Non-Gaussianities in the Cosmic Microwave Background

A Thesis

Submitted to the
Tata Institute of Fundamental Research, Mumbai
for the degree of Master of Philosophy
in Physics

by
Ishan Mata

Department of Theoretical Physics
Tata Institute of Fundamental Research, Mumbai

March, 2014
Final version submitted in August 2018

DECLARATION

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

The work was done under the guidance of Professor Sandip Trivedi, at the Tata Institute of Fundamental Research, Mumbai.

Ishan Mata

In my capacity as supervisor of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

Professor Sandip Trivedi
Date :

ACKNOWLEDGEMENTS

First of all I am highly thankful to my advisor Prof. Sandip Trivedi for patiently guiding me throughout the work. His guidance and support has been vital for my understanding of the subject. I am thankful to him and Dr. Suvrat Raju, with whom this work was carried out, for guidance, collaboration and useful discussions.

I would like to thank all the faculty members of the Dept. of Theoretical Physics. I am especially indebted to Prof. Gautam Mandal, Prof. Amol Dighe and Prof. Saumen Datta who helped me on several occasions. I consider myself to be fortunate to have had the opportunity to interact with knowledgeable faculty members like Prof. Gautam Mandal, Prof. Sunil Mukhi and Prof. Shiraz Minwalla in various courses and learn from them.

It has been a nice experience interacting with students of DTP and other departments and I feel grateful to all of them. I am especially thankful to my friends Joe Philip Ninan, Girish Kulkarni, Rickmoy Samanta and Tathamay Basu for their support. The DTP office staff and the computer administrators have always been very cooperative and approachable, I wish to take this opportunity to acknowledge their help.

I am thankful to the organizers of the ICTS 'School on Cosmology and Gravity Waves', Dec. 2011 held in Pune which added to my understanding of the subject.

Synopsis

S.1 Introduction

Observations of the sky show that the Universe is isotropic on large scales. A plausible additional assumption of homogeneity allows us to write down the metric as the FRW metric

$$d\tau^2 = dt^2 - a^2(t)[\mathbf{dx}^2 + K\frac{(\mathbf{x}.\mathbf{dx})^2}{1 - K\mathbf{x}^2}] \tag{S.1.1}$$

where $K = \pm 1$ or 0. The evolution of the scale factor is governed by the energy and pressure density of the Universe. The redshift data shows that the scale factor $a(t)$ has been increasing with time. Very far in the past the scale factor was much smaller and the energy density of the Universe was bigger. The Universe was radiation dominated (composed primarily of relativistic electrons and photons) and $a(t)$ grew as $a(t) \propto t^{\frac{1}{2}}$. As the Universe expanded, it cooled down and processes like nucleosynthesis and atom formation took place. The major contribution to the energy density was then from the nonrelativistic matter and the scale factor varied as $a(t) \propto t^{\frac{2}{3}}$. In the radiation dominated era there were frequent collisions between electrons and photons that kept them in mutual equilibrium. As the Universe expanded the rate of collisions decreased and was no longer sufficient to keep matter and radiation in equilibrium. Using the Thomson relation for scattering cross-section we can see that at $10^5 K$ the photons stopped exchanging energies of order kT with the electrons. However the rate of scattering was still quite large and each photon would be scattered many times in each doubling of the scale factor though without appreciable exchange of energy. As temperature dropped further the scattering cross section dropped due to formation of atoms and roughly at about 3000K the photons no longer interacted with the matter appreciably. This is called the time of last scattering and denoted by t_L. The present value of the radiation temperature is about 2.75K.
The CMB spectrum is remarkably uniform in all directions apart from some anisotropies of the order 1 part in 10^5. These anisotropies arise as a result of fluctuations in the spacetime metric at the time of last scattering. Suppose that there is a position dependent fluctuation in the metric at that time. Then photons coming from different locations will have experienced different amounts of redshift and thus shall have slightly different frequencies. This has the effect of shifting the temperature corresponding to the radiation spectrum and is observed as the anisotropies in the sky.
The remarkable isotropy of the CMB poses a problem however. The horizon size at the time of last scattering can be computed to be $d_H \approx H_0^{-1}(1+z_L)^{-\frac{3}{2}}$

where $H \equiv \frac{\dot{a}}{a}$, the subscript '0' indicates that the present value is to be taken and z_L is the redshift at last scattering defined by $1 + z_L = \frac{a(t_0)}{a(t_L)}$. Also the *angular diameter distance* of the surface of last scattering $d_A \equiv r_L a(t_L)$ is of the order of $H_0^{-1}(1 + z_L)^{-1}$. This means that the angle subtended by the horizon at the time of last scattering is roughly $(1 + z_L)^{-\frac{1}{2}}$ or about 1.6 degrees. Thus regions of the sky separated by more than this value of the angle should have been causally disconnected and no physical influence could have smoothened out the original fluctuations in the CMB explaining the remarkable isotropy. The hypothesis of inflation not only solves this problem but also enables us to make quantitative predictions about the anisotropies.

The hypothesis of inflation says that the Universe underwent an era of exponential expansion during which the Hubble parameter was constant prior to the radiation dominated era. If this took place for sufficiently long time the horizon size at the time of last scattering would have been large enough so as to account for the observed isotropy. Furthermore the quantum correlation functions of various perturbations can be calculated in simple models of inflation and tested against the observations. For more detailed explanations see, for example, [1]. In the simplest model inflation is driven by a single scalar field ϕ with two derivative gravity

$$S = \int d^4x \sqrt{-g} \frac{1}{16\pi G}[R - \frac{1}{2}(\nabla\phi)^2 - V(\phi)]. \tag{S.1.2}$$

In the slow roll models of inflation potential function $V(\phi)$ is chosen so that it is nearly flat and allows for a nearly constant Hubble rate till the scalar field begins to couple to other fields. The three point correlation functions in this and several other models have been determined by explicit computation, see e.g. [2].

One of the reasons why inflation is interesting is that it not only explains why the CMB is so uniform but gives quantitative predictions about the anisotropies. Furthermore if the Hubble scale was as large as the scale of the underlying fundamental theory, higher derivative corrections to (S.1.2) would become important. Thus measurement of these correlators can possibly provide us clues about the higher derivative terms. One can systematically apply the methods of effective field theory to this problem as done in [3]. However the methods of effective field theory are applicable only if the Hubble scale is lesser than the mass scale of the fundamental theory. This need not necessarily be the case in practice. However there is another way to know the correlation functions. The idea is that during inflation the Universe was approximately described by de-Sitter metric which has a large symmetry group SO(1,4). We shall refer to this group as the conformal group. The correlation functions must be invariant under the action of this group as both the

de-Sitter geometry and the initial conditions enjoy this symmetry which is respected under time evolution due to general covariance of the action. This holds even if the Hubble scale is larger than the mass scale of the fundamental theory when the effective field theory approach is no longer valid. This approach has been employed by [4] to constrain the three point graviton correlator. In the thesis we consider the constraints of conformal symmetry on the correlation function involving two scalars and one graviton . This is more subtle than the correlator considered by Maldacena because the scalar degree of freedom becomes a pure gauge transformation in the exact de-Sitter case. This has to be taken into account in the analysis in a careful manner.
The use of conformal invariance has been made to constrain correlation functions of operators in 3-dimensional Euclidean CFT in the position space [5]. For the purposes of cosmology the natural choice is to work in momentum space however. The position space answers can not be used to get the momentum space result simply by taking Fourier transforms as the resulting integrals are divergent and can not be regulated in an obvious way while preserving conformal invariance. Furthermore a related issue is to take care of the contact terms. Our approach avoids this issue by doing all analysis entirely in the momentum space from the beginning.

We now describe the setup of our problem. First we decompose the fluctuations about a given background metric into its rotational tensor components. Thereafter we discuss possible choices of gauges.

S.2 Set-Up

We consider inflation driven by a single scalar field with the action

$$S = \int d^4x\sqrt{-g}\frac{1}{16\pi G}[R - \frac{1}{2}(\nabla\phi)^2 - V(\phi) + \cdots]. \tag{S.2.1}$$

The ellipses represent higher derivative corrections. As discussed previously, during inflation the Universe is approximately described by de Sitter metric

$$ds^2 = -dt^2 + a^2(t)\sum_{i=1}^{3} dx_i dx^i, \tag{S.2.2}$$

$$a^2 = e^{2Ht}, \tag{S.2.3}$$

where H is the constant Hubble scale. The deviation from de-Sitter is captured in terms of the two parameters measuring the slow variation of the Hubble scale

$$\epsilon = -\frac{\dot{H}}{H^2}, \delta = \frac{\ddot{H}}{2H\dot{H}}, \tag{S.2.4}$$

where dot denotes derivative with respect to t. During inflation both these parameters are small and meet the slow roll conditions

$$\epsilon, \delta \ll 1. \tag{S.2.5}$$

We now examine the symmetry group of de-Sitter space. It is generated by the scale transformation :

$$x^i \to \lambda x^i, t \to t - \frac{1}{H}\log(\lambda), \tag{S.2.6}$$

three translations and rotations in the x^i coordinates and three special conformal transformations whose infinitesimal version is given by

$$x^i \to x^i - 2(b_j x^j)x^i + b^i(\sum_j (x^j)^2 - e^{-2Ht}), \tag{S.2.7}$$

$$t \to t + 2b_j x^j. \tag{S.2.8}$$

(where $b^i, i = 1, \ldots, 3$ are infinitesimal parameters.) We can see by a straightforward computation that these transformations keep the metric invariant. Note that the inflaton sector need not necessarily preserve the full conformal symmetry group.

Now we discuss the general theory of small fluctuations about a given space-time background metric. The metric is the sum of an unperturbed background metric and perturbations about the same. In general the equations governing the perturbations are quite complicated. However the perturbations can be decomposed into the rotational tensor components. The equations governing these tensor modes are relatively simpler. Having done so we fix the gauge so as to eliminate the unphysical degrees of freedom.

There is still a subtlety here. While we know the evolution of the scale factor during the period of inflation, radiation dominance and matter dominance, we do not know much about the Universe when inflation ended and it was undergoing the 'reheating' stage where the inflaton field coupled to matter and gave away its energy to it. Even if one knows the initial fluctuations produced during inflation, how could one compute the fluctuations at the time of last scattering when the equations governing the dynamics in the intervening reheating era are not known ? The answer to this question lies in a theorem due to Weinberg which says that in single field inflation the low momentum modes that were outside the horizon during the reheating era were 'frozen'. Some of these modes entered back into the horizon after the reheating era was over and the evolution of the scale factor was well known.

Let us define the fluctuations by $g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + h_{\mu\nu}$ where $\bar{g}_{\mu\nu}$ denotes the unperturbed metric. Decomposing the fluctuations as sum of rotational tensors we write

$$h_{00} = -E \tag{S.2.9}$$

$$h_{i0} = a\left[\frac{\partial F}{\partial x^i} + G_i\right] \tag{S.2.10}$$

$$h_{ij} = a^2\left[A\delta_{ij} + \frac{\partial^2 B}{\partial x_i \partial x_j} + \frac{\partial C_i}{\partial x_j} + \frac{\partial C_j}{\partial x_i} + D_{ij}\right] \tag{S.2.11}$$

where $\frac{\partial C_i}{\partial x_i} = \frac{\partial G_i}{\partial x_i} = 0$, $\frac{\partial D_{ij}}{\partial x^i} = 0$, $D_{ii} = 0$, $D_{ij} = D_{ji}$. However we have not yet fixed the gauge. In a theory of gravity alone the coordinate invariance (gauge freedom) can be used to gauge away the scalar and vector degrees of freedom. The remaining degrees of freedom are the tensor modes which represent physical gravitons. In our problem there is another scalar field i.e. the inflaton and thus one more degree of freedom compared to the pure gravity case. One can choose to work in a gauge where the inflaton field is set to zero and the scalar component is a dynamical variable or in a gauge where the scalar part of the metric fluctuation is chosen to be zero and the inflaton field is a dynamical variable. In the first gauge the perturbations in the inflaton vanish,

$$\delta\phi = 0. \tag{S.2.12}$$

The fluctuations are defined using the ADM formalism as in [2]

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{S.2.13}$$

the additional coordinate freedom can be fixed by choosing a gauge where

$$h_{ij} = a^2[(1 + 2\zeta)\delta_{ij} + \gamma_{ij}], \tag{S.2.14}$$

and γ_{ij} is required to be transverse and traceless,

$$\partial_i \gamma_{ij} = \gamma_{ii} = 0, \tag{S.2.15}$$

as discussed in [2]. The scalar and tensor perturbations are denoted by ζ and γ_{ij} respectively.

Alternatively, as mentioned above, we can choose to set ζ to vanish instead of $\delta\phi$. This second gauge is obtained by starting with the first gauge, and carrying out a time re-parameterization

$$t \to t + \frac{\zeta}{H}. \tag{S.2.16}$$

This makes ζ vanish while γ_{ij} remains unchanged. If the background value of the inflaton is

$$\phi = \bar{\phi}(t), \tag{S.2.17}$$

the resulting value for the perturbation $\delta\phi$ this gives rise to is

$$\delta\phi = -\frac{\dot{\phi}\zeta}{H}. \tag{S.2.18}$$

Assuming that the two derivative approximation is good we can express this relation as

$$\delta\phi = -\sqrt{2\epsilon}\zeta. \tag{S.2.19}$$

In our analysis we first work in the second gauge, where the leading effects of the slow-roll parameters can be incorporated and then transform to the first gauge, around the time when the mode crosses the horizon as ζ is the variable which is meaningful at all times and constant outside the horizon.

In the single field slow roll case with two derivative theory, the calculation of the three point functions of these fluctuations was carried out in [2]. In a more recent paper [4] the conformal symmetries were used to constrain the form of the three point tensor correlation function. In the thesis we ask the following question : What are the constraints imposed by the isometries of the de-Sitter space (i.e. the conformal symmetries) on the correlation function of one tensor and two scalar modes in the models of inflation which preserve the full symmetry group of the de-Sitter space ? Of course, as the scalar fluctuations become pure gauge transformations in the exact de-Sitter case, the answer that we seek refers to the leading contribution in the slow roll parameters.

S.3 Methodology and Results

To approach this problem we use rotational and translational invariance to write down the most general form of the correlator as

$$\begin{aligned}\langle O(\boldsymbol{k}_1)O(\boldsymbol{k}_2)T_{ij}(\boldsymbol{k}_3)\rangle &= [k_{1i}k_{1j}f_1(k_1,k_2,k_3) + k_{2i}k_{2j}f_1(k_2,k_1,k_3) \\ &\quad + (k_{1i}k_{2j} + k_{2i}k_{1j})f_2(k_1,k_2,k_3) \\ &\quad + \delta_{ij}f_3(k_1,k_2,k_3)](2\pi)^3\delta^3(\sum_i \boldsymbol{k}_i) \\ &\equiv M_{ij}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)(2\pi)^3\delta(\sum_i \boldsymbol{k}_i).\end{aligned} \tag{S.3.1}$$

The overall delta function is a consequence of translational invariance. We then derive the constraints of special conformal transformations and find them to be (after simplification)

$$[\boldsymbol{k}_3\cdot\boldsymbol{k}_1\Theta(k_1) + (\boldsymbol{k_3}\cdot\boldsymbol{k_2})\Theta(k_2) + k_3^2\Theta(k_3)]S(k_1,k_2,k_3) = 0, \tag{S.3.2}$$

$$-(\boldsymbol{k_2}\cdot\boldsymbol{k_3})k_1\partial_{k_1}S+(\boldsymbol{k_1}\cdot\boldsymbol{k_3})k_2\partial_{k_2}S-(k_1^2-k_2^2)S+\frac{3}{2}(k_1^3-k_2^3)=0, \tag{S.3.3}$$

and

$$(\Theta(k_1)-\Theta(k_2))S=0, \tag{S.3.4}$$

where

$$S(k_1,k_2,k_3)=\frac{1}{2}[f_1(k_1,k_2,k_3)+f_1(k_2,k_1,k_3)-2f_2(k_1,k_2,k_3)]. \tag{S.3.5}$$

We then check that these are indeed satisfied by the answer computed by Maldacena in the special case of single field slow roll two derivative theory. Thereafter we attempt to find the most general solution of these equations. This is found to be

$$\begin{aligned} S=\sum_{n_1,n_2=\pm1} m_{n_1n_2}\Big(&-n_1n_2\frac{k_2k_3k_1}{(n_1k_1+n_2k_2+k_3)^2}+n_1k_1+n_2k_2+k_3 \\ &-\frac{n_1n_2k_1k_2+n_2k_3k_2+n_1k_1k_3}{n_1k_1+n_2k_2+k_3}\Big). \end{aligned} \tag{S.3.6}$$

where the $m_{n_1n_2}$ satisfy

$$\sum_{n_1,n_2} m_{n_1,n_2}n_1^3 \;=\; 1, \tag{S.3.7}$$

$$\sum_{n_1,n_2} m_{n_1,n_2}n_2^3 \;=\; 1. \tag{S.3.8}$$

Only one of the terms, the one with both n_1 and n_2 equal to $+1$, behaves consistently in the limits when one momentum is much smaller than the other two. The normalization of our solutions is fixed by the Ward identity. We thus conclude that the answer for the correlation functions in terms of the ζ variable is given by

$$\langle\zeta(k_1)\zeta(k_2)\gamma_s(k_3)\rangle=(2\pi)^3\delta(\sum_i\boldsymbol{k}_i)\frac{1}{\Pi_i(2k_i^3)}(\frac{4H^4}{M_{pl}^4c})(\frac{H^2}{\dot{\phi}^2})e^{s,ij}k_{1i}k_{2j}S(k_1,k_2,k_3), \tag{S.3.9}$$

with

$$S(k_1,k_2,k_3)=(k_1+k_2+k_3)-\frac{\sum_{i>j}k_ik_j}{(k_1+k_2+k_3)}-\frac{k_1k_2k_3}{(k_1+k_2+k_3)^2}. \tag{S.3.10}$$

and

$$\gamma_{ij}(k_3)=\gamma_s(k_3)e^s_{ij}(k_3), \tag{S.3.11}$$

where $e^s_{ij}(k_3)$ is the transverse and traceless polarization and c is a normalization defined through the normalization conventions :

$$\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})\rangle = ck_1^3(2\pi)^3\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2}), \qquad \text{(S.3.12)}$$

$$\langle T^s(\boldsymbol{k_1})T^{s'}(\boldsymbol{k_2})\rangle = k_1^3(2\pi)^3\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2})(\frac{\delta^{ss'}}{2}). \qquad \text{(S.3.13)}$$

S.4 Conclusions

In this work we have studied the three point tensor-scalar-scalar correlator in single field slow roll models of inflation allowing for higher derivative corrections but assuming that the full conformal symmetry is preserved by the scalar sector as well. The analysis is based on the idea that the symmetries of the unperturbed metric can be used to constrain the correlation functions of the small perturbations about it. These techniques were originally employed by Maldacena [4] for his study of the three point graviton correlator. We find that our correlator is completely determined by the symmetries of de-Sitter space in such models. The assumptions that go into this derivation are that (i) the inflation is driven by a single scalar field , (ii) both the scalar and tensor sectors preserve the full isometry group of de-Sitter and that (iii) the Universe was in the Bunch-Davies vacuum to start with. Other than these three assumptions the analysis is model independent and applies to a large class of models. In particular the analysis applies even if the Hubble scale was comparable to the scale of the underlying fundamental theory when the methods of effective field theory are not valid. We note that it is possible to have models of inflation where translations, rotations and scale invariance are preserved but special conformal invariance is broken. Our analysis does not apply to these models. This work shows that the three point correlator considered here is a good way to test if special conformal invariance was preserved during inflation.

Note : The work outlined in this synopsis was done under the guidance of Prof. Sandip Trivedi and in collaboration with him and Dr. Suvrat Raju. This has been uploaded on arxiv hep-th/1211.5482 and submitted to Journal of High Energy Physics.

References

[1] S. Weinberg, *Cosmology.* Oxford University Press, 2008.

[2] J. M. Maldacena, *Non-Gaussian features of primordial fluctuations in single field inflationary models*, *JHEP* **0305** (2003) p. 013, [astro-ph/0210603].

[3] S. Weinberg, *Effective Field Theory for Inflation*, *Phys.Rev.* **D77** (2008) p. 123541, [arXiv:0804.4291].

[4] J. M. Maldacena and G. L. Pimentel, *On graviton non-Gaussianities during inflation*, arXiv:1104.2846.

[5] H. Osborn and A. Petkou, *Implications of conformal invariance in field theories for general dimensions*, *Annals Phys.* **231** (1994) pp. 311–362, [hep-th/9307010].

List of Publications

1. I.Mata, S.Raju and S.Trivedi, *CMB from CFT, JHEP* **1307** (2013) p. 15 [arxiv:1211.5482]

Contents

1 Introduction

It is consistent with observations to assume that our Universe is homogeneous and isotropic. These assumptions allow us to constrain the spacetime metric to the form $d\tau^2 = dt^2 - a^2(t)[\mathbf{dx}^2 + K\frac{(\mathbf{x}.\mathbf{dx})^2}{1-K\mathbf{x}^2}]$ where $K = \pm 1$ or 0 and $a(t)$ is called the scale factor. In order to further determine the metric we need to know the constituents of the Universe and use the Einstein equations. The very early Universe consisted of hot relativistic matter. Plugging in the appropriate stress energy tensor in Einstein equations implies that the scale factor increased as $a(t) \propto t^{\frac{1}{2}}$. As the Universe expanded it cooled and the constituent particles slowed down. In this non-relativistic matter dominated era, the scale factor varied as $a(t) \propto t^{\frac{2}{3}}$. This history of the Universe predicts a finite Horizon size. Calculations show that after a certain time, called the time of last scattering, t_L the photons have moved relatively freely and the gravitational redshift of these photons carries information about the fluctuations in the metric at the time of last scattering.

Observations find that the radiation temperature is remarkably isotropic. This raises a puzzle : Why is it that the temperatures of the radiation coming from causally separated regions so remarkably uniform ? This is called the 'Horizon problem' in cosmology and is addressed by the idea of inflation. Inflation states that our Universe underwent a period of exponentially rapid expansion in its early history. This makes the Horizon size much bigger than as computed in non-inflationary theories thereby solving the Horizon problem. What is particularly attractive is that the same exponential expansion also results in small quantum perturbations being produced which account for the observed anisotropies of the microwave background and also provide the seed perturbations for the growth of large scale structure in the Universe.

The exponentially expanding Universe during inflation is well described by the metric of de Sitter space, up to small corrections. It is well known that de Sitter space is a maximally symmetric spacetime. In four dimensions the group of isometries of de Sitter space is $SO(1,4)$ — the Lorentz group in $4+1$ dimensional flat spacetime. This large group of symmetries has ten generators, which include translations and rotations along the three space directions, scale transformations, and the three generators of special conformal transformations. We will refer to it as the conformal group below.

So far, the experimental tests of inflation, coming for example from the study of the CMB, have shown that the perturbations can be well approximated as being Gaussian. The good news is that future experiments, with improved sensitivity, will be able to probe and possibly detect evidence for non-Gaussianity in these perturbations. For example, it is hoped that the Planck experiment will be able to provide significant constraints of this sort quite soon.

A Gaussian distribution is completely determined by its two point correlation function. Any non-Gaussianity in the perturbations can therefore be characterized by the three point or higher point correlations. Considerable attention has been paid in the recent literature to the three-point function, called the bispectrum; there is also a growing body of literature on the four point function, called the trispectrum. We can refer to [1, 2, 3, 4] for a review of these developments and to [5] for background material.

There are two kinds of perturbations of the metric that are relevant for inflation: these transform as scalars and spin-2 representations of the rotation group, and are called scalar and tensor perturbations respectively. In addition each perturbation is characterized by a value for the spatial three-momentum. It is easy to see that the momentum dependence of the two-point function of the perturbations is simple and is fixed, up to small corrections, by the approximate scale invariance of de Sitter space. One the other hand, it is well known that the momentum dependence of the three point functions can be much more complicated. For example, various different shapes which characterize this momentum dependence have been obtained for the three point scalar correlation function in different models of inflation. (See [1, 6] and references there.)

The symmetries of de Sitter space need not be shared by the scalar sector in general. This happens for example in DBI inflation [7, 8] where the non-canonical kinetic energy term for the inflaton results in a speed of sound $c_s \neq 1$.[1] As a result, while scaling symmetry is preserved, the inflaton sector breaks special conformal invariance badly. Here we will assume that the full conformal group is approximately preserved by the inflationary dynamics, including both gravity and the inflaton field, and examine the resulting constraints imposed on three point functions.

In particular, we will focus on the three point function involving two scalar perturbations $\zeta(\boldsymbol{k})$ and one tensor perturbation $\gamma_{ij}(\boldsymbol{k})$, with polarization $e^{s,ij}$, denoted by,

$$\langle \zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\gamma_{ij}(\boldsymbol{k}_3)\rangle e^{s,ij}. \tag{1.0.1}$$

We will show that this correlator is completely fixed by symmetry considerations.[2] Its overall normalization is determined in terms of the two point functions of the scalar and tensor perturbations, and its momentum dependence is determined by the $SO(1,4)$ symmetry group. It turns out that the special conformal transformations play an especially important role in our

[1]Another example where the scalar sector violates special conformal symmetries is ghost inflation [9].

[2]A complete complete definition of the perturbations etc. is given in section 2.

analysis. They give rise to differential equations for the correlation function whose solution is essentially unique leading to the conclusion above. In the absence of special conformal invariance in the full theory, including the inflaton sector, our results for the correlator are not valid.

Our analysis applies to models with only one scalar field during inflation. It also assumes that the initial state was the Bunch-Davies vacuum.[3] Beyond that, our analysis only relies on the conformal group and is essentially model independent. In particular, our results also apply to models where higher derivative corrections are important and gravity or the scalar field is not well described by the two-derivative approximation. In the context of string theory, such a situation would arise if the Hubble scale H during inflation was of order the string scale M_{st}. Present bounds on H coming, for example, from the absence of any observed effects due to tensor perturbations tell us that $H <\sim 10^{16}$Gev $< M_{Pl}$. So, for example, the higher derivative corrections would be important if H and $M_{\rm st}$ are both comparable and of order the Grand unification scale $M_{\rm GUT} \sim 10^{16}$Gev. Since very little is understood about string theory in time dependent backgrounds the resulting correlation functions in such a situation cannot be calculated directly from our present knowledge of the theory. However symmetry considerations still hold and our result for the correlation function (1.0.1) is valid for such a situation as well.

The generality of our result makes the correlator given in (1.0.1) a good test, in a model independent manner, of the full symmetry group during inflation. The two-point scalar correlator, which has now been measured, is consistent with approximate scale invariance but this leaves open the possibility that the special conformal symmetries of de Sitter space are not preserved by the scalar sector. In fact, as was mentioned above, it is easy enough to construct models of inflation where this does happen and also straightforward to see that this possibility is allowed in terms of an effective field theory analysis [10]. The correlator discussed here, if observationally measured, can conclusively settle whether the special conformal symmetries were approximately preserved during inflation.

Unfortunately, experimental tests of this three point correlator are still some way away since its magnitude is small. Even the detection of the two point function for the tensor mode has not been made so far and would be a great discovery in itself. The small value that the three point scalar correlator has in conventional slow-roll inflation can be enhanced in models like DBI inflation which involve the breaking of special conformal symmetries.

[3]These boundary conditions are restated in a way more convenient for our analysis in section 2.3.

However, with the special conformal symmetries intact our analysis fixes the the overall normalization of the correlation function with two scalars and one tensor, as was mentioned above, and rules out the possibility of any such enhancement.

Therefore, we present the result of our analysis here not with any immediate experimental contact in mind, but rather with a view to the future when hopefully such contact will become possible and such model independent tests of inflation might play a useful role in sharping our understanding of the early Universe.

A second motivation for our work comes from the study of conformal field theory. The symmetry group mentioned above, $SO(4,1)$, is exactly the same as the symmetry group of a 3 dimensional Euclidean conformal field theory (CFT). This is in fact why we referred to this symmetry group as the conformal group when we first introduced it above. The problem of studying the constraints imposed by this symmetry group on the correlation functions of the scalar and tensor perturbations in de Sitter space maps in a direct way to the question of studying the constraints imposed in a 3 dimensional conformal field theory on correlation functions involving a nearly marginal scalar operator and the stress energy tensor. Thus our analysis is also of interest in the study of 3 dimensional CFTs: a subject which has also been of some considerable interest recently.[4]

The three point correlation function for two scalar operators and the stress tensor is already well known in the CFT literature [11]. However this result is in position space, while for cosmology one is interested in the answer in momentum space. It is not easy to directly Fourier transform the position space result. Moreover, the position space answer has divergences where the operators come together. It is rather subtle to regulate these divergences — which is necessary to define the Fourier transform — while preserving conformal invariance. A closely related issue is that of contact terms, which can also arise in position space. These were not determined in [11] but are important for the momentum dependence of the correlator. As our analysis shows, working directly in momentum space, the symmetry considerations are powerful enough to fix these ambiguities for the correlator and determine a unique answer.

Finally, a third motivation comes from attempts to study de Sitter space and its possible dual description in terms of a CFT [15, 16, 17, 18]. It is unclear at this point whether a precise correspondence of this type is possible. However, symmetry properties for correlators can be related between the gravity description and the CFT, as mentioned above. These are analogous

[4]For some discussion of three-point functions in 3 dimensional CFTs see [11, 12, 13, 14].

to and in fact follow after analytic continuation from the correspondence between correlators in the AdS/CFT case. Since, as our results help show, symmetry properties can significantly constrain at least some of the correlators, the correspondence in this limited sense is still of some practical benefit.

Before going further we must mention the seminal papers of Maldacena [17] and more recently Maldacena and Pimentel [19]. These papers lay out the essential ideas on which our analysis is based. The precise nature of the map between the gravity theory and the CFT using the wave function of the Universe was first discussed in [17]. And the importance of special conformal transformations was discussed in [19] where it was also shown that these symmetries significantly constrain the three point function of tensor perturbations. Our analysis is a modest extension of this approach for a correlator involving scalar perturbations as well.

Other relevant works which explore similar ideas are [20, 21, 22, 23, 24, 25, 10, 26]. Two recent papers [27, 28] contain related material.

The work presented here was carried out jointly with Prof. Sandip Trivedi and Dr. Suvrat Raju and published as a paper entitled 'CMB from CFT' in the Journal of High Energy Physics, Vol 7, 2013. The text of this thesis is taken largely from this paper with minor changes. Sections of the paper relating to the spinor helicity formalism are solely the work of other two authors and are hence not included this thesis. The organization is as follows. In §2 we discuss the basic ideas behind the analysis and background material. In §3 we set up the equations which arise due to conformal invariance. In §4 we discuss a solution to these equations and prove that it is unique. Our final results are presented in §5. We end with conclusions in §6. Three Appendices contain important supplementary material follow.

2 Basic Set-Up

We consider a theory of gravity coupled to a scalar field, the inflaton, with action

$$S = \int d^4x \sqrt{-g} \frac{1}{16\pi G} [R - \frac{1}{2}(\nabla\phi)^2 - V(\phi) + \cdots]. \qquad (2.0.2)$$

The ellipses stand for higher derivative corrections involving, in general, both gravity and the inflaton. Such corrections could be important, for example, if the Hubble scale during inflation is of order the string scale. Note that in (2.0.2) we are using conventions where the inflaton is dimensionless. Also below we will choose conventions where the Planck scale

$$M_{Pl}^2 \equiv 8\pi G = 1. \qquad (2.0.3)$$

It is well known that during inflation the Universe is approximately described by de Sitter space

$$ds^2 = -dt^2 + a^2(t)\sum_{i=1}^{3} dx_i dx^i, \tag{2.0.4}$$

$$a^2 = e^{2Ht}, \tag{2.0.5}$$

and hence undergoes exponential expansion. In (2.0.5), H is the Hubble scale which is a constant in de Sitter space. The inflationary epoch is described by de Sitter space with small corrections. These arise because of the slow variation of the Hubble scale which can be parametrized in terms of the two parameters

$$\epsilon = -\frac{\dot{H}}{H^2}, \delta = \frac{\ddot{H}}{2H\dot{H}}, \tag{2.0.6}$$

where dot denotes derivative with respect to t. During inflation both these parameters are small and meet the slow roll conditions

$$\epsilon, \delta \ll 1. \tag{2.0.7}$$

When the two-derivative approximation is good and the action can be approximated by the terms given in (2.0.2), H is given in terms of V by

$$H = \sqrt{\frac{V}{3M_{Pl}^2}}, \tag{2.0.8}$$

and the slow roll parameters can be expressed in terms of of V by

$$\epsilon = \frac{1}{2}\frac{M_{pl}^2(V')^2}{V^2}, \tag{2.0.9}$$

$$\delta = -M_{pl}^2\frac{V''}{V} + \epsilon, \tag{2.0.10}$$

where prime denotes derivatives with respect to the scalar field.[5] Also in the two-derivative theory we have

$$\epsilon = \frac{1}{2}\frac{\dot{\phi}^2}{H^2}. \tag{2.0.11}$$

When the two-derivative approximation is not valid ϵ defined in (2.0.6) and $\dot{\phi}$ will not be related by (2.0.11) in general. The slow-roll approximation then requires that besides (2.0.7) being valid,

$$\frac{\dot{\phi}}{H} \ll 1. \tag{2.0.12}$$

[5]The slow-roll parameter η which is more conventionally used is given by $\eta = M_{Pl}^2\frac{V''}{V}$.

de Sitter space is well known to be conformally invariant. For example it is easy to see that the scale transformation

$$x^i \to \lambda x^i, t \to t - \frac{1}{H}\log(\lambda), \tag{2.0.13}$$

leaves the metric (2.0.4) invariant. More generally the full isometry group of de Sitter space is $SO(1,4)$. It consists of the usual three translations and rotations in the x^i coordinates, the scale transformation, (2.0.7), and in addition three special conformal transformations. Infinitesimal special conformal transformations are of the form

$$x^i \to x^i - 2(b_j x^j)x^i + b^i(\sum_j (x^j)^2 - e^{-2Ht}), \tag{2.0.14}$$

$$t \to t + 2b_j x^j. \tag{2.0.15}$$

Here $b^i, i = 1, \ldots 3$ are infinitesimal parameters. As mentioned above de Sitter space is modified during inflation due to the time varying Hubble scale. While translations and rotations in the x^i directions are of course unbroken, this modification results in the breaking of the scaling and special conformal symmetries. However, as long as the slow roll parameters ϵ, δ, are small this breaking is small and the resulting inflationary spacetime is still approximately conformally invariant.

The inflaton sector need not preserve the full conformal group breaking the $SO(1,4)$ symmetry of de Sitter space badly and only preserving translations, rotations and scale transformations, as was mentioned in the introduction. Additional parameters enter in such a model which parameterize this breaking. For example, the speed of sound, c_s, is one such parameter. When $c_s \neq 1$ the special conformal symmetries are broken. See [10] for a more general parametrization of such effects.
We note that the Planck 2015 observations are consistent with the choice $c_s = 1$. In [31], constraints on c_s were determined for various models of inflation. One assumes a specific Lagrangian, and calculates the expression for f_{NL} in terms of c_s for this model. Observations of f_{NL} then give constraints on c_s. For DBI inflation, the estimate is $c_s^{DBI} > 0.087$ with 95% confidence. In what follows we will assume that the scalar sector also approximately preserves the full symmetry group of de Sitter space.

2.1 The Perturbations

The inflationary space-time is a solution for the system consisting of gravity and a scalar field. The rotational invariance in the x^i directions can be used to characterize perturbations about this solution. There are two kinds of perturbations which can arise, scalar and tensor perturbations. The scalar perturbations have spin zero and the tensor perturbations have spin 2.

The tensor perturbations are easy to understand — they are gravity waves in the inflationary background. The scalar perturbations essentially arise due to the presence of the inflaton field. Depending on the gauge chosen they can be thought of as perturbations in the inflaton, or in the spatial curvature or in a combination of both of these modes.

2.1.1 Gauge 1

For example, we can choose a gauge where the perturbations in the inflaton vanish,

$$\delta\phi = 0. \tag{2.1.1}$$

Starting with the form of the metric used in the ADM formalism

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{2.1.2}$$

the additional coordinate reparameterization can be fixed by choosing a gauge where

$$h_{ij} = a^2[(1 + 2\zeta)\delta_{ij} + \gamma_{ij}], \tag{2.1.3}$$

where γ_{ij} is transverse and traceless,

$$\partial_i \gamma_{ij} = \gamma_{ii} = 0, \tag{2.1.4}$$

as discussed in [17]. The tensor perturbations are given by γ_{ij}. And the scalar perturbations are given by ζ and correspond to fluctuations in the spatial curvature along the spatial directions.

2.1.2 Gauge 2

Alternatively, for the scalar perturbations, we can choose to set ζ instead of $\delta\phi$ to vanish. The perturbations are now given by fluctuations in the inflaton, $\delta\phi$. This second gauge is obtained by starting with the coordinates in which the perturbations take the form given in the previous paragraph, ζ, γ_{ij} and carrying out a time reparameterization

$$t \to t + \frac{\zeta}{H}. \tag{2.1.5}$$

It is easy to see that this sets ζ to vanish. The tensor perturbation γ_{ij} is unchanged by this coordinate transformation. If the background value of the inflaton in the inflationary solution is

$$\phi = \bar{\phi}(t), \tag{2.1.6}$$

the resulting value for the perturbation $\delta\phi$ this gives rise to is

$$\delta\phi = -\frac{\dot{\phi}\zeta}{H}. \tag{2.1.7}$$

When the two derivative approximation is good we can using (2.0.11) express this relation as

$$\delta\phi = -\sqrt{2\epsilon}\zeta. \tag{2.1.8}$$

We will find it useful to consider both gauges in our discussion below. As we will discuss further in subsection 2.3 for our purposes it will be most convenient to first work in gauge 2, where the scalar perturbation is given by $\delta\phi$ and then transform to gauge 1, where the perturbation is given by ζ, around the time when the mode crosses the horizon. This might seem conceptually confusing at first but has the advantage of allowing us to incorporate both the leading effects of the slow-roll parameters in a straightforward manner and of eventually going over to the description in terms of ζ which is the variable that it is defined for all time and also becomes constant once the mode exits the horizon.

Let us also make one more comment here. The relation (2.1.7) has corrections involving higher powers of the perturbation, $\delta\phi$. For the scalar three-point function in conventional slow-roll models, as studied in [17], the first corrections to (2.1.7) need to be kept since the leading answer is suppressed by an additional power of $\sqrt{\epsilon}$. But these corrections can be ignored for the correlator (1.0.1).

2.2 The Wave Function

The time dependence during the inflationary epoch gives rise to scalar and tensor perturbations. Our main interest here is to ask about the constraints that approximate conformal invariance imposes on the correlation functions of these perturbations. In particular we will be interested in these correlation functions at late enough times when the modes have crossed the horizon, and their wavelength, λ, has become much bigger than the Hubble scale, $\lambda \gg H^{-1}$.

At such late times the correlations functions acquire a time independent limiting form. The physical reason for this is well understood. Once the wavelength of a mode gets much longer than the Hubble scale the evolution of the mode gets dominated by Hubble friction and as a result it comes to rest.

In our discussion it will be useful to think in terms of a wavefunction which describes the state of the system at late times. The wavefunction tells us the amplitude to observe a particular perturbation and clearly encodes all information about the correlation functions. Since the correlation functions become time independent at late times the wave function also becomes time independent in this limit.[6]

The wave function will be a convenient description for our analysis since we are interested in the constraints imposed by symmetries and these can be conveniently translated to invariances of the wavefunction as we will see shortly. In turn this will allow us to map the constraints imposed by symmetries to an analysis of constraints imposed on correlators in a 3 dimensional Euclidean conformal field theory. More generally, thinking in terms of the wave function also allows us to exploit the analogy with calculations in AdS space for our purpose.

The perturbations produced during inflation are known to be Gaussian with small corrections. This allows the late time wave function to be written as a power series expansion of the form

$$\begin{aligned}\psi[\chi(\boldsymbol{x})] =& \exp\Big(-\frac{1}{2}\int d^3x d^3y \chi(\boldsymbol{x})\chi(\boldsymbol{y})\langle\hat{O}(\boldsymbol{x})\hat{O}(\boldsymbol{y})\rangle \\ &+\frac{1}{6}\int d^3x d^3y d^3z\, \chi(\boldsymbol{x})\chi(\boldsymbol{y})\chi(\boldsymbol{z})\langle\hat{O}(\boldsymbol{x})\hat{O}(\boldsymbol{y})\hat{O}(\boldsymbol{z})\rangle + \cdots\Big).\end{aligned} \tag{2.2.1}$$

Here χ stands for a generic perturbation which could be a scalar or tensor perturbation. The ellipses stand for higher order terms involving more powers of ϕ. The coefficients $\langle\hat{O}(\boldsymbol{x})\hat{O}(\boldsymbol{y})\rangle, \langle\hat{O}(\boldsymbol{x})\hat{O}(\boldsymbol{y})\hat{O}(\boldsymbol{z})\rangle$ etc. are for now just functions which determine the correlators.

The expression above is schematic. In the case at hand there are two kinds of perturbations, scalar and tensor. Working in the gauge described in subsection 2.1.2 these are $\delta\phi, \gamma_{ij}$. With a suitable choice of normalization the wave function will then take the form

$$\begin{aligned}\psi[\delta\phi, \gamma_{ij}] = \exp\Big[&\frac{M_{pl}^2}{H^2}\Big(-\frac{1}{2}\int d^3x d^3y \delta\phi(\boldsymbol{x})\delta\phi(\boldsymbol{y})\langle O(\boldsymbol{x})O(\boldsymbol{y})\rangle \\ &-\frac{1}{2}\int d^3x d^3y \gamma_{ij}(\boldsymbol{x})\gamma_{kl}(\boldsymbol{y})\langle T^{ij}(\boldsymbol{x})T^{kl}(\boldsymbol{y})\rangle \\ &-\frac{1}{4}\int d^3x d^3y d^3z \delta\phi(\boldsymbol{x})\delta\phi(\boldsymbol{y})\gamma_{ij}(\boldsymbol{z})\langle O(\boldsymbol{x})O(\boldsymbol{y})T^{ij}(\boldsymbol{z})\rangle + \cdots\Big)\Big].\end{aligned} \tag{2.2.2}$$

The ellipses stand for additional terms of various kinds involving three powers of the perturbations with appropriate coefficient functions and then higher order terms.

[6]More accurately, this happens after suitable infra-red divergences are subtracted. Physical answers do not depend on the choice of subtraction procedure.

Note, in our notation every additional power of the scalar perturbation is accompanied by an additional factor of $O(\boldsymbol{x})$ in the coefficient functions and every additional power of the tensor perturbation is accompanied by an additional factor of $T_{ij}(\boldsymbol{x})$. We will soon see that the coefficient functions transform under the symmetries in the same way as correlation functions involving a scalar operator and the stress energy tensor in a 3 dimensional Euclidean conformal field theory.

We will be interested in the last term in the RHS of (2.2.2). Together with the two point functions, this term determines the three point correlator of interest to us.

2.3 Symmetries and Their Consequences

We have seen that the wave function at late times is a functional of the late time values of the perturbations. Schematically we can write

$$\psi[\chi(\boldsymbol{x})] = \int^{\chi(\boldsymbol{x})} D\chi e^{iS}, \tag{2.3.1}$$

where χ again stands for the value a generic perturbation takes at late time and the action for any configuration is denoted by S. We would now like to derive constraints imposed by symmetries on this wavefunction.

Before doing so it is worth considering the boundary conditions in the path integral in more detail. We will consider inflation with the standard Bunch-Davies boundary conditions in the far past, when the modes of interest had a wavelength much shorter than the Hubble scale. At these early times the short wavelengths of the modes makes them insensitive to the geometry of de Sitter space and they essentially propagate as if in Minkowski spacetime. The Bunch Davies vacuum corresponds to taking the modes to be in the Minkowski vacuum at early enough time.

An elegant way to impose this boundary condition in the path integral above, as discussed in [17], is as follows. Consider de Sitter space in conformal coordinates,

$$ds^2 = \frac{1}{\eta^2}(-d\eta^2 + (dx_i)^2), \tag{2.3.2}$$

with the far past being $\eta \to -\infty$, and late time being $\eta \to 0$. Continue η so that it acquires a small imaginary part $\eta \to \eta(1 - i\epsilon), \epsilon > 0$. Then the Bunch Davies boundary condition is correctly imposed if the path integral is done over configurations which vanish at early times when $\eta \to -\infty(1 - i\epsilon)$. Note that in general the resulting path integral is over complex field configurations.

As an example, consider a free field ϕ satisfying the equation

$$\nabla^2\phi = 0. \tag{2.3.3}$$

A mode with momentum $\boldsymbol{k}$ is of the form, $\phi = f_{\boldsymbol{k}}(\eta)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$, where

$$f_{\boldsymbol{k}} = c_1(1 - ik\eta)e^{ik\eta} + c_2(1 + ik\eta)e^{-ik\eta}, \tag{2.3.4}$$

and $k \equiv |\vec{k}|$. Requiring that the solution vanish when $\eta \to -\infty(1 - i\epsilon)$, sets $c_2 = 0$ and requiring $f_{\boldsymbol{k}}$ equals the boundary value, $f_{\boldsymbol{k}} = f^0_{\boldsymbol{k}}$ at the late time $\eta = \eta_c$, gives

$$f_{\boldsymbol{k}} = f^0_{\boldsymbol{k}} \frac{(1 - ik\eta)e^{ik\eta}}{(1 - ik\eta_c)e^{ik\eta_c}}. \tag{2.3.5}$$

Since $f_{\boldsymbol{k}} \neq f^*_{-\boldsymbol{k}}$ the resulting field configuration is complex.

We are now ready to return to our discussion of the constraints imposed by symmetries on the wave function. What is important for this purpose, as far as the boundary conditions in the far past are concerned, is that the field configurations we sum over in the path integral vanish in the far past.

Consider in fact a general situation where we have a wave function of the form (2.3.1) for a general set of fields χ, with some boundary condition in the far past. Now if the system has a symmetry which keeps the action and the measure invariant and which also preserves the boundary conditions in the far past and if under the symmetry the boundary value of the field χ transforms as follows

$$\chi(\boldsymbol{x}) \to \chi'(\boldsymbol{x}), \tag{2.3.6}$$

then it follows from the definition of the wave function (2.3.1) that $\psi[\chi]$ satisfies the condition

$$\psi[\chi(\boldsymbol{x})] = \psi[\chi'(\boldsymbol{x})], \tag{2.3.7}$$

and is invariant under the symmetry.

For the case at hand where we work with de Sitter space, the symmetry group is the conformal group $SO(1,4)$ of isometries discussed above. Being isometries, the action and measure are invariant under it on account of reparameterization invariance. The boundary condition in the far past corresponding to the Bunch Davies vacuum is that the fields vanish. This is indeed preserved by the conformal transformations since the field transform homogeneously under these symmetries. For tensor perturbations this is all we need to use the general argument above. It follows that the wave function must be invariant under a change of the boundary values of the tensor perturbations which arise due to conformal transformations. As we will see shortly this implies that the coefficient functions, which we have suggestively denoted as $\langle T_{ij}T_{kl}\rangle$ etc., behave exactly like the correlations functions of the stress energy tensor of a three dimensional conformal field theory under conformal transformations. It is true, as we discussed above, that conformal invariance is broken slightly during inflation but this leads to only subleading corrections in the tensor mode correlations.

For the scalar mode the situation is a little more complicated. In pure de Sitter space, without the inflaton, the scalar perturbation in the metric ζ, (2.1.2), is pure gauge. In the presence of the inflaton there is a genuine scalar perturbation. However as (2.1.7), (2.1.8) which relates the perturbations in the two gauges discussed in section 2.1 shows, the slow roll parameter ϵ which is non-zero due to the breaking of conformal invariance is then involved in the definition of the scalar perturbation itself. This can make it confusing to apply the consequences of the small breaking of conformal invariance to the scalar sector.

The simplest way to proceed is to work in the second gauge discussed in subsection 2.1.2, where $\zeta = 0$. The scalar perturbation is then just the fluctuation in the scalar field. To leading order in the slow-roll parameters these fluctuations can be calculated in de Sitter space and the time evolution of the inflaton can be neglected for this process. As a result the full set of perturbations, scalar and tensor, with Bunch-Davies boundary conditions, then meet the conditions of the general argument given above and we learn that the wave function must be invariant under conformal transformations of the boundary values of these perturbations.

Once the results are obtained in this gauge one can always transform to other gauges, in particular the first gauge considered in subsection 2.1.1 where ζ is non-vanishing. In fact this is very convenient to do for purposes of following the evolution of the scalar mode after the end of inflation. Since ζ is related to $\delta\phi$ by (2.1.7) the resulting correlation functions will depend on the breaking of conformal invariance even to leading order but this dependence arises solely due to the relation (2.1.7) and is easily obtained.

2.4 Constraints on Coefficient Functions

Let us now work out the constraints imposed by conformal symmetries on the coefficient functions which arise in the expansion of the wave function (2.2.1) in more detail. It is easy to see that the constraints of translational invariance make the coefficient functions also translationally invariant. Under rotations in the x^i directions the wave function will be invariant if $O(\boldsymbol{x})$ transforms like a scalar and T_{ij} like a two-index tensor within coefficient functions.

Next we come to the scale transformation and special conformal transformations. Under the scale transformation (2.0.13) the scalar perturbation transforms by

$$\delta\phi(\boldsymbol{x},t) \to \delta\phi'(\boldsymbol{x},t) = \delta\phi(\frac{\boldsymbol{x}}{\lambda}, t + \frac{1}{H}\log(\lambda)). \tag{2.4.1}$$

At late times $\delta\phi$ becomes independent of t, as a result this equation becomes

$$\delta\phi(\boldsymbol{x}) \to \delta\phi'(\boldsymbol{x}) = \delta\phi(\frac{\boldsymbol{x}}{\lambda}). \tag{2.4.2}$$

In particular this is true for the boundary value of $\delta\phi$ as well.

As a result, suppressing the dependence on tensor modes for the moment, we learn that the wavefunction must satisfy the conditions

$$\psi[\delta\phi(\boldsymbol{x})] = \psi[\delta\phi'(\boldsymbol{x})] = \psi[\delta\phi(\frac{\boldsymbol{x}}{\lambda})]. \tag{2.4.3}$$

As mentioned above every additional factor of $\delta\phi(\boldsymbol{x})$ in the expansion of the wave function involves an additional factor of $O(\boldsymbol{x})$ in the corresponding coefficient function and also an integral over the spatial position of $\delta\phi(\boldsymbol{x})$. Thus schematically speaking the wave function will satisfy the condition (2.4.3) if

$$\int d^3x\delta\phi'(\boldsymbol{x})O(\boldsymbol{x}) = \int d^3x\delta\phi(\boldsymbol{x})O(\boldsymbol{x}), \tag{2.4.4}$$

where more correctly we mean the coefficient functions involving $O(\boldsymbol{x})$, rather that $O(\boldsymbol{x})$ itself. This leads to the condition

$$\int d^3x\lambda^3\delta\phi(\boldsymbol{x})O(\lambda\boldsymbol{x}) = \int d^3x\delta\phi(\boldsymbol{x})O(\boldsymbol{x}). \tag{2.4.5}$$

(In deriving this relation we first change variables in the middle expression of (2.4.4) to $\boldsymbol{y} = \frac{\boldsymbol{x}}{\lambda}$ and then change $\boldsymbol{y}$ to $\boldsymbol{x}$ since it is a dummy variable of integration.) Since (2.4.5) is true for an arbitrary function $\delta\phi(\boldsymbol{x})$ we learn that coefficient functions are invariant under the replacement

$$O(\boldsymbol{x}) \to \lambda^3 O(\lambda\boldsymbol{x}). \tag{2.4.6}$$

Or in infinitesimal form if $\lambda = 1 + \epsilon$,

$$O(\boldsymbol{x}) \to O(\boldsymbol{x}) + \epsilon\delta O(\boldsymbol{x}), \tag{2.4.7}$$

with

$$\delta O(\boldsymbol{x}) = 3O(\boldsymbol{x}) + x^i\partial_i O(\boldsymbol{x}). \tag{2.4.8}$$

This is exactly the condition that would arise due to scale invariance if the coefficient functions were the correlation functions in a conformal field theory with $O(\boldsymbol{x})$ being an operator of dimension 3. Note that in 3 dimensions this makes $O(\boldsymbol{x})$ marginal.

A similar argument for the tensor perturbation shows that under the scaling transformation, (2.0.13), the boundary value of the tensor perturbation transforms like[7]

$$\gamma_{ij}(\boldsymbol{x}) \rightarrow \gamma'_{ij}(\boldsymbol{x}) = \gamma_{ij}(\frac{\boldsymbol{x}}{\lambda}). \tag{2.4.9}$$

This is entirely analogous to (2.4.2) and a similar argument leads to the conclusion that T_{ij} must behave like an operator of dimension 3 under scaling transformations for the wave function to be invariant under it.

Finally we consider special conformal transformations. At late times when $e^{-Ht} \rightarrow 0$ we see from (2.0.14) that the x^i coordinates transform as

$$x^i \rightarrow x^i + \delta x^i, \tag{2.4.10}$$

$$\delta x^i = x^2 b^i - 2x^i(\boldsymbol{x} \cdot \boldsymbol{b}). \tag{2.4.11}$$

Henceforth we will use notation where $(\boldsymbol{a} \cdot \boldsymbol{b}) \equiv a^i b_i$ and also raise and lower indices along the spatial directions using the flat metric δ_{ij}.

The boundary value of the scalar field perturbation transforms under this as

$$\delta\phi(\boldsymbol{x}) \rightarrow \delta\phi'(\boldsymbol{x}) = \delta\phi(x^i - \delta x^i). \tag{2.4.12}$$

Arguing as in the case of the scale transformation above we then learn that for the wave function to be invariant coefficient functions must be invariant when

$$O(\boldsymbol{x}) \rightarrow O(\boldsymbol{x}) + \delta O(\boldsymbol{x}), \tag{2.4.13}$$

$$\delta O(\boldsymbol{x}) = -6(\boldsymbol{x} \cdot \boldsymbol{b}) O(\boldsymbol{x}) + D O(\boldsymbol{x}), \tag{2.4.14}$$

$$D = x^2(\boldsymbol{b} \cdot \boldsymbol{\partial}) - 2(\boldsymbol{b} \cdot \boldsymbol{x})(\boldsymbol{x} \cdot \boldsymbol{\partial}). \tag{2.4.15}$$

This is exactly the transformation of an operator of dimension 3 under special conformal transformations. Similarly from the transformation of the tensor mode we learn that the coefficient functions must be invariant when

$$T_{ij}(\boldsymbol{x}) \rightarrow T_{ij} + \delta T_{ij}, \tag{2.4.16}$$

$$\delta T_{ij} = -6(\boldsymbol{x} \cdot \boldsymbol{b}) T_{ij} + 2\hat{M}_i^k T_{kj} + 2\hat{M}_j^k T_{ik} - D T_{ij}, \tag{2.4.17}$$

$$\hat{M}_i^k \equiv 2(x^k b^i - x^i b^k). \tag{2.4.18}$$

[7]The reader might find this puzzling at first since the metric should transform as a tensor under the coordinate transformation (2.0.13). In fact the metric h_{ij}, (2.1.2), does transform like a tensor and goes to $h_{ij}(\boldsymbol{x}) \rightarrow \frac{1}{\lambda^2} h_{ij}(\frac{\boldsymbol{x}}{\lambda})$. However γ_{ij} is related to h_{ij} after multiplying by an additional factor of a^2, (2.1.3). Since t shifts, (2.0.13), the a^2 factor also changes resulting in the transformation rule (2.4.9).

These agree with the transformation rules for the stress energy tensor of a 3d CFT and also agree with eq.(4.9) in [19].

The stress energy tensor of a CFT also satisfies one additional condition — it is conserved. This gives rise to Ward identities that must be satisfied by correlations functions in the CFT involving the stress energy tensor. The same conditions also arise for the coefficient functions at hand here. The wave function must be reparameterization invariant with respect to general coordinate transformations,

$$x^i \to x^i + v^i, \tag{2.4.19}$$

under which the metric and scalar perturbations transform as

$$\gamma_{ij} \to \gamma_{ij} - \nabla_i v_j - \nabla_j v_i, \tag{2.4.20}$$
$$\delta\phi \to \delta\phi - v^k \partial_k \delta\phi. \tag{2.4.21}$$

Invariance of the wave function $\psi[\gamma_{ij}, \delta\phi]$ then leads to the condition

$$\int d^3x v^j \partial_{x^i} \langle T_{ij}(\boldsymbol{x}) \hat{O}(\boldsymbol{y_1}) \hat{O}(\boldsymbol{y_2}) \cdots \hat{O}(\boldsymbol{y_n}) \rangle = -\sum_i \langle \hat{O}(\boldsymbol{y_1}) \cdots \delta\hat{O}(\boldsymbol{y_i}) \cdots \hat{O}(\boldsymbol{y_n}) \rangle, \tag{2.4.22}$$

where $\hat{O}$ is a schematic notation standing for both T_{ij}, O, and $\delta\hat{O}(\boldsymbol{y_i})$ is the change in operator $\hat{O}(\boldsymbol{y_i})$ at the point $\boldsymbol{y_i}$. In particular when $\hat{O} = O$ is a scalar we get for the three point function

$$\begin{aligned} \partial_{x^i} \langle T_{ij}(\boldsymbol{x}) O(\boldsymbol{y_1}) O(\boldsymbol{y_2}) \rangle =& [\partial_{x^j} \delta^3(\boldsymbol{x} - \boldsymbol{y_1})] \langle O(\boldsymbol{y_1}) O(\boldsymbol{y_2}) \rangle \\ &+ [\partial_{x^j} \delta^3(\boldsymbol{x} - \boldsymbol{y_2})] \langle O(\boldsymbol{y_1}) O(\boldsymbol{y_2}) \rangle. \end{aligned} \tag{2.4.23}$$

To summarize, the coefficient functions which arise in the wave function (2.2.1) satisfy all the symmetry properties of correlations functions involving a scalar operator of dimension 3 and the stress energy tensor in a conformal field theory. Namely, they are invariant under the conformal symmetry group $SO(1,4)$ and satisfy the Ward identities due to conservation of the stress energy tensor.

3 Constraints of Conformal Invariance on the Correlation Function

In this section we will discuss how the correlation function (1.0.1) is constrained by the symmetries. This correlation function is obtained from the coefficient function, $\langle OOT_{ij}\rangle$, of the wave function in (2.2.1). We have argued in the previous section that as far as symmetries are concerned the coefficient functions behave in exactly the same manner as corresponding correlation functions of a CFT. In our discussion below we will find it convenient to adopt the language of CFT. We remind the reader that this is only a kind of short-hand for analyzing the consequences of symmetries. In particular, we will not be assuming any kind of deeper dS/CFT type relation in our analysis. We will obtain the constraints by applying the generators of conformal transformations to the momentum space correlator.

Notation:

Before proceeding let us list our conventions. We denote the three momentum by $\boldsymbol{k}$ below. Its magnitude will be denoted simply by $k \equiv |\boldsymbol{k}|$. Components will be denoted by $k_i, i = 1, ..., 3$ and indices will be raised and lowered by the flat space metric δ_{ij}.

3.1 Direct Momentum Space Analysis

In our conventions the momentum space scalar operator is given by

$$O(\boldsymbol{k}) \equiv \int d^3x O(\boldsymbol{x}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{3.1.1}$$

and similarly for $T_{ij}(\boldsymbol{k})$.

Translational and rotational invariance allows us to express the correlators in the form

$$\begin{aligned}\langle O(\boldsymbol{k}_1)O(\boldsymbol{k}_2)T_{ij}(\boldsymbol{k}_3)\rangle = [&k_{1i}k_{1j}f_1(k_1,k_2,k_3) + k_{2i}k_{2j}f_1(k_2,k_1,k_3) \\ &+ (k_{1i}k_{2j} + k_{2i}k_{1j})f_2(k_1,k_2,k_3) \\ &+ \delta_{ij}f_3(k_1,k_2,k_3)](2\pi)^3\delta^3(\sum_i \boldsymbol{k}_i).\end{aligned} \tag{3.1.2}$$

The overall delta function arises due to translational invariance. In the discussion below we will use $M_{ij}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ to denote the correlation function without the overall delta function factor,

$$\langle O(\boldsymbol{k}_1)O(\boldsymbol{k}_2)T_{ij}(\boldsymbol{k}_3)\rangle = M_{ij}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)(2\pi)^3\delta(\sum_i \boldsymbol{k}_i). \tag{3.1.3}$$

The three functions f_1, f_2, f_3 in (3.1.2) at first sight could have also depended on inner products $\boldsymbol{k_1}\cdot\boldsymbol{k_2}$ etc. However using momentum conservation these can be expressed in terms of the three scalars k_i. For example

$$\boldsymbol{k_1}\cdot\boldsymbol{k_2} = \frac{1}{2}(k_3^2 - k_1^2 - k_2^2). \tag{3.1.4}$$

The correlator is symmetric under the exchange of $\boldsymbol{k}_1 \leftrightarrow \boldsymbol{k}_2$. As a result f_2, f_3 are symmetric under the exchange of their first two arguments. Since the operators O and T_{ij} are dimension 3 in position space and thus dimension 0 in momentum space, scale invariance tells us that the f_i's are dimension 1.

Next we come to the non-trivial constraints due to special conformal transformations. The transformation in position space of the operators O and T_{ij} under an infinitesimal special conformal transformation with parameter b_i is given in (2.4.14) and (2.4.17) respectively. In momentum space these take the form,

$$\begin{aligned} \delta O(\boldsymbol{k}) &= -\tilde{D}O(\boldsymbol{k}), & (3.1.5)\\ \delta T_{ij}(\boldsymbol{k}) &= 2\tilde{M}_i^l T_{lj} + 2\tilde{M}_j^l T_{il} - \tilde{D}T_{ij}, & (3.1.6)\\ \tilde{M}_i^l &\equiv b^l \partial_{k^i} - b^i \partial_{k^l}, & (3.1.7)\\ \tilde{D} &\equiv (\boldsymbol{b}\cdot\boldsymbol{k})\partial_{k^i}\partial_{k^i} - 2k_j\partial_{k_j}(\boldsymbol{b}\cdot\boldsymbol{\partial_k}). & (3.1.8) \end{aligned}$$

These expressions agree with eq.(4.12) in [19] and in fact we have chosen essentially the same conventions to try and ensure readability.

The condition for invariance of the correlator is

$$\langle \delta O(\boldsymbol{k}_1)O(\boldsymbol{k}_2)T_{ij}(\boldsymbol{k}_3)\rangle + \langle O(\boldsymbol{k}_1)\delta O(\boldsymbol{k}_2)T_{ij}(\boldsymbol{k}_3)\rangle + \langle O(\boldsymbol{k}_1)O(\boldsymbol{k}_2)\delta T_{ij}(\boldsymbol{k}_3)\rangle = 0. \quad (3.1.9)$$

As was argued in [19] all terms involving derivatives that act on the overall momentum conserving delta function sum to zero so we will henceforth neglect the effect of the derivative operators acting on the delta function.

Defining the operator

$$\Theta(k) \equiv -\frac{2}{k}\frac{\partial}{\partial k} + \frac{\partial^2}{\partial k^2}, \quad (3.1.10)$$

where $k \equiv |\boldsymbol{k}|$ one can then show after some algebra that

$$\begin{aligned} &\langle \delta O(\boldsymbol{k}_1)O(\boldsymbol{k}_2)T_{ij}(\boldsymbol{k}_3)\rangle = -2(\boldsymbol{b}\cdot\boldsymbol{k_1})\delta_{ij}f_1 + 2(b_ik_{1j}+b_jk_{1i})(1+k_1\partial_{k_1})f_1 \\ &+ 2(b_ik_{2j}+b_jk_{2i})k_1\partial_{k_1}f_2 + (\boldsymbol{b}\cdot\boldsymbol{k_1})\Theta(k_1)[f_1k_{1i}k_{1j} + f_1^Tk_{2i}k_{2j} + f_2(k_{1i}k_{2j}+k_{2i}k_{1j}) + f_3\delta_{ij}]. \end{aligned} \quad (3.1.11)$$

Here we have omitted the overall delta function. We have also introduced the notation

$$f_1^T(k_1,k_2,k_3) \equiv f_1(k_2,k_1,k_3). \quad (3.1.12)$$

At this stage it is useful to contract the LHS of (3.1.11) with the symmetric (real) polarization tensor e_{ij}^s which is traceless and transverse to $\boldsymbol{k}_3$,

$$e_i^{s,i} = e_{ij}^s k_3^i = 0. \quad (3.1.13)$$

The s here indicates that there are two possible choices for this tensor. This gives

$$\begin{aligned}\langle \delta O(\boldsymbol{k}_1) O(\boldsymbol{k}_2) T_{ij}(\boldsymbol{k}_3)\rangle e^{s,ij} &= 4b_i k_{1j} e^{s,ij}[(1+k_1\partial_{k_1})f_1 - k_1\partial_{k_1} f_2] \\ &\quad +(\boldsymbol{b}\cdot\boldsymbol{k_1})\Theta(k_1)(2f_2 - f_1 - f_1^T)k_{1i}k_{2j}e^{s,ij},\end{aligned} \tag{3.1.14}$$

where we have used the condition

$$e^{s,ij}k_{1i} = -e^{s,ij}k_{2i} = 0. \tag{3.1.15}$$

Similarly we get

$$\begin{aligned}\langle O(\boldsymbol{k}_1)\delta O(\boldsymbol{k}_2) T_{ij}(\boldsymbol{k}_3)\rangle e^{s,ij} =& - 4b_i k_{1j} e^{s,ij}[(1+k_2\partial_{k_2})f_1^T - k_2\partial_{k_2} f_2] \\ &+ (\boldsymbol{b}\cdot\boldsymbol{k_2})\Theta(k_2)(2f_2 - f_1 - f_1^T)k_{1i}k_{2j}e^{s,ij}.\end{aligned} \tag{3.1.16}$$

And also

$$\begin{aligned}\langle O(\boldsymbol{k}_1) O(\boldsymbol{k}_2) \delta T_{ij}(\boldsymbol{k}_3)\rangle e^{s,ij} =& -\frac{4}{k_3} b_i k_{1j} e^{s,ij}[(\boldsymbol{k_3}\cdot\boldsymbol{k_1})\partial_{k_3}(f_1 - f_2) - (\boldsymbol{k_3}\cdot\boldsymbol{k_2})\partial_{k_3}(f_1^T - f_2)] \\ &+ \boldsymbol{b}\cdot\boldsymbol{k_3}\Theta(k_3)(2f_2 - f_1 - f_1^T).\end{aligned} \tag{3.1.17}$$

Adding (3.1.14), (3.1.16) and (3.1.17) and setting the total change to vanish finally gives the equation

$$\begin{aligned}&4b_i k_{1j} e^{s,ij}\Big[(1+k_1\partial_{k_1})f_1 - (1+k_2\partial_{k_2})f_1^T + (k_2\partial_{k_2} - k_1\partial_{k_1})f_2 \\ &\qquad - \frac{(\boldsymbol{k_3}\cdot\boldsymbol{k_1})}{k_3}\partial_{k_3}(f_1 - f_2) + \frac{(\boldsymbol{k_3}\cdot\boldsymbol{k_2})}{k_3}\partial_{k_3}(f_1^T - f_2)\Big] \\ &+k_{1i}k_{2j}e^{s,ij}\Big[(\boldsymbol{b}\cdot\boldsymbol{k_1})\Theta(k_1) + (\boldsymbol{b}\cdot\boldsymbol{k_2})\Theta(k_2) + (\boldsymbol{b}\cdot\boldsymbol{k_3})\Theta(k_3)\Big](2f_2 - f_1 - f_1^T) = 0.\end{aligned} \tag{3.1.18}$$

This is the main equation we will use to derive the constraints imposed by the special conformal transformations.

There are three linearly independent values that $\boldsymbol{b}$ can take in (3.1.18). Choosing $\boldsymbol{b} \propto \boldsymbol{k}_3$ gives

$$[\boldsymbol{k}_3\cdot\boldsymbol{k}_1\Theta(k_1) + (\boldsymbol{k_3}\cdot\boldsymbol{k_2})\Theta(k_2) + k_3^2\Theta(k_3)]S(k_1,k_2,k_3) = 0, \tag{3.1.19}$$

where

$$S(k_1,k_2,k_3) = \frac{1}{2}[f_1(k_1,k_2,k_3) + f_1(k_2,k_1,k_3) - 2f_2(k_1,k_2,k_3)]. \tag{3.1.20}$$

Choosing $\boldsymbol{b} \propto \boldsymbol{k}_{1\perp} = \boldsymbol{k}_1 - \boldsymbol{k}_3 \frac{(\boldsymbol{k_1} \cdot \boldsymbol{k_3})}{k_3^2}$ gives

$$4[\frac{-\boldsymbol{k_2} \cdot \boldsymbol{k_3}}{k_3^2} k_1 \partial_{k_1} S + \frac{\boldsymbol{k_1} \cdot \boldsymbol{k_3}}{k_3^2} k_2 \partial_{k_2} S - \frac{(k_1^2 - k_2^2)}{k_3^2} S + \frac{3}{2} \frac{(k_1^3 - k_2^3)}{k_3^2}]$$
$$-(k_1^2 - \frac{(\boldsymbol{k_3} \cdot \boldsymbol{k_1})^2}{k_3^2})(\Theta(k_1) - \Theta(k_2))S = 0, \quad (3.1.21)$$

as shown in Appendix B. The term inhomogeneous in S above arises due to the use of the Ward identity for conservation of the stress tensor. We take the two-point function of the scalar $O(k)$ to be normalized so that

$$\langle O(k_1) O(k_2) \rangle = (2\pi)^3 \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2) |\boldsymbol{k}_1|^3. \quad (3.1.22)$$

The Ward identity for conservation of the stress tensor, (2.4.23) then takes the form

$$M_{ij} k_3^j = -k_1^3 k_1^j - k_2^3 k_2^j, \quad (3.1.23)$$

where M_{ij} is defined in (3.1.3).

Finally we can choose $\boldsymbol{b}$ to be orthogonal to all the $\boldsymbol{k}_i$'s so that $\boldsymbol{b} \cdot \boldsymbol{k}_i = 0$. For a suitable choice of polarization $b_i k_{1j} e^{s,ij}$ will not vanish and as discussed in Appendix B (3.1.18) then becomes

$$-(\boldsymbol{k_2} \cdot \boldsymbol{k_3}) k_1 \partial_{k_1} S + (\boldsymbol{k_1} \cdot \boldsymbol{k_3}) k_2 \partial_{k_2} S - (k_1^2 - k_2^2) S + \frac{3}{2}(k_1^3 - k_2^3) = 0. \quad (3.1.24)$$

Subtracting (3.1.21) and (3.1.24) then gives

$$(\Theta(k_1) - \Theta(k_2))S = 0. \quad (3.1.25)$$

Substituting this in (3.1.19) then gives

$$(\Theta(k_1) - \Theta(k_3))S = 0. \quad (3.1.26)$$

Equations (3.1.24), (3.1.25) and (3.1.26) can be taken to be the three final equations which arise because of special conformal invariance.

Before proceeding let us note here that from (3.1.2), (3.1.15) and (3.1.20) we get that

$$\langle O(k_1) O(k_2) T_{ij}(k_3) \rangle e^{s,ij} = -2(2\pi)^3 \delta(\sum_i \boldsymbol{k}_i) e^{s,ij} k_{1i} k_{2j} S, \quad (3.1.27)$$

where e_{ij}^s is a traceless polarization tensor transverse to $\boldsymbol{k}_3$.

4 Solving the Conformal Constraints

The three point correlator involving two scalar and one tensor perturbations was calculated for a model of inflation in [17]. The answer is given in equations (4.10) and (4.11) of [17] in terms of the function

$$I = -(k_1 + k_2 + k_3) + \frac{\sum_{i>j} k_i k_j}{(k_1 + k_2 + k_3)} + \frac{k_1 k_2 k_3}{(k_1 + k_2 + k_3)^2}. \tag{4.0.28}$$

From this result we can read off the functional form for the corresponding $\langle OOT_{ij} \rangle$ coefficient. This gives

$$S = -I = -[-(k_1 + k_2 + k_3) + \frac{\sum_{i>j} k_i k_j}{(k_1 + k_2 + k_3)} + \frac{k_1 k_2 k_3}{(k_1 + k_2 + k_3)^2}]. \tag{4.0.29}$$

It is easy to check that this function solves the three equations (3.1.21), (3.1.25), (3.1.26) above.

4.1 Uniqueness

In this subsection we will see that (4.0.29) is the unique solution to (3.1.24), (3.1.25), (3.1.26) which meets all the required conditions.

We begin by noting that the set of functions

$$f_z(k) = (1 + ikz)e^{-ikz}, \tag{4.1.1}$$

with z allowed to range over both positive and negative values forms a complete set. Any function $\mathcal{H}(k)$ can be expanded in terms of this set,

$$\mathcal{H}(k) = \int_{-\infty}^{\infty} \tilde{\phi}(z) f_z(k) dz. \tag{4.1.2}$$

The point is that $\tilde{\phi}$ is a kind of souped up Fourier transform of $\mathcal{H}(k)$. Let $\phi(k)$ be the Fourier transform of $\tilde{\phi}(z)$. Then (4.1.2) gives

$$\mathcal{H}(k) = \phi(k) - k\phi'(k) = -k^2 \frac{d}{dk} \left(\frac{\phi(k)}{k} \right), \tag{4.1.3}$$

which can be solved to obtain

$$\phi(k) = -k \int^k \frac{\mathcal{H}(x)}{x^2} dx, \tag{4.1.4}$$

and correspondingly

$$\tilde{\phi}(z) = \int_{-\infty}^{\infty} - \left[k e^{ikz} \int^{k} \frac{\mathcal{H}(x)}{x^2} dx \right] \frac{dk}{2\pi}. \tag{4.1.5}$$

Note that (4.1.4) determines $\phi(k)$ up to a term proportional to k and this in turns leads to an ambiguity proportional to $\delta'(z)$ in $\tilde{\phi}(z)$, but this ambiguity drops out of the integral in (4.1.2) leading to a well defined value for $\mathcal{H}(k)$.

Thus the most general solution can be expanded as

$$\begin{aligned} S(k_1, k_2, k_3) = \int \Big[&(1 + ik_1 z_1) e^{-ik_1 z_1} (1 + ik_2 z_2) e^{-ik_2 z_2} \\ &\times (1 + ik_3 z_3) e^{-ik_3 z_3} \mathcal{M}(z_1, z_2, z_3) \Big] dz_1 dz_2 dz_3, \end{aligned} \tag{4.1.6}$$

where each z_i integral runs over $(-\infty, \infty)$.

Now note that since

$$\Theta(k) f_z(k) = -z^2 f_z(k), \tag{4.1.7}$$

the functions $f_z(k)$ are eigenvectors of the operator $\Theta(k)$.[8] It then follows that (3.1.25), (3.1.26), for S given in (4.1.6) lead to the conditions

$$z_1^2 = z_2^2 = z_3^3. \tag{4.1.8}$$

As a result an allowed solution can be written in the following form:

$$S = \sum_{n_1, n_2, n_3 = \pm 1} \int_0^{\infty} \mathcal{F}_{n_1 n_2 n 3}(z) \mathcal{M}_{n_1 n_2 n_3}(z) dz, \tag{4.1.9}$$

where $\mathcal{M}_{n_1, n_2, n_3}$ are a set of 8 functions for the 8 possible combinations of n_1, n_2, n_3 and

$$\mathcal{F}_{n_1 n_2 n_3}(z) = (1 + in_1 k_1 z) e^{-in_1 k_1 z} (1 + in_2 k_2 z) e^{-in_2 k_2 z} (1 + in_3 k_3 z) e^{-in_3 k_3 z}. \tag{4.1.10}$$

Next, we apply the dilatation constraint:

$$\left(k_1 \frac{\partial}{\partial k_1} + k_2 \frac{\partial}{\partial k_2} + k_3 \frac{\partial}{\partial k_3} \right) S = S. \tag{4.1.11}$$

[8] The functions $f_z(k)$ are in fact solutions to the massless scalar equation in de Sitter space with z being conformal time.

We notice that:

$$\left(\frac{k_1\partial}{\partial k_1}+\frac{k_2\partial}{\partial k_2}+\frac{k_3\partial}{\partial k_3}\right)S-S=\sum_{n_1,n_2,n_3=\pm1}\int_0^\infty \mathcal{M}_{n_1n_2n_3}(z)\left(z\frac{\partial}{\partial z}-1\right)\mathcal{F}_{n_1n_2n_3}(z) \tag{4.1.12}$$

$$=-\sum_{n_1,n_2,n_3=\pm1}\int_0^\infty\left(\frac{\partial}{\partial z}z+1\right)\mathcal{M}_{n_1n_2n_3}(z)\mathcal{F}_{n_1n_2n_3}(z)dz, \tag{4.1.13}$$

which leads to

$$-\frac{\partial}{\partial z}z\mathcal{M}_{n_1,n_2n_3}(z)=\mathcal{M}_{n_1,n_2,n_3}(z). \tag{4.1.14}$$

This provides us with

$$\mathcal{M}_{n_1,n_2,n_3}=\frac{m_{n_1,n_2,n_3}}{z^2}, \tag{4.1.15}$$

where m_{n_1,n_2,n_3} is an arbitrary constant. Essentially all that we are saying that the z dependence of $\mathcal{M}_{n_1,n_2,n_3}$ is fixed by noting that it must have dimension 2 and z has dimension -1.

In going from (4.1.12) to (4.1.13), we tacitly assumed that $\mathcal{M}$ was regular at the origin so that we could drop the boundary term at 0. However, the result in (4.1.15) makes (4.1.12) divergent both at 0 and at ∞. We can be more careful as follows. To define the integral at $z=\infty$, we can analytically continue the correlator to give the k_i a small imaginary part. To define the integral at $z=0$, we can define it by:

$$S=\sum_{n_1,n_2,n_3=\pm1}m_{n_1n_2n_3}\int_0^\infty\mathcal{F}_{n_1n_2n_3}(z)\frac{dz}{z^2}\equiv\sum_{n_1,n_2,n_3=\pm1}m_{n_1n_2n_3}\int_\epsilon^\infty\mathcal{M}_{n_1n_2n_3}(z)\Big|_{\epsilon^0}, \tag{4.1.16}$$

which means that we regulate the integral, by changing the range to (ϵ,∞) and then pick up the ϵ^0 term. This prescription now makes the resulting integral well defined while preserving its behaviour under scale transformations.

The prescription above leads to:

$$S=\sum_{n_1,n_2,n_3=\pm1}m_{n_1n_2n_3}\Big(-n_1n_2n_3\frac{k_2k_3k_1}{(n_1k_1+n_2k_2+n_3k_3)^2}+n_1k_1+n_2k_2+n_3k_3 \\ -\frac{n_1n_2k_1k_2+n_2n_3k_3k_2+n_1n_3k_1k_3}{n_1k_1+n_2k_2+n_3k_3}\Big). \tag{4.1.17}$$

Actually there are only four distinct terms in the sum above since the function of k_i's within the bracket on the RHS above only changes by an overall sign when the sign of all three n_i's is changed. We can use this property to fix $n_3 = +1$ so that S is given by a sum over four terms

$$S = \sum_{n_1,n_2=\pm 1} m_{n_1 n_2}\Big(- n_1 n_2 \frac{k_2 k_3 k_1}{(n_1 k_1 + n_2 k_2 + k_3)^2} + n_1 k_1 + n_2 k_2 + k_3 - \frac{n_1 n_2 k_1 k_2 + n_2 k_3 k_2 + n_1 k_1 k_3}{n_1 k_1 + n_2 k_2 + k_3}\Big). \tag{4.1.18}$$

where $m_{n_1,n_2} = m_{n_1 n_2 +1}$.

So far we have used (3.1.25), (3.1.26). It is easy to show that the remaining equation (3.1.24) acting on the solution above gives rise to the two conditions

$$\sum_{n_1,n_2} m_{n_1,n_2} n_1^3 = 1, \tag{4.1.19}$$

$$\sum_{n_1,n_2} m_{n_1,n_2} n_2^3 = 1. \tag{4.1.20}$$

4.2 Various Limits For The Momenta

In this subsection we will show by considering two different limits for the momenta that one can rule out three of the four terms which appear in the sum in (4.1.18) leaving only the term with $n_1 = n_2 = 1$. The normalization of this term is then fixed by (4.1.19),(4.1.20) leading to the unique result given in (4.0.29).

4.2.1 First Limit

First consider the limit where the momentum carried by the tensor perturbation is much smaller than that of the two scalar perturbations,

$$k_3 \ll k_1 \simeq k_2. \tag{4.2.1}$$

In this limit the scalar perturbations can be taken to be propagating in an essentially constant metric γ_{ij}. The resulting wave function (2.3.1) can be calculated in two ways. Either by working directly with the boundary values, $\gamma_{ij}, \delta\phi$. Or by first taking a boundary metric which is flat, $\gamma_{ij} = \delta_{ij}$, and then transforming the answer by a coordinate transformation to the case of the constant metric γ_{ij}. The two answers must of course agree.

This gives rise to the condition, [17], that in this limit

$$\langle T_{ij}(k_3) O(k_1) O(k_2)\rangle' e^{s,ij} = -e^{s,ij} k_{2i} k_{2j} \frac{d}{dk_2^2} \langle O(k_2) O(-k_2)\rangle', \tag{4.2.2}$$

where the superscript prime on the two sides stands for the correlator without the factor of $(2\pi)^3\delta^3(\sum \boldsymbol{k_i})$. From (A.0.10) and (3.1.27) this gives that in the limit (4.2.1)

$$S \to \frac{3}{2}k_2. \tag{4.2.3}$$

One finds that this condition rules out the two terms in (4.1.18) where n_1, n_2 have the opposite sign so that

$$S = \sum_{\{(n_1,n_2)=(+,+),(n_1,n_2)=(-,-)\}} m_{n_1 n_2}\Big(- n_1 n_2 \frac{k_2 k_3 k_1}{(n_1 k_1 + n_2 k_2 + k_3)^2} + n_1 k_1 + n_2 k_2 + k_3 - \frac{n_1 n_2 k_1 k_2 + n_2 k_3 k_2 + n_1 k_1 k_3}{n_1 k_1 + n_2 k_2 + k_3}\Big). \tag{4.2.4}$$

4.2.2 Second Limit and the OPE

Next we examine the limit where $k_2 \simeq k_3 \gg k_1$. The behaviour in this limit is most easily understood if we can appeal to the operator product expansion (OPE). We have seen that the coefficient functions which appear in the wave function (2.2.1), (2.2.2), transform under the conformal symmetries like the correlation functions of a CFT. It is well known that in a CFT operators satisfy the operator product expansion. For the arguments that follow we will assume that this is true for the coefficient functions in the wave function as well. While this assumption is quite plausibly true we do not provide a proof for it here.[9] In the next section we provide another argument for uniqueness that does not reply on the OPE.

To see how the argument goes let us first examine the limit which was studied above, where k_1, k_2 are large compared to k_3, but now using the OPE. We take

$$\boldsymbol{k}_2 = \boldsymbol{K}, \quad \boldsymbol{k}_1 = -\boldsymbol{K} + \boldsymbol{k}_3, \quad \text{with } K \equiv |\boldsymbol{K}| \gg k_3. \tag{4.2.5}$$

In position space we are considering the limit $\boldsymbol{x_1} \to \boldsymbol{x_2}$ for the correlation function

$$\langle O(\boldsymbol{x_1}) O(\boldsymbol{x_2}) T_{\mu\nu}(\boldsymbol{x_3})\rangle. \tag{4.2.6}$$

[9] In the AdS/CFT correspondence which is related by analytic continuation to the dS case one can plausibly provide an argument for the operator product expansion from the bulk using the prescription for calculating the boundary correlation functions from the bulk, the properties of the bulk to boundary propagator, etc. By analytic continuation one would expect then to be able to show this for the coefficient functions in the dS case as well.

The operator product expansion tells us that in this limit the leading contribution comes from the term

$$O(0)O(\boldsymbol{x}) = \frac{x_\mu x_\nu}{x^5} T^{\mu\nu}(\boldsymbol{x}) + \ldots, \tag{4.2.7}$$

where $\boldsymbol{x} \equiv \boldsymbol{x_2} - \boldsymbol{x_1}$.

The momentum space correlator is obtained by taking a Fourier transform of (4.2.6)

$$\begin{aligned}
&\int \langle O(\boldsymbol{x_1}) O(\boldsymbol{x_2}) T^{\mu\nu}(\boldsymbol{x_3})\rangle e^{i((\boldsymbol{k_1}\cdot\boldsymbol{x_1})+(\boldsymbol{k_2}\cdot\boldsymbol{x_2})+(\boldsymbol{k_3}\cdot\boldsymbol{x_3}))} d^3x_1 d^3x_2 d^3x_3 \\
&= \int \langle O(0) O(\boldsymbol{x_2}-\boldsymbol{x_1}) T^{\mu\nu}(\boldsymbol{x_3}-\boldsymbol{x_1})\rangle e^{i((\boldsymbol{k_1}+\boldsymbol{k_2}+\boldsymbol{k_3})\cdot\boldsymbol{x_1}+\boldsymbol{k_2}\cdot(\boldsymbol{x_2}-\boldsymbol{x_1})+\boldsymbol{k_3}(\boldsymbol{x_3}-\boldsymbol{x_1}))} d^3x_1 d^3x_2 d^3x_3 \\
&= (2\pi)^3 \delta^3(\boldsymbol{k_1}+\boldsymbol{k_2}+\boldsymbol{k_3}) \int \langle O(0) O(\boldsymbol{x_2}-\boldsymbol{x_1}) T^{\mu\nu}(\boldsymbol{x_3}-\boldsymbol{x_1})\rangle e^{i(\boldsymbol{k_2}\cdot(\boldsymbol{x_2}-\boldsymbol{x_1})+\boldsymbol{k_3}\cdot(\boldsymbol{x_3}-\boldsymbol{x_1}))} d^3x_2 d^3x_3.
\end{aligned} \tag{4.2.8}$$

In the limit (4.2.5) it follows from (4.2.7) that the momentum space correlator should go like

$$\int \frac{x^\mu x^\nu}{x^5} e^{i\boldsymbol{K}\cdot\boldsymbol{x}} d^3x \sim O(K^0). \tag{4.2.9}$$

Since the expression (3.1.27) already has a factor of K^2 outside, we learn that

$$S \sim \frac{k_3^3}{K^2}, \tag{4.2.10}$$

where we have inserted the correct factor of k_3 by dimensional analysis. It is easy to check that this only happens in the sum in (4.1.18) if n_1, n_2 have the same sign.

For example, consider the term in (4.1.18) with $n_1 = n_2 = 1$. And scale $k_3 \to \lambda k_3$ and expand in powers of λ, for small λ. We get:

$$\begin{aligned}
-S = &\frac{3k_1}{2} + \frac{3(\boldsymbol{k_1}\cdot\boldsymbol{k_3})\lambda}{4k_1} + \left(\frac{{k_3}^2}{k_1} - \frac{(\boldsymbol{k_1}\cdot\boldsymbol{k_3})^2}{4{k_1}^3}\right)\lambda^2 \\
&+ \lambda^3\left(\frac{3(\boldsymbol{k_1}\cdot\boldsymbol{k_3})^3}{16{k_1}^5} - \frac{9{k_3}^2(\boldsymbol{k_1}\cdot\boldsymbol{k_3})}{16{k_1}^3} - \frac{3{k_3}^3}{8{k_1}^2}\right).
\end{aligned} \tag{4.2.11}$$

One might naively believe that this contradicts (4.2.10). However, it is rather interesting that all the terms that grow too fast with K are actually analytic *in at least two momenta* and so lead to contact terms when transformed to position space.

For example, we have

$$\begin{aligned}
&\int k_{1i}k_{2j}\frac{3k_1}{2}\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2}+\boldsymbol{k_3})e^{-i(\boldsymbol{k_1}\cdot\boldsymbol{x_1}+\boldsymbol{k_2}\cdot\boldsymbol{x_2}+\boldsymbol{k_3}\cdot\boldsymbol{x_3})}d^3k_1d^3k_2d^3k_3\\
&=-\frac{\partial}{\partial x_1^i}\frac{\partial}{\partial x_2^j}\int\frac{3k_1}{2}e^{-i(\boldsymbol{k_1}\cdot(\boldsymbol{x_1}-\boldsymbol{x_2})+\boldsymbol{k_3}\cdot(\boldsymbol{x_3}-\boldsymbol{x_2}))}d^3k_1d^3k_3 \qquad (4.2.12)\\
&=-(2\pi)^3\frac{\partial}{\partial x_1^i}\frac{\partial}{\partial x_2^j}\delta(\boldsymbol{x}_3-\boldsymbol{x}_2)\int\frac{3k_1}{2}e^{-i(\boldsymbol{k_1}\cdot(\boldsymbol{x_1}-\boldsymbol{x_2}))}d^3k_1+\ldots
\end{aligned}$$

where $\ldots$ are subleading in λ.

The first non-analytic term in (4.2.11) is the term that goes like $\frac{k_3^3}{k_1^2}$, which is indeed of the form that we expected in (4.2.10)!

It is easy to check that if we consider a term in (4.1.18) where n_1, n_2 have opposite sign we will not get an answer consistent with the OPE. For example consider the term with $n_1 = -1, n_2 = n_3 = 1$, we have

$$-S = \frac{1}{\lambda}\frac{2k_3{k_1}^4 + (\boldsymbol{k_1}\cdot\boldsymbol{k_3}){k_1}^3}{((\boldsymbol{k_1}\cdot\boldsymbol{k_3})+k_1k_3)^2}. \qquad (4.2.13)$$

This is already non-analytic and is clearly of the wrong form.

Having considered the limit where k_1, k_2 are large compared to k_3 we can finally turn to the limit of interest where k_2, k_3 are large compared to k_1. In position space this corresponds to the case where $x_2 \to x_3$, in which case we expect the dominant OPE

$$\begin{aligned}
O(x_2)T_{\mu\nu}(x_3) =& A\frac{(x_2-x_3)_\mu(x_2-x_3)_\nu}{x_2-x_3^5}O(x_3) + B\frac{(x_2-x_3)_\mu\partial_\nu+(x_2-x_3)_\nu\partial_\mu}{x_2-x_3^4}O(x_3)\\
&+\frac{C}{x_3^3}\partial_\mu\partial_\nu O(x_3).
\end{aligned} \qquad (4.2.14)$$

We are now concerned with the limit where $k_3 = K, k_2 = -K - k_1$ and $\frac{K}{k_1}$ is large. The terms that multiply A and B might seem like they scale like K^0 in this limit, but this is deceptive. In fact, if we work through the Fourier transform, we expect that these terms give rise to

$$\frac{K_\mu K_\nu + K_{(\mu}k_{1\nu)} + k_{1\mu}k_{1\nu}}{K^2}, \qquad (4.2.15)$$

in Fourier space. Of these terms only the last one — $k_{1\mu}k_{1\nu}$ is meaningful, since the others point along K, and yield 0 when contracted with a transverse polarization tensor for the stress-tensor. A similar logic applies for the term that multiplies B. So, in fact, all the three terms in (4.2.14) should give terms that scale like $\frac{1}{K^2}$ when transformed to momentum space.

This now implies that S itself must scale like $\frac{1}{K^2}$ since the full correlator is given by S multiplied with $e^{s,ij}k_{1i}k_{2j}$ (3.1.27), and even though $\boldsymbol{k_2} = -\boldsymbol{K} - \boldsymbol{k_1}$, since $e^{s,ij}K_j = 0$, this factor scales like $O(K^0)$. It is now simple to see that of the two terms that remain in (4.2.4) the only one which gives the correct behaviour for S is the one with $n_1 = n_2 = 1$. The analysis of expanding the terms in this limit and comparing with the required behaviour is completely analogous to the one above and we will skip the details.

To summarize, by considering two limits for the momenta we learn that of the four terms which could have been present in S, (4.1.18) only one term survives giving the final result in (4.0.29).

4.3 Final Solution

As mentioned above the unique solution for S was obtained above in (4.0.29). The overall normalization followed from the use of the normalization of the two point function $\langle O(k_1)O(k_2)\rangle$ given in (3.1.22) which in turn determined the Ward identity (3.1.23).

Instead as discussed in Appendix A it is convenient to take the two point function $\langle O(k_1)O(k_2)\rangle$ to be normalized as given in (A.0.10) so that its normalization differs from (3.1.22) by a factor of[10] c. With this choice the solution for the correlator becomes

$$\langle O(k_1)O(k_2)T_{ij}(k_3)\rangle e^{s,ij} = -2(2\pi)^3 c\delta(\sum_i \boldsymbol{k}_i) e^{s,ij} k_{1i}k_{2j}S. \tag{4.3.1}$$

From the general arguments of section 2 this should be the value for the coefficient function, $\langle OOT_{ij}\rangle e^{s,ij}$, in the wave function (2.2.1).

5 Final Result

Using the wave function (2.2.1) and (2.1.7) it is now a simple matter to find the three point correlator involving two scalar perturbations $\zeta(\boldsymbol{k}_1), \zeta(\boldsymbol{k}_2)$ and one tensor perturbation $\gamma_{ij}(\boldsymbol{k}_3)$ with polarization $e^{s,ij}$.

One finds that it is given by

$$\langle \zeta(k_1)\zeta(k_2)\gamma_s(k_3)\rangle = (2\pi)^3\delta(\sum_i \boldsymbol{k}_i)\frac{1}{\Pi_i(2k_i^3)}(\frac{4H^4}{M_{pl}^4 c})(\frac{H^2}{\dot{\phi}^2})e^{s,ij}k_{1i}k_{2j}S(k_1,k_2,k_3), \tag{5.0.2}$$

[10]The constant c can be set to unity by rescaling the inflaton, but keeping it explicit allows for the normalization of the inflaton to be determined in an independent manner.

with

$$S(k_1,k_2,k_3) = (k_1+k_2+k_3) - \frac{\sum_{i>j} k_i k_j}{(k_1+k_2+k_3)} - \frac{k_1k_2k_3}{(k_1+k_2+k_3)^2}. \quad (5.0.3)$$

In this formula $\dot{\phi}$ is the time derivative of the inflaton and c is a constant which is defined from the normalization of the scalar two-point function given in (A.0.10). This constant can be set to unity by rescaling $\dot{\phi}$. When the two derivative approximation is valid, in the normalization where $c = 1$, $\dot{\phi}$ is related to the slow roll parameter ϵ by (2.0.11). γ_s is related to the tensor perturbation by

$$\gamma_{ij}(k_3) = \gamma_s(k_3)e^s_{ij}(k_3), \quad (5.0.4)$$

where $e^s_{ij}(k_3)$ is the polarization which is transverse and traceless, (3.1.13), with normalization given in (A.0.8).

Equation (5.0.2) is the main result of this thesis.

By comparing this result with the two point functions for the scalar and tensor perturbations given in (A.0.10), (A.0.11) of the appendix A we see that the normalization of the correlator is completely fixed in terms of the normalization of these two two-point functions.

For conventional slow-roll inflation the answer above agrees, up to an overall sign, with that obtained in [17], with $c = 1$ and ϕ being the canonically normalized inflaton.

It should be attempted to cast the result (5.0.2) in a form which does not involve the time derivative of the background inflaton field. This can provide a starting point for further generalization of the result to single-clock models of inflation not necessarily driven by a single scalar field.[11] To do this, we proceed as follows. In case when the two derivative approximation is valid, we have $H^2 = \frac{V(\phi)}{3M_{Pl}^2}$. Differentiating and rearranging, we get $\frac{\dot{\phi}}{H} = \frac{6M_{Pl}^2}{V'(\phi)}\dot{H}$. Eliminating V' using the relation $\epsilon = \frac{1}{2}(\frac{V'}{V})^2$, and eliminating V in favour of the Hubble parameter, we get $\frac{\dot{\phi}}{H} = (\frac{-2\dot{H}}{H^2})^{\frac{1}{2}}$. Using this, we can re-write our result (5.0.2) as
$\langle\zeta(k_1)\zeta(k_2)\gamma_s(k_3)\rangle = -(2\pi)^3\delta(\sum_i \boldsymbol{k}_i)\frac{1}{\Pi_i(2k_i^3)}(\frac{2H^6}{M_{Pl}^4 c\dot{H}})e^{s,ij}k_{1i}k_{2j}S(k_1,k_2,k_3)$.

[11] I am thankful to Prof. Anshuman Maharana, the external thesis examiner for suggesting this point to me.

6 Conclusions

In this thesis we have studied the three point function involving two scalars and one tensor perturbation. We showed that this correlator is completely fixed by the $SO(1,4)$ symmetries of de Sitter space, up to small corrections. Our final result is given in (5.0.2). The normalizations for the scalar and tensor two point functions are given in (A.0.10) and (A.0.11); we see that the normalization of the three point function is fixed in terms of the normalization of the two point functions.

Our result is based on three main assumptions. First, that the inflationary dynamics— including the scalar sector—approximately preserves the full $SO(1,4)$ conformal group of isometries of de Sitter space. Second, that there is only one scalar field during inflation. And third, that the initial state is the Bunch-Davies vacuum. Other than these assumptions the result is general and essentially model independent. In particular it should apply to models where higher derivative corrections in gravity are important, as was discussed in the introduction.

The general nature of this result means that this three point function is observationally a good way to test if the inflationary dynamics had the full conformal group including the special conformal transformations as its symmetries. It is worth emphasizing that the two point functions do not by themselves allow for a test of this feature. In conventional slow-roll inflation there is one relation between the various parameters which arises as follows. The tensor two point function allows for a determination of H^2/M_{Pl}^2 from its normalization and for ϵ, defined in (2.0.6), from its tilt. The normalization of the scalar two-point function goes like $\frac{H^2}{M_{Pl}^2}\frac{H^2}{\dot{\phi}^2}$ and is then fixed since $\frac{\dot{\phi}}{H}$ is determined in terms of ϵ by (2.0.11). However, once higher derivative corrections are included (2.0.11) need not be valid any longer even when the full conformal group is approximately preserved. For example (2.0.8) could receive corrections due to higher powers of curvature becoming important in the action (2.0.2). Thus this relation between the parameters of the two point functions does not allow us to test whether the special conformal transformations were good symmetries during inflation.

Corrections to our result for the three point function will arise from effects which break the $SO(1,4)$ symmetries. These can be of two kinds. Effects which break the special conformal symmetries but preserve scale invariance and effects which break scale invariance. Examples of the breaking of special conformal invariance include a speed of sound which is different from unity. More generally, these effects can be parameterized using the effective Lagrangian approach discussed in [10]. The breaking of scale invariance occurs because the Hubble constant and the inflaton slowly evolve during inflation and are not constant. When the momenta of the three perturbations in the correlator are of the same order of magnitude one immediate way to incorporate some of the resulting corrections is to set the parameters, $H, \dot{\phi}$ which enter in (5.0.2), to take their values at the time of horizon crossing for the modes.[12] More generally, corrections due to the breaking of scale invariance are of order the slow roll parameters and about 1% in order of magnitude.

As stated above our result applies to models of single field inflation. When more than one scalar is present both adiabatic and isocurvature perturbations can be present and it is harder to come up with model independent results. We can always still go to the gauge where $\zeta = 0$, discussed for the single scalar case in 2.1.2. And then work in a basis where the scalar field perturbations, $\delta\phi_i, i = 1, \cdots N$, have a diagonal two point function. Assuming that scalars are approximately massless we get the two-point functions to be[13]

$$\langle \delta\phi_i(\boldsymbol{k}_1)\delta\phi_j(\boldsymbol{k}_2)\rangle = \delta_{ij}(2\pi)^3\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2})\frac{H^2}{M_{pl}^2}\frac{1}{2}\frac{1}{k_1^3}. \tag{6.0.5}$$

The three point functions for the scalar and tensor perturbations then easily follows and is again diagonal in this basis of scalar perturbations and takes the model independent form (5.0.2) (with $c = 1$). The model dependence in the result enters when we try to obtain the three point function in terms of the the curvature perturbation, ζ, which is defined for all time and conserved after the modes cross the horizon. The value of this perturbation and its correlations depend on how the various scalars affect the end of inflation and this is model dependent.

The analysis here is based on earlier papers [17, 19]. In [19] it was shown that working in the de Sitter approximation the three point tensor perturbation can be significantly constrained from symmetry considerations alone. Unlike tensor perturbations when dealing with scalars the small breaking of de Sitter symmetries in the inflationary background cannot be totally ignored. However for the correlation function of interest in this thesis this breaking can be incorporated, at least to leading order in the slow-roll parameters, in a straightforward manner. As explained in section 2 one first works in the gauge where $\zeta = 0$ and calculates the correlation function in terms of the scalar perturbation $\delta\phi$, then transforms to the gauge where the $\zeta \neq 0$ using (2.1.7). The calculation in terms of $\delta\phi$ can be done in de Sitter space and the breaking of de Sitter invariance enters only in the last step

[12]In the squeezed limit, when one momentum is much smaller one can also incorporate similar effects by carrying out an analysis along the lines of section 4.2.

[13]Here we have rescaled $\delta\phi$ to set a possible constant c which appears in the normalization on the RHS to unity.

through the factor of $\frac{\dot{\phi}}{H}$ in (2.1.7). This is analogous to using the relation (**??**) in conformal perturbation theory and computing the correlation function in terms of the scalar operator O in the CFT.

It is important to try to extend this analysis to other correlation functions especially the three point scalar correlator which is observationally most significant. The analysis is more complicated here since in general one cannot get away by simply taking the breaking of the de Sitter symmetries into account in the manner described in the previous paragraph. This can be seen from the results for the conventional slow-roll case in [17] where it was found that the scalar three-point function is suppressed by an additional factor of $\sqrt{\epsilon}$ leading to an answer that goes like[14] $\frac{H^4}{M_{Pl}^4 \epsilon}$. Despite these complications, it would be worthwhile to consider a CFT which has say just the stress tensor and a scalar as its low dimension operators and ask how much the scalar correlators are constrained by CFT considerations alone along the lines of [29].

We have used both scale and special conformal invariance in deriving our result. We have already discussed the possibility that the scalar sector could break the special conformal symmetries badly. On the gravity side translations, rotations and scale invariance uniquely lead to de Sitter space, which is then also invariant under special conformal transformations. However, more generally, when higher spin fields are also excited it is conceivable that one has time dependent solutions with translations, rotations and scale invariance symmetry but without special conformal invariance. It would be worth developing an understanding of such solutions and their possible role in the early Universe.[15] The correlator studied here could be used to distinguish solutions of this type also from de Sitter space.

A The Two Point Function and Normalizations

In this appendix we discuss the two point function and related issues about normalizations of correlation functions. The wave function at quadratic order

[14] This fact also follows from CFT by noting that the three-point function of an exactly marginal operator must vanish.

[15] For a review of higher spin fields and related issues see [30].

can be read off from (2.2.2)

$$\psi = \exp\left(\frac{M_{pl}^2}{H^2}\Big[-\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\delta\phi(\boldsymbol{k})\delta\phi(\boldsymbol{k'})\langle O(-\boldsymbol{k})O(-\boldsymbol{k'})\rangle\right.$$
$$\left.-\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\gamma_s(\boldsymbol{k})\gamma_{s'}(\boldsymbol{k'})\langle T^s(-\boldsymbol{k})T^{s'}(-\boldsymbol{k'})\rangle\Big]\right). \tag{A.0.6}$$

Here the labels s, s' denote the two polarizations of the graviton. In our notation a graviton can be written as a linear combination of its two polarizations

$$\gamma_{ij}(\boldsymbol{k}) = \sum_{s=1,2}\gamma_s e_{ij}^s(\boldsymbol{k}), \tag{A.0.7}$$

where the polarization tensors are normalized so that

$$e^{s,ij}e_{ij}^{s'} = 2\delta^{s,s'}. \tag{A.0.8}$$

For the stress energy tensor we define

$$T^s(\boldsymbol{k}) \equiv T_{ij}(\boldsymbol{k})e^{s,ij}(-\boldsymbol{k}). \tag{A.0.9}$$

Translational and rotational invariance along with scaling symmetry fixes the form of the two point functions to be

$$\langle O(\boldsymbol{k_1})O(\boldsymbol{k_2})\rangle = ck_1^3(2\pi)^3\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2}), \tag{A.0.10}$$
$$\langle T^s(\boldsymbol{k_1})T^{s'}(\boldsymbol{k_2})\rangle = k_1^3(2\pi)^3\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2})(\frac{\delta^{ss'}}{2}). \tag{A.0.11}$$

A constant could have appeared on the RHS of (A.0.11) but that can be absorbed into a redefinition of H. The constant c which appears on the RHS of (A.0.10) could also have been set to unity by rescaling the operator O. However doing so also requires us to rescale the inflaton perturbation $\delta\phi$ which is the source for O. It is convenient instead to not do this rescaling and keep the constant c explicit in (A.0.10).

Substituting (A.0.10), (A.0.11) in the wave function one can easily show that the resulting two-point functions for the perturbations are

$$\langle\delta\phi(\boldsymbol{k_1})\delta\phi(\boldsymbol{k_2})\rangle = (2\pi)^3\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2})\frac{H^2}{M_{pl}^2}\frac{1}{2c}\frac{1}{k_1^3}, \tag{A.0.12}$$
$$\langle\gamma_s(\boldsymbol{k_1})\gamma_{s'}(\boldsymbol{k_2})\rangle = (2\pi)^3\delta(\boldsymbol{k_1}+\boldsymbol{k_2})\frac{H^2}{M_{pl}^2}\frac{1}{2k_1^3}(2\delta_{s,s'}). \tag{A.0.13}$$

Using (2.1.7) we get from (A.0.12) that the two point function of the scalar perturbation is

$$\langle\zeta(\boldsymbol{k_1})\zeta(\boldsymbol{k_2})\rangle = (2\pi)^3\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2})\frac{H^2}{M_{pl}^2}\frac{1}{2c}\frac{H^2}{\dot{\phi}^2}\frac{1}{k_1^3}. \tag{A.0.14}$$

(A.0.13), (A.0.14) agree with the results of the standard slow-roll two - derivative theory when $c=1$ and ϕ is the canonically normalized inflaton. More generally c can be set to unity by rescaling ϕ.

B Details of the Equations for Special Conformal Invariance

From (3.1.2) and (3.1.3) we learn that

$$\begin{aligned} M_{ij}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) &= k_{1i}k_{1j}f_1(k_1,k_2,k_3)+k_{2i}k_{2j}f_1(k_2,k_1,k_3) \\ &\quad +(k_{1i}k_{2j}+k_{2i}k_{1j})f_2(k_1,k_2,k_3)+\delta_{ij}f_3(k_1,k_2,k_3). \end{aligned} \tag{B.0.15}$$

Multiplying by $k_{3i}(k_{1j}-\frac{k_{3j}(k_1.k_3)}{k_3^2})$ we get

$$k_{3i}(k_{1j}-\frac{k_{3j}(k_1.k_3)}{k_3^2})M_{ij}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = [k_1^2-\frac{(\boldsymbol{k_1}\cdot\boldsymbol{k_3})^2}{k_3^2}]\left((\boldsymbol{k_1}\cdot\boldsymbol{k_3})(f_1-f_2)+(\boldsymbol{k_2}\cdot\boldsymbol{k_3})(f_2-f_1^T)\right). \tag{B.0.16}$$

Now, choosing $\boldsymbol{b}\propto\boldsymbol{k}_1-\frac{\boldsymbol{k_3}(\boldsymbol{k}_1\cdot\boldsymbol{k}_3)}{k_3^2}$ in (3.1.18) we get

$$\begin{aligned} &4k_{1i}k_{1j}e^{s,ij}\Big[(1+k_1\frac{\partial}{\partial k_1})f_1-(1+k_2\frac{\partial}{\partial k_2})f_1^T+(k_2\frac{\partial}{\partial k_2}-k_1\frac{\partial}{\partial k_1})f_2 \\ &\qquad -\frac{(\boldsymbol{k_1}\cdot\boldsymbol{k_3})}{k_3}\frac{\partial}{\partial k_3}(f_1-f_2)+\frac{(\boldsymbol{k_3}\cdot\boldsymbol{k_2})}{k_3}\frac{\partial}{\partial k_3}(f_1^T-f_2)\Big] \\ &+k_{1i}k_{2j}e^{s,ij}\Big[k_1^2-\frac{(\boldsymbol{k_1}\cdot\boldsymbol{k_3})}{k_3^2}\Big](\Theta(k_1)-\Theta(k_2))(2f_2-f_1-f_1^T)=0. \end{aligned} \tag{B.0.17}$$

Using $k_{2j}e^{s,ij}=-(k_{1j}+k_{3j})e^{s,ij}=-k_{1j}e^{s,ij}$, (B.0.17) reduces to

$$\begin{aligned} e^{s,ij}k_{1i}k_{1j}\Bigg\{&4[(1+k_1\frac{\partial}{\partial k_1})f_1-(1+k_2\frac{\partial}{\partial k_2})f_1^T+(k_2\frac{\partial}{\partial k_2}-k_1\frac{\partial}{\partial k_1})f_2 \\ &-\frac{(\boldsymbol{k_1}\cdot\boldsymbol{k_3})}{k_3}\frac{\partial}{\partial k_3}(f_1-f_2)+\frac{(\boldsymbol{k_2}\cdot\boldsymbol{k_3})}{k_3}\frac{\partial}{\partial k_3}(f_1^T-f_2)] \\ &-(k_1^2-\frac{(\boldsymbol{k_1}\cdot\boldsymbol{k_3})^2}{k_3^2})(\Theta(k_1)-\Theta(k_2))(2f_2-f_1-f_1^T)\Bigg\}=0. \end{aligned} \tag{B.0.18}$$

Since the polarization can be chosen so that $e^{s,ij}k_{1i}k_{1j}$ does not vanish the quantity within the curly brackets must vanish leading to

$$4\Big[(k_1\frac{\partial}{\partial k_1}-\frac{(\boldsymbol{k_1}\cdot\boldsymbol{k_3})}{k_3}\frac{\partial}{\partial k_3})(f_1-f_2)-(k_2\frac{\partial}{\partial k_2}-\frac{(\boldsymbol{k_2}\cdot\boldsymbol{k_3})}{k_3}\frac{\partial}{\partial k_3})(f_1^T-f_2)$$
$$+(f_1-f_2)-(f_1^T-f_2)\Big]-(k_1^2-\frac{(\boldsymbol{k_1}\cdot\boldsymbol{k_3})^2}{k_3^2})(\Theta(k_1)-\Theta(k_2))(2f_2-f_1-f_1^T)=0. \quad \text{(B.0.19)}$$

In terms of $S\equiv[(f_1-f_2)+(f_1^T-f_2)]/2$ and $A\equiv[(f_1-f_2)-(f_1^T-f_2)]/2$ this becomes

$$4\Big[(k_1\frac{\partial}{\partial k_1}-\frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_3}{k_3}\frac{\partial}{\partial k_3})(S+A)-(k_2\frac{\partial}{\partial k_2}-\frac{\boldsymbol{k}_2\cdot\boldsymbol{k}_3}{k_3}\frac{\partial}{\partial k_3})(S-A)+2A\Big]$$
$$-2(k_1^2-\frac{(\boldsymbol{k}_1\cdot\boldsymbol{k}_3)^2}{k_3^2})(\Theta(k_1)-\Theta(k_2))S=0. \quad \text{(B.0.20)}$$

Similarly, (B.0.16) in terms of S,A becomes,

$$k_{3i}(k_{1j}-\frac{k_{3j}(\boldsymbol{k_1}\cdot\boldsymbol{k_3})}{k_3^2})M_{ij}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)=[k_1^2-\frac{(\boldsymbol{k_1}\cdot\boldsymbol{k_3})^2}{k_3^2}]((\boldsymbol{k_1}\cdot\boldsymbol{k_3})(S+A)-(\boldsymbol{k_2}\cdot\boldsymbol{k_3})(S-A)), \quad \text{(B.0.21)}$$

which can be used to solve for A and gives

$$A=\frac{(\boldsymbol{k_1}-\boldsymbol{k_2})\cdot\boldsymbol{k_3}}{k_3^2}S-\frac{M_{ij}k_{3i}\epsilon_{\perp j}}{k_3^2\epsilon_\perp^2}, \quad \text{(B.0.22)}$$

where $\epsilon_{\perp j}\equiv k_{1j}-\frac{k_{3j}(\boldsymbol{k_1}\cdot\boldsymbol{k_3})}{k_3^2}$.(We caution the reader that this is different from the null transverse vector ϵ_3 that has appeared above.) Substituting in (B.0.20) this leads to

$$4\Big[(\frac{\boldsymbol{k}_2\cdot\boldsymbol{k}_3}{k_3^2}k_1\frac{\partial}{\partial k_1}-\frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_3}{k_3^2}k_2\frac{\partial}{\partial k_2})S+\frac{\boldsymbol{k}_3\cdot(\boldsymbol{k}_2-\boldsymbol{k}_1)}{k_3^2}S$$
$$+(k_1\frac{\partial}{\partial k_1}+k_2\frac{\partial}{\partial k_2}+k_3\frac{\partial}{\partial k_3})(\frac{M_{ij}k_{3i}\epsilon_{\perp j}}{2k_3^2\epsilon_\perp^2})+\frac{M_{ij}k_{3i}\epsilon_{\perp j}}{k_3^2\epsilon_\perp^2}\Big] \quad \text{(B.0.23)}$$
$$+(k_1^2-\frac{(\boldsymbol{k}_1\cdot\boldsymbol{k}_3)^2}{k_3^2})(\Theta(k_1)-\Theta(k_2))S=0.$$

Next, using the Ward identity, (3.1.23) we get

$$\frac{M_{ij}k_{3i}\epsilon_{\perp j}}{\epsilon_\perp^2}=-k_1^3+k_2^3. \quad \text{(B.0.24)}$$

Substituting (B.0.24) in (B.0.23) after some algebra gives (3.1.21).

Finally we consider taking $\boldsymbol{b}$ to be orthogonal to all $\boldsymbol{k}_i$ so that $\boldsymbol{b}\cdot\boldsymbol{k}=0$. We can also choose a polarization so that $b_i k_{1j} e^{s,ij} \neq 0$. (3.1.18) then gives

$$(1+k_1\partial_{k_1})f_1-(1+k_2\partial_{k_2})f_1^T+(k_2\partial_{k_2}-k_1\partial_{k_1})f_2-\frac{(\boldsymbol{k_3}\cdot\boldsymbol{k_1})}{k_3}\partial_{k_3}(f_1-f_2)$$
$$+\frac{(\boldsymbol{k_3}\cdot\boldsymbol{k_2})}{k_3}\partial_{k_3}(f_1^T-f_2)=0. \tag{B.0.25}$$

The reader will notice that the LHS above is the first two lines of the LHS of (B.0.18). Thus the analysis above when applied to (B.0.25) directly leads to (3.1.24).

References

[1] E. Komatsu, *Hunting for Primordial Non-Gaussianity in the Cosmic Microwave Background*, *Class.Quant.Grav.* **27** (2010) p. 124010, [arXiv:1003.6097].

[2] E. Komatsu, N. Afshordi, N. Bartolo, D. Baumann, J. Bond, *et. al.*, *Non-Gaussianity as a Probe of the Physics of the Primordial Universe and the Astrophysics of the Low Redshift Universe*, arXiv:0902.4759.

[3] N. Bartolo, E. Komatsu, S. Matarrese, and A. Riotto, *Non-Gaussianity from inflation: Theory and observations*, *Phys.Rept.* **402** (2004) pp. 103–266, [astro-ph/0406398].

[4] E. Komatsu, *The pursuit of non-gaussian fluctuations in the cosmic microwave background*, astro-ph/0206039.

[5] S. Weinberg, *Cosmology.* Oxford University Press, 2008.

[6] X. Chen, M.-x. Huang, S. Kachru, and G. Shiu, *Observational signatures and non-Gaussianities of general single field inflation*, *JCAP* **0701** (2007) p. 002, [hep-th/0605045].

[7] E. Silverstein and D. Tong, *Scalar speed limits and cosmology: Acceleration from D-cceleration*, *Phys.Rev.* **D70** (2004) p. 103505, [hep-th/0310221].

[8] M. Alishahiha, E. Silverstein, and D. Tong, *DBI in the sky*, *Phys.Rev.* **D70** (2004) p. 123505, [hep-th/0404084].

[9] N. Arkani-Hamed, P. Creminelli, S. Mukohyama, and M. Zaldarriaga, *Ghost inflation*, *JCAP* **0404** (2004) p. 001, [hep-th/0312100].

[10] C. Cheung, P. Creminelli, A. L. Fitzpatrick, J. Kaplan, and L. Senatore, *The Effective Field Theory of Inflation*, *JHEP* **0803** (2008) p. 14, [arXiv:0709.0293].

[11] H. Osborn and A. Petkou, *Implications of conformal invariance in field theories for general dimensions*, *Annals Phys.* **231** (1994) pp. 311–362, [hep-th/9307010].

[12] S. Giombi, S. Prakash, and X. Yin, *A Note on CFT Correlators in Three Dimensions*, arXiv:1104.4317.

[13] J. Maldacena, A. Zhiboedov, and A. Zhiboedov, *Constraining conformal field theories with a slightly broken higher spin symmetry*, arXiv:1204.3882.

[14] J. Maldacena and A. Zhiboedov, *Constraining Conformal Field Theories with A Higher Spin Symmetry*, arXiv:1112.1016.

[15] A. Strominger, *The dS / CFT correspondence*, *JHEP* **0110** (2001) p. 034, [hep-th/0106113].

[16] E. Witten, *Quantum gravity in de Sitter space*, hep-th/0106109.

[17] J. M. Maldacena, *Non-Gaussian features of primordial fluctuations in single field inflationary models*, *JHEP* **0305** (2003) p. 013, [astro-ph/0210603].

[18] D. Anninos, T. Hartman, and A. Strominger, *Higher Spin Realization of the dS/CFT Correspondence*, arXiv:1108.5735.

[19] J. M. Maldacena and G. L. Pimentel, *On graviton non-Gaussianities during inflation*, arXiv:1104.2846.

[20] I. Antoniadis, P. O. Mazur, and E. Mottola, *Conformal Invariance, Dark Energy, and CMB Non-Gaussianity*, *JCAP* **1209** (2012) p. 024, [arXiv:1103.4164].

[21] A. Bzowski, P. McFadden, and K. Skenderis, *Holographic predictions for cosmological 3-point functions*, *JHEP* **1203** (2012) p. 091, [arXiv:1112.1967].

[22] P. McFadden and K. Skenderis, *Cosmological 3-point correlators from holography*, *JCAP* **1106** (2011) p. 030, [arXiv:1104.3894].

[23] P. McFadden and K. Skenderis, *Holographic Non-Gaussianity*, *JCAP* **1105** (2011) p. 013, [arXiv:1011.0452].

[24] F. Larsen and R. McNees, *Inflation and de Sitter holography*, *JHEP* **0307** (2003) p. 051, [hep-th/0307026].

[25] F. Larsen, J. P. van der Schaar, and R. G. Leigh, *De Sitter holography and the cosmic microwave background*, *JHEP* **0204** (2002) p. 047, [hep-th/0202127].

[26] S. Weinberg, *Effective Field Theory for Inflation*, *Phys.Rev.* **D77** (2008) p. 123541, [arXiv:0804.4291].

[27] K. Schalm, G. Shiu, and T. van der Aalst, *Consistency condition for inflation from (broken) conformal symmetry*, arXiv:1211.2157.

[28] A. Bzowski, P. McFadden, and K. Skenderis, *Holography for inflation using conformal perturbation theory*, arXiv:1211.4550.

[29] I. Heemskerk, J. Penedones, J. Polchinski, and J. Sully, *Holography from Conformal Field Theory*, *JHEP* **0910** (2009) p. 079, [arXiv:0907.0151].

[30] M. Vasiliev, *Higher spin gauge theories in any dimension*, *Comptes Rendus Physique* **5** (2004) pp. 1101–1109, [hep-th/0409260].

[31] Planck Collaboration, *Planck 2015 results. XVII. Constraints on primordial non-Gaussianity*, arxiv:1502.01592