discussion under (580).

solution order by order in the $1/D$ expansion. [118]

Though the interior and exterior solutions are continuous across the membrane they are not analytic continuations of each other. In particular normal derivatives of fields are generically discontinuous across the membrane. The discontinuities in these normal derivatives determine an effective source for the wave equations that is localized on the membrane (see (584), (595) and (602)). As explained in those equations, this source is the difference between an 'exterior' current (the exterior normal derivative) and 'interior' current (the interior normal derivative). [119]

To recap, the procedure described in this section and the next allows us to constructively establish a one to one map between decaying linearized solutions to the exterior of a membrane and an auxiliary solution (which has no physical reality). The auxiliary solution agrees with the decaying solution - upto corrections of order $1/D^D$ - to the exterior of the membrane. It is constructed to ensure that it is regular everywhere in the interior of the membrane. The auxiliary solution solves the free uncharged equations everywhere to the exterior and interior of the membrane. The auxiliary solution also solves the bulk equations precisely on the membrane provided the membrane is assumed to carry a charge; in this section and the next we find precise formulae for this charge as a functional of the prescribed external solution. The discussion of this section and the next is precise (even conceptually) only in the $\frac{1}{D}$ expansion.

The starting point of the discussion of this section was a decaying external solution which was assumed to be known in the neighbourhood of the membrane surface. This original solution is - in general - not known far away from the membrane. However the analysis of this section - together with one additional piece of information - allows us to determine this asymptotic behaviour as we now explain.

Recall that the auxiliary solution obeys the linearized bulk equation, with a known charge, all over spacetime. It follows that the auxiliary solution is given *all over spacetieme* by the convolution of the membrane current with a Greens function. This statement does not, as yet, completely determine the auxiliary solution as all of the linearized equations of motion we study

[118]The fact that these boundary and regularity conditions uniquely determine our solution is true only in the $1/D$ expansion and is certainly untrue at finite D. As an example consider the minimally coupled scalar equation $\Box\phi = 0$ with the membrane manifold taken to be $S^{D-2} \times$ time and the Dirichlet boundary condition that ϕ vanish on the membrane. One solution with these boundary conditions is $\phi = 0$, but this solution is clearly not unique. In the $l = 0$ sector, for instance, we also have solutions of the form $\phi = \sum_n a_n e^{-i\omega_n t} \left(\frac{\omega_n}{r}\right)^{\frac{D-3}{2}} J_{\frac{D-3}{2}}(\omega_n r)$ where ω_n run over the set of zeroes of $J_{\frac{D-3}{2}}(\omega_n)$. Note however that at large D the first zero of this Bessel function occurs at a value of order $D/2$. It follows that the frequencies ω_n are all of order D or higher at large D. In the large D limit we disallow solutions with such high frequencies. In this extremely simple toy example it follows that the unique allowed interior solution is simply $\phi = 0$.

[119]As explained in the introduction, the interior current is neatly encoded in the action of the interior solution as a function of the metric, gauge field or scalar field on the membrane.

admit an infinite number of inequivalent Greens functions (e.g. advanced, retarded, Feynman etc). We now add an additional condition on the auxiliary solution; we demand that it is (e.g.) purely outgoing at infinity. This condition uniquely singles out one particular Green's function (e.g. the retarded Green's function) and yields a well defined - and practically useful - formula for the auxiliary solution all over spacetime. [120]

Recall, however, that the original external solution agrees with the auxiliary solution in an exterior neighbourhood of the membrane. If physical considerations inform us that the external solution obeys (e.g.) outgoing boundary conditions at infinity, it then follows that the external solution agrees with the auxiliary solution - to non perturbative accuracy - everywhere outside the membrane. It follows that the external solution is also given everywhere outside the membrane by the integral formula described in the previous paragraph.

In summary let us suppose we are given a linearized external solution in the neighbourhood of the membrane world volume that is known to be purely outgoing at infinity. The following two step procedure can be used to continue this solution to large r. In the first step we determine the 'membrane current' corresponding to our external solution. This determination is the topic of this section and the next. In the second step we convolute this current against a Greens function - this is the topic of section 4.8. The resultant expression is the continuation of the external solution to large r. In the external neighbourhood of the membrane this expression is guaranteed to agree with the configuration we started out with, upto nonperturbative corrections. The large r behaviour of this solutions yeilds the radiation field that our external solution continues to at infinity.

4.4.1 Minimally coupled scalar

We start with the case of a minimally coupled scalar equation

$$\Box\phi = -\mathcal{S}, \tag{611}$$

with the source $\mathcal{S}$ assumed to be delta function localized on the membrane.

Given the decaying part of the solution to (611) in the exterior, we wish to construct the matching interior solution. Our tactic for achieving this is very straightforward. We first construct the most general decaying solution to (611) in the vicinity of the exterior of the membrane. We then construct the most general regular solution to the same equation in the vicinity of the interior

[120] The fact that the auxiliary solution is given by the convolution of a membrane current with the Green's function depends crucially on the fact that the auxiliary solution was defined to be regular in the interior of the membrane. Had we defined the auxiliary solution differently- perhaps by allowing prescribed singularities in the interior of the membrane - we would have obtained an integral formula for this solution given by the convolution of the Greens function with all sources - those located at singularities together with those on the membrane.

of the membrane. By matching solutions in the exterior with those in the interior we produce the most general solution to (611) that is continuous across the membrane. Our construction - which uniquely pairs any external solution with an internal solution - turns out to depend on one free function on the membrane. This function can be thought of as the value of ϕ on the membrane or equally as the source 'current' $\mathcal{S}$. The construction thus gives us

- 1. An explicit classification and construction of all consistent decaying external solutions.
- 2. A one to one map between such solutions and corresponding interior solutions.
- 3. Consequently a one to one map between decaying external solutions and a source function $\mathcal{S}$ localized on the membrane.

Our construction of the exterior and interior solutions takes the form of a power series expansion in the distance s away from the membrane. The radius of convergence of this expansion is of order D/K and so this expansion is useful, from a practical point of view, only when $s \ll D/K$. The coefficients in this power series expansion are each individually determined in a power series expansion in $\frac{1}{D}$.

Given that (611) is a second order equation, the reader may wonder how it is possible that exterior and interior solutions to this equation are parametrized by one rather than two functions on the membrane. The key point here is the restriction that the exterior solution rapidly decay away from the membrane and that the interior solution be regular (in particular not grow arbitrarily large as D is taken to infinity at any point reliably captured by our approximations). These two conditions cut down the set of exterior and interior solutions each to solutions parametrized by a single function on the membrane; upon imposing continuity across the membrane we find a set of sewn solutions parametrized by a single function on the membrane.

As this point is very important, we now explain it again in a more precise and much more detailed manner.

The full set of solutions to the equation $\Box\phi = 0$ - either to the exterior or in the interior of the membrane - is indeed parametrized by two functions on the membrane world volume. Let us denote these two functions by α and β. It follows from linearity that the most general solution of the equation $\Box\phi = 0$ away from the membrane is given by

$$\phi = F_1[\alpha(x)] + F_2[\beta(x)], \tag{612}$$

where $F_{1,2}$ are linear maps from the space of functions on the membrane to functions in the flat spacetime in which the membrane is embedded. Later in this section we will explicitly construct the two functionals F_1 and F_2 (in a Taylor series expansion in distance away from the membrane)

[121] with the following two properties.

- First, on the membrane $F_1[\alpha] = \alpha$ and $F_2[\beta] = \beta$. In other words α and β are the values of ϕ restricted to the membrane. $F_1[\alpha]$ and $F_2[\beta]$ are two different continuations of the scalar field on the membrane into the bulk.

- Second F_1 decays rapidly (over a distance scale $1/D$) to the exterior of the membrane, and grows rapidly over the same distance scale on the interior of the membrane, while F_2 neither grows nor decays as we move distances of order $1/D$ away from the membrane. Instead the variation of F_2, as we move away from the membrane, occurs over length scales of order unity. [122]

We will now use the two functionals F_1 and F_2 to construct solutions $\phi(x)$ of (611) that are of the form described in the previous subsection, or, more specifically have the following properties

- $\phi(x)$ reduces to an arbitrarily prescribed function $\phi_0(x)$ on the membrane world volume.

- $\phi(x)$ is continuous across the membrane but its normal derivative is across this surface

- $\phi(x)$ decays to the exterior of the membrane, and stays regular (does not blow up) in the interior.

A moment's thought will convince the reader that the required solution is given by

$$\begin{aligned} \phi(x) &= F_1[\phi_0] \quad \text{outside}, \\ \phi(x) &= F_2[\phi_0] \quad \text{inside}. \end{aligned} \tag{613}$$

As mentioned above, in the next section we will explicitly determine the functionals F_1 and F_2 in a power series expansion in $1/D$.

Note that the solutions (613) are parameterized by a *single* membrane's function worth of data - which can be thought of either as $\phi_0(x)$ or the source function $\mathcal{S}$ on the membrane. This fact can also be understood in the following terms. Suppose we are given a source $\mathcal{S}$ localized on

[121] We determine the coefficients of this expansion order by order in $1/D$.

[122] The functionals F_1 and F_2 are effectively local functions of α and β in the following sense: it is possible to foliate spacetime around the membrane into tubes each of which cuts the membrane and is labeled by the point x_0 at which it does so. To any given order in $1/D$, F_1 and F_2 at any x_0 depend only on the distance from the membrane (which is assumed small in units of the local radius of extrinsic curvature of the membrane), the extrinsic geometry of the membrane at x_0 and a finite number of derivatives of $\alpha(x_0)$ or $\beta(x_0)$. The reason for this locality is simply that the boundary conditions of decay in the exterior and lack of blow up in the interior can each effectively be imposed at distances of order $1/D$ away from the membrane. The thinness of the region enclosed by our boundary conditions is the underlying reason for the locality of our expansion.

the world volume of the membrane. Clearly the most general solution to (611) in the presence of this source takes the form

$$\phi(x) = \int dy G(x-y)\mathcal{S}(y), \tag{614}$$

where G is a Greens function for the operator $\Box$ and the integral over y is taken over the membrane world volume. At finite D (614) does not define a unique solution to the problem, because the Greens function, G, is not unique. As we have explained in subsection 4.2.2, however, all reasonable Greens functions are identical (upto differences of order $1/D^D$) at distances of order unity around the source. It follows that the formula (614) does unambiguously define a unique solution to (611) in the neighbourhood of the membrane an expansion in $1/D$. (613) is this unique solution; i.e. (614) can be identified with (613) in the neighbourhood of the membrane for every reasonable choice of the Greens function D, even though the expressions (614) begin to depend sensitively on the choice of Greens function at large r (i.e. distances of order D). As we have explained in detail above, the 'correct' choice of Greens functions is determined by physical considerations for the problem at hand; the relevant Greens function for this chapter will always prove to be the retarded Greens function.

4.4.2 Maxwell Equation

Although it is possible to solve the Maxwell equations in a gauge invariant manner, we will find it convenient to proceed by fixing a gauge. We first define a foliation of spacetime into surfaces of constant ρ, chosen so that the surface $\rho = 1$ is the membrane. We choose the function ρ to obey the equation $\Box\left(\frac{1}{\rho^{D-3}}\right) = 0$ (see subsection 4.5.1 below). We then choose to work in a gauge in which A^ρ vanishes, i.e. the gauge $d\rho.A = 0$.

With this choice of foliation, the Maxwell equations can be divided up into the constraint equations (Maxwell equations dotted with $d\rho$) and the dynamical equations. More precisely, by a slight misuse of terminology, we will refer to the equations

$$\Pi^A_C \partial_B F^{BC} = 0, \tag{615}$$

as dynamical equations where

$$\begin{aligned} \Pi_{CA} &= \eta_{CA} - n_A n_C, \\ n_A &= \frac{\partial_A \rho}{\sqrt{\partial_D \rho \partial^D \rho}}. \end{aligned} \tag{616}$$

On the other hand we refer to

$$\mathcal{M} = 0,$$
$$\mathcal{M} \equiv n_C \partial_B F^{BC}, \tag{617}$$

as the constraint Maxwell equation

We proceed by first solving the dynamical equations defined above and then turn later to the constraint equation. The dynamical equations are very similar in character to the minimally coupled scalar equation discussed in the previous subsubsection. As in the previous subsubsection we find in general that the solutions to the dynamical Maxwell equations take the form

$$A = F_1[C_\mu(x)] + F_2[B_\mu(x)], \tag{618}$$

where A is the oneform gauge field in spacetime and C_μ and B_μ are worldvolume gauge fields on the membrane. $F_{1,2}$ are now linear maps from gauge fields on the membrane to oneform gauge fields in flat spacetime. These functional share the following properties with their scalar counterparts. First, on the membrane $F_1[C_\mu] = C_\mu$ and $F_2[B_\mu] = B_\mu$ (it makes sense to equate a spacetime gauge field with a world volume gauge field precisely because $d\rho.A$ vanishes). As for scalars F_1 decays rapidly (over a distance scale $1/D$) to the exterior of the membrane, and grows rapidly over the same distance scale on the interior of the membrane, while F_2 neither grows nor decays as we move distances of order $1/D$ away from the membrane. Instead the variation of F_2, as we move away from the membrane, occurs over length scales of order unity.

As in the case of scalars above, the boundary condition that our spacetime gauge field decays in the exterior, is regular and bounded in the interior and that the field strength restricted to the membrane is continuous on the membrane, and that it takes the value $(A_0)_\mu$ on the membrane leaves us with the solutions

$$\begin{aligned} A(x) &= F_1[(A_0)_\mu] \quad \text{outside}, \\ A(x) &= F_2[(A_0)_\mu] \quad \text{inside}. \end{aligned} \tag{619}$$

We have completed our programme of solving the dynamical equations. What remains is to solve the Maxwell constraint equations. It is a well known property of Maxwell's equations that if the dynamical equations are obeyed everywhere and the constraint equation is obeyed on a single slice then the constraint equation is obeyed everywhere. Our definition of dynamical and constraint equations are different from the usual ones (which are adapted to a foliation of spacetime into coordinate systems including ρ as a special coordinate) and it is instructive to work our our version of this standard statement. This is easily done. Note that

$$\partial_A\left(\Pi^A_B\partial_C F^{CB}\right)=\partial_C\partial_B F^{CB}-\partial_A\left(n^A n_B\partial_C F^{CB}\right)=-n.\partial\left(n_B\partial_C F^{CB}\right)-K\left(n_B\partial_C F^{CB}\right), \quad (620)$$

(where we have used the antisymmetry of F^{AB} in the last step). It follows that

$$(n.\partial)\mathcal{M}=-K\mathcal{M}-\partial_A\left(\Pi^A_B\partial_C F^{CB}\right), \quad (621)$$

(see (617) for a definition of $\mathcal{M}$). Now the last term on the RHS of (622) is the divergence of the dynamical equations and so vanishes once those equations are solved. On solutions of the dynamical equations it thus follows that

$$(n.\partial)\mathcal{M}=-K\mathcal{M}. \quad (622)$$

Integrating (622) along flow lines of the vector field n it follows that

$$\mathcal{M}(\rho)=\mathcal{M}_0 e^{-\int_1^\rho K ds}, \quad (623)$$

where $\mathcal{M}_0$ is the value of $\mathcal{M}$ at $\rho=1$ (i.e. on the membrane) and ds is the proper distance from the membrane along the integral curves of the vector field n.
Note that K, the extrinsic curvature of slices of constant ρ is positive and of order D (see subsection 4.5.1 below).

Let us assume that $\mathcal{M}_0$ is nonzero. It follows that $\mathcal{M}(\rho)$ decays rapidly to zero (over a length scale of order $1/D$) as we move away from the membrane towards the exterior. But it also follows that $\mathcal{M}(\rho)$ blows up rapidly - over a length scale of order $1/D$ - as we move away from the membrane towards the interior.

Let us now apply these results to the two special solutions $F_1[(A_0)_\mu]$ and $F_2[(A_0)_\mu]$ defined above. The solution $F_1[(A_0)_\mu]$ is defined so that it decays rapidly to the exterior of the membrane and blows up rapidly in the interior of the membrane. The fact that $\mathcal{M}$ also has the same behaviour comes as no surprise for this solution. On the other hand the solution $F_2[(A_0)_\mu]$ is defined so that it *does not* blow up in the interior of the membrane. It is thus impossible for $\mathcal{M}$ to blow up in the interior - in the manner determined by (623). It follows that $\mathcal{M}_0$ must in fact vanish on the solution $F_2[(A_0)_\mu]$.

In summary we have demonstrated that the solution $F_2[(A_0)_\mu]$ is very special; it is the solution on which the constraint equation is automatically satisfied - without the need to impose any further constraint on $(A_0)_\mu$. On the other hand the configuration $F_2[(A_0)_\mu]$ is a solution of the full Maxwell equations not for all $(A_0)_\mu$ but only for those that are constrained to obey a further condition (which we will interpret below as the condition of conservation of the membrane current).

Matching the solutions F_1 and F_2 as in (619) yields a class of solutions of Maxwell's equations parametrized by $(A_0)_\mu$ subject to the single constraint just described above. The solution (619) is a solution to Maxwell's equations with a current of the form (594) with the function J^M given in (595). This current may be rewritten as

$$J_M = J_M^{(out)} - J_M^{in}, \quad J_M^{(out)} = n^N F_{NM}^{(out)}, \quad J_M^{(in)} = n^N F_{NM}^{(in)}. \tag{624}$$

Note that the conservation of this current follows immediately from the constraint equations applied to the external and internal solutions respectively. As we have explained above this conservation is automatic for the internal solution, but imposes a constraint on the data $(A_0)_\mu$ in the case of the external solution.

The interior current $J_M^{(in)}$ is most compactly presented by evaluating the action of the interior solution $S_{in}[A_0]$. The current $J_M^{(in)}$ is then given by varying this action w.r.t A_0 using

$$\delta S_{in}[A_0] = \int \delta(A_0)_M J^{(in)M}, \tag{625}$$

(see (596)). As the interior solution $F_2[A_\mu(x)]$ is well defined for every value of the boundary gauge field $(A_0)_\mu(x)$, $S_{in}[A_0]$, is a gauge invariant functional of this boundary gauge field that also turns out to be local in the large D limit. [123]

On the other hand the external contribution to the current is simply evaluated from the definition (624), where the quantity on the RHS of that equation is evaluated on the external solution which is assumed to be known.

Let us summarize. Solutions of Maxwell's equations that obey our boundary conditions are parametrized by the membrane gauge field subject to a single constraint (the conservation of the exterior contribution to the membrane current). The full membrane current is given by adding the exterior contribution to the interior contribution which, in turn, is obtained from the variation of a gauge invariant 'counterterm' boundary action. In order to compute the current associated with a given external solution the *only* remaining nontrivial step is the determination of the counterterm action associated with the interior solution.

4.4.3 Linearized Einstein Equation

Let the metric be given by $\eta_{MN} + H_{MN}$. As in the previous subsection we work with a particular gauge choice; we impose the gauge $n^N H_{NM} = 0$.

In parallel with the previous subsection it is convenient to decompose Einstein's equations

[123]Recall that $(A_0)_\mu$ is also the gauge field on the membrane viewed from the outside and so is known.

into dynamical and constraint equations. Let us define

$$\mathcal{E}_{MN} = R_{MN} - \frac{R}{2} g_{MN} - 8\pi T_{MN}. \tag{626}$$

The Einstein equations take the form

$$\mathcal{E}_{MN} = 0. \tag{627}$$

The dynamical equations are defined to be

$$\Pi_A^M \mathcal{E}_{MN} \Pi_B^N = 0. \tag{628}$$

The constraint Einstein equations are

$$\begin{aligned} \mathcal{C}_M^E &\equiv n^A \mathcal{E}_{AM}, \\ \mathcal{C}_A^E &= 0. \end{aligned} \tag{629}$$

As in the previous subsection we first solve the dynamical Einstein equations to find a structure very similar to that for the minimally coupled scalar. The most general solution is given by

$$H = F_1[\mathfrak{h}_{\mu\nu}(x)] + F_2[\mathfrak{g}_{\mu\nu}(x)], \tag{630}$$

where the $G = \eta + H$ is the spacetime metric and $\mathfrak{h}_{\mu\nu}(x)$ and $\mathfrak{g}_{\mu\nu}(x)$ are induced metrics on the membrane. $F_{1,2}$ are now maps from the induced metric on the membrane to linearized metric fluctuations in flat spacetime. Note that the induced metric is nontrivial even in the absence of the fluctuation H_{MN}. The maps F_1 and F_2 linearly map *changes* in this induced metric to linearized fluctuations of the bulk.

As in the previous section F_1 decays rapidly (over a distance scale $1/D$) to the exterior of the membrane, and grows rapidly over the same distance scale in the interior of the membrane, while F_2 neither grows nor decays as we move distances of order $1/D$ away from the membrane.

Following the previous subsection we proceed to solve the dynamical equations subject to the boundary conditions that g_{MN} reduces to $\left[g_{\mu\nu}^{(ind)} = g_{\mu\nu}^{(ind,f)} + h_{\mu\nu}^{(0)}\right]$ on the membrane where $g_{\mu\nu}^{(ind,f)}$ is the induced metric on the membrane viewed as a submanifold of the spacetime with metric η_{MN} and $h_{\mu\nu}^{(0)}$ is arbitrary but small. Through this section we work to linearized order in $h_{\mu\nu}^{(0)}$.

Imposing the boundary conditions of fall off to the exterior and regularity in the interior and the continuity of the induced metric on the membrane as we pass from outside to inside, we find that the unique solutions to our equations are

$$
\begin{aligned}
H &= F_1[g^{(ind)}_{\mu\nu}] \quad \text{outside}, \\
H &= F_2[g^{(ind)}_{\mu\nu}] \quad \text{inside},
\end{aligned}
\tag{631}
$$

where H is a spacetime symmetric two tensor (we have omitted its indices for brevity).

As with the study of the Maxwell equation the main qualitative difference between the solutions of the linearized Einstein equations and the minimally coupled scalar equation lies in the constraint equations. However the Einstein constraint equations are of two varieties. Let

$$X_N \equiv \mathcal{C}^E_M \Pi^M_N .$$

We refer to the equation $X_M = 0$ as the momentum constraint equations. Moreover let

$$Y \equiv \mathcal{C}^E_M n^M .$$

We refer to the equation $Y = 0$ as the Hamiltonian constraint equation.

In Appendix 4.11.10 we use the identity

$$\nabla_M \left(\mathcal{E}^{MN}\right) = 0,$$

to demonstrate that the momentum and Hamiltonian Einstein constraint equations obey the equations

$$
\begin{aligned}
&\Pi^C_B (n\cdot\nabla) X_C = -K\ X_B - X^A K_{AB} - Y(n\cdot\nabla) n_B\,, \\
&n\cdot\nabla Y = -K\ Y - \nabla\cdot X + X^C (n\cdot\nabla) n_C\,.
\end{aligned}
\tag{632}
$$

As in the previous subsection, these equations determine the ρ dependence of the constraint equations in terms of their value at $\rho = 1$. Let us first consider the momentum constraint equations. The first term on the RHS of the first line of (632) is of order D while the last two terms on the RHS of this equation are of order unity and can be ignored. It follows that, as in the previous subsection, the constraint equations X_C grow exponentially as we move away from the membrane in the interior region, but decay exponentially in the exterior. As in the previous subsection this means that the constraint equations X_C must simply vanish for the interior solution, F_2 in (631). Once this result has been established for X_C, the second equation in (632) ensures that the same is true of the constraint equation Y. As in the previous subsection there is no particular reason for the constraint equations to vanish for the exterior solutions - F_1 in (631), and we will see by explicit computation below that they do not.

It follows that the interior solution F_2 is labeled by a boundary metric on the membrane.

On the other hand the external solution F_1 is labeled by the same boundary data modulo one constraint. We will later interpret this condition as the requirement that the membrane stress tensor be conserved. It follows also that the solution (631) is also labeled by membrane boundary metric subject to a single constraint.

We now turn the 'Hamiltonian' constraint equation

$$\mathcal{C}^M n_M = 0.$$

Recall that in section 4.3 we demonstrated that a stress tensor of the form (600) is conserved in spacetime provided that

- T^{MN}, viewed as a tensor on the membrane world volume is conserved.
- $T_{MN}\mathcal{K}^{MN} = 0$.

We have just argued that the 'momentum' constraint equations guarantee that the first condition is satisfied. We will now use the 'Hamiltonian' constraint equations to show that the second condition is also satisfied.

It is well known that the Hamiltonian constraint equation can be rewritten in terms of the membrane extrinsic curvature and intrinsic membrane curvatures as follows (see e.g. eqn 10.2.30. page 259, of [82])[124]

$$0 = n^A n^B E_{AB} = \frac{1}{2}\left(-\mathcal{R} + \mathcal{K}^2 - \mathcal{K}_{AB}\mathcal{K}^{AB}\right), \tag{633}$$

where $E_{AB} =$ is the Einstein Tensor, $\mathcal{R}$ is the intrinsic Ricci scalar on ($\rho = const$) slices and $\mathcal{K}^{AB}$ is the extrinsic curvature of the same slices. All indices in (633) are raised or lowered using the induced metric on $\rho = const$ slices, embedded in full space-time. As Einstein's equations are obeyed both just outside and just inside the membrane, it follows in particular that

$$\begin{aligned}
&\frac{1}{2}\left(-\mathcal{R}_{(out)} + \mathcal{K}^2_{(out)} - \mathcal{K}^{(out)}_{AB}\mathcal{K}^{AB}_{(out)}\right) = 0,\\
&\frac{1}{2}\left(-\mathcal{R}_{(in)} + \mathcal{K}^2_{(in)} - \mathcal{K}^{(in)}_{AB}\mathcal{K}^{AB}_{(in)}\right) = 0,
\end{aligned} \tag{634}$$

where all quantities with the subscript 'out' are evaluated on the special slice $\rho = 1$ (we refer to this slice as the membrane) as approached from the outside, while all quantities with the subscript 'in' are evaluated on the membrane when approached from the interior.

[124] In [82], the eqn 10.2.30 is derived for a spacelike hypersurface where the normal is timelike. But in our case the normal is spacelike and this is why the sign in the first term of our equation (633) is different from what it is there in [82]. See appendix (4.11.15) for a derivation.

Recall that the membrane world volume - viewed as a submanifold of flat space - has a nontrivial Ricci curvature tensor $R_{\mu\nu}$ and a nontrivial extrinsic curvature tensor $\mathcal{K}_{MN}$; the trace of $\mathcal{K}_{MN}$ is $\mathcal{K}$. Now $R^{(out)}_{\mu\nu}$, $\mathcal{K}^{(out)}_{MN}$ and $\mathcal{K}_{(out)}$ refer to the same quantities - but evaluated with the membrane regarded as a submanifold of $\left[g_{MN} = \eta_{MN} + h^{(out)}_{MN}\right]$. Similar remarks apply to the inside. It follows that - for instance $\mathcal{K}^{(out)}_{MN}$ differs from K_{MN} at first order in the fluctuation field h_{MN}. Let us now subtract the two equations in (634) above. Using the fact that $\mathcal{R}_{(out)} = \mathcal{R}_{(in)}$ (this follows because $\mathcal{R}$ is a function only of the induced metric on the membrane and not its normal derivative) we find

$$\begin{aligned}
0 &= n^A n^B E_{AB}|_{out} - n^A n^B E_{AB}|_{in} \\
&= K\left(\mathcal{K}_{out} - \mathcal{K}_{in}\right) - K_{AB}\left(\mathcal{K}^{AB}_{out} - \mathcal{K}^{AB}_{in}\right) \\
&= -K_{AB}\left[\left(\mathcal{K}^{AB}_{out} - \mathcal{K}^{AB}_{in}\right) - \left(\mathcal{K}_{out} - \mathcal{K}_{in}\right)\ \Pi^{AB}\right] \\
&= 8\pi K_{AB} T^{AB}.
\end{aligned} \tag{635}$$

In the second line of this equation we have worked to linear order in h_{AB}. The third line is an algebraic rearrangement of the second line and in the fourth line we have used the definition of the membrane stress tensor given in (602)

Notice that, as in the previous subsection it is useful to define

$$\begin{aligned}
T^{(out)}_{AB} &= \left(\mathcal{K}^{AB}_{(out)} - \mathcal{K}_{(out)}\ \mathfrak{p}^{AB}_{(out)}\right), \\
T^{(in)}_{AB} &= \left(\mathcal{K}^{AB}_{(in)} - \mathcal{K}_{(in)}\ \mathfrak{p}^{AB}_{(in)}\right), \\
&\text{where} \\
\mathfrak{p}^{AB}_{(out/in)} &= \text{ Projector on the membrane, embedded in outside (inside) metric.}
\end{aligned} \tag{636}$$

This implies

$$T_{AB} = -\left(\frac{1}{8\pi}\right)\left[T^{(out)}_{AB} - T^{(in)}_{AB}\right]. \tag{637}$$

In parallel with the previous subsection, the 'momentum' Einstein equations ensure that T_{AB} is conserved. [125]

As in the previous subsection, the fact that the interior solution is well defined for every value of the induced metric $g^{(ind)}_{\mu\nu}$ without restriction allows us determine $T^{(in)}_{AB}$ by first evaluating the

[125] More precisely each of $T^{(out)}_{AB}$ and $T^{(in)}_{AB}$ are separately conserved when viewed as tensor fields on the membrane with metric induced from $\eta_{MN} + h_{MN}$. Note that $T^{(out)}_{AB}$ and $T^{(in)}_{AB}$ each have a term that is zeroth order in fluctuations. However this zero order piece is common between $T^{(out)}$ and $T^{(in)}$ and so cancels in their difference. As a consequence T_{AB} is of first order in fluctuations. It follows that T_{AB} is conserved, to first order, even when viewed as a tensor field living on the membrane with undeformed induced metric $g^{(ind,f)}_{\mu\nu}$.

action S_{in} using (605) and obtaining the current using (603). Note that S_{in} is a gauge invariant function of $g^{(ind)}_{\mu\nu}$ which will also turn out to be local in the large D limit.

Counterterm Action for $T^{(in)}_{AB}$ at first order As we have seen above, the interior solution F_2 that appears in (631) is labeled by a metric on the boundary of the membrane. As we have explained in the previous section, the interior contribution to this stress tensor may be obtained as follows. We first compute the boundary action

$$S_{(in)} = -\left(\frac{1}{8\pi}\right)\int \sqrt{-g^{(ind)}}\ \ \mathcal{K}^{(in)}, \tag{638}$$

of this solution. This action should be viewed as a functional of the membrane metric that parameterizes solutions of the functional F_2. Varying the action (638) w.r.t this boundary metric then yields the contribution of the interior stress solution to the membrane stress tensor (see (603)).

It turns out that, upto first order in the expansion in $\frac{1}{D}$, the action (638) is easily evaluated as a functional of the metric on the membrane using the Gauss Codacci formalism For any Ricci-flat space, the intrinsic quantities could be related to extrinsic quantities in the following way [82] (see Appendix (4.11.15) for derivation).

$$\begin{aligned} 0 &= \mathcal{R}^{\mu\nu} - \mathcal{K}\mathcal{K}^{\mu\nu} + \mathcal{K}^{\mu\alpha}\mathcal{K}^{\nu}_{\alpha} + e^{\mu}_{A}e^{\nu}_{B}R^{ACBC'}\ \ n_C n_{C'}, \\ 0 &= \mathcal{R} - \mathcal{K}^2 + \mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu}, \end{aligned} \tag{639}$$

where $\mathcal{R}^{\mu\nu}$ and $\mathcal{R}$ is the intrinsic Ricci tensor and Ricci scalar of the membrane, $R^{ACBC'}$ is the Riemann tensor of the full space-time and and n_C is the unit normal to the membrane. e^{μ}_{A} is the matrix that relates coordinates along the membrane ($\{x^\mu\}$) to the full space-time coordinate ($\{X^A\}$) as

$$x^\mu = e^{\mu}_{A}\ X^A.$$

The following scalings with D apply to the various quantities that in equation (639) when evaluated on the interior solution F_2

$$\begin{aligned} &\mathcal{R} \sim \mathcal{O}(D^2), \quad \mathcal{R}_{\mu\nu} \sim \mathcal{O}(D), \\ &\mathcal{K}^{(in)} \sim \mathcal{O}(D), \quad \mathcal{K}^{(in)}_{\mu\nu} \sim \mathcal{O}(1), \\ &e^{\mu}_{A}e^{\nu}_{B}R^{ACBC'}\ \ n_C n_{C'} \sim \mathcal{O}(1), \end{aligned} \tag{640}$$

(the derivation of these scalings use the fact that in the interior solution F_2 the metric varies in the ρ direction on length scale unity - rather than length scale $1/D$ (as is the case for the exterior solution F_1).

The nature of these scalings allow us determine $\mathcal{K}$ in terms of intrinsic Riemann curvature tensor by solving equation (639) order by order in $\left(\frac{1}{D}\right)$ expansion.

$$\begin{aligned}
\mathcal{K}^{(in)} &= \sqrt{\mathcal{R}^{(in)}} + \frac{1}{2}\left[\frac{\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}}{\mathcal{R}^{\frac{3}{2}}_{(in)}}\right] + \mathcal{O}\left(\frac{1}{D}\right), \\
\mathcal{K}^{(in)}_{\mu\nu} &= \frac{\mathcal{R}_{\mu\nu}}{\sqrt{\mathcal{R}}} + \mathcal{O}\left(\frac{1}{D}\right).
\end{aligned} \tag{641}$$

Note that the last term in the first equation of (639) has not contributed to this order. In order to evaluate this complicated term we would need the full details of the solution F_2 developed in the next section. As this term does not contribute, however, the computation we have presented is identical to the computation of the counter term on a curved membrane surface embedded in flat-Minkowski space [126].

Substituting the first equation of (641) in equation (638) we get the form of the counter term action in terms of membrane's intrinsic curvature:

$$S_{counter} = -8\pi S_{(in)} = \int \sqrt{g^{(ind)}}\ \left[\sqrt{\mathcal{R}} + \frac{1}{2}\left(\frac{\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}}{\mathcal{R}^{\frac{3}{2}}}\right) + \mathcal{O}\left(\frac{1}{D}\right)\right]. \tag{642}$$

In Appendix 4.11.8 we have demonstrated that the stress tensor

$$-8\pi\sqrt{g^{(ind)}}T^{(in)}_{\mu\nu} = g^{(ind)}_{\mu\alpha}\left[\frac{\delta S_{counter}}{\delta g^{(ind)}_{\alpha\beta}}\right]g^{(ind)}_{\nu\beta},$$

obtained from this action is given by

$$(-8\pi)T^{(in)}_{\mu\nu} = -\left(\frac{\mathcal{R}_{\mu\nu}}{2\sqrt{\mathcal{R}}}\right) + \left(\frac{g^{(ind)}_{\mu\nu}}{2}\right)\left[\sqrt{\mathcal{R}} + \frac{1}{2}\left(\frac{\mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta}}{\mathcal{R}^{\frac{3}{2}}}\right)\right] + \mathcal{O}\left(\frac{1}{D}\right). \tag{643}$$

4.5 Membrane currents from Linearized Solutions: Detailed Construction

In this detailed technical section we present an explicit construction of the functionals F_1 and F_2 defined in the previous section, separately for the scalar, vector and linearized gravity theories ((612), (618), (630)). As explained behind we construct these functionals in a power series expansion in $\rho - 1$. Each Taylor series coefficient in this expansion is computed in an expansion

[126] Note that if we are considering the outside solution, the equation (639) is still applicable, but the scaling rules described in equation (640) are not valid. In that case, $\mathcal{K}^{(in)}_{\mu\nu}$ also scales like order $\mathcal{O}(D^2)$ and therefore the solution that we have presented in equation (641) is not valid for the space-time outside the membrane.

in $1/D$.

The results of this section will be used in the next section to read off the current and stress tensor carried by the large D gravitational membrane. The only aspect of the internal solution that will be needed for this purpose is its action; as explained in the previous section the action is given by a surface integral of the solution and its first normal derivative at the membrane. For the purposes of computing this action we are thus specially interested in the first Taylor series expansion coefficient of our solution.

As explained above we present our solutions in terms of a Taylor series in $\rho - 1$. Before proceeding to the explicit constructions we thus need to pause to give a precise definition of the function ρ and to briefly explore its properties.

4.5.1 A membrane adapted foliation of spacetime

Consider a function ρ defined in flat Minkowski space by the following conditions.

- ρ takes the value unity on the membrane world volume.
- ρ obeys the equation
$$\Box\left(\frac{1}{\rho^{D-3}}\right) = 0, \tag{644}$$
everywhere outside the membrane.
- $\frac{1}{\rho^{D-3}}$ decays at infinity and is purely outgoing there.

The conditions above uniquely define the function ρ to the exterior of the membrane at any D. [127] Once we have the solution for ρ to the exterior of the membrane, we define it in the interior of the membrane by an analytic continuation. The interior solution ρ defined in this manner continues to obey the equation (644) in the interior except at positions of potential singularities of $\frac{1}{\rho^{D-3}}$. We will see below that such singularities - which are always present - do not occur at distances $\ll \frac{D}{K}$ away from the membrane and will play no role in our analysis below.

While the requirements above uniquely determine the function ρ in principle, an explicit determination of ρ as a functional of the membrane world volume is a difficult job at finite D. The situation in this regard is much better at large D. In this subsection we explicitly determine the function ρ in a Taylor series expansion in distance away from the membrane [128]. The coefficients of this expansion are determined in a Taylor series expansion in $\frac{1}{D}$. The key

[127] This may be understood as follows. Any solution of the second order differential equation (644) is uniquely specified by two boundary conditions. In the present context the two boundary conditions are the requirement that $\rho = 1$ on the membrane world volume and the requirement that the solution is outgoing at infinity.

[128] This expansion is good at distances $\ll \frac{D}{K}$ away from the membrane

simplification at large D is that, in this limit, the function ρ turns out to be locally determined by the shape of the membrane world volume (see later in this subsection for the precise version of this statement). [129]

Consider a point in flat spacetime with coordinates x^M. To every such point we can associate a point $\hat{x}^\mu$ on the membrane by the requirement that the straight line between x^M and $\hat{x}^\mu$ is collinear with the normal at $\hat{x}^\mu$. Let $s(x^M)$ denote the distance between x^M and $\hat{x}^\mu(x^M)$ measured along this straight line. In Appendix 4.11.9 we demonstrate that

$$\begin{aligned}\rho(x) = 1 + s(x)\left(\frac{K}{D-2} + \frac{2}{K}\left(\frac{1}{2K}\hat{\nabla}^2\left(\frac{K}{D-2}\right) + \frac{K^2}{2(D-2)^2} + \frac{K_{MN}K^{\mu\nu}}{K}\right) + \mathcal{O}\left(\frac{1}{D^2}\right)\right) \\ + s(x^\mu)^2\left(\frac{1}{2K}\hat{\nabla}^2\left(\frac{K}{D-2}\right) + \frac{K^2}{2(D-2)^2} + \frac{K_{\mu\nu}K^{\mu\nu}}{K} + \mathcal{O}\left(\frac{1}{D}\right)\right) + \mathcal{O}\left(s^3\right),\end{aligned} \tag{645}$$

where all intrinsic membrane quantities (like K, $K_{\mu\nu}$ etc) are evaluated at the membrane point $\hat{x}(x)$. The quantity $\hat{\nabla}$ represents the covariant derivative along the world volume of the membrane. [130]

Later in this subsection will need to take derivatives of the function ρ. As we have expressed ρ as a function of s, it is useful to first compute relevant derivatives of the function s. It is possible

[129] A related fact is that we do not need to use the boundary condition that ρ is outgoing at infinity in order to determine ρ in the large D limit. If, in other words we were to define a new function $\tilde{\rho}$ by the conditions listed in this subsection, with the one replacement that $\tilde{\rho}$ is required to be ingoing rather than outgoing at infinity, then in the $\frac{1}{D}$ expansion $\tilde{\rho}$ would have the same Taylor series expansion around the horizon as ρ. It turns out that the two functions ρ and $\tilde{\rho}$ differ only at order $\frac{1}{D^D}$ at distances of order unity away from the black hole. The two functions begin to differ substantially from each other only at distances of order D away from the membrane. All these remarks are, of course, tightly connected to the properties of Greens functions at large D discussed in section 4.2.

[130] The structure of the equations we encounter in evaluating the function $\rho(x^M)$ in the large D expansion is as follows. At leading order in perturbation theory we are able to obtain the $\mathcal{O}(1)$ part of the coefficient of s. At next leading order we find the $\mathcal{O}(1/D)$ piece in the coefficient of s together with the $\mathcal{O}(1)$ part of the coefficient of s^2. At third order we would find the $\mathcal{O}(1/D)^2$ contribution to the coefficient of s, the $\mathcal{O}(1/D)$ contribution to the coefficient of s^2 and the $\mathcal{O}(1)$ part of the coefficient of s^3, and so on. In other words if we specialize to the case that $s(x^\mu)$ is of order $1/D$ then our perturbative expansion evaluates ρ in an expansion in $\frac{1}{D}$. In (645) have reported the result of our expansion upto second order. In the special case that $s \sim \mathcal{O}(1/D)$ we have

$$\begin{aligned}&\rho(x^\mu) - 1 = s(x^\mu)\frac{K(\hat{r}^\mu)}{D-2} + \\ &\left(\frac{2s(x^\mu)}{K} + s(x^\mu)^2\right)\left(\frac{1}{2K}\hat{\nabla}^2\left(\frac{K}{D-2}\right) + \frac{K^2}{2(D-2)^2} + \frac{K_{MN}K^{MN}}{K}\right) \\ &+\mathcal{O}\left(\frac{1}{(D-2)^3}\right),\end{aligned} \tag{646}$$

where we have arranged terms so that the first and second lines in this (646) are respectively of order $1/D$ and $1/D^2$

to verify that

$$\begin{aligned}
&\partial_M s = n_M,\\
&\Box s = K + s K_{MN} K^{MN},\\
&+ \mathcal{O}(1/D) + s \times \mathcal{O}(1) + s^2 \mathcal{O}(D),
\end{aligned} \tag{647}$$

where n_M is the vector $\partial_M \rho$ rescaled to have unit norm. [131] Using these results it may be verified that

$$\begin{aligned}
&N^2 \equiv |\partial\rho|^2 \equiv \partial_M \rho \partial^M \rho = \left(\frac{K}{D-2}\right)^2\\
&+ \frac{4}{D-2}(1+Ks)\left(\frac{2}{K}\left(\frac{1}{2K}\hat{\nabla}^2\left(\frac{K}{D-2}\right) + \frac{K^2}{2(D-2)^2} + \frac{K_{MN}K^{MN}}{K}\right)\right)\\
&+ \mathcal{O}(1/D^2) + s \times \mathcal{O}(1/D) + s^2 \times \mathcal{O}(1).
\end{aligned} \tag{648}$$

4.5.2 Membrane solutions of the minimally coupled scalar

In this subsection we will construct the solution (613) (see the previous section) both for $\rho > 1$ and $\rho < 1$. We obtain our solution in a Taylor series expansion in $\rho - 1$. The coefficients in this expansion are obtained in a power series expansion in $\frac{1}{D}$. [132]

Recall that the solution (613) is labeled by the value $\phi_0(\hat{x})$ of the scalar field on the membrane. In the special case that $\phi_0(\hat{x})$ is a constant α, it follows immediately that the solution of interest is given by $\phi_a = \frac{\alpha}{\rho^{D-3}}$ (for $\rho > 1$) and $\phi = \alpha$ (for $\rho < 1$). Note that in the exterior region ϕ varies on the length scale $1/D$ in the direction normal to the membrane. If $\phi_0(\hat{x})$ is a function that varies on length scale unity, the relative slowness of this variation suggests the following. Let $\alpha(x)$ in (649) be any smooth extension of the membrane function $\phi_0(\hat{x})$ into the bulk. Then

$$\begin{aligned}
&\phi_a(x) = \frac{\alpha(x)}{\rho^{D-3}} \quad (\rho \geq 1),\\
&\phi_a(x) = \alpha(x) \quad (\rho \leq 1),
\end{aligned} \tag{649}$$

[131] The second equation in (647) may be understood as follows. As $\partial_\mu s = n_\mu$, it follows that $\Box s$ equals K of the constant ρ slice at that point. To the appropriate order in $1/D$, $K(x^M)$ can be re-expressed in terms of curvature invariants at the corresponding $\hat{x}$ point, yielding the second equation of (647)

[132] As in the previous subsection, at leading order in our expansion we find the coefficient of the constant term in the Taylor series expansion at order unity in the expansion in $\frac{1}{D}$. At next order we find the $\mathcal{O}(1/D)$ correction to this constant together with the order unity (i.e leading) contribution to the coefficient of $(\rho - 1)$. We stop our expansion at this point. Had we gone to one higher order in the perturbative expansion we would have obtained the $\mathcal{O}(1/D^2)$ correction to the constant, the $\mathcal{O}(1/D)$ correction to the coefficient of $\rho-1$ and the order unity correction to the coefficient of $(\rho-1)^2$. In other words our expansion reduces to an honest expansion in $\left(\frac{1}{D}\right)$ provided $(\rho - 1)$ is of order $\left(\frac{1}{D}\right)$.

[133] also solves the minimally coupled scalar equations; not exactly (as was the case when α was constant), but at leading order in the expansion in $\frac{1}{D}$. We will check below that this expectation is indeed correct.

In order to proceed with our computation we need to make a particular choice for the extension of the membrane valued function $\phi_0(\hat{x})$ to the bulk function $\alpha(x)$. In the rest of this section we choose, arbitrarily, to extend the function $\phi_0(\hat{x})$ into the bulk in such a way that it obeys the 'subsidiary condition'

$$d\rho \cdot d\alpha = 0. \tag{650}$$

This requirement together with the condition that $\alpha(x)$ agrees with $\phi_0(\hat{x})$ on the membrane. [134] completely determines the bulk field in terms of the membrane valued field $\alpha(x)$.

$\phi_a(x)$ in (649) is a function of order unity which varies on length scale $\left(\frac{1}{D}\right)$. We would thus expect that the action of $\Box$ on a configuration of this sort should yield an expression of order $\mathcal{O}(D^2)$. Using (644), however, it is easily verified that

$$\begin{aligned} \Box\phi_a(x) &= \frac{\Box\alpha(x)}{\rho^{D-3}} \quad (\rho \geq 1), \\ \Box\phi_a(x) &= \Box\alpha(x) \quad (\rho \leq 1). \end{aligned} \tag{651}$$

Recall from the introduction that even though the function α varies over length scale unity, $\Box\alpha$ is generically of order $\mathcal{O}(D)$. It follows that the ansatz (649) satisfies the minimally coupled scalar equation at order D^2 - the order at which we might at first expect this equation to be violated,

Systematic procedure to correct the ansatz ϕ_a In order to proceed, we search for a systematic correction of (649). The corrections should have the property that they are subleading compared to $\phi_a(x)$ presented above when $(\rho-1)$ is of order $\mathcal{O}\left(\frac{1}{D}\right)$, and also that they are capable of canceling the RHS of (651). An ansatz that obviously satisfies the first criterion and turns out to satisfy the second is

$$\begin{aligned} \phi(x) &= \frac{\sum_{n=0}^{\infty} \alpha_n(x)(\rho-1)^n}{\rho^{D-3}} \quad (\rho \geq 1), \\ \phi(x) &= \sum_{n=0}^{\infty} \beta_n(x)(\rho-1)^n \quad (\rho \leq 1), \\ \alpha_0(x) &= \beta_0(x) = \alpha(x), \end{aligned} \tag{652}$$

[133]The subscript a in ϕ_a stands for 'ansatz'; a is not a spacetime vector index.

[134]The subsidiary condition (650) is simply one convenient way of extending α away from the membrane surface in a smooth, D independent way. The auxiliary condition (650) is convenient but essentially arbitrary. We could, for example, also have used the condition $\alpha(x^\mu) = \alpha(\hat{x}^\mu(x^\mu))$. This condition would also have served our purposes in principle but proves less convenient for actually solving the problem in practice.

with

$$n \cdot \partial\alpha_n = n \cdot \partial\beta_n = 0. \tag{653}$$

Assuming the expansion (652) and focusing on the region $\rho > 1$, a straightforward algebraic exercise demonstrates that

$$\begin{aligned}\Box\phi(x) &= \sum_{n=1}^{\infty} A_n \frac{(\rho-1)^n}{\rho^{D-3}},\\ A_n &= \Bigg(\Box\alpha_n + ((n+1)(D-2) - 2(D-3))\,\frac{(d\rho \cdot .d\rho)\ \alpha_{n+1}}{\rho}\\ &\quad + (n+2)(n+1)(d\rho \cdot .d\rho)\ \alpha_{n+2}\Bigg).\end{aligned} \tag{654}$$

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When $\rho - 1 < 1$, on the other hand, we find

$$\begin{aligned}\Box\phi(x) &= \sum_{n=1}^{\infty} B_n (\rho-1)^n,\\ B_n &= \left(\Box\alpha_n + ((n+1)(D-2))\,\frac{d\rho.d\rho\alpha_{n+1}}{\rho} + (n+2)(n+1)(d\rho \cdot .d\rho)\alpha_{n+2}\right).\end{aligned} \tag{656}$$

The coefficients A_n and B_n in the expansion above can themselves be expanded in a power series in $(\rho - 1)$. Let

$$\begin{aligned}A_n &= \sum_m A_n^m (\rho-1)^m,\\ B_n &= \sum_m B_n^m (\rho-1)^m,\end{aligned} \tag{657}$$

where

$$n.\nabla A_n^m = n.\nabla B_n^m = 0. \tag{658}$$

The equations (657) and (658) define the expansion functions A_n^m and B_n^m. The expressions for

[135] We have used the fact that

$$(D-2)\partial_\mu\rho\partial^\mu\rho = \rho\Box\rho, \tag{655}$$

(this is an expansion of the equation $\Box\frac{1}{\rho^{D-3}} = 0$) to simplify the RHS of (654).

$\Box\phi$ can be rewritten in terms of these expansion coefficients as

$$
\begin{aligned}
\Box\phi(x) &= \sum_{n=1}^{\infty} \tilde{A}_n \frac{(\rho-1)^n}{\rho^{D-3}}, \quad (\rho > 1)\\
\Box\phi(x) &= \sum_{n=1}^{\infty} \tilde{B}_n (\rho-1)^n, \quad (\rho < 1)\\
\tilde{A}_n &= \sum_{m=0}^{n} A^m_{n-m},\\
\tilde{B}_n &= \sum_{m=0}^{n} B^m_{n-m},\\
n\cdot\partial\tilde{A}_n &= n\cdot\partial\tilde{B}_n = 0.
\end{aligned}
\tag{659}
$$

The condition that ϕ is harmonic then simply reduces to the condition $\tilde{A}_n = \tilde{B}_n = 0$. We will now demonstrate that these equations are very easily solved in a power series expansion in $1/D$.

Explicit solution at low orders for $\rho > 1$ In this subsection we construct the functional F_1 defined in (612).

Let us consider the special case $n = 0$. $\tilde{A}_0 = 0$ implies that $A_0 = 0$ i.e. that

$$\Box\alpha_0 - (D-4)\frac{d\rho\cdot d\rho}{\rho}\alpha_1 + 2(d\rho\cdot d\rho)\ \alpha_2 = 0.$$

This equation is practically solvable in the large D limit because the term proportional to α_2 is subleading at large D compared to the other terms in this equation. Ignoring this term in the equation we obtain the equation

$$\alpha_1 = \frac{\rho\Box\alpha_0}{(D-4)(d\rho\cdot d\rho)}. \tag{660}$$

More precisely α_1 is given by (660) on the membrane and determined elsewhere by subsidiary conditions $n\cdot\partial\alpha_1 = 0$. [136]

At any event we are most interested in α_1 evaluated on membrane surface. The solution we have presented for α_1 on the membrane is given in terms of the spacetime d'Alembertian of α.

[136] To see why this is so recall that (660) was obtained by equating the coefficient of $(\rho-1)^0$ in (659) to zero. Clearly (660) is not the unique solution to this condition; if we add $(\rho-1)G$ to the solution for α_1 presented in (660) the coefficient of $(\rho-1)^0$ in (659) continues to vanish. In other words (660) is too strong; the correct statement is

$$\alpha_1 = \frac{\rho\Box\alpha_0}{(D-4)(d\rho\cdot d\rho)} + \mathcal{O}(\rho-1). \tag{661}$$

The ambiguity of extending α_1 off the membrane is then resolved by the condition $n.\nabla\alpha_1 = 0$.

This result may be reworded in terms of the membrane d'Alembertian acting on the membrane valued function ϕ_0 using

$$\Box\alpha = \tilde{\Box}(\phi_0) - \frac{\hat{\nabla}K \cdot \hat{\nabla}\phi_0}{K} + \mathcal{O}\left(\frac{1}{D}\right), \tag{662}$$

(here $\Box$ in (660) is the full spacetime d'Alembertian operator, $\tilde{\Box}$ is the d'Alembertian on the membrane world volume and (662) is derived using the subsidiary condition $n \cdot \partial\alpha = 0$). The dot product in the last term on the RHS of (662) is taken in the membrane world volume metric $\Pi_{MN} = \eta_{MN} - n_M n_N$. Note that the second term on the RHS of (662) is of order unity in the $1/D$ expansion, and so is subleading compared to the first term in that equation. On the membrane (i.e. on the surface $\rho = 1$ and at leading order

$$\Box\alpha = \tilde{\Box}\phi_0.$$

Using (648) it then follows that on the membrane surface $\rho = 1$

$$\begin{aligned} n \cdot \partial\phi &= \left[\frac{K}{D-2}\right]\left[-(D-3)\alpha + \left(\frac{D}{K^2}\right)\Box\alpha_0(x^\mu)\right] \\ &= -K\alpha_0\left(1 - \frac{1}{D}\right) + \frac{\Box\alpha_0(x^\mu)}{K} + \mathcal{O}\left(\frac{1}{D}\right) \\ &= -K\alpha\left(1 - \frac{1}{D}\right) + \frac{\tilde{\Box}(\phi_0)}{K} + \mathcal{O}\left(\frac{1}{D}\right). \end{aligned} \tag{663}$$

Recall from the introduction that $\Box\alpha_0$ and K are both of order D. The RHS of (663) has terms of order D and order unity.

The procedure outlined here can be generalized to all orders. The equation $\tilde{A}_1 = 0$ will now allow us to determine α_2 to leading order. Plugging this result into the equation $\tilde{A}_0 = 0$ then allows us to determine the first subleading correction to α_1 in the $1/D$ expansion. In a similar manner the equation $\tilde{A}_2 = 0$ allows us to determine α_3 to leading order; which in turn permits the determination of α_2 to first subleading and α_1 to second subleading order in $1/D$, and so on.

Explicit solution at low orders when $\rho < 1$ In this subsection we construct the functional F_2 defined in (612) at lowest nontrivial order. In order to do this we focus on the special case $n = 0$. $\tilde{B}_0 = 0$ implies that $B_0 = 0$ i.e. that

$$\Box\beta_0 + (D-2)\frac{(d\rho \cdot d\rho)\,\beta_1}{\rho} + (n+2)(n+1)\,(d\rho \cdot d\rho)\,\beta_2 = 0.$$

Once again the term proportional to β_2 is subleading at large D compared to the other terms in this equation. It follows that

$$\beta_1 = -\frac{\rho \Box \alpha_0}{(D-2)(d\rho \cdot d\rho)}. \tag{664}$$

Once again (664) is reliable only on the membrane; β_1 is extended off the membrane using the condition $n \cdot \partial \beta_1 = 0$.

We are particularly interested in this coefficient evaluated on the membrane surface. Using (662) it follows that on the membrane

$$\begin{aligned} n \cdot \partial\phi|_{\rho=1} &= -\frac{\Box \alpha_0(x^\mu)}{K} + \mathcal{O}\left(\frac{1}{D}\right) \\ &= -\left(\frac{1}{D}\right)\left[\tilde{\Box}(\phi_0) - \frac{\hat{\nabla} K \cdot \hat{\nabla} \phi_0}{K}\right] + \mathcal{O}\left(\frac{1}{D}\right) \\ &= -\hat{\nabla}_\mu \left(\frac{\hat{\nabla}^\mu \phi_0}{K}\right) + \mathcal{O}\left(\frac{1}{D}\right). \end{aligned} \tag{665}$$

According to (583), the contribution of the internal solution to the current on the membrane is given by the spacetime source

$$\begin{aligned} \mathcal{S} &= \left(\sqrt{d\rho \cdot d\rho}\right) \delta(\rho - 1)(n \cdot \partial \phi_{in}) \\ &= -\left(\sqrt{d\rho \cdot d\rho}\right) \delta(\rho - 1) \hat{\nabla}_\mu \left(\frac{\hat{\nabla}^\mu \phi}{K}\right). \end{aligned} \tag{666}$$

This current can be derived from the variation of the action for the internal solution w.r.t. ϕ_0 using the equation (587) once we identify

$$S_{in} = \frac{1}{2} \int \frac{(\hat{\nabla}\phi_0)^2}{K}, \tag{667}$$

(667) can also be obtained from (589) using (665).

Current Using the results of the previous two subsubsections it is easily verified that

$$n \cdot \partial\phi|_{out} - n \cdot \partial\phi|_{in} = -K\alpha_0(x^\mu)\left(1 - \frac{1}{D}\right) + \left(\frac{2}{K}\right) \Box \alpha_0(x^\mu).$$

In other words our field ϕ obeys the equation (583) (which we repeat here for convenience)

$$\Box \phi = \left[\left(\sqrt{d\rho \cdot d\rho}\right) \delta(\rho - 1)\right] \mathcal{J}, \tag{668}$$

$$\mathcal{J} = -K\phi_0 \left(1 - \frac{1}{D}\right) + \frac{2}{K}\left(\tilde{\Box}(\phi_0) - \frac{\hat{\nabla}K \cdot \hat{\nabla}\phi_0}{K}\right). \tag{669}$$

4.5.3 Membrane solutions of the Maxwell Equations

We will now imitate the analysis of the previous subsection to demonstrate that the most general solution of the Maxwell equations is parametrized by a conserved current living on the membrane, and explicitly construct the solution generated by any particular current.

$\rho > 1$ In this subsubsection we find the solution $F_1[A_0]$ (see (618)). We will find it convenient to slightly change notation as compared to the previous section; in particular the data for our solution - referred to as $(A_0)_\mu$ in the previous section will be taken to be $G^{(0)}_M$ below. As we explain in detail below, $G^{(0)}_M$ is a bulk spacetime gauge field whose restriction onto the membrane equals $(A_0)_\mu$ of the previous section.

Following previous subsections we assume that the gauge field $\mathcal{A}_A$ can be expanded outside the membrane as

$$\begin{aligned} \mathcal{A}_A &= \rho^{-(D-3)} G_A, \\ G_A &= \sum_{k=0}^{\infty} (\rho - 1)^k G^{(k)}_A, \end{aligned} \tag{670}$$

where each of $G^{(k)}_A$ admits further expansion in $\left(\frac{1}{D}\right)$.

As in the previous subsection, the leading term $G^{(0)}_B$ in this expansion will turn out to be the data of our solution (which we will later be able to trade for a conserved current). Below we will outline the procedure that determines all the remaining coefficient functions in terms of $G^{(0)}_A$.

In order to set up the problem we work in the gauge $\mathcal{A}_A n^A = 0$. Of course this is simply a convenient device; the gauge invariant content in our expansion lies in the field strengths. This particular gauge is convenient as our problem has a special oneform - $\partial\rho$ - at each point in spacetime. By using this oneform to fix gauge we obtain a parametrization that keeps all the symmetries of the physical problem manifest.

Our gauge condition implies

$$n^A G_A = 0, \tag{671}$$

$$n^A G^{(k)}_A = 0, \quad \text{for every } k \tag{672}$$

where n_A is the unit normal to the $\rho = \textit{constant}$ surfaces, defined by

$$\partial_A \rho = N\ n_A, \quad N = \sqrt{(\partial_A \rho)(\partial^A \rho)},$$

(recall that N was evaluated in (648) and equals $\frac{K}{D-2}$ to leading order). As in the previous subsection, we impose a subsidiary condition on the coefficient functions G_A to give our expansion meaning. The condition we impose is

$$\Pi^A_C(n.\partial)G^{(k)}_A = 0, \quad \text{for every } k \tag{673}$$

where

$$\Pi_{AB} = \eta_{AB} - n_A n_B.$$

From (671) it follows that

$$n^A(n.\partial)G_A = -G_A\big[(n.\partial)n^A\big]. \tag{674}$$

Similarly from (672) it follows that

$$n^A(n.\partial)G^{(k)}_A = -G^{(k)}_A\big[(n.\partial)n^A\big], \tag{675}$$

(the last two equations are consistent because of (672))[137].

Our discussion above has been presented in a particular gauge. However the functions G_A actually have a simple gauge invariant significance as we now explain. Note that

$$\begin{aligned} F_{AB} &= \partial_A(\rho^{-(D-3)}G_B) - \partial_B(\rho^{-(D-3)}G_A) \\ &= (\partial_A\rho^{-(D-3)})G_B - (\partial_B\rho^{-(D-3)})G_A + \rho^{-(D-3)}(\partial_A G_B - \partial_B G_A). \end{aligned}$$

Now using

$$\begin{aligned} -n_A\partial_B G^A &= -\partial_B(n_A G^A) + (\partial_B n_A)G^A \\ &= \eta^C_B(\partial_C n_A)G^A \\ &= \left(\Pi^C_B + n^C n_B\right)(\partial_C n_A)G^A \\ &= K_{BA}G^A + n_B G^A(n.\partial)n_A \\ &= K^A_B G_A - n_B n^A(n.\partial)G_A, \end{aligned} \tag{676}$$

where the projector $\Pi_{AB} = \eta_{AB} - n_A n_B$.

[137]Here all lowering, raising and contraction of indices have been done using the flat metric η_{AB}.

It follows that

$$\begin{aligned} n_A F^A{}_B &= \frac{-N(D-3)G_B}{\rho^{D-2}} + \frac{1}{\rho^{D-3}}\left[(n.\partial)G_B - n_A\partial_B G^A\right] \\ &= \frac{-(D-3)NG_B}{\rho^{D-2}} + \frac{1}{\rho^{D-3}}K^A_B G_A. \end{aligned} \tag{677}$$

Here in the last line we have used the subsidiary condition (673).
Moreover

$$\Pi^{A'}_A F_{A'B'}\Pi^{B'}_B = \left(\frac{1}{\rho^{D-3}}\right)\Pi^{A'}_A\left(\partial_{A'}G_{B'} - \partial_{B'}G_{A'}\right)\Pi^{B'}_B. \tag{678}$$

Equations (676) (and in particular (677) and (678)) are presentations of the gauge invariant significance of the functions G_A.

We now proceed to use the Maxwell equations to determine G^k_A (for $k \geq 1$) in terms of $G^{(0)}_A$. Our analysis proceeds in analogy with that of the previous subsection (scalar field) with one crucial difference. While there are $(D-1)$ unknown functions G_A we have D Maxwell equations. In order to solve for G_A we will use only the $(D-1)$ dynamical Maxwell equations (615). [138] In Appendix 4.11.11 we have presented all the algebraic details of our computation of G_A. Here we simply present our results.

Let us define

$$F^{(m)}_{AB} = \partial_A G^{(m)}_B - \partial_B G^{(m)}_A. \tag{679}$$

At first subleading order in $\frac{1}{D}$ we find

$$\begin{aligned} G^{(1)}_B &= \frac{\Pi^C_B\,\partial^A F^{(0)}_{AC}}{2(D-3)N^2 - NK} + \mathcal{O}\left(\frac{1}{D}\right) \\ &= \left(\frac{\Pi^C_B\,\partial^A F^{(0)}_{AC}}{NK}\right) + \mathcal{O}\left(\frac{1}{D}\right). \end{aligned} \tag{680}$$

Here, in the second line, we have used the fact that $K = DN + \mathcal{O}(1)$.
Note that $\Pi^C_B\,\partial^A F^{(0)}_{AC}$ could be re-expressed completely in terms of quantities and covariant

[138] As we have explained in detail in the previous section, the remaining constraint equation (617) constrains the data $G^{(0)}_A$ (which we referred to as $(A_0)_\mu$ in the previous section) that parametrizes general solutions of the Maxwell equation.

derivatives that are defined only along the membrane.

$$\begin{aligned}\Pi^C_B\ \partial^A F^{(0)}_{AC} &= \Pi^C_B \partial^A \left[n_A n^{A'} F^{(0)}_{A'C} - n_C n^{A'} F^{(0)}_{A'A} + \Pi^{A'}_A F^{(0)}_{A'C'} \Pi^{C'}_C \right] \\ &= K\ n^A F^{(0)}_{AC} \Pi^C_B + \Pi^C_B \partial^A \left[\Pi^{A'}_A F^{(0)}_{A'C'} \Pi^{C'}_C \right] + \mathcal{O}(1) \\ &= K\ n^A (\partial_C G^{(0)}_A) \Pi^C_B + \Pi^C_B \partial^A \left[\Pi^{A'}_A F^{(0)}_{A'C'} \Pi^{C'}_C \right] + \mathcal{O}(1) \\ &= K\ K^A_B G^{(0)}_A + \Pi^C_B \partial^A \left[\Pi^{A'}_A F^{(0)}_{A'C'} \Pi^{C'}_C \right] + \mathcal{O}(1)\,. \end{aligned} \tag{681}$$

In (681) all free indices are projected on the membrane and also all contracted indices and derivatives run along the membrane directions only. Similarly, because of our gauge condition, $G^{(k)}_A$ for every value of k could also be considered as a vector field ($G^{(k)}_\mu$) defined only along the membrane. Therefore It follows that $G^{(1)}_\mu$ - the first Taylor coefficient in the expansion of the gauge field off the membrane but viewed as a vector field along the membrane - can be rewritten entirely in terms of intrinsic quantities on the membrane as

$$G^{(1)}_\mu = \left(\frac{1}{N}\right) \left(K^\nu_\mu G^{(0)}_\nu + \frac{\hat{\nabla}^\nu \hat{F}_{\nu\mu}}{K} \right) + \mathcal{O}\left(\frac{1}{D}\right), \tag{682}$$

where $\hat{F}_{\mu\nu}$ is the field strength along the surface and $\hat{\nabla}_\mu$ is the covariant derivative on the membrane surface, with respect to the intrinsic metric of the membrane. Also all raising lowering and contraction of indices have been done using the intrinsic metric of the membrane as embedded in flat space.

Restricting attention to the surface $\rho = 1$ we have in particular

$$n_A F^A{}_B|_{\rho=1} = J^{(out)}_B = -(D-3) N G^{(0)}_B + N G^{(1)}_B + K^A_B G^{(0)}_A. \tag{683}$$

Using the same argument as given above and substituting equations (682) in equation (683) we get the outside current as vector field along the membrane (upto first subleading order)

$$J^{(out)}_\mu = -(D-3) N G^{(0)}_\mu + \frac{\hat{\nabla}^\nu \hat{F}_{\nu\mu}}{K} + 2 K^\nu_\mu G^{(0)}_\nu + \mathcal{O}\left(\frac{1}{D}\right), \tag{684}$$

where $\hat{F}_{\mu\nu}$ is the field strength along the surface

As explained in the previous section, the constraint Maxwell equation asserts that

$$\hat{\nabla}_\mu J^\mu_{(out)} = 0,$$

(where $\hat{\nabla}_\mu$ is the covariant derivative on the membrane surface) yielding an effective constraint on the data $G^{(0)}_\mu$ of the solution.

$\rho < 1$ In this subsection we proceed to construct the functional F_2 defined in (618). As in the previous subsection, the data for this solution will be taken to be the spacetime gauge field $\tilde{G}^{(0)}_A$ whose restriction onto the membrane defines A_0 of the previous section.

In order to proceed with our computation we proceed assuming that the solution in the region $\rho < 1$ can be expanded as

$$\tilde{G}_A = \sum_{k=0}^{\infty} (\rho - 1)^k \tilde{G}^{(k)}_A. \tag{685}$$

In order that the gauge field is continuous across the membrane we will require that the restriction of $\tilde{G}^{(0)}_B$ to the surface $\rho = 1$ agree with the restriction of $G^{(0)}_B$ on the same surface. As in the previous subsection we will use Maxwell's equations to determine the higher order terms in the expansion of the gauge field in terms of G^0_A. As in the previous subsection we adopt the gauge $n^A \tilde{G}_A = 0$ which implies that .

$$n^A \tilde{G}^{(k)}_A = 0, \quad \text{for every } k. \tag{686}$$

As in the previous subsubsection we also demand that

$$\Pi^C_B n^A \partial_A \tilde{G}^{(k)}_C = 0.$$

Again as in the previous subsubsection it follows that

$$(n.\partial)\tilde{G}^{(k)}_A = -n_A \ \tilde{G}^{(k)}_B \left[(n \cdot \partial) n^B\right].$$

The quantities $\tilde{G}^{(k)}_A$ have the following gauge invariant significance:

$$\begin{aligned} \tilde{F}_{AB} &= \partial_A \tilde{G}_B - \partial_B \tilde{G}_A, \\ \tilde{F}_{AB} &= \sum_{k=0}^{\infty} k(\rho-1)^{k-1} N \left[n_A \tilde{G}^{(k)}_B - n_B \tilde{G}^{(k)}_A \right] + \sum_{k=0}^{\infty} (\rho-1)^k \left[\partial_A \tilde{G}^{(k)}_B - \partial_B \tilde{G}^{(k)}_A \right]. \end{aligned} \tag{687}$$

Solving the equation ($\partial_A \tilde{F}^{AB} = 0$) at first subleading order we find (see Appendix 4.11.11)

$$\begin{aligned} \tilde{G}^{(1)}_B &= -\frac{\Pi^C_B \ \partial^A F^{(0)}_{AC}}{NK} + \mathcal{O}\left(\frac{1}{D}\right) \\ &= -\left(\frac{1}{N}\right) \left(K^A_B G^{(0)}_A + \frac{\Pi^C_B \partial^A \left[\Pi^{A'}_A F^{(0)}_{A'C'} \Pi^{C'}_C \right]}{K} \right) + \mathcal{O}\left(\frac{1}{D}\right), \end{aligned} \tag{688}$$

where

$$\tilde{F}^{(m)}_{AB} = \partial_A \tilde{G}^{(m)}_B - \partial_B \tilde{G}^{(m)}_A, \quad K_{AB} = \text{Extrinsic curvature}, \quad K = \eta^{AB} K_{AB}.$$

In the last line we have used equation (681).

As in previous subsection we could also express $G^{(1)}$ as a vector field defined intrinsically on the membrane

$$\tilde{G}^{(1)}_\mu = -\frac{\hat{\nabla}^\nu \hat{F}_{\nu\mu}}{NK} - \frac{K^\nu_\mu G^{(0)}_\nu}{N} + \mathcal{O}\left(\frac{1}{D}\right), \tag{689}$$

where $\hat{F}_{\mu\nu}$ is the field strength along the surface and $\hat{\nabla}_\mu$ is the covariant derivative on the membrane surface, with respect to the intrinsic metric of the membrane. Also all raising lowering and contraction of indices have been done using the intrinsic metric of the membrane as embedded in flat space.

(689) is our result for the first Taylor coefficient of the internal solution expressed entirely in terms of the gauge field $G^{(0)}_A$ restricted to the membrane (which we denote here as $G^{(0)}_\mu$).

According to (595) and (624) we have

$$J^{(in)}_B = n^A \tilde{F}_{AB}|_{\rho=1} = N\tilde{G}^{(1)}_B + K^A_B G^{(0)}_A. \tag{690}$$

Substituting equation (688) in equation (690) to first subleading order we find

$$\begin{aligned} J^{(in)}_B &= -\frac{\Pi^C_B \partial^A \left[\Pi^{A'}_A F^{(0)}_{A'C'} \Pi^{C'}_C\right]}{K} + \mathcal{O}\left(\frac{1}{D}\right) \\ &= -\Pi^C_B \; \Pi^{A''A} \; \partial_A \left[\frac{\Pi^{A'}_{A''} F^{(0)}_{A'C'} \Pi^{C'}_C}{K}\right] + \mathcal{O}\left(\frac{1}{D}\right). \end{aligned} \tag{691}$$

It follows from (691) and (594) that the contribution of the internal solution to the current on the membrane is given by the spacetime source

$$\begin{aligned} \mathcal{J}^{in}_B &= -\left(\sqrt{\nabla\rho\cdot\nabla\rho}\right)\delta(\rho-1) J^{in}_B \\ &= \left(\sqrt{\nabla\rho\cdot\nabla\rho}\right)\delta(\rho-1)\left[\Pi^C_B \; \Pi^{A''A} \; \partial_A \left(\frac{\Pi^{A'}_{A''} F^{(0)}_{A'C'} \Pi^{C'}_C}{K}\right) + \mathcal{O}\left(\frac{1}{D}\right)\right]. \end{aligned} \tag{692}$$

As before we could also view the current as a vector defined only along the membrane.

$$J^{(in)}_\mu = -\frac{\hat{\nabla}^\nu F_{\nu\mu}}{K} + \mathcal{O}\left(\frac{1}{D}\right). \tag{693}$$

This current is consistent with (596) if we define

$$S_{int} = -\frac{1}{4}\int \frac{F_{\mu\nu}F^{\mu\nu}}{K}, \tag{694}$$

where the integration is now taken only over the membrane world volume. It may be verified using (597) and (688) that (694) is indeed the action of the interior solution. As explained in the previous section, the fact that the interior current is identically conserved follows immediately from the gauge invariance of the action (694).

Membrane Current Let us summarize. We have constructed the most general decaying solution to the linearized Maxwell equations in the exterior neighbourhood of a membrane surface. This solution is parametrized by one vector field $G^{(0)}_B$ on the membrane world volume, or equivalently a conserved current on the membrane world volume. The conserved current is given in terms of $G^{(0)}_B$ by the formula

$$\begin{aligned}
J^B &= J^B_{(out)} - J^B_{(in)} \\
&= \left[-(D-3)NG^{(0)}_B + NG^{(1)}_B + K^A_B G^{(0)}_A\right] - \left[N\tilde{G}^{(1)}_B + K^A_B G^{(0)}_A\right] \\
&= -(D-3)NG^{(0)}_B + N\left[G^{(1)}_B - \tilde{G}^{(1)}_B\right] \\
&= -(D-3)NG^{(0)}_B + \left(\frac{2\ \Pi^C_B}{K}\right)\partial^A\left[\partial_A G^{(0)}_C - \partial_C G^{(0)}_A\right] + \mathcal{O}\left(\frac{1}{D}\right) + \mathcal{O}\left(\frac{1}{D}\right).
\end{aligned} \tag{695}$$

Expressed as current as a vector intrinsic to the membrane, we find

$$\begin{aligned}
J_\mu &= J_{\mu(out)} - J_{\mu(in)} \\
&= \left[-(D-3)NG^{(0)}_\mu + NG^{(1)}_\mu + K^A_B G^{(0)}_\mu\right] - \left[N\tilde{G}^{(1)}_\mu + K^\nu_\mu G^{(0)}_\nu\right] \\
&= -(D-3)NG^{(0)}_\mu + N\left[G^{(1)}_\mu - \tilde{G}^{(1)}_\mu\right] \\
&= -(D-3)NG^{(0)}_\mu + \frac{2\hat{\nabla}^\nu F_{\nu\mu}}{K} + \mathcal{O}\left(\frac{1}{D}\right).
\end{aligned} \tag{696}$$

4.5.4 Membrane solutions of the linearized Einstein Equations

In this subsection we will find the most general solution of the Einstein equation linearized around flat space-time

$$\begin{aligned} g_{AB} &= \eta_{AB} + h_{AB}, \\ R_{AB} &= \frac{1}{2}\left(\partial_C \partial_A h^C_B + \partial_C \partial_B h^C_A - \Box h_{AB} - \partial_A \partial_B h^C_C\right) + \mathcal{O}(h^2) = 0. \end{aligned} \tag{697}$$

As explained in the previous section we proceed by first solving the *dynamical* Einstein equations (628) to determine the functionals F_1 and F_2 defined in (630). We construct these two functionals - to lowest nontrivial order - in the next two subsubsections. As in the previous subsection, in this subsubsection we find it convenient to use the bulk metrics $\eta_{MN} + h^{(0)}_{MN}$ and $\eta_{MN} + \tilde{h}^{(0)}_{MN}$ (see below) as the data in terms of which we write our solutions. The restrictions of these metrics to the membrane defines the intrinsic metric $g^{(ind)}_{\mu\nu}$ used as the data for the functionals F_1 and F_2 used in (630).

As explained in the previous section, once we have solved the dynamical equations, the constraint equation is automatic for the inner solution. For the outer solution it is simply the requirement that the Brown York stress tensor is conserved on the membrane approached from the outside. Below we will find explicit expressions for the Brown York Stress tensor on the membrane approached from both the outside and the inside in our solutions.

$\rho > 1$: Let us first study the external region $\rho > 1$. In analogy with previous subsections the solution in this region takes the form

$$h_{AB} = \left[\rho^{-(D-3)} \sum_{m=0}^{\infty} (\rho - 1)^m h^{(m)}_{AB}\right]. \tag{698}$$

As in the previous subsection we adopt a gauge condition adapted to the foliation of spacetime in slices of constant ρ

$$n^A h^{(m)}_{AB} = 0. \tag{699}$$

As in the previous subsection we impose the subsidiary conditions

$$\Pi^{C'}_B \Pi^C_A \,(n.\partial) h^{(m)}_{CC'} = 0. \tag{700}$$

on the expansion coefficients of (698). These conditions together with the gauge conditions (671) make (698) a well defined expansion of the metric function.

As in the previous subsection the functions $h^{(0)}_{AB}$ may be thought of as the basic data of the solutions. The dynamical Einstein equations determine the higher order coefficients in (698) in terms of $h^{(0)}_{AB}$. We present the details for how this works in Appendix 4.11.12. To first order in the expansion in $(\rho-1)$ and at leading order in $(1/D)$ we find

$$\begin{split} h^{(1)}_{AB} =& -\Pi^{C'}_{B}\Pi^{C}_{A}\left[\frac{\partial_C\partial^M h^{(0)}_{MC'}+\partial_{C'}\partial^M h^{(0)}_{MC}-\Box h^{(0)}_{CC'}+(D-3)Nh^{(0)}K_{CC'}+\partial_C\partial_{C'}h^{(0)}}{2(D-3)N^2-NK}\right] \\ &+\mathcal{O}\left(\frac{1}{D}\right) \\ =& -\Pi^{C'}_{B}\Pi^{C}_{A}\left[\frac{\partial_C\partial^M h^{(0)}_{MC'}+\partial_{C'}\partial^M h^{(0)}_{MC}-\Box h^{(0)}_{CC'}+(D-3)Nh^{(0)}K_{CC'}+\partial_C\partial_{C'}h^{(0)}}{NK}\right] \\ &+\mathcal{O}\left(\frac{1}{D}\right), \end{split} \tag{701}$$

where $h^{(0)}=\eta^{AB}h^{(0)}_{AB}$.

As explained in the previous section, (see around (636)), the Einstein constraint equation is simply the condition that the Brown York stress tensor

$$T^{(out)}_{AB}=\mathcal{K}^{(out)}_{AB}-\mathcal{K}^{(out)}\ \mathfrak{p}_{AB}, \tag{702}$$

is conserved on the membrane, w.r.t the induced metric on the membrane. Here $\mathcal{K}_{AB}$ is the extrinsic curvature of the $\rho=1$ slice, $\mathcal{K}$ is its trace, $\mathfrak{p}_{AB}$ is the projector on the $\rho=1$ slice.

At leading nontrivial order the stress tensor evaluated at $\rho=1$ turns out to be

$$\begin{split} T^{AB}_{(out)}=&\ \left(\tilde{K}^{AB}-\tilde{K}\tilde{\Pi}^{AB}\right)+\frac{N}{2}\left(h^{AB}_{(1)}-h^{(1)}\Pi^{AB}\right) \\ &-\frac{N}{2}(D-3)\left(h^{AB}_{(0)}-h^{(0)}\Pi^{AB}\right). \end{split} \tag{703}$$

Here $\tilde{K}^{AB}$ and $\tilde{\Pi}^{AB}$ denote the extrinsic curvature and the projector respectively on the membrane embedded in the metric $\left[\eta_{AB}+h^{(0)}_{AB}\right]$ and $\tilde{K}$ is the trace of $\tilde{K}^{AB}$.

$$\begin{split} \tilde{\Pi}^{AB} &= \eta^{AB}-n^An^B-h^{AB}_{(0)}, \\ \tilde{K}^{AB} &= K^{AB}-\frac{1}{2}\left[K_{AC}h^{CB}_{(0)}+K^{B}_{C}h^{CA}_{(0)}\right], \\ \tilde{K} &= \tilde{\Pi}_{AB}\tilde{K}^{AB}=\left[\Pi_{AB}+h^{(0)}_{AB}\right]\tilde{K}^{AB}=K+\mathcal{O}(h^2), \end{split} \tag{704}$$

where K_{AB} and Π_{AB} are respectively the extrinsic curvature and projectors on the $\rho=1$ slice as embedded in flat Minkowski space-time, $K\equiv\eta^{AB}K_{AB}$.

As explained in the previous section, this 'internal' stress tensor can be rewritten more elegantly in terms of purely intrinsic geometrical quantities on the membrane (see (643)). The less elegant expression (711) will, however, prove practically more useful to us in the next subsection, as the cancellations with the outer stress tensor (703) are more manifest in this form.

Plugging (701) into (703) yields an expression for the Brown York stress tensor purely in terms of $h^{AB}_{(0)}$. The requirement that this stress tensor is conserved on the membrane yields an effective constraint on $h^{AB}_{(0)}$. [139]

$\rho < 1$: As above, in the interior of the membrane we expand the metric as

$$\begin{aligned} \text{Bulk metric} = g_{AB} &= \eta_{AB} + \tilde{h}_{AB}, \\ \tilde{h}_{AB} &= \left[\sum_{m=0}^{\infty} (\rho - 1)^m \tilde{h}^{(m)}_{AB} \right]. \end{aligned} \tag{706}$$

As above we use the gauge condition

$$n^A \tilde{h}^{(m)}_{AB} = 0. \tag{707}$$

As above we require the coefficients of the expansion (706) to obey the additional subsidiary constraints

$$\Pi^{C'}_B \Pi^C_A \ (n.\partial)\tilde{h}^{(m)}_{CC'} = 0. \tag{708}$$

As above h^0_{AB} may be regarded as data of the solutions. The dynamical Einstein equations determine all other terms in the expansion in terms of data. At leading order we find (see Appendix 4.11.12 for details)

$$\tilde{h}^{(1)}_{AB} = \Pi^{C'}_B \Pi^C_A \left[\frac{\partial_{\bar{C}} \partial^M h^{(0)}_{MC'} + \partial_{C'} \partial^M h^{(0)}_{MC} - \Box h^{(0)}_{CC'} - \partial_C \partial_{C'} h^{(0)}}{NK} \right] + \mathcal{O}\left(\frac{1}{D}\right). \tag{709}$$

As explained in the previous section, the momentum constraint equations in the interior of

[139] Note that the stress tensor (703) is non vanishing even when $h_{AB} = 0$, i.e. when the spacetime metric is flat. The conservation of this zero order stress tensor w.r.t. the zero order metric (i.e. the induced metric on the surface $\rho = 1$ viewed as a submanifold of the flat bulk spacetime with metric η_{AB}.) on the membrane is a trivial identity. The conservation of (703), when expanded to first order in h_{AB} is nontrivial. If we expand the stress tensor (703) as $T_{AB} = T^0_{AB} + T^1_{AB}$ and the world volume metric on the membrane as $P_{AB} = P^0_{AB} + P^1_{AB}$ (where superscripts denote the order of expansion in h_{AB}) then the conservation equation, expanded to first order takes the schematic form

$$(\nabla^1)^M T^0_{MN} + (\nabla^0)^M T^1_{MN} = 0 \tag{705}$$

(here we have expanded the covariant derivative as $\nabla = \nabla^0 + \nabla^1$; as above superscripts keep track of the order of h_{AB} and have used the fact that $(\nabla^0)^M T^0_{MN}$ vanishes identically.) Note that the equation (705) asserts that T^1_{MN} is not quite a conserved stress tensor on the membrane. The lack of perfect conservation of T^1_{MN} is a direct consequence of the nonvanishing of T^0_{MN}.

the membrane assert the conservation of the stress tensor

$$8\pi T^{AB}_{(in)} = \mathcal{K}^{AB}_{(in)} - \mathcal{K}_{(in)}\, \mathfrak{p}^{AB}_{(in)}, \tag{710}$$

where $\mathcal{K}^{AB}_{(in)}$ and $\mathfrak{p}^{AB}_{(in)}$ are the extrinsic curvature and the projector on the membrane embedded in the metric $\left[\eta_{AB} + \tilde{h}_{AB}\right]$. $\mathcal{K}_{(in)}$ is the trace of $\mathcal{K}^{AB}_{(in)}$. Using the expansion equation (706) we find

$$T^{AB}_{(in)} = \left(\tilde{K}^{AB} - \tilde{K}\tilde{\Pi}^{AB}\right) + \frac{N}{2}\left(\tilde{h}^{AB}_{(1)} - \tilde{h}^{(1)}\Pi^{AB}\right). \tag{711}$$

As described before, here $\tilde{K}^{AB}$ and $\tilde{\Pi}^{AB}$ denote the extrinsic curvature and the projector respectively on the membrane embedded in the metric $\left[\eta_{AB} + h^{(0)}_{AB}\right]$ and $\tilde{K}$ is the trace of $\tilde{K}^{AB}$.

$$\begin{aligned} &\tilde{\Pi}^{AB} = \eta^{AB} - n^A n^B - h^{AB}_{(0)}, \\ &\tilde{K}^{AB} = K^{AB} - \frac{1}{2}\left[K_{AC}h^{CB}_{(0)} + K^B_C h^{CA}_{(0)}\right], \\ &\tilde{K} = \tilde{\Pi}_{AB}\tilde{K}^{AB} = \left[\Pi_{AB} + h^{(0)}_{AB}\right]\tilde{K}^{AB} = K + \mathcal{O}(h^2). \end{aligned} \tag{712}$$

The conserved membrane stress tensor The full membrane stress tensor is given by

$$\begin{aligned} 8\pi T^{AB} &= -\left(T^{AB}_{(out)} - T^{AB}_{(in)}\right) \\ &= \frac{N}{2}(D-3)\left(h^{AB}_{(0)} - h^{(0)}\Pi^{AB}\right) - \frac{N}{2}\left[h^{AB}_{(1)} - \tilde{h}^{AB}_{(1)} - (h^{(1)} - \tilde{h}^{(1)})\Pi^{AB}\right]. \end{aligned} \tag{713}$$

Now from equation (701) and (709) it follows that

$$\tilde{h}^{(1)}_{AB} = -h^{(1)}_{AB} - \left(\frac{D}{K}\right)h^{(0)}K_{AB} + \mathcal{O}\left(\frac{1}{D}\right).$$

Substituting we find

$$\begin{aligned} 8\pi T^{AB} = &\frac{N}{2}(D-3)\left(h^{AB}_{(0)} - h^{(0)}\Pi^{AB}\right) - \frac{K}{D}\left[h^{AB}_{(1)} - h^{(1)}\Pi^{AB}\right] \\ &+ \left(\frac{h^{(0)}}{2}\right)K_{AB} + \mathcal{O}\left(\frac{1}{D}\right). \end{aligned} \tag{714}$$

In the next section we shall see that for our particular solution $h^{(0)} \sim \mathcal{O}\left(\frac{1}{D}\right)$. In that case the expression for the final stress tensor simplifies further and we find.

$$
\begin{aligned}
8\pi T^{AB} = \; & \frac{N}{2}(D-3)\left(h^{AB}_{(0)} - h^{(0)}\Pi^{AB}\right) - \left(\frac{K}{D}\right)\left[h^{AB}_{(1)} - h^{(1)}\Pi^{AB}\right] \\
& + \mathcal{O}\left(\frac{1}{D}\right).
\end{aligned}
\tag{715}
$$

4.6 The Charge Current and Stress Tensor for the large D black hole membrane

4.6.1 Review of the nonlinear large D charged black hole membrane solutions

As reviewed in some detail in the introduction, the authors of [1, 63, 66] found a class of asymptotically flat solutions to the Einstein Maxwell equations. The solutions obtained in [1, 63, 66] are in one to one correspondence with the configuration (shape, velocity and charge density) of a membrane in flat space, and describe the dynamics of black holes in a large number of dimensions at time and distance scales of order unity.

The spacetime metric $\mathcal{G}_{MN}$ and gauge field $\mathfrak{a}_M$ of [1, 63, 66] take the schematic form

$$
\begin{aligned}
&\mathcal{G}_{MN} = \eta_{MN} + \mathfrak{g}_{MN}, \quad \mathfrak{g}_{MN} = \sum_{n=1}^{\infty} \frac{G^n_{MN}(\rho-1)}{\rho^{n(D-3)}}, \\
&\mathfrak{a}_N = \sum_{n=1}^{\infty} \frac{A^n_N(\rho-1)}{\rho^{n(D-3)}}.
\end{aligned}
\tag{716}
$$

The functions $G^n_{MN}(\rho-1)$ and $A^n_N(\rho-1)$ each admit a power series expansion in $\rho-1$. Schematically

$$
G^n_{MN}(\rho-1) = \sum_{k=0}^{\infty} G^{nk}_{MN}\ (\rho-1)^k, \quad A^n_N(\rho-1) = \sum_{k=0}^{\infty} A^{nk}_N\ (\rho-1)^k. \tag{717}
$$

The coefficients G^{nk}_{MN} and A^{nk}_{MN} are all finite in the limit $D \to \infty$ and each themselves admit a power series expansion in $\frac{1}{D}$, whose coefficients are various derivatives of the shape, velocity and charge density fields of the membrane.

The authors of [1, 63, 66] have developed a systematic perturbative procedure to determine the coefficients G^{nk}_{MN} and A^{nk}_M. The m^{th} iteration of the perturbative procedure of [1, 63, 66] simultaneously determines the coefficients G^{nk}_{MN} A^{nk}_N upto order $\frac{1}{D^{m-k}}$ (simultaneously for all n).

It follows that the m^{th} iteration allows systematic determination of the metric and gauge field to order $\frac{1}{D^m}$ for those values of ρ for which $\rho-1$ is of order $\frac{1}{D}$. This was, in fact, the method adopted in [1, 63, 66]. The authors of those papers work with a scaled coordinate $R = D(\rho-1)$

and then, in the m^{th} order of perturbation theory, systematically determine the gauge field and metric to order $1/D^m$. The fact that the authors of [1, 63, 66] found solutions of the full nonlinear Einstein Maxwell equations is reflected in the fact that the perturbative procedure works uniformly at every value of n in the expansion (716).

Note that (717) reduces to the expansions (698)and (670) when (716) is truncated to the term with $n = 1$. This observation makes perfect sense; the terms in (716) with $n \geq 2$ are all highly subdominant compared to the leading term when $\rho - 1 \gg \frac{1}{D}$. As explained in the introduction this is precisely the matching region in which we expect the general nonlinear solution of [1, 63, 66] to reduce to a particular linearized solution of the Einstein Maxwell equations. In fact the attentive reader will have noticed that the structure of the perturbative expansion described in the previous paragraph is *precisely* the structure employed to obtain the general solution to the linearized Einstein Maxwell solutions in section 4.5. In other words the solution of [1, 63, 66] is guaranteed to reduce to a special case of the construction of section 4.5 when we truncate (716) to $n = 1$.

In this section we will see how this works in detail in a particular example. Our starting point is the *first order* solution of the perturbative procedure of [1, 63, 66] presented in [63]. [140] In the rest of this section we massage the explicit solution of the chapter 3 to put it in the form (716) and (717). We then drop all terms with $n \geq 2$ in this expansion, identify the effective solution of section 4.5 that we are left with and thereby read off the membrane charge current and stress tensor of the solution.

In the rest of this subsection we simply recall the final result for the membrane metric and gauge field determined in chapter 3 in some detail. This solution is parametrized by the shape of a metric in flat space, a velocity field u_M on the membrane and a charge density field Q on the membrane As in earlier sections in this chapter, the symbols n_M denotes the normal of the flat space membrane while K_{NM} is its extrinsic curvature and K is the trace of K_{MN} in flat space. Like in chapter 3 we define

$$O_M = n_M - u_M.$$

In terms of all these quantities the metric and gauge field, presented in the previous chapter

[140]The results presented in 4.5 have since been generalized to one higher order in [66] for the special case of uncharged membranes. As this generalization has not yet been performed for the case of charged membranes, in this chapter we restrict our attention to metrics and gauge fields at first order in the derivative expansion, leaving the extension to second order results to future work.

is given by

$$
\begin{aligned}
\mathcal{G}_{MN} &= \eta_{MN} + \mathfrak{g}_{MN}\\
\mathfrak{g}_{MN} &= F(\rho) O_M O_N + \mathfrak{g}^{(T)}_{MN} + 2 O_{(M}\mathfrak{g}^{(V)}_{N)} + \mathfrak{g}^{(S)} O_M O_N + \mathfrak{g}^{(Tr)} P_{MN},\\
\sqrt{16\pi}\ \mathfrak{a}_M &= \sqrt{2} Q\ \rho^{-(D-3)}\ O_M + \left(\mathfrak{a}^{(S)} O_M + \mathfrak{a}^{(V)}_M\right),\\
&\text{where}\\
P_{MN} &= \eta_{MN} - O_M n_N - O_N n_M + O_M O_N,\\
P^{MN}\mathfrak{g}^{(V)}_N &= P^{MN}\mathfrak{a}^{(V)}_N = 0, \quad P^{MN}\mathfrak{g}^{(T)}_{MQ} = 0, \quad P^{MN}\mathfrak{g}^{(T)}_{MN} = 0,
\end{aligned}
\tag{718}
$$

The factor of $\sqrt{16\pi}$ in the third line of (718) is a consequence of the differences in the conventions used for the gauge field in chapter 3 and this chapter (see around (879)). The various free functions appearing in equations (718) are given by

$$
\begin{aligned}
\mathfrak{a}^{(V)}_M &= -\left(\frac{\sqrt{2}}{D}\right) Q\rho^{-D}\left[D(\rho-1)V^{(1)}_M - Q^2[1+\log(1-\rho^{-D}Q^2)]V^{(2)}_M\right]\\
&\quad + \mathcal{O}\left(\frac{1}{D}\right)^2,\\
\mathfrak{a}^{(S)} &= \left(\frac{1}{D}\right)\left[\sqrt{2}\ Q\ D(\rho-1)\ \rho^{-D} S^{(1)} + 2\sqrt{2}\left(\frac{Q^3}{1-Q^2}\right)\rho^{-D}\ \Upsilon_A(\rho)\ S^{(2)}\right] + \mathcal{O}\left(\frac{1}{D}\right)^2.
\end{aligned}
\tag{719}
$$

$$
\begin{aligned}
\mathfrak{g}^{(T)}_{MN} &= \left(\frac{2}{D}\right)\log(1-Q^2\rho^{-D})\ \tau_{MN} + \mathcal{O}\left(\frac{1}{D}\right)^2,\\
\mathfrak{g}^{(V)}_M &= \left(\frac{1}{D}\right)\left[Q^2\left[(F(\rho)-\rho^{-(D-3)}) + (F(\rho)-1)\log(1-Q^2\rho^{-D})\right] V^{(2)}_M\right.\\
&\qquad\left. - D(\rho-1)F(\rho)\ V^{(1)}_M\right] + \mathcal{O}\left(\frac{1}{D}\right)^2.
\end{aligned}
\tag{720}
$$

$$
\begin{aligned}
\mathfrak{g}^{(S)} &= -\sqrt{2}Q\ \rho^{-D}\mathfrak{a}^{(S)} + \left(\frac{1}{D}\right)\left[\rho^{-(D-3)} - F(\rho)\right]\\
&\quad + \left(\frac{2}{D}\right)\rho^{-D}\left[Q^2\ D(\rho-1)\ S^{(1)} + \Upsilon_H(\rho) S^{(2)}\right] + \mathcal{O}\left(\frac{1}{D}\right)^2,
\end{aligned}
\tag{721}
$$

$$
\mathfrak{g}^{(Tr)} = \mathcal{O}\left(\frac{1}{D}\right)^3,
$$

The different functions and the derivative structures that appear in equations (719), (720) and (721) are defined as[141]

Scalars	$S^{(1)} = \left(\frac{D}{K^2}\right) \bar{\nabla}^2 Q$ $S^{(2)} = \left(\frac{D}{K}\right) \left[u^A u^B K_{AB} - \frac{(u\cdot\partial)K}{K}\right]$
Vectors	$V^{(1)}_M = \left(\frac{D}{K}\right) \left[\frac{\bar{\nabla}^2 u_N}{K} + u^C K_{CN}\right] \; P^N_M$ $V^{(2)}_M = \left(\frac{D}{K}\right) \left[\frac{\partial_N K}{K} - (u\cdot\partial) u_N\right] \; P^N_M$
Tensor	$\tau_{MN} = P^{Q_1}_M \; \left(\frac{D}{K}\right) \left[\frac{\partial_{Q_1} O_{Q_2} + \partial_{Q_2} O_{Q_1}}{2} - \eta_{Q_1 Q_2} \left(\frac{\partial\cdot O}{D-2}\right)\right] \; P^{Q_2}_N$

Table 5: A listing of the 'first order' quantities that appear in the formula for the metric and gauge field, taken from chapter 3.

$$
\begin{split}
F(\rho) &= \left[(1+Q^2)\rho^{-(D-3)} - Q^2 \rho^{-2(D-3)}\right],\\
\Upsilon_A(\rho) &= \int_0^{D(\rho-1)} dx \; \log(1 - Q^2 e^{-x}),\\
\Upsilon_H(\rho) &= \left[(\rho^D - Q^2)\log(1 - Q^2\rho^{-D}) - (1-Q^2)\log(1-Q^2) + Q^2\left(\frac{1+Q^2}{1-Q^2}\right)\Upsilon_A(\rho)\right].\\
\bar{\nabla}^2 Q &= \Pi^A_B \partial_A \left[\Pi^{BC}\partial_C Q\right], \quad \bar{\nabla}^2 u_A = \Pi_{AA'}\Pi^B_C \partial_B \left[\Pi^{CC'}\Pi^{A'A''}(\partial_{C'} u_{A''})\right].
\end{split}
\tag{722}
$$

4.6.2 The Membrane Charge Current

From equation (719) it is not difficult to read off the corresponding value of A^1_M (see (716)). Recall that A^1_M is guaranteed to be a solution for the linearized Maxwell equations around flat space. We find

$$
\sqrt{16\pi} A^1_B \equiv M_B = \sum_{k=0}^{\infty} (\rho - 1)^k M^{(k)}_B, \tag{723}
$$

[141] Here our basis for the independent boundary data (the derivatives of velocity and the shape of the membrane) is little different from what has been used in the previous chapter. The basis we have used turns out to be more convenient for our analysis later in this chapter.

with

$$
\begin{aligned}
M_B^{(0)} &= \sqrt{2}Q\ O_B + \left(\frac{\sqrt{2}}{D}\right) Q^3 \left(\frac{D}{K}\right)\left(\frac{\partial_A K}{K} - (u\cdot\partial)u_A\right) P_B^A + \mathcal{O}\left(\frac{1}{D}\right)^2 , \\
M_B^{(1)} &= \sqrt{2}\left(\frac{D}{K}\right)\left(\frac{\bar{\nabla}^2 Q}{K}\right) O_B - \sqrt{2}\left(\frac{D}{K}\right)\left[\frac{\bar{\nabla}^2 u_A}{K} + u^C K_{CA}\right] P_B^A + \mathcal{O}\left(\frac{1}{D}\right), \\
&\text{where } O_B = n_B - u_B, \quad P_{AB} = \eta_{AB} - n_A n_B + u_A u_B = \Pi_{AB} + u_A u_B, \\
&\bar{\nabla}^2 Q = \Pi_B^A \partial_A \left[\Pi^{BC}\partial_C Q\right], \quad \bar{\nabla}^2 u_A = \Pi_{AA'}\Pi_C^B \partial_B \left[\Pi^{CC'}\Pi^{A'A''}(\partial_{C'}u_{A''})\right],
\end{aligned}
\tag{724}
$$

(for notational convenience we have renamed A_A^{1k} of (717) as $M_A^{(k)}$; we have dropped the superscript unity as we will only concern ourselves with the linearized part of the solution from now on).

As we have emphasized above, the configuration (723) is guaranteed to be a linearized solution of the form presented in subsection 4.5.3. As we have explained around that subsection, every such solution may is associated with a membrane current. This current is given by $J_M = J_M^{(out)} - J_M^{(in)}$ where $J_M^{(out)}$ is simply $n^N F_{NM}$ where F_{NM} is the field strength evaluated on the solution (723) above and $J_M^{(in)}$ is given by (691) where the field strength in that expression is once again evaluated on the configuration (723) using the solution (723). The algebra required to evaluate these two components of the current is straightforward; in Appendix 4.11.13 we demonstrate that

$$
\begin{aligned}
\sqrt{16\pi} J_B^{out} = \sqrt{2}\Bigg[& Q\left(K + \frac{\bar{\nabla}^2 K}{K^2} - \frac{2K}{D}\right) + (u\cdot\partial)Q - \left(\frac{\bar{\nabla}^2 Q + Q(u\cdot\partial)K}{K}\right) \\
& + Q(u^C u^{C'} K_{CC'})\Bigg] u_B - \sqrt{2}Q\left[\left(\frac{\partial_A Q}{Q}\right) + (u\cdot\partial)u_A\right] P_B^A + \mathcal{O}\left(\frac{1}{D}\right),
\end{aligned}
\tag{725}
$$

while

$$
\sqrt{16\pi} J_B^{(in)} = \sqrt{2}\Bigg[\left(\frac{\bar{\nabla}^2 Q}{K} + Q\ u^C u^{C'} K_{CC'}\right) u_B + Q\ P_B^A \left(\frac{\bar{\nabla}^2 u_A}{K}\right) - Q\ K_A^C u_C\Bigg] + \mathcal{O}\left(\frac{1}{D}\right).
\tag{726}
$$

Here we have used the following short-hand notation for derivatives projected along the membrane.

$$
\begin{aligned}
&\bar{\nabla}^2 K \equiv \Pi^{AB}\partial_A\left[\Pi_B^{B'}\partial_{B'}K\right], \quad \bar{\nabla}^2 Q \equiv \Pi^{AB}\partial_A\left[\Pi_B^{B'}\partial_{B'}Q\right], \\
&\bar{\nabla}_A \bar{u}_B \equiv \Pi_A^{A'}\Pi_B^{B'}\partial_{A'}u_{B'}, \quad \bar{\nabla}^2 u_A \equiv \Pi^{CB}\partial_C\left[\Pi_A^{A'}\Pi_B^{B'}\partial_{B'}u_{A'}\right],
\end{aligned}
\tag{727}
$$

(725) and (726) are our final results for the internal and external contributions to the membrane

current. Putting them together we find

$$J_B = J_B^{(out)} - J_B^{(in)}.$$

Subtracting equation (725) from equation (726) we find

$$\begin{split}\sqrt{16\pi}J_B = \sqrt{2}\left[Q\left(K + \frac{\bar{\nabla}^2 K}{K^2} - \frac{2K}{D}\right) + (u\cdot\partial)Q - \left(\frac{2\bar{\nabla}^2 Q + Q(u\cdot\partial)K}{K}\right)\right]u_B \\ - \sqrt{2}Q\left[\left(\frac{\partial_A Q}{Q}\right) + (u\cdot\bar{\nabla})u_A + \left(\frac{\bar{\nabla}^2 u_A}{K}\right) - K_A^C u_C\right]P_B^A + \mathcal{O}\left(\frac{1}{D}\right).\end{split} \tag{728}$$

Note that by construction J_B is a vector tangent to the membrane and also all the derivatives that appears in the expression of J_B are all along the membrane. All these derivatives could re expressed as covariant derivatives with respect to the intrinsic metric of the membrane. In terms of the coordinates intrinsic to the membrane we write the current as

$$\begin{split}\sqrt{16\pi}J^\mu = \sqrt{2}\left[Q\left(K + \frac{\hat{\nabla}^2 K}{K^2} - \frac{2K}{D}\right) + (u\cdot\hat{\nabla})Q - \left(\frac{2\hat{\nabla}^2 Q + Q(u\cdot\hat{\nabla})K}{K}\right)\right]u^\mu \\ - \sqrt{2}Q\left[\left(\frac{\hat{\nabla}_\nu Q}{Q}\right) + (u\cdot\hat{\nabla})u_\nu + \left(\frac{\hat{\nabla}^2 u_\nu}{K}\right) - K_\nu^\alpha u_\alpha\right]p^{\nu\mu} + \mathcal{O}\left(\frac{1}{D}\right), \\ \text{where } p_{\mu\nu} = g_{\mu\nu}^{(ind,f)} + u_\mu u_\nu,\ \ \hat{\nabla}_\mu = \text{Covariant derivative w.r.t } g_{\mu\nu}^{(ind,f)}, \\ g_{\mu\nu}^{(ind,f)} = \text{Induced metric on membrane, embedded in flat space-time} \\ (\cdot) \text{ denotes contraction w. r. t } \ g_{\mu\nu}^{(ind,f)}.\end{split} \tag{729}$$

4.6.3 A consistency check

In the previous subsection we obtained the results for the membrane charge current assuming that the configuration (723) is indeed a particular case of a solution of the general solution presented in subsection 4.5.3. While this must be the case on logical grounds, it is, of course, reassuring to have a direct algebraic check of this claim. We have performed such a direct check; in this subsection we present a brief explanation of the check we have here relegating most details to Appendix 4.11.13.

In subsection 4.5.3 we argued that the most general linearized solution to the Maxwell equation is parametrized by the single function $G_A^{(0)}$, the gauge field on the membrane. The Taylor series coefficients of this gauge field off the membrane are completely determined in terms of $G_A^{(0)}$. In particular, to first order, $G_A^{(1)}$ is given in terms of $G_A^{(0)}$ by (680). We will now verify that (724) is consistent with (680).

Roughly speaking, $G_A^{(0)}$ is simply $M_B^{(0)}$ while $G_A^{(1)}$ is $M_B^{(1)}$ (see (724)). However this is not completely accurate for two reasons

- The analysis of the previous section was performed with the choice of gauge $n^B.G_B = 0$. Unfortunately the solution (723) is presented in a different gauge. In order to compute $G_M^{(0)}$ and $G_M^{(1)}$, consequently, we must either compute gauge invariants or perform a gauge transformation that puts the solution (723) into the gauge $n^B.G_B = 0$. We found it more convenient to actually perform the gauge transformation.
- The statement that $G_A^{(0)}$ is given by (724) evaluated at $\rho = 1$ is unambiguous. However the statement that $G_M^{(1)}$ is the part of (724) proportional to $(\rho - 1)$ is meaningful only once we have agreed on a set of subsidiary conditions on the coefficients of the expansion in $(\rho - 1)$. In the analysis of the previous section we assumed that all coefficient functions obeyed the subsidiary conditions (673). The coefficient functions in (724) turn out not to obey these subsidiary conditions (the coefficients in (724) obey the subsidiary conditions employed in chapter 3, which are slightly different from (673)). Consequently they have to be re-expanded in terms of quantities that do obey (673) before we can read off $G_A^{(1)}$.

In Appendix 4.11.13 we have carefully dealt with both these issues, and verified that the solution (724) does indeed take the general form presented in subsection 4.5.3 with

$$\begin{aligned}
\sqrt{16\pi}G_B^{(0)} &= -\sqrt{2}Q\ u_B + \frac{\sqrt{2}Q^3}{D}\left(\frac{D}{K}\right)\left(\frac{\partial_A K}{K} - (u\cdot\partial)u_A\right)P_B^A \\
&\quad + \sqrt{2}\Pi_B^A\left[\frac{\partial_A Q}{K} - \frac{Q\partial_A K}{K^2}\right] + \mathcal{O}\left(\frac{1}{D}\right)^2, \\
\sqrt{16\pi}G_B^{(1)} &= \left[\tilde{M}_B^{(1)} + C_B^{(0)}\right] \\
&= -\sqrt{2}\left(\frac{D}{K}\right)\left(\frac{\bar{\nabla}^2 Q}{K}\right)u_B - \sqrt{2}Q\left(\frac{D}{K}\right)\left(\frac{P_A^B\bar{\nabla}^2 u_B}{K}\right) + \mathcal{O}\left(\frac{1}{D}\right).
\end{aligned} \tag{730}$$

4.6.4 Membrane equation of motion from conservation of the charge current

In section 4.5 we have argued that any membrane constructed out of the general linearized solution of the Maxwell equations presented in that section must be automatically conserved. Earlier in this section we have used the formalism of section 4.5 to explicitly determine a charge current for the membrane spacetimes of [1, 63, 66]. Our final result, presented in (728) is given in terms of the curvatures, charge and velocity derivatives of the large D black hole membrane. If the analysis presented in this chapter is self consistent it must turn out that the charge current (728) - which can simply be algebraically determined in terms of membrane curvatures, velocity and charge derivatives - must automatically vanish using only constraints between these derivatives that were already determined in [1, 63, 66]. In this subsection we explain how this works in detail.

At leading order in the large D limit, the current (728) takes the form

$$\sqrt{16\pi}J^\mu = \sqrt{2}QKu^\mu \tag{731}$$

and is of order D [142] . The divergence of a current of order $\mathcal{O}(D)$ is generically of order $\mathcal{O}(D^2)$. In the current context the naively order $\mathcal{O}(D^2)$ term in the divergence of the leading order current is given by

$$\sqrt{16\pi}\hat{\nabla}_\mu J^\mu = \sqrt{2}QK\left(\hat{\nabla}_\mu u^\mu\right) + \mathcal{O}(D). \tag{732}$$

This expression is naively of order $\mathcal{O}(D^2)$ because K is of order $\mathcal{O}(D)$ and $\left(\nabla_B u^B\right)$ would also be of order $\mathcal{O}(D)$ if u were an unrestricted arbitrary velocity field. The fact that the divergence of the charge current must vanish tells us that u *cannot* be an unrestricted velocity field; it must, in fact, be chosen to ensure that

$$\left(\hat{\nabla}_\mu u^\mu\right) = \mathcal{O}(1). \tag{733}$$

The requirement (733) is the first of (546) and was, in fact, the starting point of the membrane construction of [1, 63, 66].

In this chapter we have systematically determined the large D membrane charge current (728) upto $\mathcal{O}(1)$. [143] As the operation of taking the divergence generically increases the order of D of a current by one power, our knowledge of the charge current (728) is sufficient to determine the divergence of this current only to order $\mathcal{O}(D)$. We have already explained that the condition (733) ensures that the divergence of the charge current vanishes at order $\mathcal{O}(D^2)$. We will now explore the requirement that this divergence also vanishes at order $\mathcal{O}(D)$.

Apart from the expression listed in (731), every term in (728) is of $\mathcal{O}(1)$ rather than order $\mathcal{O}(D)$. While a generic term in a current of order unity has a divergence of order D, it follows from (733) that any term of order unity proportional to u^M has a divergences of order unity. It follows that such terms do not contribute to the divergence of the charge current at order $\mathcal{O}(D)$. Dropping all such terms we find the simplified current

$$\sqrt{16\pi}J^{(simp)}_\mu = \sqrt{2}Q\left\{Ku_\alpha - \left[\left(\frac{\hat{\nabla}_\mu Q}{Q}\right) + (u\cdot\hat{\nabla})u_\mu + \left(\frac{\hat{\nabla}^2 u_\mu}{K}\right) - K^\nu_\mu u_\nu\right]p^\mu_\alpha\right\} + \mathcal{O}\left(\frac{1}{D}\right), \tag{734}$$

[142] This scaling is because K is of order D as explained in chapter 3 - see the introduction.

[143] The determination of the charge current to order $\mathcal{O}(1/D)$ requires knowledge of the gauge field in the solutions of [1, 63, 66] at order $\mathcal{O}(1/D)$ which has not yet been worked out.

whose divergence is given by

$$
\begin{aligned}
&\sqrt{16\pi}\hat{\nabla}_\mu J^\mu_{(simp)} \\
&= -\hat{\nabla}_\mu\Bigg\{\left[\hat{\nabla}_\nu Q + Q(u\cdot\hat{\nabla})u_\nu + Q\left(\frac{\hat{\nabla}^2 u_\nu}{K}\right) - QK^\alpha_\nu u_\alpha\right]p^{\nu\mu}\Bigg\} \\
&\quad + KQ(\hat{\nabla}\cdot u) + K(u\cdot\hat{\nabla})Q + Q(u\cdot\hat{\nabla})K + \mathcal{O}(1)
\end{aligned}
\tag{735}
$$

$$
= K\Bigg\{Q(\hat{\nabla}\cdot u) + (u\cdot\hat{\nabla})Q + Q\left[\frac{(u\cdot\hat{\nabla})K}{K}\right] - \left[\frac{\hat{\nabla}^2 Q}{K}\right] - Q\ (u^\mu u^\nu K_{\mu\nu})\Bigg\} + \mathcal{O}(1).
$$

In computing equation (735) we have used the identities (1101) and (1106). [144]

In the analysis of [1, 63, 66] it turns out that $\nabla\cdot u = \mathcal{O}(1/D)$. Moreover the 'charge' equation of motion of the previous chapter asserts that

$$
(u\cdot\hat{\nabla})Q + Q\left[\frac{(u\cdot\hat{\nabla})K}{K}\right] - \left[\frac{\hat{\nabla}^2 Q}{K}\right] - Q\ (u^\mu u^\nu K_{\mu\nu}) = \mathcal{O}(1/D). \tag{736}
$$

It follows that the last line of (735) - and so the divergence of the charge current (728) - does indeed vanish at order D.

In summary, the charge current computed in (728) is indeed divergence free; the fact that this is the case is, in fact, a restatement of the 'charge' equation of motion of chapter 3.

4.6.5 The Membrane Stress Tensor and its conservation

In the rest of this section we imitate the analysis already presented for the membrane charge current in order to obtain and analyse the large D black hole membrane stress tensor. As the logic of our construction proceeds in close analogy with the case of the charge current we keep our explanations brief.

Expanding the metric presented in (718),(720) and (721)) in the form (716), it is not difficult to show that the function G^1_{MN} in (716) (which, for notational convenience, we refer to below as M_{AB}) is given by

$$
G^1_{MN} \equiv M_{AB} = \sum_n (\rho-1)^n M^{(n)}_{AB}, \tag{737}
$$

[144] We emphasize that it is permissible to replace the full charge current J_μ by J^{simp}_μ only for the purposes of computing its divergence and not for the purposes of computing radiation.

where

$$
\begin{aligned}
M^{(0)}_{AB} &= (1+Q^2)O_AO_B + 2Q^4\left(O_AV^{(2)}_B + O_BV^{(2)}_A\right) - Q^2O_AO_B - 2Q^2\tau_{AB} \\
&\quad + \mathcal{O}\left(\frac{1}{D}\right)^2, \\
M^{(1)}_{AB} &= 2Q^2S^{(1)}O_AO_B - (1+Q^2)\left[V^{(1)}_AO_B + O_AV^{(1)}_B\right] \\
&\quad + \mathcal{O}\left(\frac{1}{D}\right),
\end{aligned}
\tag{738}
$$

with[145]

$$
\begin{aligned}
V^{(1)}_A &= \left(\frac{D}{K}\right)\left[\frac{\bar{\nabla}^2u_B}{K} + u^CK_{CB}\right]\ P^B_A, \\
V^{(2)}_A &= \left(\frac{D}{K}\right)\left[\frac{\partial_CK}{K} - (u\cdot\partial)u_C\right]P^C_A, \\
S^{(1)} &= \left(\frac{D}{K^2}\right)\bar{\nabla}^2Q, \\
\tau_{AB} &= P^{A'}_A\ \left(\frac{D}{K}\right)\left[\frac{\partial_{A'}O_{B'} + \partial_{B'}O_{A'}}{2} - \eta_{A'B'}\left(\frac{\partial\cdot O}{D-2}\right)\right]\ P^{B'}_B, \\
&\text{where} \\
\bar{\nabla}^2Q &= \Pi^A_B\partial_A\left[\Pi^{BC}\partial_CQ\right],\quad \bar{\nabla}^2u_A = \Pi_{AA'}\Pi^B_C\partial_B\left[\Pi^{CC'}\Pi^{A'A''}(\partial_{C'}u_{A''})\right].
\end{aligned}
\tag{739}
$$

The metric (737) is a particular example of the general linearized solution to the Einstein equations presented in subsection (4.5.4). As in the previous subsection we have also verified in detail that the solution (737) and (738) after appropriate transformation agrees with the general structure listed in subsection 4.5.4 provided we identify

$$
\begin{aligned}
h^{(0)}_{AB} &= (1+Q^2)\ u_Au_B \\
&\quad + \left(\frac{1}{D}\right)\Big[-2Q^4\left(u_AV^{(2)}_B + u_BV^{(2)}_A\right) - Q^2u_Au_B - 2Q^2\ \tau_{AB} \\
&\qquad + \Pi^C_A\left[\partial_C\zeta_{C'} + \partial_{C'}\zeta_C\right]\Pi^{C'}_B\Big] + \mathcal{O}\left(\frac{1}{D}\right)^2, \\
h^{(1)}_{AB} &= \left(\frac{D}{K^2}\right)\left[2Q\bar{\nabla}^2Q\ u_Au_B + (1+Q^2)\ \Pi^C_B\left(u_A\bar{\nabla}^2u_C + u_C\bar{\nabla}^2u_A\right)\right] + \mathcal{O}\left(\frac{1}{D}\right),
\end{aligned}
\tag{740}
$$

where

$$
\zeta_A = (1+Q^2)\left(\frac{D}{K}\right)\left(\frac{n_A}{2} - u_A\right). \tag{741}
$$

[145] In equation (737) and (738) we simply renamed G^{1k}_{AB} as $M^{(k)}_{AB}$ to avoid confusion.

We have, in particular, verified that the results quoted in (740) are consistent with (701).

From the first equation of (740) it follows that the trace of $h^{(0)}_{AB}(= \Pi^{AB} h^{(0)}_{AB})$ is of order $\mathcal{O}\left(\frac{1}{D}\right)$, which justifies our expression of stress tensor as given in equation (715) of previous section.

According to the general analysis of that subsection, any such solution is associated with a stress tensor, which is given by the difference between the Brown York stress tensor evaluated on the metric (737) and the expression (711) evaluated on the same solution. Also in equation (715), we have an expression for the final stress tensor explicitly in terms of $h^{(0)}_{AB}$ and $h^{(1)}_{AB}$. Substituting equation (740) in equation (715) we find the explicit expression for the stress tensor for metric (738).

At this stage to simplify our calculation of stress tensor we shall use a trick. We shall define $T^{(NT)}_{AB}$ as

$$T^{(NT)}_{AB} = \frac{N}{2}(D-3)h^{(0)}_{AB} - \left(\frac{K}{D}\right) h^{(1)}_{AB}.$$

Then from equation (715) we could clearly see that $T_{AB} - T^{(NT)}_{AB} \propto \Pi_{AB}$. We write the proportionality factor as Δ. With this notation the stress tensor could be written as

$$8\pi T_{AB} = 8\pi \left[T^{(NT)}_{AB} + \Delta\ \Pi_{AB}\right]. \tag{742}$$

Now we shall determine Δ using the condition that $K_{AB}T^{AB} = 0$ (see equation (607)).

$$K_{AB}T^{AB} = 0 \Rightarrow \Delta = -\frac{K^{AB}T^{(NT)}_{AB}}{K}.$$

Now collecting all these pieces together we finally get the explicit expression for the stress tensor

$$\begin{aligned} 8\pi T^{(NT)}_{AB} =& \left(\frac{K}{2}\right)(1+Q^2)u_A u_B + \left(\frac{1-Q^2}{2}\right)K_{AB} - \left(\frac{\bar{\nabla}_A u_B + \bar{\nabla}_B u_A}{2}\right) \\ & - \left(\frac{KQ^2}{2D} + \frac{2Q\bar{\nabla}^2 Q}{K} + Q^2 u^C u^{C'} K_{CC'}\right) u_A u_B - (u_A \mathcal{V}_B + u_B \mathcal{V}_A) + \mathcal{O}\left(\frac{1}{D}\right), \end{aligned} \tag{743}$$

$$\begin{aligned} \mathcal{V}_A =& Q\ \bar{\nabla}_A Q + Q^2(u^C K_{CA}) + \left(\frac{2Q^4 - Q^2 - 1}{2}\right)\left(\frac{\bar{\nabla}_A K}{K}\right) \\ & - \left(\frac{Q^2 + 2Q^4}{2}\right)(u\cdot\bar{\nabla})u_A + \left(\frac{1+Q^2}{K}\right)\bar{\nabla}^2 u_A \end{aligned}$$

and

$$\Delta = -\left[\left(\frac{1+Q^2}{2}\right)(u^A u^B K_{AB}) + \left(\frac{1-Q^2}{2}\right)\left(\frac{K}{D}\right) + \mathcal{O}\left(\frac{1}{D}\right)\right]. \tag{744}$$

As in the previous subsection, $\bar{\nabla}_A$ defines the projected derivative along the membrane as embedded in flat Minkowski space. See equation (727) for a more precise definition.Also in our algebra we used the fact that to leading order in $\frac{1}{D}$,

$$K^{AB}K_{AB} = \frac{K^2}{D} + \mathcal{O}(1).$$

In equations (742), (743) and (744) all derivatives and all free and contracted indices are along the membrane. Therefore we can as well re-express the stress tensor as a tensor defined completely on the membrane, where all projected derivatives are replaced by covariant derivatives, defined with respect to the membrane's intrinsic metric.

$$\begin{aligned}
8\pi T_{\mu\nu} = & \left(\frac{K}{2}\right)(1+Q^2)u_\mu u_\nu + \left(\frac{1-Q^2}{2}\right)K_{\mu\nu} - \left(\frac{\hat{\nabla}_\mu u_\nu + \hat{\nabla}_\nu u_\mu}{2}\right) \\
& - \left(\frac{KQ^2}{2D} + \frac{2Q\hat{\nabla}^2 Q}{K} + Q^2 u^\alpha u^\beta K_{\alpha\beta}\right)u_\mu u_\nu - (u_\mu \mathcal{V}_\nu + u_\nu \mathcal{V}_\mu) \\
& - \left[\left(\frac{1+Q^2}{2}\right)\left(u^\alpha u^\beta K_{\alpha\beta}\right) + \left(\frac{1-Q^2}{2}\right)\left(\frac{K}{D}\right)\right]g^{(ind,f)}_{\mu\nu} \\
& + \mathcal{O}\left(\frac{1}{D}\right),
\end{aligned} \tag{745}$$

where

$$\begin{aligned}
\mathcal{V}_\mu = & \, Q\,\hat{\nabla}_\mu Q + Q^2(u^\alpha K_{\alpha\mu}) + \left(\frac{2Q^4 - Q^2 - 1}{2}\right)\left(\frac{\hat{\nabla}_\mu K}{K}\right) \\
& - \left(\frac{Q^2 + 2Q^4}{2}\right)(u\cdot\hat{\nabla})u_\mu + \left(\frac{1+Q^2}{K}\right)\hat{\nabla}^2 u_\mu.
\end{aligned} \tag{746}$$

Conservation of the stress tensor In this subsection we shall compute the divergence of the stress tensor (742) and demonstrate that it vanishes at order $\mathcal{O}(D^2)$ and at order $\mathcal{O}(D)$ once we impose the membrane equations of motion.

As in the case of the charge current, the stress tensor has a leading order piece

$$8\pi T_{\mu\nu} = \left(\frac{K}{2}\right)(1+Q^2)u_\mu u_\nu, \tag{747}$$

which is of order $\mathcal{O}(D)$. All other terms in (742) are of order $\mathcal{O}(1)$. As in our analysis of the charge current the divergence of (747) is naively of order $\mathcal{O}(D^2)$; the requirement that the divergence vanish at this order reimposes the condition (733). As in the case of the charge current we must now impose the condition that the divergence of the stress tensor vanishes also at order $\mathcal{O}(D)$.

The order $\mathcal{O}(D)$ part of this divergence receives contributions only from those $\mathcal{O}(1)$ terms in (742) whose divergences is of order $\mathcal{O}(D)$. This criterion excludes all order $\mathcal{O}(1)$ terms proportional to $g^{(ind,f)}_{\mu\nu}$ in (742). [146] In order to compute the divergence of the stress tensor at order D, it follows that we can replace the stress tensor in (742) by the simpler effective stress tensor $T^{(eff)}_{\mu\nu}$.

$$\begin{aligned} T^{(eff)}_{\mu\nu} = & \left(\frac{K}{2}\right)(1+Q^2)u_\mu u_\nu + \left(\frac{1-Q^2}{2}\right)K_{\mu\nu} \\ & - \left(\frac{\hat{\nabla}_\mu u_\nu + \hat{\nabla}_\nu u_\mu}{2}\right) - (u_\mu \mathcal{V}_\nu + u_\nu \mathcal{V}_\mu), \end{aligned} \tag{749}$$

where $\mathcal{V}_\mu$ is defined in equation (746).

The divergence of $T^{(eff)}_{\mu\nu}$ has a free index and so can be decomposed into the part orthogonal to u^μ and the part in the direction of u^μ. We will find it convenient to give these two different pieces names. Let

$$E^\mu \equiv p^\mu_\nu \, \hat{\nabla}_\alpha T^{\alpha\nu}_{(eff)}$$

and let

$$E \equiv u_\nu \hat{\nabla}_\alpha T^{\alpha\nu}_{(eff)}.$$

We will first demonstrate that the requirement that E^μ vanish at order $\mathcal{O}(D)$ is simply a re-statement of the motion. On the other hand the requirement that E vanish at order D tells us $(\hat{\nabla}\cdot u)$ is of order $\mathcal{O}\left(\frac{1}{D}\right)$ or smaller (this is a strengthening of the condition (733)). As both these conditions were independently met in chapter 3, it follows that the stress tensor dual to the large D membrane is indeed conserved.

[146] In order to see this recall that $\hat{\nabla}_\mu g^{\mu\nu}_{(ind,f)} = 0$ identically. Therefore

$$\hat{\nabla}^\mu \left[\Delta \; g^{(ind,f)}_{\mu\nu}\right] = \hat{\nabla}_\nu \Delta = \mathcal{O}(1). \tag{748}$$

We now turn to a demonstration of the first of these assertions.

$$
\begin{aligned}
E^\mu =& -\left(\frac{K}{2}\right)(1+Q^2)(u\cdot\hat\nabla)u^\mu - \left(\frac{1-Q^2}{2}\right)p^{\mu\nu}\hat\nabla_\alpha K^\alpha_\nu \\
&+ p^{\mu\alpha}\left(\frac{\hat\nabla^2 u_\alpha + \hat\nabla_\nu\hat\nabla_\alpha u^\nu}{2}\right) + \mathcal{O}(1) \\
=& -\left(\frac{K}{2}\right)\Bigg[(1+Q^2)(u\cdot\hat\nabla)u^\mu + (1-Q^2)p^{\mu\nu}\left(\frac{\hat\nabla_\nu K}{K}\right) \\
&\qquad - p^{\mu\nu}\left(\frac{\hat\nabla^2 u_\nu}{K} + \frac{\hat\nabla_\alpha\hat\nabla_\nu u^\alpha}{K}\right)\Bigg] + \mathcal{O}(1) \\
=& -\left(\frac{K}{2}\right)\Bigg[(1+Q^2)(u\cdot\hat\nabla)u^\mu + (1-Q^2)p^{\mu\nu}\left(\frac{\hat\nabla_\nu K}{K}\right) \\
&\qquad - p^{\mu\nu}\left(\frac{\hat\nabla^2 u_\nu}{K} + K_{\nu\alpha}u^\alpha\right)\Bigg] + \mathcal{O}(1).
\end{aligned}
\tag{750}
$$

In the last step we have used identities (1103) and (1107) and also (733).

We now turn to the quantity E. After a little bit of algebra (see appendix (4.11.14) we are able to show that

$$
\begin{aligned}
E \equiv\ & u_\nu\hat\nabla_\alpha T^{\alpha\nu}_{(eff)} \\
=& \left(\frac{K}{2}\right)(1+Q^2)(\hat\nabla\cdot u) - (1+2Q^2)(u\cdot\hat\nabla)K + \frac{K}{2}(u^\mu u^\nu K_{\mu\nu})\left(1+3Q^2+2Q^4\right) \\
&+ \frac{1}{2}\left(\frac{D^2 K}{K}\right)\left(1+Q^2-2Q^4\right) + \mathcal{O}(1) \\
=& \frac{(1+2Q^2)}{2}\left[-2(u\cdot\hat\nabla)K + K(1+Q^2)(u^\mu u^\nu K_{\mu\nu}) + (1-Q^2)\left(\frac{\hat\nabla^2 K}{K}\right)\right] \\
&+ \left(\frac{K}{2}\right)(1+Q^2)(\hat\nabla\cdot u) + \mathcal{O}(1) \\
=& \left(\frac{K}{2}\right)(1-Q^2)(\hat\nabla\cdot u) - \left(\frac{1+2Q^2}{K}\right)(\hat\nabla_\mu E^\mu) + \mathcal{O}(1).
\end{aligned}
\tag{751}
$$

We have already argued above that all the E_μ are of order unity or smaller. It follows that $\nabla_\mu E^\mu$ is of order D or smaller and so (751) implies that

$$
\begin{aligned}
&\left(\frac{K}{2}\right)(1+Q^2)(\hat\nabla\cdot u) = \mathcal{O}(1) \\
\Rightarrow\ &(\hat\nabla\cdot u) = \mathcal{O}\left(\frac{1}{D}\right),
\end{aligned}
\tag{752}
$$

as we claimed above.

4.6.6 Stress Tensor and current conservation imply the membrane equations of motion

We have already demonstrated above that the membrane equations of motion are sufficient to ensure that the charge current and stress tensors dual to the solutions constructed in chapter 3 are automatically conserved. In this brief subsection we point out that the relationship between the membrane equations of motion and current conservation can be reversed. Just as the equations of motion imply current and stress tensor conservation, the conservation equations in turn imply the membrane equations of motion.

The argument is immediate. The first membrane equation is simply (750), which we have already derived as a consequence of conservation. Plugging (752) in equation (735) then yields the second membrane equation. In other words the conservation equations directly imply

$$\begin{aligned}
&(1+Q^2)(u\cdot\hat{\nabla})u^\mu + (1-Q^2)(p^{\mu\nu}\nabla_\nu K) - P^{\mu\nu}\left(\frac{\hat{\nabla}^2 u_\nu}{K} + K_{\nu\alpha}u^\alpha\right) = \mathcal{O}(1),\\
&(u\cdot\hat{\nabla})Q + Q\left[\frac{(u\cdot\hat{\nabla})K}{K}\right] - \left[\frac{\hat{\nabla}^2 Q}{K}\right] - Q\ (u^\mu u^\nu K_{\mu\nu}) = \mathcal{O}(1),
\end{aligned} \tag{753}$$

the two membrane equations of motion listed in the introduction of chapter 3.

4.6.7 Qualitative discussion of the uncharged membrane stress tensor and resulting equation of motion

In this subsection we focus our attention to the relatively simple case of an uncharged membrane. In this special case we re-discuss the structure of the membrane stress tensor and resulting equation of motion emphasizing qualitative features. The purpose of this subsection is to help the reader develop some intuition for the structure of the large D membrane.

Let us first note that the expression for the membrane stress tensor, (555), simplifies considerably when we specialize to the study of uncharged membranes. We find

$$\begin{aligned}
T_{\mu\nu} = \left(\frac{1}{8\pi}\right)\Bigg[&\left(\frac{K}{2}\right)u_\mu u_\nu + \left(\frac{1}{2}\right)K_{\mu\nu} - \left(\frac{\hat{\nabla}_\mu u_\nu + \hat{\nabla}_\nu u_\mu}{2}\right)\\
&+\left(\frac{u_\mu\hat{\nabla}_\nu K + u_\nu\hat{\nabla}_\mu K}{2K}\right) - \left(\frac{u^\alpha u^\beta K_{\alpha\beta}}{2} + \frac{K}{2D}\right)g^{(ind,f)}_{\mu\nu}\Bigg]\\
&+\mathcal{O}\left(\frac{1}{D}\right).
\end{aligned} \tag{754}$$

At leading order in the large D limit (754) simplifies to

$$T_{\mu\nu} = \frac{K}{16\pi} u_\mu u_\nu. \tag{755}$$

This term is of order $\mathcal{O}(D)$ because K is of order $\mathcal{O}(D)$; all other terms in (754) are of order $\mathcal{O}(1)$. Note, in particular, that the leading order stress tensor lacks a 'surface tension' term proportional to $g^{(ind,f)}_{\mu\nu}$. (755) appears to assert that the large D black hole membrane is made up of a collection of pressure free dust particles with density proportional to K. This slogan is misleading, as we now explain.

The divergence of (755) is given by

$$\hat{\nabla}^\nu T_{\mu\nu} = \frac{(\hat{\nabla} \cdot u)K}{16\pi} u_\mu + \frac{(u \cdot \hat{\nabla})(K u^\mu)}{16\pi}. \tag{756}$$

The first term in (756) is or order D^2 while the second term is of order $\mathcal{O}(D)$ (recall that $(\hat{\nabla} \cdot u)$ is of order D). Setting the divergence of the stress tensor to zero at order $\mathcal{O}(D^2)$ immediately yields the condition that $(\hat{\nabla} \cdot u) = 0$. We emphasize that even though (755) is the stress tensor of a collection of pressure free dust particles of (variable) density $\frac{K}{16\pi}$, one of the equations of motion that follows from the conservation of (755) asserts that the velocity flow u^μ is incompressible The reason for this apparent dissonance is that terms involving a derivative of the dust density are subleading in $\left(\frac{1}{D}\right)$ compared to the term involving the divergence of the velocity.

It might naively seem from (756) that the remaining equations of motion that follow from the requirement that the stress tensor is conserved is the equation

$$p^\mu_\nu \cdot \left(u \cdot \hat{\nabla}\right)(K u^\nu) = K \left(u \cdot \hat{\nabla}\right) u^\mu = 0, \tag{757}$$

where $p_{\mu\nu}$ represents the world volume projector orthogonal to the velocity u^μ. The equation (757), if correct, would have been the statement that the 'proper acceleration' of u^μ vanishes on the membrane world volume in the directions orthogonal to u^μ. This statement would have been consistent with the interpretation of u^μ as the velocity field of a pressure free gas of dust.

The equation (757) is in fact incorrect. This is because the expression in (757), which is of order $\mathcal{O}(D)$, is of the same order as (parts of) the divergence of the $\mathcal{O}(1)$ terms in the stress tensor (754) that were omitted in the leading order expression (755). The corrected version of (757) takes these additional terms into account, yielding the membrane equation (546) which can be rewritten for the special case of an uncharged membrane as

$$K \; p^\mu_\nu (u \cdot \hat{\nabla}) u^\nu = p^\mu_\nu \left(\hat{\nabla}^2 u^\nu + u^\alpha K^\nu_\alpha - \hat{\nabla}^\nu K\right) = 0. \tag{758}$$

The equation (758) can be thought of as an expression of Newton's force applied to the particles that make up the membrane. The LHS represents 'mass density' (K) times acceleration $(u \cdot \hat{\nabla})u$ while the RHS of (758) describes the forces that these particles are subject to. The first two term on the RHS of (758) are an expression of the of the force of shear viscosity and have their origin in the last term - the shear viscosity term - in the first line of (754). [147] The final term in the RHS of (758) has its origin in the second term - the bending or curvature energy term - the on the RHS of the first line of (754) (see (1103)). Roughly speaking this term reflects the fact membrane has a restoring force that tries to smooth out gradients of the membrane extrinsic curvature.

4.7 Membrane Entropy current

In the previous section we have found explicit formulae for the stress tensor and a charge current on the world volume of the membrane. In this section we will use a pullback of the area form on the event horizon of our the spacetimes dual to large D black hole membranes to define and determine an entropy current on the membrane. The Hawking area increase theorem guarantees that the entropy current that we define in this section has a divergence that is point wise non negative [79].

As in the case of the charge current and stress tensor, in this section we first explain the general strategy that we use to construct a membrane entropy current at every order in the $\frac{1}{D}$ expansion. We then proceed to implement our construction at low orders in this expansion, using explicit results for the spacetimes dual to large D membranes.

In previous sections we obtained results for the charge current and stress tensor on the membrane using the explicit results of chapter 3 for the spacetime solutions dual to membrane motions accurate to first order in $\frac{1}{D}$. The knowledge of the stress tensor and charge current to this order proved sufficient to test one of the most important structural features of these currents; namely that the requirement that these currents be conserved is a restatement of the membrane equations of motion. In a similar manner it is possible to obtain an entropy current to first order in the derivative expansion at first order in $\frac{1}{D}$ using the results of chapter 3. However the divergence of the current obtained in this manner turns out to vanish identically. In other words at this order we are blind to one of the most important general properties of the entropy current, namely that it is not conserved, but it's divergence is instead point wise positive definite.

In order to capture this basic qualitative feature of the membrane entropy current, in this section we work with *second* order (in $\frac{1}{D}$) metrics of [66] dual to second order membrane motions.

[147]The first term on the RHS of (758) is the classic expression of a viscous force, familiar from the Navier Stokes equations. The second term in (758) is less familiar because it vanishes when the membrane world volume is flat. This term arises because $\hat{\nabla}_\mu \hat{\nabla}_\nu u^\mu$ differs, in general, from $\hat{\nabla}_\nu \hat{\nabla}_\mu u^\mu$ by terms proportional to the curvature of the membrane (see (1107)).

[148] The disadvantage of our reliance on the results of [66] is that these results apply only to uncharged black holes. Second order spacetimes and gauge fields dual to charged large D black holes have not yet been obtained. For this reason all the explicit results presented in this section apply only to the case of uncharged membranes. The extension of this analysis to charged membranes should be a straightforward exercise once the charged version of [66] are available.

After obtaining our formula for the entropy current we turn our attention to the simplest solution of the membrane equations of motion - namely the solution for a static spherical membrane - and compute energy, charge and entropy of this solution. We demonstrate that the charges of our solution agree with those of exact large D black holes to leading order in the large D limit, demonstrating in particular, the consistency of our results for membrane currents with the first law of thermodynamics.

4.7.1 Determination of the entropy current

Consider the spacetime dual to a membrane configuration. Let the *bulk* spacetime metric at the event horizon be denoted by G_{AB}. [149] The precise definition of the membrane shape function and membrane velocity were chosen in [66] to ensure that the spacetime metric G_{MN} takes the following simple form at any point on the event horizon

$$G_{MN} = \eta_{MN} + (n-u)_M(n-u)_N + H^{(T)}_{MN} + H^{Tr}\frac{P_{MN}}{D-2}. \tag{759}$$

Here n_M is the normal oneform on the event horizon normalized so that

$$\eta^{MN} n_M n_N = 1,$$

u_M is the 'velocity' field chosen to be orthogonal to n_M (i.e. $\eta^{MN}u_M n_N = 0$) and also to be unit normalized (i.e. $\eta^{MN}u_M u_N = -1$). Moreover

$$P_{MN} = \eta_{MN} + u_M u_N - n_M n_N \tag{760}$$

and the 'tensor' field $H^{(T)}_{MN}$ is orthogonal to both u^M and n^M and is also traceless i.e.

$$H^{(T)}_{MN} n^M = H^{(T)}_{MN} u^M = H^{(T)}_{MN} P^{MN} = 0,$$

(where all indices are raised using the inverse metric η^{MN}).

[148] In fact the use of the results of [66] (rather than those of chapter 3) proves convenient for another unrelated reason. In their construction the authors of [66] have employed a natural all orders definition of the membrane shape and velocity that turn out to significantly simplify their metric in the neighbourhood of its event horizon in a manner that proves convenient for the analysis we present below.

[149] We emphasize that G_{AB} is the full spacetime metric, not the metric restricted to the event horizon.

(759) is a formula for the full D dimensional spacetime metric at any point on the event horizon. The metric (759) carries information about the inner product between any two vectors in the D dimensional tangent space to the full manifold at any point on the metric. In this section we will be primarily interested only in the metric restricted to the event horizon itself - i.e. the inner product between any two vectors, both of which lie in the $D-1$ dimensional tangent space of the event horizon. The tangent space of the event horizon is a codimension one subspace of the tangent space of the full space, consisting of those vectors whose dot product with n_M vanishes. It is easily verified that a basis for such vectors is given by the tangent vector $u^M = \eta^{MN} u_N$ together with any basis for the $D-2$ dimensional space of vectors orthogonal to both u_M and n_M.

If we are concerned only with the tangent space of the event horizon then the metric (759) is easily verified to be equivalent to

$$G^{eh}_{MN} = H^{(T)}_{MN} + \left(1 + \frac{H^{Tr}}{D-2}\right) P_{MN}, \tag{761}$$

in the sense that

$$j^M k^N G_{MN} = j^M k^N G^{eh}_{MN},$$

where j^A and k^B are arbitrary vectors in the tangent space of the event horizon. Note that $G^{eh}_{MN} n^M = G^{eh}_{MN} u^M = 0$. It follows that the metric (761) has rank $D-2$, even though the event horizon is a $D-1$ dimensional manifold, reflecting the fact that the event horizon is a null manifold.

We will now define $D-2$ dimensional 'area form' on the event horizon. Consider any point on the event horizon and consider a 'patch' of a $D-2$ dimensional sub manifold enclosed by the generalized parallelogram formed out of the $D-2$ infinitesimal vectors $\delta t_1^A \ldots \delta t_{D-2}^A$. Let the $D-2$ volume of this patch - computed using the metric induced on this patch by (761) (or equivalently (761)) -be given by δV_{D-2}. The $D-2$ area form $A_{B_1 \ldots B_{D-2}}$ on the event horizon is defined by equation

$$\delta V_{D-2} = A_{B_1 \ldots B_{D-2}} \delta t_1^{A_1} \ldots \delta t_{D-2}^{A_{D-2}}, \tag{762}$$

(this equation is required to hold for every choice of the infinitesimal vectors δt_i^A).

If one of the boundary vectors, t_1^A is chosen to be u^A then it is clear by inspection that the metric induced by (761) on the $D-2$ dimensional patch is of rank $D-3$, and so δV_{D-2} vanishes. It follows from (762) that $A_{B_1 \ldots B_{D-2}}$ must vanish when contracted with u^A. Now the area form is only well defined in its action on tangent vectors of the event horizon. However we could choose instead to generalize this area form to any $D-2$ form that can be contracted with tangent vectors on the full manifold, so long as this form has the correct action when acting on tangent vectors of the event horizon. Of course this 'uplift' of the volume form on the event horizon is not unique;

we choose a unique ‘uplift’ by arbitrarily imposing the additional requirement that the $D-2$ form vanishes when contracted with n^M. With this choice the uplifted area form on the event horizon necessarily takes the form

$$A_{A_1\ldots A_{D-2}} = \zeta \epsilon_{A_1\ldots A_{D-2}B_1B_2} u^{B_1} n^{B_2}, \tag{763}$$

where $\epsilon_{A_1\ldots A_{D-2}B_1B_2}$ is the standard volume form in flat D dimensional space with metric η_{MN} and ζ is yet to be determined.

We will now determine ζ in (763). Consider a $D-2$ dimensional parallelepiped constructed out of $D-2$ basis vectors $\delta t_1^A \ldots \delta t_{D-2}^A$ where these basis vectors are all chosen to be orthogonal to both u_A and n_A. As above we will denote the volume of this parallelepiped - constructed in the spacetime (761)- by δV_{D-2}.

Let us now consider a different problem. Consider a fictional space time with metric G'_{AB} given by

$$G'_{AB} = G^{eh}_{MN} + n_A n_B - u_A u_B. \tag{764}$$

Using (761) and (760) we find

$$G'_{MN} = \eta_{MN} + H^{(T)}_{MN} + H^{Tr} \frac{P_{MN}}{D-2}. \tag{765}$$

Working with the metric G'_{MN} we now consider the D dimensional parallelepiped bounded the vectors $\delta t_1^A \ldots \delta t_{D-2}^A$ together with the additional two vectors $\delta a\ n^M$ and $\delta b\ u^M$. Let δV_D denote the volume of this D dimensional parallelepiped. A little thought will convince the reader that (upto a sign we will not keep track of)

$$\delta V_D = \delta a\ \delta b\ \delta V_{D-2}. \tag{766}$$

However δV_D is easily independently determined. Using the fact that the volume form of the non degenerate D dimensional metric G'_{AB} is simply given by $\sqrt{-G'}\epsilon_{A_1\ldots A_D}$ we conclude that

$$\delta V_D = \sqrt{-G'}\ \delta a \delta b\ \epsilon_{A_1\ldots A_{D-2}B_1B_2} u^{B_1} n^{B_2} \tag{767}$$

Comparing (767), (766) and (763) we conclude that (upto a sign)

$$\zeta = \sqrt{-G'}, \tag{768}$$

so that

$$A_{A_1\ldots A_{D-2}} = \sqrt{-G} \epsilon_{A_1\ldots A_{D-2}B_1B_2} u^{B_1} n^{B_2}, \tag{769}$$

(769) is our final result for the area form on the world volume of the membrane.

At least for the case of uncharged black holes, it was demonstrated in [66] that $H^{(T)}_{MN}$ and H^{Tr} both vanish at leading and first subleading order in $\frac{1}{D}$ and are nonzero only at order $\mathcal{O}(1/D^2)$. As $H^{(T)}_{MN}$ is traceless, it follows that the contribution of this term to the determinant G' starts at order $\frac{1}{D^4}$. On the other hand the trace of $\frac{\mathcal{P}_{MN}}{D-2}$ is unity. It follows that upto order $\frac{1}{D^2}$

$$\begin{aligned}\sqrt{-G'} &= 1 + \frac{H^{Tr}(\rho = 1)}{2} = 1 - \frac{C}{2} + \mathcal{O}(1/D^3),\\ C &= \frac{2}{K^2}(u \cdot K - u.\nabla u)^2,\end{aligned} \tag{770}$$

where we have used the explicit result for $H^{Tr}(\psi = 1)$ at order $\frac{1}{D^2}$ (see Equation 4.16 of [66]) to obtain the explicit value for C. Note that C is of order $\frac{1}{D^2}$. [150]

The entropy current on the membrane is obtained by dualizing the area $D-2$ form and dividing by 4 [79]. We obtain

$$J^\mu_S = \sqrt{-G'}\frac{u^\mu}{4} \approx \left(1 - \frac{C}{2} + \mathcal{O}\left(1/D^3\right)\right)\frac{u^\mu}{4}. \tag{771}$$

Note in particular that at leading order in $1/D$

$$J^\mu_S = \frac{u^\mu}{4}. \tag{772}$$

The first correction to this leading order result occurs at order $1/D^2$.

The divergence of this entropy current (771) is easily computed;

$$\hat{\nabla}_\mu J^\mu_S = \frac{\hat{\nabla}_\mu u^\mu}{4} - \frac{u^\mu \hat{\nabla}_\mu C}{8} + \mathcal{O}\left(1/D^3\right), \tag{773}$$

$\nabla_A u^A$ was evaluated in [66] with the result

$$\hat{\nabla}_\mu u^\mu = \frac{p^{\mu\nu} p^{\alpha\beta} \hat{\nabla}_{(\mu} u_{\alpha)} \hat{\nabla}_{(\nu} u_{\beta)}}{2K} + \mathcal{O}(1/D^2). \tag{774}$$

Note in particular that $\nabla_A u^A$ is of order $\frac{1}{D}$. As C is of order $\frac{1}{D^2}$, if follows from (773) that

$$\hat{\nabla}_\mu J^\mu_S = \frac{p^{\mu\nu} p^{\alpha\beta} \hat{\nabla}_{(\mu} u_{\alpha)} \hat{\nabla}_{(\nu} u_{\beta)}}{8K} + \mathcal{O}(1/D^2). \tag{775}$$

Note that the RHS of (775) is positive definite. As we have explained earlier in this section

[150] We emphasize again that the explicit result for C reported in the second line of (770) is accurate only for uncharged black holes; the computations required to determine C have not yet been performed for charged black holes. We leave the determination of C with nonzero charge as a task for the future.

this positivity could have been anticipated on general grounds using the Hawking area increase theorem [79].

4.7.2 Thermodynamics of spherical membranes

The simplest solution of the membrane equations of motion (546) is a static spherical bubble of radius r_0 with $u = -dt$ and $Q = Q_0 = \text{const}$. In this brief subsection we compute the charges of this solution and match these with the thermodynamic charges of black holes.

At leading order in the large D limit it follows from (555) that T_{00} for this solution is given by

$$T_{00} = \frac{(D-2)(1+Q_0^2)}{16\pi r_0}.$$

It follows that the mass m of this solution is given by

$$m = \Omega_{D-2} r_0^{D-2} T_{00} = \frac{\Omega_{D-2}(D-2) r_o^{D-3}(1+Q_0^2)}{16\pi}. \tag{776}$$

Note that m in (776) agrees with the mass of the black hole (885) at large D (recall that $c_D = 1$ in (886) the large D limit).

The static membrane solution described above has a gravitational tail at infinity. It follows from (1001) and (998) that the curvature of this tail is given by

$$R_{0i0j} = -\frac{8\pi}{(D-2)\Omega_{D-2}} \nabla_i \nabla_j \left(\frac{m}{r^{D-3}} \right), \tag{777}$$

in agreement with (882), supporting our identification of $\int T_{00}$ with the mass of the membrane.

It may be verified that (777) agrees with the curvature of the black hole solution (885) at large r and large D.

In a similar manner the charge density of our solution is given by

$$J^0 = \frac{Q_0(D-2)}{2\sqrt{2}\pi r_0}.$$

It follows that the charge of our membrane configuration is given by

$$q = \Omega_{D-2} r_0^{D-2} J^0 = \frac{\Omega_{D-2}(D-2) Q_0 r_o^{D-3}}{2\sqrt{2}\pi}. \tag{778}$$

Once again the q in (778) agrees with the charge of the black hole (885) at large D (see (886)).

At large r our charged static membrane solution sources an electric field given by

$$E^i = F^{i0} = -\frac{1}{(D-2)\Omega_{D-2}} \nabla_i \left(\frac{q}{r^{D-3}} \right), \tag{779}$$

in agreement with (883).

It follows from this analysis of metric and field strength tails at infinity that the spherical membranes studied in this section are dual to static black holes (885) of mass m and charge Q_0.

Finally it follows from (772) that the entropy S of our static solution is given by the area of the membrane divided by 4, i.e.

$$S = \frac{\Omega_{D-2} r_0^{D-2}}{4}, \tag{780}$$

in agreement with the entropy (889) of a black hole with the same mass and charge.

In the study of black hole physics we define the black hole temperature and chemical potential via the formulae (887) and (888). These definitions ensure that black holes obey the first law of thermodynamics (890). As the spherical membranes of this subsection are dual to the corresponding black holes, it is natural to assign them the same temperature and chemical potentials

$$\begin{aligned} T &= \frac{(1-Q_0^2)K}{4\pi}, \\ \mu &= \frac{Q}{\sqrt{8\pi}}. \end{aligned} \tag{781}$$

With these definitions the equation (890) can be viewed as the assertion that the spherical membranes of this subsection obey the first law of thermodynamics.

In the spirit of the equations of hydrodynamics, the identification (781) can be made locally for any membrane configuration, allowing us to discuss the evolution of the local black hole temperature and chemical potential in the course of a dynamical evolution.

4.8 Radiation in general dimensions

Earlier in this chapter we have determined the explicit form of the stress tensor and charge current carried by a large D black hole membrane. As the membrane undergoes a dynamical motion these currents source electromagnetic and gravitational radiation. The resultant radiation field is determined by plugging these currents into radiation formulae: the formulae that determine radiation fields in terms of currents. In this section we review radiation formulae in arbitrary dimensions.

For completeness - and clarity of presentation - we begin this section with a discussion of the formulae for the radiation response of a massless minimally coupled field to a scalar source, even though this theory is not needed in order to analyse the black hole membrane. We then turn to the analysis of the cases of real interest; the radiation response of a Maxwell field to an arbitrary conserved current and the analysis of the radiation response of the linearized gravitational field to an arbitrary conserved stress tensor. In the next section we will apply the formulae developed in this section to a particular situation, namely to the study of small fluctuations about a static membrane.

In a certain abstract sense the radiation formulae are extremely simple; they take the schematic form

$$R(x) = \int d^D x' G(x - x') J(x'), \tag{782}$$

where J is the source, R the radiation response and G a retarded Greens function. From this point of view the theory of radiation ends with the computation of the appropriate Greens function, a topic we have already discussed in section 4.2.

Let us now, however, specialize to situations in which the 'centre of mass' of the sources is at rest and localized in a shell of radius R about a particular spatial point x'. If we are interested in the radiation response at points x whose distance from x' is much larger than R, the resultant radiation formulae will clearly be most transparent when expressed in spherical polar coordinates with x' as origin. In this coordinate system the sources and radiation fields are both naturally expanded in a basis of scalar, vector and tensor spherical harmonics. (782) then turns into an integral transform that expresses radiation fields a particular symmetry property (say, e.g. radiation fields in the l^{th} vector spherical harmonic) as an integral over sources in the same representation. The resultant final expressions are much more explicit - and so much more transparent - than (782).

The starting point for the derivation of the formulae presented in this section is the expansion of the retarded scalar Greens function of section 4.2 in spherical coordinates. In (565) the Greens function was already presented in polar coordinates in the special case of the source at the origin. In Appendix 4.11.6 we demonstrate that when the source is displaced away from the origin, the generalization of (565) is given by (we assume $|\vec{r}| > |\vec{r}'|$)

$$G(\omega, |\vec{r} - \vec{r}'|) = \frac{i\pi}{2} \sum_{l=0}^{\infty} \frac{1}{(r'r)^{\frac{D-3}{2}}} H^{(1)}_{\frac{D-3+2l}{2}}(\omega r) J_{\frac{D-3+2l}{2}}(\omega r') \mathcal{P}_l(\theta, \theta'), \tag{783}$$

where $\mathcal{P}_l(\theta, \theta')$ is the projector onto the space of functions whose angular dependence is a linear combinations of l^{th} scalar spherical harmonics; see around (894) in the Appendix for more details) and θ and θ' are the angular locations of $\vec{r}$ and $\vec{r}'$ respectively. [151]

As mentioned above, all the results of this section are presented in terms of scalar, vector and tensor spherical harmonics. We define and study spherical harmonics in arbitrary dimensions in Appendix 4.11.4.

[151] While the formulae developed in this section are standard extensions of textbook treatments of radiation to arbitrary dimensions, we were unable to locate a reference with all formulae presented in a clear and systematic manner and so chose to undertake the exercise ourselves. All the formulae developed in this section are derived for arbitrary values of D ; however we also emphasize special simplifications that occur at large D.

4.8.1 Scalar Radiation

Consider a minimally coupled scalar ϕ which is zero at early times. The scalar is subsequently kicked out of its 'vacuum' state by coupling to a source according to the equation

$$-\Box\phi = \mathcal{S}. \tag{784}$$

where $\mathcal{S}$ is an arbitrary function of space and time. [152] We further assume that the sourcce $\mathcal{S}(x)$ is spatially localized about a particular point in space at all times in a particular Lorentz frame. We choose to work in this Lorentz frame, and choose the this point as the origin of our spatial coordinates. At large enough distance from the origin the equation of motion for ϕ simplifies to $-\Box\phi = 0$.

Spherical Expansion of outgoing radiation The most general solution to the minimally coupled scalar equation that is outgoing radiation at infinity takes the form (see (966))

$$\phi(\omega, \vec{x}) = \sum_l \alpha_l(\omega, \theta) \frac{H_{\frac{D+2l-3}{2}}(\omega r)}{r^{\frac{D-3}{2}}}. \tag{786}$$

The functions $\alpha_l(\omega, \theta)$ are angular functions in the l^{th} spherical harmonic sector for scalars, i.e. they obey the equation

$$\mathcal{P}_l \alpha_{l'} = \delta_{l,l'} \alpha_l.$$

[153]

Radiation in terms of sources The response of the field ϕ to the source function $\mathcal{S}$ is given by the formula

$$\phi(\omega, \vec{x}) = \int d^{D-1}\vec{x}' G(\omega, |\vec{x} - \vec{x}'|) \mathcal{S}(\omega, \vec{x}'). \tag{788}$$

[152] This equation of motion follows from the action

$$S = \int d^D x \sqrt{-G} \left(-\frac{1}{2} (\nabla\phi)^2 + \mathcal{S}\phi \right). \tag{785}$$

[153] This is an abbreviated form of the equation

$$\int d\Omega'_{D-2} P_l(\theta, \theta') \alpha_{\omega,l'}(\theta') = \delta_{l,l'} \alpha_{\omega,l}(\theta). \tag{787}$$

Here $G(\omega, |\vec{x} - \vec{x}'|)$ is the retarded Greens function determined in (565) and

$$\mathcal{S}(\omega, \vec{x}') = \int e^{i\omega t} \mathcal{S}(x'^{\mu}) dt. \tag{789}$$

In other words $\mathcal{S}(\omega, \vec{x}')$ is the source function Fourier transformed in time.

It is useful to decompose the source into its distinct angular momentum components

$$\mathcal{S}(\omega, \vec{x}) = \sum_l \mathcal{S}_l(\omega, r', \theta'), \tag{790}$$

where

$$\mathcal{P}_l \mathcal{S}_{l'} = \delta_{l,l'} \mathcal{S}_l. \tag{791}$$

In other words $\mathcal{S}_l(\omega, r', \theta')$ is the part of $\mathcal{S}(\omega, \vec{x}')$ that transforms in the l^{th} spherical harmonic representation. Inserting the expansion (783) for the Greens function in (788) and specializing that formula to large r, it is easily verified that (788) reduces to (786) with

$$\alpha_l(\omega, \theta) = \frac{i\pi}{2} \int dr' J_{\frac{D-3+2l}{2}}(\omega r')\ r'^{\frac{D-1}{2}} \mathcal{S}_l(\omega, r', \theta), \tag{792}$$

(792) is our final formula for scalar radiation. In the rest of this subsection we will study limits and properties of the formula (792).

The static limit Recall that

$$J_n(x) \approx \frac{\left(\frac{x}{2}\right)^n}{\Gamma(n+1)}, \quad (x^2 \ll n). \tag{793}$$

This observation may be used to simplify (792) in two different physical situations. The first is the static limit $\omega \to 0$ taken at finite D. The second - of particular interest to this chapter - is the limit in which ωR is held fixed as D is taken to infinity (here R is an estimate of the spatial size of the support for the scalar source $\mathcal{S}$ which is assumed to be of finite extent). In either limit we obtain the simplified formula

$$\alpha_l(\omega, \theta) = \tilde{\alpha}_l \omega^{l+\frac{D-3}{2}} \int dr' (r')^{l+D-2} \mathcal{S}_l(\omega, r', \theta), \tag{794}$$

with

$$\tilde{\alpha}_l = \frac{i\pi}{2^{l+\frac{D-1}{2}}} \frac{1}{\Gamma\left(l + \frac{D-1}{2}\right)}. \tag{795}$$

In this subsubsection we study the static limit, postponing our stdy of the large D limit to the next subsubsection.

In the static limitthere is a further simplification. As ω is taken to zero the Hankel function in (786) may be approximated by its small argument expansion (566) and we find

$$\phi(0,r,\theta) = \frac{1}{r^{D-3}} \sum_{l=0}^{\infty} \left(\frac{1}{2}\right) \frac{2}{2l+D-3} \int dr' r'^{D-2} \left(\frac{r'}{r}\right)^l S_l(0,r',\theta), \tag{796}$$

(796) is simply the multipole expansion of the solution to the Euclidean equation $\nabla^2\phi = -\mathcal{S}$ in $D-1$ Euclidean dimensions, and may directly be obtained by inserting (962) into (788). Note that the static field falls off much faster at large r than the radiation field does; this is the generalization of the familiar fact that the Coulomb field in $D=4$ falls off like $\frac{1}{r^2}$ while a radiation field decays more slowly, like $\frac{1}{r}$.

Large D limit Let us now turn to the large D limit at finite ω. The Sterling approximation allows us to simplify the expression $\tilde{\alpha}_l$; we find

$$\tilde{\alpha}_l \approx \frac{i\sqrt{\pi}}{2D^{\frac{D}{2}}} \left(\frac{e^{l+\frac{D-3}{2}}}{D^{l-1}}\right). \tag{797}$$

We would now like to estimate how fast our system loses 'charge' via radiation at large D. In order to make this question precise, let us slightly generalize the discussion of this subsection to the case of a complex scalar field ϕ and source $\mathcal{S}$. All the formulae derived above continue to apply. The advantage is that our scalar field now carries a current given by $J_M = i(\partial_M\phi^*\phi - \phi^*\partial_M\phi)$. Let us assume that the source function is nonzero only in a shell of radius of order R and vanishes outside the shell. In the external region the current J_M is conserved. We will now estimate first the integrated density of this charge contained in the field ϕ to the exterior of the shell of $\mathcal{S}$ and second the rate of loss of charge to infinity by radiation. The ratio of these two quantities will give us an estimate of the rate of loss of charge due to radiation per unit time.

Using (786) we see that the charge carried by our configuration in the l^{th} mode is of order

$$\frac{RD^D}{(\omega R)^{2l+D-4}} \int_{S^{D-2}} |\alpha_l(\theta)^2|,$$

where the source is assumed to be of size R we have retained only leading order terms in the limit of large D. On the other hand the rate of energy lost due to radiation is of order

$$\int_{S^{D-2}} |\alpha_l(\theta)^2|.$$

It follows that the fractional loss of charge by radiation per unit time is of order $\frac{(\omega R)^{2l+D-4}}{RD^D}$, and so is extremely small at large D.

4.8.2 Electromagnetic Radiation

In this section we will find the solution to the Maxwell equation

$$\nabla^M F_{MN} = \mathcal{J}_N, \tag{798}$$

sourced by an arbitrary localized charge current $\mathcal{J}_M$. It follows from (798) that

$$\Box F_{MN} = -(\nabla_N \mathcal{J}_M - \nabla_M \mathcal{J}_N). \tag{799}$$

In particular the electric field defined by

$$E_i = F_{0i}, \tag{800}$$

obeys the equation

$$\begin{aligned} \Box \vec{E} &= -\vec{\mathcal{J}}_{eff}, \\ \vec{\mathcal{J}}_{eff} &= \vec{\nabla} \mathcal{J}_0 - \partial_0 \vec{\mathcal{J}}. \end{aligned} \tag{801}$$

In order to determine the radiation response to a current it is sufficient to determine the electric field at large distances; all other components of the field strength may be obtained rather simply from the electric field using the Bianchi identity. To see how this works recall that the Bianchi identity with free indices $0, i, j$ takes the form

$$\begin{aligned} &\partial_0 F_{ij} = \nabla_i E_j - \nabla_j E_i, \quad \text{i.e} \\ &F_{ij}(\omega, \vec{x}) = \frac{i(\nabla_i E_j(\omega, \vec{x}) - \nabla_j E_i(\omega, \vec{x}))}{\omega}. \end{aligned} \tag{802}$$

It follows that the F_{ij} is completely determined in terms of $\vec{E}$ at every nonzero ω.

Free outgoing solutions of Maxwell's equations At large distances where the source $\mathcal{J}^M$ vanishes, (801) reduces to

$$\Box \vec{E} = 0. \tag{803}$$

Now $\vec{E}$ is a vector field in spacetime. As we have explained in Appendix 4.11.4, any such field can be written in terms of two scalar fields and one divergenceless, purely tangential (to the sphere) vector field. This tangential divergenceless vector field can be expanded in vector spherical harmonics while the two scalars are expanded in scalar spherical harmonics. A useful basis for this decomposition is listed in (915) in the Appendix. In this basis the action of ∇^2 is diagonal and is listed in (921).

It follows immediately from (921) that the most general solution to (803) is given by a vector of the form (915) where the radial dependence of the coefficients α_l, $\vec{\gamma}_l$ and β_l respectively is the same as that of the $(l-1)^{th}$, l^{th} and $(l+1)^{th}$ angular momentum component of the modes in (786). In other words the most general harmonic solution for $\vec{E}$ is given by

$$\begin{aligned}\vec{E}(\omega,\vec{x}) = &\sum_{l=0}^{\infty}\left(\frac{H_{\frac{D+2l-5}{2}}(\omega r)}{r^{\frac{D-3}{2}}}\vec{\mathcal{A}}^-[S_l^-(\omega,\theta)] + \frac{H_{\frac{D+2l-1}{2}}(\omega r)}{r^{\frac{D-3}{2}}}\vec{\mathcal{A}}^+[S_l^+(\omega,\theta)]\right)\\ &+\sum_{l=1}^{\infty}\left(\frac{H_{\frac{D+2l-3}{2}}(\omega r)}{r^{\frac{D-3}{2}}}\vec{V}_l\right),\end{aligned} \tag{804}$$

where $S_l^\pm$ are arbitrary r independent scalar functions in the l^{th} scalar spherical harmonic sector and $\vec{V}_l$ is an arbitrary vector function in the l^{th} vector spherical harmonic sector, normalized so that each of the Cartesian components of $\vec{V}_l$ is a function only of the angles and is independent of r. Here $\vec{\mathcal{A}}^-[S_l^-(\omega,\theta)$ and $\vec{\mathcal{A}}^+[S_l^+(\omega,\theta)]$ are the maps from scalar to vector functions in R^{D-1} defined in (917).

The equation

$$\vec{\nabla}\cdot\vec{E} = 0, \tag{805}$$

(which also holds in the absence of sources) further constrains radiation fields. Using (923) and appropriate recursion relations for Hankel functions we demonstrate in subsection 4.11.7 below that

$$lS_l^- = (l+D-3)S_l^+, \tag{806}$$

(804) with the constraint (806) is the most general solution to the source free Maxwell equations.

At very large distances, $\omega r \gg D^2$, (804) simplifies to

$$\begin{aligned}\vec{E}(\omega,\vec{x}) &= \sqrt{\frac{2}{\pi\omega}}\frac{e^{ir\omega}}{r^{\frac{D-2}{2}}}\sum_{l=0}^{\infty}e^{\frac{-i(D+2l)\pi}{4}}\left(\vec{\mathcal{A}}^+[S_l^+(\omega,\theta)] - \vec{\mathcal{A}}^-[S_l^-(\omega,\theta)]\right)\\ &\quad + i\sqrt{\frac{2}{\pi\omega}}\frac{e^{ir\omega}}{r^{\frac{D-2}{2}}}\sum_{l=1}^{\infty}e^{\frac{-i(D+2l)\pi}{4}}\vec{V}_l\\ &= \sqrt{\frac{2}{\pi\omega}}\frac{e^{ir\omega}}{r^{\frac{D-2}{2}}}\sum_{l=0}^{\infty}e^{\frac{-i(D+2l)\pi}{4}}\left(\hat{r}\left(lS_l^- - (l+D-3)S_l^+\right) - r\vec{\bar{\nabla}}\left(S_l^- + S_l^+\right)\right)\\ &\quad + i\sqrt{\frac{2}{\pi\omega}}\frac{e^{ir\omega}}{r^{\frac{D-2}{2}}}\sum_{l=1}^{\infty}e^{\frac{-i(D+2l)\pi}{4}}\vec{V}_l.\end{aligned} \tag{807}$$

Where we have used (917) in the last step. Note, in particular, that the radiation electric field is orthogonal to $\hat{r}$ - and so is finally well approximated by a local plane wave - at these distances.

Radiation in terms of sources In order to determine the radiation field sourced by an arbitrary current we expand the effective source $\vec{\mathcal{J}}_{eff}$ in the form (915). In particular let

$$\vec{\mathcal{J}}_{eff} = \left(\vec{\mathcal{A}}^{-}[\mathfrak{a}] + \vec{\mathcal{A}}^{+}[\mathfrak{b}] + \vec{\mathfrak{c}}\right), \tag{808}$$

where $\mathfrak{a}$, $\mathfrak{b}$ and $\vec{\mathfrak{c}}$ respectively play the role of α, β and $\vec{\gamma}$ in (915). In Appendix 4.11.6 we have determined the action of the retarded Greens function on an arbitrary vector field expanded in the basis employed in (808). Using (977), (978) and (979) of the Appendix we find that the electric field at large r takes the form (804) with

$$\begin{aligned}
S_l^- &= \frac{i\pi}{2} \int dr' J_{\frac{D-3+2(l-1)}{2}}(\omega r')\ r'^{\frac{D-1}{2}} \mathfrak{a}_l(\omega, r', \theta), \\
S_l^+ &= \frac{i\pi}{2} \int dr' J_{\frac{D-3+2(l+1)}{2}}(\omega r')\ r'^{\frac{D-1}{2}} \mathfrak{b}_l(\omega, r', \theta), \\
\vec{V}_l &= \frac{i\pi}{2} \int dr' J_{\frac{D-3+2l}{2}}(\omega r')\ r'^{\frac{D-1}{2}} \vec{\mathfrak{c}}_l(\omega, r', \theta).
\end{aligned} \tag{809}$$

The conservation of the electromagnetic current $\vec{\mathcal{J}}$ can be used to show that the effective current obeys the following equation

$$\vec{\nabla} \cdot \vec{\mathcal{J}}^{eff} = \Box \mathcal{J}_0. \tag{810}$$

This relation can be used to verify that the coefficients (809) obey (806). [154]

Special limits As in the previous subsection (793) may be used to simplify (809) in both the static and the the large D limits. In either limit we obtain the simplified formula

$$\begin{aligned}
S_l^-(\omega, \theta) &= \tilde{S}_l^- \omega^{l+\frac{D-5}{2}} \int dr' (r')^{l+D-3} \mathfrak{a}(\omega, r', \theta), \\
S_l^+(\omega, \theta) &= \tilde{S}_l^+ \omega^{l+\frac{D-1}{2}} \int dr' (r')^{l+D-1} \mathfrak{b}(\omega, r', \theta), \\
V_l(\omega, \theta)_i &= \tilde{V}_l \omega^{l+\frac{D-3}{2}} \int dr' (r')^{l+D-2} \mathfrak{c}(\omega, r', \theta),
\end{aligned} \tag{812}$$

[154] At the formal level it is obvious that this had to work.

$$\nabla \cdot E = -\mathcal{J}_0. \tag{811}$$

This is simply the Gauss law, and ensures that the Electric field is divergence free in the absence of a source. The fact that the actual formulae (809) obey (806) may be regarded as a check on our algebra.

with

$$\begin{aligned}\tilde{S}_l^- &= \frac{i\pi}{2^{l+\frac{D-3}{2}}}\frac{1}{\Gamma\left(l+\frac{D-3}{2}\right)},\\ \tilde{S}_l^+ &= \frac{i\pi}{2^{l+\frac{D+1}{2}}}\frac{1}{\Gamma\left(l+\frac{D+1}{2}\right)},\\ \tilde{V}_l &= \frac{i\pi}{2^{l+\frac{D-1}{2}}}\frac{1}{\Gamma\left(l+\frac{D-1}{2}\right)}.\end{aligned} \tag{813}$$

As in the previous subsection in the large D limit at fixed ω we use the Sterling approximation to further simplify $\tilde{\alpha}_l$; we find

$$\begin{aligned}\tilde{S}_l^- &\approx \frac{i\sqrt{\pi}}{2D^{\frac{D}{2}}}\left(\frac{e^{l+\frac{D-5}{2}}}{D^{l-2}}\right),\\ \tilde{S}_l^+ &\approx \frac{i\sqrt{\pi}}{2D^{\frac{D}{2}}}\left(\frac{e^{l+\frac{D-1}{2}}}{D^{l}}\right),\\ \tilde{V}_l &\approx \frac{i\sqrt{\pi}}{2D^{\frac{D}{2}}}\left(\frac{e^{l+\frac{D-3}{2}}}{D^{l-1}}\right).\end{aligned} \tag{814}$$

As in the previous subsection (814) does not apply in the static limit at fixed D. In this limit, however, the small argument expansion of the Hankel function leads to simplifications. In Appendix 4.11.7 we demonstrate that in the limit $\omega \to 0$ the radiation formulae (809) yield results consistent with the familiar formulae of electrostatics

$$\begin{aligned}\vec{E} &= -\nabla\Phi_E,\\ \nabla^2\Phi_E &= \mathcal{J}_0(r'),\\ F_{ij} &= \partial_i A_j - \partial_j A_i,\\ \nabla^2\vec{A} &= \vec{\mathcal{J}}.\end{aligned} \tag{815}$$

4.8.3 Gravitational Radiation

In this section we will find the unique purely outgoing solution to the linearized Einstein equation; i.e. the linearized version of

$$R_{MN} = 8\pi\mathcal{T}_{MN}, \tag{816}$$

as a functional of an arbitrarily specified conserved $\mathcal{T}_{MN}$.

It follows from (816) that, to linear order in an expansion around flat space

$$\begin{aligned}\Box R_{MNPQ} =& 8\pi\left(\partial_M\partial_P T_{NQ} - \partial_M\partial_Q T_{NP} - \partial_N\partial_P T_{MQ} + \partial_N\partial_Q T_{MP}\right)\\ &- \frac{8\pi}{D-2}\left(\eta_{MP}\partial_N\partial_Q T - \eta_{MQ}\partial_N\partial_P T - \eta_{NP}\partial_M\partial_Q T + \eta_{NQ}\partial_M\partial_P T\right).\end{aligned} \tag{817}$$

In particular

$$\begin{aligned}&\Box R_{0i0j} = -(\mathcal{T}_{eff})_{ij},\\ &(\mathcal{T}_{eff})_{ij} = 8\pi\left(\omega^2\mathcal{T}_{ij} - i\omega(\partial_i\mathcal{T}_{0j} + \partial_j\mathcal{T}_{0i}) - \partial_i\partial_j\left(\mathcal{T}_{00} + \frac{\mathcal{T}}{D-2}\right) - \eta_{ij}\omega^2\frac{\mathcal{T}}{D-2}\right).\end{aligned} \tag{818}$$

As in the previous subsection it is sufficient to consider R_{0i0j} as all other curvature components are easily obtained from this one by use of the Bianchi identity. [155] One way to understand this statement is to work in the $h_{0M} = 0$. In this gauge and in Fourier space

$$h_{ij} = \frac{-2}{\omega^2}R_{0i0j}. \tag{820}$$

As all gauge invariants can be built out of h_{ij}, it follows that all gauge invariant information is also contained in R_{0i0j} except in the special limit $\omega \to 0$.

Parametrization of vacuum solutions When all source currents vanish (818) reduces to

$$\Box R_{0i0j} = 0. \tag{821}$$

As in the previous subsection the most general tensor field R_{0i0j} can be decomposed into four scalars, two divergence free tangential vector fields and one divergence free, traceless tangential tensor field - the later can be decomposed in tensor spherical harmonics. The form of this expansion is given in (924). Away from all sources the equation (821) determines the radial dependence of all the coefficient functions in (924). It follows from (930) that the radial dependence of κ_l, γ_l and χ_l^{ij} (in the decomposition (924) applied to R_{0i0j}) is precisely that of the coefficient of the mode α_l in the equation (786). On the other hand the radial dependence of α_l, $\vec{\phi}_l$, $\vec{\psi}_l$ and β_l is that of the modes with angular momentum $l-2$, $l-1$, $l+1$ and $l+2$ respectively in (786). It

[155] The Bianchi identity yields

$$\begin{aligned}R_{0ijk} &= \frac{-i}{\omega}(\partial_k R_{0i0j} - \partial_j R_{0i0k}),\\ R_{ijpq} &= \frac{-i}{\omega}(\partial_q R_{0pij} - \partial_p R_{0qij}).\end{aligned} \tag{819}$$

thus follow that away from all sources

$$
\begin{aligned}
R_{0i0j}(\omega,\vec{x}) = \sum_{l=0}^{\infty} \Bigg(& \frac{H_{\frac{D+2l-7}{2}}(\omega r)}{r^{\frac{D-3}{2}}}(\mathcal{C}^-)_{ij}[S_l^-(\omega,\theta)] + \frac{H_{\frac{D+2l+1}{2}}(\omega r)}{r^{\frac{D-3}{2}}}(\mathcal{C}^+)_{ij}[S_l^+(\omega,\theta)] \\
& + \frac{H_{\frac{D+2l-3}{2}}(\omega r)}{r^{\frac{D-3}{2}}}\left((\mathcal{C}^0)_{ij}[S_l^0(\omega,\theta)] + \delta_{ij}S_l^{Tr}(\omega,\theta)\right)\Bigg) \\
& + \sum_{l=1}^{\infty}\left(\frac{H_{\frac{D+2l-5}{2}}(\omega r)}{r^{\frac{D-3}{2}}}(\mathcal{B}^-)_{ij}[V_l^-(\omega,\theta)] + \frac{H_{\frac{D+2l-1}{2}}(\omega r)}{r^{\frac{D-3}{2}}}(\mathcal{B}^+)_{ij}[V_l^+(\omega,\theta)]\right) \\
& + \sum_{l=2}^{\infty}\left(\frac{H_{\frac{D+2l-3}{2}}(\omega r)}{r^{\frac{D-3}{2}}}(X_l)_{ij}\right),
\end{aligned}
\tag{822}
$$

where $S_l^{\pm}$, S_l^0 and S_l^{Tr} are arbitrary r independent scalar functions in the l^{th} scalar spherical harmonic sector, $\vec{V}_l^{\pm}$ is an arbitrary vector function in the l^{th} vector spherical harmonic sector, normalized so that each of the Cartesian components of $\vec{V}_l^{\pm}$ are functions only of the angles and are independent of r, and X_l is an arbitrary symmetric, divergencelsess, traceless tensor function in the l^{th} vector spherical harmonic sector, normalized so that each of the Cartesian components of X_l are functions only of the angles and are independent of r, and all functionals (e.g. $(\mathcal{C})_{ij}$) were defined in (925).

(822) is the most general solution to the linearized dynamical Einstein equations; however the general solution (822) does not automatically solve the Einstein constraint equations. Using

$$
\nabla^i(\mathcal{T}_{eff})_{ij} = 8\pi\Box\left(i\omega\mathcal{T}_{0j} + \partial_j\left(\mathcal{T}_{00} + \frac{\mathcal{T}}{D-2}\right)\right), \tag{823}
$$

we find the linearized gravity analogue of the electromagnetic Gauss law of the previous subsection

$$
\nabla^i R_{0i0j} = -8\pi\left(i\omega\mathcal{T}_{0j} + \partial_j\left(\mathcal{T}_{00} + \frac{\mathcal{T}}{D-2}\right)\right). \tag{824}
$$

In particular, in the absence of sources we have

$$
\nabla^i R_{0i0j} = 0. \tag{825}
$$

Using (825), (923) and appropriate recursion relations for Hankel functions we find

$$
\begin{aligned}
(l-1)S_l^- &= \frac{(l+D-3)S_l^0}{2(2l+D-3)}\left((2l+D-3) - \frac{4l}{D-1}\right), \\
(l+D-2)S_l^+ &= \frac{lS_l^0}{2(2l+D-3)}\left((2l+D-3) - \frac{4(l+D-3)}{D-1}\right), \\
(l-1)\vec{V}_l^- &= (l+D-2)\vec{V}_l^+.
\end{aligned}
\tag{826}
$$

Moreover it is easily verified that

$$(\mathcal{T}_{eff})^i_i = 8\pi\Box\left(\mathcal{T}_{00} + \frac{\mathcal{T}}{D-2}\right), \tag{827}$$

so that

$$R^i{}_{0i0} = -8\pi\left(\mathcal{T}_{00} + \frac{\mathcal{T}}{D-2}\right). \tag{828}$$

The equation (828) implies that R_{0i0j} is traceless in the absence of sources; this sets

$$S_l^{Tr} = 0. \tag{829}$$

(822) with the constraint (826) and (829) is the most general source free solution of the linearized Einstein equations. Notice that the general radiation field is parametrized by a single scalar function, a single divergence free vector function and a single traceless divergence free tensor function on the unit sphere.
In the large distance limit $\omega r \gg D^2$ (822) simplifies to

$$\begin{aligned}
R_{0i0j}(\omega,\vec{x}) &= i\sqrt{\frac{2}{\pi\omega}}\frac{e^{ir\omega}}{r^{\frac{D-2}{2}}}\sum_{l=0}^{\infty}e^{\frac{-i(D+2l)\pi}{4}}\left(\mathcal{C}^0_{ij}[S^0_l(\omega,\theta)] - \mathcal{C}^+_{ij}[S^+_l(\omega,\theta)] - \mathcal{C}^-_{ij}[S^-_l(\omega,\theta)]\right)\\
&+ \sqrt{\frac{2}{\pi\omega}}\frac{e^{ir\omega}}{r^{\frac{D-2}{2}}}\sum_{l=1}^{\infty}e^{\frac{-i(D+2l)\pi}{4}}\left(\mathcal{B}^+_{ij}[\vec{V}^+_l(\omega,\theta)] - \mathcal{B}^-_{ij}[\vec{V}^-_l(\omega,\theta)]\right)\\
&+ i\sqrt{\frac{2}{\pi\omega}}\frac{e^{ir\omega}}{r^{\frac{D-2}{2}}}\sum_{l=2}^{\infty}e^{\frac{-i(D+2l)\pi}{4}}(X_l)_{ij}\\
&= i\sqrt{\frac{2}{\pi\omega}}\frac{e^{ir\omega}}{r^{\frac{D-2}{2}}}\sum_{l=0}^{\infty}e^{\frac{-i(D+2l)\pi}{4}}\\
&\Bigg(\hat{r}_i\hat{r}_j\left(l(l+D-3)\left(\frac{D-3}{D-1}\right)S^0_l - (l+D-3)(l+D-2)S^+_l - l(l-1)S^-_l\right)\\
&+ r\hat{r}_i\tilde{\nabla}_j\left(\frac{D-3}{2}S^0_l + (l+D-2)S^+_l - (l-1)S^-_l\right) + \{i\leftrightarrow j\}\\
&- r^2\tilde{\nabla}_{ij}\left(S^0_l + S^+_l + S^-_l\right)\\
&- \Pi_{ij}\left(\frac{2l(l+D-3)}{D-1}S^0_l + (l+D-3)S^+_l - lS^-_l\right)\Bigg)\\
&+ \sqrt{\frac{2}{\pi\omega}}\frac{e^{ir\omega}}{r^{\frac{D-2}{2}}}\sum_{l=0}^{\infty}e^{\frac{-i(D+2l)\pi}{4}}\Bigg(\hat{r}_i\left((l-1)(V_l)^-_j - (l+D-2)(V_l)^+_j\right)\\
&- r\tilde{\nabla}_i\left((V_l)^-_j + (V_l)^+_j\right) + \{i\leftrightarrow j\}\Bigg)\\
&+ i\sqrt{\frac{2}{\pi\omega}}\frac{e^{ir\omega}}{r^{\frac{D-2}{2}}}\sum_{l=1}^{\infty}e^{\frac{-i(D+2l)\pi}{4}}(X_l)_{ij}.
\end{aligned} \tag{830}$$

Where we have expanded the expressions of $\mathcal{C}$s and $\mathcal{B}$s as given in (925) using (927) in the last step. Note that the radiation field is polarized orthogonal to the line of sight from the observation to the source point in this limit.

Radiation in terms of sources These scalar, vector and tensor functions on the unit sphere that characterize radiation may be determined in terms of the $(\mathcal{T}_{eff})_{ij}$ as follows. The tensor field $(\mathcal{T}_{eff})_{ij}$ may be decomposed along the lines of (924) as

$$\begin{aligned}(\mathcal{T}_{eff})_{ij} = &\Big(\mathcal{C}^-_{ij}[\mathfrak{a}] + \mathcal{C}^+_{ij}[\mathfrak{b}] + \mathcal{C}^0_{ij}[\mathfrak{c}] + \delta_{ij}\mathfrak{d}\Big) \\ &+ \Big(\mathcal{B}^-_{ij}[\vec{\mathfrak{u}}] + \mathcal{B}^+_{ij}[\vec{\mathfrak{v}}]\Big) + \mathfrak{z}_{ij},\end{aligned} \tag{831}$$

where $\mathcal{C}^-_{ij}[\mathfrak{a}]$, $\mathcal{C}^+_{ij}[\mathfrak{b}]$ and $\mathcal{C}^0_{ij}[\mathfrak{c}]$ are the maps from scalars to tensors in R^{D-1} defined in (925) in the Appendix.

In what follows we use obvious notation to denote the l^{th} spherical harmonic components of the scalar, tensor and vector functions on the unit sphere that appear in (831). For instance $\mathfrak{a}_l$ denotes the projection of the scalar function $\mathfrak{a}$ to the l^{th} scalar spherical harmonic sector, while $(\mathfrak{z}_l)_{ij}$ denotes the projection of the tensor field $\mathfrak{z}_{ij}$ to the l^{th} tensor harmonic sector. As in the previous subsection, the action of the retarded Greens function on an arbitrary tensor field (831) takes the for, (981) and (982) and we obtain

$$\begin{aligned}
S^-_l &= \frac{i\pi}{2}\int dr' J_{\frac{D-3+2(l-2)}{2}}(\omega r')\ r'^{\frac{D-1}{2}}\mathfrak{a}_l(\omega, r', \theta),\\
S^+_l &= \frac{i\pi}{2}\int dr' J_{\frac{D-3+2(l+2)}{2}}(\omega r')\ r'^{\frac{D-1}{2}}\mathfrak{b}_l(\omega, r', \theta),\\
S^0_l &= \frac{i\pi}{2}\int dr' J_{\frac{D-3+2l}{2}}(\omega r')\ r'^{\frac{D-1}{2}}\mathfrak{c}_l(\omega, r', \theta),\\
\vec{V}^-_l &= \frac{i\pi}{2}\int dr' J_{\frac{D-3+2(l-1)}{2}}(\omega r')\ r'^{\frac{D-1}{2}}\vec{\mathfrak{u}}_l(\omega, r', \theta),\\
\vec{V}^+_l &= \frac{i\pi}{2}\int dr' J_{\frac{D-3+2(l+1)}{2}}(\omega r')\ r'^{\frac{D-1}{2}}\vec{\mathfrak{v}}_l(\omega, r', \theta),\\
(X_l)_{ij} &= \frac{i\pi}{2}\int dr' J_{\frac{D-3+2l}{2}}(\omega r')\ r'^{\frac{D-1}{2}}(\mathfrak{z}_l)_{ij}(\omega, r', \theta),\\
S^{Tr}_l &= \frac{i\pi}{2}\int dr' J_{\frac{D-3+2l}{2}}(\omega r')\ r'^{\frac{D-1}{2}}\mathfrak{d}_l(\omega, r', \theta) = 0.
\end{aligned} \tag{832}$$

Although it may not be apparent from a casual glance, the solution (832) obeys the constraints (829) and (826). (829) is obeyed after a partial integration simply because $\mathfrak{d}$ equals the operator $\square$ acting on another function (see (823)). Moreover the expressions for $\mathfrak{a}_l$, $\mathfrak{b}_l$ and $\mathfrak{c}_l$ in (832) may also be shown to obey (826) by using (932), integrating by parts and using an appropriate recursion relation (see subsection 4.11.7 for details).

Special limits As in the previous subsection (793) may be used to simplify (832) in both the static and the the large D limits. In either limit we obtain the simplified formula

$$
\begin{aligned}
S_l^- &= \tilde{S}_l^- \omega^{l+\frac{D-7}{2}} \int dr' (r')^{l+D-4} \mathfrak{a}_l(\omega, r', \theta), \\
S_l^+ &= \tilde{S}_l^+ \omega^{l+\frac{D+1}{2}} \int dr' (r')^{l+D} \mathfrak{b}_l(\omega, r', \theta), \\
S_l^0 &= \tilde{S}_l^0 \omega^{l+\frac{D-3}{2}} \int dr' (r')^{l+D-2} \mathfrak{c}_l(\omega, r', \theta), \\
\vec{V}_l^- &= \tilde{V}_l^- \omega^{l+\frac{D-5}{2}} \int dr' (r')^{l+D-3} \vec{\mathfrak{u}}(\omega, r', \theta), \\
\vec{V}_l^+ &= \tilde{V}_l^+ \omega^{l+\frac{D-1}{2}} \int dr' (r')^{l+D-1} \vec{\mathfrak{v}}(\omega, r', \theta), \\
(X_l)_{ij} &= \tilde{X}_l \omega^{l+\frac{D-3}{2}} \int dr' (r')^{l+D-2} (\mathfrak{z})_{ij}(\omega, r', \theta),
\end{aligned}
\tag{833}
$$

with

$$
\begin{aligned}
\tilde{S}_l^- &= \frac{i\pi}{2^{l+\frac{D-5}{2}}} \frac{1}{\Gamma\left(l+\frac{D-5}{2}\right)}, \\
\tilde{S}_l^+ &= \frac{i\pi}{2^{l+\frac{D+3}{2}}} \frac{1}{\Gamma\left(l+\frac{D+3}{2}\right)}, \\
\tilde{S}_l^0 &= \frac{i\pi}{2^{l+\frac{D-1}{2}}} \frac{1}{\Gamma\left(l+\frac{D-1}{2}\right)}, \\
\tilde{V}_l^- &= \frac{i\pi}{2^{l+\frac{D-3}{2}}} \frac{1}{\Gamma\left(l+\frac{D-3}{2}\right)}, \\
\tilde{V}_l^+ &= \frac{i\pi}{2^{l+\frac{D+1}{2}}} \frac{1}{\Gamma\left(l+\frac{D+1}{2}\right)}, \\
\tilde{X}_l &= \frac{i\pi}{2^{l+\frac{D-1}{2}}} \frac{1}{\Gamma\left(l+\frac{D-1}{2}\right)}.
\end{aligned}
\tag{834}
$$

As in the previous subsection in the large D limit at fixed ω we use the Sterling approximation to further simplify $\tilde{\alpha}_l$; we find

$$
\begin{aligned}
\tilde{S}_l^- &\approx \frac{i\sqrt{\pi}}{2D^{\frac{D}{2}}}\left(\frac{e^{l+\frac{D-7}{2}}}{D^{l-3}}\right),\\
\tilde{S}_l^+ &\approx \frac{i\sqrt{\pi}}{2D^{\frac{D}{2}}}\left(\frac{e^{l+\frac{D+1}{2}}}{D^{l+1}}\right),\\
\tilde{S}_l^0 &\approx \frac{i\sqrt{\pi}}{2D^{\frac{D}{2}}}\left(\frac{e^{l+\frac{D-3}{2}}}{D^{l-1}}\right),\\
\tilde{V}_l^- &\approx \frac{i\sqrt{\pi}}{2D^{\frac{D}{2}}}\left(\frac{e^{l+\frac{D-5}{2}}}{D^{l-2}}\right),\\
\tilde{V}_l^+ &\approx \frac{i\sqrt{\pi}}{2D^{\frac{D}{2}}}\left(\frac{e^{l+\frac{D-1}{2}}}{D^{l}}\right),\\
\tilde{X}_l &\approx \frac{i\sqrt{\pi}}{2D^{\frac{D}{2}}}\left(\frac{e^{l+\frac{D-3}{2}}}{D^{l-1}}\right).
\end{aligned}
\tag{835}
$$

As in the previous subsection (835) does not apply in the static limit at fixed D. In this limit, however, the small argument expansion of the Hankel function leads to simplifications. In Appendix 4.11.7 we demonstrate that the radiation formulae (822), in this limit yield results consistent with the equations

$$
\begin{aligned}
R_{0i0j} &= \nabla_i\nabla_j\Phi^G,\\
\nabla^2\phi^G &= -8\pi\left(\mathcal{T}_{00} + \frac{\mathcal{T}}{D-2}\right),\\
R_{0ijk} &= -\nabla_i\left(\nabla_j A_k^G - \nabla_k A_j^G\right),\\
\nabla^2 A_i^G &= -8\pi\mathcal{T}_{0i},\\
R_{ijkl} &= \nabla_i\nabla_k\mathfrak{T}_{jm} + \nabla_j\nabla_m\mathfrak{T}_{ik} - \nabla_j\nabla_k\mathfrak{T}_{im} - \nabla_i\nabla_m\mathfrak{T}_{jk},\\
\nabla^2\mathfrak{T}_{ij}^G &= 8\pi\mathcal{T}_{ij}.
\end{aligned}
\tag{836}
$$

4.9 Radiation from linearized fluctuations about spherical membranes

4.9.1 Electromagnetic Radiation

As we have explained in the previous section, the simplest solution of the charged membrane equations of motions is a static spherical membrane whose world volume is $S^{D-2}\times$ time. This solution is dual to a static charged black hole. The spectrum of linearized membrane fluctuations about this simple solution was determined in chapter 3. These linearized solutions are dual to the light quasinormal modes around the dual stationary black holes. In this section we will compute the radiation sourced by these linearized membrane modes. The radiation fields we compute have the bulk interpretation as the 'outgoing' pieces of the corresponding quasinormal modes.

We begin this section by briefly recalling the linearized solutions of chapter 3. As in chapter 3 we choose our background solution to be a charged black hole of unit radius (as explained in chapter 3, the scale invariance of the Einstein Maxwell equations ensures that this choice involves no loss of generality). We work to linearized order about this static solution. In other words the membrane configurations we study are

$$\begin{aligned} r &= 1 + \delta r(t,\theta), \\ Q &= Q_0 + \delta Q(t,\theta), \\ u &= -dt + \delta u_\mu(t,\theta) dx^\mu. \end{aligned} \tag{837}$$

As we have demonstrated earlier in this chapter, the charge current associated with any membrane configuration is given, in terms of arbitrary coordinates on the membrane world volume, by

$$\begin{aligned} J^\mu = &\left(\frac{Q}{2\sqrt{2\pi}}\right)\left[K u^\mu - \left(\frac{p^{\nu\mu}\hat{\nabla}_\nu Q}{Q}\right) - (u\cdot\hat{\nabla})u^\mu - \left(\frac{\hat{\nabla}^2 u^\mu}{K}\right) + K^{\alpha\mu}u_\alpha\right] \\ &+ \mathcal{Q}\, u^\mu + \mathcal{O}\left(\frac{1}{D}\right), \end{aligned} \tag{838}$$

where

$$\mathcal{Q} = \left(\frac{Q}{2\sqrt{2\pi}}\right)\left[\frac{\hat{\nabla}^2 K}{K^2} - \frac{2K}{D} - \frac{(u\cdot\hat{\nabla})K}{K} - \left(\frac{2\hat{\nabla}^2 Q + K(u\cdot\hat{\nabla})Q}{Q\ K}\right) + \left(u^\alpha u^\beta K_{\alpha\beta}\right)\right]. \tag{839}$$

We will now evaluate the current J_μ listed in (838) for the special case of small fluctuations around the spherical membrane ((837)) to first order in fluctuations. For this purpose we use the angular coordinates on the unit S^{D-2} and time as coordinates on the membrane world volume. Note that, to linear order in fluctuations, the metric on the membrane world volume is given by

$$ds^2 = -dt^2 + (1+2\delta r)dr^2. \tag{840}$$

All covariant derivatives in (838) must be evaluated on this metric. However, following chapter 3, we will find it most convenient to view our fluctuation fields δr and u_μ as living on the undeformed unit sphere. In all formulae below the symbol ∇_a will refer to the covariant derivative on this round sphere (a are the angular directions on the sphere). [156] Adopting these conventions the

[156] In order to present all our formulae in terms of covariant derivatives w.r.t the unit sphere, we sometimes need to rewrite covariant derivatives w.r.t. the metric (840) in terms of covariant derivatives on the unit

formulae

$$
\begin{aligned}
n_r &= 1, \\
n_\mu &= -\partial_\mu \delta r, \\
K_{tt} &= -\partial_t^2 \delta r, \\
K_{ta} &= -\partial_t \nabla_a \delta r, \\
K_{ab} &= -\nabla_a \nabla_b \delta r + (1+\delta r) g_{ab}, \\
\delta u_t &= 0, \qquad\qquad (u\cdot u = -1) \\
(u\cdot K)_t &= K_{tt} = -\partial_t^2 \delta r, \\
(u\cdot K)_a &= -\partial_t \nabla_a \delta r + \delta u_a, \\
K &= K_A{}^A = D\left(1 - \left(1 + \frac{\nabla^2}{D}\right)\delta r\right),
\end{aligned}
\tag{841}
$$

(which we have borrowed from chapter 3) allow us to explicitly evaluate all components of the membrane world volume current in terms of the linearized fluctuations in (837); we find

$$
\begin{aligned}
J_t =& \frac{1}{2\sqrt{2\pi}}\Bigg(- DQ_0 + \left(DQ_0 \left(1 + \frac{\nabla^2}{D}\right)\delta r - D\delta Q - \partial_t \delta Q - Q_0 \partial_t^2 \delta r + \frac{\nabla^2}{D}\delta Q\right)\Bigg), \\
J_i =& \frac{1}{2\sqrt{2\pi}}\left(Q_0 \delta u_i - Q_0 \partial_t \delta u_i - \partial_i \delta Q - Q_0 \partial_i \partial_t \delta r - Q_0 \frac{\nabla^2}{D}\delta u_i \right).
\end{aligned}
\tag{842}
$$

Note that we have presented our current with lower indices, i.e. as a oneform field. This oneform field lives of the membrane world volume whose metric is given by (840).

Recall that the membrane current is conserved, i.e.

$$
\nabla \cdot J = 0, \tag{843}
$$

Explicitly evaluating this conservation equation for the current (842) we obtain the equation

$$
\left(\frac{\nabla^2}{D} - \partial_t\right)\delta Q = Q_0 \left(\partial_t^2 - \partial_t \left(\frac{\nabla^2}{D} + 1\right)\right)\delta r + \mathcal{O}(1/D). \tag{844}
$$

Note that (844) is precisely the linearized 'charge' membrane equation presented in chapter 3. We view this agreement as a consistency check on the algebra that led to (842)

The expression (842) is the current evaluated on the membrane world volume in our particular choice of world volume coordinates. Radiation is sourced by the current viewed as a distributional vector field in spacetime. We obtained the spacetime current as follows. We first converted the

sphere.

oneform field J into a vector field on the membrane world volume using its metric (840) [157]. We then converted the vector field on the membrane to a vector field in spacetime using the formulae

$$\begin{aligned} J^a_{ST} &= \delta(r+\delta r-1)J^a, \\ J^t_{ST} &= \delta(r+\delta r-1)J^t, \\ J_{ST}r &= \delta(r+\delta r-1)\left(J^t\partial_t\delta r + J^a\partial_a\delta r\right) = \delta(r+\delta r-1)J^t\partial_t\delta r. \end{aligned} \tag{845}$$

The equality in the last line of this equation holds to linear order in fluctuations as J^a vanishes on the static membrane. Note that we have also omitted the measure factor $\sqrt{1+(\nabla\delta r)^2}$ in the spacetime current (see e.g. (594)) as this term is unity to linear order. We find the following expression for the spacetime current

$$\begin{aligned} J^a_{ST} &= \frac{1}{2\sqrt{2\pi}}\delta(r+\delta r-1)\left(DQ_0\delta u^a - Q_0\partial_t\delta u^a - \partial^a\delta Q - Q_0\partial^a\partial_t\delta r - Q_0\frac{\nabla^2}{D}\delta u^a\right), \\ J^t_{ST} &= \frac{1}{2\sqrt{2\pi}}\delta(r+\delta r-1)\Bigg(DQ_0 - \left(DQ_0\left(1+\frac{\nabla^2}{D}\right)\delta r - D\delta Q \right. \\ &\quad \left. - \partial_t\delta Q - Q_0\partial_t^2\delta r + \frac{\nabla^2}{D}\delta Q\right)\Bigg), \\ J^r_{ST} &= \frac{1}{2\sqrt{2\pi}}\delta(r+\delta r-1)\left(DQ_0\partial_t\delta r\right), \end{aligned} \tag{846}$$

As was explained in chapter 3 the linearized solutions take the form

$$\begin{aligned} \delta r &= \sum_{l,m} a_{lm}Y_{lm}e^{-i\omega^r_l t}, \\ \delta Q &= \sum_{l,m} a_{lm}\frac{i\omega^r_l Q_0\left(l-1-i\omega^r_l\right)}{l-i\omega^r_l}Y_{lm}e^{-i\omega^r_l t} + \sum_{l,m} q_{lm}Y_{lm}e^{-i\omega^Q_l t}, \\ \delta u_i &= \sum_{l,m}\frac{-i\omega^r_l}{l}a_{lm}\nabla_i Y_{lm}e^{-i\omega^r_l t} + \sum_{l,m} b_{lm}V^{lm}_i e^{-i\omega^v_l t}, \end{aligned} \tag{847}$$

(the summation over l involving a_{lm} in the last line excludes $l=0$). The coefficients b_{lm} parametrize the 'velocity fluctuations' of chapter 3; note that these fluctuations affect only the velocity field. The coefficients q_{lm} parametrize the 'charge fluctuations' of chapter 3; note that they affect only the charge field. The coefficients a_{lm} parametrize the 'shape' fluctuations of chapter 3. These are the most complicated quasinormal modes, as they affect the shape δr, the charge δQ and the velocity δu. In the rest of this subsection we will determine the radiation field

[157] However the term proportional to δr does not contribute to leading order in fluctuations as J_a vanishes for the stationary membrane. In effect, thus, consequently we raise all indices using the metric of the unit sphere

sourced by each of these fluctuations in turn.

Radiation from Charge Fluctuations Let us first restrict our attention to the l^{th} spherical harmonic mode of 'charge' fluctuations (i.e. mode in (847) that is proportional to q_{lm}). In this special case the spacetime current (846) reduces to

$$\begin{aligned}
J^a_{ST} &= -\frac{1}{2\sqrt{2\pi}}\delta(r-1)q_{lm}e^{-i\omega_l^Q t}\left(\partial^a Y_{lm}\right), \\
J^t_{ST} &= \frac{D}{2\sqrt{2\pi}}\delta(r-1)\left(q_{lm}Y_{lm}e^{-i\omega_l^Q t}\right), \\
J^r_{ST} &= 0.
\end{aligned} \tag{848}$$

It follows that the quantities $\mathfrak{b}$ and $\mathfrak{c}$ relevant for (808) are given by

$$\begin{aligned}
\mathfrak{b} &= \frac{l}{2\sqrt{2\pi}}\delta(r-1)q_{lm}Y_{lm}e^{-i\omega_l^Q t} - \frac{1}{2\sqrt{2\pi}}q_{lm}Y_{lm}e^{-i\omega_l^Q t}\partial_r\delta(r-1), \\
\mathfrak{c} &= 0,
\end{aligned} \tag{849}$$

(at $\omega = \omega_l^Q = -il$). In writing our result (848) we have taken the large D limit and retained only leading order terms. It follows from (812) that

$$\begin{aligned}
S_l^+ &= \frac{D}{2\sqrt{2\pi}}\tilde{S}_l^2\omega^{l+\frac{D-1}{2}}q_{lm}Y_{lm}e^{-i\omega_l^Q t}, \\
\vec{V}_l &= 0,
\end{aligned} \tag{850}$$

where the coefficients $\tilde{S}_l^+$ are defined in (813). The S_l^+ can be computed from the constraint equation (806) which in the large D limit reduces to

$$S_l^- = \frac{D}{l}S_l^+.$$

The electromagnetic radiation field associated with the l^{th} 'charge' fluctuation quasinormal mode is given by plugging these results into (804).

Radiation from Velocity Fluctuations Let us now restrict our attention to the l^{th} spherical harmonic mode of 'velocity' fluctuations (i.e. mode in (847) that is proportional to b_{lm}). In this special case the spacetime current (846) reduces to

$$\begin{aligned}
J^a_{ST} &= \frac{DQ_0}{2\sqrt{2\pi}}\delta(r-1)b_{lm}V^a_{lm}e^{-i\omega^v_l t},\\
J^t_{ST} &= 0,\\
J^r_{ST} &= 0.
\end{aligned} \tag{851}$$

It follows that the quantities $\mathfrak{b}$ and $\mathfrak{c}$ relevant for (808) are given by

$$\begin{aligned}
\mathfrak{b} &= 0,\\
\mathfrak{c} &= \frac{iDQ_0\omega^v_l}{2\sqrt{2\pi}}\delta(r-1)b_{lm}V^a_{lm}e^{-i\omega^v_l t},
\end{aligned} \tag{852}$$

(at $\omega = \omega^v_l = \frac{-i(l-1)}{1+Q_0^2}$). It follows from (809) that

$$\vec{V}_l = \frac{iDQ_0}{2\sqrt{2\pi}}(\omega^v_l)^{\frac{D-1}{2}+l}\tilde{V}_l b_{lm}\vec{V}_{lm}e^{-i\omega^v_l t}, \tag{853}$$

where the coefficients $\tilde{V}_l$ is defined in (813). The electromagnetic radiation field associated with the l^{th} 'velocity' fluctuation quasinormal mode is obtained by plugging (853) into (804).

Radiation from Shape Fluctuations Let us first restrict our attention to the l^{th} spherical harmonic mode of 'shape' fluctuations (i.e. mode in (847) that is proportional to a_{lm}). The radiation due to the shape fluctuation is little complicated compared to the 'charge ' fluctuation or the 'velocity ' fluctuation , since the small perturbation in the shape turns on both the charge and the velocity fluctuation(847) . In this special case the spacetime current (846) reduces to

$$\begin{aligned}
J^a_{ST} &= -i\frac{\omega^r_l}{l}\frac{DQ_0}{2\sqrt{2\pi}}\delta(r-1)a_{lm}\nabla^a Y_{lm}e^{-i\omega^r_l t},\\
J^t_{ST} &= \delta(r-1)\frac{DQ_0}{2\sqrt{2\pi}}\left(a_{lm}\frac{i\omega^r_l\,(l-1-i\omega^r_l)}{l-i\omega^r_l}Y_{lm}e^{-i\omega^r_l t}\right) + \frac{DQ_0}{2\sqrt{2\pi}}a_{lm}Y_{lm}\partial_r\delta(r-1)e^{-i\omega^r_l t}\\
&\quad - \frac{DQ_0(-l+1)}{2\sqrt{2\pi}}\delta(r-1)a_{lm}Y_{lm}e^{-i\omega^r_l t},\\
J^r_{ST} &= -\frac{i\omega^r_l DQ_0}{2\sqrt{2\pi}}\delta(r-1)a_{lm}Y_{lm}e^{-i\omega^r_l t}.
\end{aligned} \tag{854}$$

It follows that the quantities $\mathfrak{b}$ and $\mathfrak{c}$ relevant for (808) are given by

$$\begin{aligned}
\mathfrak{b} &= \frac{Q_0}{2\sqrt{2\pi}}a_{lm}Y_{lm}\partial_r^2\delta(r-1)e^{-i\omega^r_l t},\\
\mathfrak{c} &= 0,
\end{aligned} \tag{855}$$

(at $\omega = \omega_l^r$). It follows from (809) that

$$S_l^+ = \tilde{S}_l^+ \omega^{l+\frac{D-1}{2}} \frac{D^2 Q_0}{2\sqrt{2\pi}} a_{lm} Y_{lm} e^{-i\omega_l^r t},$$
$$\vec{V}_l = 0, \tag{856}$$

where the coefficient $\tilde{S}_l^+$ are defined in (813). The S_l^+ can be computed from the constraint equation (806) which in the large D limit reduces to

$$S_l^- = \frac{D}{l} S_l^+.$$

The electromagnetic radiation field associated with the l^{th} 'shape' fluctuation quasinormal mode is obtained by plugging (804)).

4.9.2 Gravitational Radiation

In this subsection we compute the gravitational radiation emitted by the quasinormal modes described earlier in this section. As we demonstrated earlier in this chapter, the stress tensor on the world volume of the large D black hole membrane is given by

$$\begin{aligned} T_{\mu\nu} = & \left(\frac{1}{8\pi}\right) \Bigg[\left(\frac{K}{2}\right)(1+Q^2) u_\mu u_\nu + \left(\frac{1-Q^2}{2}\right) K_{\mu\nu} - \left(\frac{\hat{\nabla}_\mu u_\nu + \hat{\nabla}_\nu u_\mu}{2}\right) \\ & - \left(\frac{KQ^2}{2D} + \frac{2Q\hat{\nabla}^2 Q}{K} + Q^2 u^\alpha u^\beta K_{\alpha\beta}\right) u_\mu u_\nu - (u_\mu \mathcal{V}_\nu + u_\nu \mathcal{V}_\mu) \\ & - \left[\left(\frac{1+Q^2}{2}\right)\left(u^\alpha u^\beta K_{\alpha\beta}\right) + \left(\frac{1-Q^2}{2}\right)\left(\frac{K}{D}\right)\right] g^{(ind,f)}_{\mu\nu} \Bigg] \\ & + \mathcal{O}\left(\frac{1}{D}\right), \end{aligned} \tag{857}$$

where

$$\begin{aligned} \mathcal{V}_\mu = & \, Q\, \hat{\nabla}_\mu Q + Q^2 (u^\alpha K_{\alpha\mu}) + \left(\frac{2Q^4 - Q^2 - 1}{2}\right)\left(\frac{\hat{\nabla}_\mu K}{K}\right) \\ & - \left(\frac{Q^2 + 2Q^4}{2}\right)(u \cdot \hat{\nabla}) u_\mu + \left(\frac{1+Q^2}{K}\right) \hat{\nabla}^2 u_\mu. \end{aligned} \tag{858}$$

The stress tensor is conserved upto the membrane equation of motion (546) and the divergencelessness of the velocity field.

$$\nabla_\mu T^\mu_\nu = 0. \tag{859}$$

The most general form of the fluctuation of the stress tensor about the RN background takes the form

$$-8\pi T_{tt} = -\frac{D}{2}(1+Q_0^2) + \frac{Q_0^2}{2} - \Bigg(- (1+Q_0^2)\left(1+\frac{\nabla^2}{D}\right)\left(\frac{D}{2}+\partial_t\right)\delta r + (D+1)Q_0\delta Q$$

$$+ Q_0^2\left(1+\frac{\nabla^2}{D}\right)(Q_0^2\partial_t - 1/2)\delta r + 2Q_0\left(\frac{\nabla^2}{D}-\partial_t\right)\delta Q + 2Q_0^2\partial_t^2\delta r\Bigg)$$
$$+ \frac{1+Q_0^2}{2}\partial_t^2\delta r - \frac{1-Q_0^2}{2}\left(1-\left(1+\frac{\nabla^2}{D}\right)\delta r\right) + \frac{1-Q_0^2}{2}\partial_t^2\delta r,$$
$$-8\pi T_{ta} = \Bigg(\frac{D(1+Q_0^2)-Q_0^2}{2}\delta u_a + \frac{1-Q_0^2}{2}\partial_t\nabla_a\delta r + \frac{1+Q_0^4}{2}\partial_t\delta u_a - Q_0\partial_a\delta Q$$
$$- \frac{1-Q_0^2-Q_0^4}{2}(1+\frac{\nabla^2}{D})\nabla_a\delta r - \left(\frac{1+Q_0^2}{2}\right)\frac{\nabla^2}{D}\delta u_a + 2Q_0^2\partial_t\nabla_a\delta r - 2Q_0^2\delta u_a\Bigg),$$
$$-8\pi T_{ab} = -\frac{1+Q_0^2}{2}\partial_t^2\delta r g_{ab} + \Bigg(\frac{1-Q_0^2}{2}(\nabla_a\nabla_b\delta r - g_{ab}\delta r) + Q_0 g_{ab}\delta Q + \frac{\nabla_a\delta u_b + \nabla_b\delta u_a}{2}$$
$$+ g_{ab}\partial_t\delta r\Bigg) - \frac{1-Q_0^2}{2}\left(1+\frac{\nabla^2}{D}\right)\delta r g_{ab}.$$

The velocity fluctuation and the shape fluctuation along the angular direction follows the following second order differential equation

$$\left(\left(1+\frac{\nabla^2}{D}\right) - (1+Q_0^2)\partial_t\right)\delta u_a = -\left((1-Q_0^2)\nabla_a\left(1+\frac{\nabla^2}{D}\right) - \partial_t\nabla_a\right)\delta r, \tag{860}$$

and along the t direction the velocity fluctuation and the shape fluctuation follows the constraint equation[158]

$$\nabla_a\delta u^a = -(D-2)\partial_t\delta r \approx -D\partial_t\delta r. \tag{861}$$

We have, so far, been working with tensor fields living on the world volume of the membrane. In order to determine the source for radiation we are really interested in the stress tensor viewed as a distributional tensor field living in spacetime. Apart from the delta functions that localize the spacetime quantities to the membrane (and which we will explicitly present in later formulae) the relationship between these two structures is given in general by the following translation formulae

[158]The expression below can be obtained either by use the expansion of the divergence in terms of the Christoffel symbol or use the form

$$(\nabla.T)_i = \frac{1}{\sqrt{g}}\partial_k(\sqrt{g}T_i^k) - \frac{1}{2}\partial_i g_{kl}T^{kl}.$$

between the membrane field $A^{\mu\nu}$ and the spacetime field A_{ST}^{MN}

$$
\begin{aligned}
&A_{ST}^{tt} = A^{tt}, \qquad A_{ST}^{ab} = A^{ab},\\
&A_{ST}^{rr} = \mathcal{O}(\epsilon^2), \qquad A_{ST}^{ta} = A^{ta},\\
&A_{ST}^{tr} = \left(A^{tt}\partial_t\delta r + A^{ta}\partial_a\delta r\right), A_{ST}^{ar} = \Big(A^{at}\partial_t\delta r + A^{ab}\partial_b\delta r\Big).
\end{aligned}
$$

It follows that to linear order in fluctuations

$$
\begin{aligned}
&T_{ST}^{tt} = \delta(r+\delta r-1)T^{tt},\\
&T_{ST}^{ab} = \delta(r+\delta r-1)T^{ab},\\
&T_{ST}^{ta} = \delta(r+\delta r-1)T^{ta},\\
&T_{ST}^{tr} = \delta(r+\delta r-1)\left(T^{tt}\partial_t\delta r + T^{ta}\partial_a\delta r\right) = \delta(r+\delta r-1)T^{tt}\partial_t\delta r,\\
&T_{ST}^{rr} = 0,\\
&T_{ST}^{ar} = 0.
\end{aligned}
\tag{862}
$$

Explicitly we find

$$
\begin{aligned}
-8\pi T^{tt}_{ST} &= \delta(r+\delta r-1)\Bigg(-\frac{D}{2}(1+Q_0^2) - \Bigg(-(1+Q_0^2)\left(1+\frac{\nabla^2}{D}\right)\left(\frac{D}{2}+\partial_t\right)\delta r \\
&\quad + (D+1)Q_0\delta Q + \frac{1-Q_0^2}{2}\partial_t^2\delta r \\
&\quad + Q_0^2\left(1+\frac{\nabla^2}{D}\right)(Q_0^2\partial_t - 1/2)\delta r + 2Q_0\left(\frac{\nabla^2}{D}-\partial_t\right)\delta Q + 2Q_0^2\partial_t^2\delta r\Bigg) \\
&\quad + \frac{1+Q_0^2}{2}\partial_t^2\delta r - \frac{1-Q_0^2}{2}\left(1-\left(1+\frac{\nabla^2}{D}\right)\delta r\right)\Bigg), \\
-8\pi T^{ta}_{ST} &= -\delta(r+\delta r-1)\Bigg(\frac{D(1+Q_0^2)-Q_0^2}{2}\delta u_a + \frac{1-Q_0^2}{2}\partial_t\nabla_a\delta r + \\
&\quad \frac{1+Q_0^4}{2}\partial_t\delta u_a - \frac{1-Q_0^2-Q_0^4}{2}(1+\frac{\nabla^2}{D})\nabla_a\delta r - Q_0\partial_a\delta Q - \left(\frac{1+Q_0^2}{2}\right)\frac{\nabla^2}{D}\delta u_a \\
&\quad + 2Q_0^2\partial_t\nabla_a\delta r - 2Q_0^2\delta u_a\Bigg), \\
-8\pi T^{ab}_{ST} &= \delta(r+\delta r-1)\Bigg(-\frac{1+Q_0^2}{2}\partial_t^2\delta r g_{ab} + \Bigg(\frac{1-Q_0^2}{2}(\nabla_a\nabla_b\delta r - g_{ab}\delta r) + Q_0 g_{ab}\delta Q \\
&\quad + \frac{\nabla_a\delta u_b + \nabla_b\delta u_a}{2} + g_{ab}\partial_t\delta r\Bigg) - \frac{1-Q_0^2}{2}\left(1+\frac{\nabla^2}{D}\right)\delta r g_{ab}\Bigg) \\
-8\pi T^{tr}_{ST} &= \delta(r+\delta r-1)\left(-\frac{D}{2}(1+Q_0^2)\partial_t\delta r\right), \\
-8\pi T^{rr}_{ST} &= 0, \\
-8\pi T^{ar}_{ST} &= 0.
\end{aligned}
$$

Gravitational Radiation from Charge fluctuation Let us first restrict our attention to the l^{th} spherical harmonic mode of 'charge' fluctuations (i.e. mode in (847) that is proportional to q_{lm}). In this special case the stress tensor (863) reduces to

$$
\begin{aligned}
-8\pi T^{tt}_{ST} &= -\delta(r-1)DQ_0\delta Q, \\
-8\pi T^{ta}_{ST} &= \delta(r-1)Q_0\partial^a\delta Q, \\
-8\pi T^{ab}_{ST} &= \delta(r-1)Q_0 g^{ab}\delta Q, \\
-8\pi T^{tr}_{ST} &= 0, \\
-8\pi T^{rr}_{ST} &= 0, \\
-8\pi T^{ar}_{ST} &= 0,
\end{aligned}
\tag{863}
$$

where δQ is given by the part of (847) proportional to q_{lm}.

It follows that we can read of the relevant quantity $\mathfrak{b}$ from the effective stress tensor (818) and (831) is given by

$$\mathfrak{b} = \frac{Q_0}{D}(\partial_r^2 \delta(r-1)) q_{lm} Y_{lm} e^{-i\omega_l^Q t} \tag{864}$$

(at $\omega = \omega_l^Q$). It follows from (809) that

$$S_l^+ = \tilde{S}_l^+ D Q_0 (\omega_l^Q)^{l+\frac{D+1}{2}} q_{lm} Y_{lm} e^{-i\omega_l^Q t}, \tag{865}$$

where $\tilde{S}_l^+$ is given by (835). The other components can be read of using the constraint equation (826),which in the large D limit can be simplified as

$$S_l^0 = \frac{2D}{l} S_l^+, \quad S_l^- = \frac{D^2}{l(l-1)} S_l^+. \tag{866}$$

The contribution to the radiation due to the charge fluctuation to the vector sector and the tensor sector vanishes in the linear order.

The explicit formula for gravitational radiation from the charge fluctuations is given by plugging (865) and (866) into (822).

Gravitational Radiation from Velocity fluctuation We now turn our attention to the l^{th} spherical harmonic mode of 'velocity' fluctuations (i.e. mode in (847) that is proportional to b_{lm}). In this special case the spacetime current (846) evaluates to

$$\begin{aligned}
-8\pi T_{ST}^{tt} &= 0, \\
-8\pi T_{ST}^{ta} &= \delta(r-1) Q_0 \frac{D(1+Q_0)^2}{2} \delta u^a, \\
-8\pi T_{ST}^{ab} &= \delta(r-1) \left(\frac{\nabla^a \delta u^b + \nabla^b \delta u^a}{2} \right), \\
-8\pi T_{ST}^{tr} &= 0, \\
-8\pi T_{ST}^{rr} &= 0, \\
-8\pi T_{ST}^{ar} &= 0,
\end{aligned} \tag{867}$$

where δu^a is obtained from the part of (847) proportional to b_{lm}. We can read of the relevant quantity $\vec{\mathfrak{v}}$ from the effective stress tensor (818) and (831). We find

$$\vec{\mathfrak{v}} = \frac{-i\omega_l^v (1+Q_0^2)}{2} (\partial_r \delta(r-1)) b_{lm} V_i^{lm} e^{-i\omega_l^v t}, \tag{868}$$

(at $\omega = \omega_l^v$). It follows from (809) that

$$\vec{V}_l^{+} = iD\tilde{V}_l^{+}(\omega_l^u)^{l+\frac{D-1}{2}}\frac{1+Q_0^2}{2}b_{lm}V_i^{lm}e^{-i\omega_l^v t}, \tag{869}$$

where $\tilde{V}_l^{+}$ is given by (835). The other components can be read of using the constraint equation (826),which in the large D limit can be simplified as

$$\vec{V}_l^{-} = \frac{D}{(l-1)}\vec{V}_l^{+}. \tag{870}$$

The contribution to the radiation due to the velocity fluctuation to the scalar sector and the tensor sector vanishes in the linear order. The gravitational radiation associated with the l^{th} 'velocity' fluctuation quasinormal mode is given by plugging (869) and (870) into (822).

Gravitational Radiation from Shape fluctuation Finally, we turn to the l^{th} spherical harmonic mode of 'shape' fluctuations (i.e. mode in (847) that is proportional to a_{lm}). In this special case the spacetime current (846) reduces to

$$\begin{aligned}
-8\pi T_{ST}^{tt} &= \delta(r-1)\left(\frac{1+Q_0^2}{2}\left(1+\frac{\nabla^2}{D}\right)\frac{D}{2}\delta r - DQ_0\delta Q\right) - \frac{D(1+Q_0^2)}{2}\delta r\partial_r\delta(r-1),\\
-8\pi T_{ST}^{ta} &= \delta(r-1)Q_0\frac{D(1+Q_0)^2}{2}\delta u^a,\\
-8\pi T_{ST}^{ab} &= \delta(r-1)\Bigg(-\frac{1+Q_0^2}{2}\partial_t^2\delta r g_{ab} + \Bigg(\frac{1-Q_0^2}{2}\left(\nabla_a\nabla_b\delta r - g_{ab}\delta r\right) + Q_0 g_{ab}\delta Q\\
&\quad + \frac{\nabla_a\delta u_b + \nabla_b\delta u_a}{2} + g_{ab}\partial_t\delta r\Bigg) - \frac{1-Q_0^2}{2}\left(1+\frac{\nabla^2}{D}\right)\delta r g_{ab}\Bigg)\\
-8\pi T_{ST}^{tr} &= \delta(r-1)\Bigg(-\frac{D}{2}(1+Q_0^2)\partial_t\delta r\Bigg),\\
-8\pi T_{ST}^{rr} &= 0,\\
-8\pi T_{ST}^{ar} &= 0,
\end{aligned} \tag{871}$$

where all fluctuation fields are obtained from the part of (847) proportional to a_{lm}. It follows that we can read of the relevant quantity $\mathfrak{b}$ from the effective stress tensor (818) and (831) and is given by

$$\mathfrak{b} = \frac{1+Q_0^2}{2D}\left(\partial_r^2\delta(r-1)a_{lm}Y_{lm}e^{-i\omega_l^r t} - \partial_r\delta(r-1)a_{lm}Y_{lm}e^{-i\omega_l^r t}\right), \tag{872}$$

(at $\omega = \omega_l^r$). It follows from (809) that

$$S_l^{+} = \tilde{S}_l^{+}D^2\frac{1+Q_0^2}{2}(\omega_l^r)^{l+\frac{D+1}{2}}a_{lm}Y_{lm}e^{-i\omega_l^r t}, \tag{873}$$

where $\tilde{S}_l^+$ is given by (835). The other components can be read of using the constraint equation (826),which in the large D limit can be simplified as

$$S_l^0 = \frac{2D}{l} S_l^+, \quad S_l^- = \frac{D^2}{l(l-1)} S_l^+.$$

The contribution to the radiation due to the charge fluctuation to the vector sector and the tensor sector vanishes in the linear order.The radiation field associated with the l^{th} 'shape' fluctuation quasinormal mode is obtained by plugging these results into (822).

4.10 Discussion

In this chapter we have obtained explicit formulae for the stress tensor, charge current and entropy current that live on the world volume of the large D black hole membrane of [1, 63, 66]. We have demonstrated that the membrane stress tensor and charge current are conserved. When written in terms of membrane variables, the requirement of conservation is simply a restatement of the membrane equations of motion of [1, 63, 66]. In contrast to the charge current and the stress tensor, the entropy current on the membrane world volume is not conserved; $\nabla_M J_S^M$ is nonvanishing at order $\frac{1}{D}$. We have used the Hawking area increase theorem to demonstrate that the divergence of this entropy current is point wise positive definite. At lowest nontrivial order (order $\frac{1}{D}$) we have demonstrated that this divergence is proportional to the square of the shear tensor.

In this chapter we have also derived explicit formulae for linearized radiation response of the metric and the electromagnetic field to an arbitrary stress tensor and a charge current. Plugging the our explicit results for the membrane stress tensor and charge current into these general formulae yields a formula for the radiation emitted from a large D black hole membrane as it undergoes any particular solution of the large D membrane equations. A central qualitative result of this chapter is that the fractional energy lost to radiation as the large D black hole membrane moves, oscillates and vibrates around is of order $\frac{1}{D^D}$. The smallness of radiation is a simple kinematical consequence of the nature of Greens functions in large D, and results in the decoupling of membrane motion from asymptotic low energy gravitons at large D. It also ensures that the 'radiation reaction' on large D black hole membranes can be ignored when working to any fixed order in $\frac{1}{D}$.

The results of this chapter could be generalized in many ways. First, the membrane stress tensor has been derived in this chapter at first subleading order in $\frac{1}{D}$. It should be straightforward to use the explicit results of [66] to generalize this stress tensor to second order in the large D expansion, and verify that the conservation of this improved stress tensor leads to the second order membrane equations of motion derived in [66]. Second it would be interesting to generalize the construction of [66] to the study of charged membranes and thereby obtain the formula for the leading entropy production for charged membranes.

It was demonstrated in [69] that the 'black brane' equations of Emparan, Suzuki and Tanabe (EST) and collaborators are a special scaling limit of the membrane equations of [1, 63, 66]. It should be straightforward to take the same scaling limit of the stress tensor derived in this chapter and compare the result with the 'black brane stress tensor' constructed by EST and collaborators.

It would be interesting (and may be possible) to use the formulae derived in this chapter - especially the formula for the divergence of the entropy current - to classify all stationary solutions of the membrane equations of motion.

The membrane equation of motion (546) and the formula for the membrane stress tensor (555) apply, at first order in $\frac{1}{D}$, note only to membranes in flat space but also to membranes propagating in any slowly varying solution of the vacuum Einstein equations $R_{MN} = 0$, e.g. a gravitational wave. Using this fact the membrane equations of motion together with the formulae for the membrane stress tensor and charge current of this chapter can be used to study how external gravitational waves 'polarize' large D black holes. The induced polarization will set the black hole oscillating, and the black oscillation will in turn radiate gravitational and electromagnetic waves in accordance with the formulae derived in this chapter. It should be straightforward to work out the details of this process in order to compute the ω dependent analogues of the 'Love Numbers' for black holes described, for instance, in [83]. [159]

It would be interesting to generalize the construction of the membrane entropy current to higher to the large D black hole membrane for higher derivative theories of gravity. The study of this subject should make contact with ongoing attempts to establish the second law of thermodynamics in higher derivative theories of gravity.

The RHS of the formula (775) for the divergence of the entropy is of order $\frac{1}{D}$. At least naively, this fact suggests that the fractional rate of entropy production in black hole motion is of order $\frac{1}{D}$. This conclusion appears to lead to a paradox.

Consider the head on collision of two non rotating black holes, each of which is moving at a substantial fraction of the speed of light. If the energy lost as radiation in this collision process is very small - as suggested by the discussion of this chapter - then almost all of the initial energy of this configuration must find its way into the black hole that is formed out of this

[159] It is interesting to understand how energy conservation works when a gravitational wave is incident on a black hole at large D. Consider, for instance, a spherical wave of amplitude A incident on a black hole. Only a fraction ϵA of this amplitude reaches the membrane (where ϵ is a small number of order $\frac{1}{D^{\frac{D}{2}}}$). This part of the wave excites the membrane into a motion of amplitude proportional to ϵA and so of energy proportional to $(\epsilon A)^2$. The membrane oscillation set up by this process result in radiation, and so a back scattered wave of amplitude of order $\epsilon^2 A$. The interference of this back scattered wave with the initial incident wave reduces the energy of the initial wave by an amount proportional to $\epsilon^2 A \times A = \epsilon^2 A^2$, accounting for the energy deposited into membrane vibrations.

collision. It follows that the mass of this daughter black hole is substantially larger than the sum of masses of the initial colliding black holes implying that the entropy of the final black hole is also substantially larger than the sum of the entropies of the original colliding black holes. In other words the collision of two black holes at large D appears to lead to fractional entropy production of order unity, in apparent contradiction with the claim of the previous paragraph.

We do not have a clear resolution to the puzzle described above. Note, however, that there is a time period of order $\frac{1}{D}$ when the colliding black holes first come very near to each other, when the membrane description of [1, 63, 66] breaks down. It is possible that the solution over this time period is a rather violent one, leading to the emission of a substantial amount of radiation over the short time scale of order $\frac{1}{D}$, invalidating the claim the energy lost in radiation at large D is rather small. This discussion suggests that the solution describing the collision of two black holes may be rather interesting when the black holes first touch. It is possible that the details of this solution are amenable to an analytical analysis of some sort. We hope to return to this fascinating question in the future.

4.11 Appendices for Chapter 4

4.11.1 Conventions and notation

Table 6: Different indices

Minkowski Spacetime indices	Capital Latin (A, B, M, N)
Indices in the membrane	Small Greek (α, β, μ, ν)
Cartesian Space indices	Small Latin (i, j, k, m)
Angle indices on S^{D-2}	Small Latin (a, b, c, d)

- For our case because of the continuity of the metric $\mathfrak{p}^{(in)}_{AB} = \mathfrak{p}^{(out)}_{AB}$. So sometimes we have denoted it by just $\mathfrak{p}_{AB}$.
- In all sections we have used (in) and (out) both as superscript and subscript, in a way so that it does not clutter the notation mixing with other raised or lowered indices. The same is true for the superscipt (or sometimes subscript) (k), used to denote the kth coefficient in an expansion around $\rho = 1$.
- In most places $\hat{\nabla}$ denotes covariant derivative with respect to $g^{(ind,f)}_{\mu\nu}$. But in some sections (e.g., in appendix (4.11.8)) it denotes covariant derivative with respect to $g^{(ind)}_{\mu\nu}$. What we mean will be clear from the context.

Table 7: Gauge fields

Full (nonlinear) Gauge field (as read off from chapter 3)	$\mathfrak{a}_M$
Linearized part from $\mathfrak{a}_B$ (not satisfying gauge conditions of this chapter)	$\mathcal{A}_B = \rho^{-(D-3)} M_B$
Coefficient of kth term in expansion of M_B around $\rho = 1$	$M_B^{(k)}$
Linearized and outside the membrane (satisfying gauge conditions of this chapter)	G_A
Coefficient of kth term in expansion of G_A around $\rho = 1$	$G_A^{(k)}$
Linearized and inside the membrane	$\tilde{G}_A$
Coefficient of kth term in expansion of $\tilde{G}_A$ around $\rho = 1$	$\tilde{G}_A^{(k)}$

Table 8: Different metrics

Full (nonlinear) metric: (as read off from chapter 3)	$\mathcal{G}_{AB} = \eta_{AB} + \mathfrak{g}_{AB}$
Linearized part from $\mathcal{G}_{AB}$ (not satisfying gauge conditions of this chapter)	$\eta_{AB} + \rho^{-(D-3)} M_{AB}$
Coefficient of kth term in expansion of M_{AB} around $\rho = 1$	$M_{AB}^{(k)}$
Linearized Metric Outside the membrane:	$g_{AB} = \eta_{AB} + h_{AB} = \eta_{AB} + \frac{\mathfrak{h}_{AB}}{\rho^{D-3}}$
Linearized Metric Inside the membrane:	$\tilde{g}_{AB} = \eta_{AB} + \tilde{h}_{AB}$
Coefficient of kth term in expansion of h_{AB} around $\rho = 1$	$h_{AB}^{(k)}$
Coefficient of kth term in expansion of $\tilde{h}_{AB}$ around $\rho = 1$	$\tilde{h}_{AB}^{(k)}$
Induced Metric from Full space-time:	$g_{\mu\nu}^{(ind)}$
Induced Metric from flat space-time:	$g_{\mu\nu}^{(ind,f)}$

Table 9: Differential operators

w.r.t induced metric on the membrane	$\hat{\nabla}$
w.r.t full space-time metric	∇
w.r.t Minkowski metric	∂
d'Alembertian	$\Box$
d'Alembertian w.r.t $g_{\mu\nu}^{(ind,f)}$	$\tilde{\Box}$

Table 10: Different projectors

On the membrane as embedded in flat space-time	$\Pi_{AB} = \eta_{AB} - n_A n_B.$
On the membrane as embedded in space time with metric $g_{AB} = \eta_{AB} + h_{AB}$	$\mathfrak{p}^{(out)}_{AB}$
On the membrane as embedded in space time with metric $\tilde{g}_{AB} = \eta_{AB} + \tilde{h}_{AB}$	$\mathfrak{p}^{(in)}_{AB}$
Projector orthogonal to both the normal as embedded in flat space and the velocity	$P_{AB} = \eta_{AB} - n_A n_B + u_A u_B.$
Projector orthogonal to velocity along the as membrane as embedded in flat space	$p_{\mu\nu} = g^{(ind,f)}_{\mu\nu} + u_\mu u_\nu.$
Projector orthogonal to membrane as embedded in space with metric $\eta_{AB} + h^{(0)}_{AB}$	$\tilde{\Pi}_{AB} = \eta_{AB} + h^{(0)}_{AB} - n_A n_B.$
Projector on the membrane as dependence of the l^{th}spherical harmonic	$\mathcal{P}_l$

Table 11: Extrinsic curvature

when embedded in $g_{AB} = \eta_{AB} + h_{AB}$	$\mathcal{K}^{(out)}_{AB}$
when embedded in $\tilde{g}_{AB} = \eta_{AB} + \tilde{h}_{AB}$	$\mathcal{K}^{(in)}_{AB}$
when embedded in $\eta_{AB} + h^{(0)}_{AB}$	$\bar{K}_{AB}$
when embedded in η_{AB}	K_{AB}
$g^{AB}\mathcal{K}^{(out)}_{AB}$	$\mathcal{K}^{(out)}$
$\tilde{g}^{AB}\mathcal{K}^{(in)}_{AB}$	$\mathcal{K}^{(in)}$
$\left[\eta^{AB} - h^{AB}_{(0)}\right]\bar{K}_{AB}$	$\bar{K}$
$\eta^{AB}K_{AB}$	K

- We have used ∇ for covariant derivative with respect to both g_{AB} and $\tilde{g}_{AB}$. What we mean, will be clear from the context.

- In section (4.9) and section 4.8 and appendices from (4.11.4) to (4.11.6), ∇_i denotes covariant derivative in flat space-time, but not necessarily in Cartesian coordinates and $\hat{\nabla}_a$ denotes covariant along unit sphere.

- Throughout this chapter we employ the mostly positive sign convention.

Table 12: Intrinsic curvature and Field strength

Riemann Tensor for full space time (for general analysis)	R_{ABCD}
Ricci Tensor for $g_{AB} = \eta_{AB} + h_{AB}$	$R^{(out)}_{AB}$
Ricci Tensor for $\tilde{g}_{AB} = \eta_{AB} + \tilde{h}_{AB}$	$R^{(in)}_{AB}$
Ricci Scalar for $g_{AB} = \eta_{AB} + h_{AB}$	$R^{(out)}$
Ricci Scalar for $\tilde{g}_{AB} = \eta_{AB} + \tilde{h}_{AB}$	$R^{(in)}$
Ricci Tensor for $g^{(ind)}_{\mu\nu}$	$\mathcal{R}_{\mu\nu}$
Ricci Scalar for $g^{(ind)}_{\mu\nu}$	$\mathcal{R}$
Field strength for G_A	F_{AB}
$\partial_A G^{(k)}_B - \partial_B G^{(k)}_A$	$F^{(k)}_{AB}$
Field strength for $\tilde{G}_A$	$\tilde{F}_{AB}$
$\partial_A \tilde{G}^{(k)}_B - \partial_B \tilde{G}^{(k)}_A$	$\tilde{F}^{(k)}_{AB}$
Field strength along the membrane	$\hat{F}_{\mu\nu}$

Table 13: Different Sources

$\mathcal{T}_{AB}$	Space-time Stress tensor
$\mathcal{J}_A$	Space-time Current
T_{AB}	Defined through $\mathcal{T}_{AB} = \sqrt{d\rho \cdot d\rho}\ \delta(\rho - 1) T_{AB}$
J_A	Defined through $\mathcal{J}_A = \sqrt{d\rho \cdot d\rho}\ \delta(\rho - 1) J_A$
$T_{\mu\nu}$	Stress tensor along the membrane
J_μ	Current along the membrane
$T^{(out/in)}_{AB}$	$\mathcal{K}^{(out/in)}_{AB} - \mathcal{K}^{(out/in)} \mathfrak{p}_{AB}$
$J^{(out)}_A$	$n^B F_{BA}$
$J^{(in)}_A$	$n^B \tilde{F}_{BA}$

Table 14: other notations

Fourier transform defined as	$\psi(t) = \int e^{-i\omega t} \widetilde{\psi}(\omega) \frac{d\omega}{2\pi}$
Outgoing wave represented by	$e^{-i\omega(t-r)}$.
Greens function defined as	$\Box\ G(x, y) = -\delta^D(x - y)$
N	$\sqrt{d\rho \cdot d\rho}$
n_A	$\frac{\partial_A \rho}{N}$

4.11.2 Linearized Solutions for point masses and charges

In this brief Appendix - whose purpose is largely to fix conventions for the normalization of mass and charge, we solve the linearized Einstein and Maxwell equations in the presence of a point mass and charge at the origin.

Conventions for the action and equations of motion employed in this chapter In this chapter we work with the metric and gauge field governed by the action

$$S = \frac{1}{16\pi} \int \sqrt{-g} \left(R - 4\pi F_{MN} F^{MN} \right). \tag{874}$$

As explained in the text we will sometimes study this action coupled to a classical source - at linearized order. The resultant linearized equations can be obtained from the action -

$$\text{Action} = \frac{1}{16\pi} \int \sqrt{-g} \left(R - 4\pi F_{MN} F^{MN} \right) - \int \left(\frac{1}{2} h^{MN} \mathcal{T}_{MN} + \mathcal{J}^M A_M \right), \tag{875}$$

where

$$F_{MN} = \nabla_M A_N - \nabla_N A_M. \tag{876}$$

The linearized equations of motion (about flat space and zero gauge field) that follow from this action is

$$\begin{aligned} & R_{MN} - \frac{R g_{MN}}{2} = 8\pi \mathcal{T}_{MN}, \\ & \nabla^M F_{MN} = \mathcal{J}_N \end{aligned} \tag{877}$$

(the LHS of the first equation in (877) should be linearized).

Comparison with the conventions used in earlier work In contrast with the conventions employed in this chapter, the previous chapter used the action

$$S = \frac{1}{16\pi} \int \sqrt{-g} \left(R - \frac{1}{4} F_{MN} F^{MN} \right). \tag{878}$$

It follows that the gauge fields of this chapter are related to the gauge fields of chapter 3 by the map

$$A^{here} = \frac{A^{there}}{\sqrt{16\pi}}. \tag{879}$$

Solutions for point sources We will now find solutions to the linearized version of (877) in the presence of point sources

$$\mathcal{T}_{00} = m \delta^{D-1}(\vec{x}), \quad \mathcal{J}^0 = q \delta^{D-1}(\vec{x})$$

(with all other components zero). By definition, the spacetime that arises in response to these sources will be taken to have mass m and charge q.

In order to solve the linearized versions of (877) we first employ the gauge

$$\nabla^M \left(h_{MN} - \frac{\eta_{MN} h}{2} \right) = 0$$

and

$$\nabla^M A_M = 0.$$

We find that the solution to the linearized version of (877) is given by

$$\begin{aligned} h_{00} &= \frac{16\pi m}{(D-2)\Omega_{D-2} r^{D-3}}, \\ h_{ii} &= \frac{16\pi m}{((D-3)(D-2)\Omega_{D-2} r^{D-3}}, \\ A^0 &= -A_0 = -\frac{q}{(D-3)\Omega_{D-2} r^{D-3}}. \end{aligned} \tag{880}$$

The corresponding linearized metric and gauge field takes the form

$$\begin{aligned} ds^2 &= -dt^2 \left(1 - \frac{16\pi m}{(D-2)\Omega_{D-2} r^{D-3}} \right) + dy^i dy^i \left(1 + \frac{16\pi m}{(D-3)(D-2)\Omega_{D-2} r^{D-3}} \right), \\ A_0 &= \frac{q}{(D-3)\Omega_{D-2} r^{D-3}} dt, \end{aligned} \tag{881}$$

where $r^2 = y^i y_i$. It may be verified that the curvature component R_{0i0j} evaluated on this solution is given by

$$R_{0i0j} = -\frac{8\pi}{(D-2)\Omega_{D-2}} \nabla_i \nabla_j \left(\frac{m}{r^{D-3}} \right). \tag{882}$$

In a similar manner the field strength F_{0i} evaluated on this solution is given by

$$F_{0i} = -\frac{1}{(D-3)\Omega_{D-2}} \nabla_i \frac{q}{r^{D-3}}. \tag{883}$$

The coordinate change

$$y^i = x^i \left(1 - \frac{8\pi m}{(D-3)(D-2)\Omega_{D-2} r^{D-3}} \right),$$

turns (881) into

$$\begin{aligned} ds^2 &= -dt^2 \left(1 - \frac{16\pi m}{(D-2)\Omega_{D-2} \tilde{r}^{D-3}} \right) + \frac{d\tilde{r}^2}{\left(1 - \frac{16\pi m}{(D-2)\Omega_{D-2} \tilde{r}^{D-3}} \right)}, \\ A_0 &= \frac{q}{(D-3)\Omega_{D-2} r^{D-3}}, \end{aligned} \tag{884}$$

where $\tilde{r}^2 = x^i x_i$ [160] . The solution in (884) is presented in the Schwarzschild gauge most convenient for comparing with the Reissner Nordstrom black holes of the next section.

4.11.3 Reisnner Nordstorm Black Holes and their thermodynamics

In this Appendix we present the solutions for Reisnner Naordstorm black holes in arbitrary dimensions and also review their thermodynamics. The material reviewed here is, of course, well known. Our main purpose is to establish conventions.

The system (874) admits the following two parameter set of exact Reissner Nordstrom solutions

$$\begin{aligned} ds^2 &= -dt^2 f(r) + \frac{dr^2}{f(r)} + r^2 d\Omega_{D-2}^2; \\ f(r) &= \left(1 - \frac{(1+c_D Q^2) r_0^{D-3}}{r^{D-3}} + \frac{c_D Q^2 r_0^{D-3}}{r^{2(D-3)}}\right), \\ A &= \frac{Q}{\sqrt{8\pi}} \left(\frac{r_0}{r}\right)^{D-3} dt, \end{aligned} \tag{885}$$

where [161]

$$c_D = \frac{D-3}{D-2}.$$

The mass and charge of these solutions are easily read off by comparison with (884); we find

$$\begin{aligned} m &= \frac{(D-2)(1+c_D Q^2) r_0^{D-3} \Omega_{D-2}}{16\pi}, \\ q &= \frac{1}{\sqrt{8\pi}} (D-3) Q r_0^{D-3} \Omega_{D-2}. \end{aligned} \tag{886}$$

The inverse temperature of these solutions is obtained by continuing to Euclidean space and identifying the periodicity of the time circle that keeps the solution regular at the outer event horizon; this procedure gives

$$T = \frac{(1-c_D Q^2)(D-3)}{4\pi r_0}. \tag{887}$$

The chemical potential μ of this solution is given by A_0 evaluated at infinity minus A_0 evaluated at the outer event horizon and equals

$$\mu = -\frac{1}{\sqrt{8\pi}} Q. \tag{888}$$

[160](884) is equivalent to (881) under the coordinate change listed above only at large r. More precisely these two terms are equivalent when we keep corrections to the flat space metric of order $\frac{1}{r^{D-3}}$ but ignore terms of order $\frac{1}{r^{2(D-3)}}$

[161]Note that the solution (885) agrees with the solution reported in chapter 3 after using (879).

Finally, the entropy of the black hole is the area of its outer event horizon divided by 4 and is given by

$$S = \frac{\Omega_{D-2} r_0^{D-2}}{4}. \tag{889}$$

It is easily verified that

$$\begin{aligned} TdS =& \frac{\Omega_{D-2}(D-2)(D-3)}{16\pi}(1 - c_D Q^2) r_0^{D-4} dr_0, \\ dm =& \frac{\Omega_{D-2}(D-2)}{16\pi}\Big((1 + c_D Q^2)(D-3) r_0^{D-4} dr_0 + 2c_D Q r - 0^{D-3} dQ\Big), \\ \mu dq =& -\frac{Q\Omega_{D-2}(D-3)}{8\pi}\Big((D-3)Q r - 0^{D-4} dr_0 + r_0^{D-3} dQ\Big) \end{aligned} \tag{890}$$

It is easily verified that these expressions are consistent with the first law of thermodynamics

$$TdS = dm + \mu dq. \tag{891}$$

4.11.4 Spherical Harmonics

In this appendix we review various properties of scalar vector and tensor spherical harmonics that will prove useful to us in the rest of this chapter.

Scalar Spherical Harmonics Scalar spherical harmonics form a basis for functions on the unit S^{D-2}. Every scalar spherical harmonic may be obtained as the restriction of a polynomial function in R^{D-1} to the unit sphere. Distinct polynomials that have the same restriction to the unit sphere define the same spherical harmonic. In other words spherical harmonics may be thought of as equivalence classes of polynomials in R^{D-1}. In each equivalence class it is possible to find a unique representative polynomial which given by a linear combination of monomials of the form

$$S_l = C_{i_1 \ldots i_l} x^{i_1} \ldots x^{i_l}, \tag{892}$$

where the coefficients $C_{i_1 \ldots i_l}$ are symmetric and traceless.

Monomials of the form (892) of degree l define a basis for l^{th} scalar spherical harmonics. Such monomials transform in the representation $(l, 0, \ldots 0)$ of $SO(D-1)$, where we label $SO(D-1)$ representations by highest weights under rotations in orthogonal two planes of R^{D-1}. It follows from the tracelessness of $C_{i_1 \ldots i_l}$ that $\nabla^2 S_l = 0$ where ∇^2 is the Laplacian in R^{D-1}. Transforming this equation to spherical polar coordinates we deduce that

$$-\nabla^2 Y_l = l(D + l - 3) Y_l, \tag{893}$$

where Y_l is the restriction of S_l onto the unit sphere and ∇^2 on the LHS of (893) is the Laplacian on the unit sphere.

Projectors onto spaces of scalar spherical harmonics We use notation in which the angles on the unit S^{D-2} are collectively denoted by θ. For some purposes it is useful to define $\mathcal{P}_l$. $\mathcal{P}_l$ acts on the space of functions on the unit sphere as a projector onto the l^{th} spherical harmonic sector. In other words

$$\int d\Omega'_{D-2}\mathcal{P}_l(\theta,\theta')Y_{l'}(\theta') = \delta_{ll'}Y_l(\theta). \tag{894}$$

It is not difficult to find an explicit expression for the projector $\mathcal{P}_l(\theta,\theta')$. In order to do this first note that

$$\mathcal{P}_l(\theta,\theta')R\left[Y_l'(\theta')\right] = R\left[\mathcal{P}_l(\theta,\theta')Y_l'(\theta')\right],$$

where R is any $SO(D-1)$ rotation operator (this equation follows because the action of a rotation operator on any k^{th} spherical harmonic is another k^{th} spherical harmonic). It follows, in other words, that $\mathcal{P}_l(\theta,\theta')$ is invariant under simultaneous rotations of θ and θ'. Let $\hat{r}$ denote the unit vector in the direction of θ and $\hat{r}'$ denote the unit vector in the direction of θ'. It follows that

$$\mathcal{P}_l(\theta,\theta') = f_l(\hat{r}.\hat{r}'),$$

where $f_l(x)$ is an as yet undetermined function of a single real variable x.

In order to determine $f_l(x)$ we note that

$$\nabla^2\mathcal{P}_l(\theta,\theta') = \nabla'^2\mathcal{P}_l(\theta,\theta') = -l(D+l-3)\mathcal{P}_l(\theta,\theta'). \tag{895}$$

Let us now specialize to the case that the vector $\hat{r}'$ points along the x^{D-1} axis. In this case $\hat{r}.\hat{r}'$ is simply the cosine of the angle (let us call it θ) that $\hat{r}$ makes with the x^{D-1} axis. It follows from (895) that $f_l(\cos\theta)$ is an l^{th} spherical harmonic. Notice that $f_l(\cos\theta)$ depends only on the angle with the x^{D-1} axis and so is rotational invariant under $SO(D-2)$ rotations that leave x^{D-1} unchanged. The unique spherical harmonic with these properties is proportional to the unique regular solution to the differential equation

$$\frac{1}{(\sin(\theta))^{D-3}}\partial_\theta\left((\sin(\theta))^{D-3}f_l(\cos\theta)\right) = -l(D+l-3)f_l(\cos\theta).$$

Solving the equation we find

$$f_l(\cos\theta) = N_l(\sin\theta)^{-\frac{D-4}{2}}P^{\frac{D}{2}-2}_{\frac{D}{2}+l-2}(\cos\theta), \tag{896}$$

where $P^{\frac{D}{2}-2}_{\frac{D}{2}+l-2}(x)$ is an associated Legendre function and N_l is an as yet undetermined constant.

In order to determine N_l we use the equation (894) for the special case that $\hat{r}$ points along the x^{D-1} axis, and the function it acts on (Y_l' in (894)) is chosen to be $(\sin\theta')^{-\frac{D-4}{2}} P^{\frac{D}{2}-2}_{\frac{D}{2}+l-2}(\cos\theta')$ where θ' is the angle of $\hat{r}'$ with the x^{D-1} axis. It follows from (894) that

$$\begin{aligned}\lim_{\theta\to 0}\left((\sin\theta')^{-\frac{D-4}{2}} P^{\frac{D}{2}-2}_{\frac{D}{2}+l-2}(\cos\theta)\right) &= N_l\Omega_{D-3}\int (sin\theta)^{D-3}\left((\sin\theta)^{-\frac{D-4}{2}} P^{\frac{D}{2}-2}_{\frac{D}{2}+l-2}(\cos\theta)\right)^2 \\ &= N_l\Omega_{D-3}\int \sin\theta \left(P^{\frac{D}{2}-2}_{\frac{D}{2}+L-2}(\cos\theta)\right)^2,\end{aligned} \tag{897}$$

where Ω_{D-3} is the volume of the unit $D-3$ sphere. The integral on the RHS of (897) is standard in the theory of Legendre functions and is given by

$$\int \sin\theta' \left(P^{\frac{D}{2}-2}_{\frac{D}{2}+l-2}(\cos\theta')\right)^2 = \frac{2(l+D-4)!}{(2l+D-3)l!}.$$

Moreover the limit on the LHS is given by

$$\lim_{\theta\to 0}\left((\sin\theta')^{-\frac{D-4}{2}} P^{\frac{D}{2}-2}_{\frac{D}{2}+l-2}(\cos\theta)\right) = \left(\frac{-1}{2}\right)^{\frac{D-4}{2}} \frac{(l+D-4)!}{l!\Gamma(\frac{D-2}{2})}.$$

These relations determine N_l; plugging in the value we obtain

$$f_l(\cos\theta) = \left(\frac{-1}{2}\right)^{\frac{D}{2}} \frac{2l+D-3}{\pi^{\frac{D-2}{2}}} (\sin\theta')^{-\frac{D-4}{2}} P^{\frac{D}{2}-2}_{\frac{D}{2}+l-2}(\cos\theta). \tag{898}$$

In particular we have

$$\mathcal{P}_l(0) = \lim_{\theta\to 0} f_l(\cos\theta) = \frac{1}{2^{D-2}\pi^{\frac{D-2}{2}}} \frac{(l+D-4)!}{l!} \frac{(2l+D-3)}{\Gamma\left(\frac{D-2}{2}\right)}. \tag{899}$$

Vector spherical harmonics Vector spherical harmonics form a basis for the set of divergence free vector fields on the unit sphere S^{D-2}. In this brief section we will describe how vector spherical harmonics can be obtained as the restriction of polynomial valued vector fields in R^{D-1}. We will also use this description to compute some of the properties of these harmonics.

Consider a vector field in R^{D-1} of the form

$$W_i^l = V_{i,i_1\ldots i_l} x^{i_1}\ldots x^{i_l}. \tag{900}$$

We will be interested in the restriction of this vector field onto the unit sphere. As in the previous subsection different expressions of the form (900) that restrict to the same vector field on the unit sphere will be considered equivalent; in other words vector spherical harmonics are identified with equivalence classes of expressions of the form (900). The indices $i_1 \dots i_l$ are clearly symmetric. As the normal component of the vector field W_i^l has no restriction to the sphere it is convenient to set this component to zero. The requirement $x^i W_i^l = 0$ is equivalent to the condition that the $V_{i,i_1 \dots i_l}$ vanishes under symmetrization between i and (say) i_1. As in the previous section one can find a representative in any equivalence class with the property that the coefficient functions $V_{i,i_1 \dots i_l}$ vanish upon tracing, say, i_1 and i_2. The set of coefficient functions with these properties transform in the $(l, 1, \dots 0)$ representation of $SO(D-1)$ (see the previous subsection for an explanation of our labelling of representations).

It follows from all the conditions we have imposed that

$$\nabla.W^l = 0, \tag{901}$$

where the divergence is taken in the embedding R^{D-1}. Translating this equation to polar coordinates we also find

$$\nabla.W^l = 0, \tag{902}$$

where W^l is now thought of as a vector field on the unit sphere and ∇ is now regarded as the covariant derivative on the unit sphere.

The set of vector fields W_i^l - when restricted to the sphere - define a basis for l^{th} vector spherical harmonics on S^{D-2}. We use the symbol V_l^α to denote l^{th} vector spherical harmonics on S^{D-2}. We will sometimes also use the symbol V_l^α to denote a vector function in the full embedding R^{D-1} defined by

$$V_i^l = V_{i,i_1 \dots i_l} \frac{x^{i_1} \dots x_l^i}{r^l}, \tag{903}$$

where the coefficients $V_{i,i_1 \dots i_l}$ are constants independent of r. With this normalization each Cartesian component of the vector field V^l is independent of r.

Note that for any fixed i (where i is a Cartesian coordinate) W_i^l is a polynomial of the form (892). It follows that $\nabla^2 W_i^l = 0$ (where ∇^2 is the Laplacian acting on R^{D-1}). In a similar manner for any fixed i the function V_i^l defined in (903) is an r independent scalar spherical harmonic of degree l and so it follows that

$$-\nabla^2 V_i^l = \frac{l(D+l-3)}{r^2} V_i^l, \tag{904}$$

where, once again, the Laplacian is taken in the embedding R^{D-1}.

Consider a sphere of radius r centered about the origin of R^{D-1}. The restriction of V_i^l onto this sphere defines a vector field on the sphere. We will now compute the eigenvalue of the

Laplacian $\hat{\nabla}^2$ on this sphere acting on this vector field. Using standard formulae

$$\begin{aligned}
&\hat{\nabla}^2(V_i(\theta) \\
&= \partial_m(\Pi_{jn}\partial_m V_j)\Pi_{in} \\
&= -\frac{1}{r}\Pi_{im}\hat{r}_j\partial_m V_j + \Pi_{ij}\partial_m\partial_n V_j \\
&= \frac{1}{r^2}V_i - \frac{l(l+D-3)}{r^2}V_i \\
&= -\frac{l(l+D-3)-1}{r^2}V_i,
\end{aligned} \tag{905}$$

(where Π_{ij} denotes the projector onto the unit sphere). Here we have used some identities

$$\begin{aligned}
\partial_i\hat{r}_j &= \frac{1}{r}\Pi_{ij}, \\
\partial_k\partial_k\hat{r}_i &= -\frac{D-2}{r^2}\hat{r}_i, \\
\partial_i\Pi_{ij} &= -\frac{D-2}{r}\hat{r}_j, \\
\partial_k\Pi_{ij} &= -\frac{1}{r}\left(\Pi_{ki}\hat{r}_j + \Pi_{kj}\hat{r}_i\right), \\
\partial_k\partial_k\Pi_{ij} &= -\frac{2}{r^2}\left(\Pi_{ij} - (D-2)\hat{r}_i\hat{r}_j\right).
\end{aligned} \tag{906}$$

Upon setting $r = 1$ (905) gives the eigenvalue Laplacian (viewed as a vector field acting on vectors on the unit sphere) acting on the l^{th} vector spherical harmonic.

As in the previous subsection we define the linear operator $\mathcal{P}_l^V$ which acts on vector fields on the unit sphere and projects onto the sector of vector field spanned by l^{th} vector spherical harmonics.

$$\begin{aligned}
&\mathcal{P}_l^V[V_{l'}] = \delta_{ll'}V_{l'}, \\
&\mathcal{P}_l^V[\partial\chi] = 0.
\end{aligned} \tag{907}$$

It is possible to work out an explicit form for the projector $\mathcal{P}_l^V$; however we will not have need for the explicit expression in this chapter and so will not pause to do so.

Tensor Spherical harmonics Mimicking the analysis of the previous section, a basis for traceless, divergenceless symmetric tensor fields on the unit sphere is given by the restriction of the polynomial expressions

$$B^l_{ij} = T_{ij,i_1\ldots i_l}x^{i_1}\ldots x^{i_l}, \tag{908}$$

onto the unit sphere. The coefficient function $T_{ij,i_1\ldots i_l}$ is chosen to have the following properties.

- It is symmetric in the indices $i_1 \dots i_l$ and separately in i, j.
- It vanishes under tracing any two of the indices.
- It vanishes under the symmetrization of (say) i with (say) i_1.

The coefficient functions $T_{ij,i_1 \dots i_l}$ transform in the $(l, 2, 0 \dots 0)$ representation of $SO(D-1)$. The restriction of B^l_{ij} to the unit sphere yields a set of symmetric traceless, divergenceless tensor fields on the unit sphere that form the basis for the set of l^{th} tensor spherical harmonics.

Note that any fixed Cartesian component of B^l_{ij} is a function of the form (892), and so $\nabla^2 B^l_{ij} = 0$, where ∇^2 is the Laplacian on R^{D-1}.

As in the previous subsection we sometimes have used B^l_{ij} for a tensor spherical harmonic field that is defined in all of R^{D-1}. Rather than the function B^l_{ij} defined above, we find it convenient to use the normalized tensor fields

$$T^l_{ij} = T_{ij,i_1 \dots i_l} \frac{x^{i_1} \dots x^{i_l}}{r^l}. \tag{909}$$

As in the previous section we may restrict T^l_{ij} to the surface of a sphere of radius r. The Laplacian of T^l viewed as a tensor field on this restricted surface is easily computed; we have

$$\begin{aligned} \hat{\nabla}^2 (T_{ij}(\theta) &= \partial_m (\Pi_{pn} \Pi_{qk} \partial_m T_{pq}) \Pi_{in} \Pi_{jk} \\ &= \frac{2}{r^2} T_{ij} - \frac{l(l+D-3)}{r^2} T_{ij} \\ &= -\frac{l(l+D-3)-2}{r^2} T_{ij}. \end{aligned} \tag{910}$$

As in previous subsections we define the linear operator $\mathcal{P}^T_l$, which acts on traceless symmetric tensor on the unit sphere and projects onto the sector of tensor fields spanned by l^{th} tensor spherical harmonics.

$$\begin{aligned} &\mathcal{P}^T_l [(T_{ij})_{l'}] = \delta_{ll'} (T_{ij})_{l'}, \\ &\mathcal{P}^T_l [\text{anything else}] = 0; \end{aligned} \tag{911}$$

where 'anything else' refers to tensors formed out of derivatives acting on scalar or vector spherical harmonics. It should be possible to work out an explicit form for the projector $\mathcal{P}^T_l$; however we will not have need for the explicit expression in this chapter and so will not pause to do so.

Decomposition of the general vector field on R^{D-1} in a spherical basis As we have mentioned above, the most general vector field on R^{D-1} can be constructed out of two scalar fields and one divergenceless purely angular vector field. The decomposition takes the form

$$\vec{A} = \hat{r}a + \nabla b + \vec{\gamma}, \tag{912}$$

where a and b are arbitrary scalar fields and $\vec{\gamma}$ is an arbitrary divergence free, purely angular vector field. We emphasize that the scalars a and b and the vector field $\vec{\gamma}$ are arbitrary functions of the radial coordinate r.

In (912) we have arbitrarily chosen a basis for the two scalar fields in the problem; of course any two linearly independent linear combinations of a and b would form as good a basis. We will now find a geometrically natural basis for the problem. Let α and β be the two scalar functions and let α_l and β_l respectively represent the projection of these functions into the space of l^{th} spherical harmonics i.e.

$$\alpha = \sum_{l=0}^{\infty} \alpha_l, \quad \beta = \sum_{l=0}^{\infty} \beta_l, \quad \mathcal{P}_{l'}\alpha_l = \delta_{ll'}\alpha_l, \quad \mathcal{P}_{l'}\beta_l = \delta_{ll'}\beta_l, \tag{913}$$

where $\mathcal{P}_l$, the projector onto the l^{th} scalar spherical harmonic was defined in (894). Let $\vec{\gamma}$ represent the non radial and divergence free vector field and let $\vec{\gamma}_l$ represent the projection of this field onto the space of l^{th} vector spherical harmonics i.e. let

$$\vec{\gamma} = \sum_{l=1}^{\infty} \vec{\gamma}_l, \quad \mathcal{P}^V_{l'}\vec{\gamma}_l = \delta_{ll'}\vec{\gamma}_l, \tag{914}$$

where $\mathcal{P}^V_{l'}$ was defined in (907). As emphasized above α_l , β_l and $\vec{\gamma}_l$ are all arbitrary functions of r. The most general vector field $\vec{J}_{\text{eff}}$ can be parametrized in terms of α, β and $\vec{\gamma}$ by

$$\vec{J_{\text{eff}}} = \left(\vec{\mathcal{A}}^-[\alpha] + \vec{\mathcal{A}}^+[\beta] + \vec{\gamma}\right), \tag{915}$$

where[162]

$$\begin{aligned}\vec{\mathcal{A}}^-[\alpha] &= \sum_{l=0}^{\infty}\left(l\hat{r}\alpha_l + r\vec{\nabla}^p\alpha_l\right),\\ \vec{\mathcal{A}}^+[\beta] &= \sum_{l=0}^{\infty}\left((l+D-3)\hat{r}_i\beta_l - r\vec{\nabla}^p\beta_l\right).\end{aligned} \tag{917}$$

[162] We have defined the projected derivative ∇^p as follows

$$\begin{aligned}&\text{Scalar}: \nabla^p_i\alpha = \Pi^j_i\partial_j\alpha,\\ &\text{Vector}: \nabla^p_i\beta_j = \Pi^k_i\Pi^l_j\partial_k(\Pi^m_l\beta_m).\end{aligned} \tag{916}$$

and so on for the tensor .

The linear combinations in (917) are special because they are 'diagonal' under the action of $\mathcal{P}_{l'}$, the projector onto scalar spherical harmonics acting separately on each Cartesian components. Specifically we have

$$
\begin{aligned}
\mathcal{P}_{l'}\left(\vec{\mathcal{A}}^{-}[\alpha]\right) &= \vec{\mathcal{A}}^{-}[\mathcal{P}_{l'+1}\alpha], \\
\mathcal{P}_{l'}\left(\vec{\mathcal{A}}^{+}[\beta)]\right) &= \vec{\mathcal{A}}^{+}[\mathcal{P}_{l'-1}\beta].
\end{aligned}
\tag{918}
$$

[163]

The action of the scalar projector on individual Cartesian components of vector spherical harmonics is automatically diagonal and is very simple

$$
\mathcal{P}_l\left(\vec{\gamma}\right) = \mathcal{P}_l^V\left(\vec{\gamma}\right) = \vec{\gamma}_l, \tag{920}
$$

where $\mathcal{P}_l^V$ represents the projector onto the space of l^{th} vector spherical harmonics.

It is now easy to deduce the action of the R^{D-1} Laplacian ∇^2 on the vector field $\vec{J}_{\text{eff}}$. Using the fact the Laplacian in Cartesian coordinates acts on each component of a vector field as if it were a scalar, it follows immediately from (920) and (918)

$$
\begin{aligned}
&\nabla^2\vec{\mathcal{A}}^{-}[\alpha] = \vec{\mathcal{A}}^{-}[\tilde{\alpha}], \\
&\nabla^2\vec{\mathcal{A}}^{+}[\beta] = \vec{\mathcal{A}}^{+}[\tilde{\beta}], \\
&\nabla^2\vec{\gamma} = \tilde{\vec{\gamma}},
\end{aligned}
\tag{921}
$$

where

$$
\begin{aligned}
\tilde{\alpha} &= \sum_l \left(\frac{1}{r^{D-2}}\partial_r\left(r^{D-2}\partial_r\alpha_l\right) - \frac{(l-1)(l-1+D-3)}{r^2}\alpha_l\right), \\
\tilde{\beta} &= \sum_l \left(\frac{1}{r^{D-2}}\partial_r\left(r^{D-2}\partial_r\beta_l\right) - \frac{(l+1)(l+1+D-3)}{r^2}\beta_l\right), \\
\tilde{\vec{\gamma}} &= \sum_l \left(\frac{1}{r^{D-2}}\partial_r\left(r^{D-2}\partial_r\vec{\gamma}_l\right) - \frac{(l)(l+D-3)}{r^2}\vec{\gamma}_l\right).
\end{aligned}
\tag{922}
$$

In words, ∇^2 acts on α_l as it would on the $(l-1)^{th}$ component of a scalar field. ∇^2 acts on β_l as it would on the $(l+1)^{th}$ component of a scalar field. ∇^2 acts on $\vec{\gamma}_l$ as it would on the $(l)^{th}$ component of a scalar field; the last statement reflects the fact that the l^{th} scalar and vector

[163] This equation can be restated as

$$
\begin{aligned}
\mathcal{P}_{l'}(\vec{\mathcal{A}}^{-}[\alpha_l]) &= \delta_{l',l-1}(\vec{\mathcal{A}}^{-}[\alpha_l]), \\
\mathcal{P}_{l'}(\vec{A}^{+}[\beta_l]) &= \delta_{l',l+1}(\vec{\mathcal{A}}^{+}[\beta_l]).
\end{aligned}
\tag{919}
$$

spherical harmonics have equal eigenvalues under the the action of the Laplacian on the unit sphere.

It is also not difficult to verify that

$$\begin{aligned} \vec{\nabla}\cdot\vec{\mathcal{A}}^-[\alpha] &= \sum_l l r^{l-1}\partial_r\left(\frac{\alpha_l}{r^{l-1}}\right), \\ \vec{\nabla}\cdot\vec{\mathcal{A}}^+[\beta] &= \sum_l \frac{l+D-3}{r^{l+D-2}}\partial_r\left(r^{l+D-2}\beta_l\right), \\ \vec{\nabla}\cdot\vec{\gamma} &= 0. \end{aligned} \tag{923}$$

An expansion of the arbitrary tensor field in a spherically adapted basis The most general symmetric tensor field in R^{D-1} can be split into its trace - which is a decoupled scalar - and a traceless symmetric tensor. We ignore the trace part in what follows. The most general traceless symmetric tensor field is parametrized by three scalar fields, two angular divergence free vector fields and one angular divergence free tensor field. As in the previous subsection, it is dynamically convenient to choose a basis for the vectors and the scalars that diagonalizes the action of ∇^2. The logic and algebra is very similar to the previous subsection and we only present final results.

Following the previous subsection we use obvious notation to denote the projection of any of these quantities to their l^{th} spherical harmonic sector. For instance α_l represents the projection of α to the l^{th} scalar spherical harmonic sector, while $\vec{\phi}_l$ represents the projection of $\vec{\phi}$ to the l^{th} vector spherical harmonic sector, etc. [164] A general tensor field T_{ij} is given in terms of this data by the decomposition

$$\begin{aligned} T_{ij} = &\left(\mathcal{C}^-_{ij}[\alpha] + \mathcal{C}^+_{ij}[\beta] + \mathcal{C}^0_{ij}[\gamma] + \delta_{ij}\kappa\right) \\ &+ \left(\mathcal{B}^-_{ij}[\phi] + \mathcal{B}^+_{ij}[\psi]\right) + \chi_{ij}, \end{aligned} \tag{924}$$

[164] The index l runs from 0 to ∞ in the case of scalars, from 1 to ∞ in the case of vectors and from 2 to ∞ in the case of tensors

where

$$
\begin{aligned}
(\mathcal{C}^-)_{ij}[\alpha] =& \mathcal{A}_i^-\left(\mathcal{A}_j^-[\alpha]\right) = \mathcal{A}_j^-\left(\mathcal{A}_i^-[\alpha]\right),\\
(\mathcal{C}^+)_{ij}[\beta] =& \mathcal{A}_i^+\left(\mathcal{A}_j^+[\beta]\right) = \mathcal{A}_j^+\left(\mathcal{A}_i^+[\beta]\right),\\
(\mathcal{C}^0)_{ij}[\gamma] =& \frac{1}{4}\left(\mathcal{A}_i^-\left(\mathcal{A}_j^+[\gamma]\right) + \mathcal{A}_j^-\left(\mathcal{A}_i^+[\gamma]\right) + \mathcal{A}_i^+\left(\mathcal{A}_j^-[\gamma]\right) + \mathcal{A}_j^+\left(\mathcal{A}_i^-[\gamma]\right)\right)\\
&- \delta_{ij}\sum_l\left(\frac{2l(l+D-3)}{D-1} + \frac{D-3}{2}\right)\gamma_l,\\
\delta_{ij}\kappa =& \sum_l \frac{\mathcal{A}_i^-\left(\mathcal{A}_j^+[\kappa_l]\right) + \mathcal{A}_j^-\left(\mathcal{A}_i^+[\kappa_l]\right) - \mathcal{A}_i^+\left(\mathcal{A}_j^-[\kappa_l]\right) - \mathcal{A}_j^+\left(\mathcal{A}_i^-[\kappa_l]\right)}{2(2l+D-3)},\\
(\mathcal{B}^-)_{ij}[\vec{\phi}] =& \mathcal{A}_i^-[\phi_j] + \mathcal{A}_j^-[\phi_i],\\
(\mathcal{B}^+)_{ij}[\vec{\psi}] =& \mathcal{A}_i^+[\psi_j] + \mathcal{A}_j^+[\psi_i].
\end{aligned}
\tag{925}
$$

Quantities like $\mathcal{A}_i^-\left(\mathcal{A}_j^-[\alpha]\right)$ that appear in the equation above have the following meaning; the operator $\mathcal{A}_i^-$ acts on each Cartesian component of $\mathcal{A}_j^-[\alpha]$ as if it were a scalar (i.e. according to the formula (917)). [165] More generally the action of the $\mathcal{A}_i^-$ or $\mathcal{A}_i^+$ on any vector field is that these operators act on each of the Cartesian components of the corresponding vector field as they would on a scalar (i.e. according to (917)). Using the fact that each Cartesian component of an l^{th} vector harmonic is an l^{th} scalar harmonic , it follows that the action of these operators on tangential divergenceless vector fields (those that can be expanded in vector harmonics) is given

[165] Here we explicitly write the expressions for the action of two $\vec{\mathcal{A}}'s$ on a scalar

$$
\begin{aligned}
\mathcal{A}_i^-\mathcal{A}_j^-[\alpha] &= \left((l-1)\hat{r}_i + r\nabla_i^p\right)\left(l\hat{r}_j + r\nabla_j^p\right)[\alpha]\\
&= l(l-2)\hat{r}_i\hat{r}_j\alpha + (l-1)(r\hat{r}_i\nabla_j^p + r\hat{r}_j\nabla_i^p)\alpha + l\delta_{ij}\alpha + r^2\nabla_{ij}\alpha,\\
\mathcal{A}_i^+\mathcal{A}_j^+[\alpha] &= \left((l+D-2)\hat{r}_i - r\nabla_i^p\right)\left((l+D-3)\hat{r}_j - r\nabla_j^p\right)[\alpha]\\
&= (l+D-3)(l+D-1)\hat{r}_i\hat{r}_j\alpha - (l+D-2)(r\hat{r}_i\nabla_j^p + r\hat{r}_j\nabla_i^p)\alpha - (l+D-3)\delta_{ij}\alpha + r^2\nabla_{ij}\alpha,\\
\mathcal{A}_i^-\mathcal{A}_j^+[\alpha] &= \left((l+1)\hat{r}_i + r\nabla_i^p\right)\left((l+D-3)\hat{r}_j - r\nabla_j^p\right)[\alpha]\\
&= l(l+D-3)\hat{r}_i\hat{r}_j\alpha - (l+1)r\hat{r}_i\nabla_j^p\alpha + (l+D-2)r\hat{r}_j\nabla_i^p\alpha + (l+D-3)\delta_{ij}\alpha - r^2\nabla_{ij}\alpha,\\
\mathcal{A}_i^+\mathcal{A}_j^-[\alpha] &= \left((l+D-4)\hat{r}_i - r\nabla_i^p\right)\left(l\hat{r}_j + r\nabla_j^p\right)[\alpha]\\
&= l(l+D-3)\hat{r}_i\hat{r}_j\alpha + (l+D-4)r\hat{r}_i\nabla_j^p - (l-1)r\hat{r}_j\nabla_i^p\alpha - l\delta_{ij}\alpha - r^2\nabla_{ij}\alpha,
\end{aligned}
\tag{926}
$$

where we have defined ∇_{ij} as the complete projected derivative of two ∇_i and is given by

$$\nabla_{ij}\alpha = \Pi_i^k\Pi_j^m\partial_m\left(\Pi_k^n\partial_n\alpha\right).$$

It can be easily shown that ∇_{ij} is symmetric under the exchange of $i \longleftrightarrow j$.

by

$$
\begin{aligned}
\vec{\mathcal{A}}^-[\vec{\phi}] &= \sum_{l=0}^{\infty} \left(l\hat{r}\vec{\phi}_l + r\vec{\nabla}^p\vec{\phi}_l \right), \\
\vec{\mathcal{A}}^+[\vec{\psi}] &= \sum_{l=0}^{\infty} \left((l+D-3)\hat{r}_i\vec{\psi}_l - r\vec{\nabla}^p\vec{\psi}_l \right).
\end{aligned}
\tag{927}
$$

Of course the linear combinations in (925) are 'diagonal' under the action of $\mathcal{P}_{l'}$, the projector onto scalar spherical harmonics acting separately on each Cartesian component.

$$
\begin{aligned}
\mathcal{P}_l \left((\mathcal{C}^-)_{ij}[\alpha] \right) &= (\mathcal{C}^-)_{ij}[\mathcal{P}_{l+2}\alpha], \\
\mathcal{P}_l \left((\mathcal{C}^+)_{ij}[\beta] \right) &= (\mathcal{C}^+)_{ij}[\mathcal{P}_{l-2}\beta], \\
\mathcal{P}_l \left((\mathcal{C}^0)_{ij}[\gamma] \right) &= (\mathcal{C}^0)_{ij}[\mathcal{P}_l\gamma], \\
\mathcal{P}_l \left((\mathcal{B}^-)_{ij}[\vec{\phi}] \right) &= (\mathcal{B}^-)_{ij}[\mathcal{P}^V_{l+1}\vec{\phi}], \\
\mathcal{P}_l \left((\mathcal{B}^+)_{ij}[\vec{\psi}] \right) &= (\mathcal{B}^+)_{ij}[\mathcal{P}^V_{l-1}\vec{\psi}]
\end{aligned}
\tag{928}
$$

(recall $\mathcal{P}^V_l$ projects onto the subspace of l^{th} vector spherical harmonics).

The action of the scalar projector on individual Cartesian components of tensor spherical harmonics is automatically diagonal and is very simple

$$
\mathcal{P}_l\,(\chi)_{ij} = \mathcal{P}^T_l\,(\chi)_{ij} = (\chi_l)_{ij}\,, \tag{929}
$$

where $\mathcal{P}^T_l$ represents the projector onto the space of l^{th} tensor spherical harmonics. Equation (929) simply asserts that each Cartesian component of modes in the l^{th} tensor spherical harmonic is a scalar spherical harmonic of degree l.

The action of the operator ∇^2 is also diagonal - and rather simple - in this basis

$$
\begin{aligned}
&\nabla^2 \left((\mathcal{C}^-)_{ij}[\alpha] \right) = (\mathcal{C}^-)_{ij}[\tilde{\alpha}], \\
&\nabla^2 \left((\mathcal{C}^+)_{ij}[\beta] \right) = (\mathcal{C}^+)_{ij}[\tilde{\beta}], \\
&\nabla^2 \left((\mathcal{C}^0)_{ij}[\gamma] \right) = (\mathcal{C}^0)_{ij}[\tilde{\gamma}], \\
&\nabla^2 \left((\mathcal{B}^-)_{ij}[\vec{\phi}] \right) = (\mathcal{B}^-)_{ij}[\tilde{\vec{\phi}}], \\
&\nabla^2 \left((\mathcal{B}^+)_{ij}[\vec{\psi}] \right) = (\mathcal{B}^+)_{ij}[\tilde{\vec{\psi}}], \\
&\nabla^2\chi_{ij} = \tilde{\chi}_{ij}, \\
&\nabla^2\kappa = \tilde{\kappa},
\end{aligned}
\tag{930}
$$

where

$$
\begin{aligned}
\tilde{\alpha} &= \sum_l \left(\frac{1}{r^{D-2}} \partial_r \left(r^{D-2} \partial_r \alpha_l \right) - \frac{(l-2)(l-2+D-3)}{r^2} \alpha_l \right), \\
\tilde{\beta} &= \sum_l \left(\frac{1}{r^{D-2}} \partial_r \left(r^{D-2} \partial_r \beta_l \right) - \frac{(l+2)(l+2+D-3)}{r^2} \beta_l \right), \\
\tilde{\gamma} &= \sum_l \left(\frac{1}{r^{D-2}} \partial_r \left(r^{D-2} \partial_r \gamma_l \right) - \frac{l(l+D-3)}{r^2} \gamma_l \right), \\
\tilde{\vec{\phi}} &= \sum_l \left(\frac{1}{r^{D-2}} \partial_r \left(r^{D-2} \partial_r \vec{\phi}_l \right) - \frac{(l-1)(l-1+D-3)}{r^2} \vec{\phi}_l \right), \\
\tilde{\vec{\psi}} &= \sum_l \left(\frac{1}{r^{D-2}} \partial_r \left(r^{D-2} \partial_r \vec{\psi}_l \right) - \frac{(l+1)(l+1+D-3)}{r^2} \vec{\psi}_l \right), \\
\tilde{\chi}_{ij} &= \sum_l \left(\frac{1}{r^{D-2}} \partial_r \left(r^{D-2} \partial_r (\chi_l)_{ij} \right) - \frac{l(l+D-3)}{r^2} (\chi_l)_{ij} \right), \\
\tilde{\kappa} &= \sum_l \left(\frac{1}{r^{D-2}} \partial_r \left(r^{D-2} \partial_r \kappa_l \right) - \frac{l(l+D-3)}{r^2} \kappa_l \right).
\end{aligned}
\tag{931}
$$

It is also not difficult to verify that

$$
\begin{aligned}
\nabla_i \mathcal{C}^-_{ij}[\alpha] &= \mathcal{A}^-_j \left[(l-1)(r)^{l-2} \partial_r \left(\frac{\alpha_l}{(r)^{l-2}} \right) \right], \\
\nabla_i \mathcal{C}^+_{ij}[\beta] &= \mathcal{A}^+_j \left[(l+D-2) \frac{\partial_r \left((r)^{l+D-1} \beta_l \right)}{(r)^{l+D-1}} \right], \\
\nabla_i \mathcal{C}^0_{ij}[\gamma] &= \mathcal{A}^+_j \left[\frac{l}{2(2l+D-3)} \left((2l+D-3) - \frac{4(l+D-3)}{D-1} \right) (r)^l \partial_r \left(\frac{\gamma_l}{(r)^l} \right) \right] \\
&+ \mathcal{A}^-_j \left[\frac{(l+D-3)}{2(2l+D-3)} \left((2l+D-3) - \frac{4l}{D-1} \right) \frac{\partial_r \left((r)^{l+D-3} \gamma_l \right)}{(r)^{l+D-3}} \right], \\
\nabla_i \mathcal{B}^-_{ij}[\vec{\phi}] &= \sum_l (l-1) r^{l-1} \partial_r \left(\frac{(\phi_l)_j}{r^{l-1}} \right), \\
\nabla_i \mathcal{B}^-_{ij}[\vec{\psi}] &= \sum_l \frac{l+D-2}{r^{l+D-2}} \partial_r \left(r^{l+D-2} (\psi_l)_j \right), \\
\nabla_i \chi_{ij} &= 0.
\end{aligned}
\tag{932}
$$

4.11.5 Scalar Greens Functions

Retarded Greens Functions in position space In this subsection we obtain explicit expressions for the Greens function in position space, starting from the exact Fourier space result (565).

- *Even D*

When D is even the argument of the Hankel function that appears in (565) is half integral. Now Hankel functions of half integral argument have an amazing property; their large argument expansion truncates at a finite order. In equations

$$H^{(1)}_{m+1/2}(\omega r) = \sqrt{\frac{2}{\pi\omega r}}(-i)^{m+1}e^{i\omega r}\sum_{k=0}^{m}\frac{(m+k)!}{k!(m-k)!}\frac{i^k}{(2\omega r)^k}, \tag{933}$$

m (which is $m = \frac{D-4}{2}$ in our context) is an integer. As this expression takes the form $e^{i\omega r}$ times a polynomial in ω. It follows that $G(r,t)$ defined by

$$G(r,t) = \int \frac{d\omega}{2\pi} G_\omega(r) e^{-i\omega t}, \tag{934}$$

is a linear sum of a finite number of derivatives of $\delta(r-t)$. We find

$$G(r,t) = \frac{1}{2}\left(\frac{1}{2\pi r}\right)^{m+1}\sum_{k=0}^{m}\frac{(m+k)!}{k!(m-k)!}\int d\omega \ \frac{(-i\omega)^{m-k}e^{i\omega(r-t)}}{(2r)^k} \tag{935}$$

$$= -\frac{1}{2}\left(\frac{-1}{2\pi r}\right)^{m+1}\sum_{k=0}^{m}\frac{(m+k)!}{k!(m-k)!}\frac{\partial_r^{m-k}\delta(t-r)}{(-2r)^k}. \tag{936}$$

It may be verified that (935) resums to

$$G(r,t) = \frac{\theta(X^0)}{2}\left(\frac{1}{\pi}\right)^{\frac{D-2}{2}}\delta^{\left(\frac{D-4}{2}\right)}(r^2-t^2) = \frac{\theta(X^0)}{2}\left(\frac{1}{\pi}\right)^{\frac{D-2}{2}}\left(\frac{X^M\partial_M}{2X^NX_N}\right)^{\frac{D-2}{2}}\delta(X^MX_M). \tag{937}$$

[166] We have checked the equivalence of (937) and (935) on Mathematica for $D \leq 14$.

- *Odd D*

In order to obtain an explicit expression for the Greens function in odd D we found it convenient to start with the explicit expression for the Greens function in ω and $\vec{k}$. Transforming to polar coordinates in $\vec{k}$ space we have

$$G_D(r,t) = -\Omega_{D-3}\int \frac{d\omega}{(2\pi)^D}d\theta(\sin\theta)^{D-3}\frac{k^{D-2}dk}{(\omega+i\epsilon k)^2-k^2}e^{-i(\omega t - kr\cos\theta)} \tag{938}$$

(here ϵ is an infinitesimal dimensionless number and the positive factor of k in front of ϵ has been inserted for future convenience). For $t<0$ we can close the contour in the upper half plane. As

[166] Recall that $\delta^m(\alpha)$ is the m^{th} derivative of the delta function w.r.t. α. In the case at hand α is $r^2 - t^2 = X^MX_M$ and partial derivatives w.r.t. α can be converted into partial derivatives w.r.t. X^M using the chain rule $\partial_\alpha = \frac{X^M\partial_M}{2X^NX_N}$.

the integrand of (938) is analytic here the integral vanishes, as expected for a retarded correlator. On the other hand for $t > 0$ we close the contour in the lower half plane and pick up contributions from the two poles in the integrand. Doing the ω integral we find

$$\begin{aligned}
G_D(r,t) &= i\Omega_{D-3}\int \frac{1}{(2\pi)^D} d\theta(\sin\theta)^{D-3}k^{D-2}dk\frac{e^{-ik(t-i\epsilon)}-e^{ik(t+i\epsilon)}}{2k}e^{ikr\cos\theta} \\
&= \frac{i\Omega_{D-3}}{(2\pi)^{D-3}}\int \frac{1}{(2\pi)^2}(k^2-k^2\cos\theta)^{\frac{D-3}{2}}\frac{e^{-ik(t-i\epsilon)}-e^{ik(t+i\epsilon)}}{2}e^{ikr\cos\theta}d\theta dk \\
&= \frac{i\Omega_{D-3}}{(2\pi)^{D-3}}(-\partial_t^2+\partial_r^2)^{\frac{D-3}{2}}\int \frac{1}{(2\pi)^2}\frac{e^{-ik(t-i\epsilon)}-e^{ik(t+i\epsilon)}}{2}e^{ikr\cos\theta}d\theta dk \\
&= \frac{i\Omega_{D-3}}{(2\pi)^{D-3}}(-\partial_t^2+\partial_r^2)^{\frac{D-3}{2}}G_3(r,t), \qquad (939)
\end{aligned}$$

where

$$G_3(r,t) = \int \frac{e^{-ik(t-i\epsilon)}-e^{ik(t+i\epsilon)}}{2}e^{ikr\cos\theta}d\theta\frac{dk}{(2\pi)^2}. \qquad (940)$$

We now proceed to explicitly evaluate integral in (940). Evaluating the integral over k in that expression we find

$$\begin{aligned}
G_3(r,t) &= \frac{-i}{2}\int\frac{d\theta}{2\pi}\left(\frac{1}{t-r\cos(\theta)-i\epsilon}+\frac{1}{t+r\cos(\theta)+i\epsilon}\right) \\
\therefore G_3(r,t) &= -i\frac{I_1+I_2}{4\pi i}, \qquad (941)
\end{aligned}$$

where I_1 and I_2 are defined as :

$$I_1 = \oint \frac{2idz}{r\left(z-\frac{t+\sqrt{t^2-r^2}}{r}\right)\left(z-\frac{t-\sqrt{t^2-r^2}}{r}\right)}, \qquad (942)$$

$$I_2 = \oint \frac{-2idz}{r\left(z-\frac{-t+\sqrt{t^2-r^2}}{r}\right)\left(z-\frac{-t-\sqrt{t^2-r^2}}{r}\right)}, \qquad (943)$$

where we have defined $e^{i\theta} = z$, and the contour integrals above are taken anticlockwise over the unit circle. When $r > t$ the poles in z in I_1 and I_2 both lie on the unit circle. This integral can be defined by the principal value and simply vanishes. When $t^2 > r^2$, on the other hand, the second pole in I_1 lies within the unit circle while the first pole lies outside. The situation is reversed for I_2; the first pole lies within the unit circle while the second one lies outside. Evaluating the integrals by contours we find

$$\begin{aligned}
I_1 &= \frac{-i\theta(t^2-r^2)}{\sqrt{t^2-r^2}}, \\
I_2 &= \frac{-i\theta(t^2-r^2)}{\sqrt{t^2-r^2}}.
\end{aligned} \qquad (944)$$

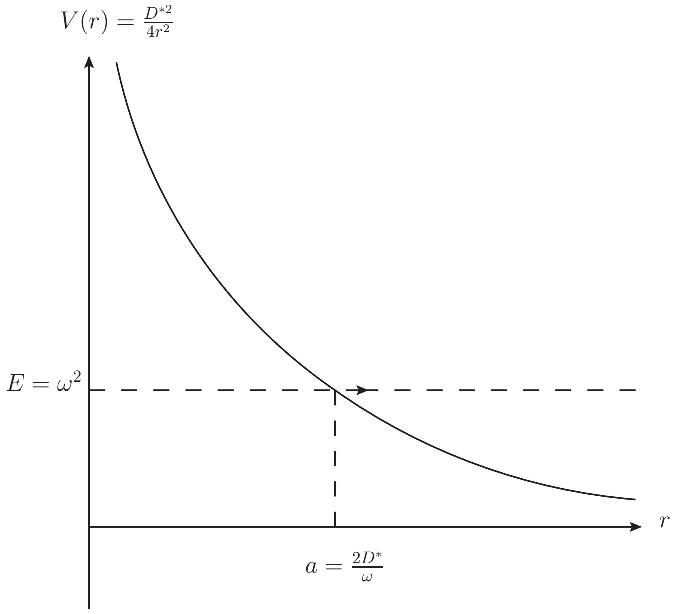

Figure 21: Potential for allowed and disallowed regions .

Using the fact that G_3 vanishes for negative t it follows that

$$G_3(r,t) = \frac{-2\pi i\theta(t-r)}{\sqrt{t^2-r^2}}. \tag{945}$$

From (939) it follows that

$$G_D(r,t) = \frac{\Omega_{D-3}}{(2\pi)^{D-4}}(-\partial_t^2+\partial_r^2)^{\frac{D-3}{2}}\left(\frac{\theta(t-r)}{\sqrt{t^2-r^2}}\right). \tag{946}$$

Large D expansion of the Greens Function using WKB As we have explained in the main text, the large D limit of the Greens function is given by the solution of an effective Schrodinger equation which takes the form

$$-\psi''(\omega,r)+\frac{D^{*2}}{4r^2}\psi(\omega,r)=\omega^2\psi(\omega,r), \quad \text{where } D^*=\sqrt{(D-2)(D-4)}. \tag{947}$$

The most general WKB solution to this equation in the classically disallowed region is given by

$$
\begin{aligned}
\psi(\omega,r) = &\frac{\sqrt{D^*}B}{\sqrt{2}\left(\frac{D^{*2}}{4r^2}-\omega^2\right)^{\frac{1}{4}}}\left(\frac{D^*}{2\omega r}-\sqrt{\frac{D^{*2}}{4\omega^2 r^2}-1}\right)^{-\frac{D^*}{2}}\left(\frac{D^*}{\omega}\right)^{-\frac{D^*}{2}} e^{\frac{D^*}{2}-\sqrt{\frac{D^{*2}}{4}-\omega^2 r^2}} \\
&+\frac{\sqrt{D^*}A}{\sqrt{2}\left(\frac{D^{*2}}{4r^2}-\omega^2\right)^{\frac{1}{4}}}\left(\frac{D^*}{2\omega r}-\sqrt{\frac{D^{*2}}{4\omega^2 r^2}-1}\right)^{\frac{D^*}{2}}\left(\frac{D^*}{\omega}\right)^{-\frac{D^*}{2}} e^{\frac{D^*}{2}+\sqrt{\frac{D^{*2}}{4}-\omega^2 r^2}}.
\end{aligned}
\tag{948}
$$

In the limit $2r\omega \ll D^*$ this solution reduces to

$$
\psi(\omega,r) = \frac{B}{r^{\frac{D-4}{2}}} + A\left(\frac{e\omega}{D}\right)^{D-3} r^{\frac{D-2}{2}}, \tag{949}
$$

in agreement with (575). In order to obtain (949) , we have used $D^* = (D-3) + \mathcal{O}(1/D)$ and have ignored this higher order correction. We have also used

$$
\Omega_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}. \tag{950}
$$

In the classically allowed region, on the other hand,

$$
\psi(\omega,r) = \frac{E e^{i\frac{D^*\pi}{4}} e^{-i\frac{D^*}{2}\sin^{-1}\left(\frac{D^*}{2\omega r}\right)} e^{-i\sqrt{\omega^2 r^2-\frac{D^{*2}}{4}}} + C e^{-i\frac{D^*\pi}{4}} e^{i\frac{D^*}{2}\sin^{-1}\left(\frac{D^*}{2\omega r}\right)} e^{i\sqrt{\omega^2 r^2-\frac{D^{*2}}{4}}}}{\left(\omega^2-\frac{D^{*2}}{4r^2}\right)^{\frac{1}{4}}}. \tag{951}
$$

In the limit $2\omega r \gg D^*$ (951) reduces to

$$
\psi(\omega,r) = \frac{1}{\sqrt{\omega}}\left(E e^{i\frac{D^*\pi}{4}} e^{-i\omega r} + C e^{-i\frac{D^*\pi}{4}} e^{i\omega r}\right), \tag{952}
$$

in agreement with (577).

The usual WKB crossing formulae relate the four constants A B C E. Using Equation 7.35 of [84] we have

$$
\begin{aligned}
C &= e^{-\frac{i\pi}{4}}\left(A+\frac{iB}{2}\right)\sqrt{\frac{D^*}{2}}\left(\frac{D^*}{\omega}\right)^{-\frac{D^*}{2}} e^{\frac{D^*}{2}}, \\
E &= e^{\frac{i\pi}{4}}\left(A-\frac{iB}{2}\right)\sqrt{\frac{D^*}{2}}\left(\frac{D^*}{\omega}\right)^{-\frac{D^*}{2}} e^{\frac{D^*}{2}}.
\end{aligned}
\tag{953}
$$

As we have explained in the main text, the constant B is universal and is given by $B = \frac{1}{(D-3)\Omega_{D-2}}$. If we now specialize to the case of the retarded Greens function we have $E = 0$. From

(953) it follows that

$$
\begin{aligned}
C &= \frac{(1+i)}{\sqrt{2}} B \sqrt{\frac{D^*}{2}} \left(\frac{D^*}{\omega}\right)^{-\frac{D^*}{2}} e^{\frac{D^*}{2}} = \frac{(1+i)}{\sqrt{2}} (2)^{-\frac{D}{2}} \frac{\omega^{\frac{D-3}{2}}}{\pi^{\frac{D-2}{2}}}, \\
A &= \frac{iB}{2} = \frac{i}{2(D-3)\Omega_{D-2}}.
\end{aligned}
\tag{954}
$$

[167] It follows that in the classically allowed region

$$
\psi(\omega, r) = -(2i)^{-\frac{D^*}{2}} \frac{\omega^{\frac{D-3}{2}}}{\pi^{\frac{D-2}{2}}} \frac{e^{i\left(\sqrt{\omega^2 r^2 - \frac{D^{*2}}{4}} + \frac{D^*}{2} \sin^{-1}\left(\frac{D^*}{2\omega r}\right)\right)}}{\left(\omega^2 - \frac{D^{*2}}{4r^2}\right)^{\frac{1}{4}}}.
\tag{955}
$$

It is easily verified that (955) matches both the leading and first subleading terms in (567) when expanded at large r.

In the classically disallowed region , where $D^* > 2\omega r$ we find the explicit formula

$$
\begin{aligned}
\psi(\omega, r) = \frac{1}{(D-3)\Omega_{D-2}} &\left(\frac{e^{\frac{D^*}{2} - \sqrt{\frac{D^{*2}}{4} - \omega^2 r^2}}}{\left(\frac{D^{*2}}{4r^2} - \omega^2\right)^{\frac{1}{4}}} \left(\frac{D^*}{\omega}\left(\frac{D^*}{2\omega r} - \sqrt{\frac{D^{*2}}{4\omega^2 r^2} - 1}\right)\right)^{-\frac{D^*}{2}} \right. \\
&\left. + \frac{i}{2} \frac{e^{\frac{D^*}{2} + \sqrt{\frac{D^{*2}}{4} - \omega^2 r^2}}}{\left(\frac{D^{*2}}{4r^2} - \omega^2\right)^{\frac{1}{4}}} \left(\frac{1}{2r} - \sqrt{\frac{1}{4r^2} - \frac{\omega^2}{D^{*2}}}\right)^{\frac{D^*}{2}} \right) \\
= &\frac{1}{(D-3)\Omega_{D-2}} \left(\frac{1}{r^{\frac{D-4}{2}}} + \frac{i}{2} \left(\frac{e\omega}{D}\right)^{D-3} r^{\frac{D-2}{2}} \right),
\end{aligned}
\tag{956}
$$

where the second expression applies at small ωr. So the Greens Function can be written as

$$
G(\omega, r) = \frac{1}{(D-3)\Omega_{D-2} r^{\frac{D-3}{2}}} \left(\frac{1}{r^{\frac{D-3}{2}}} + \frac{i}{2} \left(\frac{e\omega}{D}\right)^{D-3} r^{\frac{D-3}{2}} \right).
\tag{957}
$$

It may be verified that this result matches the small r asymptotics of the exact formula (565) in the following sense. From (565) the exact Greens function is given by

$$
G(\omega, r) = \frac{i}{4} \left(\frac{\omega}{2\pi r}\right)^{\frac{D-3}{2}} \left(J_{\frac{D-3}{2}}(\omega r) + i N_{\frac{D-3}{2}}(\omega r) \right),
\tag{958}
$$

[167] We have used the large D approximations $D^* \approx D-3$ and

$$
\Omega_{D-2} \approx 2^{-\frac{D-3}{2}} \pi^{\frac{D-2}{2}} e^{\frac{D-3}{2}} D^{-\frac{D-2}{2}}.
$$

where N_n is the Neumann function. At small ωr we use the small ωr expansion of the Bessel and Neumann functions to obtain

$$\begin{aligned} G(\omega, r) &= \frac{i}{4} \frac{1}{(D-3)\Omega_{D-2}} \left(\frac{\omega}{2\pi r}\right)^{\frac{D-2}{2}} \left(-4i\left(\frac{2\pi}{\omega r}\right)^{\frac{D-3}{2}} + \frac{2}{e}\left(\frac{2\pi e\omega r}{D}\right)^{\frac{D-3}{2}}\right) \\ &= \frac{1}{(D-3)\Omega_{D-2} r^{\frac{D-3}{2}}} \left(\frac{1}{r^{\frac{D-3}{2}}} + \frac{i}{2e}\left(\frac{e\omega}{D}\right)^{D-3} r^{\frac{D-3}{2}}\right), \end{aligned} \tag{959}$$

in agreement with (956).

We will now explain in what sense the WKB approximation may be thought of as the first term in a systematic large D approximation of the Greens function. The first correction to any WKB approximation is of order of the fractional change in the wavenumber over a distance scale of order one wavelength. In formulae, the first correction to this approximation is of order $\frac{1}{k(r)}\partial_r \ln k(r)$ where $k(r)$ is the local WKB wave number. In the classically allowed region $k(r) = \sqrt{\omega^2 - \frac{D^2}{4r^2}}$. So the fractional correction, $E(r)$, to the WKB approximation can be estimated to be of order

$$E(r) = \frac{\frac{D^{*2}}{r^3}}{\left(\sqrt{\omega^2 - \frac{D^{*2}}{4r^2}}\right)^3}.$$

Provided that $\sqrt{\omega^2 - \frac{D^{*2}}{4r^2}}$ is of order unity (i.e. provided we don't get too near the turning point) it follows that $E(r) = \mathcal{O}(1/D)$ (recall that $\omega r > D/2$). This conclusion works all the way down to $\omega r - \frac{D}{2} \sim \frac{1}{D^{\frac{2}{3}}}$. In a similar manner the fractional error to the WKB approximation in the classically disallowed region is once again estimated as

$$E(r) \sim \frac{1}{D\sqrt{1 - \frac{4\omega^2 r^2}{D^2}}},$$

and is once again of order $\frac{1}{D}$ provided we stay away from the turning point. In summary, the WKB approximation provides an excellent approximation to the Greens function at large D except within a distance of order $\frac{1}{D^{\frac{2}{3}}}$ of the turning point.

We end this subsection with a qualitative description of the retarded Green's function in the large D limit. There are four qualitatively distinct regions in the Greens function. Deep into the classically allowed region, for $r\omega \gg D^2$, the Greens function is in the radiation zone. In this regime (567) applies, and the modulus of Greens function is proportional to $\frac{e^{i\omega r}}{r^{\frac{D-2}{2}}}$. It follows that the mod squared Greens function is proportional to the inverse volume of the $D-2$ sphere of radius r in this region, and so represents radiation whose integrated flux is independent of r.

Moving further in we reach the intermediate radiation zone $\frac{D}{2} < \frac{\omega r}{2} \ll D^2$. In this region the

Greens function represents an oscillating radiation field that has not propagated far enough to settle into its large r asymptotic value.

Moving to smaller r we pass the turning point of the potential and enter the classically forbidden region. In this intermediate static regime $\sqrt{D} \ll \frac{\omega r}{2} \ll \frac{D}{2}$, ψ no longer oscillates as a function of r. Instead the Greens function turns into a sum of a term that grows as r increases and another that decays as r decreases. The decaying and growing pieces are comparable in magnitude near the turning point. However the decaying term grows towards small r and quickly dominates.

Moving to still smaller r we reach the static zone $\frac{\omega r}{2} \ll \sqrt{D}$. In this region the first of (567) applies, and $G(\omega, r)$ becomes independent of ω (justifying the name static zone). The decaying term in (956) is much larger than the growing term in this region; in particular the the ratio of the growing term to the decaying term of order $\frac{1}{D^D}$ when ωr is of order unity.

4.11.6 Action of the Greens function on scalars, vectors and tensors in a spherical basis

Results for the off centred Green's function In our analysis of radiation we will will find it useful to have a generalization of the exact expression (565) to a Greens function whose source point is displaced away from the origin. In the next subsection we demonstrate that

$$G(\omega, |\vec{r} - \vec{r}'|) = \frac{i\pi}{2} \sum_{l=0}^{\infty} \frac{1}{(r'r)^{\frac{D-3}{2}}} H^{(1)}_{\frac{D-3+2l}{2}}(\omega r) J_{\frac{D-3+2l}{2}}(\omega r') \mathcal{P}_l(\theta, \theta'). \tag{960}$$

This result applies provided $|r| > |r'|'$, i.e. provided that the observation point is located further from the origin than the source point. The Hankel function $H^{(1)}(r)$ which appears in (783) is the unique solution to the Bessel function that is is purely outgoing at infinity. On the other hand the Bessel function $J(r')$ that also appears in this expression is the unique solution to the Bessel equation that is regular at the origin. θ collectively denotes all the angles of the point $\vec{r}$ on S^{D-2}, θ' similarly denotes all angles of the point $\vec{r}'$ and $\mathcal{P}_l(\theta, \theta')$ is the projector onto the angular dependence of the l^{th} spherical harmonic defined in (894).

It is easily verified that (783) reduces to (565) in the limit $r' \to 0$. As a consistency check on this formula we have explicitly verified that the expansion (783) is translationally invariant, i.e. that

$$(\partial_{\vec{r}} + \partial_{\vec{r}'}) G(\omega, |\vec{r} - \vec{r}'|) = 0. \tag{961}$$

Note also that in the limit $\omega \to 0$,

$$G(0, |\vec{r} - \vec{r}'|) = \frac{1}{r^{D-3}} \sum_{l=0}^{\infty} \left(\frac{r'}{r}\right)^l \frac{1}{2l + D - 3} \mathcal{P}_l(\theta, \theta'). \tag{962}$$

Derivation The expression (783) may be derived as follows. Provided that $\vec{r} \neq \vec{r}'$ (and so in particular when $|\vec{r}| > |\vec{r}'|$) the Greens function is annihilated by the action of

$$\left(\omega^2 + \vec{\nabla}^2\right),$$

separately on the variables $\vec{r}$ and $\vec{r}'$. The most general solution of the equation

$$\left(\omega^2 + \vec{\nabla}^2\right) \phi(\omega, \vec{r}) = 0, \tag{963}$$

is a linear superposition of modes of the form $\phi_l(\omega, r) Y_{lm}(\theta)$, where Y_{lm} represents an arbitrary scalar spherical harmonic [168] in the representation $(l, 0, 0, \ldots, 0)$ of $SO(D-1)$. Using the fact that

$$\nabla^2 Y_{lm} = -l(l + D - 3) Y_{lm}, \tag{964}$$

where the Laplacian is taken on the unit sphere, see (893)) it follows from (963) that

$$\left(\omega^2 + \frac{1}{r^{D-2}} \partial_r (r^{D-2} \partial_r) - \frac{l(l+D-3)}{r^2}\right) \phi_l(\omega, r) = 0. \tag{965}$$

Solving this equation we find that

$$\phi_l(\omega, r) = \frac{1}{r^{\frac{D-3}{2}}} \left(A_{l,\omega} H^{(1)}_{\frac{D-3+2l}{2}}(\omega r) + B_{l,\omega} J_{\frac{D-3+2l}{2}}(\omega r) \right). \tag{966}$$

The boundary conditions on our Greens function require it to be regular at every finite value of $\vec{r}'$ other than $\vec{r}$; and requires the Greens function to be an outgoing function of $\vec{r}$; these considerations force us to use the Hankel function with argument r and the Bessel function with argument r'. The Greens function must also be rotationally invariant under simultaneous rotations of θ and θ'. As we have explained above, the unique rotationally invariant function of two angles constructed using functions only in in the l^{th} spherical harmonic sector is the projector $\mathcal{P}_l$ defined in (894). It follows from all these considerations that the Greens function must be given by an expression of the form

$$G(\omega, |\vec{r} - \vec{r}'|) = \sum_{l=0}^{\infty} \frac{a_l}{(r'r)^{\frac{D-3}{2}}} H^{(1)}_{\frac{D-3+2l}{2}}(\omega r) J_{\frac{D-3+2l}{2}}(\omega r') \mathcal{P}_l(\theta, \theta'), \tag{967}$$

for some as yet unknown coefficients a_l. We will now demonstrate that

$$a_l = \frac{i\pi}{2}; \quad \text{for all } l \tag{968}$$

[168] See Appendix 4.11.4 for a discussion of Spherical harmonics and their properties in arbitrary dimensions.

using which (783) follows.

In order to obtain (968) we use the large argument expansion of the Hankel function (566). (967) simplifies to

$$G(\omega, |\vec{r}-\vec{r}'|) \approx i\sqrt{\frac{\pi}{2}}\left(\frac{-i\omega}{r}\right)^{\frac{D-2}{2}} e^{i\omega r}\sum_l J_{\frac{D-3+2l}{2}}(\omega r')(\omega r')^{-\frac{D-3}{2}}\mathcal{P}_l(\theta,\theta') \tag{969}$$

We also specialize (967) to the case in which the source and observation points are at the same angle. In this special case the LHS of (783) is simply $G(\omega,(r-r'))$ (see (565)) and (967) reduces to

$$G(\omega,(r-r')) = \sum_{l=0}^{\infty}\frac{a_l\mathcal{P}_l(0)}{(r'r)^{\frac{D-3}{2}}}H^{(1)}_{\frac{D-3+2l}{2}}(\omega r)J_{\frac{D-3+2l}{2}}(\omega r') \tag{970}$$

(where $G(\omega,r)$ is defined in (565) and $\mathcal{P}_l(0)$ is presented in (899)). In order to determine the coefficients a_l it is sufficient to further specialize (970) to large r and retain only leading order terms on both sides in the $\frac{1}{r}$ expansion. (970) reduces to

$$\begin{aligned}
&\frac{i}{4}\left(\frac{\omega}{2\pi r}\right)^{\frac{D-3}{2}}\left(i^{-\frac{D-2}{2}}\sqrt{\frac{2}{\pi\omega r}}e^{i\omega(r-r')}\right)\\
&= i^{-\frac{D-2}{2}}\sqrt{\frac{2}{\pi\omega r}}\left(\frac{\omega}{r}\right)^{\frac{D-3}{2}}\frac{e^{i\omega r}}{\omega}\sum_l i^{-l}a_l\mathcal{P}_l(0)J_{\frac{D-3+2l}{2}}(\omega r')(\omega r')^{-\frac{D-3}{2}}\\
&\text{i.e.}\quad e^{ix} = -4i(2\pi)^{\frac{D-3}{2}}\sum_l a_l\mathcal{P}_l(0)J_{\frac{D-3+2l}{2}}(x)(x)^{\frac{D-3}{2}}
\end{aligned} \tag{971}$$

(where we have used (567), and $x=\omega r'$). Taylor expanding the LHS and RHS in x about $x=0$ and using the well known series expansion for the Bessel function

$$J_{\frac{D-3+2l}{2}}(x)(x)^{\frac{D-3}{2}} = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma\left(l+m+\frac{D-1}{2}\right)}\frac{(x)^{l+2m}}{2^{l+2m+\frac{D-3}{2}}}, \tag{972}$$

we find the following recursion relations

$$\tilde{a}_n = \Gamma\left(n+\frac{D-1}{2}\right)\left(\frac{2^n}{n!} - \sum_{m=1}^{\lfloor\frac{n}{2}\rfloor}\frac{\tilde{a}_{n-2m}}{m!\;\Gamma\left(n-m+\frac{D-1}{2}\right)}\right), \tag{973}$$

where

$$\tilde{a}_l = -4i\pi^{\frac{D-3}{2}}\mathcal{P}_l(0)a_l. \tag{974}$$

Using the explicit value of $\mathcal{P}_l(0)$ listed in (899) it may be verified that

$$a_l = \frac{i\pi}{2}, \tag{975}$$

solves the recursion relation (973), establishing (968).

Action of the retarded Greens function on the arbitrary spherically decomposed vector field We will now use the results the previous subsections together with those of Appendix 4.11.4 to present the general solution to the equation

$$\left(-\nabla^2 - \omega^2\right) \vec{E} = \vec{J}_{\text{eff}}, \tag{976}$$

where the field J_{eff} is a general vector field in R^{D-1} that admits the expansion (915). We search for the unique solution to this problem subject to the restriction that it behaves as $e^{i\omega r}$ at infinity.

In Cartesian coordinates the solution to this problem is simply given by

$$\vec{E}(\vec{r}) = \int dr' G(\omega, |\vec{r} - \vec{r}'|) \vec{J}_{\text{eff}}(r'), \tag{977}$$

where the Green's function $G(\omega, |\vec{r} - \vec{r}'|)$ was defined in (783). Using (918) the solution (977) can be rewritten in terms of a spherical decomposition as

$$\vec{E}(\omega, \vec{r}) = \vec{\mathcal{A}}^{-}[\xi_\alpha] + \vec{\mathcal{A}}^{+}[\xi_\beta] + \vec{v}_\gamma, \tag{978}$$

where

$$\begin{aligned}
\xi_\alpha(\omega, \vec{r}) &= \sum_l \frac{i\pi}{2} \frac{H^{(1)}_{\frac{D-3+2(l-1)}{2}}(\omega r)}{r^{\frac{D-3}{2}}} \int dr' J_{\frac{D-3+2(l-1)}{2}}(\omega r')\, r'^{\frac{D-1}{2}} \alpha_l(\omega, r', \theta), \\
\xi_\beta(\omega, \vec{r}) &= \sum_l \frac{i\pi}{2} \frac{H^{(1)}_{\frac{D-3+2(l+1)}{2}}(\omega r)}{r^{\frac{D-3}{2}}} \int dr' J_{\frac{D-3+2(l+1)}{2}}(\omega r')\, r'^{\frac{D-1}{2}} \beta_l(\omega, r', \theta), \\
\vec{v}_\gamma(\omega, \vec{r}) &= \sum_l \frac{i\pi}{2} \frac{H^{(1)}_{\frac{D-3+2l}{2}}(\omega r)}{r^{\frac{D-3}{2}}} \int dr' J_{\frac{D-3+2l}{2}}(\omega r')\, r'^{\frac{D-1}{2}} \vec{\gamma}_l(\omega, r', \theta).
\end{aligned} \tag{979}$$

Action of the retarded Greens function on the arbitrary spherically decomposed tensor field In this brief subsection we study the equation

$$\left(-\nabla^2 - \omega^2\right) \mathcal{H}_{ij} = T_{ij}, \tag{980}$$

where T_{ij} is a given symmetric tensor field. We will find the unique solution to (980) subject to the condition that $\mathcal{H}_{ij}$ is outgoing at infinity.

Let the source function T_{ij} have the spherical decomposition listed in (924). Proceeding as in the previous subsection, it is easy to verify that the unique outgoing solution to (980) is given by

$$\mathcal{H}_{ij}(\omega,\vec{r}) = \mathcal{C}^{-}_{ij}[\xi_\alpha] + \mathcal{C}^{+}_{ij}[\xi_\beta] + \mathcal{C}^{0}_{ij}[\xi_\gamma] + \delta_{ij}\xi_\kappa + \mathcal{B}^{-}_{ij}[\vec{v}_\phi] + \mathcal{B}^{+}_{ij}[\vec{v}_\psi] + \tau^{\chi}_{ij}, \tag{981}$$

where

$$\begin{aligned}
\xi_\alpha(\omega,\vec{r}) &= \sum_l \frac{i\pi}{2}\frac{H^{(1)}_{\frac{D-3+2(l-2)}{2}}(\omega r)}{r^{\frac{D-3}{2}}}\int dr' J_{\frac{D-3+2(l-2)}{2}}(\omega r')\ r'^{\frac{D-1}{2}}\alpha_l(\omega,r',\theta),\\
\xi_\beta(\omega,\vec{r}) &= \sum_l \frac{i\pi}{2}\frac{H^{(1)}_{\frac{D-3+2(l+2)}{2}}(\omega r)}{r^{\frac{D-3}{2}}}\int dr' J_{\frac{D-3+2(l+2)}{2}}(\omega r')\ r'^{\frac{D-1}{2}}\beta_l(\omega,r',\theta),\\
\xi_\gamma(\omega,\vec{r}) &= \sum_l \frac{i\pi}{2}\frac{H^{(1)}_{\frac{D-3+2l}{2}}(\omega r)}{r^{\frac{D-3}{2}}}\int dr' J_{\frac{D-3+2l}{2}}(\omega r')\ r'^{\frac{D-1}{2}}\gamma_l(\omega,r',\theta),\\
\xi_\kappa(\omega,\vec{r}) &= \sum_l \frac{i\pi}{2}\frac{H^{(1)}_{\frac{D-3+2l}{2}}(\omega r)}{r^{\frac{D-3}{2}}}\int dr' J_{\frac{D-3+2l}{2}}(\omega r')\ r'^{\frac{D-1}{2}}\kappa_l(\omega,r',\theta),\\
\vec{v}_\phi(\omega,\vec{r}) &= \sum_l \frac{i\pi}{2}\frac{H^{(1)}_{\frac{D-3+2(l-1)}{2}}(\omega r)}{r^{\frac{D-3}{2}}}\int dr' J_{\frac{D-3+2(l-1)}{2}}(\omega r')\ r'^{\frac{D-1}{2}}\vec{\phi}_l(\omega,r',\theta),\\
\vec{v}_\psi(\omega,\vec{r}) &= \sum_l \frac{i\pi}{2}\frac{H^{(1)}_{\frac{D-3+2(l+1)}{2}}(\omega r)}{r^{\frac{D-3}{2}}}\int dr' J_{\frac{D-3+2(l+1)}{2}}(\omega r')\ r'^{\frac{D-1}{2}}\vec{\chi}_l(\omega,r',\theta),\\
\tau^{\chi}_{ij}(\omega,\vec{r}) &= \sum_l \frac{i\pi}{2}\frac{H^{(1)}_{\frac{D-3+2l}{2}}(\omega r)}{r^{\frac{D-3}{2}}}\int dr' J_{\frac{D-3+2l}{2}}(\omega r')\ r'^{\frac{D-1}{2}}(\chi_l)_{ij}(\omega,r',\theta).
\end{aligned}\tag{982}$$

4.11.7 Details relating to the general theory of radiation

Static Limit of Electromagnetic Radiation It is useful to separately consider the scalar part of the electric field (first line of (804)) and the vector part (second line of (804)). Let us first focus on the scalar part of this field. Using (806) and the fact that $H_n(x) \sim x^{-n}$, we see that the first term in the first line of (804) is negligible compared to the second term in the same line and at small ω and we find

$$\begin{aligned}\vec{E}(\omega,\vec{x}) &= \sum_{l=0}^{\infty}\left(\frac{H_{\frac{D+2l-1}{2}}(\omega r)}{r^{\frac{D-3}{2}}}\vec{\mathcal{A}}^{+}[S_l^{+}(\omega,\theta)]\right)\\ &= \frac{-i}{\pi}\sum_{l=0}^{\infty}\left(2^{\frac{2l+D-1}{2}}\frac{\Gamma\left(\frac{2l+D-1}{2}\right)\left((l+D-3)\hat{r}S_l^{+}-r\vec{\tilde{\nabla}}S_l^{+}\right)}{\omega^{\frac{2l+D-1}{2}}r^{l+D-2}}\right)\\ &= -\vec{\nabla}\Phi^{E},\end{aligned} \tag{983}$$

where

$$\Phi^{E} = \frac{-i}{\pi}\sum_{l=0}^{\infty}\left(2^{\frac{2l+D-1}{2}}\frac{\Gamma\left(\frac{2l+D-1}{2}\right)}{\omega^{\frac{2l+D-1}{2}}}\frac{S_l^{+}}{r^{l+D-3}}\right), \tag{984}$$

$$\begin{aligned}S_l^{+} &= \frac{i\pi}{2}\int dr'(r')^{\frac{D-1}{2}}J_{\frac{2l+D-1}{2}}(\omega r')\mathfrak{b}_l(\omega,r',\theta)\\ &= \frac{i\pi}{2}\frac{\omega^{\frac{2l+D-1}{2}}}{2^{\frac{2l+D-1}{2}}\Gamma\left(\frac{2l+D+1}{2}\right)}\int dr'(r')^{l+D-1}\mathfrak{b}_l(\omega,r',\theta).\end{aligned} \tag{985}$$

[169]

(983) is simply the statement that the electric field in the stationary limit is the gradient of a scalar potential. There is, of course, a simple explanation and interpretation of this fact. Recall that t he effective source $\vec{\mathcal{J}}_{eff}$ - from which $\mathfrak{b}$ is built - is a linear combination of two terms. One of the two terms is the time derivative of the spatial current, and is subleading compared to the other term (the spatial derivative of the charge current) in the small ω limit. In this limit, consequently, the formula for the electric field reduces to

$$\begin{aligned}\vec{E} &= \int G(\omega\to 0,|\vec{r}-\vec{r'}|\nabla'\mathcal{J}_0(r')\\ &= \nabla\left(\int G(\omega\to 0,|\vec{r}-\vec{r'}|\mathcal{J}_0(r')\right)\\ &= -\nabla\Phi_E,\\ \Phi_E &= -\int G(\omega\to 0,|\vec{r}-\vec{r'}|\mathcal{J}_0(r').\end{aligned} \tag{986}$$

This is simply Coulomb's law. Indeed it may directly be verified that Φ_E defined in (984) and

[169]In going from the first to the second lines of (983) we replaced the Hankel function by its leading term in a small argument expansion (this is appropriate in the small ω limit). The equations (984) and (985) are expressions for the effective potential. In going from the first to the second line of (985) we have used the fact that ω is small to replace the Bessel function by the leading piece in a small argument expansion.

(986) agree with each other. [170].

In the static limit magnetic field can be written using the Bianchi identity as:

$$F_{ij}(0,r,\theta) = \lim_{\omega\to 0} \frac{i(\nabla_i \vec{E}_j(\omega,r,\theta) - \nabla_j \vec{E}_i(\omega,r,\theta))}{\omega}. \tag{988}$$

The only term in (808) that contributes to the RHS of (988) is the pure vector piece $\mathfrak{c}$, which only receives contributions from the the term in $\vec{\mathcal{J}}_{eff}$ equal to $\partial_0 \vec{\mathcal{J}}$.

In the limit of small ω $\tilde{\vec{\mathfrak{c}}}$ and $\vec{\mathfrak{c}}$ differ only by a factor of ω at it is easily verified that

$$\begin{aligned} F_{ij}(\omega,r,\theta) &= (\nabla_i \vec{A}_j(\omega,r,\theta) - \nabla_j \vec{A}_i(\omega,r,\theta)), \\ \vec{A} &= \lim_{\omega\to 0} \frac{i\vec{E}}{\omega} \\ \therefore \vec{A} &= \sum_l \frac{i}{(2l+D-3)r^{D-3}} \int dr'(r')^{D-2} \left(\frac{r'}{r}\right)^l (\tilde{\vec{\mathfrak{c}}}_l(0,r',\theta)). \end{aligned} \tag{989}$$

In other words the magnetic field is given by dA where $\nabla^2 \vec{A} = \vec{\mathcal{J}}$, and we recover the usual formulae of magnetostatics.

Constraints from current conservation and $\nabla \cdot E = 0$ As we have explained in the main text the fact that $\nabla.\vec{E}$ vanishes in vacuum implies that the scalar functions $\mathcal{S}^{\pm}$ that characterize a general radiation field (see (804)) are not independent but are related by (806). In (809), however, we have presented separate formulae for $\mathcal{S}^{\pm}$ in terms of integrals over scalar components of charge currents. The consistency of (809) requires that these results for $\mathcal{S}^{\pm}$ automatically obey (806). We will now demonstrate that this is indeed the case.

At the structural level the way this works is very simple. If we take the divergence of (801) and use (810) we obtain (811), which guarantees that $\nabla.E$ vanishes in vacuum. In this section we will rerun this structural argument on the explicit formulae (809). The fact that we land on our feet serves as a consistency check of the algebra that led to (809) and (806).

In the rest of this subsection we proceed to algebraically demonstrate that

- The equation $\nabla \cdot E = 0$ implies that the coefficient functions in (804) obey (806)

[170] In order to perform this verification it is useful to note in the limit of small ω

$$\vec{\mathcal{J}}_{eff} \to \nabla \mathcal{J}_0$$

$$\mathfrak{b}_l = \frac{(r')^l}{2l+D-3} \partial_{r'} \left(\frac{\mathcal{J}_{0l}}{(r')^l}\right) \tag{987}$$

where $\mathcal{J}_{0l}$ is the l^{th} spherical harmonic piece of the charge current $\mathcal{J}_0$.

- That the relations (809) automatically obey (806) once we account for the fact that the current is conserved.

Demonstration that $\nabla \cdot E = 0$ *implies* (806)

According to (804) the vacuum electromagnetic solution is given by

$$\vec{E} = \sum_l \left(\frac{H^{(1)}_{l+\frac{D-5}{2}}(\omega r)}{r^{\frac{D-3}{2}}} \vec{\mathcal{A}}^{-}[S^-_l(\omega,\theta)] + \frac{H^{(1)}_{l+\frac{D-1}{2}}(\omega r)}{r^{\frac{D-3}{2}}} \vec{\mathcal{A}}^{+}[S^+_l(\omega,\theta)] + \frac{H^{(1)}_{l+\frac{D-3}{2}}(\omega r)}{r^{\frac{D-3}{2}}} \vec{V}_l(\omega,\theta) \right). \tag{990}$$

Using

$$\begin{aligned} \nabla \cdot \vec{\mathcal{A}}^{-}[S^-_l(\omega,\theta)] &= l r^{l-1} \partial_r \left(\frac{H^{(1)}_{l+\frac{D-5}{2}}(\omega r)}{r^{l+\frac{D-5}{2}}} \right) S^-_l(\omega,\theta) \\ &= -l S^-_l(\omega,\theta) \frac{H^{(1)}_{l+\frac{D-3}{2}}(\omega r)}{r^{\frac{D-3}{2}}} \\ \nabla \cdot \vec{\mathcal{A}}^{+}[S^+_l(\omega,\theta)] &= \frac{l+D-3}{r^{l+D-2}} \partial_r \left(H^{(1)}_{l+\frac{D-1}{2}}(\omega r) r^{l+\frac{D-1}{2}} \right) S^+_l(\omega,\theta) \\ &= +(l+D-3) S^+_l(\omega,\theta) \frac{H^{(1)}_{l+\frac{D-3}{2}}(\omega r)}{r^{\frac{D-3}{2}}} \\ \nabla \cdot \vec{V}_l(\omega,\theta) &= 0, \end{aligned} \tag{991}$$

it follows that $\nabla \cdot E = 0$ provided

$$-l S^-_l(\omega,\theta) + (l+D-3) S^+_l(\omega,\theta) = 0. \tag{992}$$

Proof that current conservation satisfies this requirement

Recall that according to (808) the effective current admits the following decomposition

$$\vec{\mathcal{J}}_{eff} = \sum_l \left(\mathcal{A}^{-}[\mathfrak{a}_l(\omega,r',\theta')] + \mathcal{A}^{+}[\mathfrak{b}_l(\omega,r',\theta')] + \vec{\mathfrak{c}}_l(\omega,r',\theta') \right). \tag{993}$$

Using (923) yields

$$\nabla' \cdot \vec{\mathcal{J}}_{eff} = \sum_l \left(l r'^{l-1} \partial_{r'} \left(\frac{\mathfrak{a}_l(\omega,r',\theta')}{r'^{l-1}} \right) + \frac{l+D-3}{r'^{l+D-2}} \partial_{r'} \left(\mathfrak{b}_l(\omega,r',\theta') r'^{l+D-2} \right) \right). \tag{994}$$

From (810) we conclude that

$$lr'^{l-1}\partial_{r'}\left(\frac{\mathfrak{a}_l(\omega,r',\theta')}{r'^{l-1}}\right)+\frac{l+D-3}{r'^{l+D-2}}\partial_{r'}\left(\mathfrak{b}_l(\omega,r',\theta')r'^{l+D-2}\right)-\Box'(J_0)_l=0. \tag{995}$$

Multiplying by $\frac{i\pi}{2}J_{l+\frac{D-3}{2}}(\omega r')r'^{\frac{D-1}{2}}$ integrating by parts w.r.t. r' and noting also that $\int dr'\Box'\left(J_{l+\frac{D-3}{2}}(\omega r')r'^{\frac{D-1}{2}}\right.$ 0 we get ,

$$\frac{i\pi l}{2}\int dr' J_{l+\frac{D-5}{2}}(\omega r')r'^{\frac{D-1}{2}}\mathfrak{a}_l(\omega,r',\theta')=\frac{i\pi(l+D-3)}{2}\int dr' J_{l+\frac{D-1}{2}}(\omega r')r'^{\frac{D-1}{2}}\mathfrak{b}_l(\omega,r',\theta') \tag{996}$$

Which from (809) translates to

$$lS_l^-(\omega,\theta)=(l+D-3)S_l^+(\omega,\theta). \tag{997}$$

Static Limit of Gravitational Radiation It is useful to separately consider the scalar vector and tensor parts of the curvature given in (822).

Focusing first on the scalar part we see from (822), (826) and the fact that $h_n(x)\sim x^{-n}$, that in the limit $\omega\to 0$ the scalar part of (822) reduces to

$$\begin{aligned}
R_{0i0j}(\omega,\vec{x}) &= \lim_{\omega\to 0}\sum_{l=0}^{\infty}\left(\frac{H_{\frac{D+2l+1}{2}}(\omega r)}{r^{\frac{D-3}{2}}}\mathcal{C}^+_{ij}[S_l^+(\omega,\theta)]\right)\\
&= \frac{-i}{\pi}\lim_{\omega\to 0}\sum_{l=0}^{\infty}\left(2^{\frac{2l+D+1}{2}}\frac{\Gamma\left(\frac{2l+D+1}{2}\right)\left((l+D-2)\hat{r}-r\vec{\vec{\nabla}}\right)\left((l+D-3)\hat{r}S_l^+-r\vec{\vec{\nabla}}S_l^+\right)}{\omega^{\frac{2l+D+1}{2}}r^{l+D-1}}\right)\\
&= \nabla_i\nabla_j\Phi^G,
\end{aligned} \tag{998}$$

where

$$\begin{aligned}
\Phi^G &= \frac{-i}{\pi}\sum_{l=0}^{\infty}\left(2^{\frac{2l+D+1}{2}}\frac{\Gamma\left(\frac{2l+D+1}{2}\right)}{r^{l+D-3}}\right)\lim_{\omega\to 0}\frac{S_l^+}{\omega^{\frac{2l+D+1}{2}}}\\
&= \sum_{l=0}^{\infty}\frac{1}{(2l+D+1)r^{l+D-3}}\int dr'(r')^{l+D}\mathfrak{b}_l(0,r',\theta),
\end{aligned} \tag{999}$$

where we have used

$$\lim_{\omega\to 0}\frac{S_l^+}{\omega^{\frac{2l+D+1}{2}}}=\frac{i\pi}{2}\frac{1}{2^{\frac{2l+D+1}{2}}\Gamma\left(\frac{2l+D+3}{2}\right)}\int dr'(r')^{l+D}\mathfrak{b}_l(0,r',\theta). \tag{1000}$$

(999) is simply the statement that R_{0i0j} in the stationary limit is the double gradient of a

suitably scaled version of the Newtonian potential ϕ^G. Indeed it is easily verified that ϕ^G is given by

$$\nabla^2\phi^G = -8\pi\left(\mathcal{T}_{00} + \frac{\mathcal{T}}{D-2}\right) \tag{1001}$$

[171]

Let us now turn to the vector part of R_{0i0j}. Once again using (822), (826) and the small argument expansion of the Hankel function we see that the vector part of R_{0i0j} simplifies to

$$\begin{aligned} R_{0i0j}(\omega, \vec{x}) &= \sum_{l=0}^{\infty}\left(\frac{H_{\frac{D+2l-1}{2}}(\omega r)}{r^{\frac{D-3}{2}}}\mathcal{B}^{+}_{ij}[V_l^{+}(\omega,\theta)]\right) \\ &= \frac{-i}{\pi}\sum_{l=0}^{\infty}\left(2^{\frac{2l+D-1}{2}}\frac{\Gamma\left(\frac{2l+D-1}{2}\right)\left((l+D-3)\hat{r}_i(V_l^{+})_j - r\tilde{\nabla}_i(V_l^{+})_j\right)}{\omega^{\frac{2l+D-1}{2}}r^{l+D-2}}\right) + \{i\leftrightarrow j\} \\ &= -i\omega\left(\nabla_i A^G_j + \nabla_j A^G_i\right), \end{aligned} \tag{1003}$$

so that (using the Bianchi identity)

$$\begin{aligned} R_{0ijk} &= \frac{i}{\omega}\left(\nabla_j R_{0i0k} - \nabla_k R_{0i0j}\right) \\ &= -\nabla_i\left(\nabla_j A^G_k - \nabla_k A^G_j\right), \end{aligned} \tag{1004}$$

where

$$A^G_i = \frac{1}{2\pi}\left(\frac{2}{\omega}\right)^{\frac{2l+D+1}{2}}\Gamma\left(\frac{2l+D-1}{2}\right)\frac{(V_l^{+})_i}{r^{l+D-3}}. \tag{1005}$$

It is easily verified that

$$\nabla^2 A^G_i = -8\pi T_{0i}.$$

[172] Note that $\vec{A}^G$ obeys the same equation obeyed by the 'vector potential' magnetostatics with the role of the current being played by T_{0i}. Indeed (1004) asserts that R_{0ijk} is proportional to

[171]To see this note that, in the strict limit $\omega \to 0$ the effective stress tensor reduces to

$$\mathcal{T}^{eff}_{ij} = 8\pi\nabla'_i\nabla'_j\left(\mathcal{T}_{00} + \frac{\mathcal{T}}{D-2}\right).$$

It follows that

$$\mathfrak{b}_l = \frac{8\pi(r')^{l+1}}{(2l+D-1)(2l+D-3)}\partial_{r'}\left(\frac{1}{r'}\partial_{r'}\left(\frac{\mathcal{T}_{00}+\frac{\mathcal{T}}{D-2}}{(r')^l}\right)\right). \tag{1002}$$

[172] To see this note that the term in $(\mathcal{T}_{eff})_{ij}$ (see (818)) that contributes to the the vector in (822)

$$i\omega(\partial_i\mathcal{T}_{0j} + \partial_j\mathcal{T}_{0i}).$$

$\nabla_i F^G_{jk}$ where F^G_{jk} is the magnetic field constructed from the effective vector potential A^G_i.

Finally we turn to the tensor part of R_{0i0j}. It follows immediately from

$$\nabla^2 R_{0i0j} = 8\pi\omega^2 T_{ij}$$

In the small ω limit the contribution of tensor sources to to R_{ijkm} takes the form

$$\begin{aligned} R_{0ijk} &= \frac{i}{\omega}\left(\nabla_j R_{0i0k} - \nabla_k R_{0i0j}\right), \\ R_{ijkm} &= \frac{i}{\omega}\left(\nabla_i R_{0jkm} - \nabla_j R_{0ikm}\right) \\ &= \nabla_i\nabla_k \mathfrak{T}_{jm} + \nabla_j\nabla_m \mathfrak{T}_{ik} - \nabla_j\nabla_k \mathfrak{T}_{im} - \nabla_i\nabla_m \mathfrak{T}_{jk}, \\ \mathfrak{T}_{ij} &= \frac{-R_{0i0j}}{\omega^2}. \end{aligned} \tag{1006}$$

The tensor contribution from the source is

$$\mathfrak{z}_{ij} = -8\pi\omega^2\left(\mathcal{T}_{ij} - \delta_{ij}\frac{\mathcal{T}^k_k}{D-2}\right). \tag{1007}$$

The scalar sector also contributes to R_{ijkm}, but its closed form is a bit ugly unlike the other beautiful results in this section, and that can be obtained from the scalar contribution to R_{0i0j}. We don't present it here.

Tracelessness and divergenceless of gravitational radiation In this subsection we rerun some of the discussion of section 4.11.7 but this time in the context of gravitational radiation. In particular we will explain how the explicit gravitational radiation formulae ensure that gravitational radiation is traceless and divergence free. At the formal level these results follow immediately once we use that fact that when a box of something (e.g. $\Box\zeta$) is convoluted with Green's function, the resulting integral vanishes. We will now use this fact to demonstrate

Result 1: Gravitational Radiation is traceless.

$$h_{ij}(\omega,\vec{x}) = -\frac{2}{\omega^2}\int G(\omega,|\vec{x}-\vec{x}'|)\hat{\mathcal{T}}_{ij}(\omega,\vec{x'})d^{D-1}x'. \tag{1008}$$

Hence

$$h_{ij}\eta^{ij}(\omega,\vec{x}) = -\frac{2}{\omega^2}\int G(\omega,|\vec{x}-\vec{x}'|)(\eta^{ij}\hat{\mathcal{T}}_{ij}(\omega,\vec{x'}))d^{D-1}x'. \tag{1009}$$

It follows that

$$\mathfrak{v}_i = -\frac{8\pi i\omega r'^l}{2l+D-3}\partial_{r'}\left(\frac{T_{0i}}{r'^l}\right).$$

.

Using Conservation of stress tensor, we have

$$\eta^{ij}\hat{\mathcal{T}}_{ij} = -\Box' \left(\mathcal{T}_{00} + \frac{\mathcal{T}}{D-2}\right), \tag{1010}$$

hence the integration vanishes, i.e. $h_{ij}\eta^{ij} = 0$

Result 2: Gravitational Radiation is divergenceless.

Taking spatial divergence of (1008),

$$\begin{aligned}\partial_i h_{ij}(\omega, \vec{x}) &= -\frac{2}{\omega^2}\partial_i \int G(\omega, |\vec{x} - \vec{x}'|)\hat{\mathcal{T}}_{ij}(\omega, \vec{x'})d^{D-1}x' \\ &= -\frac{2}{\omega^2}\int G(\omega, |\vec{x} - \vec{x}'|)\left(\partial_i\hat{\mathcal{T}}_{ij}(\omega, \vec{x'})\right)d^{D-1}x'.\end{aligned} \tag{1011}$$

Using Conservation of stress tensor, we have

$$\partial_i'\hat{\mathcal{T}}_{ij} = -\Box' \left(i\omega\mathcal{T}_{0j} + \mathcal{T}_{00} + \frac{\mathcal{T}}{D-2}\right), \tag{1012}$$

hence the integration vanishes, i.e. $\partial_i h_{ij} = 0$.

4.11.8 Variation of the first order gravitational counterterm action

In this brief Appendix we demonstrate that the variation of (642) yields the stress tensor (643).

Varying the first term inside the bracket in (642) we find

$$\begin{aligned}\int \delta\sqrt{\mathcal{R}} &= \int \frac{\delta\mathcal{R}}{2\sqrt{\mathcal{R}}} \\ &= \int \frac{1}{2\sqrt{\mathcal{R}}}\left[-\mathcal{R}_{\mu\nu}\delta g^{\mu\nu} + \hat{\nabla}_\mu\hat{\nabla}_\nu\delta g^{\mu\nu} - \hat{\nabla}^2\delta g\right] \\ &= \int \frac{1}{2}\left[-\left(\frac{\mathcal{R}_{\mu\nu}}{\sqrt{\mathcal{R}}}\right) + \left(\hat{\nabla}_\mu\hat{\nabla}_\nu - g^{(in)}_{AB}\hat{\nabla}^2\right)\left(\frac{1}{\sqrt{\mathcal{R}}}\right)\right]\delta g^{\mu\nu} \\ &= \int \frac{1}{2}\left[-\left(\frac{\mathcal{R}_{\mu\nu}}{\sqrt{\mathcal{R}}}\right) - g^{(in)}_{\mu\nu}\hat{\nabla}^2\left(\frac{1}{\sqrt{\mathcal{R}}}\right) + \mathcal{O}\left(\frac{1}{D}\right)\right]\delta g^{\mu\nu} \\ &= \int \frac{1}{2}\left[-\left(\frac{\mathcal{R}_{\mu\nu}}{\sqrt{\mathcal{R}}}\right) + g^{(in)}_{\mu\nu}\left(\frac{\hat{\nabla}^2\mathcal{R}}{2\mathcal{R}^{\frac{3}{2}}}\right) + \mathcal{O}\left(\frac{1}{D}\right)\right]\delta g^{\mu\nu},\end{aligned} \tag{1013}$$

where for convenience, we have used the notation $\delta g^{(ind)}_{\mu\nu} = \delta g_{\mu\nu}$ and we have used the formula

$$\begin{aligned}\delta\mathcal{R}_{\mu\nu} &= \frac{1}{2}\left[\hat{\nabla}_\alpha\hat{\nabla}_\mu\delta g^\alpha_\nu + \hat{\nabla}_\alpha\hat{\nabla}_\nu\delta g^\alpha_\mu - \hat{\nabla}_\mu\hat{\nabla}_\nu\delta g - \hat{\nabla}^2\delta g_{\mu\nu}\right] \\ \Rightarrow\ \delta\mathcal{R} &= -\delta g^{\mu\nu}\mathcal{R}_{\mu\nu} + \left(\hat{\nabla}_\mu\hat{\nabla}_\nu - g^{(ind)}_{\mu\nu}\hat{\nabla}^2\right)h^{AB},\end{aligned} \tag{1014}$$

where $\delta g = g^{\mu\nu}_{(ind)}\delta g_{\mu\nu}$, $\delta g^{\mu\nu} = g^{\mu\alpha}_{(ind)}\ \delta g_{\alpha\beta}\ g^{\alpha\nu}_{(ind)}$.

Varying the second term inside the bracket in (642) we find

$$\begin{aligned}
&\frac{1}{2}\ \delta\left(\frac{\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}}{\mathcal{R}^{\frac{3}{2}}}\right) \\
&= -\frac{3}{4}\mathcal{R}^{-\frac{5}{2}}\ \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}\ \delta\mathcal{R} - \frac{\mathcal{R}_{\mu\alpha}\mathcal{R}^{\alpha}_{\nu}\ \delta g^{\mu\nu}}{\mathcal{R}^{\frac{3}{2}}} + \frac{\mathcal{R}^{\mu\nu}\ \delta\mathcal{R}_{\mu\nu}}{\mathcal{R}^{\frac{3}{2}}}.
\end{aligned} \tag{1015}$$

Now from equation (1014) it follows that

$$\begin{aligned}
&\int \mathcal{R}^{-\frac{5}{2}}\ \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}\ \delta R \\
=\ &\int \mathcal{R}^{-\frac{5}{2}}\ \mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta}\left[-\delta g^{\mu\nu}\mathcal{R}_{\mu\nu} + \left(\hat{\nabla}_\mu\hat{\nabla}_\nu - g^{(ind)}_{\mu\nu}\hat{\nabla}^2\right)\delta g^{\mu\nu}\right] \\
=\ &\int \delta g^{\mu\nu}\left[-\mathcal{R}^{-\frac{5}{2}}\ \mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta}\mathcal{R}_{\mu\nu} + \left(\hat{\nabla}_\mu\hat{\nabla}_\nu - g_{\mu\nu}\hat{\nabla}^2\right)\left(\mathcal{R}^{-\frac{5}{2}}\ \mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta}\right)\right]. \\
=\ &\int \delta g^{\mu\nu}\left[\mathcal{O}\left(\frac{1}{D}\right)\right]
\end{aligned} \tag{1016}$$

Similarly the second term in equation (1015) is also of order $\mathcal{O}\left(\frac{1}{D}\right)$. In the third term of equation (1015) if we substitute the formula equation (1014) we get one term which is of order $\mathcal{O}(1)$.

$$\begin{aligned}
&\int \frac{\mathcal{R}^{\mu\nu}\ \delta\mathcal{R}_{\mu\nu}}{\mathcal{R}^{\frac{3}{2}}} \\
=\ &\int \left[-g^{(ind)}_{\mu\nu}\left(\frac{\hat{\nabla}_\alpha\hat{\nabla}_\beta\mathcal{R}^{\alpha\beta}}{R^{\frac{3}{2}}_{(in)}}\right) + \mathcal{O}\left(\frac{1}{D}\right)\right]\delta g^{\mu\nu} \\
=\ &\int \left[-g^{(ind)}_{\mu\nu}\left(\frac{\hat{\nabla}^2\mathcal{R}}{2\mathcal{R}^{\frac{3}{2}}}\right) + \mathcal{O}\left(\frac{1}{D}\right)\right]\delta g^{\mu\nu}.
\end{aligned} \tag{1017}$$

Using equation (1013), (1015), (1016) and (1017) we find the equation (643)

4.11.9 Perturbative solution for ρ

In this section we find the solution of (645). In order to do this we find it convenient to use the following coordinate system. Choose any point on the membrane. We treat this point as the origin of our coordinate system. We now erect a Cartesian coordinate system about this point, making sure to orient a special coordinate, z, so that the normal to the membrane at that point is dz. Let the remaining Cartesian coordinates (which are all orthogonal to each other and to z but are otherwise arbitrary) be denoted by x^μ). It follows that, in this coordinate system, the equation of the membrane takes the following form

$$z(y_\mu) = -\frac{K_{\mu\nu}}{2}y_\mu y_\nu - \frac{C_{\mu\nu\sigma}}{3}y_\mu y_\nu y_\sigma - \frac{D_{\mu\nu\sigma\rho}}{4}y_\mu y_\nu y_\sigma y_\rho + \cdots \tag{1018}$$

Now consider a point outside the membrane whose coordinates are (z, x_μ). At least in a neighbourhood of the membrane any such point may uniquely be associated with a point $(z(y_\mu), y_\mu)$ on the membrane by the requirement that a straight line drawn normal through this membrane point passes through (z, x_μ).

Let $y_\mu(z, x_\mu)$ denote the coordinates of the membrane point associated with an arbitrary bulk point in this manner, and let $s(z, x_\mu)$ denote the distance along this line from the given bulk point to the membrane. We will now determine $y_\mu(z, x_\mu)$ and $s(z, x_\mu)$ in a Taylor series expansion in x_μ.

Working in a Taylor expansion in y_μ, the normal at any point on the membrane is given by

$$n = \frac{dz + (K_{\mu\nu} y_\nu + C_{\mu\nu\sigma} y_\nu y_\sigma + D_{\mu\nu\sigma\rho} y_\nu y_\sigma y_\rho)\, dy_\mu}{\mathcal{N}}, \tag{1019}$$

where the normalization $\mathcal{N}$ is chosen to ensure that $n.n = 1$. To solve for y_μ in terms of the x_μ and z we note that, by definition

$$\begin{aligned} \frac{x_\mu - y_\mu}{z - z_0} &= \frac{n_\mu}{n_z}, \\ \frac{x_\mu - y_\mu}{z - z_0} &= (K_{\mu\nu} y_\nu + C_{\mu\nu\sigma} y_\nu y_\sigma + D_{\mu\nu\sigma\rho} y_\nu y_\sigma y_\rho)\,. \end{aligned} \tag{1020}$$

These equations are easily solved in a Taylor series expansion in x_μ (but to all orders in z). To the cubic order in x_μ we have

$$\begin{aligned} y_\mu = (Px)_\mu &- z(P \cdot C)_{\mu\nu\sigma}(Px)_\nu (Px)_\sigma + 2z^2 (P.C.C)_{\mu\nu\sigma\rho}(Px)_\nu (Px)_\sigma (Px)_\rho \\ &- z(P.D))_{\mu\nu\sigma\rho}(Px)_\nu (Px)_\sigma (Px)_\rho - z(P.K)_{\sigma\rho} K_{\mu\nu} Px)_\nu (Px)_\sigma (Px)_\rho, \end{aligned} \tag{1021}$$

where we have defined

$$P_{\mu\nu} = \left(\frac{1}{1 + zK}\right)_{\mu\nu}.$$

We now turn to the determination of $s(x_\mu, z)$. First note that

$$\begin{aligned} s(x_\mu, z) &= \sqrt{(z - z_0)^2 + (x - y)^2} \\ &= (z - z_0)\sqrt{(1 + \frac{(x - y)^2}{(z - z_0)^2}}. \end{aligned} \tag{1022}$$

Using (1020) and retaining terms to cubic order in y we obtain

$$s(x_\mu, z) = z + \frac{1}{2}\left(K_{\mu\nu} + z(K \cdot K)_{\mu\nu}\right) y_\mu y_\nu + \left(\frac{1}{3} C_{\mu\nu\sigma} + z(K \cdot C)_{\mu\nu\sigma}\right) y_\mu y_\nu y_\sigma + \cdots \tag{1023}$$

Substituting the expansion of y in (1023) and retaining terms to the cubic order in x

$$\begin{aligned} s(x_\mu, z) = z + \frac{1}{2}\left(K_{\mu\nu} + z(K\cdot K)_{\mu\nu}\right)\left((Px)_\mu (Px)\nu - 2z(P\cdot C)_{\mu\sigma\rho}(Px)_\nu (Px)_\sigma (Px)_\rho\right) \\ + \left(\frac{1}{3}C_{\mu\nu\sigma} + z(K\cdot C)_{\mu\nu\sigma}\right)(Px)_\mu (Px)_\nu (Px)_\sigma + \cdots \end{aligned} \tag{1024}$$

We now turn to the determination of the function ρ. Using the Cartesian coordinate system employed in this Appendix it is not difficult to solve for ρ in a Taylor series expansion in x_μ. Once that is achieved one can re-express the result in terms of y_μ and s using (1021) and (1024). The algebra involved is tedious and we omit all details. Our final result for ρ is

$$\begin{aligned} &\rho(x^\mu) - 1 = s(x^\mu)\frac{K(y^\mu)}{D-2} + \\ &\left(\frac{2s(x^\mu)}{K} + s(x^\mu)^2\right)\left(\frac{1}{2K}\hat{\nabla}^2\left(\frac{K}{D-2}\right) + \frac{K^2}{2(D-2)^2} + \frac{K_{MN}K^{MN}}{K}\right) \\ &+ \mathcal{O}\left(\frac{1}{(D-2)^3}\right). \end{aligned} \tag{1025}$$

4.11.10 Evolution of the Einstein Constraint Equations

In this Appendix we derive the equation (632) assuming that the dynamical Einstein equations hold everywhere.
Since all components of the Einstein equation are already linear in the metric fluctuation, in this appendix we would simply replace all covariant derivatives ∇ by partial derivatives ∂.

Now the dynamical equations are true everywhere and therefore their divergence also vanishes and we find

$$\begin{aligned} 0 &= \partial_A\left[E^A_B - n^A X_B - n_B X^A - n^A n_B Y\right] \\ &= -K\ X_B - (n\cdot\partial)X_B - X^A K_{AB} - n_B(\partial\cdot X) \\ &\quad - n_B\left[K\ Y + (n\cdot\partial Y\right] - Y(n\cdot\partial)n_B. \end{aligned} \tag{1026}$$

Simplifying (1026) further using the expression for $(\nabla\cdot X)$.

$$\begin{aligned} \partial\cdot X &= \partial_A\left[\Pi^{AC}E_{CC'}n^{c'}\right] \\ &= \partial_A\left[E^A_C n^C - n^A Y\right] \\ &= E^{AC}\partial_A n_C - \left[K\ Y + (n\cdot\partial)Y\right] \\ &= X^C(n\cdot\partial)n_C - \left[K\ Y + (n\cdot\partial)Y\right]. \end{aligned} \tag{1027}$$

Substituting equation (1027) into equation (1026) we obtain

$$
\begin{aligned}
0 &= \partial_A \hat{E}^{AB} = \partial_A \left[\Pi^{CA}\Pi^{C'}_B E_{CC'}\right] \\
&= \partial_A \left[\Pi^{CA}\Pi^{C'}_B E_{CC'}\right] \\
&= -K\ X_B - (n\cdot\partial)X_B - X^A K_{AB} - n_B\left[X^C(n\cdot\partial)n_C\right] - Y(n\cdot\partial)n_B \\
&= -K\ X_B - (n\cdot\partial)X_B - X^A K_{AB} + n_B\left[n^C(n\cdot\partial)X_C\right] - Y(n\cdot\partial)n_B \\
&= -K\ X_B - \Pi^C_B(n\cdot\partial)X_C - X^A K_{AB} - Y(n\cdot\partial)n_B.
\end{aligned}
\tag{1028}
$$

It follows that

$$
\partial\cdot X + [K\ Y + (n\cdot\partial)Y] - X^C(n\cdot\partial)n_C = 0, \tag{1029}
$$

$$
K\ X_B + \Pi^C_B(n\cdot\partial)X_C + X^A K_{AB} + Y(n\cdot\partial)n_B = 0 \tag{1030}
$$

These are the equations (632).

4.11.11 Derivation of the large D foliation adapted solution to Maxwell's equations and Charge Current

In this Appendix we present the derivation of some of the results reported in subsections 4.5.3 and 4.5.3.

$\rho > 1$ As reported in (670), the Maxwell field in the region $\rho > 1$ is assumed to take the form

$$
\mathcal{A}_M = \rho^{-(D-3)} G_M = \rho^{-(D-3)} \sum_k (\rho-1)^k G^{(k)}_M.
$$

Maxwell's equations take the form[173]

$$
0 = \partial_A F^A{}_B = 2(\partial_A \rho^{-(D-3)})(\partial^A G_B) - (\partial_B \rho^{-(D-3)})(\partial\cdot G) + \rho^{-(D-3)}\partial_A(\partial^A G_B - \partial_B G^A). \tag{1031}
$$

To derive expression for $\partial_A F^A{}_B$ in (1031) we have used the subsidiary condition (673), the gauge choice (671) and the harmonicity condition (644). Note also that

$$
n^A G_A = 0 \Rightarrow (\partial_A\partial_B\rho)G^A = -(\partial_A\rho)(\partial_B G^A). \tag{1032}
$$

[173]In this subsection all raising, lowering and contraction of indices have been done using the flat Minkowski metric η_{AB}.

It is convenient to rewrite the Maxwell equation (1031) in the form

$$T_B^{(1)} + T_B^{(2)} + T_B^{(3)} = 0, \tag{1033}$$

where

$$\begin{aligned}
T_B^{(1)} =& 2(\partial^A \rho^{-(D-3)})(\partial_A G_B) \\
=& -\frac{2(D-3)}{\rho^{D-3}}\left[\sum_{k=0}^{\infty} k\frac{(\rho-1)^{k-1}}{\rho} N^2 G_B^{(k)} + \sum_{k=0}^{\infty}\frac{(\rho-1)^k}{\rho} N(n.\partial)G_B^{(k)}\right], \\
T_B^{(2)} =& -\left(\partial_B \rho^{-(D-3)}\right)\left(\partial_A G^A\right) = \frac{(D-3)}{\rho^{D-2}}(N n_B)\sum_{k=0}^{\infty}(\rho-1)^k(\partial^A G_A^{(k)}), \\
T_B^{(3)} =& \rho^{-(D-3)}\partial_A(\partial^A G_B - \partial_B G^A) \\
=& \rho^{-(D-3)}\sum_{k=0}^{\infty}\Bigg\{ k(\rho-1)^{(k-1)}\bigg\{ \Big((n\cdot\partial)\left(N G_B^{(k)}\right) + KNG_B^{(k)} + 2Nn_B(\partial\cdot G^{(k)})\Big) \\
& - n_B(G^{(k)}\cdot\partial)N - Nn_B(\partial\cdot G^{(k)})\bigg\} + (\rho-1)^k\partial^A F_{AB}^{(k)}\Bigg\}, \\
& \text{where} \quad F_{AB}^{(k)} = \partial_A G_B^{(k)} - \partial_B G_A^{(k)}.
\end{aligned} \tag{1034}$$

We now simply plug (1034) into (1033) and equate the coefficients of distinct powers of $(\rho-1)$. As explained in the main text, at this stage we are only interested in solving the dynamical Maxwell equations (615). We find the first nontrivial constraint by equating to zero the coefficient of $(\rho-1)^0$ in the projected version ((615)) of the Maxwell equation (1033). This procedure yields the equation

$$\begin{aligned}
0 =& \rho^{(D-3)}\Pi_B^C\partial^A F_{AC} \\
=& -2(D-3)N^2 G_B^{(1)} + \left[KN + (n\cdot\partial)N\right] G_B^{(1)} + \Pi_B^C\partial^A F_{AC}^{(0)} + 2N^2 G_B^{(2)} \\
& + \mathcal{O}(\rho-1).
\end{aligned} \tag{1035}$$

Solving equation (1035) at leading order in $\left(\frac{1}{D}\right)$ we get

$$\begin{aligned}
G_B^{(1)} =& \frac{\Pi_B^C\,\partial^A F_{AC}^{(0)}}{2(D-3)N^2 - NK} + \mathcal{O}\left(\frac{1}{D}\right) \\
=& \frac{\Pi_B^C\,\partial^A F_{AC}^{(0)}}{NK} + \mathcal{O}\left(\frac{1}{D}\right).
\end{aligned} \tag{1036}$$

In the second line we have used the fact that $K = DN + \mathcal{O}(1)$.

As we have explained around (661), the solution (1036) for $G_B^{(1)}$ is only valid on the membrane i.e. at $\rho = 1$. $G_B^{(1)}$ can be determined off the membrane using (672) to evolve the result (1036) off the membrane.

Exterior current The exterior current for the solution determined above is given by

$$J_B = n_A F^A{}_B \Big|_{\rho=1} . \tag{1037}$$

In order to explicitly evaluate this current we note that

$$\begin{aligned} n_A F^A{}_B &= -(D-3)\left(\frac{\rho^{-(D-3)}}{\rho}\right) N G_B + \rho^{-(D-3)}\left[(n.\partial)G_B - n_A \partial_B G^A\right] \\ &= -(D-3)\left(\frac{\rho^{-(D-3)}}{\rho}\right) N \sum_{k=0}^{\infty} (\rho-1)^k G_B^{(k)} + \rho^{-(D-3)} \sum_{k=0}^{\infty} k(\rho-1)^{(k-1)} N G_B^{(k)} \\ &+ \rho^{-(D-3)} \sum_{k=0}^{\infty} (\rho-1)^{(k)} K^A{}_B G_A^{(k)}. \end{aligned} \tag{1038}$$

In the derivation of the last equation we have used

$$\begin{aligned} -n_A \partial_B G^A &= -\partial_B (n_A G^A) + (\partial_B n_A) G^A \\ &= \delta_B^C (\partial_C n_A) G^A \\ &= \left(\Pi_B^C + n^C n_B\right)(\partial_C n_A) G^A \\ &= K_{BA} G^A + n_B G^A (n.\partial) n_A \\ &= K^A{}_B G_A - n_B n^A (n.\partial) G_A. \end{aligned} \tag{1039}$$

Setting $\rho = 1$ in (1038) we obtain

$$J_B^{(out)} = -(D-3) N G_B^{(0)} + N G_B^{(1)} + K_B^A G_A^{(0)} \tag{1040}$$

$\rho < 1$ For $\rho < 1$ the form of the gauge field is given by

$$A_M^{(in)} = \tilde{G}_M = \sum_k (\rho-1)^k \tilde{G}_M^{(k)}.$$

Maxwell equation takes the form

$$
\begin{aligned}
\partial^A F^{(in)}_{AB} = \sum_{k=0}^{\infty} \Big[k(\rho-1)^{(k-1)} \Big\{ \tilde{G}^{(k)}_B\,(n\cdot\partial)\,N + 2N(n\cdot\partial)\tilde{G}^{(k)}_B + KN\tilde{G}^{(k)}_B + Nn_B(\partial\cdot\tilde{G}^{(k)}) \Big\} \\
+ k(k-1)(\rho-1)^{(k-2)}N^2\tilde{G}^{(k)}_B + (\rho-1)^k\partial^A\tilde{F}^{(k)}_{AB} \Big], \\
\text{where}\ \ \tilde{F}^{(k)}_{AB} = \partial_A\tilde{G}^{(k)}_B - \partial_B\tilde{G}^{(k)}_A.
\end{aligned}
\tag{1041}
$$

Here also we could determine $\tilde{G}^{(k)}_B,\quad k>0$ in terms of $\tilde{G}^{(0)}_B = G^{(0)}_B$ using equation (1041) projected in the direction perpendicular to n_B. The leading order $\tilde{G}^{(1)}_B$ could be determined from $\left(\Pi^C_B\partial^A F^{(in)}_{AC}\right)$ by setting the coefficient of $(\rho-1)^0$ to zero :

$$
[KN + (n\cdot\partial)N]\,\tilde{G}^{(1)}_B + \Pi^C_B\partial^A F^{(0)}_{AC} + N^2\tilde{G}^{(2)}_B = 0. \tag{1042}
$$

Here all lowering and raising of indices have been done using the flat metric η_{AB} .

$$
\tilde{G}^{(1)}_B = -\frac{\Pi^C_B\ \partial^A F^{(0)}_{AC}}{NK} + \mathcal{O}\left(\frac{1}{D}\right). \tag{1043}
$$

Inside current The inside current on the $\rho=1$ surface is given as

$$
J^{(in)}_B = n^A F^{(in)}_{AB}\Big|_{\rho=1}, \tag{1044}
$$

so that

$$
n^A F^{(in)}_{AB} \ = \ \sum_{k=0}^{\infty} k(\rho-1)^{(k-1)}N\tilde{G}^{(k)}_B + \sum_{k=0}^{\infty}(\rho-1)^{(k)}K^A{}_B\tilde{G}^{(k)}_A. \tag{1045}
$$

Here also to simplify we have used the equation (1039). Substituting $\rho=1$ in equation (1045) we find the inside current

$$
J^{(in)}_B = N\tilde{G}^{(1)}_B + K^A{}_B G^{(0)}_A. \tag{1046}
$$

4.11.12 Derivation of the large D foliation adapted solution to Einstein's equations

$\rho > 1$ As explained in subsection 4.5.4, the metric in the region $\rho > 1$ is assumed to take the form (698) which we repeat here for convenience

$$g_{AB} = \eta_{AB} + \rho^{-(D-3)}\mathfrak{h}_{AB} = \eta_{AB} + \rho^{-(D-3)}\sum_k (\rho-1)^k h^{(k)}_{AB}. \tag{1047}$$

Einstein's equations (linearized around η_{AB}) take the form :

$$0 = R^{(out)}_{AB} = t^{(1)}_{AB} + t^{(2)}_{AB} + t^{(3)}_{AB}, \tag{1048}$$

where

$$\begin{aligned}
t^{(1)}_{AB} &= \partial_A\left[\left(\partial_C\rho^{-(D-3)}\right)\ \mathfrak{h}^C_B + \rho^{-(D-3)}\partial_C\mathfrak{h}^C_B\right] + (A\leftrightarrow B)\\
&= \left[\partial_A\rho^{-(D-3)}\right]\left[\partial_C\mathfrak{h}^C_B\right] + \rho^{-(D-3)}\partial_A\partial_C\mathfrak{h}^C_B + (A\leftrightarrow B), \qquad (1049)\\
t^{(2)}_{AB} &= -\partial^2\left[\rho^{-(D-3)}\mathfrak{h}_{AB}\right]\\
&= -2\left[\partial_C\rho^{-(D-3)}\right]\left[\partial^C\mathfrak{h}_{AB}\right] - \rho^{-(D-3)}\partial^2\mathfrak{h}_{AB}, \qquad (1050)\\
t^{(3)}_{AB} &= -\partial_A\partial_B\left[\rho^{-(D-3)}\mathfrak{h}\right]\\
&= -[\partial_A\partial_B\rho^{-(D-3)}]\mathfrak{h} - (\partial_A\rho^{-(D-3)})\partial_B\mathfrak{h}\\
&\quad -(\partial_B\rho^{-(D-3)})\partial_A\mathfrak{h} - \rho^{-(D-3)}(\partial_A\partial_B\mathfrak{h}). \qquad (1051)
\end{aligned}$$

In deriving equations (1049), (1050) and (1051) we have used (699), (700) and (644).

We now substitute equation (1047) in equation (1048) and expand it in powers of $(\rho-1)$. Equating powers of $\rho-1$ in the dynamical equation allows us to solve for the unknown coefficients $h^{(k)}_{AB},\ \ k>0$ in terms of $h^{(0)}_{AB}$, order by order in $\left(\frac{1}{D}\right)$. In particular $h^{(1)}_{AB}$ is determined at leading order in $\left(\frac{1}{D}\right)$ by equating terms of order $(\rho-1)^0$ on both sides of the projected Einstein equation

$$\begin{aligned}
&0 = \rho^{D-3}\Pi^C_B\Pi^{C'}_A R^{(out)}_{CC'}\\
= \ &\left(\frac{D-3}{2}\right)NK_{AB}h^{(0)} + (D-3)N^2h^{(1)}_{AB} - \left(\frac{1}{2}\right)\left((n.\partial)N + NK\right)h^{(1)}_{AB}\\
&+\frac{1}{2}\Pi^C_B\Pi^{C'}_A\left[\partial_C\partial^M h^{(0)}_{MC'} + \partial_{C'}\partial^M h^{(0)}_{MC} - \partial^2 h^{(0)}_{CC'} - \partial_C\partial_{C'}h^{(0)}\right]\\
&+\mathcal{O}(\rho-1).
\end{aligned} \tag{1052}$$

Solving equation (1052) at leading order in $\left(\frac{1}{D}\right)$ we get ,

$$\begin{split}
&h^{(1)}_{AB}\\
=&-\Pi^C_B\Pi^{C'}_A\left[\frac{\partial_C\partial^M h^{(0)}_{MC'}+\partial_{C'}\partial^M h^{(0)}_{MC}-\partial^2 h^{(0)}_{CC'}-\partial_C\partial_{C'}h^{(0)}+(D-3)h^{(0)}K_{CC'}}{2(D-3)N^2-NK}\right]\\
&+\mathcal{O}\left(\frac{1}{D}\right)\\
=&-\Pi^C_B\Pi^{C'}_A\left[\frac{\partial_C\partial^M h^{(0)}_{MC'}+\partial_{C'}\partial^M h^{(0)}_{MC}-\partial^2 h^{(0)}_{CC'}-\partial_C\partial_{C'}h^{(0)}+Dh^{(0)}K_{CC'}}{NK}\right]\\
&+\mathcal{O}\left(\frac{1}{D}\right).
\end{split}\tag{1053}$$

In equation (1053) naively it seems that the last term is of order $\mathcal{O}(D)$. But we shall see that for our case $h^{(0)}$ is actually of order $\mathcal{O}\left(\frac{1}{D}\right)$, so the last two terms do not even contribute to the leading solution for $h^{(1)}_{AB}$.

External stress tensor The stress tensor T^{out}_{AB} is given by

$$T^{(out)}_{AB}=\left[\mathcal{K}^{(out)}_{AB}-\mathcal{K}^{(out)}\ \mathfrak{p}^{(out)}_{AB}\right],\tag{1054}$$

where $\mathcal{K}^{(out)}_{AB}$ is the extrinsic curvature of the $(\rho=1)$ surface (approached from the outside) viewed as a submanifold of the full space-time with bulk metric $g_{AB}=\eta_{AB}+\rho^{-(D-3)}\mathfrak{h}_{AB}$. The trace of $\mathcal{K}^{(out)}_{AB}$ is denoted by $\mathcal{K}^{(out)}$ and $\mathfrak{p}^{(out)}_{AB}$ is the projector onto the surface $(\rho=1)$. Let the normal to the surface is denoted by $n^{(out)}_A=\frac{\partial_A\rho}{\sqrt{g^{AB}(\partial_A\rho)(\partial_B\rho)}}$. It follows from the gauge condition (699) that the denominator of this expression - the norm of the one-form $\partial_A\rho$ in the metric g_{AB} - differs from the norm of the same oneform in the metric η_{AB} only at quadratic order in h_{AB}. If we work only to linear order in h_{AB} it follows that $(n^{(out)}_A=n_A)$ and also since $n_A\mathfrak{h}^{AB}=0$, it implies $g^{AB}n^{(out)}_B=g^{AB}n_B=n^A$.
It thus also follows that

$$\begin{split}
&\mathfrak{p}^{(out)}_{AB}\equiv g_{AB}-n^{(out)}_A n^{(out)}_B=g_{AB}-n_An_B=\Pi_{AB}+\rho^{-(D-3)}\mathfrak{h}_{AB},\\
&[\mathfrak{p}^{(out)}]^C_A=\delta^C_A-n^Cn_A=\Pi^C_A.
\end{split}$$

Where in the last step we have used the definition $\Pi_{AB}=\eta_{AB}-n_An_B$ and the definition $g_{AB}=\eta_{AB}+\frac{\mathfrak{h}_{AB}}{\rho^{D-3}}$.

The extrinsic curvature evaluates to [174]

$$\begin{aligned}\mathcal{K}^{(out)}_{AB} &= [\mathfrak{p}^{(out)}]^C_A[\mathfrak{p}^{(out)}]^{C'}_B \nabla_C n_{C'}|_{\rho=1}\\ &= \Pi^C_A\ \Pi^{C'}_B \left(\partial_C n_{C'} - n_q \Gamma^q_{CC'}\right)|_{\rho=1}\\ &= K_{AB} - \Pi^C_A\Pi^{C'}_B\ (n_q\Gamma^q_{CC'})|_{\rho=1},\end{aligned} \tag{1055}$$

where K_{AB} is the extrinsic curvature of $(\rho = 1)$ surface as embedded in flat Minkowski space-time η_{AB}. The last term in equation (1055) can be evaluated by determining the Christoffel symbol with respect to the metric g_{AB} to linear order in $\mathfrak{h}_{AB}$. We find

$$\begin{aligned}&-\Pi^C_A\Pi^{C'}_B\ n_q\Gamma^q_{CC'}|_{\rho=1}\\ &= -\left(\frac{\Pi^C_A\Pi^{C'}_B}{2}\right) n^q \left[\partial_C\left(\rho^{-(D-3)}\mathfrak{h}_{C'q}\right) + \partial_{C'}\left(\rho^{-(D-3)}\mathfrak{h}_{Cq}\right) - \partial_q\left(\rho^{-(D-3)}\mathfrak{h}_{C'C}\right)\right]_{\rho=1}\\ &= \left[\frac{N}{2}h^{(1)}_{AB} - \frac{N}{2}(D-3)h^{(0)}_{AB} + \frac{1}{2}\left(h^{(0)}_{Aq}K^q_B + h^{(0)}_{Bq}K^q_A\right)\right]_{\rho=1}\end{aligned} \tag{1056}$$

In the last step of equation (1056) we have used the following manipulation :

$$\begin{aligned}\Pi^{AC}\Pi^{BC'} n_q\partial_C h^q_{C'} &= -\Pi^{AC}\Pi^{BC'}\mathfrak{h}^q_{C'}(\partial_C n_q)\\ &= -\Pi^{AC}\Pi^{BC'}\mathfrak{h}^q_{C'}K_{Cq}\\ &= -h^{qB}K^A_q.\end{aligned} \tag{1057}$$

Substituting equation (1056) in equation (1055) we finally get

$$\mathcal{K}^{(out)}_{AB} = K_{AB} + \left[\frac{N}{2}h^{(1)}_{AB} - \frac{N}{2}(D-3)h^{(0)}_{AB} + \frac{1}{2}\left(h^{(0)}_{Aq}K^q_B + h^{(0)}_{Bq}K^q_A\right)\right]_{\rho=1}. \tag{1058}$$

It follows that the trace of the trace of the external extrinsic curvature is given by

$$\mathcal{K}^{(out)} = \left[\eta^{AB} - h^{AB}_{(0)}\right]\mathcal{K}^{(out)}_{AB} = K + \left[\frac{N}{2}h^{(1)} - \frac{N}{2}(D-3)h^{(0)}\right]_{\rho=1}, \tag{1059}$$

where $K = \eta^{AB}K_{AB}$ = Trace of the extrinsic curvature of $(\rho = 1)$ surface as embedded in flat space-time and $h^{(k)}$ denotes $\left[\eta^{AB}h^{(k)}_{AB}\right]$.
Note, if we assume the membrane to be embedded in an auxiliary space with metric $(\eta_{AB}+h^{(0)}_{AB})$ and denote the extrinsic curvature as $\tilde{K}_{AB}$, then $\mathcal{K}^{(out)}_{AB}$ and $\mathcal{K}^{(out)}$ could simply be written as

$$\mathcal{K}^{(out)}_{AB} = \tilde{K}_{AB} + \frac{N}{2}\left[h^{(1)}_{AB} - (D-3)h^{(0)}_{AB}\right],\quad \mathcal{K}^{(out)} = \tilde{K} + \frac{N}{2}\left[h^{(1)} - (D-3)h^{(0)}\right] \tag{1060}$$

[174] In this section ∇ means covariant derivative with respect to full linearised space-time from outside.

Substituting equations (1058), (1059) and (1060) in equation (1054) we obtain our final expression for the stress tensor from outside ($\rho = 1$) surface as given in (703).

$\rho < 1$ For $\rho < 1$ the metric is assumed to take the form (706) which we reproduce for convenience

$$\tilde{g}_{AB} = \eta_{AB} + \tilde{h}_{AB} = \eta_{AB} + \sum_k (\rho - 1)^k \tilde{h}^{(k)}_{AB}.$$

Einstein equation takes the form

$$R^{(in)}_{AB} = \left(\frac{1}{2}\right)\left[\partial_C \partial_A \tilde{h}^C_B + \partial_C \partial_B \tilde{h}^C_A - \partial^2 \tilde{h}_{AB} - \partial_A \partial_B \tilde{h}\right] = 0. \tag{1061}$$

As in the previous subsection $\tilde{h}^{(k)}_{AB}$, $k > 0$ can be determined in terms of $\tilde{h}^{(0)}_{AB} = h^{(0)}_{AB}$ using the dynamical Einstein equations. In particular $\tilde{h}^{(1)}_{AB}$ may be determined from the coefficient of $(\rho - 1)^0$ in

$$\begin{aligned} 0 = \left(\Pi^C_B \Pi^{C'}_A R^{(in)}_{CC'}\right) = &\left(\frac{\Pi^C_A \Pi^{C'}_B}{2}\right)\left[\partial^M \partial_{C'} \tilde{h}^{(0)}_{MC} + \partial^M \partial_C \tilde{h}^{(0)}_{MC'} - \partial^2 \tilde{h}^{(0)}_{CC'} - \partial_C \partial_{C'} \tilde{h}^{(0)}\right] \\ &- \left(\frac{1}{2}\right)[NK + (n \cdot \partial)N]\ \tilde{h}^{(1)}_{AB} - K_{AB}\ \tilde{h}^{(1)} + \mathcal{O}(\rho - 1) \end{aligned} \tag{1062}$$

(Here all lowering and raising of indices have been done using the flat metric η_{AB}). Solving equation (1062) in leading order in $\mathcal{O}\left(\frac{1}{D}\right)$ we find :

$$\tilde{h}^{(1)}_{AB} = \left(\frac{\Pi^C_A \Pi^{C'}_B}{NK}\right)\left[\partial^M \partial_{C'} \tilde{h}^{(0)}_{MC} + \partial^M \partial_C \tilde{h}^{(0)}_{MC'} - \partial^2 \tilde{h}^{(0)}_{CC'} - \partial_C \partial_{C'} \tilde{h}^{(0)}\right] + \mathcal{O}\left(\frac{1}{D}\right). \tag{1063}$$

Interior stress tensor The interior stress tensor is given by

$$T^{(in)}_{AB} = \mathcal{K}^{(in)}_{AB} - \mathcal{K}^{(in)} \mathfrak{p}^{(in)}_{AB}\Big|_{\rho=1}, \tag{1064}$$

where $\mathcal{K}^{(in)}_{AB}$ is the extrinsic curvature of the $\rho = 1$ surface (as approached from the interior) viewed as a submanifold of the full space-time with bulk metric $\tilde{g}_{AB} = \eta_{AB} + \tilde{h}_{AB}$. The trace of $\mathcal{K}^{(in)}_{AB}$ is denoted by $\mathcal{K}^{(in)}$ and $\mathfrak{p}^{(in)}_{AB}$ is the projector onto the surface ($\rho = 1$). As in the previous subsection, working to linear order in the metric fluctuations

$$n^{(in)}_A = n_A; \quad \mathfrak{p}^{(in)}_{AB} = \Pi_{AB} + \tilde{h}_{AB}.$$

The extrinsic curvature evaluates to

$$\begin{aligned}\mathcal{K}^{(in)}_{AB} &= [\mathfrak{p}^{(in)}]^C_A[\mathfrak{p}^{(in)}]^{C'}_B \nabla_C \hat{n}_{C'}|_{\rho=1} \\ &= \Pi^C_A\Pi^{C'}_B\left(\partial_C n_{C'} - n_q\Gamma^q_{CC'}\right)|_{\rho=1} \\ &= K_{AB} - \Pi^C_A\Pi^{C'}_B n_q\Gamma^q_{CC'}|_{\rho=1},\end{aligned} \tag{1065}$$

where K_{AB} is the extrinsic curvature of $(\rho = 1)$ surface as embedded in flat Minkowski space-time η_{AB}. The last term in equation (1065) is simplified further by evaluating the Christoffel symbol as :

$$\begin{aligned}& - \Pi^C_A\Pi^{C'}_B n_q\Gamma^q_{CC'}|_{\rho=1} \\ =& -\left(\frac{1}{2}\right)\Pi^C_A\Pi^{C'}_B n^q\left[\partial_C\tilde{h}_{C'q} + \partial_{C'}\tilde{h}_{Cq} - \partial_q\tilde{h}_{C'C}\right]_{\rho=1} \\ =& \left[\frac{N}{2}\tilde{h}^{(1)}_{AB} + \frac{1}{2}\left(\tilde{h}^{(0)}_{Aq}K^q_B + \tilde{h}^{(0)}_{Bq}K^q_A\right)\right]_{\rho=1} \\ =& \left[\frac{N}{2}\tilde{h}^{(1)}_{AB} + \frac{1}{2}\left(h^{(0)}_{Aq}K^q_B + h^{(0)}_{Bq}K^q_A\right)\right]_{\rho=1}.\end{aligned} \tag{1066}$$

Substituting equation (1066) in equation (1065) we finally get

$$\begin{aligned}\mathcal{K}^{(in)}_{AB} &= K_{AB} + \left[\frac{N}{2}\tilde{h}^{(1)}_{AB} + \frac{1}{2}\left(h^{(0)}_{Aq}K^q_B + h^{(0)}_{Bq}K^q_A\right)\right]_{\rho=1} \\ &= \tilde{K}_{AB} + \left(\frac{N}{2}\right)\tilde{h}^{(1)}_{AB}.\end{aligned} \tag{1067}$$

The trace of the extrinsic curvature is given by

$$\mathcal{K}^{(in)} = \left[\eta^{AB} - \tilde{h}^{AB}_{(0)}\right]\mathcal{K}^{(in)}_{AB} = \left[\tilde{K} + \left(\frac{N}{2}\right)\tilde{h}^{(1)}\right]_{\rho=1}, \tag{1068}$$

where $\tilde{K} = \left(\eta^{AB} - h^{AB}_{(0)}\right)\tilde{K}_{AB}$ and $\tilde{h}^{(k)}$ denotes $\left[\eta^{AB}\tilde{h}^{(k)}_{AB}\right]$.

Substituting equations (1067) and (1068) in equation (1064) we get the final expression for the stress tensor from interior of the $(\rho = 1)$ surface as given in equation (711).

4.11.13 Details Related to the Large D black hole membrane current

In this Appendix we first perform the consistency check described in subsection 4.6.3. We then go onto supply some of the algebraic details of the derivation of the final form of the charge current on the large D black hole membrane (728).

Details of the consistency check described in subsection 6.3

Gauge Transformation In this subsection we gauge transform the gauge field presented in (723) to put it in the gauge employed in subsection 4.5.3.

Let us apply a gauge transformation parametrized by the gauge function Λ on the gauge field of (723), where

$$\begin{aligned} &\Lambda = \rho^{-(D-3)}\left[\Lambda^{(0)} + (\rho-1)\Lambda^{(1)} + (\rho-1)^2\Lambda^{(2)} + \cdots\right], \\ &\tilde{M}_B = \partial_B\Lambda + M_B, \quad 0 = n^B\tilde{M}_B = n^B M_B + (n\cdot\partial)\Lambda \\ &\Rightarrow (n\cdot\partial)\Lambda = -n^B M_B. \end{aligned} \tag{1069}$$

Equating different powers of $(\rho-1)$ on both sides of the last equation in (1069) we get the following equations for $\Lambda^{(0)}$, $\Lambda^{(1)}$ and $\Lambda^{(2)}$.

$$\begin{aligned} &-(D-3)N\Lambda^{(0)} + (n\cdot\partial)\Lambda^{(0)} + N\Lambda^{(1)} = -\sqrt{2}Q \\ &-(D-3)N[\Lambda^{(1)} - \Lambda^{(0)}] + (n\cdot\partial)\Lambda^{(1)} + N\Lambda^{(2)} = -\sqrt{2}\left(\frac{D}{K}\right)\left(\frac{\bar{\nabla}^2 Q}{K}\right), \end{aligned} \tag{1070}$$

where $\bar{\nabla}^2 Q = \Pi^{AB}\partial_A\left[\Pi^C_B\partial_C Q\right]$. Solving equation (1070)

$$\begin{aligned} \Lambda^{(0)} =& \left(\frac{1}{D-3}\right)\frac{\sqrt{2}Q}{N} + \left(\frac{1}{D}\right)^2\left(\frac{\sqrt{2}}{N}\right)\left[Q + (n\cdot\partial)\left(\frac{Q}{N}\right) + \left(\frac{D}{K}\right)\left(\frac{\bar{\nabla}^2 Q}{K}\right)\right] \\ &+ \mathcal{O}\left(\frac{1}{D}\right)^3, \\ \Lambda^{(1)} =& \left(\frac{1}{D-3}\right)\left(\frac{\sqrt{2}}{N}\right)\left[Q + \left(\frac{D}{K}\right)\left(\frac{\bar{\nabla}^2 Q}{K}\right)\right] + \mathcal{O}\left(\frac{1}{D}\right)^2. \end{aligned} \tag{1071}$$

Now after applying the gauge transformation

$$\begin{aligned} \tilde{M}_B =& M_B + \partial_B\Lambda = \rho^{-(D-3)}\left[\tilde{M}^{(0)}_B + (\rho-1)\tilde{M}^{(1)}_B + \cdots\right]; \\ \tilde{M}^{(0)}_B =& -\sqrt{2}Q\ u_B + \frac{\sqrt{2}Q^3}{D}\left(\frac{D}{K}\right)\left(\frac{\partial_A K}{K} - (u\cdot\partial)u_A\right)P^A_B \\ &+ \frac{\sqrt{2}}{D}\Pi^A_B\left[\frac{\partial_A Q}{N} - \frac{Q\partial_A N}{N^2}\right] + \mathcal{O}\left(\frac{1}{D}\right)^2, \\ \tilde{M}^{(1)}_B =& -\sqrt{2}\left(\frac{D}{K}\right)\left(\frac{\bar{\nabla}^2 Q}{K}\right)u_B - \sqrt{2}Q\left(\frac{D}{K}\right)\left(\frac{\bar{\nabla}^2 u_A}{K} + u^C K_{CA}\right)p^A_B + \mathcal{O}\left(\frac{1}{D}\right), \end{aligned} \tag{1072}$$

where

$$\bar{\nabla}^2 u_A \equiv \Pi^{CB}\partial_C\left[\Pi^{A'}_A\Pi^{B'}_B\partial_{B'}u_{A'}\right], \quad \bar{\nabla}^2 Q \equiv \Pi^{AB}\partial_A\left[\Pi^{B'}_B\partial_{B'}Q\right] \tag{1073}$$

Note that $\tilde{M}_B$ now satisfies the gauge condition of the previous section, i.e., $n^B \tilde{M}_B = 0$.

Change in the subsidiary condition In this section we re-expand the coefficients of the gauge field of the previous subsection so that these coefficients obey the subsidiary conditions of subsection 4.5.3.

$\tilde{M}_B$ satisfies the gauge condition imposed in the previous section, and consequently can be identified with the field G_B of (670). However the expansion coefficients $\tilde{M}_B^{(k)}$ cannot yet be identified with the expansion coefficients $G_B^{(k)}$ of (670) as $\tilde{M}_B^{(k)}$ do not obey (673), i.e.

$$P_A^B (n \cdot \partial) \tilde{M}_B^{(k)} \neq 0.$$

The coefficient functions $G_B^{(k)}$ are easily extracted from the expansion of $\sqrt{16\pi} G_B = \tilde{M}_B$ by following a recursive procedure we now outline. On the surface $\rho = 1$, the quantity $\sqrt{16\pi} G_B{}^{(0)}$ simply equals $\tilde{M}_B^{(0)}$. Away from $\rho = 1$, $\sqrt{16\pi} G_B{}^{(0)}$ (which no longer agrees with $\tilde{M}_B^{(0)}$) can be determined from knowledge of its value on the $\rho = 1$ surface using the equation

$$P_A^B (n \cdot \partial) G_B^{(0)} = 0.$$

Now that $G_B^{(0)}$ is known everywhere consider

$$G - G_B^{(0)}$$

This expression is a known power series expansion in $(\rho - 1)$ which starts at $(\rho - 1)^1$. On the surface $\rho = 1$ the quantity $G_B^{(1)}$ is simply the coefficient of the linear term in $(\rho - 1)$ in this expansion. We have thus determined $G_B^{(1)}$ at $\rho = 1$. However this information together with the subsidiary condition

$$P_A^B (n \cdot \partial) G_B^{(1)} = 0,$$

determines $G_B^{(1)}$ everywhere.

Now that we know $G_B^{(1)}$ also everywhere consider the quantity

$$G - G_B^{(0)} - G_B^1 (\rho - 1).$$

This quantity is a known power series that starts at order $(\rho - 1)^2$. The coefficient of $(\rho - 1)^2$ is simply $G_B^{(2)}$ evaluated at $\rho = 1$ ..., and so on. We can thus proceed to evaluate $G_B^{(n)}$ for all n.

As the black hole membrane solution is known only to a very low order, we need to implement the recursive procedure described above only to very low order. This is very easily done. Clearly

$$\sqrt{16\pi}G^{(0)}_B = \tilde{M}^{(0)}_B - (\rho-1)C^{(0)}_B + \mathcal{O}(\rho-1)^2,$$

for some as yet unknown function $C^{(0)}_B$. Now the operator $P^A_B(n\cdot\partial)$ annihilates the LHS so it must also kill the RHS. Applying this operator to both sides of this equation, Taylor expanding in $\rho-1$ and equating the coefficient of $(\rho-1)^0$ to zero we find

$$C^{(0)}_A = \frac{1}{N}P^B_A(n\cdot\partial)\tilde{M}^{(0)}_B.$$

It follows that

$$\sqrt{16\pi}G_B = G^{(0)}_B + \left(\tilde{M}^{(1)}_B + C^{(0)}_B\right)(\rho-1).$$

so that on the surface $\rho = 1$

$$G^{(1)}_B = \left(\tilde{M}^{(1)}_B + C^{(0)}_B\right). \tag{1074}$$

From the explicit black hole membrane solution we know $\tilde{M}^{(1)}_B$ only to leading order in the $1/D$ expansion (though we know $\tilde{M}^{(0)}_B$ and so $C^{(0)}_B$ to first subleading order). It follows that our current knowledge of the black hole membrane solution is detailed enough only to allow to determine $G^{(1)}_B$ only at leading order in $1/D$ on the membrane surface. [175]

We now turn to simplifying the expression for $C^{(0)}_B$. Plugging in the actual value of $\tilde{M}^{(0)}_B$ for the black hole membrane we may simplify this expression as follows :

$$\begin{aligned} C^{(0)}_A &= \frac{1}{N}\Pi^B_A(n\cdot\partial)\tilde{M}^{(0)}_B \\ &= -\frac{\sqrt{2}Q}{N}\Pi^B_A(n\cdot\partial)u_B + \mathcal{O}\left(\frac{1}{D}\right) \\ &= -\frac{\sqrt{2}Q}{N}P^B_A(n\cdot\partial)u_B + \mathcal{O}\left(\frac{1}{D}\right) \\ &= \frac{\sqrt{2}Q}{N}P^B_A(u\cdot\partial)n_B + \mathcal{O}\left(\frac{1}{D}\right) \\ &= \frac{\sqrt{2}Q}{N}u^C K_{CB}\; P^B_A + \mathcal{O}\left(\frac{1}{D}\right), \end{aligned} \tag{1075}$$

where we have plugged in the explicit expressions listed in (1072). In the second and last line of equation (1075) we have used the fact that the membrane charge density and velocity field in

[175] As explained above, once $G^{(1)}_B$ has been determined on the surface $\rho = 1$ it is easily continued away from this surface. We will, however, have no need for this continuation.

chapter 3 obey the subsidiary conditions

$$(n\cdot\partial)Q=0,\quad P_A^B(n\cdot\partial)u_B+p_A^B(u\cdot\partial)n_B=0.$$

From equation (1072) and (1075) it is not difficult to read off the values of $\tilde{M}_B^{(1)}$ and $C_B^{(0)}$. Using

$$K=DN+\mathcal{O}(1)\quad\text{and}\quad\frac{\bar{\nabla}^2u_A}{K}=\frac{P_A^B\bar{\nabla}^2u_B}{K}+\mathcal{O}\left(\frac{1}{D}\right),$$

we find

$$\begin{aligned}\sqrt{16\pi}G_B^{(0)}&=-\sqrt{2}Q\ u_B+\frac{\sqrt{2}Q^3}{D}\left(\frac{D}{K}\right)\left(\frac{\partial_AK}{K}-(u\cdot\partial)u_A\right)p_B^A\\&\quad+\frac{\sqrt{2}}{D}\Pi_B^A\left[\frac{\partial_AQ}{N}-\frac{Q\partial_AN}{N^2}\right]+\mathcal{O}\left(\frac{1}{D}\right)^2,\\\sqrt{16\pi}G_B^{(1)}&=\left[\tilde{M}_B^{(1)}+C_B^{(0)}\right]=-\sqrt{2}\left(\frac{D}{K}\right)\left(\frac{\bar{\nabla}^2Q}{K}\right)u_B-\sqrt{2}Q\left(\frac{D}{K}\right)\left(\frac{\bar{\nabla}^2u_A}{K}\right)+\mathcal{O}\left(\frac{1}{D}\right).\end{aligned}\tag{1076}$$

Consistency In the previous subsubsections we have transformed the linearized part of the large D black gauge field into gauge and subsidiary conditions used in subsubsection 4.5.3, and have thus managed to read off the expressions for the quantities $G_B^{(0)}$ and $G_B^{(1)}$ listed in that subsection. However, according to the analysis of subsubsection 4.5.3 the quantities $G_B^{(0)}$ and $G_B^{(1)}$ are not independent. In fact $G_B^{(1)}$ is given in terms of $G_B^{(0)}$ by the equations (680) and (682).

In other words the linearized part of the large D black hole metric is fits into the general framework of subsubsection 4.5.3 if and only if the explicit results (1076) obey (680) upto corrections of order $\mathcal{O}\left(\frac{1}{D}\right)$. We have explicitly verified that this is indeed the case. This completes our check of the consistency of the large D black hole solutions at linearized order.

Details of the derivation of equation 6.13

The Membrane Current from Outside Now that we have recast the solution (723) in the form of the solutions presented in subsection 4.5.3 we can use any of the formulae of that subsection to evaluate the membrane current. The external contribution to the current, J^{out}, is most simply obtained from the equation (683) which we quote again here for convenience

$$J_B^{out}=-(D-3)NG_B^{(0)}+NG_B^{(1)}+K_B^AG_A^{(0)}.$$

Substituting $G^{(0)}_B$ and $G^{(1)}_B$ from equation (1076) we find that upto corrections of order $\mathcal{O}\left(\frac{1}{D}\right)$,

$$\begin{aligned}\sqrt{16\pi}J^{out}_B = &\sqrt{2}Q\left[(1-Q^2)\left(\frac{\partial_A K}{K}\right) + (1+Q^2)(u\cdot\partial)u_A - \left(\frac{\bar{\nabla}^2 u_A}{K}\right) - \ K^C_A u_C\right]P^A_B \\ &+ \sqrt{2}\left[(D-3)NQ + (u\cdot\partial)Q - \left(\frac{\partial^2 Q + Q(u\cdot\partial)K}{K}\right) + Q(u\cdot K\cdot u)\right]u_B \\ &- \sqrt{2}Q\left[\left(\frac{\partial_A Q}{Q}\right) + (u\cdot\partial)u_A\right]P^A_B + \mathcal{O}\left(\frac{1}{D}\right).\end{aligned} \tag{1077}$$

In the next subsection we shall see that the first line in the final expression of J^{out}_B (the third step) vanishes as consequence of the stress tensor conservation equation on the membrane. So the final form of the outside current after removing the first line

$$\begin{aligned}\sqrt{16\pi}J^{out}_B = \sqrt{2}\Bigg[&Q\left(K + \frac{\bar{\nabla}^2 K}{K^2} - \frac{2K}{D}\right) + (u\cdot\partial)Q \\ &- \left(\frac{\bar{\nabla}^2 Q + Q(u\cdot\partial)K}{K}\right) + Q(u\cdot K\cdot u)\Bigg]u_B \\ &- \sqrt{2}Q\left[\left(\frac{\partial_A Q}{Q}\right) + (u\cdot\partial)u_A\right]P^A_B + \mathcal{O}\left(\frac{1}{D}\right).\end{aligned} \tag{1078}$$

To simplify in equation (1078) we have used the identities (see equations (1110), (1111), (1112), (1113) and (1114) for derivation) that

$$(D-3)N = K + \frac{\bar{\nabla}^2 K}{K^2} - \frac{2K}{D} + \mathcal{O}\left(\frac{1}{D}\right).$$

The membrane current from inside In order to compute J^{in}_B we use (690) which we quote here again for convenience

$$J^{in}_B = N\tilde{G}^{(1)}_B + K^A_B G^{(0)}_A. \tag{1079}$$

By comparing (688) and (680) we see that it is a general feature of the solutions obtained in subsections 4.5.3 and 4.5.3 that

$$\tilde{G}^{(1)}_B = -G^{(1)}_B + \mathcal{O}\left(\frac{1}{D}\right).$$

It follows that (1079) can be rewritten as

$$J^{in}_B = -NG^{(1)}_B + K^A_B G^{(0)}_A. \tag{1080}$$

Using (1076) it follows that

$$\sqrt{16\pi}J_B^{(in)} = \sqrt{2}\left[\left(\frac{\bar{\nabla}^2 Q}{K}\right)u_B + Q\left(\frac{P_B^A\bar{\nabla}^2 u_A}{K}\right) - QK_B^A\ u_A\right] + \mathcal{O}\left(\frac{1}{D}\right). \tag{1081}$$

4.11.14 Details Related to the large D black hole Membrane Stress Tensor

This Appendix mirrors the previous one except for the fact that it focuses on the membrane stress tensor rather than the charge current. In the first part of this Appendix we check that the large D black hole metrics - upon linearization - do indeed fit into the general structure of linearized solutions to Einstein's equations at large D developed in this chapter. In the second part of the Appendix we provided details of our computation of the precise form of the large D black hole stress tensor.

Consistency As we have described above, the large D black hole metric of chapter 3 simplifies in the 'matching' region to the linearized form (737) with (738). For the convenience of the reader we reproduce those equations here:

$$G_{AB} = \eta_{AB} + \rho^{-(D-3)}M_{AB} = \eta_{AB} + \rho^{-(D-3)}\sum_n(\rho-1)^n M_{AB}^{(n)}, \tag{1082}$$

where

$$\begin{aligned}
M_{AB}^{(0)} &= (1+Q^2)O_AO_B + 2Q^4\left(O_AV_B^{(2)} + O_BV_A^{(2)}\right) - Q^2O_AO_B - 2Q^2\tau_{AB} \\
&\quad + \mathcal{O}\left(\frac{1}{D}\right)^2, \\
M_{AB}^{(1)} &= 2Q^2S^{(1)}O_AO_B - (1+Q^2)\left[V_A^{(1)}O_B + O_AV_B^{(1)}\right] \\
&\quad + \mathcal{O}\left(\frac{1}{D}\right),
\end{aligned} \tag{1083}$$

with

$$
\begin{aligned}
V_A^{(1)} &= \left(\frac{D}{K}\right)\left[\frac{\bar{\nabla}^2 u_B}{K} + u^C K_{CB}\right]\ P_A^B, \\
V_A^{(2)} &= \left(\frac{D}{K}\right)\left[\frac{\partial_C K}{K} - (u\cdot\partial)u_C\right] P_A^C, \\
S^{(1)} &= \left(\frac{D}{K^2}\right)\bar{\nabla}^2 Q, \\
\tau_{AB} &= P_A^{A'}\ \left(\frac{D}{K}\right)\left[\frac{\partial_{A'}O_{B'} + \partial_{B'}O_{A'}}{2} - \eta_{A'B'}\left(\frac{\partial\cdot O}{D-2}\right)\right]\ P_B^{B'}, \\
&\text{where} \\
\bar{\nabla}^2 Q &= \Pi_B^A \partial_A\left[\Pi^{BC}\partial_C Q\right],\quad \bar{\nabla}^2 u_A = \Pi_{AA'}\Pi_C^B\partial_B\left[\Pi^{CC'}\Pi^{A'A''}(\partial_{C'}u_{A''})\right].
\end{aligned}
\tag{1084}
$$

In this section we will recast the results (1082) and (1083) into the general form obtained subsection 4.5.4. As in the previous Appendix, this requires us to perform first a coordinate (gauge) transformation on the solution (1082), (1083). We then read off the expansion coefficients of the general solution described in subsection 4.5.4 by imposing the subsidiary conditions defined in that subsection.

Gauge transformation Starting with the solution (1082) and (1083) we perform the infinitesimal coordinate transformation

$$x_A \to x^A + \rho^{-(D-3)}\xi^A,$$

which recasts the solution into the form

$$\tilde{M}_{AB} = M_{AB} + \rho^{(D-3)}\partial_A\left(\rho^{-(D-3)}\xi_B\right) + \partial_B\left(\rho^{-(D-3)}\xi_A\right). \tag{1085}$$

We wish to choose our coordinate transformation to ensure that $\hat{h}_{ab}$ satisfies the gauge condition of subsubsection 4.5.4, namely

$$n^A\tilde{M}_{AB} = 0. \tag{1086}$$

It follows that the infinitesimal coordinate transformation must be chosen to ensure that

$$-n^A M_{AB} = (n\cdot\partial)\left[\rho^{-(D-3)}\xi_B\right] + n^A\partial_B\left[\rho^{-(D-3)}\xi_A\right]. \tag{1087}$$

Our general strategy for determining the vector field ξ_A that satisfied (1087) is to assume that like h_{AB}, the vector ξ_A generating the coordinate transformation also admits an expansion in the powers of $(\rho-1)$:

$$\xi_A = \sum_{m=0}^{\infty} (\rho-1)^m \xi_A^{(m)}. \tag{1088}$$

We then substitute the expansion equations (737) and (1088) into (1087) and determine the expansion coefficients $\xi^{(m)}$, order by order in the $\left(\frac{1}{D}\right)$ expansion by equating powers of $(\rho-1)$ on both sides for the equation (1087).

For the practical purposes of this chapter we only need to implement this programme to the first couple of orders. Equating the coefficient of $(\rho-1)^0$ on both sides of equation (1087) we find

$$\begin{aligned} n^A M_{AB}^{(0)} = {} & (D-3)N\left[\xi_B^{(0)} + n_B(n\cdot\xi^{(0)})\right] - \left[(n\cdot\partial)\xi_B^{(0)} + n^A\partial_B\xi_A^{(0)}\right] \\ & - N\left[\xi_B^{(1)} + n_B(n\cdot\xi^{(1)})\right]. \end{aligned} \tag{1089}$$

Similarly equating the coefficient of $(\rho-1)^1$ we find

$$\begin{aligned} n^A M_{AB}^{(1)} = {} & (D-3)N\left[(\xi_B^{(1)} - \xi_B^{(0)}) + n_B(\xi_A^{(1)} - \xi_A^{(0)})n^A\right] \\ & - \left[(n\cdot\partial)\xi_B^{(1)} + n^A\partial_B\xi_A^{(1)}\right] - 2N\left[\xi_B^{(2)} + n_B(n\cdot\xi^{(2)})\right]. \end{aligned} \tag{1090}$$

Solving equation (1089) and (1090) simultaneously we find ,

$$\begin{aligned} \xi_A^{(1)} = {} & \left[\frac{1}{(D-3)N}\right]\left[n^B[M_{AB}^{(1)} + M_{AB}^{(0)}] - \left(\frac{n_A}{2}\right)\left(n\cdot[M^{(1)} + M^{(0)}]\cdot n\right)\right] \\ & + \mathcal{O}\left(\frac{1}{D}\right), \\ \xi_A^{(0)} = {} & \xi_A^{(0,1)} + \left(\frac{1}{D}\right)\xi_A^{(0,2)} + \left(\frac{1}{D}\right)^2, \\ & \text{where} \\ \xi_A^{(0,1)} = {} & \left[\frac{D}{(D-3)N}\right]\left[n^B M_{AB}^{(0)} - \left(\frac{n_A}{2}\right)\left(n\cdot[M^{(0)}]\cdot n\right)\right], \\ \xi_A^{(0,2)} = {} & \left[\frac{1}{(D-3)N}\right]\left[n^B\left(\partial_A\xi_B^{(0,1)} + \partial_B\xi_A^{(0,1)}\right) - n_A\left(n^C[\partial_C\xi_{C'}^{(0,1)}]n^{C'}\right)\right] \\ & + \left[\frac{D}{(D-3)^2N^2}\right]\left[n^B[M_{AB}^{(1)} + M_{AB}^{(0)}] - \left(\frac{n_A}{2}\right)\left(n\cdot[M^{(1)} + M^{(0)}]\cdot n\right)\right]. \end{aligned} \tag{1091}$$

After substituting equation (1091) in equation (1085) we find

$$\tilde{M}_{AB} = \Pi_A^C\Pi_B^{C'}\left[M_{CC'}^{(0)} + \partial_C\xi_{C'}^{(0)} + \partial_{C'}\xi_C^{(0)} + (\rho-1)M_{CC'}^{(1)} + \mathcal{O}(\rho-1)^2\right]. \tag{1092}$$

Change in subsidiary condition In order to extract the expansion coefficients $h^{(m)}_{MN}$ defined in subsection 4.5.4 we need to 'Taylor' expand the metric (1092) in a power series expansion in ρ while ensuring that the Taylor coefficients of this expansion obey the subsidiary conditions (700). This is easily accomplished using the method outlined in the previous Appendix for the case of the gauge field. Let $\tilde{M}^{(j)}_{MN}$ represent the expansion coefficients of the metric (1092) where these coefficients don't necessarily obey the subsidiary condition (700), i.e.

$$\Pi^C_A \Pi^{C'}_B \ (n \cdot \partial) \ \tilde{M}^{(k)}_{CC'} \neq 0.$$

It must be that

$$h^{(0)}_{AB} = \tilde{M}^{(0)}_{AB} - (\rho - 1) C^{(0)}_{AB} + \mathcal{O}(\rho - 1)^2,$$

for some as yet unknown function $C^{(0)}_{AB}$. Now the operator $\Pi^{C'}_A \Pi^C_B (n \cdot \partial)$ annihilates the LHS so it must also kill the RHS. Applying this operator to both sides of this equation, Taylor expanding in $\rho - 1$ and equating the coefficient of $(\rho - 1)^0$ to zero we find

$$C^{(0)}_{AB} = \frac{1}{N} \Pi^{C'}_A \Pi^C_B \ (n \cdot \partial) \tilde{M}^{(0)}_{CC'}.$$

It follows that

$$h_{AB} = h^{(0)}_{AB} + \left(\tilde{M}^{(1)}_{AB} + C^{(0)}_{AB} \right) (\rho - 1),$$

so that on the surface $\rho = 1$

$$h^{(1)}_{AB} = \left(\tilde{M}^{(1)}_{AB} + C^{(0)}_{AB} \right). \tag{1093}$$

Using equations (1092) and (1093) it follows that the coefficients $h^{(0)}_{AB}$ and $h^{(1)}_{AB}$ corresponding to metric (1092) are given by :

$$\begin{aligned}
h^{(0)}_{AB} &= (1 + Q^2) \ u_A u_B \\
&\quad + \left(\frac{1}{D} \right) \Bigg[- 2Q^4 \left(u_A V^{(2)}_B + u_B V^{(2)}_A \right) - Q^2 u_A u_B - 2Q^2 \ \tau_{AB} \\
&\qquad\qquad + \Pi^C_A \left[\nabla_C \xi_{C'} + \nabla_{C'} \xi_C \right] \Pi^{C'}_B \Bigg] + \mathcal{O} \left(\frac{1}{D} \right)^2, \\
h^{(1)}_{AB} &= \left(\frac{D}{K^2} \right) \left[2Q \bar{\nabla}^2 Q \ u_A u_B + (1 + Q^2) \ \Pi^C_B \Pi^{C'}_A \left(u_{C'} \bar{\nabla}^2 u_C + u_C \nabla^2 u_{C'} \right) \right] \\
&\quad + \mathcal{O} \left(\frac{1}{D} \right),
\end{aligned} \tag{1094}$$

where

$$\begin{aligned}
\xi_A &= (1+Q^2)\left(\frac{D}{K}\right)\left(\frac{n_A}{2} - u_A\right), \\
V_A^{(2)} &= \left(\frac{D}{K}\right)\left[\frac{\partial_C K}{K} - (u.\partial)u_C\right] P^C_A, \\
\tau_{AB} &= \left(\frac{D}{K}\right) P^C_A \Bigg[K_{CD} - \left(\frac{\partial_C u_D + \partial_D u_C}{2}\right) \\
&\qquad\qquad - \eta_{CD}\left(\frac{K-(\partial\cdot u)}{D-3}\right)\Bigg] P^D_B.
\end{aligned} \tag{1095}$$

Here $\Big[p_{AB} = \eta_{AB} - n_A n_B + u_A u_B\Big]$ and ∇ denotes covariant derivative with respect to the intrinsic metric on the membrane as embedded in flat space.
Note that the trace of $h^{(0)}_{AB}$ vanishes till order $\mathcal{O}(1)$ in our $\left(\frac{1}{D}\right)$ expansion.

$$\begin{aligned}
\therefore h^{(0)} = \eta^{AB} h^{(0)}_{AB} &= -(1+Q^2) + \frac{\Pi^{AB}\nabla_A \xi_B}{D} + \mathcal{O}\left(\frac{1}{D}\right) \\
&= -(1+Q^2) + 2\left(\frac{1+Q^2}{D}\right)\left(\frac{D}{K}\right)\nabla_A\left(\frac{n^A}{2} - u^A\right) + \mathcal{O}\left(\frac{1}{D}\right) \\
&= \mathcal{O}\left(\frac{1}{D}\right).
\end{aligned} \tag{1096}$$

Consistency As in the previous Appendix, it is not difficult to verify that the second equation in (1094) is consistent with (701) upto corrections of order $\mathcal{O}\left(\frac{1}{D}\right)$.

Derivation of equation 6.36

$$
\begin{aligned}
E \equiv\ & u^{\mu}\hat{\nabla}_{\nu}[T^{(NT)}]^{\nu}_{\mu} \\
=\ & \left(\frac{K}{2}\right)(1+Q^2)(\hat{\nabla}\cdot u) + \left(\frac{1+Q^2}{2}\right)(u\cdot\hat{\nabla})K + \left(\frac{K}{2}\right)(u\cdot\hat{\nabla})Q^2 \\
& - \left(\frac{1-Q^2}{2}\right) u_{\mu}\hat{\nabla}_{\nu}K^{\mu\nu} + u_{\nu}\hat{\nabla}_{\mu}\left(\frac{\hat{\nabla}^{\nu}u^{\mu}+\hat{\nabla}^{\mu}u^{\nu}}{2}\right) - (\hat{\nabla}\cdot\mathcal{V}) + \mathcal{O}(1) \\
=\ & \left(\frac{K}{2}\right)(1+Q^2)(\hat{\nabla}\cdot u) + Q^2(u\cdot\hat{\nabla})K + \left(\frac{K}{2}\right)(u\cdot\hat{\nabla})Q^2 \\
& + K(u^{\alpha}K_{\alpha\beta}u^{\beta}) - (\hat{\nabla}\cdot\mathcal{V}) + \mathcal{O}(1) \\
=\ & \left(\frac{K}{2}\right)(1+Q^2)(\hat{\nabla}\cdot u) - (1+Q^2)(u\cdot\hat{\nabla})K + \left(\frac{K}{2}\right)(u\cdot\hat{\nabla})Q^2 \\
& - Q\hat{\nabla}^2Q - \left(\frac{2Q^4-Q^2-1}{2}\right)\left(\frac{\hat{\nabla}^2K}{K}\right) + \left(1+\frac{Q^2+2Q^4}{2}\right)K\ (u^{\alpha}K_{\alpha\beta}u^{\beta}) \\
& + \mathcal{O}(1).
\end{aligned}
\tag{1097}
$$

In the second last line we have used identities (1103) and (1107). In the last line we have used identity (1109).
Now we could simplify equation (1097) further by using the current conservation equation equation (735). For convenience we are quoting the equation here.

$$
\hat{\nabla}^2Q = QK(\hat{\nabla}\cdot u) + K(u\cdot\hat{\nabla})Q + Q(u\cdot\hat{\nabla})K - QK(u^{\alpha}K_{\alpha\beta}u^{\beta}) + \mathcal{O}(1). \tag{1098}
$$

Substituting equation (1098) in equation (1097) we find

$$
\begin{aligned}
E =\ & -\left(\frac{1+2Q^2}{2}\right)\left[2(u\cdot\hat{\nabla}K) - (1-Q^2)\left(\frac{\hat{\nabla}^2K}{K}\right) - (1+Q^2)K(u^{\alpha}K_{\alpha\beta}u^{\beta})\right]. \\
& + \left(\frac{K}{2}\right)(1-Q^2)(\hat{\nabla}\cdot u) + \mathcal{O}(1)
\end{aligned}
\tag{1099}
$$

Now we shall show that the term in the first line of equation (1099) could be re-expressed as $\left[-\left(\frac{1+2Q^2}{K}\right)(\hat{\nabla}_{\mu}E^{\mu})\right]$, where E^{μ} is the projection of stress tensor conservation equation in the direction perpendicular to u^μ.

$$E^A = -\left(\frac{K}{2}\right)\Bigg[(1+Q^2)(u\cdot\nabla)u^A + (1-Q^2)p^{AC}\left(\frac{\nabla_C K}{K}\right) \\ - p^{AC}\left(\frac{\nabla^2 u_C}{K} + K_{CB}u^B\right)\Bigg] + \mathcal{O}(1).$$

Taking the divergence of the above equation we find

$$\begin{aligned}\hat{\nabla}_\mu E^\mu = &-\left(\frac{K}{2}\right)\hat{\nabla}_\mu\Bigg[(1+Q^2)(u\cdot\hat{\nabla})u^\mu + (1-Q^2)p^{\mu\nu}\left(\frac{\hat{\nabla}_\nu K}{K}\right) \\ &- p^{\mu\nu}\left(\frac{\hat{\nabla}^2 u_\nu}{K} + K_{\nu\alpha}u^\alpha\right)\Bigg] + \mathcal{O}(D) \\ = &-\left(\frac{K}{2}\right)\Bigg[(1+Q^2)K(u^\alpha K_{\alpha\beta}u^\beta) + (1-Q^2)\left(\frac{\hat{\nabla}^2 K}{K}\right) - 2(u\cdot\hat{\nabla})K\Bigg] + \mathcal{O}(D).\end{aligned} \tag{1100}$$

Here in the last line we have used identities (1103), (1108) and (1105). Substituting equation (1100) in equation (1099) we get equation (751).

4.11.15 Identities

In this appendix we shall prove several identities and equations that we have used at different steps in our calculations.

Membrane embedded in flat-spacetime In this subsection all identities are derived on $\rho = 1$ hypersurface as embedded in flat space-time. Usually all contractions (often denoted by '$\cdot$') are with respect to flat Minkowski metric η_{AB}. In few cases we have to use contraction and covariant derivative with respect to the induced metric on the membrane. In those cases we have used Greek indices and the covariant derivatives are denoted as $\hat{\nabla}$. Sometimes we have used $\bar{\nabla}_A$ to denote $\hat{\nabla}$ in the language of the embedding space. For example,

$$\hat{\nabla}_\mu u_\nu \to \bar{\nabla}_A u_B \equiv \Pi_A^{A'}\Pi_B^{B'}\nabla_{A'}u_{B'},$$

where Π_{AB} is the projector on the membrane.[176]

[176] Most of the identities that are derived here involve indices, functions and derivatives that are defined entirely along the membrane. Therefore they could be very easily re expressed in the language of the intrinsic geometry of the membrane, (by simply replacing $\bar{\nabla} \to \hat{\nabla}$, $\{A, B\} \to \{\mu, \nu\}$, $\Pi_{AB} \to g^{(ind,f)}_{\mu\nu}$). In the main text we have often used these identities with such replacement.

Identity-1:

$$\begin{aligned}
&\hat{\nabla}_\mu \left[(u^\nu \hat{\nabla}_\nu) u^\mu\right] \\
&= \partial_B[\Pi^{AB}(u\cdot\partial)u_A] - n_B(n\cdot\partial)[\Pi^{AB}(u\cdot\partial)u_A] \\
&= \partial_B\left[(u\cdot\partial)u^B\right] - \partial_B\left[n^B n^C(u\cdot\partial)u^c\right] + \mathcal{O}(1) \\
&= \partial_B\left[(u\cdot\partial)u^B\right] + \partial_B\left[n^B u^C(u\cdot\partial)n^c\right] + \mathcal{O}(1) \\
&= \partial_B\left[(u\cdot\partial)u^B\right] + \partial_B\left[n^B u^C(u\cdot\partial)n^c\right] + \mathcal{O}(1) \\
&= (u\cdot\partial)\left[\partial\cdot u\right] + (\partial_A u^B)(\partial_B u^A) + \partial_B\left[n^B(u^A u^{A'} K_{AA'})\right] + \mathcal{O}(1) \\
&= K(u^A u^{A'} K_{AA'}) + \mathcal{O}(1).
\end{aligned} \tag{1101}$$

Here $\hat{\nabla}_\mu$ denotes covariant derivative with respect to the induced metric on the membrane as embedded in the flat space, $g^{(ind,f)}_{\mu\nu}$.

Identity-2:

$$\begin{aligned}
n^A\partial^2 u_A &= \partial_C(n^A\partial^C u_A) + \mathcal{O}(1) \\
&= -\,\partial_C(u^A\partial^C n_A) + \mathcal{O}(1) \\
&= -\,\partial_C(n^C u^k(n\cdot\partial)n_k + K^C_A u^A) + \mathcal{O}(1) \\
&= -\,(u\cdot\partial)K - \partial_A(K^C_A u^A) + \mathcal{O}(1) \\
&= -\,(u\cdot\partial)K - \partial_A(K^C_A)u^A + \mathcal{O}(1) \\
&= -\,2(u\cdot\partial)K + \mathcal{O}(1).
\end{aligned} \tag{1102}$$

Identity-3:

$$\begin{aligned}
&\Pi^{A'}_A\partial_{A'}\left[K^{AB} - K\Pi^{AB}\right] = 0 \\
&\Rightarrow \Pi^{A'}_A\partial_{A'}K^{AB} = \Pi^{AB}\partial_A K.
\end{aligned} \tag{1103}$$

Identity-4:

$$\begin{aligned}
&u_A\bar{\nabla}^2 u^A = -\Pi^{BB'}(\partial_B u_A)(\partial_{B'}u^A) = \mathcal{O}(1), \\
&\text{since } \Pi^{AB}\partial_A u_B \sim \mathcal{O}\left(\frac{1}{D}\right).
\end{aligned} \tag{1104}$$

Here $\bar{\nabla}^2 u_A$ denotes the following.

$$\bar{\nabla}^2 u_A \equiv \Pi^{A'}_A \Pi^{BB'}\partial_B\left(\Pi^{B''}_{B'}\Pi^{C}_{A'}\partial_{B''}u_C\right).$$

Identity-5:

$$
\begin{aligned}
\bar\nabla_A \bar\nabla^2 u^A &= \Pi^A_{A'}\partial_A \left[\Pi^{A'A''}\bar\nabla^2 u_{A''}\right] \\
&= -K n^A \bar\nabla^2 u_A + \mathcal{O}(D) \\
&= -K \left[\partial_B(n^A\partial^B u_A) - (\partial_B n^A)\partial^B u_A)\right] + \mathcal{O}(D) \\
&= -K \left[\partial_B(n^A\partial^B u_A)\right] + \mathcal{O}(D) \\
&= K \left[\partial_B(u^A\partial^B n_A)\right] + \mathcal{O}(D) \\
&= K \left[\partial_B(K^B_A u^A)\right] + \mathcal{O}(D) \\
&= K \left[(u\cdot\partial)K\right] + \mathcal{O}(D).
\end{aligned}
\tag{1105}
$$

In the last line we have used identity (1103).

Identity-6:

$$
\begin{aligned}
&\Pi^{B'}_B\partial_{B'}\left[p^{AB}Q\left(\frac{\bar\nabla^2 u_A}{K} - K^C_A u_C\right)\right] \\
&= Q\ \Pi^{B'}_B\partial_{B'}\left[p^{AB}\left(\frac{\bar\nabla^2 u_A}{K} - K^C_A u_C\right)\right] + \mathcal{O}(1) \\
&= \mathcal{O}(1).
\end{aligned}
\tag{1106}
$$

Here p_{AB} denotes the projector perpendicular to both n_A and u_A.

$$
p_{AB} = \eta_{AB} - n_A n_B + u_A u_B.
$$

In the last step of equation (1106) we have used the identities (1102), (1103), (1104) and (1105).

Identity-7:

$$
\begin{aligned}
\bar\nabla_A \bar\nabla_B u^A &\equiv \Pi^{B'}_B \Pi^A_{A'}\partial_A\left[\Pi^{A'A''}\Pi^{B''}_{B'}\left(\partial_{B''}u_{A''}\right)\right] \\
&= -K\left[\Pi^{B'}_B n^A \partial_{B'} u_A\right] + \mathcal{O}(1) \\
&= K\left[\Pi^{B'}_B u^A \partial_{B'} n_A\right] + \mathcal{O}(1) \\
&= K(u^A K_{BA}) + \mathcal{O}(1).
\end{aligned}
\tag{1107}
$$

Identity-8:

$$
\begin{aligned}
\bar\nabla_A (u\cdot\bar\nabla)u^A &\equiv \Pi^{A'}_A\partial_{A'}\left[\Pi^{AA''}(u^B\partial_B)u_{A''}\right] \\
&= -K\ n^A(u\cdot\partial)u_A + \mathcal{O}(1) \\
&= K\ (u\cdot K\cdot u) + \mathcal{O}(1).
\end{aligned}
\tag{1108}
$$

Identity-9:

$$
\begin{aligned}
\mathcal{V}_A = Q\ \Pi^B_A \partial_B Q + Q^2(u^C K_{CA}) + \left(\frac{2Q^4 - Q^2 - 1}{2}\right)\left(\frac{\Pi^B_A \partial_B K}{K}\right) \\
- \left(\frac{Q^2 + 2Q^4}{2}\right)(u \cdot \partial) u_A + \left(\frac{1 + Q^2}{K}\right)\bar{\nabla}^2 u_A.
\end{aligned}
\tag{1109}
$$

$$
\begin{aligned}
\therefore \Pi^{AB}\partial_A \mathcal{V}_B = Q\bar{\nabla}^2 Q + (1 + Q^2)(u \cdot \partial)K + \left(\frac{2Q^4 - Q^2 - 1}{2}\right)\left(\frac{\bar{\nabla}^2 K}{K}\right) \\
- \left(\frac{Q^2 + 2Q^4}{2}\right) K\ (u^A u^B K_{AB}) + \mathcal{O}(1).
\end{aligned}
$$

Here $\bar{\nabla}^2 Q$ and $\bar{\nabla}^2 K$ denote

$$
\bar{\nabla}^2 Q = \Pi^{AB}\partial_A \partial_B Q, \qquad \bar{\nabla}^2 K = \Pi^{AB}\partial_A \partial_B K.
$$

In the last line of (1109) we have used identities (1103), (1108) and (1105).

Identity-10:

$$
\begin{aligned}
& \partial^2 \rho^{-(D-3)} = 0 \\
& \Rightarrow \partial_A \left[\rho^{-(D-2)} N n^A\right] = 0 \\
& \Rightarrow KN - \frac{(D-2)N^2}{\rho} + (n \cdot \partial)N = 0 \\
& \Rightarrow KN - (D-2)N^2 + (n \cdot \partial)N = 0 \qquad \because\ \rho = 1 \\
& \Rightarrow (D-3)N = K - N + \frac{(n \cdot \partial)N}{N} \\
& \Rightarrow (D-3)N = K - \frac{K}{D} + \frac{(n \cdot \partial)K}{K} + \mathcal{O}\left(\frac{1}{D}\right).
\end{aligned}
\tag{1110}
$$

Identity-11

$$
\begin{aligned}
\partial_A N = \frac{\partial_A\left[(\partial_B \rho)(\partial^B \rho)\right]}{2N} = \frac{(\partial^B \rho)\partial_A \partial_B \rho}{N} \\
= \frac{(\partial^B \rho)\partial_B \partial_A \rho}{N} = (n \cdot \partial)(N n_A) \\
\Rightarrow\ (n \cdot \partial) n_A = \frac{\Pi^B_A \partial_B N}{N} = \frac{\Pi^B_A \partial_B K}{K} + \mathcal{O}\left(\frac{1}{D}\right).
\end{aligned}
\tag{1111}
$$

Identity-12:

$$
\begin{aligned}
(n\cdot\partial)K &= n^A\partial_A\partial_B n^B \\
&= n^A\partial_B\partial_A n^B \\
&= \partial_B\left[(n\cdot\partial)n^B\right] - (\partial_B n^A)(\partial_A n^B) \\
&= \partial_B\left(\frac{\Pi^{BA}\partial_A K}{K}\right) - K_{AB}K^{AB} \\
&= \frac{\bar{\nabla}^2 K}{K} - \frac{K^2}{D} + \mathcal{O}(1).
\end{aligned}
\tag{1112}
$$

Here in the last line we have used identity (1111). Combining (1110). (1111) and (1112) we find

Identity-13:

$$
(D-3)N = K + \left(\frac{\bar{\nabla}^2 K}{K^2}\right) - 2\left(\frac{K}{D}\right) + \mathcal{O}\left(\frac{1}{D}\right). \tag{1113}
$$

Identity-14:

$$
\begin{aligned}
\partial^2 Q &= \partial_A\left(\Pi^{AB}\partial_B Q\right) \qquad \because\ (n\cdot\partial)Q = 0 \\
&= \bar{\nabla}^2 Q + \mathcal{O}(1).
\end{aligned}
\tag{1114}
$$

Relating intrinsic and extrinsic curvature of membrane with curvature of embedding space-time Here we shall relate the intrinsic curvatures of a timelike membrane with the extrinsic curvature of the membrane and the curvatures of the full space-time. For our derivation we shall follow [82].

Define the coordinates along the full-space time as

$$
\{X^A\} \equiv \{\rho, x^\mu\}, \quad A = \{1,2,\cdots,D\}, \quad \mu = \{2,\cdots,D\}
$$

The equation of the membrane is given by $(\rho = 1)$. $\{x^\mu\}$ are the coordinates that can vary along the membrane. The unit normal to the surface is denoted as n_A .

Suppose ω_A is a vector tangent to the membrane. $\hat{\nabla}_A$ denotes the covariant derivative with respect to the intrinsic metric of the membrane and ∇_A denotes the covariant derivative with respect to the full space-time metric. It follows that

$$
\begin{aligned}
[\hat{\nabla}_A, \hat{\nabla}_B]\omega_C &= \mathcal{R}^P{}_{CBA}\,\omega_P \\
[\nabla_A, \nabla_B]\omega_C &= R^P{}_{CBA}\,\omega_P,
\end{aligned}
\tag{1115}
$$

where $\mathcal{R}^P{}_{CBA}$ denotes the intrinsic Riemann tensor of the membrane and $R^P{}_{CBA}$ is the Riemann tensor of the full space-time. We shall use $\mathfrak{p}_{AB}$ as the projector on the membrane surface.

$$
\begin{aligned}
&\hat{\nabla}_A \hat{\nabla}_B \omega_C \\
&= \mathfrak{p}_A^{A'} \mathfrak{p}_B^{B'} \mathfrak{p}_C^{C'} \nabla_{A'} \left(\mathfrak{p}_{B'}^{B''} \mathfrak{p}_{C'}^{C''} \nabla_{B''} \omega_{C''} \right) \\
&= \mathfrak{p}_A^{A'} \mathfrak{p}_B^{B'} \mathfrak{p}_C^{C'} \nabla_{A'} \nabla_{B'} \omega_{C'} + \mathfrak{p}_A^{A'} \mathfrak{p}_B^{B'} \mathfrak{p}_C^{C'} \nabla_{A'} \left(\mathfrak{p}_{B'}^{B''} \mathfrak{p}_{C'}^{C''} \right) (\nabla_{B''} \omega_{C''}) \\
&= \mathfrak{p}_A^{A'} \mathfrak{p}_B^{B'} \mathfrak{p}_C^{C'} \nabla_{A'} \nabla_{B'} \omega_{C'} + \mathcal{K}_{AC} \mathcal{K}_{BC'} \omega^{C'} - \mathcal{K}_{AB} \left[(n \cdot \nabla) \omega_{C'} \right] \mathfrak{p}_C^{C'} .
\end{aligned}
\tag{1116}
$$

Here in the last line we have used the fact that $n^C \omega_C = 0$
Using equations (1115) and (1116) we find

$$
\mathcal{R}_{PCBA}\, \omega^P = \mathfrak{p}_A^{A'} \mathfrak{p}_B^{B'} \mathfrak{p}_C^{C'} R_{PC'B'A'}\, \omega^P + \left[\mathcal{K}_{AC} \mathcal{K}_{BP} - \mathcal{K}_{AP} \mathcal{K}_{BC} \right] \omega^P . \tag{1117}
$$

Since equation (1117) is true for any ω^P we find

$$
\mathcal{R}_{PCBA} = \mathfrak{p}_A^{A'} \mathfrak{p}_B^{B'} \mathfrak{p}_C^{C'} R_{PC'B'A'} + \left[\mathcal{K}_{AC} \mathcal{K}_{BP} - \mathcal{K}_{AP} \mathcal{K}_{BC} \right] . \tag{1118}
$$

Contracting equation (1118) with $\mathfrak{p}^{AC}$ and $\mathfrak{p}^{AC} \mathfrak{p}^{BP}$ we find

$$
\begin{aligned}
&\mathfrak{p}_A^C \ \mathfrak{p}_B^{C'} R_{CC'} = \mathcal{R}_{AB} - \mathcal{K} \mathcal{K}_{AB} + \mathcal{K}_{AC} \mathcal{K}_B^C + R_{AkBk'}\ n^k n^{k'} , \\
&R = \mathcal{R} + 2 R_{CC'}\ n^C n^{C'} - \mathcal{K}^2 + \mathcal{K}_{AB} \mathcal{K}^{AB} .
\end{aligned}
\tag{1119}
$$

Note that the second equation of (1119) could be rewritten as

$$
\left[R_{CC'} - \frac{R}{2} G_{CC'} \right] n^C n^{C'} \equiv n^C n^{C'} \mathcal{E}_{CC'} = -\mathcal{R} + \mathcal{K}^2 - \mathcal{K}_{AB} \mathcal{K}^{AB} . \tag{1120}
$$

Note also that for Ricci flat geometries equation (1119) reduces to

$$
\begin{aligned}
&0 = \mathcal{R}_{AB} - \mathcal{K} \mathcal{K}_{AB} + \mathcal{K}_{AC} \mathcal{K}_B^C + R_{AkBk'}\ n^k n^{k'} \\
&\therefore 0 = \mathcal{R} - \mathcal{K}^2 + \mathcal{K}_{AB} \mathcal{K}^{AB} .
\end{aligned}
\tag{1121}
$$

5 Conclusion

The work presented in this Thesis is divided into two parts each addressing a different problem. In the first part more light is shed on the famous level-rank duality between two theories, namely a Chern-Simons theory coupled with critical bosons and a Chern-Simons theory coupled with fundamental fermions in the 't Hooft large N limit. The work deals with the duality between the S-matrices of these two theories, and while doing that it also produces a surprise, namely the modification of the conventional channel crossing symmetry in these theories due to a non-analyticity in one of the scattering channels, which has anyonic character. Although this modification is only conjectured on the basis of the unitarity requirement, it has found multiple evidences. A special nonrelativistic limit of the S-matrix in this anyonic channel exactly reduces to the self-adjoint extension of the Aharonov- Bohm scattering matrix, which is reported in this thesis. In addition, a later work on Supersynnetric Chern-Simons theory ($\mathcal{N} = 1$ and $\mathcal{N} = 2$) has revealed that a similar conjecture has to be made in these theories as well. However, it will be really interesting to find the version of this conjecture for finite N and k theories, as well as in the supersymmetric theories with bifundamental matter.

The second part is about 'a membrane paradigm at large D', which states that in large number of spacetime dimensions D the black hole dynamics reduces to the dynamics of a codimenison-1 membrane in a flat spacetime with the same number of dimensions. This dynamics is governed by the 'membrane equations of motion', which are actually the equations of conservation of a stress tensor and a charge current defined on this membrane. This stress tensor and charge current is coupled to the gravitational and electromagnetic radiation that this membrane emits, and this radiation is nonperturbatively small. It would be amazing if this program is extended to the processes like a collision of two black holes and some general lessons can be learnt out of it.

My own work

Since no part of my work is a single author publication (By putting 'we' and 'our' all over the place I wasn't just being modest, I was being honest), the list of my own direct contribution to the work presented in this thesis is as follows.

Chapter 1:

- Setting up and solving the Euclidean Schwinger-Dyson equation to obtain the bosonic 4-point function
- Taking the onshell limit in different channels to find the bosonic S-matrices
- Setting up and solving the Euclidean Schwinger-Dyson equation to obtain the fermionic

4-point function

- Taking the onshell limit in different channels to find the fermionic S-matrices
- Checking the duality between the bosonic and fermionic S-matrices in non-anyonic channels

Chapter 2:

- Taking the near threshold limit of the conjectured S-matrix
- Solving the Schrodinger equation with self-adjoint boundary condition and verifying the result with [22]
- Comparing the two results

Chapter 3:

- Converting the metric, gauge field and membrane equations into geometric form
- Determining the light quasinormal spectrum of RN black hole

Chapter 4:

- Determining and analyzing Greens function in general dimensions
- Constructing membrane current and stress tensor from linearized solution
- Showing that membrane entropy current is proportional to the velocity field
- Analysing spherical harmonics in general dimensions
- Deriving the formulae for radiation in general dimensions and relating it to the sources

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