Black Holes in Large D and Scattering in $\mathcal{N} = 1$ Susy Matter Chern Simons Theories

A Thesis

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By

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Declaration

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions. The work was done under the guidance of Prof. Shiraz Minwalla, at the Tata Institute of Fundamental Research, Mumbai.

Subhajit Mazumdar.
Subhajit Mazumdar

In my capacity as supervisor of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

Prof. Shiraz Minwalla
Date: 22nd April 2019

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I would like to thank Yogesh Dandekar, Suman Kundu, Amiya Mishra, Arunabha Saha, Sachin Jain, Shuichi Yokoyama, Karthik Inbasekar, Anandita De and V Umesh for wonderful collaborations.

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To my Parents

LIST OF PUBLICATIONS

Papers relevant to the thesis work:

- Y. Dandekar, S. Mazumdar, S. Minwalla, A. Saha, " Unstable 'black branes' from scaled membranes at large D,"
  arXiv:1609.02912 , JHEP 1612 (2016) 140

- Y. Dandekar, A. De, S. Mazumdar, S. Minwalla, A. Saha, " The large D black hole Membrane Paradigm at first subleading order,"
  arXiv:1607.06475, JHEP 1612 (2016) 113

- K. Inbasekar, S. Jain, S. Mazumdar, S. Minwalla, V. Umesh, S. Yokoyama , " Unitarity, crossing symmetry and duality in the scattering of $\mathcal{N} = 1$ susy matter Chern-Simons theories,"
  arXiv:1505.06571, JHEP 1510 (2015) 176

Other papers not relevant to the thesis work:

- An Action for and Hydrodynamics from the improved Large D membrane Y. Dandekar, S. Kundu, S. Mazumdar, S. Minwalla, A. Mishra, A. Saha, "An Action for and Hydrodynamics from the improved Large D membrane,"
  arXiv:1712.09400

COLLABORATIONS AND SPECIFICATIONS OF MY CONTRIBUTIONS

As mentioned above, my thesis is based on three papers [1–3] written in collaboration with other authors. As part of these collaborations I participated actively in the development of all of the material in each of these papers. In the interests of clarity I have presented all the results obtained in each of these papers in this thesis. I would like however to note that I played a particularly central role in the development of the following sections of my thesis:

Chapter 1:

- All the material in this Chapter (based on [1]).

Chapter 2:

- Section 3.5 of this Chapter (based on [2]).

Chapter 3:

- Sections 4.3, 4.4 and 4.5 of this Chapter (based on [3]).

In addition to the work presented above, in section 1.4 I had included a preliminary unpublished analysis of "The Polarizibilities of Black Holes" in my thesis synopsis. I have not presented this material in my thesis as I have subsequently realized that the analysis presented there misses a subtlety and needs more work (which is currently ongoing) to make solid.

Contents

1 Introduction

This thesis is divided into two parts. In the first part of this thesis follows up on the recent discovery [4] that the dynamics of black holes in a large number of dimensions can be reformulated as the dynamics of a non gravitational membrane propagating in flat space. In Part A (Chapter 1 and Chapter 2) of this thesis I present work that further develops this black hole - membrane correspondence, and in particular present two nontrivial checks and applications of the correspondence.

In Part B (Chapter 3) of this thesis I present on the exact computation of S matrices of supersymmetric matter Chern Simons theories in the large N limit. I obtain these S matrices by solving for offshell four point functions of this theory in lightcone gauge and then taking the onshell limit. I focus on the physical properties of the S channel S matrix, and demonstrate that it has a bound state pole which goes massless at a particular value of the coupling.

1.1 Black Holes in Large D

Black holes are simultaneously the best understood and most enigmatic solutions of Einstein's equations of general relativity. Black holes are well understood in the following sense; several stationary black hole solutions (like the Schwarzschild, Kerr and Reisnner Nordstrom solutions) are exactly known in every dimension and generally take a reasonably simple form. Even at the classical level, however, the simplicity of stationary black holes is deceptive. Black holes can undergo extremely complicated dynamical motions - like the collisions recently observed by the LIGO experiment - which are very hard to capture analytically.

The classical study of black holes has also played a key role in studies of the AdS/CFT correspondence of string theory. This fascinating correspondence has allowed us to compute all kinds of properties of strongly coupled field theories - ranging from the detailed equations that govern their hydrodynamics to entanglement properties and the study of thermalization in these theories - all by studying the classical dynamics of black holes in Anti-de Sitter (AdS) space.

Although my work on black holes so far has focussed on their classical aspects, I would like to parenthetically note that once quantum mechanics is added to the game, the mysteries of black holes multiply. Even stationary black holes carry more entropy (and so in that sense are more complex) than any other object of comparable volume. The contrast between the classical simplicity and quantum complexity of black holes leads to the fascinating - and in my view, as yet, unresolved - information paradox of black hole physics, whose resolution may turn out to hold the key for the next revolution in the study of quantum gravity.

Returning to the classical domain, it is clear that good control over the classical dynamics of black holes in sufficiently complicated situations can teach us a great deal about diverse aspects of theoretical physics. However the complicated nature of Einstein's equations make analytically controlling a violently dynamical process - like a black hole collision - next to impossible. In this situation, the natural instinct of a theoretical physicist is to search for a parameter in which to set up a perturbative expansion of the problem. However Einstein's equations in vacuum are parameter free.

It has recently been noted that the classical dynamics of black holes simplifies in the limit of a large number of dimensions. The key observation - first made by Emparan, Suzuki, Tanabe and collaborators in [5–11] - is that black holes at large D have two effective length scales. The first of these, r_0, is the size of the black holes. The second is the thickness of the black hole's gravitational tail, i.e. the distance beyond the black hole event horizon after which the gravitational potential rapidly decays to zero. In four dimensions the black hole size and thickness are comparable. In the large D limit, however, the thickness of the gravitational tail turns out to scale like r_0/D [5] and so is much smaller than the the black hole size.

This observation suggests the possibility of an effective 'dimensional reduction' of black hole dynamics to the membrane region; a slab of spacetime of thickness $1/D$ centered around the codimension one event horizon. Motivated by this observation, several papers written over the last three years have demonstrated that black hole physics at large D can be reformulated in terms of dual non gravitational equations. In broad terms there have been two different approaches to this problem. In this part of this introduction I will first summarize the approach to this problem is called the 'membrane paradigm' approach. In this subsection I will first proceed to give a detailed introduction to the approach to this problem that I follow in my work- namely the membrane paradigm approach. I will then briefly review a second approach to this problem (one developed by Emparan, Suzuki Tanabe and collaborators) called the black brane approach. Finally I will give a brief summary of the original work on this topic contained in this thesis.

1.1.1 The large D membrane Paradigm

The first of these approaches is laid out in the 'membrane paradigm' papers of [2, 4, 12] (see also [13–15] [1]) . The authors of these papers have demonstrated that nonlinear black hole dynamics can be reformulated in terms of the equations of motion of a non gravitational membrane that lives in flat space. The variables of this problem are the shape of the membrane and a velocity field on this membrane[2]. Einstein's equations force the membrane variables to obey a set of equations of motion. There are as many equations of motion as variables, so the membrane description defines a good initial value problem. We emphasize that the membrane equations of [2, 4, 12] apply to arbitrarily nonlinear and completely

[1] These papers worked out the equations that govern the shape of the membrane, described later in this paragraph, for stationary configurations. Atleast in the absence of a cosmological constant, these equations may be shown to follow from the more general dynamical membrane equations of [2, 4, 12] upon inserting an appropriate stationary ansatz, and so are special cases of the general membrane equations.

[2] The variables of the membrane also include a charge field for charged black holes. In this chapter, however, we focus solely on solutions of the vacuum Einstein equations $R_{MN} = 0$.

dynamical black hole motions. There are, in particular, no restrictions on the initial shape of the membrane which can be chosen to be any sufficiently smooth codimension one submanifold of flat spacetime; the evolution of this shape (and the membrane velocity fields) in time is, of course, governed by the membrane equations of motion. We now explain how this works in more detail.

Consider a class of D dimensional metrics of the form

$$g_{MN} = \eta_{MN} + \frac{(n_M - u_M)(n_N - u_N)}{\psi^{D-3}} \tag{1}$$

The metrics (1) are parametrized by a smooth D dimensional function ψ and a smooth oneform field u_M. n_M in (1) is the normal field to surfaces of constant ψ, (i.e. $n_M = \frac{\partial_M \psi}{\sqrt{\partial_P \psi \partial_Q \psi \eta^{PQ}}}$). The oneform field u_M is assumed to be unit normalized (i.e. $u_N u_M \eta^{MN} = -1$) and tangent to surfaces of constant ψ (i.e. $u_M n_N \eta^{MN} = 0$).

In order to gain intuition for space times of the form (1) it is useful to first consider a special case. Working with coordinates in which the metric on Minkowski space takes the form

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{D-2}^2,$$

the choice $u = -dt$ and $\psi = \frac{r}{r_0}$ turns (1) into the metric of a Schwarzschild black hole of radius r_0 in the so called Kerr Schild coordinates.

Note $\psi = 1$ is the event horizon of the Schwarzschild black hole. More generally the surface $\psi = 1$ is easily verified to be a null submanifold of (1) for every choice of ψ and u. This null manifold coincides with the event horizon of the (1) provided that ψ and u are chosen such that the metric (1) settles down into a collection of stationary black holes at late times. Following [4, 12] we refer to the submanifold $\psi = 1$ as the membrane world volume. [3]

[3]Through this chapter we assume that ψ in (1) is chosen to ensure that the membrane surface is a smooth codimension one surface that is timelike when viewed as a submanifold of flat space (we have emphasized above that this surface is a null submanifold of the metric (1)). We also assume that ψ is

Note that as ψ increases past unity $\frac{1}{\psi^{D-3}}$ decays to zero very rapidly. This decay is exponential in D once $\psi - 1 \gg \frac{1}{D}$. It follows that (1) represents a class of asymptotically flat spacetimes with the following property; the spacetime outside the event horizon deviates significantly from flat space only in a slab of thickness $\frac{1}{D}$ around the event horizon. We will refer to this as the membrane region.

[4, 12] set out to characterize solutions of the vacuum Einstein equations, $R_{MN} = 0$, that reduce to metrics of the form (1) in the large D limit, with corrections in a power series in $\frac{1}{D}$. As we have reviewed above, when $\psi - 1 \gg \frac{1}{D}$ the spacetimes (1) reduce to flat space. Deviations from flatness are nonperturbatively small in the $\frac{1}{D}$ expansion. Thus Einstein's equations are automatically solved at all order in $1/D$ outside the membrane region. In order to obtain a true solution of Einstein's equations, the solution (1) needs to be corrected order by order in the $\frac{1}{D}$ expansion only in the membrane region.

Consider a region of size $\frac{1}{D}$ centered around any point x_0 on the event horizon of (1). It may be shown that the metric of this ball is closely approximated by the metric in an equivalent small region centered around the appropriate event horizon point of *some* boosted Schwarzschild black hole provided that

$$\nabla^2 \left(\frac{1}{\psi^{D-3}} \right) = 0, \quad \nabla . u = 0, \tag{2}$$

(the contraction of all indices is achieved by use of the metric η_{MN} in the equations above) [4]

chosen to ensure that $\frac{1}{\psi^{D-3}}$ decays at spatial infinity.

[4] When an expression like ∇^2 acts on $\frac{1}{\psi^{D-3}}$ we get two distinct terms of order D^2 in two ways. The first term is $\propto (D-3)(D-2)\frac{(\nabla\psi)^2}{\psi^{D-1}}$. The second term is $\propto (D-3)\frac{\nabla^2\psi}{\psi^{D-2}}$. Though the second term has one less explicit factor of D than the first, it actually contributes at the same order in the $1/D$ expansion - i.e. at leading order - because of the contraction of indices in ∇^2. This is the reason that (1) solves the leading order equations only if $\nabla^2\psi$ takes the same value as it does in a Schwarzschild black hole, leading to the first requirement listed in (2). In a similar manner worldvolume derivatives of the horizon shape and velocity field - which are of order unity - compete with derivatives acting on $\frac{1}{\psi^{D-3}}$ only if their order is enhanced by the contraction of a worldvolume index. The only first derivative expression involving the black hole velocity that has such a contraction is $\nabla . u$. It follows that (1) satisfies the leading order equations only if $\nabla . u$ takes the same value as it does on a Schwarzschild black hole. This leads to the

. These equations need only be satisfied at leading order in D and can be violated at subleading orders. As Schwarzschild black holes are exact solutions to Einstein's equations, it follows as a consequence that the spacetimes (1) *almost* solve Einstein's equations in the membrane region, provided that (2) is satisfied at every point on the membrane.

The statement that Einstein's equations are 'almost' solved in the membrane region has the following precise meaning. When evaluated in the membrane region the four derivative scalar $R_{AB}R^{AB}$ is in general of order D^4. This estimate follows immediately from the fact that the metric varies on a length scale of order $1/D$ in the membrane region. Once we impose (2), on the other hand, $R_{AB}R^{AB}$ turns out to be of order D^2, i.e. In a coordinate system in which all components of the metric are of order unity, R_{AB} is of order D; one order lower than the generic order suggested by a dimensional estimate. In other words (2) ensures that Einstein's equations are obeyed to leading order - but are generically violated at first subleading order. Consequently the metrics (1) - with the conditions (2) imposed at leading order- are plausible starting points for the construction of true solutions of Einstein's equations in a power series in $\frac{1}{D}$.

The authors of [4, 12] were able to carry out this perturbative expansion to first subleading order in $\frac{1}{D}$ (see below for a review). Interestingly they discovered that arbitrary metrics of the form (1) could *not* be corrected to yield regular solutions to Einstein's equations at next order in $\frac{1}{D}$. It turns out to be possible to correct (1) at first order in $1/D$ only when the fields ψ and u obey an integrability constraint - a membrane equation of motion - that we will describe in considerable detail below. Whenever this condition is obeyed, a regular correction (of order $1/D$) to the metric (1) was found in [4, 12]. The corrected metric obeys $R_{AB} = \mathcal{O}(1)$ [5] ; i.e. once the corrections are taken into account, Einstein's equations are solved at leading *and first subleading order* in $\frac{1}{D}$.

We now turn to a description of the integrability constraints mentioned in the

second of (2).

[5]More precisely, $R_{AB} = \mathcal{O}(1)$ in coordinates in which all metric components are of order unity. More generally, $R_{AB}R^{AB}$ is of order unity.

previous paragraph. Consider the surface $\psi = 1$, viewed as a submanifold of flat space with metric η_{MN}; we refer to this submanifold as the membrane. Let K_{MN} represent the extrinsic curvature of this (generically timelike) submanifold. Recall also that the velocity oneform field u_M on the membrane surface is tangent to the membrane and so may be regarded as a oneform field in the membrane world volume. The authors of [4, 12] found that the metric (1) could be corrected to a regular [6] solution of Einsteins equations at first order if and only if the following constraints are obeyed

$$\left(\frac{\nabla^2 u_A}{\mathcal{K}} - \frac{\nabla_A \mathcal{K}}{\mathcal{K}} + u_C K^C_A - u.\nabla u_A \right) \mathcal{P}^A_B = 0 \tag{3}$$

where $\mathcal{P}^A_B = \delta^A_B + u^A u_B$ is the projector orthogonal to the velocity vector on the membrane world volume, and all covariant derivatives are taken with respect to the induced metric on the membrane. The quantity $\mathcal{K}$ is the trace of the extrinsic curvature of the membrane worldvolume.

The integrability conditions (3) have an interesting interpretation. They may be thought of as a set of $D-2$ equations for $D-2$ variables (one of these variables is the shape of the membrane, and the other $D-3$ variables are the components of the unit normalized, divergence free velocity field). In other words the equations (3) define an initial value problem for membrane dynamics. As every configuration that obeys (3) gives rise to a metric that obeys Einstein's equations to the appropriate order in $1/D$, it follows that solutions of the membrane equations (3) are in one to one correspondence with asymptotically flat dynamical black hole configurations that solve Einstein's equations to first subleading order in $1/D$.

[6]By a regular solution we mean a solution with a smooth event horizon that is regular everywhere outside the event horizon.

1.1.2 The large D 'black brane' approach

A second approach is that of the 'scaled black brane' papers of [16, 17] (see also [18, 19]). These papers study small fluctuations about the p dimensional 'black brane'; a spacetime given by the direct product of the Schwarzschild solution in $R^{D-p-1,1}$ and R^p. The authors of [16] consider fluctuations that preserve $SO(D-p-1)$ isometry but vary in the R^p direction over length scales of order $\frac{1}{\sqrt{D}}$ and time scales of order unity. Focusing attention on wiggles of the event horizon of amplitude $\frac{1}{D}$ and on boost velocities of the horizon of order $\frac{1}{\sqrt{D}}$, the authors of [16] were able to derive a set of effective non gravitational nonlinear equations that completely reproduce black brane dynamics in the scaled large D limit described above. This scaling limit is of particular interest because it turns out to capture the Gregory-Laflamme instability of black branes at large D. [7] [8]

1.1.3 Original Results in this Thesis

In this thesis I report two separate sets of original results pertaining to the dynamics of black holes at large D. The first result pertains to the relationship between the two approaches - namely the membrane paradigm and the black brane approaches reviewed above. In Chapter 1 below I demonstrate that the equations of the 'black brane' approach follow as a special case (a particular scaling limit) of the more general membrane paradigm equations. This observation unifies the two different approaches to large D black hole physics. The material contained in chapter 1, though obtained in collaboration, is principally my own work. The second result presented in Chapter 2 concerns a generalization of the derivation of the membrane equations - previously presented only to leading order in $1/D$ - to arbitrary order (in principal) and first subleading order (in practice), and the use of these results to obtain the quasinormal spectrum of black holes corrected

[7] [16] has subsequently been generalized to the the study of charged black branes in [17]. As mentioned above, however, in this chapter we focus attention on uncharged black holes and black branes.

[8] Additional recent studies of black hole physics at large D include [20–25]).

to first subleading order in 1/D. The work presented in Chapter 2 was also obtained in collaboration; though I collaborated in working out every aspect of the work presented in the Chapter, the results of section 3.5 are principally my own work.

1.2 Scattering in $\mathcal{N} = 1$ Susy Matter Chern Simons Theories

Non-Abelian $U(N)$ gauge theories in three spacetime dimensions are dynamically rich. At low energies parity preserving gauge self interactions are generically governed by the Yang-Mills action

$$\frac{1}{g_{YM}^2}\int d^3x \ \mathrm{Tr}\ F_{\mu\nu}^2\ . \tag{4}$$

As g_{YM}^2 has the dimensions of mass, gluons are strongly coupled in the IR. In the absence of parity invariance the gauge field Lagrangian generically includes an additional Chern-Simons term and schematically takes the form

$$\frac{i\kappa}{4\pi}\int \mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right) - \frac{1}{4g_{YM}^2}\int d^3x \ \mathrm{Tr}\ F_{\mu\nu}^2\ . \tag{5}$$

The Lagrangian (5) describes a system of *massive* gluons; with mass $m \propto \kappa g_{YM}^2$. At energies much lower than g_{YM}^2 (5) has no local degrees of freedom. The effective low energy dynamics is topological, and is governed by the action (5) with the Yang-Mills term set to zero. This so called pure Chern-Simons theory was solved over twenty five years ago by Witten [26]; his beautiful and nontrivial exact solution has had several applications in the study of two dimensional conformal field theories and the mathematical study of knots on three manifolds.

Let us now add matter fields with standard, minimally coupled kinetic terms, (in any representation of the gauge group) to (5). The resulting low energy dynamics is particularly simple in the limit in which all matter masses are parametrically smaller than g_{YM}^2. In order to focus on this regime we take the limit $g_{YM}^2 \to \infty$ with masses of matter fields held fixed. In this limit the Yang-Mills term in (5) can be ignored and we obtain a Chern-Simons self coupled gauge theory minimally coupled to matter fields. While the gauge fields are non propagating, they mediate nonlocal interactions between matter fields.

In order to gain intuition for these interactions it is useful to first consider the special case $N = 1$, i.e. the case of an Abelian gauge theory interacting with a unit charge scalar field. The gauge equation of motion

$$\kappa \varepsilon^{\mu\nu\rho} F_{\nu\rho} = 2\pi J^{\mu} \tag{6}$$

ensures that each matter particle traps $\frac{1}{\kappa}$ units of flux (where $i \int F = 2\pi$ is defined as a single unit of flux). It follows as a consequence of the Aharonov-Bohm effect that exchange of two unit charge particles results in a phase $\frac{\pi}{\kappa}$; in other words the Chern-Simons interactions turns the scalars into anyons with anyonic phase $\pi\nu = \frac{\pi}{\kappa}$.

The interactions induced between matter particles by the exchange of non-abelian Chern-Simons gauge bosons are similar with one additional twist. In close analogy with the discussion of the previous paragraph, the exchange of two scalar matter quanta in representations R_1 and R_2 of $U(N)$ results in the phase $\frac{\pi T_{R_1} \cdot T_{R_2}}{\kappa}$ where T_R is the generator of $U(N)$ in the representation R. The new element in the non-abelian theory is that the phase obtained upon interchanging two particles is an operator (in $U(N)$ representation space) rather than a number. The eigenvalues of this operator are given by

$$\nu'_R = \frac{c_2(R_1) + c_2(R_2) - c_2(R')}{2\kappa} \tag{7}$$

where $c_2(R)$ is the quadratic Casimir of the representation R and R' runs over the finite set of representations that appear in the Clebsh-Gordon decomposition of the tensor product of R_1 and R_2. In other words the interactions mediated by non-abelian Chern-Simons coupled gauge fields turns matter particles into non-abelian anyons.

In some ways anyons are qualitatively different from either bosons or fermions.

For example anyons (with fixed anyonic phases) are never free: there is no limit in which the multi particle anyonic Hilbert space can be regarded as a 'Fock space' of a single particle state space. Thus while matter Chern-Simons theories are regular relativistic quantum field theories from a formal viewpoint, it seems possible that they will display dynamical features never before encountered in the study of quantum field theories. This possibility provides one motivation for the intensive study of these theories.

Over the last few years matter Chern-Simons theories have been intensively studied in two different contexts. The $\mathcal{N} = 6$ supersymmetric ABJ and ABJM theories [27, 28] have been exhaustively studied from the viewpoint of the AdS/CFT correspondence [29, 30]. Several other supersymmetric Chern-Simons theories with $\mathcal{N} \geq 2$ supersymmetry have also been intensively studied, sometimes motivated by brane constructions in string theory. The technique of supersymmetric localization has been used to perform exact computations of several supersymmetric quantities [31–36] (indices, supersymmetric Wilson loops, three sphere partition functions). These studies have led, in particular, to the conjecture and detailed check for 'Seiberg like' Giveon-Kutasov dualities between Chern-Simons matter theories with $\mathcal{N} \geq 2$ supersymmetry [37, 38]. Most of these impressive studies have, however, focused on observables [9] that are not directly sensitive to the anyonic nature of of the underlying excitations and have exhibited no qualitative surprises.

Qualitative surprises arising from the effectively anyonic nature of the matter particles seem most likely to arise in observables built out of the matter fields themselves rather than gauge invariant composites of these fields. There exists a well defined gauge invariant observable of this sort; the S matrix of the matter fields. While this quantity has been somewhat studied for highly supersymmetric Chern-Simons theories, the results currently available (see e.g. [39–45]) have all be obtained in perturbation theory. Methods based on supersymmetry have not yet proved powerful enough to obtain results for S

[9]These observables include partition functions, indices, Wilson lines and correlation functions of local gauge invariant operators. Note that gauge invariant operators do not pick up anyonic phases when they go around each other precisely because they are gauge singlets.

matrices at all orders in the coupling constant, even for the maximally supersymmetric ABJ theory. For a very special class of matter Chern-Simons theories, however, it has recently been demonstrated that large N techniques are powerful enough to compute S matrices at all orders in a ‘t Hooft coupling constant, as we now pause to review.

Consider large N Chern-Simons coupled to a finite number of matter fields in the fundamental representation of $U(N)$. [10] It was realized in [46] that the usual large N techniques are roughly as effective in these theories as in vector models even in the absence of supersymmetry (see [47–68] for related works). In particular large N techniques have recently been used in [61] to compute the $2 \to 2$ S matrices of the matter particles in purely bosonic/fermionic fundamental matter theories coupled to a Chern-Simons gauge field.

Before reviewing the results of [61] let us pause to work out the effective anyonic phases for two particle systems of quanta in the fundamental/ antifundamental representations at large N. [11] Following [61] we refer to any matter quantum that transforms in the (anti)fundamental of $U(N)$ a(n) (anti)particle. A two particle system can couple into two representations R' (see (7)); the symmetric representation (two boxes in the first row of the Young Tableaux) and the antisymmetric representation (two boxes in the first column of the Young Tableaux). It is easily verified that the anyonic phase $\nu_{R'}$ (see (7)) is of order $\frac{1}{N}$ (and so negligible in the large N limit) for both choices of R'. On the other hand a particle - antiparticle system can couple into R' which is either the adjoint of the singlet. $\nu_{R'}$ once again vanishes in the large N limit when R' is the adjoint. However when R' is the singlet representation it turns out that $\nu_{sing} = \frac{N}{\kappa} = \lambda$ and so is of order

[10] These theories were initially studied because of their conjectured dual description in terms of Vasiliev equations of higher spin gravity.

[11] The application of large N techniques to these theories has led to conjectures for strong weak coupling dualities between classes of these theories. The simplest such duality relates a Chern-Simons theory coupled to a single fundamental bosonic multiplet to another Chern-Simons theory coupled to a single fermionic multiplet. This duality was first clearly conjectured in [51], building on the results of [48, 49], and following up on an earlier suggestion in [46]. The discovery of a three dimensional Bose-Fermi duality was the first major qualitative surprise in the study of Chern-Simons matter theories, and is intimately connected with the effectively anyonic nature of the matter excitations, as explained, for instance, in [61].

unity in the large N limit. In summary two particle systems are always non anyonic in the large N limit of these special theories. Particle - antiparticle systems are also non anyonic in the adjoint channel. However they are effectively anyonic - with an interesting finite anyonic phase- in the singlet channel. See [61] for more details. This preparation makes clear that qualitative surprises related to anyonic physics in the two quantum scattering in these theories might occur only in particle - antiparticle scattering in the singlet sector.

The authors of [61] used large N techniques to explicitly evaluate the S matrices in all three non-anyonic channels in the theories they studied (see below for more details of this process). They also used a mix of consistency checks and physical arguments involving crossing symmetry to conjecture a formula for the particle - antiparticle S matrix in the singlet channel. The conjecture of [61] for the S matrix in the singlet channel has two unexpected novelties related to the anyonic nature of the two particle state

- 1. The singlet S matrix in both the bosonic and fermion theories has a contact term localized on forward scattering. In particular the S matrix is not an analytic function of momenta.
- 2. The analytic part of the singlet S matrix is given by the analytic continuation of the S matrix in any of the other three channels $\times \frac{\sin \pi\lambda}{\pi\lambda}$. In other words the usual rules of crossing symmetry to the anyonic channel are modified by a factor determined by the anyonic phase.

The modification of the usual rules of analyticity and crossing symmetry in the anyonic channel of 2×2 scattering was a major surprise of the analysis of [61]. The authors of [61] offered physical explanations - involving the anyonic nature of scattering in the singlet channel for both these unusual features of the S matrix. The simple (though non rigorous) explanations proposed in [61] are universal in nature; they should apply equally well to all large N Chern-Simons theories coupled to fundamental matter, and not just the particular theories studied in [61]. This fact suggests a simple strategy for testing the conjectures of [61] which we employ in chapter 3.

1.2.1 Original Results in this thesis

In this thesis I present the generalization of the scattering computations described above to $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric matter Chern Simons theories. We find explicit all orders formulae for the S matrices in these theories; our explicit results provide a verification of the surprising crossing symmetry properties reviewed above. In order to obtain these results we needed to develop extensive technical machinery. Though I collaborated in every aspect of work presented in the Chapter, the work presented in 4.3, 4.4 and 4.5 of Chapter 3 is primarily my own work.

Part A

2 Chapter 1: Unstable 'black branes' from scaled membranes at large D

(*This chapter is based on the published paper written in collaboration with Y. Dandekar, S. Minwalla, A. Saha, " Unstable 'black branes' from scaled membranes at large D," arXiv:1609.02912 , JHEP 1612 (2016) 140*)

2.1 Introduction

In this chapter we derive the 'black brane' equations of [16] starting from the membrane equations of [2, 4, 12]. The starting point of our analysis is the simple exact solution to the membrane equations of motion that is dual to the p dimensional 'black brane' described in the previous paragraph. This solution is static, which means that the membrane velocity field is simply given by $u = -dt$. The shape of the membrane on this solution is $S^{D-p-2} \times R^{p,1}$. We then proceed to study the scaling limit of [16] directly within the membrane picture. In other words we study fluctuations of the membrane that preserve $SO(D-p-1)$ isometry but vary in the R^p direction over length scales of order $\frac{1}{\sqrt{D}}$ and time scales of order unity. We then focus on wiggles of the shape of the membrane with amplitude of order $\frac{1}{D}$ and on membrane velocities of order $\frac{1}{\sqrt{D}}$. At leading order in the large D limit we obtain a simple set of scaled equations of membrane dynamics which (after the appropriate field redefinitions) turn out to agree exactly with the equations of [16]. We view our derivation of the (uncharged) black brane equations from the membrane equations as a unification of these two approaches to horizon dynamics at large D. Note it follows, in particular, that the dynamics of the Gregory Laflamme instability is captured by scaling limit of membrane equations described above.

The limit of the previous paragraph is loosely reminiscent of the scaling limit that yields the nonrelativistic Navier-Stokes equations starting from the more general relativistic equations [69]. The membrane equations may also admit other interesting scaling limits. We leave the investigation of this point to future work.

2.2 A scaling limit of the membrane equations

In this chapter we study the equations of motion [2, 4, 12] of an uncharged large D membrane propagating in flat Minkowski spacetime. To leading order in $\frac{1}{D}$ these equations take the form

$$\left[\frac{\nabla^2 u_A}{\mathcal{K}} - \frac{\nabla_A \mathcal{K}}{\mathcal{K}} + u^B K_{BA} - u^B \nabla_B u_A\right] \mathcal{P}^A_C = 0 \tag{8}$$

with ,

$$\nabla . u = 0. \tag{9}$$

Here K_{AB} is the extrinsic curvature of the membrane, $\mathcal{K}$ is its trace and u is the local world volume velocity field of the membrane. All covariant derivatives in (8) and (9) are defined with respect to the induced metric on the membrane. Also

$$\mathcal{P}^{AB} = \hat{g}^{AB} + u^A u^B \tag{10}$$

where $\hat{g}^{AB}$ is the metric induced from the ambient flat space on the world volume of the membrane. In other words $\mathcal{P}^{AB}$ is the projector, on the membrane world volume, orthogonal to the velocity field u.

2.2.1 Linearized Fluctuations

In our study we will find it useful to use coordinates in which the flat space D dimensional metric takes the form

$$ds^2 = -dt^2 + d\tilde{x}^a d\tilde{x}^a + dr^2 + r^2 d\Omega_n^2 \tag{11}$$

where

$$n = D - p - 2.$$

and $a = 1 \dots p$ label the spatial directions on the black brane. A simple solution to the equations (8) and (9) is given by the membrane shape $r = 1$ and constant static velocity field $u = -dt$. [12]

The solution of the membrane equations described in the previous paragraph is dual to a 'black brane' - the solution of general relativity given by the direct product of R^p and the Schwarschild black hole in $D - p$ dimensions. It is well known that this solution of general relativity is unstable in an arbitrary number of dimensions. We will now use the membrane equations to exhibit this instability, by linearizing these equations about the simple solution. The Gregorry Laflamme instability of black branes is known to preserve the $SO(n+1)$ symmetry of the sphere but to break translational invariance along R^p, so we study fluctuations with the same property. In other words we set

$$\begin{aligned} r &= 1 + \tilde{\delta r}(t, \tilde{x}^a) \\ u &= -dt + \tilde{\delta u}_a(t, \tilde{x}^a) d\tilde{x}^a \end{aligned} \tag{12}$$

Note that our velocity fluctuations lie entirely in the black brane directions and none of our fluctuations fields depend on the angular variables on S^n.

Following the method described in section 5 of [12], it is not difficult to linearize

[12] The choice $r = 1$ involves no loss of generality, as the scale invariance of the classical Einstein equations relate the solution with $r = 1$ to the solution with $r = r_0$ for any constant r_0.

the membrane equations around the 'black brane' solution. The equation (9) reduces to

$$n\ \partial_t \tilde{\delta r} + \tilde{\partial}_a \tilde{\delta u^a} = 0 \tag{13}$$

(recall $n = D - p - 2$). [13] The equation with a free index in the (spatial) R^p direction turns out to take the form

$$\left(\tilde{\partial}_a \tilde{\delta r} - \partial_t \tilde{\partial}_a \tilde{\delta r} - \partial_t \tilde{\delta u_a}\right) + \left(\frac{-\partial_t^2 + \tilde{\partial}_b \tilde{\partial}^b}{n}\right)\left(\tilde{\delta u_a} + \tilde{\partial}_a \tilde{\delta r}\right) = 0 \tag{14}$$

where $\tilde{\partial}_a$ is the derivative with respect to the coordinate $\tilde{x}^a$ defined in (11). When all spatial and time derivatives are of order unity or smaller, the term

$$\left(\frac{-\partial_t^2 + \tilde{\partial}_b \tilde{\partial}^b}{n}\right)\left(\tilde{\delta u_a} + \tilde{\partial}_a \tilde{\delta r}\right)$$

in (14) is subleading in the $\frac{1}{n}$ expansion and so can naively be dropped at leading order. However we will soon find ourselves interested in configurations with spatial derivatives of order $\sqrt{n}$ but time derivatives of order unity. For such configurations the term proportional to time derivatives in (14) is indeed subleading in $\frac{1}{n}$. On the other hand the term proportional to the spatial laplacian is comparable to the other terms in (14) and so must be retained. Over the parameter ranges of interest to this chapter, therefore, we can replace (14) with the slightly simpler equation

$$\left(\tilde{\partial}_a \tilde{\delta r} - \partial_t \tilde{\partial}_a \tilde{\delta r} - \partial_t \tilde{\delta u_a}\right) + \left(\frac{\tilde{\partial}_b \tilde{\partial}^b}{n}\right)\left(\tilde{\delta u_a} + \tilde{\partial}_a \tilde{\delta r}\right) = 0 \tag{15}$$

[14]

[13] The factor of n, which plays a crucial role in the analysis below, has its origin in the fact that the induced metric on the world volume of the membrane is given, to leading order in fluctuations by

$$ds^2 = -dt^2 + d\tilde{x}^a d\tilde{x}_a + (1 + 2\tilde{\delta r}) d\Omega_n^2$$

so that $\sqrt{g} = 1 + n\tilde{\delta r}$ in these coordinates.

[14] The membrane equations with free index in sphere direction is trivially satisfied, while the equation

The equations (15) and (13) are easily analysed. Substituting the plane wave expansion

$$\begin{aligned}\tilde{\delta r}(t, \tilde{x}^a) &= \delta r^0 e^{-i\omega t} e^{i\tilde{k}_a \tilde{x}^a} \\ \tilde{\delta u}_a(t, \tilde{x}^a) &= \delta u^0_a e^{-i\omega t} e^{i\tilde{k}_a \tilde{x}^a}\end{aligned} \tag{16}$$

into (15) and (13) turns these equations into eigenvalue equations for the fluctuation frequencies ω. Solving the resultant cubic equation in ω we find find that the most general solution to these equations is given by

$$\begin{aligned}&\tilde{\delta r}(t, \tilde{x}^a) = \delta r^0_1 e^{-i\omega_1 t} e^{i\tilde{k}_a \tilde{x}^a} + \delta r^0_2 e^{-i\omega_2 t} e^{i\tilde{k}_a \tilde{x}^a} \\ &\tilde{\delta u}_a(t, \tilde{x}^a) = \delta r^0_1 \tilde{k}_a \left(-i + \frac{\sqrt{n}}{\tilde{k}}\right) e^{-i\omega_1 t} e^{i\tilde{k}_a \tilde{x}^a} + \delta r^0_2 \tilde{k}_a \left(-i - \frac{\sqrt{n}}{\tilde{k}}\right) e^{-i\omega_2 t} e^{i\tilde{k}_a \tilde{x}^a} + v_a e^{-i\omega_3 t} e^{i\tilde{k}_a \tilde{x}^a} \\ &w_1 = i\left(\frac{\tilde{k}}{\sqrt{n}} - \frac{\tilde{k}^2}{n}\right), \quad w_2 = i\left(-\frac{\tilde{k}}{\sqrt{n}} - \frac{\tilde{k}^2}{n}\right), \quad w_3 = -i\frac{\tilde{k}^2}{n}, \quad \text{where} \quad \tilde{k}^2 = \tilde{k}_a \tilde{k}^a\end{aligned} \tag{17}$$

(17) is a solution to the linearized membrane equations for arbitrary constant values of δr^0_1 and δr^0_2 and for any constant vector v_a s.t. $\tilde{k}^a v_a = 0$.

Note that the mode proportional to δr^0_1 - i.e. the mode with frequency ω_1 - is unstable when $\tilde{k} < \sqrt{n}$. This IR instability (i.e. an instability that occurs at distance scales lareger than a minimum) is the membrane dual of the Gregory Laflamme instability. When $\tilde{k}$ is of order unity time scale associated with this frequency is of order $\sqrt{n}$ and so is very large. The minimum time scale for an instability, however, occurs at $\tilde{k} = \frac{\sqrt{n}}{2}$. At this wavelength the time scale of the instability is order unity. [15]

At the level of the linearized equations the Gregory - Laflamme unstable modes simply grow forever. Nonlinear effects, however, stabilize these modes. The discussion

in the time direction is also a triviality (this is a consequence of the projector in (8)).

[15] The expression for the unstable mode w_1 was conjectured earlier from fluid/gravity methods in [70]. See also [71] and [5] for further evidence for the above proposal.

of the previous paragraph makes it clear that the length scale relevant to this physics is $\frac{1}{\sqrt{n}}$. We will now proceed to find the effective nonlinear theory within which the Gregory-Laflamme instability and its end point can be reliably studied.

2.2.2 Scaled nonlinear equations

In order to restrict attention to distance of order $\frac{1}{\sqrt{n}}$ in the spatial black brane directions we work with the scaled coordinate x^a defined by $\tilde{x}^a = \frac{x^a}{\sqrt{n}}$. Unstable modes with finite wavelength in this new coordinate have frequencies of order unity. The background flat space metric now takes the form

$$ds^2 = -dt^2 + dr^2 + \frac{1}{n} dx_a dx^a + r^2 d\Omega_n^2 \tag{18}$$

As our fluctuations field all vary over distances of order unity and time scales of order unity in scaled coordinates, the velocity field u^a should thus also be of order unity. This implies that $u_a \sim \mathcal{O}(\frac{1}{n})$. Translating back to unscaled coordinates it follows that $\tilde{u}^a = \mathcal{O}(\frac{1}{\sqrt{n}})$. In order to ensure this scaling in our solution (17) we must choose $v_a \sim \mathcal{O}(\frac{1}{\sqrt{n}})$, $\delta r_1^0 \sim \delta r_2^0 \sim \mathcal{O}(\frac{1}{n})$. These choices, in turn, ensure that $\tilde{\delta r} \sim \mathcal{O}(\frac{1}{n})$ (see (17)). It is thus natural to make the further coordinate change

$$r = 1 + \frac{y}{n} \tag{19}$$

The flat space metric is now given by

$$ds^2 = -dt^2 + \frac{dy^2}{n^2} + \frac{1}{n} dx_a dx^a + \left(1 + \frac{y}{n}\right)^2 d\Omega_n^2 \tag{20}$$

With our scalings now in place we focus attention on membrane configurations of the form

$$\begin{aligned} y &= y(x^a, t) \\ u^a &= u^a(x^a, t) \end{aligned} \tag{21}$$

where the functions $y(x^a, t)$ and $u^a(x^a, t)$ are independent of n. We then evaluate the membrane equations (9) and (8) for such configurations propagating on the metric (20). Retaining only terms of leading order at large n we find that the equation (9) (which we call E^s below) and the a components of (8) (which we call E^v_a below) reduce to

$$\begin{aligned} E^s &\equiv u^b \partial_b y + \partial^b u_b + \partial_t y = 0 \\ E^v_a &\equiv \partial^b \partial_b u_a + \partial_a y - u^b \partial_b u_a + \partial^b y \partial_b u_a - u^b \partial_b \partial_a y \\ &\quad + \partial^b y \partial_b \partial_a y + \partial^b \partial_b \partial_a y - \partial_t u_a - \partial_t \partial_a y = 0 \end{aligned} \tag{22}$$

Note that the equations (22) are nonlinear. If we linearize these equations around the background $y = u^a = 0$ we obtain the linearized equations

$$\begin{aligned} &\partial^b \delta u_b + \partial_t \delta r = 0 \\ &\partial^b \partial_b \delta u_a + \partial_a \delta r + \partial^b \partial_b \partial_a \delta r - \partial_t \delta u_a - \partial_t \partial_a \delta r = 0 \end{aligned} \tag{23}$$

The first and second of (23) are simply (13) and (15) expressed in scaled variables. It follows that (22) are nonlinear generalizations of the linearized fluctuation equations of the previous subsection. The (23) are exact at large n within the scaling limit described in this section.

The nonlinear equations (22) capture both the linear exponential growth as well as the nonlinear settling down of the Gregory Laflamme instability. We do not need to perform the analysis of this fact, however, because it has already been done! We will now demonstrate that the equations (22) are equivalent to those that Emparan Suzuki and Tanabe [16] derived to study large D 'black branes' - and used to perform an extensive study of the Gregory Laflamme instability.

In order to make contact with the work of [16] we make the following field redefinitions

$$\begin{aligned} y(t,x^a) &= \log m(t,x^a) \\ u_a(t,x^a) &= \frac{p_a(t,x^a) - \partial_a\left(m(t,x^a)\right)}{m(t,x^a)} \end{aligned} \tag{24}$$

and work with the following linear combinations of (22)

$$E_1 = m(t,x^a)E^s \quad \text{and} \quad E_a = p_a(t,x^a)E^s - m(t,x^a)E^v_a. \tag{25}$$

It is easily verified that E_1 and E_a take the form

$$\begin{aligned} E_1 &= \partial_t m - \partial_b\partial^b m + \partial_b p^b = 0 \\ E_a &= \partial_t p_a - \partial_b\partial^b p_a - \partial_a m + \partial_b\left(\frac{p_a p^b}{m}\right) = 0 \end{aligned} \tag{26}$$

The equations (26) are precisely the nonlinear black brane equations (11) and (12) of [16]. It follows that these black brane equations are simply a particular scaled limit of the general leading order (in an expansion in $\frac{1}{D}$) equations (9) and (8).

2.3 Discussion

In this chapter we have demonstrated by explicit computation that the uncharged 'black brane' equations of [16] may be obtained from a scaling limit of the general membrane equations (9) and (8). The reader may, at first, find herself puzzled at this agreement, given the scaling limit described in this chapter focuses on length scales of order $\frac{1}{\sqrt{D}}$ while that the membrane equations (9) and (8) were derived as the first term in a systematic expansion in $\frac{1}{D}$ under the assumption that the horizon and velocity fields all vary on length scale unity. We will now explain why this agreement was infact to be expected despite the apparent conflict of regimes of validity.

The equations (9) and (8) would fail to accurately capture dynamics at leading order in the large D limit if the explicit factors of D in the metric (20) ensured that a higher order term[16] were to contribute to the equations at same (or higher) order in $\frac{1}{D}$ as the terms in (9) and (8). We will now explain that this never happens. Potentially dangerous terms are those that contain one or more factors of the inverse metric g^{ab} where the indices a and b are spatial black brane directions. These terms are potentially dangerous as g^{ab} (see (20)) is of order D. However these factors never actually lead to a mixing of orders because the extra indices a and b each need to contract with something. When these indices contract with u_a the extra factor of D is nullified by the fact that u_a is of order $\frac{1}{D}$. When these indices contract with a derivative, the derivative acts on some quantity built out of fluctuation fields. However all such quantities are of order $\frac{1}{D}$ (recall, for instance, that every fluctuation component of the extrinsic curvature is proportional to $\tilde{\delta}r$ which is of order $\frac{1}{D}$). The smallness of fluctuations in our scaling limit once again counteracts the potential enhancement of powers of D. It follows that leading order equations (9) and (8) is infact sufficient to capture the leading order large D dynamics of the scaling limit described in this chapter despite the fact that the scaling limit zooms in on distance scales of order $\frac{1}{\sqrt{D}}$.

It should not be difficult to generalize the discussion of this chapter to obtain the first corrections, in an expansion in $\frac{1}{D}$, to the black brane equations of (26). These corrections have been obtained from 'scaled black brane' approach in [18, 22, 23]. The starting point for such an analysis would be the first order corrected membrane equations derived in [2]. It would also be interesting to check whether the analysis of this chapter generalizes to a derivation of the charged 'black brane' equations of [17] starting with the charged membrane equations of [12]. [17] We leave a study of these issues to future work.

We end this chapter by reiterating that we have demonstrated that the black

[16]i.e. a term that appears at higher order in the expansion in $\frac{1}{D}$ in the membrane equations of [2]

[17]The discussion of the last paragraph suggests that this is guaranteed to work only if the scaling limit of [17] turns on membrane charge fluctuations that scale like $\frac{1}{D}$.

brane equations of [16] can be derived as a special case of the more general membrane equations of [2, 4, 12], leading to a satisfying unification recent attempts to reformulate large D horizon dynamics in non gravitational terms.

3 Chapter 2: Spectrum of Quasinormal modes from subleading Large D Membrane Paradigm

(*This chapter is based on the paper written in collaboration with Y. Dandekar, A. De, S. Minwalla, A. Saha, " The large D black hole Membrane Paradigm at first subleading order," arXiv:1607.06475, JHEP 1612 (2016) 113*)

3.1 Introduction

In this chapter we further develop the general nonlinear dynamical construction of [4, 12]. In particular we demonstrate that the reduction of black hole dynamics to membrane dynamics, worked out to leading nontrivial order in the $1/D$ expansion in [4, 12], can be systematically generalized to every order in $1/D$. As an application of this systematic framework we explicitly work out the first subleading corrections to the membrane equations of motion in the $1/D$ expansion, and also determine the spacetimes dual to any particular membrane solution at next subleading order in the $1/D$ expansion.

3.1.1 The membrane paradigm at higher orders in $1/D$

In this chapter we demonstrate that first order perturbative procedure outlined above extends systematically to arbitrary orders in the expansion in $\frac{1}{D}$. We will now very briefly outline our inductive argument. We assume that the perturbative procedure has been implemented upto n^{th} order, i.e. that corrections to the metric (1) have been determined upto n^{th} order in the $1/D$ expansion in such a manner that R_{MN} evaluated on the corrected solution is of order D^{1-n}. We then add further corrections of order $1/D^{n+1}$ to the metric (see (35) and (38)). At order D^{n-1} we demonstrate that the Einstein constraint equations are independent of the new unknown correction functions when evaluated on the event horizon $\psi = 1$. These equations determine the correction to the membrane equations (and the divergence condition on the velocity) at order $1/D^{n+1}$. Moving away from the horizon we argue that the order D^{1-n} part of R_{MN} takes the form listed in table 2. Setting the expressions in this table yields a set of inhomogeneous linear differential equations that can be used to determine order $1/D^{n+1}$ corrections to the metric. Explicit expressions for the sources in these differential equations can only be obtained by grinding through the perturbative procedure, but we use a contracted Bianchi identity to demonstrate that the sources that occur in these equations are not all independent, but obey certain relations (see (53)) at every order of perturbation theory. Using these relations we are able to integrate the inhomogeneous differential equations for any source functions and obtain an explicit and unique expressions for the metric corrections at order $1/D^{n+1}$ (see Section 3.3) that are manifestly regular and obey all required boundary conditions.

As an illustration of the general method outlined above we explicitly implement the perturbative procedure to second subleading order in $\frac{1}{D}$. We find that the modified membrane equations take the form

$$
\begin{aligned}
&\left[\frac{\nabla^2 u_A}{\mathcal{K}} - \frac{\nabla_A \mathcal{K}}{\mathcal{K}} + u^B K_{BA} - u\cdot\nabla u_A\right]\mathcal{P}^A_C \\
+ &\left[\left(-\frac{u^C K_{CB} K^B_A}{\mathcal{K}}\right) + \left(\frac{\nabla^2\nabla^2 u_A}{\mathcal{K}^3} - \frac{u\cdot\nabla\mathcal{K}\nabla_A\mathcal{K}}{\mathcal{K}^3} - \frac{\nabla^B\mathcal{K}\nabla_B u_A}{\mathcal{K}^2} - 2\frac{K^{CD}\nabla_C\nabla_D u_A}{\mathcal{K}^2}\right)\right. \\
+ &\left(-\frac{\nabla_A\nabla^2\mathcal{K}}{\mathcal{K}^3} + \frac{\nabla_A\left(K_{BC}K^{BC}\mathcal{K}\right)}{\mathcal{K}^3}\right) + 3\frac{(u\cdot K\cdot u)(u\cdot\nabla u_A)}{\mathcal{K}} - 3\frac{(u\cdot K\cdot u)(u^B K_{BA})}{\mathcal{K}} \\
- &\left. 6\frac{(u\cdot\nabla\mathcal{K})(u\cdot\nabla u_A)}{\mathcal{K}^2} + 6\frac{(u\cdot\nabla\mathcal{K})(u^B K_{BA})}{\mathcal{K}^2} + \frac{3}{(D-3)}u\cdot\nabla u_A - \frac{3}{(D-3)}u^B K_{BA}\right]\mathcal{P}^A_C = 0
\end{aligned}
\tag{27}
$$

while the divergence free condition on the velocity field is modified, at second subleading order, to the equation

$$\nabla \cdot u = \frac{1}{2\mathcal{K}} \left(\nabla_{(A}u_{B)}\nabla_{(C}u_{D)}\mathcal{P}^{BC}\mathcal{P}^{AD}\right) \tag{28}$$

Note that the first line in (27) is simply a rewriting of (3); the 2nd-4th lines of this equations represent corrections to (3). There is a well defined sense (see below) in which each of these correction terms is of order $\frac{1}{D}$ relative to the leading order terms in the first line. It follows that the equations (27) represent small corrections to the leading order equations (3). The first order corrected membrane equation of motion (27) and (28) are the main result of this chapter.

We then present explicit expressions for the second order sources for all the inhomogeneous differential equations (see table 6). Plugging these sources into the general equations for the metric corrections at any order we obtain explicit results for the second order correction to the spacetime metric dual to any particular solution of the membrane equations of motion.

The second order corrected membrane equations (27) admit a simple solution; a spherical membrane at rest. This solution is dual to the Schwarzschild black hole. As a check of our second order corrections to the membrane equations we use (27) to compute the spectrum of small fluctuations about this simple solutions. This spectrum is easy to obtain, and turns out to be in perfect agreement with the second order corrected spectrum of quasinormal modes obtained by Emparan Suzuki and Tanabe in [10], providing confidence in the correctness of (27).

3.2 Perturbation theory: general structure

3.2.1 A more detailed description of the starting ansatz

As we have explained in the introduction, the starting point of our perturbative construction of large D solutions to Einstein's equations is the metric (1). In the introduction we noted that the metrics (1) are parameterized by the D dimensional function ψ and the oneform field u. We assume these fields have a good large D limit, i.e. that the length scale of variation in ψ and u is of order unity. Following [4, 12], however, consider two different functions ψ with the same membrane surface (i.e. with coincident zero sets for $\psi - 1$). These two functions define metrics (1) that coincide (outside the event horizon) at leading order in $1/D$ but differ at subleading orders in $1/D$. Similarly u functions that agree on the membrane but differ off it lead to metrics (1) that differ only at subleading order in $1/D$.

Any two metrics (1) that differ only at subleading orders in $1/D$ constitute equivalent starting points for the perturbative construction of solutions in the following sense: the end result of perturbation theory starting from the two different starting points will be the same. In order to construct all distinct final metrics we need only consider one member of each 'equivalence class' of metrics (1). As explained above the equivalence classes are labeled by the zero set of the function $\psi - 1$ (the membrane world volume) and the value of the velocity field on the membrane world volume. In order to pick a representative from each equivalence class that we can use to set up our perturbation theory we invent an arbitrary way of constructing the full function ψ from its zero set, and the full velocity field u from its values on the membrane. Following [4, 12] we refer to the (essentially arbitrary) rule for achieving this construction as a subsidiary condition on the functions ψ and u.

For technical reasons, in this chapter we utilize the subsidiary conditions of [4] rather than that of [12]. We now describe these conditions in detail.

Consider a given timelike membrane submanifold in flat space. At each point on the manifold consider a geodesic that shoots outwards from the manifold along its normal vector. The resultant collection of curves [18] is a spacefilling congruence of spacelike geodesics; caustics of this congruence, if any, only occur at distances of order unity (rather than $1/D$) away from the membrane. [19] We define the scalar function B in the neighborhood of the membrane as follows; B at any point is defined to be the signed proper distance, along the geodesic that passes through it, to the membrane. This distance is defined to be positive outside the membrane and negative inside the membrane. Note that B vanishes on the membrane. We define

$$n_M = \nabla_M B \tag{29}$$

It follows from our construction above that

$$n.n = 1 \tag{30}$$

n_A is the normal oneform to surfaces of constant B. We use the symbol K_{MN} denote the extrinsic curvature of surfaces of constant B. Note of course that $n^A K_{AB} = 0$. We also define $\mathcal{K} = K^A_A$. We then proceed to define the function ψ as

$$\psi = 1 + \frac{\mathcal{K}B}{D-3} \tag{31}$$

In a similar manner we use the velocity function on the membrane to define a velocity oneform field in spacetime simply by parallel transport along our congruence of

[18]These 'curves' are actually straight lines as they are all geodesics in flat space. We use the term 'curve' to bring to mind the obvious generalization of this construction when the membrane is embedded in a curved spacetime.

[19]The quantity $\frac{D}{\mathcal{K}}$ gives a rough estimate for the distance away from the membrane at which the geodesics caustic. Below we explain that $\mathcal{K}$ is of order D so that this caustic length scale is of order unity.

geodesics. It follows from our definitions above that

$$\begin{aligned} n.\nabla n_A &= 0 \\ n.\nabla u_A &= 0 \end{aligned} \tag{32}$$

The first line of (32) follows upon differentiating 0 (30), using (29) and interchanging derivatives. This equation is in fact simply the geodesic equations for the congruence of geodesics that defines B. The equation on the second line of (32) follows from the fact that u is defined off the membrane by parallel transport. It follows from (32) that

$$K_{AB} = \left(\eta_A^C - n_A n^C\right)\left(\nabla_C n_D\right)\left(\eta_B^D - n^D n_B\right) = \left(\nabla_A - n_A(n.\nabla)\right) n_B = \nabla_A n_B = \nabla_A \nabla_B B \tag{33}$$

Note that our definition of n_A in this section, and the rest of this chapter, differs slightly from the definition given in the introduction. The two definitions agree at leading order (which was all that was required in the discussion around (1)) but differ at subleading orders in $1/D$. The vector n_A defined in this section - rather than the normal vector defined in the introduction - will be used through the rest of this chapter.

Using (31) it is easily verified that on the submanifold $B = 0$

$$\begin{aligned} \psi \nabla^2 \psi &= \frac{\mathcal{K}^2}{D-3} + 2\frac{n.\nabla \mathcal{K}}{D-3} \\ (D-2)\nabla\psi.\nabla\psi &= \frac{D-2}{D-3}\frac{\mathcal{K}^2}{D-3} \end{aligned} \tag{34}$$

As we explain below, in the large D limit taken in this chapter $2\frac{n.\nabla\mathcal{K}}{D-3}$ is of order unity while $\frac{\mathcal{K}^2}{D-3}$ is order D. It follows that to leading order in D

$$(D-2)\nabla\psi.\nabla\psi = \psi\nabla^2\psi, \quad i.e. \nabla^2\left(\frac{1}{\psi^{D-3}}\right) = 0$$

In other words our construction satisfies the first equation of (2). We satisfy the second equation in (2) by construction; we simply choose our u oneform on the membrane such that its divergence vanishes at leading order in D. The divergence of u will turn out not to vanish at a subleading order.

3.2.2 Coordinate Choice for the correction metric

In this chapter we search for solutions of Einstein's equations in a power series expansion in $\frac{1}{D}$

$$
\begin{aligned}
G_{MN} &= \eta_{MN} + h_{MN}, \\
h_{MN} &= \sum_{n=0}^{\infty} \frac{h^{(n)}_{MN}}{(D-3)^n}\,, \\
\text{with, } h^{(0)}_{MN} &= \frac{O_M O_N}{\psi^{D-3}},
\end{aligned}
\tag{35}
$$

Here

$$
O_M = n_M - u_M \tag{36}
$$

We fix coordinate redefinition ambiguities by demanding

$$
h_{MN} O^N = 0, \tag{37}
$$

Consider any point in the metric (1). The tangent space built about this point has two special vectors; the vector n and the vector u. All the other $D-2$ directions orthogonal to n and u are equivalent and can be rotated into each other. It is thus useful to parameterize the most general fluctuation field h_{MN} (subject to the gauge condition

(37)) in the form

$$h^{(n)}_{MN} = H^{(S,n)} O_M O_N + O_{(M} H^{(V,n)}_{N)} + H^{(T,n)}_{MN} + H^{(Tr,n)} \mathcal{P}_{MN},$$

where,

$$\begin{aligned}\mathcal{P}_{MN} =& \eta_{MN} - O_M n_N - O_N n_M + O_M O_N, \\ O^N H^{(V,n)}_N = 0, \quad n^N H^{(V,n)}_N &= 0, \quad O^M H^{(T,n)}_{MN} = 0, \quad n^M H^{(T,n)}_{MN} = 0, \quad \mathcal{P}^{MN} H^{(T,n)}_{MN} = 0,\end{aligned} \tag{38}$$

The superscripts S, V and T stand for scalar, vector and tensor respectively, and denote the transformation properties of the relevant symbol under the $SO(D-2)$ rotations in tangent space that leave n and u fixed. The superscript Tr stands for trace, and labels a second scalar.

3.2.3 Orders of D

As we have explained above, in this chapter we solve Einstein's equations in a systematic expansion in $\frac{1}{D}$. In order for this process to be well defined, we need to be able to unambiguously estimate the scaling with D of various terms that appear in the metric and in the membrane equation of motion. Such an estimation is only unambiguous within subclasses of solutions, as we will now explain with an example.

Consider a membrane whose world volume is a $D-2$ sphere (of radius R) times time. The trace of extrinsic curvature, $\mathcal{K}$, of this surface is easily shown to be $\frac{D-2}{R}$ and so is of order D (assuming R is of order unity). On the other hand the surface $S^p \times R^{D-2-p}$ times time has $\mathcal{K} = \frac{p}{R}$. If p and R are both held fixed as D is taken to infinity, $\mathcal{K}$ is of order unity for this surface. It follows that $\mathcal{K}$ cannot unambiguously be assigned a scaling with D without making further assumptions. The same holds true of various other quantities (e.g. $\nabla^2 u_M$) that enter the metric and equation of motion.

In this chapter we follow [4, 12] and estimate the D scalings of all terms as follows.

We assume that

- Our starting ansatz is constructed by sewing together bits of the event horizon of black holes of radii R and timelike velocity u^M where R and u^M are everywhere finite and of order unity.
- Our starting configuration (and so our full solution) preserves an $SO(D-p-2)$ rotational invariance with p held fixed as D is taken to infinity

As explained in [12], these assumptions unambiguously specify the scaling with D of all quantities of interest (in particular they force $\mathcal{K}$ to be of order D).

We emphasize that in this chapter we use the assumptions listed above only to estimate the scalings of D of various quantities. When the assumptions listed in the previous paragraph are obeyed, the membrane equations and metrics listed in this chapter certainly apply. However the formulae of this chapter apply more generally to any spacetime whose variables scale with D in the same manner in which they would if the assumptions above were obeyed - a much larger class of configurations.

3.2.4 All orders definition of the membrane surface and velocity

As explained in subsection 3.2.1, the metric (1) - the starting point of our perturbative expansion - is completely determined by the shape of a membrane and a velocity field on the membrane. To what precision can this procedure be reversed? In other words if we are given a solution to Einstein's equations of the appropriate kind, how precisely can we read off the corresponding 'shape' and 'velocity' of the membrane?

We could attempt to identify the membrane shape and velocity field by simply expanding the exact solution in powers of $1/D$ and focusing attention on the leading order term. By comparing with (1) we could then read off the membrane shape and velocity field. While this procedure is simple, a moment's thought will convince the reader that it is ambiguous at all orders in $1/D$ save the leading order. [20] In other words the requirement that our solution reduce to (1) defines the membrane shape and velocity only at leading order, leaving the subleading corrections to these quantities ambiguous. In this subsection we will fix this ambiguity by adopting a more precise definition of the shape and velocity field. This definition agrees with that of (1) at leading order, but is precise at all orders. We use this precise definition in the computations presented in the rest of this chapter.

We define the membrane shape to be the location of the event horizon of our spacetime, and will choose higher order corrections to the metric (1) to ensure that this event horizon coincides with the surface $\psi = 1$.

Turning to the velocity field, let G^{AB} denote the full spacetime inverse metric. Let n_A be the oneform normal to the event horizon. We define the velocity field on the membrane by the requirement that

$$u^A = G^{AB} n_B \tag{39}$$

on the event horizon (i.e. at $\psi = 1$). In other words the velocity field is a tangent vector to the generators of the event horizon. It is easily verified that (39) is a true equation for the starting point of perturbation theory (1). We will choose corrections to the perturbative ansatz to ensure that (39) holds at all orders in $1/D$.

The requirement (39) together with the requirement that $\psi = 1$ is the exact event horizon of our spacetime are easily seen to be satisfied provided that

$$\begin{aligned} H^{(S)}(\psi = 1) = 0 \\ H^{(V)}_M(\psi = 1) = 0 \end{aligned} \tag{40}$$

[20] For instance, the velocity redefinition $u^\mu \to u^\mu + \delta u^\mu / D$ does not change the metric at leading order in $1/D$.

The first condition ensures that $G^{MN}\partial_M\psi\partial_N\psi = 0$, i.e. $d\psi$ is null at $\psi = 1$ while the second condition then ensures that the full spacetime metric on the event horizon takes the form

$$\eta_{MN} + O_M O_N + H^{(T)}_{MN} + H^{Tr}\mathcal{P}_{MN}$$

Let us write this metric in a the local basis of oneforms (n, u, Y_a) where Y_a is any $D-2$ dimensional basis of oneforms chosen orthogonal to n and u. In this basis the metric takes a block diagonal form with a 2×2 block (with basis n and u) and a $D-2\times D-2$ block (with basis Y_a). It follows that the inverse metric also has this block diagonal structure. Note that the 2×2 block is universal, i.e. it is the same at every order in perturbation theory. This block is the only one that contributes in (39). As (39) holds at leading order, it follows that the conditions (40) ensure that (39) holds at every order in perturbation theory.

Recall that according to (2) the velocity field used in (1) is divergence free at leading order in $\frac{1}{D}$. As we will see below, the divergence of the velocity field defined in this subsection will not, in general, vanish at subleading orders in $1/D$.

3.2.5 Structure of the equations of perturbation theory

Our perturbative procedure proceeds as follows. We assume that our solution takes the form (35) together with (37) and (38). The Ricci tensor of this metric - evaluated in a slab of spacetime of thickness $1/D$ around $\psi = 1$ - takes the schematic form

$$R_{MN} = \sum_n D^{2-n} R^n_{MN} \tag{41}$$

Let us imagine that we have implemented our perturbative procedure to order $n-1$, i.e. that we have determined $h^{(m)}_{MN}$ for $m = 1\ldots n-1$ in a manner that ensures that $R^{(m)}_{MN} = 0$ for $m = 0\ldots n-1$. In order to go to one higher order in perturbation theory we must solve for $h^{(n)}_{MN}$ to ensure that R^n_{MN} also vanishes.

Schematically

$$R^{(n)}_{MN} = C^{PQ}_{MN} h^{(n)}_{PQ} + \mathcal{S}^{(n)}_{MN}$$

where C^{PQ}_{MN} is a linear differential operator with derivatives only in the ψ direction and $\mathcal{S}^{(n)}_{MN}$ is a source function. As $h^{(n)}_{PQ}$ is already of order n, the differential operator C^{PQ}_{MN} is built entirely out of the zero order background metric (1), and so is the same at every order. On the other hand the source function $\mathcal{S}^{(n)}_{MN}$ is proportional to expressions of n^{th} order in $1/D$ built out of derivatives of the membrane velocity and shape function, and is different at every order.

At every point of the event horizon of the ansatz metric (1) there are two distinguished vectors; n^A and u^A. Let

$$\mathcal{P}_{AB} = \eta_{AB} - n_A n_B + u_A u_B$$

denote the projector orthogonal to these two vectors (all dot products taken in flat space). Instead of dealing directly with the components of R_{MN} we find it more convenient to use a basis adopted to u^A and n^A listed in table 1.

Table 1: Basis of components of R_{MN}

Scalar sector	Vector sector	Tensor sector
$R^{S_1} = O^M R_{MN} O^N$	$R^{V_1}_L = O^M R_{MN} \mathcal{P}^N_L$	$R^T_{AB} = \mathcal{P}^M_A R_{MN} P^N_B - \frac{\mathcal{P}_{AB}}{D-2} \mathcal{P}^{MN} R_{MN}$
$R^{S_2} = O^M R_{MN} u^N$	$R^{V_2}_L = u^M R_{MN} \mathcal{P}^N_L$	
$R^{S_3} = u^M R_{MN} u^N$		
$R^{S_4} = R_{MN} \mathcal{P}^{MN}$		

By explicit computation (plugging (35) into the formula for the Ricci tensor) we find that the linear combinations listed in Table 1 of the curvature components R^n_{MN} (see

(41)) are given by the expressions listed in Table 2. [21]

In table 2, fluctuation fields H^S, H^{Tr} H^V_A and H^T_{MN} are taken to be of n^{th} order and all source functions (e.g. $\mathcal{S}^{S_1}$) also understood to be n^{th} order sources. All appearances of $\nabla.u$ [22] in the table 2 should also be understood as follows. Naively $\nabla.u$ is of order D. For that reason we expand

$$\nabla.u = (D-3)\left(\sum_{n=0}^{\infty}\frac{(\nabla.u)_n}{(D-3)^n}\right) \tag{42}$$

Every appearance of $\nabla.u$ in table 2 should actually be replaced by $(\nabla.u)_n$. We have already seen in the introduction that $(\nabla.u)_0 = 0$. We will see below that $(\nabla.u)_1$ also vanishes, but that $(\nabla.u)_2$ is nonzero.

In order to obtain Table 2 we have worked in the neighbourhood of the surface $\psi = 1$ and the variable R is defined by $R = (D-3)(\psi - 1)$. [23]

[21]We evaluated the curvature components listed in Table 1 - including explicit results for sources presented later - using Mathematica as follows. Following [4, 12] we first focused on the special case in which the metric preserves $SO(D-p-2)$ isometry with p held fixed as D is taken to infinity. Working with $p = 2, 3$ we used Mathematica to explicitly evaluate the needed curvature components. We then uplifted these explicit results to the unique consistent covariant expressions listed in table 2, 3. We followed this procedure merely for convenience- in order to put the computation in a form in which it could be programmed into Mathematica. Were we to proceed by hand - as the authors of [72] are currently doing - we would have directly obtained the covariant results of table 2, 3.

[22]$\nabla.u$ is the divergence of the velocity field thought of as a vector field in $R^{D-1,1}$. On the surface $\psi = 1$, however, $\nabla.u$ coincides with the membrane worldvolume divergence of velocity field (this follows upon using the second of (32)).

[23]We will explain below that the sources listed in Table 2 are not completely independent, but are constrained by the well known relation

$$\nabla^M\left(R_{MN} - \frac{\tilde{R}}{2}G_{MN}\right) = 0 \tag{43}$$

.

Table 2: Expressions for basis of R_{MN}

Scalar sector
$R^{S_1} = \left(\frac{-\mathcal{K}^2}{2(D-3)^2}\right) \frac{d^2 H^{(Tr)}}{dR^2} + \mathcal{S}^{S_1}(R)$
$R^{S_2} = \left(\frac{\mathcal{K}^2}{2(D-3)^2}\right) e^{-R} \frac{d}{dR} \left(e^R \frac{d}{dR} H^{(S)}\right) - \frac{\mathcal{K}^2}{4(D-3)^2} e^{-R} \frac{d}{dR} H^{(Tr)} + \frac{\mathcal{K}}{2(D-3)} \nabla^M H^{(V)}_M$ $+\mathcal{S}^{S_2}(R) + \frac{\mathcal{K}}{2(D-3)} e^{-R}\ \nabla.u$
$R^{S_3} = \left(\frac{\mathcal{K}^2}{2(D-3)^2}\right) e^{-2R}(1-e^R) \frac{d}{dR}(e^R \frac{dH^{(S)}}{dR})$ $-\left(\frac{\mathcal{K}^2}{4(D-3)^2}\right) e^{-2R}(1-e^R) \frac{dH^{(Tr)}}{dR} - \frac{\mathcal{K}}{2(D-3)} e^{-R} \nabla^M H^{(V)}_M + \mathcal{S}^{S_3}(R) + \frac{\mathcal{K}}{2(D-3)}\ e^{-2R}\ \nabla.u$
$R^{S_4} = \left(\frac{\mathcal{K}^2}{(D-3)^2}\right) e^{-R} \frac{d}{dR}(e^R H^{(S)}) + \left(\frac{\mathcal{K}^2}{2(D-3)^2}\right) e^{-2R}(1-e^R) \frac{d}{dR} \left(e^R \frac{d}{dR} H^{(Tr)}\right)$ $-\left(\frac{\mathcal{K}^2}{2(D-3)^2}\right) \frac{dH^{(Tr)}}{dR} + \frac{\mathcal{K}}{D-3} \nabla^M H^{(V)}_M + \frac{2\mathcal{K}}{D-3} \frac{d}{dR} \nabla^M H^{(V)}_M + \nabla^M \nabla^N H^{(T)}_{MN} + \mathcal{S}^{S_4}(R) - \frac{\mathcal{K}}{(D-3)} e^{-R} \nabla.u$
Vector sector
$R^{V_1}_M = \left(\frac{\mathcal{K}^2}{2(D-3)^2}\right) e^{-R} \frac{d}{dR}(e^R \frac{d}{dR} H^{(V)}_M) + \frac{1}{2} \frac{\mathcal{K}}{(D-3)} \frac{d}{dR} \left(\nabla^N H^{(T)}_{NM}\right) + \mathcal{S}^{V_1}_M(R)$
$R^{V_2}_M = \left(\frac{\mathcal{K}^2}{2(D-3)^2}\right) e^{-2R}(1-e^R) \frac{d}{dR}(e^R \frac{d}{dR} H^{(V)}_M) + \mathcal{S}^{V_2}_M(R)$
Tensor sector
$R^T_{AB} = \left(\frac{-\mathcal{K}^2}{2(D-3)^2}\right) e^{-R} \frac{d}{dR} \left((e^R - 1)\ \frac{dH^{(T)}_{AB}}{dR}\right) + \mathcal{S}^T_{AB}(R)$

3.2.6 The Einstein Constraint Equations

In the process of solving for the fluctuation fields $h^{(n)}_{MN}$ we will find the Einstein constraint equations (relevant to the foliation of our spacetime in slices of constant ψ) particularly useful. We will now provide a careful definition of these equations.

Let us define

$$E_{MN} \equiv R_{MN} - \tilde{R} \frac{G_{MN}}{2} \tag{44}$$

where $\tilde{R}$ is the Ricci scalar. The constraint equations are defined by the relations

$$E^{(ec)}_M = E_{MN} G^{NL} n_L \tag{45}$$

We have a total of D constraint equations. These equations decompose into two scalars and one vector under local $SO(D-2)$ rotations.

Let us imagine we have solved for our membrane metric at $(n-1)^{th}$ order in perturbation theory, and are now attempting to solve for the metric correction at n^{th} order. If, in this process, we evaluate the constraint equation (45) and retain terms only up to n^{th} order then we need use G^{NL} on the RHS of (45) only at zero order (i.e. from the metric (1)), because E_{MN} is already of n^{th} order. It follows that the n^{th} order scalar and vector constraint equations are simply linear combinations of the n^{th} order scalars and vectors listed in table 1. We will now determine the relevant linear combinations. In order to to this we first determine the n^{th} order Ricci scalar $\tilde{R}$ as a linear combination of the scalars in table 1.

$$\tilde{R} = R_{AB}G^{AB} = \left(R^{AB}P_{AB} + O.R.O(1-e^{-R}) + 2O.R.u\right) = (R^{S_4} + (1-e^{-R})R^{S_1} + 2R^{S_2}) \tag{46}$$

Using this equation we find

$$\begin{aligned} E_M^{(ec)} &= \left(R_{MN} - \frac{\tilde{R}}{2}G_{MN}\right) G^{NL} n_L \\ &= R_{MN}O^N(1-e^{-R}) + R_{MN}u^N - \frac{1}{2}\tilde{R}\, n_M \end{aligned} \tag{47}$$

By dotting (47) with n and u or by projecting it orthogonal to these vectors we finally obtain the n^{th} order constraint equations written as linear combinations of the scalars and vectors in table 1.

$$\begin{aligned} E^{S_1} &= E_M^{(ec)}u^M = (1-e^{-R})R^{S_2} + R^{S_3} \\ E^{S_2} &= E_M^{(ec)}O^M = \frac{1}{2}\left((1-e^{-R})R^{S_1} - R^{S_4}\right) \\ E_L^V &= E_N^{(ec)}\mathcal{P}_L^N = (1-e^{-R})R_L^{V1} + R_L^{V_2} \end{aligned} \tag{48}$$

The explicit form of the n^{th} order constraint equations is listed in table 3 below

As in table 1, all fluctuation fields in table 3 should be taken to be of n^{th} order. The source

Table 3: Listing of constraint equations

Vector constraint
$E^V_M = E^{(ec)}_N \mathcal{P}^N_M = (1-e^{-R})R^{V_1}_M + R^{V_2}_M$ $= \frac{1}{2}\frac{\mathcal{K}}{(D-3)}(1-e^{-R})\frac{d}{dR}\left(\nabla^A H^{(T)}_{AM}\right) + \mathcal{V}^V_M(R)$
Scalar constraint 1
$E^{S_1} = E^{(ec)}_M u^M = (1-e^{-R})R^{S_2} + R^{S_3}$ $= \frac{\mathcal{K}}{2(D-3)}(1-e^{R})\frac{d}{dR}\left(\nabla^M H^{(V)}_M\right) - \frac{\mathcal{K}}{2(D-3)}e^{-R}\nabla^M H^{(V)}_M + \mathcal{V}^{S1}(R) + \frac{\mathcal{K}}{2(D-3)}\ e^{-R}\ \nabla.u$
Scalar constraint 2
$E^{S_2} = E^{(ec)}_M O^M = \frac{1}{2}\left((1-e^{-R})R^{S_1} - R^{S_4}\right) = -\frac{\mathcal{K}}{2(D-3)}\frac{d}{dR}\left(\nabla^M H^{(V)}_M\right) - \frac{\mathcal{K}}{(D-3)}\nabla^M H^{(V)}_M$ $+\frac{\mathcal{K}^2}{4(D-3)^2}(2-e^{-R})\frac{d}{dR}H^{(Tr)} - \frac{\mathcal{K}^2}{2(D-3)^2}\left(\frac{d}{dR}H^{(S)} + H^{(S)}\right) - \frac{1}{2}\nabla_M\nabla_N H^{(T)}_{MN} + \mathcal{V}^{S2}(R) + \frac{\mathcal{K}}{2(D-3)}e^{-R}\ \nabla.u$

functions in table 3 are also of n^{th} order and are given in terms of the sources in table 1 and the as yet unknown quantity $\nabla.u$ by

$$\begin{aligned}
\mathcal{V}^{S_1}(R) &= (1-e^{-R})\mathcal{S}^{S_2}(R) + \mathcal{S}^{S_3}(R) \\
\mathcal{V}^{S_2}(R) &= \frac{1}{2}\left[(1-e^{-R})\mathcal{S}^{S_1}(R) - \mathcal{S}^{S_4}(R)\right] \\
\mathcal{V}^V_L(R) &= (1-e^{-R})\mathcal{S}^{V_1}_L(R) + \mathcal{S}^{V_2}_L(R)
\end{aligned} \tag{49}$$

Now it is well known that the Einstein tensor obeys the identity

$$\nabla_M E^{MN} = 0 \tag{50}$$

It is also well known (and easy to see) that this identity ensures that the 'normal' derivative of the constraint equations is a linear combination of the 'in plane' derivatives of Einstein's equations. [24] Within the perturbation theory of interest to this chapter the equation (50) may be evaluated and projected onto its scalar and vector sectors and shown to be

[24]This is the fact that ensures that if all Einstein constraint equations are solved on one 'time' slice then they are automatically solved on the next 'time' slice. In other words, in order to solve Einstein's equations you need only solve the constraint equations on one time slice provided you solve the other equations - lets call them the dynamical equations - everywhere.

equivalent to the following relations

$$\begin{aligned}
&\frac{d}{dR}E^V_M + E^V_M + \frac{(D-3)}{\mathcal{K}}\nabla^N R^T_{NM} = 0 \\
&\frac{d}{dR}E^{S_1} + E^{S_1} + \frac{(D-3)}{\mathcal{K}}\nabla^N R^{V_2}_N = 0 \\
&\frac{d}{dR}E^{S_2} + E^{S_2} + \left(\frac{1}{2}R^{S_1} + R^{S_2} + \frac{1}{2}R^{S_4}\right) + \frac{(D-3)}{\mathcal{K}}\nabla^N R^{V_1}_N = 0
\end{aligned} \tag{51}$$

Using (48) the RHS of these relations may be recast in the equivalent form

$$\begin{aligned}
&\frac{d}{dR}E^V_M + (1-e^{-R})R^{V_1}_M + R^{V_2}_M + \frac{(D-3)}{\mathcal{K}}\nabla^N R^T_{NM} = 0 \\
&\frac{d}{dR}E^{S_1} + (1-e^{-R})R^{S_2} + R^{S_3} + \frac{(D-3)}{\mathcal{K}}\nabla^N R^{V_2}_N = 0 \\
&\frac{d}{dR}E^{S_2} + \frac{1}{2}e^{-R}R^{S_1} + (1-e^{-R})R^{S_1} + R^{S_2} + \frac{(D-3)}{\mathcal{K}}\nabla^N R^{V_1}_N = 0
\end{aligned} \tag{52}$$

In either form these equations express the R derivatives of the Einstein constraint equations (48) in terms of linear combinations of the Einstein equations. Using the explicit expressions in tables 2 and 3, it is possible to verify that the equations (51) are indeed obeyed, provided that the scalar and vector sources in table 2 and 3 are not all independent but are constrained by the following relations

$$\begin{aligned}
&\frac{d}{dR}\mathcal{V}^V_M + \mathcal{V}^V_M + \frac{(D-3)}{\mathcal{K}}\nabla^N \mathcal{S}^T_{NM} = 0 \\
&\frac{d}{dR}\mathcal{V}^{S_1} + \mathcal{V}^{S_1} + \frac{(D-3)}{\mathcal{K}}\nabla^N \mathcal{S}^{V_2}_N = 0 \\
&\frac{d}{dR}\mathcal{V}^{S_2} + \mathcal{V}^{S_2} + \left[\frac{1}{2}\mathcal{S}^{S_1} + \left(\mathcal{S}^{S_2} + \frac{\mathcal{K}}{2(D-3)}e^{-R}\ \nabla.u\right) + \frac{1}{2}\left(\mathcal{S}^{S_4} - \frac{\mathcal{K}}{(D-3)}e^{-R}\nabla.u\right)\right] \\
&+ \frac{(D-3)}{\mathcal{K}}\nabla^N \mathcal{S}^{V_1}_N = 0
\end{aligned} \tag{53}$$

Note that we have two relations between the four scalar sources and one relation between the two vector sources in table 2. Note that the relations also involve the as yet unknown quantity $\nabla.u$. Later in this chapter we will explicitly verify that the sources that appear in the first and second order calculation obey the relations (53). However we would like to emphasize here that these relations are necessarily obeyed at every order in perturbation theory.

3.2.7 Choice of basis for the constraint and dynamical equations

Because we have the linear relationship between constraint and dynamical equations we use the following basis for solving the scalar, vector and tensor fluctuations

$$\begin{aligned} &\text{Tensor:} \quad R^T_{AB} \\ &\text{Vector:} \quad R^{V_2}_M, \quad E^V_M \\ &\text{Scalar:} \quad R^{S_1}, \quad R^{S_2}, \quad E^{S_1}, \quad E^{S_2} \end{aligned} \tag{54}$$

From now on we write every expression in this basis. The expressions that we get from Bianchi identities i.e. equations (51),(52) can be converted to the basis (54) as

$$\begin{aligned} &\frac{d}{dR}E^V_M + E^V_M + \frac{(D-3)}{\mathcal{K}}\nabla^N R^T_{NM} = 0 \\ &\frac{d}{dR}E^{S_1} + E^{S_1} + \frac{(D-3)}{\mathcal{K}}\nabla^N R^{V_2}_N = 0 \\ &\frac{d}{dR}E^{S_2} + (1-\frac{1}{2}e^{-R})R^{S_1} + R^{S_2} + \frac{1}{1-e^{-R}}\frac{(D-3)}{\mathcal{K}}\nabla^M\left(E^V_M - R^{V_2}_M\right) = 0 \end{aligned} \tag{55}$$

The corresponding relationship between the sources is given by

$$\begin{aligned}
&\frac{d}{dR}\mathcal{V}^V_M + \mathcal{V}^V_M + \frac{(D-3)}{\mathcal{K}}\nabla^N \mathcal{S}^T_{NM} = 0 \\
&\frac{d}{dR}\mathcal{V}^{S_1} + \mathcal{V}^{S_1} + \frac{(D-3)}{\mathcal{K}}\nabla^N \mathcal{S}^{V_2}_N = 0 \\
&\frac{d}{dR}\mathcal{V}^{S_2} + (1-\frac{1}{2}e^{-R})\mathcal{S}^{S_1} + \mathcal{S}^{S_2} + \frac{1}{1-e^{-R}}\frac{(D-3)}{\mathcal{K}}\nabla^N\left(\mathcal{V}^V_N - \mathcal{S}^{V_2}_N\right) = 0
\end{aligned} \tag{56}$$

3.3 Perturbation theory at first order

In this section we will explicitly solve for the first order correction metric $h^{(1)}_{MN}$. However we will perform our analysis in a manner that makes the generalization to higher orders obvious.

3.3.1 Listing first order source functions

As we have explained in the previous section, the components of R^1_{MN} are given in terms of $h^{(1)}_{MN}$ by the expressions in Table 2 with particular values for the source functions in that table. By explicit calculation at first order we find that these source functions are given by the values listed in the table 4.

Moreover the constraint equations take the form listed in Table 3 with first order source functions listed in Table 5. We list the corresponding sources to the constraint equations at 1st order in table 5. We have verified that our explicit expressions for the sources obey the constraints (53).

We now proceed to solve the metric corrections at 1st order i.e. $h^{(1)}_{MN}$. We impose

Table 4: Sources of R_{MN} equations at 1st order

Scalar sector
$\mathcal{S}^{S_1}(R) = 0$ $\mathcal{S}^{S_2}(R) = \frac{\mathcal{K}}{2(D-3)} e^{-R} u.K.u - \frac{e^{-R}(-1+R)}{2} \frac{u.\nabla\mathcal{K}}{(D-3)} - \frac{\mathcal{K}^2}{2(D-3)^2} e^{-R}(-3+2R)$ $\mathcal{S}^{S_3}(R) = \frac{1}{2\mathcal{K}(D-3)} R e^{-R} \nabla^2\mathcal{K} - \frac{e^{-2R}(-2+2e^R+R)}{2} \frac{u.\nabla\mathcal{K}}{(D-3)} + \frac{\mathcal{K}^2}{2(D-3)^2} e^{-2R}\left(3e^R(R-1) - 2R + 3\right)$ $\mathcal{S}^{S_4}(R) = e^{-R}(-1+R) \frac{u.\nabla\mathcal{K}}{(D-3)} + \frac{\mathcal{K}^2}{(D-3)^2} e^{-R}(-1+2R)$
Vector sector
$\mathcal{S}^{V_1}_A(R) = \frac{\mathcal{K}}{2(D-3)} e^{-R}\left(u^M K_{MN} - u^M \nabla_M u_N\right) \mathcal{P}^N_A$ $\mathcal{S}^{V_2}_A(R) = \frac{\mathcal{K}}{2(D-3)} e^{-2R}\left(u^M K_{MN} - u^M \nabla_M u_N\right) \mathcal{P}^N_A + \frac{e^{-R}}{2}\left(\frac{\nabla^2 u_A}{(D-3)} - \frac{\nabla_A \mathcal{K}}{(D-3)}\right)$
Tensor sector
$\mathcal{S}^T_{AB}(R) = 0$

Table 5: Sources to constraint equations at 1st order

Vector constraint source
$\mathcal{V}^V_M(R) = \frac{e^{-R}}{2}\left(\frac{\nabla^2 u_M}{(D-3)} - \frac{\nabla_M \mathcal{K}}{(D-3)} + \frac{\mathcal{K}}{(D-3)}(u^A K_{AM} - u.\nabla u_M)\right)$
Scalar constraint 1 source
$\mathcal{V}^{S_1}(R) = \frac{1}{2\mathcal{K}(D-3)} R e^{-R} \nabla^2\mathcal{K} - \frac{-e^{-2R}+e^{-R}(1+R)}{2} \frac{u.\nabla\mathcal{K}}{(D-3)}$ $+\frac{\mathcal{K}}{2(D-3)} e^{-R}(1-e^{-R}) u.K.u + R e^{-R} \frac{\mathcal{K}^2}{2(D-3)^2}$
Scalar constraint 2 source
$\mathcal{V}^{S_2}(R) = \frac{e^{-R}}{2}\left(\frac{\mathcal{K}^2}{(D-3)^2}(1-2R) + \frac{u.\nabla\mathcal{K}}{(D-3)}(1-R)\right)$

the conditions (40) as discussed in section 3.2.4.

3.3.2 Tensor sector

In this sector we have a single equation for the single variable $H^{(T)}_{MN}$. This equation is obtained by equating the last line of Table 2 to zero and takes the form

$$R^T_{AB} = e^{-R} \frac{d}{dR}\left(\left(e^R - 1\right) \frac{dH^{(T)}_{AB}}{dR}\right)\left(\frac{-\mathcal{K}^2}{2(D-3)^2}\right) + \mathcal{S}^T_{AB}(R) = 0 \qquad (57)$$

where $\mathcal{S}^T_{AB}(R)$ is the source for the tensor sector. At first order it turns out that $\mathcal{S}^T_{AB}(R) = 0$ (see Table 5). In order to facilitate generalizations to higher orders however, in this subsection we will solve (57) for an arbitrary source function, and substitute $\mathcal{S}^T_{AB}(R) = 0$ only at the end of the calculation.

Integrating (57) once we find

$$\frac{d}{dR}(H^{(T)}_{AB}) = \left(\frac{-2(D-3)^2}{\mathcal{K}^2}\right)\frac{-1}{e^R - 1}\int_0^R e^x \mathcal{S}^T_{AB}(x)dx \tag{58}$$

The condition that $H^{(T)}_{AB}$ (and so RHS of (58)) is regular at $R = 0$ fixes the lower limit of the integral in (58). Integrating a second time we find

$$\begin{aligned} H^{(T)}_{AB} &= \left(\frac{-2(D-3)^2}{\mathcal{K}^2}\right)\int_R^\infty \frac{dy}{e^y - 1}\int_0^y e^x \mathcal{S}^T_{AB}(x)dx \\ &= \left(\frac{2(D-3)^2}{\mathcal{K}^2}\right)\left[\log(1-e^{-R})\int_0^R e^x S^T_{AB}(x)dx + \int_R^\infty \log(1-e^{-x})e^x S^T_{AB}(x)\right] \end{aligned} \tag{59}$$

where the upper limit in the outer integral in (59) is fixed by the requirement that $H^{(T)}_{AB}$ decay at large R.

In summary, the tensor fluctuation $H^{(T)}_{AB}$ is given at any order, in terms of the tensor source function $\mathcal{S}^T_{AB}(x)$ at that order, by the expression (59). Note that $H^{(T)}_{AB}$ is uniquely determined by its source function; requirements of regularity at $R = 0$ and decay at infinity unambiguously fix all integration constants in (57).

As we have mentioned above, at first order $\mathcal{S}^{T,1}_{AB}(R) = 0$. It follows from (59) that the first order tensor fluctuation $H^{(T)}_{AB}$ also vanishes and so

$$H^{(T,1)}_{AB} = 0 \tag{60}$$

3.3.3 Vector Sector

Constraint Equation and the Membrane Equation of Motion In the vector sector we have two equations for the single variable $H_M^{(V)}$. The two equations may be chosen to be the vector constraint equation E_M^V (see the first line of Table 3) and the equation $R_L^{V_2} = 0$ (see Table 2).

One cannot, of course, solve two equations for a single variable unless one linear combination of the two equations is an identity. Indeed the first equation of (55)

$$\frac{d}{dR}E_M^V + E_M^V + \frac{(D-3)}{\mathcal{K}}\nabla^N R_{NM}^T = 0 \tag{61}$$

asserts that the vector constraint equation is automatically solved at all values of R if its solved at one value of R (we use here that we have already solved the tensor equation so that $R_{AB}^T = 0$).

We will find it convenient to solve the vector constraint equation at $R = 0$. From Table 3 we see that

$$E_M^V = \frac{1}{2}\frac{\mathcal{K}}{(D-3)}(1-e^{-R})\frac{d}{dR}\left(\frac{\nabla^M H_{MN}^{(T)}}{(D-3)}\right) + \mathcal{V}_M^V(R)$$

At $R = 0$

$$E_M^V = \mathcal{V}_M^V(0)$$

It follows that the constraint equation is solved at $R = 0$ if and only if $\mathcal{V}_M^V(0)$ vanishes (here we use the fact that $H_{MN}^{(T)}$ is regular at $R = 0$; see the previous subsection) . This requirement is a statement of the membrane equations of motion.

We would like to reemphasize that the membrane equations of motion at n^{th} order are obtained simply by evaluating the n^{th} order vector constraint equation at $R = 0$. At $R = 0$ this equation is independent of all the unknown n^{th} order fluctuation fields. As a consequence the membrane equations of motion may be obtained at n^{th} order *before* solving for the fluctuation fields at n^{th} order, as in studies of the fluid gravity correspondence.

The analysis presented in this subsection so far has been valid at every order in perturbation theory. Specializing now to the first order, we read off the value of $\mathcal{V}_M^V(0)$ from Table 5. Equating this expression to zero we find the first order membrane equation of motion

$$\left(\frac{\nabla^2 u_A}{\mathcal{K}} - \frac{\nabla_A \mathcal{K}}{\mathcal{K}} + u_C K^C_A - u.\nabla u_A\right)\mathcal{P}^A_B = 0 \tag{62}$$

While all fields in (62) live in the full bulk spacetime $R^{D-1,1}$, and all derivatives in that equation are bulk spacetime derivatives, the equation (62) itself holds only on the membrane surface $\psi = 1$. Using the subsidiary conditions (32) it is possible to rewrite (62) as an equation restricted to the membrane. As demonstrated in [12] the equation of motion of motion turns out to take exactly the same form as (62) in this language. In other words (62) also holds true if we think of K_{MN} and u_M as membrane world volume fields, and regard every derivative in that equation as a covariant derivative on the membrane world volume.

Solving for the vector fluctuation As we have explained in the previous subsubsection, the constraint vector equation is automatically solved at every R provided the membrane equation is obeyed. Assuming this is the case, we have already solved one of the two vector equations.

In order to solve for the unknown function, $H_M^{(V)}$, in the vector sector, we now turn to the second vector equation $R_L^{V_2} = 0$. This equation takes the form

$$\left(\frac{-\mathcal{K}^2}{2(D-3)^2}\right) e^{-2R}(-1+e^R)\frac{d}{dR}(e^R \frac{d}{dR} H_M^{(V)}) + \mathcal{S}_M^{V_2}(R) = 0 \tag{63}$$

As in the previous subsection we will proceed to solve (63) for an arbitrary source function, plugging in the first order result for the source

$$\mathcal{S}_A^{V_2,1}(R) = -\frac{\mathcal{K}}{2(D-3)} e^{-2R}(-1+e^R)\left(u^M K_{MN} - u^M \nabla_M u_N\right)\mathcal{P}_A^N \tag{64}$$

only at the end of the computation.

Notice that the LHS of (63) vanishes at $R=0$. It follows that (63) admits regular solutions if and only if $\mathcal{S}_M^{V_2}(R)$ also vanishes at $R=0$. It would naively seem that this requirement imposes a new constraint on membrane data, independent of (62). [25] However it turns out that the vanishing of $\mathcal{S}_M^{V_2}(R)$ is automatic; indeed it follows from (48) that $R_M^{V_2}$ is simply identical to the vector constraint equation E_M^V at $R=0$. It follows as a consequence that $\mathcal{S}_M^{V_2}(R)$ is proportional to the LHS of (62) at $R=0$. [26].

Using the fact that $\mathcal{S}_M^{V_2,1}(0)$ vanishes, we integrate (63) once to find

$$e^R \frac{d}{dR} H_M^{(V)} = \left(\frac{-2(D-3)^2}{\mathcal{K}^2}\right)\left[\int_0^R \left(\frac{-e^y}{1-e^{-y}}\right)\mathcal{S}_M^{V_2}(y)dy + C_M^{V_2}\right] \tag{65}$$

where $C_M^{V_2}$ is an as yet undetermined integration constant. Integrating a second time we find

$$H_M^{(V)} = \left(\frac{2(D-3)^2}{\mathcal{K}^2}\right)\int_R^\infty e^{-x}\left[\int_0^x \left(\frac{-e^y}{1-e^{-y}}\right)\mathcal{S}_M^{V_2}(y)dy\right]dx - C_M^{V_2}e^{-R} \tag{66}$$

The upper limit on the the outer integral of (66) has been determined from the requirement that $H_M^{(V)}$ vanishes at large R. The expression for H_M^V may be simplified by integrating by parts; we find

$$H_M^{(V)}(R) = \left(\frac{2(D-3)^2}{\mathcal{K}^2}\right)\left(e^{-R}\int_0^R \left(\frac{-e^x}{1-e^{-x}}\right)\mathcal{S}_M^{V_2}(x)dx - \int_R^\infty \frac{\mathcal{S}_M^{V_2}(x)}{1-e^{-x}}\right) - C_M^{V_2}e^{-R} \tag{67}$$

[25]Had this step of the programme imposed a new constraint, we would have obtained a new membrane equation - and so obtained more membrane equations than membrane variables, leading to an inconsistent dynamical system.

[26] To see this we note that (63) reduces to $\mathcal{S}_M^{V_2}(R)$ at $R=0$ while E_M^V reduces to the LHS of (62) at $R=0$.

In particular that

$$H_M^{(V)}(0) = -\left(\frac{2(D-3)^2}{\mathcal{K}^2}\right)\int_0^\infty \frac{\mathcal{S}_M^{V_2}(x)}{1-e^{-x}} - C_M^{V_2} \tag{68}$$

It follows (see (40)) that

$$C_M^{V_2} = -\left(\frac{2(D-3)^2}{\mathcal{K}^2}\right)\int_0^\infty \frac{\mathcal{S}_M^{V_2}(x)}{1-e^{-x}} \tag{69}$$

so that

$$H_M^{(V)}(R) = \left(\frac{2(D-3)^2}{\mathcal{K}^2}\right)\left(e^{-R}\int_0^R \left(\frac{-e^x}{1-e^{-x}}\right)\mathcal{S}_M^{V_2}(x)dx - \int_R^\infty \frac{\mathcal{S}_M^{V_2}(x)}{1-e^{-x}} + e^{-R}\int_0^\infty \frac{\mathcal{S}_M^{V_2}(x)}{1-e^{-x}}\right) \tag{70}$$

The expression (70) is our final expression for $H_M^{(V)}(R)$ at any order in perturbation theory in terms of the source function at that order. Note that $H_M^{(V)}(R)$ is uniquely determined in terms of its source function; the integration constants in (63) are uniquely determined by the requirement that $H_M^{(V)}(R)$ vanish at infinity and that (40) is obeyed at $R = 0$.

Plugging the first order expression for the source (64) into (70), at first order we find

$$H_M^{(V,1)} = \frac{(D-3)}{\mathcal{K}} Re^{-R}\left(u^A K_{AN} - u^A \nabla_A u_N\right) P_M^N \tag{71}$$

3.3.4 Scalar sector

In the scalar sector we have four equations for the two variables $H^{(Tr)}$ and $H^{(S)}$. As a basis for the four equations we find it convenient to use the two scalar constraint equations E^{S_1} and E^{S_2} (see Table 3) together with the two additional equations $R^{S_1} = 0$ and $R^{S_2} = 0$ (see Table 1).

Constraint Equations and $\nabla.u$ As in the previous subsection it is consistent to have four equations for two variables only if two of the four equations are identities. The last two equations in (55)

$$\begin{aligned}&\frac{d}{dR}E^{S_1}+E^{S_1}+\frac{(D-3)}{\mathcal{K}}\nabla^N R^{V_2}_N=0\\&\frac{d}{dR}E^{S_2}+(1-\frac{1}{2}e^{-R})R^{S_1}+R^{S_2}+\frac{(D-3)}{\mathcal{K}}\frac{1}{1-e^{-R}}\nabla^M\left(E^V_M-R^{V_2}_M\right)=0\end{aligned} \tag{72}$$

assert that this is indeed the case. As we have already solved the vector sector at n^{th} order $R^{V_2}_N$ vanishes. It follows that the first equation in (72) asserts that if E^{S_1} is solved at any R it is automatically solved at every R. When evaluated at $R=0$ this equation reduces to the condition

$$\mathcal{V}^{S_1}(0)+\frac{\mathcal{K}}{2(D-3)}\ \nabla.u=0 \tag{73}$$

Recall that at leading order $\nabla.u=0$. (73) determines the correction to this statement at subleading orders.

As in the previous subsection we emphasize that the expression for $\nabla.u$ at n^{th} order is determined simply by evaluating the n^{th} order constraint equation E^{S_1} at $R=0$. In order to obtain this correction we do not need to solve for any of the n^{th} order fluctuation fields, all of which drop out in E^{S_1} evaluated at $R=0$.

The analysis of this subsection has, so far, been valid at every order in perturbation theory. Specializing to first order it is easily verified from Table 5 that $\mathcal{V}^{S_1}(0)=0$. It follows that the zero order relation $\nabla.u=0$ is uncorrected at first order (since $(\nabla.u)_0=\mathcal{V}^{S_1}(0)=0$). As we will see in the next section, the situation is different at second order.

The constraint equation E^{S_2} plays a distinct logical role from E^{S_1} in our perturbative programme. Once the tensor and vector equations had been solved, (72) assured us that $E^{S_1}(R)$ obeys a homogeneous differential equation in R (see (51) which makes no reference to any of the other equations in the scalar sector. On the other hand the differential equation obeyed by E^{S_2} involves the other scalar equations (see the last equation in (52)). The most useful way to view the last equation in (52) is as follows. It might, a priori, have seemed that we have 4 equations in the scalar sector. We have already dealt with E^{S_1} above leaving behind a three dimensional space of equations. A useful basis for this space is given by E^{S_2}, R^{S_1} and R^{S_2}. The last equation in (52) allows us to eliminate R^{S_2} from this basis. In order to complete solving in the scalar sector we need only solve the equations E^{S_2}, R^{S_1}. In other words the constraint equation E^{S_2} does not constrain data: instead it may be used to solve for the scalar fluctuation. We turn to this task in the next subsubsection.

Solving for the scalar fluctuations The equation R^{S_1}

$$R^{S_1} = \left(\frac{-\mathcal{K}^2}{2(D-3)^2}\right)\frac{d^2 H^{(Tr)}}{dR^2} + \mathcal{S}^{S_1}(R) = 0 \tag{74}$$

is easily solved. Integrating the above equation once we get

$$\frac{dH^{(Tr)}}{dR} = \left(\frac{-2(D-3)^2}{\mathcal{K}^2}\right)\int_R^\infty dx\ \mathcal{S}^{S_1}(x) \tag{75}$$

Where we have fixed the boundary condition from the requirement that $H^{(Tr)}$ and so its derivative $\frac{dH^{(Tr)}}{dR} = 0$ vanish at large R. Integrating this equation once again we have

$$\begin{aligned} H^{(Tr)} &= \left(\frac{2(D-3)^2}{\mathcal{K}^2}\right)\int_R^\infty dy \int_y^\infty dx\ \mathcal{S}^{S_1}(x) \\ &= \left(\frac{2(D-3)^2}{\mathcal{K}^2}\right)\left[-R\int_R^\infty dx\ \mathcal{S}^{S_1}(x) + \int_R^\infty dx\ x\ \mathcal{S}^{S_1}(x)\right] \end{aligned} \tag{76}$$

where, once again we have fixed the integration constant from the requirement that $H^{(Tr)} = 0$ at large R.

Specializing now to first order we note $\mathcal{S}^{S_1,1} = 0$ so that

$$H^{(Tr,1)} = 0 \tag{77}$$

The equation E^{S_2} takes the form

$$\begin{aligned}
&\frac{d}{dR}(H^{(S)}e^R) = \frac{2(D-3)^2}{\mathcal{K}^2}e^R\mathcal{S}_S(R) \quad \text{where,} \\
&\mathcal{S}_S(R) = -\frac{\mathcal{K}}{2(D-3)}\frac{d}{dR}\left(\nabla^M H_M^{(V)}\right) - \frac{\mathcal{K}}{(D-3)}\nabla^M H_M^{(V)} \\
&+ \frac{\mathcal{K}^2}{4(D-3)^2}(2-e^{-R})\frac{d}{dR}H^{(Tr)} - \frac{1}{2}\nabla^M\nabla^N H_{MN}^{(T)} + \mathcal{V}^{S_2}(R) + \frac{\mathcal{K}}{2(D-3)}e^{-R}\ \nabla.u
\end{aligned} \tag{78}$$

Plugging in the already obtained expressions of $H_M^{(V)}$, $H_{MN}^{(T)}$, $H^{(Tr)}$ (see (70),(76) and (59)) and using (56), the source function $\mathcal{S}_S(R)$ can be rewritten as a linear functional of the elementary sources $\mathcal{S}^{S_1}$, $\mathcal{S}^{S_2}$ and $\mathcal{V}^{S_1}$ [27]. Upon simplifying (by integrating by parts on several occasions) we find

$$\begin{aligned}
\mathcal{S}_S(R) = &\int_R^\infty \mathcal{S}^{S_2}(x)dx + \frac{1}{2}\int_R^\infty (2-e^{-x})\mathcal{S}^{S_1}(x)dx - \frac{1}{2}(2-e^{-R})\int_R^\infty \mathcal{S}^{S_1}(x)dx \\
&- \left(1-e^{-R}\right)\int_R^\infty \left(\frac{e^x\left({\mathcal{V}^{S_1}}'(x) + \mathcal{V}^{S_1}(x)\right)}{(e^x-1)}dx\right)dy - \mathcal{V}^{S_1}(R) + e^{-R}\mathcal{V}^{S_1}(0) \\
&+ \log(1-e^{-R})\left({\mathcal{V}^{S_1}}'(0) + \mathcal{V}^{S_1}(0)\right) + (\nabla\cdot u)\frac{\mathcal{K}e^{-R}}{2(D-3)}
\end{aligned} \tag{79}$$

We note that $\mathcal{S}_S$ is analytic at $R=0$ if and only if

$${\mathcal{V}^{S_1}}'(0) + \mathcal{V}^{S_1}(0) = 0 \tag{80}$$

This condition is, in fact, automatic. It follows from the second of (56) that the LHS of (80) is proportional to $\nabla^N\mathcal{S}_N^{V_2}(0)$. We have already argued, however, that $\mathcal{S}_N^{V_2}$ vanishes at

[27] It turns out that all dependence on the fourth independent scalar source, $\mathcal{V}^{S_2}$ cancels.

$R = 0$. Since this condition holds at every point on the membrane, it follows also that $\nabla^N \mathcal{S}_N^{V_2}(0) = 0$ establishing (80). [28]

Plugging (79) into (78), integrating (and simplifying using integration by parts) we find

$$
\begin{aligned}
H_S(R) = & \frac{2(D-3)^2}{\mathcal{K}^2} e^{-R} \Bigg(\frac{(\mathcal{K}(\nabla \cdot u))R}{2(D-3)} + e^R \int_R^\infty \mathcal{S}^{S_2}(x)dx - \int_0^\infty \mathcal{S}^{S_2}(x)dx + \int_0^R e^x \mathcal{S}^{S_2}(x)dx \\
& + \frac{e^R}{2} \int_R^\infty (2 - e^{-x}) \mathcal{S}^{S_1}(x)dx + \frac{1}{2} \int_0^R e^x (2 - e^{-x}) \mathcal{S}^{S_1}(x)dx - \frac{1}{2} \int_0^\infty (2 - e^{-x}) \mathcal{S}^{S_1}(x)dx \\
& - \frac{1}{2}(2e^R - R) \int_R^\infty \mathcal{S}^{S_1}(x)dx + \int_0^\infty \mathcal{S}^{S_1}(x)dx - \frac{1}{2} \int_0^R (2e^y - y) \mathcal{S}^{S_1}(x)dx \\
& - \int_0^R (e^y - 1) \int_y^\infty \left(\frac{e^x \left({\mathcal{V}^{S_1}}'(x) + \mathcal{V}^{S_1}(x) \right)}{(e^x - 1)} dx \right) dy - \int_0^R e^x \mathcal{V}^{S_1}(x)dx + R\mathcal{V}^{S_1}(0) \Bigg)
\end{aligned}
\tag{81}
$$

Explicitly at first order

$$
H^{(S,1)} = \frac{D-3}{\mathcal{K}} R e^{-R} \left(R \left(-\frac{\mathcal{K}}{D-3} - \frac{u \cdot \nabla \mathcal{K}}{\mathcal{K}} + \frac{u \cdot K \cdot u}{2} \right) + \left(\frac{\mathcal{K}}{D-3} + u \cdot K \cdot u \right) \right)
\tag{82}
$$

3.3.5 Final Result for the first order metric

After integrating the ordinary differential equations corresponding to Einstein's equations and imposing the condition that the metric is regular at the horizon, matches flat space at the end of the membrane region and (40), we get the following solutions for the various

[28] In studies of the fluid gravity correspondence a derivative of the equation of the n^{th} order equation contributes to sources only at $(n+1)^{th}$ order in the derivative expansion. In the large D expansion of this chapter, however, the suppression in order resulting from using an extra derivative can be compensated for by an enhancement in order resulting from the contraction of a spacetime index. Consequently the equation of motion and its contracted derivatives are of the same order in the large D expansion.

components of the metric correction.

$$
\begin{aligned}
H^{(T,1)}_{MN} &= 0 \\
H^{(Tr,1)} &= 0 \\
H^{(V,1)}_{M} &= \frac{(D-3)}{\mathcal{K}} R e^{-R} \left(u^A K_{AL} - u^A \nabla_A u_L \right) \mathcal{P}^L_M \\
H^{(S,1)} &= \frac{D-3}{\mathcal{K}} R e^{-R} \left(R \left(-\frac{\mathcal{K}}{D-3} - \frac{u \cdot \nabla \mathcal{K}}{\mathcal{K}} + \frac{u \cdot K \cdot u}{2} \right) + \left(\frac{\mathcal{K}}{D-3} + u \cdot K \cdot u \right) \right)
\end{aligned}
\tag{83}
$$

Thus we can write the 1st order corrected metric as

$$
\begin{aligned}
& g_{MN} = \eta_{MN} + \frac{O_M O_N}{\psi^{D-3}} \\
& + \frac{1}{D-3} \Bigg[\frac{D-3}{\mathcal{K}} R e^{-R} \left(R \left(-\frac{\mathcal{K}}{D-3} - \frac{u \cdot \nabla \mathcal{K}}{\mathcal{K}} + \frac{u \cdot K \cdot u}{2} \right) + \left(\frac{\mathcal{K}}{D-3} + u \cdot K \cdot u \right) \right) O_M O_N \\
& + \frac{(D-3)}{\mathcal{K}} R e^{-R} \left(u^A K_{AL} - u^A \nabla_A u_L \right) P^L_{(M} O_{N)} \Bigg]
\end{aligned}
\tag{84}
$$

3.4 2nd order solution

The metric (84) solves Einstein equation to first subleading order. In this section we implement the perturbative procedure to one higher order. In other words we determine the correction $H^{(2)}_{MN}$ in a way that ensures that R_{AB} evaluated on the corrected metric is of order $1/D$ (more precisely that $R_{AB}R^{AB}$ is of order $1/D^2$).

The procedure we follow is exactly that of the previous section: in fact second order corrections to the metric are given directly by the formulae of the previous subsection with one modification: we need to use the second order rather than first order source functions. In other words the computation at second order boils down entirely to determining the second order sources.

In order to determine the sources at second order we plug the first order corrected metric (84) together with an as yet undetermined second order correction h^2_{MN} into Einstein's equations. We use the fact that the shape and velocity functions in the first order corrected metric obey the equation of motion

$$\left(\frac{\nabla^2 u_A}{\mathcal{K}} - \frac{\nabla_A \mathcal{K}}{\mathcal{K}} + u_C K^C_A - u.\nabla u_A\right) \mathcal{P}^A_B + \frac{1}{D}\mathcal{E}_A \mathcal{P}^A_B = 0 \tag{85}$$

where $\mathcal{E}_B$ is an as yet undetermined '2nd order' correction to the equations of motion. As in the previous subsection we solve the equations in the neighbourhood of a particular point on the event horizon. In our analysis, however, we use the fact that the membrane equations of motion (85) are obeyed not just at the particular point we are expanding about but everywhere on the membrane. In other words we use the fact that the derivative of (85) vanishes at the point of interest. Finally we also use the fact that $\nabla.u$ is an as yet undetermined quantity of order $1/D$.

We find by explicit computation that the curvature components listed table 1 do indeed take the form listed in table 2,3 once all metric fluctuation fields in that table are identified with second order fluctuations. Our explicit computations also yield explicit expressions for all the second order source functions. We present an explicit listing of these source functions in Tables 6 and 7 in the Appendix.

In the rest of this section we obtain the second order correction to the metric by inserting the second order sources listed above into the general integral formulae of the previous section and performing all integrals.

3.4.1 Constraints on membrane data

Correction to the membrane equations from the vector sector As in the previous subsection (61) guarantees that the vector constraint equation $E^V_M = 0$ is solved at any R if the equation is obeyed at $R = 0$. As in the previous subsection the constraint equation at $R = 0$ is independent of the second order fluctuation fields. From table 7 we see that this constraint equation at $R = 0$ determines $-\frac{1}{D}\mathcal{E}_A\mathcal{P}^A_B$ - the second order correction to the membrane equation of motion - in terms of appropriate expressions involving the membrane extrinsic curvature and velocity fields. Adding these correction terms to the first order membrane equation (3) we recover the second order corrected membrane equation

$$
\begin{aligned}
&\left[\frac{\nabla^2 u}{\mathcal{K}} - \frac{\nabla\mathcal{K}}{\mathcal{K}} + u\cdot K - (u\cdot\nabla)u\right]\cdot\mathcal{P} + \left[\frac{\nabla^2\nabla^2 u}{\mathcal{K}^3} - \frac{\nabla(\nabla^2\mathcal{K})}{\mathcal{K}^3}\right. \\
+ \;\; & 3\frac{(u\cdot K\cdot u)(u\cdot\nabla u)}{\mathcal{K}} - 3\frac{(u\cdot K\cdot u)(u\cdot\nabla n)}{\mathcal{K}} - 6\frac{(u\cdot(\nabla^2 n))(u\cdot\nabla u)}{\mathcal{K}^2} \\
+ \;\; & \left. 6\frac{(u\cdot(\nabla^2 n))(u\cdot\nabla n)}{\mathcal{K}^2} + \frac{3}{D-3}u\cdot\nabla u - \frac{3}{D-3}u\cdot\nabla n\right]\cdot\mathcal{P} = 0
\end{aligned}
\tag{86}
$$

where

$$\mathcal{P}^{AB} = \eta^{AB} - n^A n^B + u^A u^B \tag{87}$$

The 1st square bracket in (86) is simply the 1st order equation of motion while the 2nd square bracket represents subleading corrections. [29]

We would like, however, to emphasize an important technical point. All the fields in (86) are assumed to live in all of the embedding flat spacetime; they are extended off the surface of the membrane by the subsidiary conditions listed earlier in this chapter. While all covariant derivatives listed in (86) are evaluated on the surface of the membrane, they act on fields defined in all of spacetime.

As the membrane equations of motion are intrinsic to the membrane, it is clearly

[29]Note that we can write the equation (86) in a nicer looking form by using the subsidiary conditions (32), divergence of first order membrane equation of motion (3) and divergence of velocity condition (28). The form is

$$\left(\frac{\nabla^2 O}{\nabla.O} + O.\nabla O\right)\cdot\mathcal{P} + \left(\frac{\nabla^2\nabla^2 O}{(\nabla.O)^3} + 3\frac{\nabla^2(\nabla.O)}{(\nabla.O)^3}O.\nabla O\right)\cdot\mathcal{P} = 0 \tag{88}$$

unnatural to write them in terms of spacetime derivatives of an essentially arbitrary extension of membrane fields into the embedding spacetime. The equation of motion (86) can be rewritten so that all fields in that equation are purely membrane world volume fields, and every derivative in the equation is a covariant derivative on the membrane world volume. We now explain how this is done.

The relationship between the bulk covariant derivatives of tensors (e.g. u_M) and membrane worldvolume derivatives of the same quantities is quite straightforward when no more than one derivative acts on the same object. The spacetime covariant derivative is obtained from the corresponding bulk quantity by projecting every index (not just the derivative indices) onto the membrane world volume. However this relationship is more complicated when we have two or more derivatives acting on the same object; the reason for the additional complication is that the formula for multiple worldvolume covariant derivatives involves inserting projectors at each step (when you define the first derivative in terms of bulk derivatives, then again when you define the second derivative in terms of bulk derivatives etc); when such expressions are opened out, outer derivatives act on projectors used to define the inner derivatives. Tracing through the required algebra we find that the corrected second order membrane equation of motion, written in terms of fields and covariant derivatives that live purely on the membrane world volume, takes the form

$$
\begin{aligned}
&\left[\frac{\nabla^2 u_A}{\mathcal{K}} - \frac{\nabla_A \mathcal{K}}{\mathcal{K}} + u^B K_{BA} - u\cdot\nabla u_A\right]\mathcal{P}^A_C \\
+&\left[\left(-\frac{u^C K_{CB} K^B_A}{\mathcal{K}}\right) + \left(\frac{\nabla^2\nabla^2 u_A}{\mathcal{K}^3} - \frac{u\cdot\nabla\mathcal{K}\nabla_A\mathcal{K}}{\mathcal{K}^3} - \frac{\nabla^B\mathcal{K}\nabla_B u_A}{\mathcal{K}^2} - 2\frac{K^{CD}\nabla_C\nabla_D u_A}{\mathcal{K}^2}\right)\right. \\
+&\left(-\frac{\nabla_A\nabla^2\mathcal{K}}{\mathcal{K}^3} + \frac{\nabla_A\left(K_{BC}K^{BC}\mathcal{K}\right)}{\mathcal{K}^3}\right) + 3\frac{(u\cdot K\cdot u)(u\cdot\nabla u_A)}{\mathcal{K}} - 3\frac{(u\cdot K\cdot u)(u^B K_{BA})}{\mathcal{K}} \\
-&\left.6\frac{(u\cdot\nabla\mathcal{K})(u\cdot\nabla u_A)}{\mathcal{K}^2} + 6\frac{(u\cdot\nabla\mathcal{K})(u^B K_{BA})}{\mathcal{K}^2} + \frac{3}{D-3}u\cdot\nabla u_A - \frac{3}{D-3}u^B K_{BA}\right]\mathcal{P}^A_C = 0
\end{aligned}
\tag{89}
$$

The projector $\mathcal{P}^{AB}$ used in this equation

$$\mathcal{P}^{AB} = \tilde{g}^{AB} + u^A u^B \tag{90}$$

where $\tilde{g}^{AB}$ is the induced metric on the world volume of the membrane.

The equation (89) can be slightly simplified as follows. Let us first note that (89) takes the schematic form

$$F^A + \frac{S^A}{\mathcal{K}} = 0 \tag{91}$$

where F^A is the first order contribution to the equation of motion (the first line of (89)) while $\frac{S^A}{\mathcal{K}}$ is the second order contribution (the second-fourth lines of (89)). F^A and S^A are each vector fields of order unity.

Let us now consider the modified equation of motion

$$F^A + \frac{S^A}{\mathcal{K}} + \nabla.F\frac{\zeta^A}{\mathcal{K}^2} = 0 \tag{92}$$

where ζ^A is any vector field of order unity. As $\nabla.F$ is naively of order D, the difference between the equations (92) and (91) is naively of order $\frac{1}{D}$ suggesting that (91) and (92) differ at first subleading order. This is not the case. By taking a divergence of either (91) or (92), the reader can easily convince herself that, onshell, $\nabla.F$ is of order unity (rather than the naive estimate of order D). If follows that (92) and (91) actually differ only at second subleading order ($\frac{1}{D^2}$) and are equivalent at first subleading order. We are thus allowed to simplify (89) by adding any expression of the form $\nabla.F\frac{\zeta^A}{\mathcal{K}^2}$ to it.

Now it was demonstrated in [12] that

$$\frac{\nabla.F}{\mathcal{K}} = \frac{\nabla^2\mathcal{K}}{\mathcal{K}^2} - 2\,\frac{u.\nabla\mathcal{K}}{\mathcal{K}} + u.K.u \tag{93}$$

Using this relation and making the the choice

$$\zeta^A = -3\left((u.\nabla u)_A - u_B K^B_A\right) \tag{94}$$

we find that (89) is equivalent to (92) whose explicit form is

$$\begin{aligned}
&\left[\frac{\nabla^2 u_A}{\mathcal{K}} - \frac{\nabla_A \mathcal{K}}{\mathcal{K}} + u^B K_{BA} - u\cdot\nabla u_A\right]\mathcal{P}^A_C \\
+ &\left[\left(-\frac{u^C K_{CB} K^B_A}{\mathcal{K}}\right) + \left(\frac{\nabla^2\nabla^2 u_A}{\mathcal{K}^3} - \frac{u\cdot\nabla\mathcal{K}\nabla_A\mathcal{K}}{\mathcal{K}^3} - \frac{\nabla^B\mathcal{K}\nabla_B u_A}{\mathcal{K}^2} - 2\frac{K^{CD}\nabla_C\nabla_D u_A}{\mathcal{K}^2}\right)\right. \\
+ &\left(-\frac{\nabla_A\nabla^2\mathcal{K}}{\mathcal{K}^3} + \frac{\nabla_A\left(K_{BC}K^{BC}\mathcal{K}\right)}{\mathcal{K}^3}\right) - 3\frac{\nabla^2\mathcal{K}\ u\cdot\nabla u_A}{\mathcal{K}^3} + 3\frac{\nabla^2\mathcal{K}\ u^B K_{BA}}{\mathcal{K}^3} \\
+ &\left.\frac{3}{D-3}u\cdot\nabla u_A - \frac{3}{D-3}u^B K_{BA}\right]\mathcal{P}^A_C = 0
\end{aligned} \tag{95}$$

Divergence of velocity from a scalar constraint As we have explained in the previous section, the Einstein constraint equation E^{S_1} is satisfied at all R if it is satisfied at $R = 0$. As explained in the previous subsection, the equation at $R = 0$ simply asserts that

$$\nabla.u_2 = -\frac{2(D-3)}{\mathcal{K}}\mathcal{V}^{S_1}(0)$$

Reading off the value of $\mathcal{V}^{S_1}(0)$ from the table 7 we find

$$\nabla\cdot u = \frac{(\nabla.u)_2}{D-3} = \frac{1}{2\mathcal{K}}\left(\nabla_{(A}u_{B)}\nabla_{(C}u_{D)}\mathcal{P}^{BC}\mathcal{P}^{AD}\right) \tag{96}$$

3.4.2 Second order corrections to the metric

Tensor Sector The metric correction in the tensor sector is given by (59)

$$
\begin{aligned}
H^{(T)}_{AB} &= \left(\frac{-2(D-3)^2}{\mathcal{K}^2}\right)\int_R^\infty \frac{dy}{e^y-1}\int_0^y e^x \mathcal{S}^T_{AB}(x)dx \\
&= \left(\frac{2(D-3)^2}{\mathcal{K}^2}\right)\left[\log(1-e^{-R})\int_0^R e^x S^T_{AB}(x)dx + \int_R^\infty \log(1-e^{-x})e^x S^T_{AB}(x)\right]
\end{aligned}
\tag{97}
$$

where $\mathcal{S}^T_{AB}$ is the second order source listed in table 6. All the integrals that appear in the final answer can easily be performed analytically, but the final results (given in terms of polylogs) are not very illuminating; we prefer to leave our final result in terms of an explicit integral.

Vector Sector The solution for $H^{(V)}_M(R)$ at second order is given by (70)

$$
H^{(V)}_M(R) = \left(\frac{2(D-3)^2}{\mathcal{K}^2}\right)\left(e^{-R}\int_0^R\left(\frac{-e^x}{1-e^{-x}}\right)\mathcal{S}^{V2}_M(x)dx - \int_R^\infty \frac{\mathcal{S}^{V2}_M(x)}{1-e^{-x}} + e^{-R}\int_0^\infty \frac{\mathcal{S}^{V2}_M(x)}{1-e^{-x}}\right)
\tag{98}
$$

with all sources read off at 2nd order from table 6. As in the tensor sector, all integrals that appear in (98) can be explicitly performed in terms of polylogs, but we find the expression (98) in terms of explicit integrals more illuminating.

Scalar Sector Equation R^{S_1} is decoupled equation for $H^{(Tr)}$. The integrated form is given by (76) which we write again

$$
\begin{aligned}
H^{(Tr)} &= \left(\frac{2(D-3)^2}{\mathcal{K}^2}\right)\int_R^\infty dy\int_y^\infty dx\ \mathcal{S}^{S_1}(x) \\
&= \left(\frac{2(D-3)^2}{\mathcal{K}^2}\right)\left[-R\int_R^\infty dx\ \mathcal{S}^{S_1}(x) + \int_R^\infty dx\ x\ \mathcal{S}^{S_1}(x)\right]
\end{aligned}
\tag{99}
$$

The source $\mathcal{S}^{S_1}$ for 2nd order is given in table 6. Substituting this we get the final form of the metric correction

$$H^{(Tr,2)} = -\left(\frac{2(D-3)^2}{\mathcal{K}^2}\right) e^{-R}(1+R)\left((u \cdot K - u \cdot \nabla u) \cdot \mathcal{P} \cdot (u \cdot K - u \cdot \nabla u)\right) \tag{100}$$

In a similar manner the fluctuation H^S can is given by (81) upon plugging in the explicit values of the second order sources from Tables 6,7.

3.5 The spectrum of small fluctuations around a spherical membrane

The simplest solution of the second order membrane equations of motion is a static spherical membrane dual to a Schwarzschild Black hole. In this section we compute the spectrum of small fluctuations about this solution. Our answers agree perfectly with earlier results for the spectrum of light quasinormal modes obtained by direct gravitational analysis, in [10]. We regard this detailed agreement as a nontrivial consistency check of the second order membrane equations of motion derived in this chapter.

The computation presented in this section is a straightforward extension of the first order computation presented in section 5 of [12]. We have kept the discussion of this section brief. We refer the reader to section 5 of [12] for a fuller discussion of the logic behind our computation.

We work in standard spherical polar coordinates (see Eq 5.1 of [12]). The static spherical membrane is given by

$$r = 1, \quad u = -dt, \tag{101}$$

We study the small fluctuations

$$r = 1 + \epsilon\ \delta r(t,\theta),$$
$$u = -dt + \epsilon\ \delta u_\mu(t,\theta) dx^\mu. \tag{102}$$

about this solution and work to linear order in ϵ. As explained in [12], to linear order the metric on membrane worldvolume is given by

$$ds^2 = -dt^2 + (1 + 2\epsilon\delta r)\, d\Omega^2_{D-2} \quad . \tag{103}$$

As in [12] we find it convenient to work with covariant derivatives with respect to the unperturbed spherical metric

$$ds^2 = -dt^2 + d\Omega^2_{D-2} \quad , \tag{104}$$

The derivatives appearing from now on are all with respect to metric (104). We use the following notation for the laplacian with respect to this fixed metric

$$\overline{\nabla}^2 = \nabla_\mu \nabla^\mu = -\partial_t^2 + \nabla_a \nabla^a = -\partial_t^2 + \nabla^2$$

3.5.1 The divergence condition

The RHS of (28) is quadratic in ϵ, and so vanishes upon linearizing in ϵ. At linear order, therefore, (28) reduces to $\nabla.u = 0$ (where the divergence is taken along the dynamical membrane world volume). As explained in [12], this equation can be rewritten as

$$\nabla_\mu \delta u^\mu = -(D-2)\partial_t \delta r, \tag{105}$$

where, the covariant derivatives (105) are now taken w.r.t. the fixed metric (104). u^0 deviates from unity only at quadratic order in ϵ. For the linearized considerations of this section, therefore, the LHS of (105) is simply the spatial divergence of the velocity

$$\nabla_a \delta u^a = -(D-2)\partial_t \delta r. \tag{106}$$

As in [12], (106) may be solved by separating u into its gradient and curl parts, i.e. by setting

$$\delta u_a = \nabla_a \Phi + \delta v_a, \tag{107}$$

with

$$\nabla \cdot \delta v = 0. \tag{108}$$

It follows from (106) that

$$\nabla^2 \Phi = -(D-2)\partial_t \delta r. \tag{109}$$

3.5.2 Linearized equation of motion

In order to obtain the linearized membrane equations of motion we use Eq 5.7 of [12] together with

$$\begin{aligned}
\frac{u^E K_{EB} K^B_a}{\mathcal{K}} &= -\epsilon \frac{(\nabla_a \partial_t \delta r - \delta u_a)}{D-2} \\
\frac{\nabla^2 \nabla^2 u_a}{\mathcal{K}^3} &= \epsilon \frac{\bar{\nabla}^2 \bar{\nabla}^2 \delta u_a + \bar{\nabla}^2 \nabla_a \partial_t \delta r}{(D-2)^3} \\
\frac{K^{CD} \nabla_C \nabla_D u_a}{\mathcal{K}^2} &= \epsilon \frac{\bar{\nabla}^2 \delta u_a - \nabla_a \partial_t \delta r}{(D-3)(D-2)} \\
\frac{\nabla_a \nabla^2 \mathcal{K}}{\mathcal{K}^3} &= -\epsilon \frac{\nabla_a \bar{\nabla}^2 (\bar{\nabla}^2 \delta r + \delta r (D-2))}{(D-2)^3} \\
\frac{\nabla_a (K^{BC} K_{BC} \mathcal{K})}{\mathcal{K}^3} &= \epsilon \frac{3 \nabla_a (-\bar{\nabla}^2 \delta r - \delta r (D-2))}{(D-3)(D-2)}
\end{aligned}$$

(the equations above are accurate only to linear order in ϵ and all covariant derivatives are taken with respect to (104)). The linearized membrane equation is given by

$$
\begin{aligned}
&\left[\left(1+\frac{\overline{\nabla}^2}{D-2}\right)\delta u_a+\nabla_a\left(1+\frac{\overline{\nabla}^2}{D-2}\right)\delta r-\partial_t\nabla_a\delta r\left(1-\frac{1}{D-2}\right)-\partial_t\delta u_a\right]\\
&+\left[\frac{\nabla_a\partial_t\delta r-\delta u_a}{D-2}+\frac{\overline{\nabla}^2\overline{\nabla}^2\delta u_a+\overline{\nabla}^2\nabla_a\partial_t\delta r}{(D-2)^3}+2\frac{-\overline{\nabla}^2\delta u_a+\nabla_a\partial_t\delta r}{(D-3)(D-2)}+\frac{\nabla_a\overline{\nabla}^2(\overline{\nabla}^2\delta r+(D-2)\delta r)}{(D-2)^3}\right.\\
&\left.+3\frac{\nabla_a(-\overline{\nabla}^2\delta r-(D-2)\delta r)}{(D-3)(D-2)}+3\frac{\partial_t\delta u_a}{(D-3)}+3\frac{\partial_t\nabla_a\delta r-\delta u_a}{(D-3)}\right]=0.
\end{aligned}
\tag{110}
$$

((110) generalizes equation (5.9) of [12]). Substituting (107) into (110) we find the generalized version of of (5.15) of [12],

$$
\begin{aligned}
&\left(\frac{\nabla^2}{D-2}+1-\partial_t+\frac{\overline{\nabla}^2\overline{\nabla}^2}{(D-2)^3}-\frac{2(\nabla^2)}{(D-2)^2}+\frac{3\partial_t}{(D-3)}-\frac{3}{(D-3)}\right)\delta v_a=\\
&-\left(\frac{\partial_t\nabla_a}{D-2}+\frac{\nabla_a\nabla^2}{D-2}+\nabla_a-\nabla_a\partial_t+\frac{2\nabla_a\partial_t}{(D-2)^2}-\frac{\nabla_a\overline{\nabla}^2(\nabla^2+(D-2))}{(D-2)^3}\right.\\
&\left.-\frac{9\nabla_a((D-2)^2-(D-2)(9\nabla^2-\partial_t^2))}{3(D-2)^3}+\frac{3\partial_t\nabla_a}{(D-3)}\right)\delta r\\
&-\left(\frac{\nabla^2}{D-2}+1-\partial_t+\frac{\overline{\nabla}^2\overline{\nabla}^2}{(D-2)^3}-\frac{2(\nabla^2)}{(D-2)^2}+\frac{3\partial_t}{(D-3)}-\frac{3}{(D-3)}\right)\nabla_a\Phi
\end{aligned}
\tag{111}
$$

3.5.3 Scalar quasinormal modes

Using (106) and (109) we take the divergence of (111) to obtain

$$
\begin{aligned}
&-(\overline{\nabla}^2+D-3)\partial_t\delta r+\frac{\partial_t\nabla^2\delta r}{D-2}+\frac{\nabla^2\bar{\nabla}^2\delta r}{(D-2)}+\nabla^2\delta r-\partial_t\nabla^2\delta r-(D-2)\partial_t\delta r+(D-2)\partial_t^2\delta r\\
&+\frac{\nabla^2\partial_t\delta r+(D-2)\partial_t\delta r}{D-2}-\frac{(\overline{\nabla}^2+D-3)^2(D-2)\partial_t\delta r+(\overline{\nabla}^2+D-3)\nabla^2\partial_t\delta r}{(D-2)^3}\\
&+2\frac{(\overline{\nabla}^2+D-3)(D-2)\partial_t\delta r+\nabla^2\partial_t\delta r}{(D-2)^2}+\frac{\nabla^2\bar{\nabla}^2(\bar{\nabla}^2\delta r+\delta r(D-2))}{(D-2)^3}\\
&-\frac{\nabla^2(3\nabla^2\delta r-\partial_t^2\delta r+3\delta r(D-2))}{(D-2)^2}-3\frac{D-2}{(D-3)}\partial_t^2\delta r+\frac{3}{(D-3)}(\partial_t\nabla^2\delta r+(D-2)\partial_t\delta r)=0
\end{aligned}
\tag{112}
$$

As in [12] we expand

$$\delta r=\sum_{l,m}a_{lm}Y_{lm}e^{-i\omega_l^r t} \quad . \tag{113}$$

where the spherical harmonics Y_{lm} obey

$$-\nabla^2_{S^{D-2}}Y_{lm}=l(D+l-3)Y_{lm}. \tag{114}$$

Inserting (113) into (112) we obtain

$$\omega_l^r=\pm\sqrt{l-1}-i(l-1)+\frac{1}{D}\left(\pm\sqrt{l-1}\left(\frac{3l}{2}-2\right)-i(l-1)(l-2)\right) \tag{115}$$

The result (115) is in perfect agreement with the result obtained by EST in Equations (5.30) and (5.31) of [10].

As explained in [12], the modes with $l=0$ and $l=1$ are special. At $l=0$ the formula (115) yields $\omega=0,2i-\frac{4i}{D}$. The second solution is, however, spurious (see [12]). The first solution is the zero mode corresponding to rescaling the black hole; this is an exact zero mode at all orders in $1/D$.

At $l = 1$ (115) yields the frequencies $\omega = 0, 0$. As explained in [12] these two modes correspond to translations and boosts of the membrane.

3.5.4 Vector quasinormal modes

We expand the velocity fluctuations in a basis of vector spherical harmonic

$$\delta v_a = \sum_{l,m} b_{lm} Y_a^{lm} e^{-i\omega_l^v t} \tag{116}$$

Where, $l = 1, 2, 3, ...$ The vector spherical harmonics satisfy the property

$$\nabla^2 V = -[(D + l - 3)l - 1]V \tag{117}$$

Plugging (116) into (111), using (117) and equating the coefficients of independent vector spherical harmonics (see [12] for more discussion) we obtain

$$\omega_l^v = -i(l-1) - \frac{i}{D}(l-1)^2. \tag{118}$$

(118) is in perfect agreement with the formula (5.22) of [10] derived earlier by EST by purely gravitational analysis. Note that the mode with $l = 1$ has vanishing frequency. As explained in [12] $l = 1$ is the exact zero mode corresponding to setting the black hole spinning.

3.6 Discussion

In this chapter we have worked out the duality between the dynamics of black holes in a large number of dimensions and the motion of a non gravitational membrane in flat space to second subleading order in $1/D$. Our work generalizes the analysis of [4, 12]. The concrete new results of this chapter are

- The second order corrected membrane equations of motion listed in (27).
- The formula (28) for the divergence of the velocity field (which vanished at first order).
- The explicit form of the second order corrected metric dual to any given membrane motion (see subsection 3.4.2

In addition to obtaining the new results listed above we have also achieved an improved understanding of the structure of the perturbative expansion in $1/D$. We have demonstrated that the perturbative programme, implemented to first nontrivial order in [4, 12], can systematically be extended to every order in the $1/D$ expansion. In particular we have shown that the algebraically nontrivial 'integrability' properties that allowed for the existence of a first order solution in [4, 12] are actually automatic at all orders as as a consequence of the well known equation (50).

We have also explained that the membrane equations may directly be obtained by evaluating the Einstein constraint equation on the event horizon. In particular the membrane equations at $(n+1)^{th}$ order in $1/D$ are obtained by evaluating the constraint equations on n^{th} order metric, without needing to solve for the $(n+1)^{th}$ order metric. We have also explained that the assumption of $SO(D-p-2)$ isometry, made in [12], is not necessary; the membrane equations can be derived under much more general conditions

The fact our membrane equations arise from the Einstein constraint equations at the event horizon is strongly reminiscent of the 'traditional' membrane paradigm of black hole physics. It would be very interesting to better understand the relationship between the the large D membrane and the traditional membrane paradigm. [73–75].

As black holes are thermodynamical objects, the black hole membrane studied in [4, 12] and this chapter should carry an entropy current. At leading order in $1/D$ it turns out (see [76]) that this entropy current is given simply by a constant times u^M. The divergence of this entropy current is thus proportional to $\nabla.u$. It follows that the RHS of the formula (28) gives an expression for the rate of entropy production on the membrane. It would be interesting to further investigate this observation and its consequences.

On a related note, it would be interesting to derive the most general stationary solution of the second order corrected equations of motion derived in this chapter and compare our results with those of [14].

In this chapter we have focused our attention on black holes propagating in an otherwise perfectly flat spacetime. It would be interesting to generalize our study to the motion of black holes propagating in any vacuum solution of Einstein's equations, e.g. a gravity wave. Such a generalization would allow us, for instance, to study the absorption of gravity waves by black holes at large D. At first order in the derivative expansion we expect the generalized effective membrane equation to be given simply by covariantizing first order flat space equations of motion. At second order, however, the equations of motion could receive genuinely new contributions from the background Riemann tensor of the space in which the black hole propagates [30]. It would be interesting to work this out in detail.

Finally, it would be interesting to put the membrane equations derived in this chapter to practical use to allow us to learn new things about black holes. One possible direction would be to test out how well the large D expansion does in astrophysical contexts (i.e. when $D = 4$). Another direction would be to use the formalism developed herein to address interesting unanswered structural questions about gravity, e.g. questions about the second law of thermodynamics in higher derivative gravity. We leave such investigations for the future.

[30]Something similar happens in the study of forced fluids in the fluid gravity correspondence [77]

3.7 Appendices for Chapter 2

3.7.1 Method of calculation

In this Appendix we outline the method we have employed to obtain the results quoted in tables 2, 3, 4, 5, 6, 7.

As we have mentioned in the main text, our starting point is the metric listed in (35),(36),(37),(38). In order to obtain the equations of motion listed in table 2 (see also table 3) we simply plugged this metric into the vacuum Einstein equations. Assuming these equations are already obeyed at $n-1$ order we then obtained the form of the n^{th} order equations. As emphasized in table 2, each of these equations have a 'homogeneous' contribution and a 'source' contribution. The homogeneous contribution is linear in the (as yet unknown) n^{th} order fluctuation, and takes the same form at all orders. In order to evaluate the homogeneous contribution to all equations of motion, consequently, it is sufficient to work at first order.

While the first order computation is straightforward to perform analytically in principle, in practice the computations involved are rather lengthy . In order to guard against error we employed Mathematica in our computations using the following device. Following [4, 12] we specialized to the particular case of metrics that preserve an $SO(D-p-2)$ isometry. Such special metrics effectively depend only on $p+3$ variables. For small values of p, therefore, all computations can be effectively performed on Mathematica (see [12] for a detailed explanation of how this is done). The first order computation performed in this manner yields the homogeneous part of the differential equations listed in tables 2 and 3 in a straightforward manner. Note that the homogeneous part of the equations are differential operators only in the variable R. They are 'ultra-local' on the membrane. Consequently, even though the assumption of isometry was used as a trick to facilitate the computation of the homogeneous part of the equation, the final result obtained for the structure of the equations listed in tables 2 and 3 is valid assuming only that all background quantities (e.g. $\mathcal{K}$) scale in the manner assumed in the text. In particular the homogeneous contribution to these equations are independent of p. By repeating all of our computations for $p = 2$ and $p = 3$ we have explicitly checked that this is the case.

Apart from the homogeneous pieces, the equations listed in tables 2 and 3 also have contributions from sources. Source terms are different at different orders in the computation. We obtained our explicit results for the first order sources listed in tables 4, 5 and second order sources listed in tables 6, 7 as follows. Working separately in the scalar, vector and tensor channels we first explicitly listed all possible source structures that could appear in any given equation both at first and second order in perturbation theory. The source structures that appear in our classification are the analogues of the 'geometrical' quantities listed in the LHS of Table 4 in [12]. At any given order, it follows that the sources that appear in the equations of tables 2 and 3 are linear combinations of these structures with coefficients that are as yet unknown functions of R. We then worked out the analogue of the RHS of Table 4 of [12], i.e. we explicitly evaluated each of these basis source terms in terms of 'reduced source data' - the analogue of the expressions listed in table 1 of [12].

Using our explicit computations on Mathematica we read off the coefficients of all reduced sources in all of the equations listed in table 2 and 3. We then used our reduction formulae for 'geometrical sources in terms of reduced sources' (analogue of Table 4 in [12]) to determine the coefficients of all source terms in the original geometrical basis of possible source terms. The last step (determination of geometrical sources from the known coefficients of reduced sources) is unambiguous provided the map between geometrical and reduced sources in invertible, i.e. provided there does not exist a nontrivial linear combination of geometrical sources that maps to zero when re expressed in terms of reduced sources (i.e. vanishes under the the assumption of isometry). We have verified that this condition is met at first order provided $p \geq 2$ and at second order provided that $p \geq 3$. [31].

[31] It is easy to understand the inequalities listed here. When $p = 1$, for instance, a potential source term proportional to the shear of the velocity field trivially vanishes just because fluids in one spatial dimension

This is the reason we performed our computations at $p = 3$. [32]

3.7.2 Sources at second order

In this Appendix we present an explicit listing of all the sources that appear in the second order computation.By explicit computation we find that the sources listed in tables 1 and 2) ' are given at second order by the expressions we list in table 6 below

do not have a transverse direction in which to shear.

[32] We also performed all computations in $p = 2$ and verified that we obtained the same results for all sources from this computation - except in the case of a single second order source that vanished at $p = 2$ but not at $p = 3$. The coefficient of this term was left undetermined at $p = 2$ but we determined at $p = 3$.

Table 6: Sources of R_{MN} equations at 2nd order

Scalar sector
$\mathcal{S}^{S_1}(R) = e^{-R}(1-R)\left((u\cdot K - u\cdot\nabla u)\cdot\mathcal{P}\cdot(u\cdot K - u\cdot\nabla u)\right)$
$\mathcal{S}^{S_2}(R) = -\frac{1}{2}e^{-R}(R-2)\left(K_{MN}K_{PQ}P^{NP}P^{MQ} - \frac{\mathcal{K}^2}{D-3}\right) + \frac{1}{2}e^{-R}(R+2)\left(\nabla_M u_N \nabla_P u_Q P^{NP}P^{MQ}\right)$
$-\frac{e^{-R}}{2}\left(\nabla_{[M}u_{N]}\nabla_{[P}u_{Q]}P^{NP}P^{MQ}\right) - e^{-R}R\left(\nabla_M u_N K_{PQ}P^{NP}P^{MQ}\right)$
$+\frac{1}{\mathcal{K}}\frac{e^{-R}(R-2)R}{4}\nabla^A\left(\frac{D-3}{\mathcal{K}}\left(\frac{D-3}{\mathcal{K}^3}(\nabla\nabla^2\mathcal{K} - \nabla^2\nabla^2 u) + 8(u\cdot K - u\cdot\nabla u) + u\cdot K + \frac{\nabla^2 u}{\mathcal{K}}\right)_B \mathcal{P}^B_A\right)$
$-\frac{e^{-R}(R-2)R}{4}\frac{\nabla^2\nabla^2\mathcal{K}}{\mathcal{K}^3} + \frac{1}{4}e^{-2R}\left(e^R(R^2+2R-4) - 2(R-2)R\right)(u.\nabla u_M)(u.\nabla u_N)\mathcal{P}^{MN}$
$+\frac{1}{2}e^{-2R}\left(2e^R(R-1) - (R-2)R\right)(u^A K_{AM})(u^B K_{BN})\mathcal{P}^{MN} + e^{-2R}(R-2)R(u.\nabla u_M)(u^C K_{CN})P^{MN}$
$+\frac{1}{4}e^{-R}(R-2)R\left(\frac{\nabla^2 u_M}{\mathcal{K}}\right)\left(\frac{\nabla^2 u_N}{\mathcal{K}}\right) CP^{MN} - \frac{e^{-R}(R-2)R}{2}\left(\frac{\nabla^2 u_M}{\mathcal{K}}\right)(u.\nabla u_N)\mathcal{P}^{MN}$
$+\frac{1}{4}e^{-R}R\left(2R^2-3R-6\right)\frac{(u.\nabla\mathcal{K})^2}{\mathcal{K}^2} - \frac{e^{-R}\left(R^3-14R^2+20R+4\right)}{4}u.K.u\frac{\mathcal{K}}{(D-3)}$
$+\frac{e^{-R}\left(3R^3-38R^2+62R-4\right)}{4}\frac{\mathcal{K}}{(D-3)}\frac{u.\nabla\mathcal{K}}{\mathcal{K}} - \frac{1}{4}e^{-R}R\left(R^2-6\right)u.K.u\frac{u.\nabla\mathcal{K}}{\mathcal{K}} + e^{-R}(R-1)\frac{\mathcal{K}^2}{(D-3)^2}$
$-\frac{1}{4}e^{-R}\left(\nabla_{(A}u_{B)}\nabla_{(C}u_{D)}\mathcal{P}^{BC}\mathcal{P}^{AD}\right)$
$\mathcal{S}^{S_3}(R) = \mathcal{V}^{S_1}(R) - (1-e^{-R})\mathcal{S}^{S_2}(R)$
$\mathcal{S}^{S_4}(R) = (1-e^{-R})\mathcal{S}^{S_1}(R) - 2\mathcal{V}^{S_2}(R)$
Vector sector
$\mathcal{S}^{V_1}_M(R) = \frac{1}{(1-e^{-R})}\left(\mathcal{V}^V_L(R) - \mathcal{S}^{V_2}_L(R)\right)$
$\mathcal{S}^{V_2}_A(R) = \frac{\mathcal{K}^2}{2(D-3)^2}\Bigg[-e^{-2R}\left(e^R-1\right)(R^2-2)\frac{3}{2}\frac{D-3}{\mathcal{K}}\left(1 + 2\frac{u\cdot\nabla\mathcal{K}\ (D-3)}{\mathcal{K}^2} - \frac{u\cdot K\cdot u\ (D-3)}{\mathcal{K}}\right)(u\cdot\nabla u - u\cdot K)_B$
$-e^{-2R}\left(e^R-1\right)(R-1)\frac{(D-3)}{\mathcal{K}}\left(\frac{(D-3)}{\mathcal{K}^3}(\nabla\nabla^2\mathcal{K} - \nabla^2\nabla^2 u) + 8(u\cdot K - u\cdot\nabla u) + u\cdot K + \frac{\nabla^2 u}{\mathcal{K}}\right)_B$
$+Re^{-R}\left(-2\frac{(D-3)^2}{\mathcal{K}^2}\left(\frac{\nabla_M\mathcal{K}}{\mathcal{K}} - u^D K_{DM}\right)P^{MN}(\nabla_N u_B - K_{NB}) + \frac{(D-3)}{\mathcal{K}}\left(u^C K_{CB} - \frac{\nabla^2 u_B}{\mathcal{K}}\right)\right)\Bigg]\mathcal{P}^B_A$
$-\frac{e^{-R}}{2}\frac{\mathcal{K}}{(D-3)}\Bigg[-\mathcal{E}_M + D\frac{\nabla^2\nabla^2 u_M}{\mathcal{K}^3} - D\frac{\nabla_M(\nabla^2\mathcal{K})}{\mathcal{K}^3} + 3D\frac{(u\cdot K\cdot u)(u\cdot\nabla u_M)}{\mathcal{K}} - 3D\frac{(u\cdot K\cdot u)(u^A K_{AM})}{\mathcal{K}}$
$-6D\frac{(u\cdot\nabla\mathcal{K})(u\cdot\nabla u)}{\mathcal{K}^2} + 6D\frac{(u\cdot\nabla\mathcal{K})(u^A K_{AM})}{\mathcal{K}^2} + 3u\cdot\nabla u - 3u^A K_{AM}\Bigg]\mathcal{P}^M_L$
Tensor sector
$\mathcal{S}^T_{LP}(R) = \Bigg[e^{-R}\frac{\mathcal{K}}{(D-3)}\left((K_{MN} - \nabla_{(M}u_{N)}) - \frac{\mathcal{P}_{MN}}{D}(K_{AB} - \nabla_{(A}u_{B)})\mathcal{P}^{AB}\right)$
$-e^{-R}\left((K_{MC} - \nabla_C u_M)\mathcal{P}^{CD}(K_{DN} - \nabla_D u_N) - \frac{\mathcal{P}_{MN}}{D}(K_{AC} - \nabla_C u_A)\mathcal{P}^{CD}(K_{DB} - \nabla_D u_B)\mathcal{P}^{AB}\right)$
$-\frac{1}{2}e^{-2R}\left(R^2 - 4R + 2e^R(R-1) + 2\right)\Bigg((u_C K^C_M - u.\nabla u_M)(u_C K^C_N - u.\nabla u_N)$
$-\frac{\mathcal{P}_{MN}}{D}(u_C K^C_A - u.\nabla u_A)(u_C K^C_B - u.\nabla u_B)\mathcal{P}^{AB}\Bigg)\Bigg]\mathcal{P}^M_L\mathcal{P}^N_P$

Table 7: Sources to constraint equations at 2nd order

Vector constraint source
$\mathcal{V}^V_L(R) = \frac{1}{(D-3)}\nabla^P\Big[e^{-R}R\frac{D}{\mathcal{K}}\Big((K_{MC}-\nabla_C u_M)\mathcal{P}^{CD}(K_{DN}-\nabla_D u_N) - \frac{\mathcal{P}_{MN}}{(D-3)}(K_{AC}-\nabla_C u_A)\mathcal{P}^{CD}(K_{DB}-\nabla_D u_B)\mathcal{P}^{AB}\Big)\mathcal{P}^M_L\mathcal{P}^N_P - Re^{-R}\left((K_{MN}-\nabla_{(M}u_{N)}) - \frac{\mathcal{P}_{MN}}{(D-3)}(K_{AB}-\nabla_{(A}u_{B)})\mathcal{P}^{AB}\right)\mathcal{P}^M_L\mathcal{P}^N_P + (e^{-2R}\left(e^R-1\right)(R-2)R)\frac{(D-3)}{2\mathcal{K}}\Big((u_C K^C_M - u.\nabla u_M)(u_C K^C_N - u.\nabla u_N) - \frac{\mathcal{P}_{MN}}{(D-3)}(u_C K^C_A - u.\nabla u_A)(u_C K^C_B - u.\nabla u_B)\mathcal{P}^{AB}\Big)\mathcal{P}^M_L\mathcal{P}^N_P\Big] - \frac{e^{-R}}{2}\frac{\mathcal{K}}{(D-3)}\Big[-\mathcal{E}_M + D\frac{\nabla^2\nabla^2 u_M}{\mathcal{K}^3} - D\frac{\nabla_M(\nabla^2\mathcal{K})}{\mathcal{K}^3} + 3D\frac{(u\cdot K\cdot u)(u\cdot\nabla u_M)}{\mathcal{K}} - 3D\frac{(u\cdot K\cdot u)(u^A K_{AM})}{\mathcal{K}} - 6D\frac{(u\cdot\nabla\mathcal{K})(u\cdot\nabla u)}{\mathcal{K}^2} + 6D\frac{(u\cdot\nabla\mathcal{K})(u^A K_{AM})}{\mathcal{K}^2} + 3u\cdot\nabla u - 3u^A K_{AM}\Big]\mathcal{P}^M_L$
Scalar constraint 1 source
$\mathcal{V}^{S_1}(R) = \frac{\left(e^{-2R}R\left(e^R\left(R^2-6\right)+3(R+2)\right)\right)\mathcal{K}}{6(D-3)}\nabla^M\left(\frac{3}{2}\frac{(D-3)}{\mathcal{K}}\left(1+2\frac{u\cdot\nabla\mathcal{K}\ (D-3)}{\mathcal{K}^2} - \frac{u\cdot K\cdot u\ (D-3)}{\mathcal{K}}\right)(u\cdot\nabla u - u\cdot K)_B\,\mathcal{P}^B_M\right) + \frac{\left(e^{-2R}\left(e^R(R-2)+2\right)R\right)}{4\mathcal{K}}\nabla^M\left(\frac{(D-3)}{\mathcal{K}}\left(\frac{(D-3)}{\mathcal{K}^3}(\nabla\nabla^2\mathcal{K}-\nabla^2\nabla^2 u) + 8(u\cdot K - u\cdot\nabla u) + u\cdot K + \frac{\nabla^2 u}{\mathcal{K}}\right)_B\mathcal{P}^B_M\right) + \frac{\left(-e^{-R}R^2\right)}{4\mathcal{K}}\nabla^M\left(\left(-2\frac{(D-3)^2}{\mathcal{K}^2}\left(\frac{\nabla_M\mathcal{K}}{\mathcal{K}} - u^D K_{DM}\right)\mathcal{P}^{MN}(\nabla_N u_B - K_{NB}) + \frac{(D-3)}{\mathcal{K}}\left(u^C K_{CB} - \frac{\nabla^2 u_B}{\mathcal{K}}\right)\right)\mathcal{P}^B_M\right) - \frac{1}{4}e^{-R}\left(\nabla_{(A}u_{B)}\nabla_{(C}u_{D)}\mathcal{P}^{BC}\mathcal{P}^{AD}\right) + \frac{Re^{-R}}{2}\nabla^M\mathcal{E}_M$
Scalar constraint 2 source
$\mathcal{V}^{S_2}(R) = -\frac{1}{2}e^{-R}(R-1)\left(K_{MN}K_{PQ}P^{NP}P^{MQ} - \frac{\mathcal{K}^2}{D-3}\right) + \frac{1}{2}e^{-R}(3+R)\left(\nabla_M u_N\nabla_P u_Q P^{NP}P^{MQ}\right) + \frac{1}{2}\left(-e^{-R}\right)\left(\nabla_{[M}u_{N]}\nabla_{[P}u_{Q]}P^{NP}P^{MQ}\right) - e^{-R}(1+R)\left(\nabla_M u_N K_{PQ}P^{NP}P^{MQ}\right) + \frac{1}{\mathcal{K}}\frac{\left(e^{-R}(R+2)R\right)}{4}\nabla^M\left[\frac{(D-3)}{\mathcal{K}}\left(\frac{(D-3)}{\mathcal{K}^3}(\nabla\nabla^2\mathcal{K}-\nabla^2\nabla^2u) + 8(u\cdot K - u\cdot\nabla u) + u\cdot K + \frac{\nabla^2 u}{\mathcal{K}}\right)_B\mathcal{P}^B_M\right] - \frac{\left(e^{-R}R^2\right)}{4}\frac{\nabla^2\nabla^2\mathcal{K}}{\mathcal{K}^3} + \frac{1}{4}\left(e^{-2R}R\left(2+R(e^R-1)\right)\right)(u.\nabla u_M)(u.\nabla u_N)\mathcal{P}^{MN} - \frac{1}{4}\left(e^{-2R}R(R-2)\right)(u^A K_{AM})(u^D K_{DN})\mathcal{P}^{MN} + \frac{1}{4}e^{-R}R^2\frac{\nabla^2 u_M}{\mathcal{K}}\frac{\nabla^2 u_N}{\mathcal{K}}\mathcal{P}^{MN} - \frac{\left(e^{-R}R^2\right)}{2}\frac{\nabla^2 u_M}{\mathcal{K}}u.\nabla u_N\mathcal{P}^{MN} + \frac{1}{2}\left(e^{-2R}R(-2+4e^R+R)\right)(u.\nabla u_M)(u^C K_{CN})\mathcal{P}^{MN} + \frac{\left(e^{-R}R\left(2R^2-R-12\right)\right)}{4}\frac{(u.\nabla\mathcal{K})^2}{\mathcal{K}^2} - \frac{\left(e^{-R}\left(R^3-14R^2-8R+2\right)\right)}{4}u.K.u\frac{\mathcal{K}}{(D-3)} + \frac{\left(e^{-R}R\left(3R^2-32R-2\right)\right)}{4}\frac{u.\nabla\mathcal{K}}{(D-3)} - \frac{\left(e^{-R}R\left(R^2-2R-18\right)\right)}{4}\frac{u.\nabla\mathcal{K}}{\mathcal{K}}u.K.u + e^{-R}R\frac{\mathcal{K}^2}{(D-3)^2} - \frac{1}{4}e^{-R}\left(\nabla_{(A}u_{B)}\nabla_{(C}u_{D)}\mathcal{P}^{BC}\mathcal{P}^{AD}\right)$

Part B

4 Chapter 3: Scattering in $\mathcal{N} = 1$ Susy matter Chern Simons Theory

(*This chapter is based on the published paper written in collaboration with K. Inbasekar, S. Jain, S. Minwalla, V. Umesh, S. Yokoyama , " Unitarity, crossing symmetry and duality in the scattering of* $\mathcal{N} = 1$ *susy matter Chern-Simons theories", arXiv:1505.06571, JHEP 1510 (2015) 176*)

4.1 Introduction

In this chapter we redo the S matrix computations of [61] in a different class of Chern-Simons theory coupled to fundamental matter and check that the conjectures of [61] - unmodified in all details - indeed continue to yield sensible results (i.e. results that pass all necessary consistency checks) in the new system. We now describe the system we study and the nature of our results in much more detail.

The theories we study are the most general power counting renormalizable $\mathcal{N} = 1$ $U(N)$ gauge theories coupled to a single fundamental multiplet (see (119) below). In order to study scattering in these theories we imitate the strategy of [61]. The authors of [61] worked in lightcone gauge; in this chapter we work in a supersymmetric generalization of lightcone gauge (4.3.1). In this gauge (which preserves manifest offshell supersymmetry) the gauge self interaction term vanishes. This fact - together with planarity at large N - allows us to find a manifestly supersymmetric Schwinger-Dyson equation for the exact propagator of the matter supermultiplet. This equation turns out to be easy to solve; the solution gives simple exact expression for the all orders propagator for the matter supermultiplet (see subsection §4.3.3).

With the exact propagator in hand, we then proceed to write down an exact Schwinger-Dyson equation for the offshell four point function of the matter supermultiplet. The resultant integral equation is quite complicated; as in [61] we have been able to solve this equation only in a restricted kinematic range ($q_{\pm} = 0$ in the notation of fig 4). In this kinematic regime, however, we have been able to find a completely explicit (if somewhat complicated) solution of the resulting equation(see subsection §4.3.5-§4.3.6).

In order to evaluate the S matrices we then proceed to take the onshell limit of our explicit offshell results. As explained in detail in [61], the 3 vector q^{μ} has the interpretation of momentum transfer for both channels of particle- particle scattering and also for particle antiparticle scattering in the adjoint channel. In these channels the fact that we know the offshell four point amplitudes only when $q_{\pm} = 0$ forces us to study scattering in a particular Lorentz frame; any frame in which momentum transfer happens along the spatial q^3 direction. In any such frame we obtain explicit results for all 2×2 scattering matrices in these three channels. The results are then covariantized to formulae that apply to any frame. Following this method we have obtained explicit results for the S matrices in these three channels. Our results are presented in detail in subsections §4.3.7 - §4.3.11. As we explain in detail below, our explicit results have exactly the same interplay with the proposed strong weak coupling self duality of the set of $\mathcal{N} = 1$ Chern-Simons fundamental matter theories (see subsection 4.2.2) as that described in [61]; duality maps particle - particle S matrices in the symmetric and antisymmetric channels to each other, while it maps the particle - antiparticle S matrix in the adjoint channel to itself.

As in [61] our explicit offshell results do not permit a direct computation of the S matrix for particle - antiparticle scattering in the singlet channel. This is because the three vector q^{μ} is the center of mass momentum for this scattering process and so must be timelike, which is impossible if $q^{\pm} = 0$. Our explicit results for the S matrices in the other channels, together with the conjectured modified crossing symmetry rules of [61], however, yield a conjectured formula for the S matrix in this channel.

In section 4.4 we subject our conjecture for the particle - antiparticle S matrix to a very stringent consistency check; we verify that it obeys the nonlinear unitarity equation (178)[33]. From the purely algebraic point of view the fact that our complicated S matrices are unitary appears to be a minor miracle- one that certainly fails very badly for the S matrix obtained using the usual rules of crossing symmetry. We view this result as very strong evidence for the correctness of our formula, and indirectly for the modified crossing symmetry rules of [61].

Our proposed formula for particle - antiparticle scattering in the singlet channel has an interesting analytic structure. As a function of s (at fixed t) our S matrix has the expected two particle cut starting at $s = 4m^2$. In a certain range of interaction parameters it also has poles at smaller (though always positive) values of s. These poles represent bound states; when they exist these bound states must be absolutely stable even at large but finite N, simply because they are the lightest singlet sector states (baring the vacuum) in the theory; recall that our theory has no gluons. Quite remarkably it turns out that the mass of this bound state supermultiplet vanishes at $w = w_c(\lambda)$ where w is the superpotential interaction parameter of our theory (see (119)) and $w_c(\lambda)$ is the simple function listed in (331). In other words a one parameter tuning of the superpotential is sufficient to produce massless bound states in a theory of massive 'quarks'; we find this result quite remarkable. Scaling w to w_c permits a parametric separation between the mass of this bound state and all other states in the theory. In this limit there must exist a decoupled QFT description of the dynamics of these light states even at large but finite N; it seems likely to us that this dynamics is governed by a $\mathcal{N} = 1$ Wilson-Fisher fixed point. We leave the detailed investigation of this suggestion to future work.

The S matrices computed and conjectured in this chapter turn out to simplify dramatically at $w = 1$, at which point the system (119) turns out to enjoy an enhanced

[33] At large N this equation may be shown to close on 2×2 scattering.

$\mathcal{N} = 2$ supersymmetry. In the three non-anyonic channels our S matrix reduces simply to its tree level counterpart at $w = 1$. It follows, in other words, that the S matrix is not renormalized as a function of λ in these channels. This result illustrates the conflict between naive crossing symmetry and unitarity in a simple setting. Naive crossing symmetry would yield a singlet channel S matrix that is also tree level exact. However tree level S matrices by themselves can never obey the unitarity equations (they do not have the singularities needed to satisfy the Cutkosky's rules obtained by gluing them together). The resolution to this paradox appears simply to be that the naive crossing symmetry rules are wrong in the current context. Applying the conjectured crossing symmetry rules of [61] we find a singlet channel S matrix that continues to be very simple, but is not tree level exact, and in fact satisfies the unitarity equation.

In this chapter we have limited our attention to the study of $\mathcal{N} = 1$ theories with a single fundamental matter multiplet. Were we to extend our analysis to theories with two multiplets we would encounter, in particular, the $\mathcal{N} = 3$ theory. Extending to the study of a theory with four multiplets (and allowing for the the gauging of a $U(1)$ global symmetry) would allow us to study the $\mathcal{N} = 6$ $U(N) \times U(1)$ ABJ theory. We believe it would not be difficult to adapt the methods of this chapter to find explicit all orders results for the S matrices of all these theories at leading order in large N. We expect to find scattering matrices that are unitary precisely because they transform under crossing symmetry in the unusual manner conjectured in [61]. It would be particularly interesting to find explicit results for the $\mathcal{N} = 6$ theory in order to facilitate a detailed comparison with the perturbative computations of S matrices in ABJM theory [39–45], which appear to report results that are crossing symmetric but (at least naively) conflict with unitarity.

4.2 Review of Background Material

4.2.1 Renormalizable $\mathcal{N}=1$ theories with a single fundamental multiplet

In this chapter we study 2×2 scattering in the most general renormalizable $\mathcal{N}=1$ supersymmetric $U(N)$ Chern-Simons theory coupled to a single fundamental matter multiplet. Our theory is defined in superspace by the Euclidean action [78, 79]

$$\begin{aligned}\mathcal{S}_{\mathcal{N}=1} = -\int d^3x d^2\theta \bigg[\frac{\kappa}{4\pi} Tr\Big(-\frac{1}{4} D^\alpha \Gamma^\beta D_\beta \Gamma_\alpha + \frac{i}{6} D^\alpha \Gamma^\beta \{\Gamma_\alpha, \Gamma_\beta\} + \frac{1}{24}\{\Gamma^\alpha, \Gamma^\beta\}\{\Gamma_\alpha, \Gamma_\beta\}\Big) \\ -\frac{1}{2}(D^\alpha \bar{\Phi} + i\bar{\Phi}\Gamma^\alpha)(D_\alpha \Phi - i\Gamma_\alpha \Phi) + m_0 \bar{\Phi}\Phi + \frac{\pi w}{\kappa}(\overline{\Phi}\Phi)^2\bigg] \ .\end{aligned} \tag{119}$$

The integration in (119) is over the three Euclidean spatial coordinates and the two anticommuting spinorial coordinates θ^α (the $SO(3)$ spinorial indices α range over two allowed values $\pm$). The fields Φ and Γ^α in (119) are, respectively, complex and real superfields [34]. They may be expanded in components as

$$\begin{aligned}&\Phi = \phi + \theta\psi - \theta^2 F \ , \\ &\bar{\Phi} = \bar{\phi} + \theta\bar{\psi} - \theta^2 \bar{F} \ , \\ &\Gamma^\alpha = \chi^\alpha - \theta^\alpha B + i\theta^\beta A_\beta{}^\alpha - \theta^2(2\lambda^\alpha - i\partial^{\alpha\beta}\chi_\beta) \ ,\end{aligned} \tag{120}$$

where Γ_α is an $N\times N$ matrix in color space, while Φ is an N dimensional column.

The superderivative D_α in (119) is defined by

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\theta^\beta \partial_{\alpha\beta} \ , D^\alpha = C^{\alpha\beta} D_\beta \ , \tag{121}$$

where $C^{\alpha\beta}$ is the charge conjugation matrix. See Appendix §4.7.1 for notations and con-

[34] See Appendix §4.7.1 for our conventions for superspace

ventions.

The theories (119) are characterized by one dimensionless coupling constant w, a dimensionful mass scale m_0, and two integers N (the rank of the gauge group $U(N)$) and κ, the level of the Chern-Simons theory. [35] In the large N limit of interest to us in this chapter, the 't Hooft coupling $\lambda = \frac{N}{\kappa}$ is a second effectively continuous dimensionless parameter.

The action (119) enjoys invariance under the super gauge transformations

$$\begin{aligned} \delta\Phi &= iK\Phi \ , \\ \delta\bar{\Phi} &= -i\bar{\Phi}K \ , \\ \delta\Gamma_\alpha &= D_\alpha K + [\Gamma_\alpha, K] \ , \end{aligned} \tag{122}$$

where K is a real superfield (it is an $N \times N$ matrix in color space).

(119) is manifestly invariant under the two supersymmetry transformations generated by the supercharges Q_α

$$Q_\alpha = i(\frac{\partial}{\partial\theta^\alpha} - i\theta^\beta\partial_{\beta\alpha}) \tag{123}$$

that act on Φ and Γ_α as

$$\begin{aligned} \delta_\alpha\Phi &= Q_\alpha\Phi \ , \\ \delta_\alpha\Gamma_\beta &= Q_\alpha\Gamma_\beta \ . \end{aligned} \tag{124}$$

[35]The precise definition of κ is defined as follows. Let k denote the level of the WZW theory related to Chern-Simons theory after all fermions have been integrated out. κ is the related to k by $\kappa = k + \text{sgn}(k)N$

The differential operators Q_α and D_α obey the algebra

$$\begin{aligned}
\{Q_\alpha, Q_\beta\} &= 2i\partial_{\alpha\beta} \ , \\
\{D_\alpha, D_\beta\} &= 2i\partial_{\alpha\beta} \ , \\
\{Q_\alpha, D_\beta\} &= 0 \ .
\end{aligned} \tag{125}$$

At the special value $w = 1$, the action (119) actually has enhanced supersymmetry; it is $\mathcal{N} = 2$ (four supercharges) supersymmetric. [36]

The physical content of the theory (119) is most transparent when the Lagrangian is expanded out in component fields in the so called Wess-Zumino gauge - defined by the requirement

$$B = 0 \ , \chi = 0 \ . \tag{126}$$

Imposing this gauge, integrating over θ and eliminating auxiliary fields we obtain

[36] This may be confirmed, for instance, by checking that (129) at $w = 1$ is identical to the $\mathcal{N} = 2$ superspace Chern-Simons action coupled to a single chiral multiplet in the fundamental representation with no superpotential (see Eq 2.3 of [80]) expanded in components in Wess-Zumino gauge.

the component field action [37]

$$\mathcal{S}_{\mathcal{N}=1} = \int d^3x \bigg(\frac{i\kappa}{4\pi}\epsilon^{\mu\nu\rho}\mathrm{Tr}\big(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\big) + \mathcal{D}^\mu\bar{\phi}\mathcal{D}_\mu\phi + m_0^2\bar{\phi}\phi - \bar{\psi}(i\not{\mathcal{D}} + m_0)\psi$$
$$+ \frac{4\pi w m_0}{\kappa}(\bar{\phi}\phi)^2 + \frac{4\pi^2 w^2}{\kappa^2}(\bar{\phi}\phi)^3 - \frac{2\pi}{\kappa}(1+w)(\bar{\phi}\phi)(\bar{\psi}\psi) - \frac{2\pi w}{\kappa}(\bar{\psi}\phi)(\bar{\phi}\psi)$$
$$+ \frac{\pi}{\kappa}(1-w)\big((\bar{\phi}\psi)(\bar{\phi}\psi) + (\bar{\psi}\phi)(\bar{\psi}\phi)\big)\bigg) \tag{129}$$

displaying that (119) is the action for one fundamental boson and one fundamental fermion coupled to a Chern-Simons gauge field. Supersymmetry sets the masses of the bosonic and fermionic fields equal, and imposes several relations between a priori independent coupling constants.

4.2.2 Conjectured Duality

It has been conjectured [58] that the theory (119) enjoys a strong weak coupling self duality. The theory (119) with ‘t Hooft coupling λ and self coupling parameter w is conjectured to be dual to the theory with ‘t Hooft coupling λ' and self coupling w' where

$$\lambda' = \lambda - \mathrm{Sgn}(\lambda)\ ,\ w' = \frac{3-w}{1+w} \quad m_0' = \frac{-2m_0}{1+w}\ . \tag{130}$$

[37] Our trace conventions are

$$Tr(T^aT^b) = \frac{1}{2}\delta^{ab}\ ,\ \sum_a (T^a)_i^{\ j}(T^a)_k^{\ l} = \frac{1}{2}\delta_i^{\ l}\delta_k^{\ j}\ . \tag{127}$$

The gauge covariant derivatives in (129) are

$$\mathcal{D}^\mu\bar{\phi} = \partial^\mu\bar{\phi} + i\bar{\phi}A^\mu\ ,\ \mathcal{D}_\mu\phi = \partial_\mu\phi - iA_\mu\phi\ ,$$
$$\not{\mathcal{D}}\bar{\psi} = \gamma^\mu(\partial_\mu\bar{\psi} + i\bar{\psi}A_\mu)\ ,\ \not{\mathcal{D}}\psi = \gamma^\mu(\partial_\mu\psi - iA_\mu\psi). \tag{128}$$

As we will explain below, the pole mass for the matter multiplet in this theory is given by

$$m = \frac{2m_0}{2 + (-1+w)\lambda \ \mathrm{Sgn(m)}} \ . \tag{131}$$

It is easily verified that under duality

$$m' = -m \ . \tag{132}$$

The concrete prior evidence for this duality is the perfect matching of S^2 partition functions of the two theories. This match works provided [58]

$$\lambda m(m_0, w) \geq 0 \ , \tag{133}$$

Through this chapter we will assume that (133) is obeyed. Note that the condition (133) is preserved by duality (i.e. a theory and its conjectured dual either both obey or both violate (133)).

Note that $w = 1$ is a fixed point for the duality map (130); this was necessary on physical grounds (recall that our theory has enhanced $\mathcal{N} = 2$ supersymmetry only at $w = 1$). In the special case $w = 1$ and $m_0 = 0$, the duality conjectured in this subsection reduces to the previously studied duality [37] (a variation on Giveon- Kutasov duality [38]). Over the last few years this supersymmetric duality has been subjected to (and has successfully passed) several checks performed with the aid of supersymmetric localization, including the matching of three sphere partition function, superconformal indices and Wilson loops on both sides of the duality [31–36, 55] .

4.2.3 Properties of free solutions of the Dirac equation

In subsequent subsections we will investigate the constraints imposed supersymmetry on the S matrices of the the theory (119). Our analysis will make heavy use of the properties of the free solutions to Dirac's equations, which we review in this subsection.

Let u_α and v_α are positive and negative energy solutions to Dirac's equations with mass m. Let $p^\mu = (\sqrt{m^2+\mathbf{p}^2}, \mathbf{p})$. Then u_α and v_α obey

$$\begin{aligned}(\not{p} - m)u(p) &= 0 \ , \\ (\not{p} + m)v(p) &= 0 \ .\end{aligned} \tag{134}$$

We choose to normalize these spinors so that

$$\begin{aligned}\bar{u}(\mathbf{p}) \cdot u(\mathbf{p}) &= -2m & \bar{v}(\mathbf{p}) \cdot v(\mathbf{p}) &= 2m \\ u(\mathbf{p})u^*(\mathbf{p}) &= -(\not{p} + m)C & v(\mathbf{p})v^*(\mathbf{p}) &= -(\not{p} - m)C \ .\end{aligned} \tag{135}$$

C in (135) is the charge conjugation matrix defined to obey the equation

$$C\gamma^\mu C^{-1} = -(\gamma^\mu)^T \ . \tag{136}$$

Throughout this chapter we use γ matrices that obey the algebra[38]

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu} \ . \tag{137}$$

We also choose all three γ^μ matrices to be purely imaginary [39] and to obey

$$(\gamma^\mu)^\dagger = -\eta^{\mu\mu}\gamma^\mu \quad \text{no sum} \tag{138}$$

[38]We use the mostly plus convention for $\eta_{\mu\nu}$, the corresponding Euclidean algebra obeys $\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu}$. See Appendix §4.7.1 for explicit representations of the γ matrices and charge conjugation matrix C.

[39]This is possible in 3 dimensions; recall the unconventional choice of sign in (137).

with these conventions it is easily verified that $C = \gamma^0$ obeys (136) and so we choose

$$C = \gamma^0 \ .$$

Using the conventions spelt out above, it is easily verified that $u(\mathbf{p})$ and $v^*(\mathbf{p})$ obey the same equation (i.e. complex conjugation flips the two equations in (134)), and have the same normalization. It follows that it is possible to pick the (as yet arbitrary) phases of $u(\mathbf{p})$ and $v(\mathbf{p})$ to ensure that

$$u_\alpha(\mathbf{p}) = -v^*_\alpha(\mathbf{p}), \quad v_\alpha(\mathbf{p}) = -u^*_\alpha(\mathbf{p}) \tag{139}$$

[40]. We will adopt the choice (139) throughout our chapter.

Notice that the replacement $m \to -m$ interchanges the equations for u and v. It follows that $u(m) \propto v(-m)$. Atleast with the choice of phase that we adopt in this chapter (see below) we find

$$u(m, p) = -v(-m, p), \quad v(m, p) = -u(-m, p) \ . \tag{140}$$

To proceed further it is useful to make a particular choice of γ matrices and to adopt a particular choice of phase for u. We choose the γ^μ matrices listed in §4.7.1 and

[40] Note that $\bar{u}^\alpha = u^{*\alpha} = C^{\alpha\beta} u^*_\beta$ and not $(u^\alpha)^*$. Thus, $(u^{*\alpha})^* = -u^\alpha$, where we have used the fact that $C = \gamma^0$ is imaginary. Similarly $(u^\alpha)^* = -u^{*\alpha}$. Likewise for v. Care should be taken while complex conjugating dot products of spinors, for instance $(v^*(\mathbf{p}_i) v^*(\mathbf{p}_j))^* = -(v(\mathbf{p}_i) v(\mathbf{p}_j))$, $(u(\mathbf{p}_i) u(\mathbf{p}_j))^* = -(u^*(\mathbf{p}_i) u^*(\mathbf{p}_j))$, and so on.

take $u(\mathbf{p})$ and $v(\mathbf{p})$ to be given by

$$
\begin{aligned}
u(\mathbf{p}) &= \begin{pmatrix} -\sqrt{p^0 - p^1} \\ \frac{p^3 + im}{\sqrt{p^0 - p^1}} \end{pmatrix} \ , \ \bar{u}(\mathbf{p}) = \begin{pmatrix} \frac{ip^3 + m}{\sqrt{p^0 - p^1}} & i\sqrt{p^0 - p^1} \end{pmatrix} \ , \\
v(\mathbf{p}) &= \begin{pmatrix} \sqrt{p^0 - p^1} \\ \frac{-p^3 + im}{\sqrt{p^0 - p^1}} \end{pmatrix} \ , \ \bar{v}(\mathbf{p}) = \begin{pmatrix} \frac{-ip^3 + m}{\sqrt{p^0 - p^1}} & -i\sqrt{p^0 - p^1} \end{pmatrix} \ ,
\end{aligned}
\tag{141}
$$

where

$$
p^0 = +\sqrt{m^2 + \mathbf{p}^2} \ .
$$

Notice that the arguments of the square roots in (141) are always positive; the square roots in (141) are defined to be positive (i.e. $\sqrt{x^2} = |x|$). It is easily verified that the solutions (141) respect (140) as promised.

In the rest of this section we discuss an analytic rotation of the spinors to complex (and in particular negative) values of the p^μ (and in particular p^0). This formal construction will prove useful in the study of the transformation properties of the S matrix under crossing symmetry.

Let us define

$$
\sqrt{ae^{i\alpha}} = |\sqrt{a}|e^{i\frac{\alpha}{2}}.
$$

Clearly our function is single valued only on a double cover of the complex plane. In other words our square root function is well defined if α is specified modulo 4π, but is not well defined if α is specified modulo 2π. We define

$$
\begin{aligned}
u(\mathbf{p},\alpha) = u(e^{i\alpha}p^{\mu}) &= \begin{pmatrix} -e^{i\frac{\alpha}{2}}\sqrt{p^0-p^1} \\ \frac{p^3 e^{i\frac{\alpha}{2}}+ime^{-i\frac{\alpha}{2}}}{\sqrt{p^0-p^1}} \end{pmatrix} \ , \\
v(\mathbf{p},\alpha) = v(e^{i\alpha}p^{\mu}) &= -\begin{pmatrix} -e^{i\frac{\alpha}{2}}\sqrt{p^0-p^1} \\ \frac{p^3 e^{i\frac{\alpha}{2}}-ime^{-i\frac{\alpha}{2}}}{\sqrt{p^0-p^1}} \end{pmatrix} \ , \\
u^*(\mathbf{p},\alpha) &= \begin{pmatrix} -e^{-i\frac{\alpha}{2}}\sqrt{p^0-p^1} \\ \frac{p^3 e^{-i\frac{\alpha}{2}}-ime^{i\frac{\alpha}{2}}}{\sqrt{p^0-p^1}} \end{pmatrix} \ , \\
v^*(\mathbf{p},\alpha) &= -\begin{pmatrix} -e^{-i\frac{\alpha}{2}}\sqrt{p^0-p^1} \\ \frac{p^3 e^{-i\frac{\alpha}{2}}+ime^{+i\frac{\alpha}{2}}}{\sqrt{p^0-p^1}} \end{pmatrix} \ ,
\end{aligned}
\tag{142}
$$

with $\alpha \in [0, 4\pi)$. It follows immediately from these definitions that

$$
\begin{aligned}
u(\mathbf{p},\alpha+\pi) = &\ -iv(\mathbf{p},\alpha) \ , \ v(\mathbf{p},\alpha+\pi) &= -iu(\mathbf{p},\alpha) \ , \\
u(\mathbf{p},\alpha-\pi) = &\ \ iv(\mathbf{p},\alpha) \ , \ v(\mathbf{p},\alpha-\pi) &= iu(\mathbf{p},\alpha) \ , \\
u^*(\mathbf{p},\alpha) = &\ -v(\mathbf{p},-\alpha) \ , \ v^*(\mathbf{p},\alpha) &= -u(\mathbf{p},-\alpha) \ .
\end{aligned}
\tag{143}
$$

Notice, in particular, that the choice $\alpha = \pi$ and $\alpha = -\pi$ both amount to the replacement of p^μ with - p^μ. Note also that the complex conjugation of $u(p,\alpha)$ is equal to the function $u^*(p)$ with p rotated by $-\alpha$.

4.2.4 Constraints of supersymmetry on scattering

In this chapter we will study 2×2 scattering of particles in an $\mathcal{N} = 1$ supersymmetric field theory. In this subsection we set up our conventions and notations and explore the constraints of supersymmetry on scattering amplitudes.

Let us consider the scattering process

$$1 + 2 \to 3 + 4 \tag{144}$$

where $1, 2$ represent initial state particles and $3, 4$ are final state particles. Let the i^{th} particle be associated with the superfield Φ_i. As a scattering amplitude represents the transition between free incoming and free outgoing onshell particles, the initial and final states of Φ_i are effectively subject to the free equation of motion

$$\left(D^2 + m_i\right) \Phi_i = 0 \tag{145}$$

where $D^2 = \frac{1}{2} D^\alpha D_\alpha$. The general solution to this free equation of motion is

$$\begin{aligned}\Phi(x, \theta) = \int \frac{d^2 p}{\sqrt{2p^0}(2\pi)^2} \Bigg[& \Big(a(\mathbf{p})(1 + m\theta^2) + \theta^\alpha u_\alpha(\mathbf{p}) \alpha(\mathbf{p}) \Big) e^{ip.x} \\ & + \Big(a^{c\dagger}(\mathbf{p})(1 + m\theta^2) + \theta^\alpha v_\alpha(\mathbf{p}) \alpha^{c\dagger}(\mathbf{p}) \Big) e^{-ip.x} \Bigg]\end{aligned} \tag{146}$$

where $a/a^\dagger$ are annihilation/creation operator for the bosonic particles and $\alpha/\alpha^\dagger$ are annihilation/creation operators for the fermionic particles respectively [41]. The bosonic and fermionic oscillators obey the commutation relations

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^2 \delta^2(\mathbf{p} - \mathbf{p}'), \quad [a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^2 \delta^2 \left(\mathbf{p} - \mathbf{p}'\right) \ . \tag{147}$$

(a^c and α^c obey analogous commutation relations).

[41] Similarly $a^c/a^{c\dagger}$ and $\alpha^c/\alpha^{c\dagger}$ are the annihilation/creation operators for the bosonic and fermionic anti-particles respectively.

The action of the supersymmetry operator on a free onshell superfield is simple

$$
\begin{aligned}
&[Q_\alpha, \Phi_i] = \\
&\quad Q_\alpha \Phi_i = i \int \frac{d^2 p}{(2\pi)^2 \sqrt{2p^0}} \Bigg[\Bigg(u_\alpha(\mathbf{p})(1 + m\theta^2)\alpha(\mathbf{p}) + \theta^\beta(-u_\beta(\mathbf{p}) u^*_\alpha(\mathbf{p}))a(\mathbf{p}) \Bigg) e^{ip.x} \\
&\qquad + \Bigg(v_\alpha(\mathbf{p})(1 + m\theta^2)\alpha^{c\dagger}(\mathbf{p}) + \theta^\beta(v_\beta(\mathbf{p}) v^*_\alpha(\mathbf{p}))a^{c\dagger}(\mathbf{p}) \Bigg) e^{-ip.x} \Bigg] \ .
\end{aligned}
$$

In other words, the action of the supersymmetry generator on onshell superfields is given by

$$
\begin{aligned}
-iQ_\alpha &= u_\alpha(\mathbf{p}_i) \left(a\partial_\alpha + a^c \partial_{\alpha^c}\right) + u^*_\alpha(\mathbf{p}_i) \left(-\alpha\partial_a + \alpha^c \partial_{a^c}\right) \\
&\quad + v_\alpha(\mathbf{p}_i) \left(a^\dagger \partial^\dagger_\alpha + (a^c)^\dagger \partial_{(\alpha^c)^\dagger}\right) + v^*_\alpha(\mathbf{p}_i) \left(\alpha^\dagger \partial^\dagger_a + (\alpha^c)^\dagger \partial_{(a^c)^\dagger}\right) \ .
\end{aligned} \tag{148}
$$

The explicit action of Q_α on onshell superfields may be repackaged as follows. Let us define a superfield of annihilation operators, and another superfield for creation operators:

$$
\begin{aligned}
A_i(\mathbf{p}) &= a_i(\mathbf{p}) + \alpha_i(\mathbf{p})\theta_i \ , \\
A^\dagger_i(\mathbf{p}) &= a^\dagger_i(\mathbf{p}) + \theta_i \alpha^\dagger_i(\mathbf{p}) \ .
\end{aligned} \tag{149}
$$

Here θ_i is a new formal superspace parameter (θ_i has nothing to do with the θ_α that appear in the superfield action (119)). It follows from (148) and (149) that

$$
\begin{aligned}
[Q_\alpha, A_i(\mathbf{p}_i, \theta_i)] &= Q^1_\alpha A_i(\mathbf{p}_i, \theta_i) \\
[Q_\alpha, A^\dagger_i(\mathbf{p}_i, \theta_i)] &= Q^2_\alpha A^\dagger_i(\mathbf{p}_i, \theta_i)
\end{aligned} \tag{150}
$$

where

$$
\begin{aligned}
Q^1_\beta &= i\left(-u_\beta(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta} + u^*_\beta(\mathbf{p})\theta\right) \\
Q^2_\beta &= i\left(v_\beta(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta} + v^*_\beta(\mathbf{p})\theta\right) \ .
\end{aligned}
\tag{151}
$$

We are interested in the S matrix

$$
\begin{aligned}
S(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4)\sqrt{(2p^0_1)(2p^0_2)(2p^0_3)(2p^0_4)} = \\
\langle 0|A_4(\mathbf{p}_4,\theta_4)A_3(\mathbf{p}_3,\theta_3)UA^\dagger_2(\mathbf{p}_2,\theta_2)A^\dagger_1(\mathbf{p}_1,\theta_1)|0\rangle
\end{aligned}
\tag{152}
$$

where U is an evolution operator (the RHS denotes the transition amplitude from the in state with particles 1 and 2 to the out state with particles 3 and 4).

The condition that the S matrix defined in (152) is invariant under supersymmetry follows from the action of supersymmetries on oscillators given in (148). The resultant equation for the S matrix may be written in terms of the operators defined in (151) as

$$
\begin{aligned}
\Big(&\overrightarrow{Q}^1_\alpha(\mathbf{p}_1,\theta_1) + \overrightarrow{Q}^1_\alpha(\mathbf{p}_2,\theta_2) \\
&+ \overrightarrow{Q}^2_\alpha(\mathbf{p}_3,\theta_3) + \overrightarrow{Q}^2_\alpha(\mathbf{p}_4,\theta_4)\Big) S(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4) = 0 \ .
\end{aligned}
\tag{153}
$$

We have explicitly solved (153); the solution[42] is given by

$$
\begin{aligned}
S(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4) &= \mathcal{S}_B + \mathcal{S}_F\ \theta_1\theta_2\theta_3\theta_4 + \left(\frac{1}{2}C_{12}\mathcal{S}_B - \frac{1}{2}C^*_{34}\mathcal{S}_F\right)\ \theta_1\theta_2 \\
&+ \left(\frac{1}{2}C_{13}\mathcal{S}_B - \frac{1}{2}C^*_{24}\mathcal{S}_F\right)\ \theta_1\theta_3 + \left(\frac{1}{2}C_{14}\mathcal{S}_B + \frac{1}{2}C^*_{23}\mathcal{S}_F\right)\ \theta_1\theta_4 + \left(\frac{1}{2}C_{23}\mathcal{S}_B + \frac{1}{2}C^*_{14}\mathcal{S}_F\right)\ \theta_2\theta_3 \\
&+ \left(\frac{1}{2}C_{24}\mathcal{S}_B - \frac{1}{2}C^*_{13}\mathcal{S}_F\right)\ \theta_2\theta_4 + \left(\frac{1}{2}C_{34}\mathcal{S}_B - \frac{1}{2}C^*_{12}\mathcal{S}_F\right)\ \theta_3\theta_4
\end{aligned}
\tag{154}
$$

where

$$
\begin{aligned}
&\frac{1}{2}C_{12} = -\frac{1}{4m}v^*(\mathbf{p}_1)v^*(\mathbf{p}_2) && \frac{1}{2}C_{23} = -\frac{1}{4m}v^*(\mathbf{p}_2)u^*(\mathbf{p}_3) \\
&\frac{1}{2}C_{13} = -\frac{1}{4m}v^*(\mathbf{p}_1)u^*(\mathbf{p}_3) && \frac{1}{2}C_{24} = -\frac{1}{4m}v^*(\mathbf{p}_2)u^*(\mathbf{p}_4) \\
&\frac{1}{2}C_{14} = -\frac{1}{4m}v^*(\mathbf{p}_1)u^*(\mathbf{p}_4) && \frac{1}{2}C_{34} = -\frac{1}{4m}u^*(\mathbf{p}_3)u^*(\mathbf{p}_4)
\end{aligned}
\tag{155}
$$

and

$$
\begin{aligned}
&\frac{1}{2}C^*_{12} = \frac{1}{4m}v(\mathbf{p}_1)v(\mathbf{p}_2) && \frac{1}{2}C^*_{23} = \frac{1}{4m}v(\mathbf{p}_2)u(\mathbf{p}_3) \\
&\frac{1}{2}C^*_{13} = \frac{1}{4m}v(\mathbf{p}_1)u(\mathbf{p}_3) && \frac{1}{2}C^*_{24} = \frac{1}{4m}v(\mathbf{p}_2)u(\mathbf{p}_4) \\
&\frac{1}{2}C^*_{14} = \frac{1}{4m}v(\mathbf{p}_1)u(\mathbf{p}_4) && \frac{1}{2}C^*_{34} = \frac{1}{4m}u(\mathbf{p}_3)u(\mathbf{p}_4)
\end{aligned}
\tag{156}
$$

Note that the general solution to (153) is given in terms of two arbitrary functions $\mathcal{S}_B$ and $\mathcal{S}_F$ of the four momenta; (153) determines the remaining six functions in the general expansion of the S matrix in terms of these two functions. See Appendix 4.7.2 for a check of these relations from another viewpoint (involving offshell supersymmetry of the effective action, see section §4.3.4)

[42] The superspace S matrix (154) encodes different processes allowed by supersymmetry in the theory. In particular, the presence of grassmann parameters indicates fermionic in (θ_1,θ_2) and fermionic out (θ_3,θ_4) states. The absence of grassmann parameter indicates a bosonic in/out state. Thus, the no θ term $\mathcal{S}_B$ encodes the $2\to 2$ S matrix for a purely bosonic process, while the four θ term $\mathcal{S}_F$ encodes the $2\to 2$ S matrix of a purely fermionic process. Note in particular that S matrices corresponding to all other $2\to 2$ processes that involve both bosons and fermions are completely determined in terms of the S matrices $\mathcal{S}_B$ and $\mathcal{S}_F$ together with (155) and (156).

Although we are principally interested in $\mathcal{N}=1$ supersymmetric theories in this chapter, we will sometimes study the special limit $w=1$ in which (119) enjoys an enhanced $\mathcal{N}=2$ supersymmetry. In this case the additional supersymmetry further constrains the S matrix. In Appendix 4.7.3 we demonstrate that the additional supersymmetry determines $\mathcal{S}_B$ in terms of $\mathcal{S}_F$. In the $\mathcal{N}=2$ case, in other words, all components of the S matrix are determined by supersymmetry in terms of the four boson scattering matrix.

4.2.5 Supersymmetry and dual supersymmetry

The strong weak coupling duality we study in this chapter is conjectured to be a Bose-Fermi duality. In other words

$$a^D = \alpha, \quad \alpha^D = a \tag{157}$$

together with a similar exchange of bosons and fermions for creation operators (the superscript D stands for 'dual'). Suppose we define

$$\begin{aligned} A_i^D(\mathbf{p}) &= a_i^D(\mathbf{p}) + \alpha_i^D(\mathbf{p})\theta_i \ , \\ (A^D)_i^\dagger(\mathbf{p}) &= (a^D)_i^\dagger(\mathbf{p}) + \theta_i(\alpha_i^D)^\dagger(\mathbf{p}) \ . \end{aligned} \tag{158}$$

The dual supersymmetries must act in the same way on A^D and $(A^D)^\dagger$ as ordinary supersymmetries act on A and A^D. In other words the action of dual supersymmetries on A^D and $(A^D)^\dagger$ is given by

$$\begin{aligned} [Q_\alpha^D, A_i^D(\mathbf{p}_i,\theta_i)] &= (Q^D)^1_\alpha A_i^D(\mathbf{p}_i,\theta_i) \ , \\ [Q_\alpha^D, (A^D)_i^\dagger(\mathbf{p}_i,\theta_i)] &= (Q^D)^2_\alpha (A^D)_i^\dagger(\mathbf{p}_i,\theta_i) \ , \end{aligned} \tag{159}$$

where

$$
\begin{aligned}
(Q^D)^1_\beta &= i\left(-u_\beta(\mathbf{p},-m)\frac{\overrightarrow{\partial}}{\partial\theta} - v_\beta(\mathbf{p},-m)\theta\right) \ , \\
(Q^D)^2_\beta &= i\left(v_\beta(\mathbf{p},-m)\frac{\overrightarrow{\partial}}{\partial\theta} - u_\beta(\mathbf{p},-m)\theta\right) \ .
\end{aligned} \tag{160}
$$

The spinors in (160) are all evaluated at $-m$ as duality flips the sign of the pole mass.

The action of the dual supersymmetries on A and $A^\dagger$ is obtained from (160) upon performing the interchange $\theta \leftrightarrow \partial_\theta$ (this accounts for the interchange of bosons and fermions). Using also (140) we find that

$$
\begin{aligned}
[Q^D_\alpha, A_i(\mathbf{p}_i,\theta_i)] &= -Q^1_\alpha A^D_i(\mathbf{p}_i,\theta_i) \ , \\
[Q^D_\alpha, A^\dagger_i(\mathbf{p}_i,\theta_i)] &= Q^2_\alpha (A^D)^\dagger_i(\mathbf{p}_i,\theta_i) \ .
\end{aligned} \tag{161}
$$

It follows, in particular, that an S matrix invariant under the usual supersymmetries is automatically invariant under dual supersymmetries. In other words onshell supersymmetry 'commutes' with duality.

4.2.6 Naive crossing symmetry and supersymmetry

Let us define the analytically rotated supersymmetry operators [43]

$$
\begin{aligned}
Q^1_\beta(\mathbf{p},\alpha,\theta) &= i\left(-u_\beta(\mathbf{p},\alpha)\frac{\overrightarrow{\partial}}{\partial\theta} + u^*_\beta(\mathbf{p},-\alpha)\theta\right) \ , \\
Q^2_\beta(\mathbf{p},\alpha,\theta) &= i\left(v_\beta(\mathbf{p},\alpha)\frac{\overrightarrow{\partial}}{\partial\theta} + v^*_\beta(\mathbf{p},-\alpha)\theta\right) \ .
\end{aligned} \tag{162}
$$

[43] Note that the notation $u^*_\beta(\mathbf{p},-\alpha)$ means that the analytically rotated function of u^* in (142) is evaluated at the phase $-\alpha$.

It is easily verified from these definitions that

$$Q^2_\alpha(\mathbf{p}, 0, -i\theta) = Q^1_\alpha(\mathbf{p}, \pi, \theta)\ . \tag{163}$$

Using (163) the equation (153) may equivalently be written as

$$\Big(\overrightarrow{Q}^1_\alpha(\mathbf{p}_1, 0, \theta_1) + \overrightarrow{Q}^1_\alpha(\mathbf{p}_2, 0, \theta_2) \\ + \overrightarrow{Q}^1_\alpha(\mathbf{p}_3, \pi, \theta_3) + \overrightarrow{Q}^1_\alpha(\mathbf{p}_4, \pi, \theta_4)\Big) S(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, -i\theta_3, \mathbf{p}_4, -i\theta_4) = 0 \tag{164}$$

with $p_1 + p_2 = p_3 + p_4$.

The constraints of supersymmetry on the S matrix are consistent with (naive) crossing symmetry. In order to make this manifest, we define a 'master' function S_M

$$S_M(\mathbf{p}_1, \phi_1, \theta_1, \mathbf{p}_2, \phi_2, \theta_2, \mathbf{p}_3, \phi_3, \theta_3, \mathbf{p}_4, \phi_4, \theta_4)\ .$$

The master function S_M is defined so that

$$S(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, -i\theta_3, \mathbf{p}_4, -i\theta_4) = S_M(\mathbf{p}_1, 0, \theta_1, \mathbf{p}_2, 0, \theta_2, \mathbf{p}_3, \pi, \theta_3, \mathbf{p}_4, \pi, \theta_4) \tag{165}$$

In other words S_M is S with the replacement $-i\theta_3 \to \theta_3$, $-i\theta_4 \to \theta_4$, analytically rotated to general values of the phase ϕ_1, ϕ_2, ϕ_3 and ϕ_4. It follows from (164) that the master equation S_M obeys the completely symmetrical supersymmetry equation

$$\Big(\overrightarrow{Q}^1_\alpha(\mathbf{p}_1, \phi_1, \theta_1) + \overrightarrow{Q}^1_\alpha(\mathbf{p}_2, \phi_2, \theta_2) + \overrightarrow{Q}^1_\alpha(\mathbf{p}_3, \phi_3, \theta_3) \\ + \overrightarrow{Q}^1_\alpha(\mathbf{p}_4, \phi_4, \theta_4)\Big) S_M(\mathbf{p}_1, \phi_1, \theta_1, \mathbf{p}_2, \phi_2, \theta_2, \mathbf{p}_3, \phi_3, \theta_3, \mathbf{p}_4, \phi_4, \theta_4) = 0 \tag{166}$$

The function S_M encodes the scattering matrices in all channels. In order to extract the S matrix for $p_i + p_j \to p_k + p_m$ with $p_i + p_j = p_k + p_m$ (with (i, j, k, m) being any permutation of (1, 2, 3, 4)) we simply evaluate the function S_M with ϕ_i and ϕ_j set to zero, ϕ_k and ϕ_m set to π, θ_i and θ_j left unchanged and θ_k and θ_m replaced by $i\theta_k$ and $i\theta_m$. The fact that the master equation obeys an equation that is symmetrical in the labels $1, 2, 3, 4$ is the statement of (naive) crossing symmetry.

The solution to the differential equation (166) is

$$\begin{aligned} S_M(\mathbf{p}_1, \phi_1, \theta_1, &\mathbf{p}_2, \phi_2, \theta_2, \mathbf{p}_3, \phi_3, \theta_3, \mathbf{p}_4, \phi_4, \theta_4) = \tilde{\mathcal{S}}_B + \tilde{\mathcal{S}}_F \theta_1\theta_2\theta_3\theta_4 \\ &+ \frac{\tilde{\mathcal{S}}_B}{4} \sum_{i,j=1}^{4} D_{ij}(\mathbf{p}_i, \phi_i, \mathbf{p}_j, \phi_j)\theta_i\theta_j - \frac{\tilde{\mathcal{S}}_F}{8} \sum_{i,j,k,l=1}^{4} \epsilon^{ijkl} \tilde{D}_{ij}(\mathbf{p}_i, \phi_i, \mathbf{p}_j, \phi_j)\theta_k\theta_l \end{aligned} \tag{167}$$

where

$$\begin{aligned} \frac{1}{2} D_{ij}(\mathbf{p}_i, \phi_i, \mathbf{p}_j, \phi_j) &= -\frac{1}{4m} u^*(\mathbf{p}_i, -\phi_i) u^*(\mathbf{p}_j, -\phi_j) \ , \\ \frac{1}{2} \tilde{D}_{ij}(\mathbf{p}_i, \phi_i, \mathbf{p}_j, \phi_j) &= \frac{1}{4m} u(\mathbf{p}_i, \phi_i) u(\mathbf{p}_j, \phi_j) \ . \end{aligned} \tag{168}$$

In the above equations '$*$' means complex conjugation and the spinor indices are contracted from NW-SE as usual. To summarize, S_M obeys the supersymmetric ward identity and is completely solved in terms of two analytic functions $\tilde{\mathcal{S}}_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$ and $\tilde{\mathcal{S}}_F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$ of the momenta.

As we have explained under (166), the S matrix corresponding to scattering processes in any given channel can be simply extracted out of S_M. For example, let S denote the the S matrix in the channel with p_1, p_2 as in-states and p_3, p_4 as out-states. Then

$$S(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) = S_M(\mathbf{p}_1, \pi, i\theta_1, \mathbf{p}_2, \pi, i\theta_2, \mathbf{p}_3, 0, \theta_3, \mathbf{p}_4, 0, \theta_4) \ . \tag{169}$$

It is easily verified that (168) together with (143) imply (155).

Notice that (169) maps $\tilde{\mathcal{S}_B}$ to $\mathcal{S}_B$ while $\tilde{\mathcal{S}_F}$ is mapped to $-\mathcal{S}_F$ [44]. The minus sign in the continuation of $\mathcal{S}_F$ has an interesting explanation. The four fermion amplitude $\mathcal{S}_F$ has a phase ambiguity. This ambiguity follows from the fact that $\mathcal{S}_F$ is the overlap of initial and final fermions states. These initial and final states are written in terms of the spinors u_α and v_α, which are defined as appropriately normalized solutions of the Dirac equation are inherently ambiguous upto a phase. It is easily verified that the quantity

$$(u^*(\mathbf{p}_1, -\phi_1)u(\mathbf{p}_3, \phi_3))\,(u^*(\mathbf{p}_2, -\phi_2)u(\mathbf{p}_4, \phi_4))$$

has the same phase ambiguity as $\mathcal{S}_F$. If we define an auxiliary quantity $\tilde{\mathcal{S}_f}$ by the equation

$$\tilde{\mathcal{S}_F} = -\frac{1}{4m^2}\,(u^*(\mathbf{p}_1, -\phi_1)u(\mathbf{p}_3, \phi_3))\,(u^*(\mathbf{p}_2, -\phi_2)u(\mathbf{p}_4, \phi_4))\,\tilde{\mathcal{S}_f} \tag{170}$$

and $\mathcal{S}_f$ by

$$\mathcal{S}_F = -\frac{1}{4m^2}\,(u^*(\mathbf{p}_1)u(\mathbf{p}_3))\,(u^*(\mathbf{p}_2)u(\mathbf{p}_4))\,\mathcal{S}_f \tag{171}$$

then the phases of $\mathcal{S}_f$ and $\tilde{\mathcal{S}_f}$ are unambiguous and so potentially physical. As the quantity

$$(u^*(\mathbf{p}_1, -\phi_1)u(\mathbf{p}_3, \phi_3))\,(u^*(\mathbf{p}_2, -\phi_2)u(\mathbf{p}_4, \phi_4))$$

picks up a minus sign under the phase rotation that takes us from S_M to S. It follows that $\tilde{\mathcal{S}_f}$ rotates to $\mathcal{S}_f$ with no minus sign.

[44]Of course $\tilde{\mathcal{S}_B}$ and $\tilde{\mathcal{S}_F}$ are evaluated at $\phi_1 = \phi_2 = \pi$ while $\mathcal{S}_B$ and $\mathcal{S}_F$ are evaluated at $\phi_1 = \phi_2 = 0$; roughly speaking this amounts to the replacement $p_1^\mu \to -p_1^\mu$, $p_2^\mu \to -p_2^\mu$.

4.2.7 Properties of the convolution operator

Like any matrices, S matrices can be multiplied. The multiplication rule for two S matrices, S_1 and S_2, expressed as functions in onshell superspace is given by

$$S_1 \star S_2 \equiv \int d\Gamma S_1(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{k}_3, \phi_1, \mathbf{k}_4, \phi_2) \exp(\phi_1\phi_3 + \phi_2\phi_4) 2k_1^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_3 - \mathbf{k}_1)$$
$$2k_2^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_4 - \mathbf{k}_2) S_2(\mathbf{k}_1, \phi_3, \mathbf{k}_2, \phi_4, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) \tag{172}$$

where the measure $d\Gamma$ is

$$d\Gamma = \frac{d^2k_3}{2k_3^0(2\pi)^2} \frac{d^2k_4}{2k_4^0(2\pi)^2} \frac{d^2k_1}{2k_1^0(2\pi)^2} \frac{d^2k_2}{2k_2^0(2\pi)^2} d\phi_1 d\phi_3 d\phi_2 d\phi_4 \ . \tag{173}$$

It is easily verified that the onshell superfield I

$$I(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) = \exp(\theta_1\theta_3 + \theta_2\theta_4) I(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$$
$$I(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = 2p_3^0(2\pi)^2\delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_3) 2p_4^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_4) \tag{174}$$

is the identity operator under this multiplication rule, i.e.

$$S \star I = I \star S = S \tag{175}$$

for any S. It may be verified that I defined in (174) obeys (153) and so is supersymmetric.

In Appendix §4.7.4 we demonstrate that if S_1 and S_2 are onshell superfields that obey (153), then $S_1 \star S_2$ also obeys (153). In other words the product of two supersymmetric S matrices is also supersymmetric.

The onshell superfield corresponding to $S^\dagger$ is given in terms of the onshell superfield corresponding to S by the equation

$$S^{\dagger}(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4) = S^{*}(\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4,\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2)\ . \tag{176}$$

The equation satisfied by $S^{\dagger}$ can be obtained by complex conjugating and interchanging the momenta in the supersymmetry invariance condition for S (see (415)). It follows from the anti-hermiticity of Q that

$$\left(Q^{*}_{u(\mathbf{p}_1)} + Q^{*}_{u(\mathbf{p}_2)} + Q_{u(\mathbf{p}_3)} + Q_{u(\mathbf{p}_4)}\right) S^{*}(\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4,\mathbf{p}_1,\theta_3,\mathbf{p}_2,\theta_4) = 0 \tag{177}$$

which implies $[Q, S^{\dagger}] = 0$. Thus $S^{\dagger}$ is supersymmetric if and only if S is supersymmetric.

4.2.8 Unitarity of Scattering

The unitarity condition

$$SS^{\dagger} = \mathbb{I} \tag{178}$$

may be rewritten in the language of onshell superfields as

$$(S \star S^{\dagger} - I) = 0\ . \tag{179}$$

45

It follows from the general results of the previous subsection that the LHS of (179) is supersymmetric, i.e it obeys (153). Recall that any onshell superfield that obeys (153) must take the form (154) where $\mathcal{S}_B$ and $\mathcal{S}_F$ are the zero theta and 4 theta terms in the expansion of the corresponding object. In particular, in order to verify that the LHS of (179) vanishes, it is sufficient to verify that its zero and 4 theta components vanish.

[45] As explained in [61], the unitarity equation for 2×2 does not receive contributions from $2\times n$ scattering in the large N limits studied in the current chapter as well.

Inserting the explicit solutions for S and $S^\dagger$, one finds that the no-theta term of (179) is proportional to (we have used that $k_3 \cdot k_4 = p_3 \cdot p_4$ onshell)

$$\begin{aligned}
&\int \frac{d^2k_3}{2k_3^0(2\pi)^2}\frac{d^2k_4}{2k_4^0(2\pi)^2}\,[\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4) \\
&-\frac{1}{16m^2}\left(2(p_3\cdot p_4+m^2)\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\right. \\
&+u^*(\mathbf{k}_3)u^*(\mathbf{k}_4)\ v^*(\mathbf{p}_3)v^*(\mathbf{p}_4)\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4) \\
&+v(\mathbf{p}_1)v(\mathbf{p}_2)\ u(\mathbf{k}_3)u(\mathbf{k}_4)\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4) \\
&\left.+v(\mathbf{p}_1)v(\mathbf{p}_2)\ v^*(\mathbf{p}_3)v^*(\mathbf{p}_4)\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\right)] \\
&= 2p_3^0(2\pi)^2\delta^{(2)}(\mathbf{p}_1-\mathbf{p}_3)2p_4^0(2\pi)^2\delta^{(2)}(\mathbf{p}_2-\mathbf{p}_4)\ .
\end{aligned} \tag{180}$$

The four theta term in (179) is proportional to

$$\begin{aligned}
&\int \frac{d^2k_3}{2k_3^0(2\pi)^2}\frac{d^2k_4}{2k_4^0(2\pi)^2}\,[-\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4) \\
&+\frac{1}{16m^2}\left(2(p_3\cdot p_4+m^2)\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\right. \\
&+u(\mathbf{k}_3)u(\mathbf{k}_4)\ v(\mathbf{p}_3)v(\mathbf{p}_4)\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4) \\
&+v^*(\mathbf{p}_1)v^*(\mathbf{p}_2)\ u^*(\mathbf{k}_3)u^*(\mathbf{k}_4)\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4) \\
&\left.+v^*(\mathbf{p}_1)v^*(\mathbf{p}_2)\ v(\mathbf{p}_3)v(\mathbf{p}_4)\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\right)] \\
&= -2p_3^0(2\pi)^2\delta^{(2)}(\mathbf{p}_1-\mathbf{p}_3)2p_4^0(2\pi)^2\delta^{(2)}(\mathbf{p}_2-\mathbf{p}_4)\ .
\end{aligned} \tag{181}$$

The equations (180) and (181) are necessary and sufficient to ensure unitarity.

(180) and (181) may be thought of as constraints imposed by unitarity on the four boson scattering matrix $\mathcal{S}_B$ and the four fermion scattering matrix $\mathcal{S}_F$. These conditions are written in terms of the onshell spinors u and v (rather than the momenta of the scattering particles for a reason we now pause to review. Recall that the Dirac equation and normalization conditions define u_α and v_α only upto an undetermined phase (which could be a function of momentum). An expression built out of u's and v's can be written unambiguously in terms of onshell momenta if and only if all undetermined phases cancel

out. The phases of terms involving $\mathcal{S}_F$ in (180) and (181) do not cancel. This might at first appear to be a paradox; surely the unitarity (or lack) of an S matrix cannot depend on the unphysical choice of an arbitrary phase. The resolution to this 'paradox' is simple; the function $\mathcal{S}_F$ is itself not phase invariant, but transforms under phase transformations like $(u(\mathbf{p}_1)u(\mathbf{p}_2))\,(v(\mathbf{p}_3)v(\mathbf{p}_4))$. It is thus useful to define

$$\mathcal{S}_F = \frac{1}{4m^2}\,(u(\mathbf{p}_1)u(\mathbf{p}_2))\,(v(\mathbf{p}_3)v(\mathbf{p}_4))\,\mathcal{S}_f\ . \tag{182}$$

The utility of this definition is that $\mathcal{S}_f$ does not suffer from a phase ambiguity. Rewritten in terms of $\mathcal{S}_B$ and $\mathcal{S}_f$, the unitarity equations may be written entirely in terms of participating momenta (with no spinors) [46]. In terms of the quantity

$$Y(\mathbf{p}_3, \mathbf{p}_4) = \frac{2(p_3 \cdot p_4 + m^2)}{16m^2} \tag{183}$$

and

$$d\Gamma' = \frac{d^2k_3}{2k_3^0(2\pi)^2}\frac{d^2k_4}{2k_4^0(2\pi)^2}$$

$$\begin{aligned}&\int d\Gamma'\Big[\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\\ &-Y(\mathbf{p}_3,\mathbf{p}_4)\Big(\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)+4Y(\mathbf{p}_1,\mathbf{p}_2)\mathcal{S}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\Big)\\ &\Big(\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)+4Y(\mathbf{p}_3,\mathbf{p}_4)\mathcal{S}_f^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\Big)\Big] = 2p_3^0(2\pi)^2\delta^{(2)}(\mathbf{p}_1-\mathbf{p}_3)2p_4^0(2\pi)^2\delta^{(2)}(\mathbf{p}_2-\mathbf{p}_4)\end{aligned} \tag{184}$$

[46]See §4.7.5 for a derivation of this result.

and

$$\int d\Gamma' \Big[-16Y^2(\mathbf{p}_3,\mathbf{p}_4)\mathcal{S}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_f^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)$$
$$+Y(\mathbf{p}_3,\mathbf{p}_4)\Big(\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)+4Y(\mathbf{p}_1,\mathbf{p}_2)\mathcal{S}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\Big)$$
$$\Big(\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)+4Y(\mathbf{p}_3,\mathbf{p}_4)\mathcal{S}_f^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\Big)\Big] = -2p_3^0(2\pi)^2\delta^{(2)}(\mathbf{p}_1-\mathbf{p}_3)2p_4^0(2\pi)^2\delta^{(2)}(\mathbf{p}_2-\mathbf{p}_4). \tag{185}$$

The equations (184) and (185) followed from (178). It is useful to rephrase the above equations in terms of the "T matrix" that represents the actual interacting part of the "S matrix". Using the definition of the Identity operator (174) we can write a superfield expansion to define the "T matrix" as

$$\begin{aligned}S(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{k}_3,\theta_3,\mathbf{k}_4,\theta_4) =& I(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{k}_3,\theta_3,\mathbf{k}_4,\theta_4)\\ &+ i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)T(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{k}_3,\theta_3,\mathbf{k}_4,\theta_4)\ .\end{aligned} \tag{186}$$

The identity operator is defined in (174) is a supersymmetry invariant. It follows that the " T matrix" is also invariant under supersymmetry. In other words the "T matrix" obeys (153) and has a superfield expansion [47]

$$\begin{aligned}T(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4) =& \mathcal{T}_B + \mathcal{T}_F\ \theta_1\theta_2\theta_3\theta_4 + \left(\frac{1}{2}C_{12}\mathcal{T}_B - \frac{1}{2}C_{34}^*\mathcal{T}_F\right)\ \theta_1\theta_2\\ +\left(\frac{1}{2}C_{13}\mathcal{T}_B - \frac{1}{2}C_{24}^*\mathcal{T}_F\right)\ \theta_1\theta_3 &+ \left(\frac{1}{2}C_{14}\mathcal{T}_B + \frac{1}{2}C_{23}^*\mathcal{T}_F\right)\ \theta_1\theta_4 + \left(\frac{1}{2}C_{23}\mathcal{T}_B + \frac{1}{2}C_{14}^*\mathcal{T}_F\right)\ \theta_2\theta_3\\ &+\left(\frac{1}{2}C_{24}\mathcal{T}_B - \frac{1}{2}C_{13}^*\mathcal{T}_F\right)\ \theta_2\theta_4 + \left(\frac{1}{2}C_{34}\mathcal{T}_B - \frac{1}{2}C_{12}^*\mathcal{T}_F\right)\ \theta_3\theta_4\end{aligned} \tag{187}$$

[47] The matrices $\mathcal{T}_B$ and $\mathcal{T}_F$ correspond to the T matrices of the four boson and four fermion scattering respectively.

where

$$\mathcal{T}_F = \frac{1}{4m^2}\left(u(\mathbf{p}_1)u(\mathbf{p}_2)\right)\left(v(\mathbf{p}_3)v(\mathbf{p}_4)\right)\mathcal{T}_f\ . \tag{188}$$

and the coefficients C_{ij} are given as before in (155) and (156).

It follows from (186) that

$$\begin{aligned}
\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\ ,\\
\mathcal{S}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\ .
\end{aligned} \tag{189}$$

Substituting the definitions (189) into (184) and (185) the unitarity conditions can be rewritten as

$$\begin{aligned}
&\int d\tilde{\Gamma}\Big[\mathcal{T}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{T}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\\
&-Y(\mathbf{p}_3,\mathbf{p}_4)\Big(\mathcal{T}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)+4Y(\mathbf{p}_1,\mathbf{p}_2)\mathcal{T}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\Big)\\
&\Big(\mathcal{T}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)+4Y(\mathbf{p}_3,\mathbf{p}_4)\mathcal{T}_f^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\Big)\Big] = i(\mathcal{T}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)-\mathcal{T}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{p}_1,\mathbf{p}_2))
\end{aligned} \tag{190}$$

and

$$\begin{aligned}
&\int d\tilde{\Gamma}\Big[-16Y^2(\mathbf{p}_3,\mathbf{p}_4)\mathcal{T}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{T}_f^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\\
&+Y(\mathbf{p}_3,\mathbf{p}_4)\Big(\mathcal{T}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)+4Y(\mathbf{p}_1,\mathbf{p}_2)\mathcal{T}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\Big)\\
&\Big(\mathcal{T}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)+4Y(\mathbf{p}_3,\mathbf{p}_4)\mathcal{T}_f^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\Big)\Big]\\
&= 4iY(\mathbf{p}_3,\mathbf{p}_4)\left(\mathcal{T}_f^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{p}_1,\mathbf{p}_2)-\mathcal{T}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\right)
\end{aligned} \tag{191}$$

where

$$d\tilde{\Gamma} = (2\pi)^3\delta^3(p_1 + p_2 - k_3 - k_4)\frac{d^2k_3}{2k_3^0(2\pi)^2}\frac{d^2k_4}{2k_4^0(2\pi)^2} \ .$$

The equations (190) and (191) can be put in a more user friendly form by going to the center of mass frame with the definition

$$\begin{aligned}
&p_1 = \left(\sqrt{p^2+m^2}, p, 0\right) \ , \ p_2 = \left(\sqrt{p^2+m^2}, -p, 0\right) \\
&p_3 = \left(\sqrt{p^2+m^2}, p\cos(\theta), p\sin(\theta)\right) \ , \ p_4 = \left(\sqrt{p^2+m^2}, -p\cos(\theta), -p\sin(\theta)\right)
\end{aligned} \tag{192}$$

where θ is the scattering angle between p_1 and p_3. In terms of the Mandelstam variables

$$\begin{aligned}
&s = -(p_1+p_2)^2 \ , t = -(p_1-p_3)^2, \ u = (p_1-p_4)^2, \ s+t+u = 4m^2 \ , \\
&s = 4(p^2+m^2) \ , t = -2p^2(1-\cos(\theta)) \ , u = -2p^2(1+\cos(\theta)) \ .
\end{aligned} \tag{193}$$

Using the definitions we see that (183) becomes

$$Y = \frac{2(p_3 \cdot p_4 + m^2)}{16m^2} = \frac{-s+4m^2}{16m^2} = Y(s) \ . \tag{194}$$

Then (190) and (191) can be put in the form (See for instance eq 2.58-eq 2.59 of [61])

$$\begin{aligned}
\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(& -Y(s)(\mathcal{T}_B(s,\theta) + 4Y(s)\mathcal{T}_f(s,\theta))(\mathcal{T}_B^*(s,-(\alpha-\theta)) + 4Y(s)\mathcal{T}_f^*(s,-(\alpha-\theta))) \\
& + \mathcal{T}_B(s,\theta)\mathcal{T}_B^*(s,-(\alpha-\theta))\Big) = i(\mathcal{T}_B^*(s,-\alpha) - \mathcal{T}_B(s,\alpha))
\end{aligned} \tag{195}$$

$$\begin{aligned}
\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(& Y(s)(\mathcal{T}_B(s,\theta) + 4Y(s)\mathcal{T}_f(s,\theta))(\mathcal{T}_B^*(s,-(\alpha-\theta)) + 4Y(s)\mathcal{T}_f^*(s,-(\alpha-\theta))) \\
& -16Y(s)^2\mathcal{T}_f(s,\theta)\mathcal{T}_f^*(s,-(\alpha-\theta))\Big) = i4Y(s)(-\mathcal{T}_f(s,\alpha) + \mathcal{T}_f^*(s,-\alpha))
\end{aligned} \tag{196}$$

In a later section §4.4 we will use the simplified equations (195) and (196) for the unitarity analysis.

4.3 Exact computation of the all orders S matrix

In this section we will present results and conjectures for the the 2×2 S matrix of the general $\mathcal{N} = 1$ theory (198) at all orders in the t'Hooft coupling. In §4.3.2 we recall the action for our theory and determine the bare propagators for the scalar and vector superfields. At leading order in the $\frac{1}{N}$ the vector superfield propagator is exact (it is not renormalized). However the propagator of the scalar superfield does receive corrections. In §4.3.3, we determine the all orders propagator for the superfield Φ by solving the relevant Schwinger-Dyson equation. We will then turn to the determination of the exact offshell four point function of the superfield Φ. As in [61], we demonstrate that this four point function is the solution to a linear integral equation which we explicitly write down in §4.3.5. In a particular kinematic regime we present an exact solution to this integral equation in §4.3.6. In order to obtain the S matrix, in §4.3.7 we take the onshell limit of this answer. The kinematic restriction on our offshell result turns out to be inconsistent with the onshell limit in one of the four channels of scattering (particle - antiparticle scattering in the singlet channel) and so we do not have an explicit computation of the S matrix in this channel. In the other three channels, however, we are able to extract the full S matrix (with no kinematic restriction) albeit in a particular Lorentz frame. In §4.3.7 we present the unique covariant expressions for the S matrix consistent with our results. In §4.3.8 we report our result that the covariant S matrix reported in §4.3.7 is duality invariant. We present explicit exact results for the S matrices in the T and U channels of scattering in §4.3.9. In §4.3.10 we present the explicit conjecture for the S matrix in the singlet (S) channel. In §4.3.11 we report the explicit S matrices for the $\mathcal{N} = 2$ theory.

4.3.1 Supersymmetric Light Cone Gauge

We study the general $\mathcal{N} = 1$ theory (119). Wess-Zumino gauge, employed in subsection §4.2.1 to display the physical content of our theory, is inconvenient for actual computations as it breaks manifest supersymmetry. In other words if Γ_α is chosen to lie in Wess-Zumino gauge, it is in general not the case that $Q_\beta \Gamma_\alpha$ also respects this gauge condition. In all calculations presented in this chapter we will work instead in 'supersymmetric light cone gauge' [48]

$$\Gamma_- = 0 \tag{197}$$

As Γ_- transforms homogeneously under supersymmetry (see (124)) it is obvious that this gauge choice is supersymmetric. It is also easily verified that all gauge self interactions in (119) vanish in our lightcone gauge and the action (119) simplifies to

$$\begin{aligned} S_{tree} = -\int d^3x d^2\theta \bigg[&\frac{\kappa}{16\pi} Tr(\Gamma^- i\partial_{--}\Gamma^-) - \frac{1}{2} D^\alpha \bar{\Phi} D_\alpha \Phi - \frac{i}{2}\Gamma^-(\bar{\Phi} D_- \Phi - D_- \bar{\Phi}\Phi) \\ &+ m_0 \bar{\Phi}\Phi + \frac{\pi w}{\kappa}(\bar{\Phi}\Phi)^2 \bigg] . \end{aligned} \tag{198}$$

Note in particular that (198) is quadratic in Γ_+.

The condition (197) implies, in particular, that the component gauge fields in Γ_α obey

$$A_- = A_1 + iA_2 = 0$$

(see Appendix §4.7.6 for more details and further discussion about this gauge). In other words the gauge (197) is a supersymmetric generalization of ordinary lightcone gauge.

[48]We would like to thank S. Ananth and W. Siegel for helpful correspondence on this subject.

4.3.2 Action and bare propagators

The bare scalar propagator that follows from (198) is

$$\langle\bar{\Phi}(\theta_1,p)\Phi(\theta_2,-p')\rangle = \frac{D^2_{\theta_1,p} - m_0}{p^2+m_0^2}\delta^2(\theta_1-\theta_2)(2\pi)^3\delta^3(p-p') \ . \tag{199}$$

where m_0 is the bare mass. We have chosen the convention for the momentum flow direction

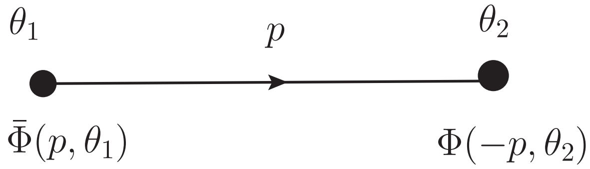

Figure 1: Scalar superfield propagator

to be from $\bar{\Phi}$ to Φ (see fig 1). Our sign conventions are such that the momenta leaving a vertex have a positive sign. The notation $D^2_{\theta_1,p}$ means that the operator depends on θ_1 and the momentum p, the explicit form for D^2 and some useful formulae are listed in §4.7.1. The gauge superfield propagator in momentum space is

$$\langle\Gamma^-(\theta_1,p)\Gamma^-(\theta_2,-p')\rangle = -\frac{8\pi}{\kappa}\frac{\delta^2(\theta_1-\theta_2)}{p_{--}}(2\pi)^3\delta^3(p-p') \tag{200}$$

where $p_{--} = -(p_1+ip_2) = -p_-$. Inserting the expansion (120) into the LHS of (200) and

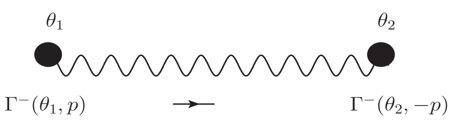

Figure 2: Gauge superfield propagator, the arrow indicates direction of momentum flow

matching powers of θ, we find in particular that

$$\langle A_+(p)A_3(-p')\rangle = \frac{4\pi i}{\kappa}\frac{1}{p_-}(2\pi)^3\delta^3(p-p') \ , \ \langle A_3(p)A_+(-p')\rangle = -\frac{4\pi i}{\kappa}\frac{1}{p_-}(2\pi)^3\delta^3(p-p') \ , \tag{201}$$

is in perfect agreement with the propagator of the gauge field in regular (non-supersymmetric) lightcone gauge (see Appendix A ,Eq A.7 of [46])

4.3.3 The all orders matter propagator

Constraints from supersymmetry The exact propagator of the matter superfield Φ enjoys invariance under supersymmetry transformations which implies that

$$(Q_{\theta_1,p} + Q_{\theta_2,-p})\langle\bar{\Phi}(\theta_1,p)\Phi(\theta_2,-p)\rangle = 0 \tag{202}$$

where the supergenerators $Q_{\theta_1,p}$ were defined in (123). This constraint is easily solved. Let the exact scalar propagator take the form

$$\langle\bar{\Phi}(p,\theta_1)\Phi(-p',\theta_2)\rangle = (2\pi)^3\delta^3(p-p')P(\theta_1,\theta_2,p) \ . \tag{203}$$

The condition (202) implies that the function P obeys the equation

$$\left[\frac{\partial}{\partial\theta_1^\alpha} + \frac{\partial}{\partial\theta_2^\alpha} - p_{\alpha\beta}(\theta_1^\beta - \theta_2^\beta)\right]P(\theta_1,\theta_2,p) = 0 \ . \tag{204}$$

The most general solution to (204) is

$$C_1(p^\mu)\exp(-\theta_1^\alpha p_{\alpha\beta}\theta_2^\beta) + C_2(p^\mu)\delta^2(\theta_1-\theta_2) \tag{205}$$

or equivalently

$$P(\theta_1,\theta_2,p) = \exp(-\theta_1^\alpha p_{\alpha\beta}\theta_2^\beta)\left(C_1(p^\mu) + C_2(p^\mu)\delta^2(\theta_1-\theta_2)\right) \tag{206}$$

[49] where $C_1(p^\mu)$ is an arbitrary function of p^μ of dimension m^{-2}, while $C_2(p^\mu)$ is another function of p^μ of dimension m^{-1}.

It is easily verified using the formulae (368) that the bare propagator (199) can be recast in the form (206) with

$$C_1 = \frac{1}{p^2 + m_0^2} \ , \ C_2 = \frac{m_0}{p^2 + m_0^2} \ . \tag{207}$$

In a similar manner supersymmetry constrains the terms quadratic in Φ and $\bar{\Phi}$ in the quantum effective action. In momentum space the most general supersymmetric quadratic effective action takes the form

$$S = -\int \frac{d^3p}{(2\pi)^3} d^2\theta \bar{\Phi}(p,\theta) \left(A(p)D^2 + B(p)\right) \Phi(-p,\theta) \tag{208}$$

$$= -\int \frac{d^3p}{(2\pi)^3} d^2\theta_1 d^2\theta_2 \bar{\Phi}(p,\theta_1) \exp(-\theta_1^\alpha p_{\alpha\beta}\theta_2^\beta)(A(p) + B(p)\delta^2(\theta_1 - \theta_2))\Phi(-p,\theta_2) \tag{209}$$

[50] The tree level quadratic action of our theory is clearly of the form (208) with $A(p) = 1$ and $B(p) = m_0$.

All orders two point function Let the exact 1PI quadratic effective action take the form

$$S_2 = \int \frac{d^3p}{(2\pi)^3} d^2\theta_1 d^2\theta_2 \bar{\Phi}(-p,\theta_1) \left(\exp(-\theta_1^\alpha p_{\alpha\beta}\theta_2^\beta) + m_0\delta^2(\theta_1 - \theta_2) + \Sigma(p,\theta_1,\theta_2)\right) \Phi(p,\theta_2) \ . \tag{210}$$

It follows from (208) that the supersymmetric self energy Σ is of the form

$$\Sigma(p,\theta_1,\theta_2) = C(p)\exp(-\theta_1^\alpha p_{\alpha\beta}\theta_2^\beta) + D(p)\delta^2(\theta_1 - \theta_2) \tag{211}$$

[49]The equivalence of (206) and (205) follows from the observation that $\theta^a A_{ab}\theta^b$ vanishes if A_{ab} is symmetric in a and b.

[50]In going from the first line to the second line of (208) we have integrated by parts and used the identity (368). See Appendix §4.7.1 for the expressions of superderivatives and operator D^2 in momentum space.

where $C(p)$ and $D(p)$ are as yet unknown functions of momenta.

Imitating the steps described in section 3 of [46], the self energy Σ defined in

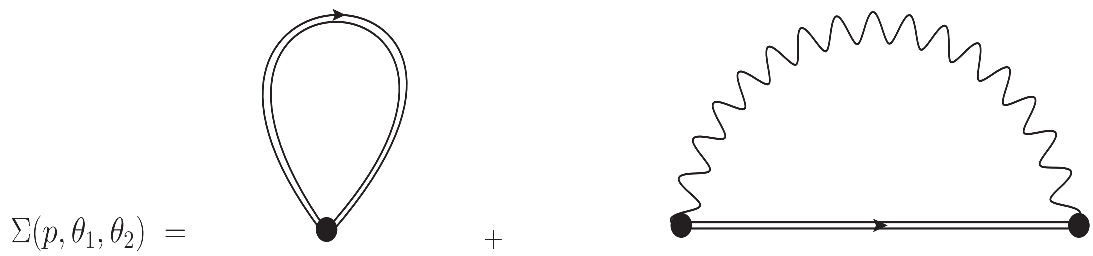

Figure 3: Integral equation for self energy

(210) may be shown to obey the integral equation [51]

$$
\begin{aligned}
\Sigma(p,\theta_1,\theta_2) = & \ 2\pi\lambda w \int \frac{d^3r}{(2\pi)^3}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2) \\
& - 2\pi\lambda \int \frac{d^3r}{(2\pi)^3} D_-^{\theta_2,-p} D_-^{\theta_1,p}\left(\frac{\delta^2(\theta_1-\theta_2)}{(p-r)_{--}}P(r,\theta_1,\theta_2)\right) \\
& + 2\pi\lambda \int \frac{d^3r}{(2\pi)^3}\frac{\delta^2(\theta_1-\theta_2)}{(p-r)_{--}} D_-^{\theta_1,r} D_-^{\theta_2,-r} P(r,\theta_1,\theta_2)
\end{aligned}
\tag{212}
$$

where $P(p,\theta_1,\theta_2)$ is the exact superfield propagator. [52] Note that the propagator P depends on Σ (in fact P is obtained by inverting quadratic term in effective action (210)). In other words Σ appears both on the LHS and RHS of (212); we need to solve this equation to determine Σ.

Using the equations (369), the second and third lines on the RHS if (212) may

[51]We work at leading order in the large N limit

[52]The first line in the RHS of (212) comes from the quartic interaction in Fig 3 while the second and third lines in (212) comes from the gaugesuperfield interaction in Fig 3 . Note that each vertex in the diagram corresponding to the gaugesuperfield interaction in Fig 3 contains one factor of D, resulting in the two powers of D in the second and third line of (212).

be considerably simplified (see Appendix §4.7.7) and we find

$$\begin{aligned}\Sigma(p,\theta_1,\theta_2) = 2\pi\lambda w &\int \frac{d^3r}{(2\pi)^3}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2) \\ &- 2\pi\lambda\int \frac{d^3r}{(2\pi)^3}\frac{p_{--}}{(p-r)_{--}}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2) \\ &+ 2\pi\lambda\int \frac{d^3r}{(2\pi)^3}\frac{r_{--}}{(p-r)_{--}}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2)\end{aligned} \tag{213}$$

Combining the second and third lines on the RHS of (213) we see that the factors of p_{--} and r_{--} cancel perfectly between the numerator and denominator, and (213) simplifies to

$$\Sigma(p,\theta_1,\theta_2) = 2\pi\lambda(w-1)\int \frac{d^3r}{(2\pi)^3}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2)\ . \tag{214}$$

Notice that the RHS of (214) is independent of p, so it follows that

$$\Sigma(p,\theta_1,\theta_2) = (m-m_0)\delta^2(\theta_1-\theta_2)$$

for some as yet undetermined constant m. It follows that the exact propagator P takes the form of the tree level propagator with m_0 replaced by m i.e.

$$P(p,\theta_1,\theta_2) = \frac{D^2-m}{p^2+m^2}\delta^2(\theta_1-\theta_2)\ . \tag{215}$$

Plugging (215) into (214) and simplifying we find the equation

$$m-m_0 = 2\pi\lambda(w-1)\int \frac{d^3r}{(2\pi)^3}\frac{1}{r^2+m^2}\ . \tag{216}$$

The integral on the RHS diverges. Regulating this divergence using dimensional regularization, we find that (216) reduces to

$$m-m_0 = \frac{\lambda|m|}{2}(1-w) \tag{217}$$

and so

$$m = \frac{2m_0}{2 + (-1 + w)\lambda \operatorname{Sgn(m)}} \ . \tag{218}$$

Let us summarize. The *exact* 1PI quadratic effective action for the Φ superfield has the same form as the tree level effective action but with the bare mass m_0 replaced by the exact mass m given in (218). [53] As explained in §4.2.2 the exact mass (218) is duality invariant.

Note also that the $\mathcal{N} = 2$ point, $w = 1$ there is no renormalization of the mass, and the bare propagator is exact and the bare mass (which equals the pole mass) is itself duality invariant.

4.3.4 Constraints from supersymmetry on the offshell four point function

Much as with the two point function, the offshell four point function of matter superfields is constrained by the supersymmetric Ward identities. Let us define

$$\begin{aligned}\langle\bar{\Phi}((p+q+\frac{l}{4}),\theta_1)\Phi(-p+\frac{l}{4},\theta_2)\Phi(-(k+q)+\frac{l}{4},\theta_3)\bar{\Phi}(k+\frac{l}{4},\theta_4)\rangle \\ = (2\pi)^3\delta(l)V(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k).\end{aligned} \tag{219}$$

It follows from the invariance under supersymmetry that

$$(Q_{\theta_1,p+q} + Q_{\theta_2,-p} + Q_{\theta_3,-k-q} + Q_{\theta_4,k})V(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k) = 0 \ . \tag{220}$$

[53] Note that propagator for the fermion in the superfield Φ is the usual propagator for a relativistic fermion of mass m. Recall, of course, that the propagator of Φ is not gauge invariant, and so its form depends on the gauge used in the computation. If we had carried out all computations in Wess-Zumino gauge (which breaks offshell supersymmetry) we would have found the much more complicated expression for the fermion propagator reported in section 2.1 of [58]. Note however that the gauge invariant physical pole mass m of (218) agrees perfectly with the pole mass (reported in eq 1.6 of [58]) of the complicated propagator of [58]. The agreement of gauge invariant quantities in these rather different computations constitutes a nontrivial consistency check of the computations presented in this subsection.

The general solution to (220) is easily obtained (see Appendix §4.7.8). Defining

$$\begin{aligned}
X &= \sum_{i=1}^{4} \theta_i \ , \\
X_{12} &= \theta_1 - \theta_2 \ , \\
X_{13} &= \theta_1 - \theta_3 \ , \\
X_{43} &= \theta_4 - \theta_3 \ . \qquad (221)
\end{aligned}$$

we find

$$V = \exp\left(\frac{1}{4}X.(p.X_{12} + q.X_{13} + k.X_{43})\right)F(X_{12}, X_{13}, X_{43}, p, q, k) \ . \qquad (222)$$

where F is an unconstrained function of its arguments. In other words supersymmetry fixes the transformation of V under a uniform shift of all θ parameters $\theta_i \to \theta_i + \gamma$. (for $i = 1 \ldots 4$ where γ is a constant Grassman parameter). The undetermined function F is a function of shift invariant combinations of the four θ_i.

Let us now turn to the structure of the exact 1PI effective action for scalar superfields in our theory. The most general effective action consistent with global $U(N)$ invariance and supersymmetry takes the form

$$\begin{aligned}
S_4 = \frac{1}{2}\int & \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}d^2\theta_1 d^2\theta_2 d^2\theta_3 d^2\theta_4 \\
& \big(V(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k)\Phi_m(-(p+q),\theta_1)\bar{\Phi}^m(p,\theta_2)\bar{\Phi}^n(k+q,\theta_3)\Phi_n(-k,\theta_4)\big) \ . \qquad (223)
\end{aligned}$$

It follows from the definition (223) that the function V may be taken to be invariant under the Z_2 symmetry

$$\begin{aligned}
& p \to k+q, k \to p+q, q \to -q \ , \\
& \theta_1 \to \theta_4, \theta_2 \to \theta_3, \theta_3 \to \theta_2, \theta_4 \to \theta_1 \ . \qquad (224)
\end{aligned}$$

As in the case of two point functions, it is easily demonstrated that the invariance of this action under supersymmetry constraints the coefficient function V that appears in (223) to obey the equation (220). As we have already explained above, the most general solution to this equation is given in equation (222) for a general shift invariant function F.

4.3.5 An integral equation for the offshell four point function

The coefficient function V of the quartic term of the exact IPI effective action may be shown to obey the integral equation (see Fig 4 for a diagrammatic representation of this equation)

$$\begin{aligned} V(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k) &= V_0(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k) \\ &+ \int \frac{d^3r}{(2\pi)^3} d^2\theta_a d^2\theta_b d^2\theta_A d^2\theta_B \Big(N V_0(\theta_1,\theta_2,\theta_a,\theta_b,p,q,r) \\ &P(r+q,\theta_a,\theta_A) P(r,\theta_B,\theta_b) V(\theta_A,\theta_B,\theta_3,\theta_4,r,q,k)\Big) \end{aligned} \tag{225}$$

In (225) V_0 is the tree level contribution to V. V_0 receives contributions from the two diagrams depicted in Fig 4. The explicit evaluation of V_0 is a straightforward exercise and we find (see Appendix §4.7.8 for details)

$$\begin{aligned} V_0(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k) = &\exp\left(\frac{1}{4}X.(p.X_{12}+q.X_{13}+k.X_{43})\right) \\ &\Big(-\frac{i\pi w}{\kappa} X^-_{12}X^+_{12}X^-_{13}X^+_{13}X^-_{43}X^+_{43} \\ &- \frac{4\pi i}{\kappa(p-k)_{--}} X^+_{12}X^+_{13}X^+_{43}(X^-_{12}+X^-_{34})\Big) . \end{aligned} \tag{226}$$

In the above, the first term in the bracket is the delta function from the quartic interaction, the second term is from the tree diagram due to the gauge superfield exchange computed in §4.7.8.

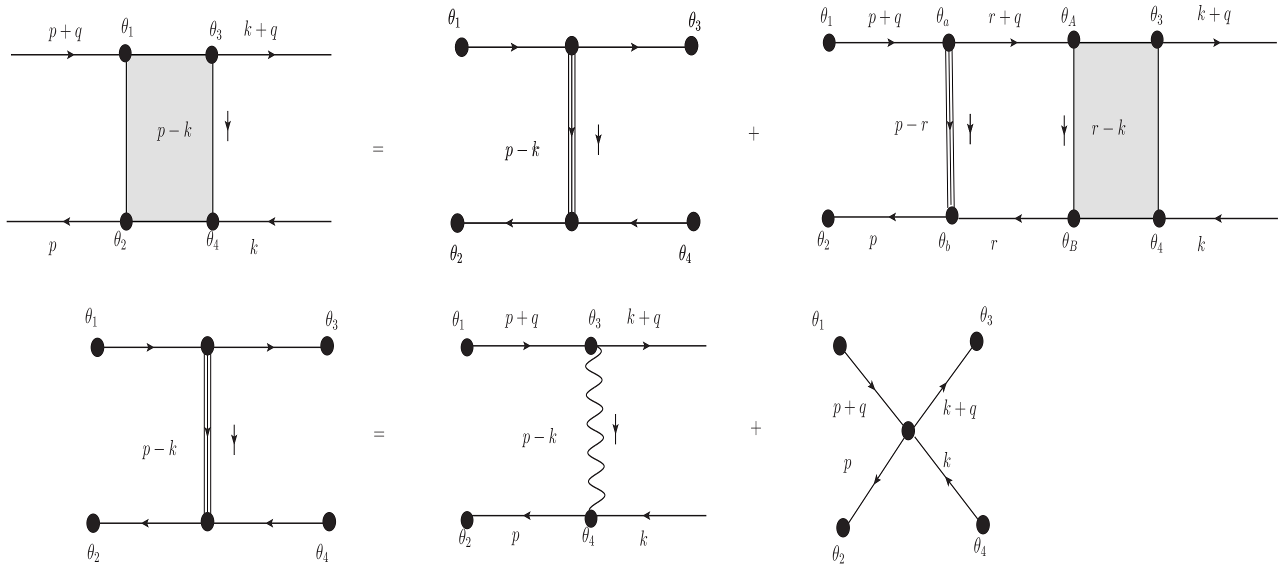

Figure 4: The diagrams in the first line pictorially represents the Schwinger-Dyson equation for offshell four point function (see (225)). The second line represents the tree level contributions from the gauge superfield interaction and the quartic interactions.

We now turn to the evaluation of the coefficient V in the exact 1PI effective action. There are 2^6 linearly independent functions of the six independent shift invariant Grassman variables $X_{12}^{\pm}$, $X_{13}^{\pm}$ and $X_{43}^{\pm}$. Consequently the most general V consistent with supersymmetry is parameterized by 64 unknown functions of the three independent momenta. V (and so F) is necessarily an even function of these variables. It follows that the most general function F can be parameterized in terms of 32 bosonic functions of p, k and q. In principle one could insert the most general supersymmetric F into the integral equation (225) and equate equal powers of θ_i on the two sides of (225) to obtain 32 coupled integral equations for the 32 unknown complex valued functions. One could, then, attempt to solve this system of equations. This procedure would obviously be very complicated and difficult to implement in practice. Focusing on the special kinematics $q^{\pm} = 0$ we were able to shortcircuit this laborious process, in a manner we now describe.

After a little playing around we were able to demonstrate that V of the form [54]

$$V(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k) = \exp\left(\frac{1}{4}X.(p.X_{12}+q.X_{13}+k.X_{43})\right)F(X_{12},X_{13},X_{43},p,q,k)$$
$$F(X_{12},X_{13},X_{43},p,q,k) = \begin{array}{l} X_{12}^+X_{43}^+\Big(A(p,k,q)X_{12}^-X_{43}^-X_{13}^+X_{13}^- + B(p,k,q)X_{12}^-X_{43}^- \\ \qquad + C(p,k,q)X_{12}^-X_{13}^+ + D(p,k,q)X_{13}^+X_{43}^-\Big) \end{array},$$
$$(227)$$

is closed under the multiplication rule induced by the RHS of (225) (see Appendix §4.7.8). Plugging in the general form of V (227) in the integral equation (225) and performing the grassmann integration, (225) turns into to the following integral equations for the coefficient functions A, B, C and D:

$$\begin{aligned} &A(p,k,q) + \frac{2\pi i w}{\kappa} \\ &+ i\pi\lambda \int \frac{d^3r_E}{(2\pi)^3} \frac{2A(q_3p_- + 2(q_3-im)r_-) + (q_3r_- + 2imp_-)(2Bq_3 + Ck_-) - Dr_-(q_3p_- + 2imr_-)}{(r^2+m^2)((r+q)^2+m^2)(p-r)_-} \\ &- i\pi\lambda w \int \frac{d^3r_E}{(2\pi)^3} \frac{4iAm + 2Bq_3^2 + Cq_3k_- + 2D(q_3+im)r_-}{(r^2+m^2)((r+q)^2+m^2)} = 0 \end{aligned} \tag{228}$$

$$\begin{aligned} B(p,k,q) &+ i\pi\lambda \int \frac{d^3r_E}{(2\pi)^3} \frac{2A(p+r)_- + 4B(q_3r_- + im(p-r)_-) - Ck_-(p+r)_- - Dr_-(p-3r)_-}{(r^2+m^2)((r+q)^2+m^2)(p-r)_-} \\ &- i\pi\lambda w \int \frac{d^3r_E}{(2\pi)^3} \frac{2A + 4imB - Ck_- - Dr_-}{(r^2+m^2)((r+q)^2+m^2)} = 0 \end{aligned} \tag{229}$$

$$\begin{aligned} C(p,k,q) - \frac{4\pi i}{\kappa(p-k)_-} &+ i\pi\lambda \int \frac{d^3r_E}{(2\pi)^3} \frac{2C\big(q_3(p+3r)_- + 2im(p-r)_-\big)}{(r^2+m^2)((r+q)^2+m^2)(p-r)_-} \\ &- i\pi\lambda w \int \frac{d^3r_E}{(2\pi)^3} \frac{2C(q_3+2im)}{(r^2+m^2)((r+q)^2+m^2)} = 0 \end{aligned} \tag{230}$$

[54]The variables X, X_{ij} are defined in terms of θ_i in (221).

$$
\begin{aligned}
&D(p,k,q) - \frac{4\pi i}{\kappa(p-k)_-} \\
&+ i\pi\lambda \int \frac{d^3 r_E}{(2\pi)^3} \frac{-A(4q_3 - 8im) + (q_3 - 2im)(4Bq_3 + 2Ck_-) + 2D(3q_3 + 2im)r_-}{(r^2+m^2)((r+q)^2+m^2)(p-r)_-} = 0 \ .
\end{aligned}
\tag{231}
$$

We will sometimes find it useful to view the four integral equations above as a single integral equation for a four dimensional column vector E whose components are the functions A, B, C, D, i.e.

$$
E(p,k,q) = \begin{pmatrix} A(p,k,q) \\ B(p,k,q) \\ C(p,k,q) \\ D(p,k,q) \end{pmatrix} \ . \tag{232}
$$

The integral equations take the schematic form

$$
E = R + IE \tag{233}
$$

where R is a 4 column of functions and I is a matrix of integral operators acting on E. The integral equation (233) may be converted into a differential equation by differentiating both sides of (233) w.r.t p_+. Using (476) and performing all d^3r integrals (using (474) for the integral over r_3) we obtain the differential equations

$$
\partial_{p_+} E(p,k,q) = S(p,k,q) + H(p,k_-,q)E(p,k,q) \ , \tag{234}
$$

where

$$
S(p,k,q) = -\frac{8i\pi^2}{\kappa}\delta^2((p-k)_-,(p-k)_+) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \tag{235}
$$

$$H(p,k_-,q_3)=\frac{1}{a(p_s,q_3)}\begin{pmatrix}(6q_3-4im)p_- & 2q_3(2im+q_3)p_- & (2im+q_3)k_-p_- & -(2im+q_3)p_-^2\\ 4p_- & 4q_3p_- & -2k_-p_- & 2p_-^2\\ 0 & 0 & 8q_3p_- & 0\\ 8im-4q_3 & 4q_3(q_3-2im) & 2(q_3-2im)k_- & (4im+6q_3)p_-\end{pmatrix} \tag{236}$$

and

$$a(p_s,q_3)=\frac{\sqrt{m^2+p_s^2}\,(4m^2+q_3^2+4p_s^2)}{2\pi}\ . \tag{237}$$

As we have explained above, the exact vertex V enjoys invariance under the $\mathbb{Z}_2$ transformation (224). In terms of the functions A, B, C, D, the $\mathbb{Z}_2$ action is given by

$$E(p,k,q)=TE(k,p,-q)\ , \tag{238}$$

where

$$T=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&-1\\0&0&-1&0\end{pmatrix}\ . \tag{239}$$

The differential equations (234) do not manifestly respect the invariance (238). In fact in Appendix §4.7.8 we have demonstrated that the differential equations (234) admit solutions that enjoy the invariance (238) if and only if the following consistency condition is obeyed:

$$[H(p,k_-,q),TH(k,p_-,-q)T]=0\ . \tag{240}$$

In the same Appendix we have also explicitly verified that this integrability condition is in fact obeyed; this is a consistency check on (234) and indirectly on the underlying integral equations.

4.3.6 Explicit solution for the offshell four point function

In this subsection, we solve the system of integral equations for the unknown functions A, B, C, D presented in the previous subsection. We propose the ansatz

$$\begin{aligned}
A(p,k,q) &= A_1(p_s,k_s,q_3) + \frac{A_2(p_s,k_s,q_3)k_-}{(p-k)_-} \ , \\
B(p,k,q) &= B_1(p_s,k_s,q_3) + \frac{B_2(p_s,k_s,q_3)k_-}{(p-k)_-} \ , \\
C(p,k,q) &= -\frac{C_2(p_s,k_s,q_3) - C_1(p_s,k_s,q_3)k_+p_-}{(p-k)_-} \ , \\
D(p,k,q) &= -\frac{D_2(p_s,k_s,q_3) - D_1(p_s,k_s,q_3)k_-p_+}{(p-k)_-} \ .
\end{aligned} \tag{241}$$

Our ansatz (241) [55] fixes the solution in terms of 8 unknown functions of p_s, k_s and q_3.

Plugging the ansatz (241) into the integral equations (228)-(231), one can do the angle and r_3 integrals (using the formulae (475) and (474) respectively) leaving only the r_s integral to be performed. Differentiating this expression w.r.t. to p_s turns out to kill the r_s integral yielding differential equations in p_s for the eight equations above. [56] The resulting differential equations turn out to be exactly solvable. Assuming that the solution respects the symmetry (238), it turns out to be given in terms of two unknown functions of k_s and q_3. These can be thought of as the integration constants that are not fixed by the symmetry requirement (238). Plugging the solutions back into the integral equations we were able to determine these two integration functions of k_s and q_3 completely. We now report our results.

[55] We were able to arrive at this ansatz by first explicitly computing the one loop answer and observing the functional forms. Moreover, in previous work a very similar ansatz was already used to solve the integral equations for the fermions (see Appendix F of [61]).

[56] Another way to obtain these differential equations is to plug the ansatz (241) directly into the differential equations (234).

The solutions for A and B are

$$
\begin{aligned}
A_1(p_s,k_s,q_3) =& e^{-2i\lambda \tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}}\Bigg(G_1(k_s,q_3) \\
&+ \frac{2\pi(w-1)(2m-iq_3)e^{2i\lambda\left(\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}+\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}\right)}}{\kappa(e^{\frac{i\pi\lambda q_3}{|q_3|}}(q_3(w+3)-2im(w-1))+i(w-1)(2m+iq_3)e^{2i\lambda\tan^{-1}\frac{2|m|}{q_3}})}\Bigg) \ , \\
A_2(p_s,k_s,q_3) =& e^{-2i\lambda\tan^{-1}\left(\frac{2\sqrt{m^2+p_s^2}}{q_3}\right)}G_2(k_s,q_3) \ ,
\end{aligned}
$$

$$
\begin{aligned}
B_1(p_s,k_s,q_3) =& \frac{2\pi A_1(p_s,k_s,q_3)}{q_3} \\
&+ \frac{2\pi}{b_1 b_2}\Bigg(-i(w-1)^2(4m^2+q_3^2)e^{i\lambda\left(\frac{\pi q_3}{|q_3|}-2\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}+4\tan^{-1}\frac{2|m|}{q_3}\right)} \\
&+ i(w-1)^2(-4m^2+8imq_3+3q_3^2)e^{i\lambda\left(\frac{\pi q_3}{|q_3|}+2\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}\right)} \\
&- 8iq_3^2(w+1)e^{i\lambda\left(\frac{\pi q_3}{|q_3|}+2(\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}-\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}+\tan^{-1}\frac{2|m|}{q_3})\right)} \\
&+ (w-1)(q_3+2im)(2m(w-1)+iq_3(w+3)) + e^{2i\lambda(\frac{\pi q_3}{|q_3|}-\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}+\tan^{-1}\frac{2|m|}{q_3})} \\
&+ (w-1)(2m-3iq_3)(q_3(w+3)+2im(w-1)) + e^{2i\lambda\left(\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}+\tan^{-1}\frac{2|m|}{q_3}\right)}\Bigg) \ ,
\end{aligned}
$$

$$
\begin{aligned}
B_2(p_s,k_s,q_3) =& \frac{A_2(p_s,k_s,q_3)}{q_3} \ , \\
G_1(k_s,q_3) =& -\frac{2\pi}{\kappa}\frac{1}{g_1}\Bigg(-8iq_3^2(w+1)e^{i\lambda\left(\frac{\pi q_3}{|q_3|}+2(\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}+\tan^{-1}\frac{2|m|}{q_3})\right)} \\
&+ i(w-1)^2(q_3-2im)^2 e^{i\lambda\left(\frac{\pi q_3}{|q_3|}+4\tan^{-1}\frac{2|m|}{q_3}\right)} \\
&- (w-1)(q_3-2im)(2m(w-1)+iq_3(w+3))e^{2i\lambda\left(\frac{\pi q_3}{|q_3|}+\tan^{-1}\frac{2|m|}{q_3}\right)}\Bigg) \ , \\
G_2(k_s,q_3) =& 0 \ , \qquad (242)
\end{aligned}
$$

where we have defined some parameters as given below for ease of presentation.

$$
\begin{aligned}
g_1 =&(w-1)(q_3+2im)e^{\frac{2i\pi\lambda q_3}{|q_3|}}(q_3(w+3)-2im(w-1)) \ , \\
&+(w-1)(4m^2(w-1)-8imq_3+q_3^2(w+3))e^{4i\lambda\tan^{-1}\frac{2|m|}{q_3}} \ , \\
&-2(4m^2(w-1)^2+q_3^2(w^2+2w+5))e^{i\lambda(\frac{\pi q_3}{|q_3|}+2\tan^{-1}\frac{2|m|}{q_3})} \ , \\
b_1 =&\kappa q_3((w-1)(q_3+2im)e^{\frac{i\pi\lambda q_3}{|q_3|}}+(-q_3(w+3)-2im(w-1))e^{2i\lambda\tan^{-1}\frac{2|m|}{q_3}}) \ , \\
b_2 =&e^{\frac{i\pi\lambda q_3}{|q_3|}}(q_3(w+3)-2im(w-1))+i(w-1)(2m+iq_3)e^{2i\lambda\tan^{-1}\frac{2|m|}{q_3}} \ ,
\end{aligned}
\tag{243}
$$

The solutions for C and D are

$$
\begin{aligned}
C_1(p_s,k_s,q_3) =&\frac{4\pi(q_3+2im)(e^{2i\lambda\tan^{-1}\frac{2|m|}{q_3}}-e^{2i\lambda\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}})e^{i\lambda(\frac{\pi q_3}{|q_3|}-2\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3})}}{\kappa k_s^2(i(q_3+2im)e^{\frac{i\pi\lambda q_3}{|q_3|}}+(2m-iq_3\left(\frac{w+3}{w-1}\right))e^{2i\lambda\tan^{-1}\frac{2|m|}{q_3}})} \ , \\
C_2(p_s,k_s,q_3) =&\frac{4\pi e^{2i\lambda(\tan^{-1}\frac{2|m|}{q_3}-\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3})}((q_3+2im)e^{\frac{i\pi\lambda q_3}{|q_3|}}-(q_3\left(\frac{w+3}{w-1}\right)+2im)e^{2i\lambda\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}})}{\kappa(i(q_3+2im)e^{\frac{i\pi\lambda q_3}{|q_3|}}+(2m-iq_3\left(\frac{w+3}{w-1}\right))e^{2i\lambda\tan^{-1}\frac{2|m|}{q_3}})} \ ,
\end{aligned}
$$

$$
\begin{aligned}
D_1(p_s,k_s,q_3) =&C_1(k_s,p_s,-q_3) \ , \\
D_2(p_s,k_s,q_3) =&C_2(k_s,p_s,-q_3) \ .
\end{aligned}
\tag{244}
$$

It is straightforward to show that the above solutions satisfy the various symmetry requirements that follow from (238).

Although the solutions (242) and (244) are quite complicated, a drastic simplification occurs at the $\mathcal{N}=2$ point $w=1$

$$
\begin{aligned}
A &= -\frac{2i\pi e^{2i\lambda\left(\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}-\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}\right)}}{\kappa}\ , \\
B &= 0\ , \\
C &= -\frac{4i\pi e^{2i\lambda\left(\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}-\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}\right)}}{\kappa(k-p)_-}\ , \\
D &= -\frac{4i\pi e^{2i\lambda\left(\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}-\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}\right)}}{\kappa(k-p)_-}\ .
\end{aligned} \tag{245}
$$

It is satisfying that the complicated results of the general $\mathcal{N}=1$ theory collapse to an extremely simple form at the $\mathcal{N}=2$ point.

4.3.7 Onshell limit and the S matrix

The explicit solution for the functions A, B, C and D, presented in the previous subsection, completely determine V in (223), and so the quadratic part of the exact (large N) IPI effective action. The most general 2×2 S matrix may now be obtained from (223) as follows. We simply substitute the onshell expressions

$$
\begin{aligned}
\Phi(p,\theta) = (2\pi)\delta(p^2+m^2)\Big[&\theta(p^0)\Big(a(\mathbf{p})(1+m\theta^2)+\theta^\alpha u_\alpha(\mathbf{p})\alpha(\mathbf{p})\Big) \\
&+\theta(-p^0)\Big(a^{c\dagger}(-\mathbf{p})(1+m\theta^2)+\theta^\alpha v_\alpha(-\mathbf{p})\alpha^{c\dagger}(-\mathbf{p})\Big)\Big]
\end{aligned} \tag{246}
$$

into (223) (here a and α are the effectively free oscillators that create and destroy particles at very early or very late times; these oscillators obey the commutation relations (147)). Performing the integrals over θ^α reduces (223) to a quartic form (let us call it L) in bosonic and fermionic oscillators. The S matrix is obtained by sandwiching the resultant expression between the appropriate in and out states, and evaluating the resulting matrix elements using the commutation relations (147).

It may be verified that the quartic form in oscillators takes the form [57]

$$L = \sum_{\phi_i=0,\pi} \int \prod_{i=1}^{4} d\theta_i \frac{d^3p_i}{((2\pi)^3)^4} \delta(p_i^2+m^2) S_M(p_1,\phi_1,\theta_1,p_2,\phi_2,\theta_2,p_3,\phi_3,\theta_3,p_4,\phi_4,\theta_4)$$
$$\left(\delta_{\phi_i,0}\theta(p_i^0)A(p_i,\phi_i,\theta_i)+\delta_{\phi_i,\pi}\theta(-p_i^0)\tilde{A}(-p_i,\phi_i,\theta_i)\right)(2\pi)^3\delta^3(p_1+p_2+p_3+p_4)$$

where

$$A(p_i,\phi_i,\theta_i) = a(\mathbf{p}_i) + \alpha(\mathbf{p}_i)e^{-\frac{i\phi_i}{2}}\theta_i \ ,$$
$$\tilde{A}(p_i,\phi_i,\theta_i) = a^\dagger(\mathbf{p}_i) + e^{-\frac{i\phi_i}{2}}\theta_i\alpha^\dagger(\mathbf{p}_i) \ , \tag{247}$$

where the one component fermionic variables θ_i are the fermionic variables that parameterize onshell superspace (see §4.2.4) and the master formula is defined in (167). Note that the phase variables ϕ_i are summed over two values 0 and π; the symbol $\delta_{\phi,0}$ is unity when $\phi = 0$ but zero when $\phi = \pi$ and $\delta_{\phi,\pi}$ has an analogous definition. (247) compactly identifies the coefficient of every quartic form in oscillators. For instance it asserts that the coefficient of $a_1a_2a_3^\dagger a_4^\dagger$ is the S matrix for scattering bosons with momentum p_1, p_2 to bosons with momenta p_3, p_4, while the the coefficient of $\alpha_2\alpha_4a_1^\dagger a_3^\dagger$ is minus the S matrix for scattering fermions with momentum p_2, p_4 to bosons with momentum p_1, p_3, etc.

We can use the δ function in (247) to perform the integral over one of the four momenta; the integral over the remaining momenta may be recast as an integral over the momenta p k and q employed in the previous section; specifically (see Fig 4)

$$p_1 = p+q \ , \ p_2 = -k-q \ , \ p_3 = -p \ , \ p_4 = k \ . \tag{248}$$

From the explicit results we get by substituting (246) into (223) we can read off all S matrices at $q_\pm = 0$.

To start with, let us restrict our attention to the bosonic sector. From direct

[57]The definition of A and $\tilde{A}$ reduces to the definition (149) for $\phi = 0$. While for $\phi = \pi$, it reduces to (149) together with the identification $\theta \to i\theta$. With these definitions $\tilde{A} = A^\dagger$ both at $\phi = 0, \pi$.

computation [58] we find that in this sector (247) reduces to

$$L_B = \sum_{\phi_i=0,\pi} \int \frac{d^3p}{(2\pi)^3} \frac{dq_3}{(2\pi)} \frac{d^3k}{(2\pi)^3} \delta((p+q)^2+m^2)\delta((k+q)^2+m^2)$$
$$\delta(p^2+m^2)\delta(k^2+m^2)\mathcal{T}_B(p,k,q_3)$$
$$\big(\delta_{\phi_i,0}\theta(p^0)a(\mathbf{p}+\mathbf{q}) + \delta_{\phi_i,\pi}\theta(-p^0)a^\dagger(-\mathbf{p}-\mathbf{q})\big)$$
$$\big(\delta_{\phi_i,0}\theta(-k^0)a(-\mathbf{k}-\mathbf{q}) + \delta_{\phi_i,\pi}\theta(k^0)a^\dagger(\mathbf{k}+\mathbf{q})\big)$$
$$\big(\delta_{\phi_i,0}\theta(-p^0)a(-\mathbf{p}) + \delta_{\phi_i,\pi}\theta(p^0)a^\dagger(\mathbf{p})\big)$$
$$\big(\delta_{\phi_i,0}\theta(k^0)a(\mathbf{k}) + \delta_{\phi_i,\pi}\theta(-k^0)a^\dagger(-\mathbf{k})\big) \tag{250}$$

while for the purely fermionic sector (247) reduces to

$$L_F = \sum_{\phi_i=0,\pi} \int \frac{d^3p}{(2\pi)^3} \frac{dq_3}{(2\pi)} \frac{d^3k}{(2\pi)^3} \delta((p+q)^2+m^2)\delta((k+q)^2+m^2)$$
$$\delta(p^2+m^2)\delta(k^2+m^2)\mathcal{T}_F(p,k,q_3)$$
$$\big(\delta_{\phi_i,0}\theta(p^0)\alpha(\mathbf{p}+\mathbf{q}) + \delta_{\phi_i,\pi}\theta(-p^0)\alpha^\dagger(-\mathbf{p}-\mathbf{q})\big)$$
$$\big(\delta_{\phi_i,0}\theta(-k^0)\alpha(-\mathbf{k}-\mathbf{q}) + \delta_{\phi_i,\pi}\theta(k^0)\alpha^\dagger(\mathbf{k}+\mathbf{q})\big)$$
$$\big(\delta_{\phi_i,0}\theta(-p^0)\alpha(-\mathbf{p}) + \delta_{\phi_i,\pi}\theta(p^0)\alpha^\dagger(\mathbf{p})\big)$$
$$\big(\delta_{\phi_i,0}\theta(k^0)\alpha(\mathbf{k}) + \delta_{\phi_i,\pi}\theta(-k^0)\alpha^\dagger(-\mathbf{k})\big) \tag{251}$$

[58] Note that the onshell delta functions in the equations (250) and (251) ensure that

$$p_3 = k_3 = -\frac{q_3}{2}\ ,\ p_s = k_s\ ,\ k_s = \frac{i}{2}\sqrt{q_3^2+4m^2}\ . \tag{249}$$

where [59]

$$\mathcal{T}_B = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2} + J_B(q,\lambda) \ , \quad (252)$$

$$\mathcal{T}_F = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2} + J_F(q,\lambda) \ , \quad (253)$$

where the J functions[60] are

$$\begin{aligned} J_B(q,\lambda) &= \frac{4\pi q}{\kappa}\frac{N_1N_2+M_1}{D_1D_2} \ , \\ J_F(q,\lambda) &= \frac{4\pi q}{\kappa}\frac{N_1N_2+M_2}{D_1D_2} \ , \end{aligned} \quad (254)$$

[59]Our actual computations gave the functions J_B and J_F in the special case $q^\pm = 0$. We obtained the answers reported in (252) and (253) by determining the unique covariant expression that reduce to our answers for our special kinematics. While this procedure is completely correct (with standard conventions) for J_B, it is a bit inaccurate for J_F. The reason for this is that J_F is Lorentz invariant only upto a phase. As we have explained around (170), the phase of J_F depends on the (arbitrary) phase of the u and v spinors of the particles in the scattering process. The accurate answer is obtained by covariantizing the unambiguous $\mathcal{S}_f$ defined in (171). $\mathcal{S}_F$ is obtained by multiplying this result by the quadrilinear term in spinor wavefunctions as defined in (182). This gives an explicit but cumbersome expression for $\mathcal{S}_F$, which agrees with the result presented above upto an overall convention dependent phase. This phase vanishes near identity scattering (where it could have interfered with identity), and we have dealt with this issue carefully in deriving the unitarity equation. In the equation above we have simply ignored the phase in order to aid readability of formulas.

[60]The J functions are quite complicated and can be written in many avatars. In this section we have written the most elegant form of the J function, the other forms are reported in Appendix §4.7.9

where

$$
\begin{aligned}
N_1 =& \left(\left(\frac{2|m| + iq}{2|m| - iq} \right)^{-\lambda} (w-1)(2m+iq) + (w-1)(2m-iq) \right) \ , \\
N_2 =& \left(\left(\frac{2|m| + iq}{2|m| - iq} \right)^{-\lambda} (q(w+3) + 2im(w-1)) + (q(w+3) - 2im(w-1)) \right) \ , \\
M_1 =& -8mq((w+3)(w-1) - 4w) \left(\frac{2|m| + iq}{2|m| - iq} \right)^{-\lambda} \ , \\
M_2 =& -8mq(1+w)^2 \left(\frac{2|m| + iq}{2|m| - iq} \right)^{-\lambda} \ , \\
D_1 =& \left(i \left(\frac{2|m| + iq}{2|m| - iq} \right)^{-\lambda} (w-1)(2m+iq) - 2im(w-1) + q(w+3) \right) \ , \\
D_2 =& \left(\left(\frac{2|m| + iq}{2|m| - iq} \right)^{-\lambda} (-q(w+3) - 2im(w-1)) + (w-1)(q+2im) \right) \ . \qquad (255)
\end{aligned}
$$

The equations (252) and (253) capture purely bosonic and purely fermionic S matrices in all channels (particle-particle scattering in the symmetric and antisymmetric channels as well as particle-antiparticle scattering in the adjoint channel) restricted to the kinematics $q_\pm = 0$. Recall that supersymmetry (see §4.2.4) determines all other scattering amplitudes in terms of the four boson and four fermion amplitudes, so the formulae (252) and (253) are sufficient to determine all $2 \to 2$ scattering processes restricted to our special kinematics. In other words S_M in (247) is completely determined by (252) and (253) together with (167).

4.3.8 Duality of the S matrix

Under the duality transformation (see (130))

$$
w' = \frac{3-w}{w+1}, \lambda' = \lambda - \mathrm{sgn}(\lambda), m' = -m, \kappa' = -\kappa \qquad (256)
$$

we have verified that

$$
\begin{aligned}
J_B(q, \kappa', \lambda', w', m') &= -J_F(q, \kappa, \lambda, w, m) \ , \\
J_F(q, \kappa', \lambda', w', m') &= -J_B(q, \kappa, \lambda, w, m) \ .
\end{aligned} \tag{257}
$$

provided (133) is respected. In other words duality maps the purely bosonic and purely fermionic S matrices into one another. It follows that (252) and (253) map to each other under duality upto a phase. As we have explained in subsection §4.2.5, this result is sufficient to guarantee that the full S matrix (including, for instance, the S matrix for Bose-Fermi scattering) is invariant under duality, once we interchange bosons with fermions.

4.3.9 S matrices in various channels

In this subsection we explicitly list the purely bosonic and purely fermionic S matrices in every channel, as functions of the Mandelstam variables of that channel. These results are, of course, easily extracted from (250) and (251). There is a slight subtlety here; even though (252) and (253) are manifestly Lorentz invariant, it is not possible to write them entirely in terms of Mandelstam variables. [61] This is because (as was noted in [61]) $2+1$ dimensional kinematics allows for an additional Z_2 valued invariant (in addition to the Mandelstam variables)

$$
E(q, p-k, p+k) = \text{Sign}\left(\epsilon_{\mu\nu\rho} q^\mu (p-k)^\nu (p+k)^\rho\right) \ . \tag{259}
$$

[62] The sign of the first term in (252) and (253) is given by this new invariant as we will see in more detail below.

[61] We define the Mandelstam variables as usual

$$
s = -(p_1 + p_2)^2 \ , \ t = -(p_1 - p_3)^2 \ , \ u = -(p_1 - p_4)^2 \ . \tag{258}
$$

[62] Note, in particular that the expression (259) changes sign under the interchange of any two vectors.

U channel For particle-particle scattering

$$P_i(p_1) + P_j(p_2) \to P_i(p_3) + P_j(p_4)$$

we have the direct scattering referred to as the U_d (Symmetric) channel. [63] Our momenta assignments (see LHS of fig 4) are

$$p_1 = p + q \ , \ p_2 = k \ , \ p_3 = p \ , \ p_4 = k + q \ . \tag{260}$$

In terms of the Mandelstam variables

$$s = -(p + q + k)^2 \ , \ t = -q^2 \ , \ u = -(p - k)^2 \ , \tag{261}$$

the U_d channel T matrices for the boson-boson and fermion-fermion scattering are

$$\begin{aligned} \mathcal{T}_B^{U_d} &= E(q, p - k, p + k)\frac{4\pi i}{k}\sqrt{\frac{ts}{u}} + J_B(\sqrt{-t}, \lambda) \ , \\ \mathcal{T}_F^{U_d} &= E(q, p - k, p + k)\frac{4\pi i}{k}\sqrt{\frac{ts}{u}} + J_F(\sqrt{-t}, \lambda) \ . \end{aligned} \tag{262}$$

For the exchange scattering, referred to as the U_e (Antisymmetric) channel the momenta assignments are (see LHS of fig 4)

$$p_1 = k \ , \ p_2 = p + q \ , \ p_3 = p \ , \ p_4 = k + q \ . \tag{263}$$

In terms of the Mandelstam variables

$$s = -(p + q + k)^2 \ , \ t = -(p - k)^2 \ , \ u = -q^2 \ , \tag{264}$$

[63] We adopt the terminology of [61] in specifying scattering channels; we refer the reader to that chapter for a more complete definition of the U_d, U_e, T, and S channels that we will repeatedly refer to below.

the U$_e$ channel T matrices for the boson-boson and fermion-fermion scattering are

$$\mathcal{T}_B^{U_e} = E(q, p-k, p+k)\frac{4\pi i}{k}\sqrt{\frac{us}{t}} + J_B(\sqrt{-u}, \lambda) \ ,$$
$$\mathcal{T}_F^{U_e} = E(q, p-k, p+k)\frac{4\pi i}{k}\sqrt{\frac{us}{t}} + J_F(\sqrt{-u}, \lambda) \ . \tag{265}$$

T channel For particle-antiparticle scattering

$$P_i(p_1) + A^j(p_2) \to P_i(p_3) + A^j(p_4)$$

S matrix in the adjoint channel is referred to as the T channel. The momentum assignments are (see LHS of fig 4)

$$p_1 = p + q \ , \ p_2 = -k - q \ , \ p_3 = p \ , \ p_4 = -k \ . \tag{266}$$

In terms of the Mandelstam variables

$$s = -(p-k)^2 \ , \ t = -q^2 \ , \ u = -(p+q+k)^2 \ , \tag{267}$$

the T channel T matrices for the boson-boson and fermion-fermion scattering are

$$\mathcal{T}_B^{T} = E(q, p-k, p+k)\frac{4\pi i}{k}\sqrt{\frac{tu}{s}} + J_B(\sqrt{-t}, \lambda) \ ,$$
$$\mathcal{T}_F^{T} = E(q, p-k, p+k)\frac{4\pi i}{k}\sqrt{\frac{tu}{s}} + J_F(\sqrt{-t}, \lambda) \ . \tag{268}$$

In particle-anti particle scattering there is also the singlet channel that we describe below.

4.3.10 The Singlet (S) channel

We now turn to the most interesting scattering process; the scattering of particles with antiparticles in the S (singlet) channel. In this channel the external lines on the LHS of Fig. 4 are assigned positive energy (and so represent initial states) while those on the right of the diagram are assigned negative energy (and so represent final states). It follows that we must make the identifications

$$p_1 = p + q \ , \ p_2 = -p \ , \ p_3 = k + q \ , p_4 = -k \ , \tag{269}$$

so that the Mandelstam variables for this scattering process are

$$s = -q^2 \ , \ t = -(p-k)^2 \ , \ u = -(p+k)^2 \ . \tag{270}$$

Note, in particular, that $s = -q^2$, and so is always negative when $q^\pm = 0$. As we have been able to evaluate the offshell correlator V (see (227)) only for $q^\pm = 0$, it follows that we cannot specialize our offshell computation to an onshell scattering process in the S channel in which $s \geq 4m^2$. In other words we do not have a direct computation of S channel scattering in any frame.

It is nonetheless tempting to simply assume that (252) and (253) continue to apply at every value of q^μ and not just when $q^\pm = 0$; indeed this is what the usual assumptions of analyticity of S matrices (and crossing symmetry in particular) would inevitably imply. Provisionally proceeding with this 'naive' assumption, it follows upon performing the appropriate analytic continuation ($q^2 \to -s$ for positive s; see sec 4.4 of [61]) that

$$\begin{aligned} \mathcal{T}_B^{S;\text{naive}} &= E(q, p-k, p+k) 4\pi i \lambda \sqrt{\frac{su}{t}} + J_B(\sqrt{s}, \lambda) \ , \\ \mathcal{T}_F^{S;\text{naive}} &= E(q, p-k, p+k) 4\pi i \lambda \sqrt{\frac{su}{t}} + J_F(\sqrt{s}, \lambda) \ , \end{aligned} \tag{271}$$

where

$$
\begin{aligned}
J_B(\sqrt{s},\lambda) =& -4\pi i\lambda\sqrt{s}\frac{N_1N_2+M_1}{D_1D_2} \ , \\
J_F(\sqrt{s},\lambda) =& -4\pi i\lambda\sqrt{s}\frac{N_1N_2+M_2}{D_1D_2} \ ,
\end{aligned}
\tag{272}
$$

where

$$
\begin{aligned}
N_1 =& \left((w-1)(2m+\sqrt{s}) + (w-1)(2m-\sqrt{s})e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^{\lambda}\right) \ , \\
N_2 =& \left((-i\sqrt{s}(w+3)+2im(w-1)) + (-i\sqrt{s}(w+3)-2im(w-1))e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^{\lambda}\right) \ , \\
M_1 =& 8mi\sqrt{s}((w+3)(w-1)-4w)e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^{\lambda} \ , \\
M_2 =& 8mi\sqrt{s}(1+w)^2e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^{\lambda} \ , \\
D_1 =& \left(i(w-1)(2m+\sqrt{s}) - (2im(w-1)+i\sqrt{s}(w+3))e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^{\lambda}\right) \ , \\
D_2 =& \left((\sqrt{s}(w+3)-2im(w-1)) + (w-1)(-i\sqrt{s}+2im)e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^{\lambda}\right) \ .
\end{aligned}
\tag{273}
$$

Including the identity factors, the naive S channel S matrix that follows from the usual rules of crossing symmetry are

$$
\begin{aligned}
\mathcal{S}_B^{S;\text{naive}}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \ &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_B^{S;\text{naive}}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \ , \\
\mathcal{S}_F^{S;\text{naive}}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \ &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_F^{S;\text{naive}}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \ ,
\end{aligned}
\tag{274}
$$

where the identity operator is defined in (174).

We pause here to note a subtlety. The quantity $\mathcal{S}_F^{S;\text{naive}}$ quoted above equals the S matrix in the S channel only upto phase. In order to obtain the fully correct S matrix we analytically continue the phase unambiguous quantity $\mathcal{S}_f^{S;\text{naive}}$ [64]. The result of that continuation is given by

$$\mathcal{S}_f^{S;\text{naive}} = \frac{\mathcal{S}_F^{S;\text{naive}}}{X(s)} \tag{275}$$

where[65]

$$X(s) = -\frac{-s+4m^2}{4m^2} = -4Y(s) \ . \tag{277}$$

The full four fermion amplitude in the S channel, including phase is then given by

$$A_F^{S;\text{naive}} = \mathcal{S}_f^{S;\text{naive}} X(p,k,q)$$

where[66]

$$X(p,k,q) = \frac{1}{4m^2}\left(u(p+q)u(-p)\right)\left(v(k+q)v(-k)\right) \ . \tag{278}$$

It is not difficult to check that

$$|X(p,k,q)| = X(s) \ .$$

It follows that the S channel 4 fermion amplitude agrees with $\mathcal{S}_F$ upto a convention dependent phase. This phase factor may be shown to vanish near the identity momentum configuration ($p_1 = p_3$, $p_2 = p_4$) and so does not affect the interference with identity, and in general has no physical effect; it follows we would make no error if we simply regarded $\mathcal{S}_F$ as the four fermion scattering amplitude. At any rate we have been careful to express the unitarity relation in terms of the phase unambiguous quantity $\mathcal{S}_f$ given unambiguously by (171).

[64]Indeed it does not make sense to analytically continue $\mathcal{S}_F$ as the ambiguous phases of this quantity are not necessarily Lorentz invariant, and so are not functions only of the Mandelstam variables.

[65]The factor of $X(s)$ is the analytic continuation of (see (171))

$$\left(\bar{u}(p)u(p+q)\right)\left(\bar{v}(-k-q)v(-k)\right) = X(q) = -\frac{q^2+4m^2}{4m^2} \ . \tag{276}$$

The analytic continuation of the above formula is same as $-4Y(s)$ (see (194).)

[66]The spinor quadrilinear is as defined in (182) with momentum assignments corresponding to the S channel (269).

The naive S channel S matrix (274) is not duality (130) invariant. In later section, we also show that it also does not obey the constraints of unitarity, leading to an apparent paradox.

A very similar paradox was encountered in [61] where it was conjectured that the usual rules of crossing symmetry are modified in matter Chern-Simons theories. It was conjectured in [61] that the correct transformation rule under crossing symmetry for *any* matter Chern-Simons theory with fundamental matter in the large N limit is given by

$$\begin{aligned}
\mathcal{S}_B^S(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_B^S(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\ , \\
\mathcal{S}_F^S(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_F^S(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\ ,
\end{aligned} \tag{279}$$

where

$$\begin{aligned}
\mathcal{T}_B^S(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= -i(\cos(\pi\lambda)-1)I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + \frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}_B^{S;\text{naive}}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\ , \\
\mathcal{T}_F^S(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= -i(\cos(\pi\lambda)-1)I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + \frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}_F^{S;\text{naive}}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\ ,
\end{aligned} \tag{280}$$

where (271) defines the T matrices obtained from naive crossing rules. In the center of mass frame the conjectured S matrix (279) has the form

$$\begin{aligned}
\mathcal{S}_B^S(s,\theta) &= 8\pi\sqrt{s}\delta(\theta) + i\mathcal{T}_B^S(s,\theta)\ , \\
\mathcal{S}_F^S(s,\theta) &= 8\pi\sqrt{s}\delta(\theta) + i\mathcal{T}_F^S(s.\theta)\ ,
\end{aligned} \tag{281}$$

where

$$\begin{aligned}
\mathcal{T}_B^S(s,\theta) &= -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + \frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}_B^{S;\text{naive}}(s,\theta)\ , \\
\mathcal{T}_F^S(s,\theta) &= -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + \frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}_F^{S;\text{naive}}(s,\theta)\ .
\end{aligned} \tag{282}$$

The naive analytically continued T matrices are

$$\mathcal{T}_B^{S;\text{naive}}(s,\theta) = 4\pi i\lambda\sqrt{s}\cot(\theta/2) + J_B(\sqrt{s},\lambda) \ ,$$
$$\mathcal{T}_F^{S;\text{naive}}(s,\theta) = 4\pi i\lambda\sqrt{s}\cot(\theta/2) + J_F(\sqrt{s},\lambda) \ , \tag{283}$$

where the J functions are as defined in (272). In other words the conjectured S matrix takes the following form

$$\mathcal{S}_B^S(s,\theta) = 8\pi\sqrt{s}\cos(\pi\lambda)\delta(\theta) + i\frac{\sin(\pi\lambda)}{\pi\lambda}\left(4\pi i\lambda\sqrt{s}\cot(\theta/2) + J_B(\sqrt{s},\lambda)\right) \ ,$$
$$\mathcal{S}_F^S(s,\theta) = 8\pi\sqrt{s}\cos(\pi\lambda)\delta(\theta) + i\frac{\sin(\pi\lambda)}{\pi\lambda}\left(4\pi i\lambda\sqrt{s}\cot(\theta/2) + J_F(\sqrt{s},\lambda)\right) \ . \tag{284}$$

It was demonstrated in [61] that the conjecture (280) yields an S channel S matrix that is both duality invariant and consistent with unitarity in the the systems under study in that paper. In this chapter we will follow [61] to conjecture that (280) continues to define the correct S channel S matrix for the theories under study. In the next section we will demonstrate that (280) obeys the nonlinear unitarity equations (190) and (191). We regard this fact as highly nontrivial evidence in support of the conjecture (280). As (280) appears to work in at least two rather different classes of large N fundamental matter Chern-Simons theories (namely the purely bosonic and fermionic theories studied in [61] and the supersymmetric theories studied in this chapter) it seems likely that (280) applies universally to all Chern-Simons fundamental matter theories, as suggested in [61].

Straightforward non-relativistic limit The conjectured S channel S matrix has a simple non-relativistic limit leading to the known Aharonov-Bohm result (see section 2.6 of [61] for details). In this limit we take (in the center of mass frame) $\sqrt{s} \to 2m$ in the

T matrix (282) with all other parameters held fixed. In this limit we find

$$\begin{aligned}
\mathcal{T}_B^S(s,\theta) &= -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + 4\sqrt{s}\sin(\pi\lambda)\left(i\cot(\theta/2)-1\right) \ , \\
\mathcal{T}_F^S(s,\theta) &= -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + 4\sqrt{s}\sin(\pi\lambda)\left(i\cot(\theta/2)+1\right) \ . \qquad (285)
\end{aligned}$$

The non-relativistic limit also coincides with the $\mathcal{N}=2$ limit of the S matrix (279) as we show in the following subsection. In §4.5.5 we describe a slightly modified non-relativistic limit of the S matrix.

4.3.11 S matrices in the $\mathcal{N}=2$ theory

As discussed in §4.2.1 the $\mathcal{N}=1$ theory (119) has an enhanced $\mathcal{N}=2$ supersymmetric regime when the Φ^4 coupling constant takes a special value $w=1$. We have already seen that the momentum dependent functions in the offshell four point function simplify dramatically (245), and so it is natural to expect that the S matrices at $w=1$ are much simpler than at generic w. This is indeed the case as we now describe.

By taking the limit $w \to 1$ in the S matrix formulae presented in (252) and (253), we find that the four boson and four fermion $\mathcal{N}=2$ S matrices take the very simple form [67]

$$\mathcal{T}_B^{\mathcal{N}=2} = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2} - \frac{8\pi m}{\kappa} \ , \qquad (286)$$

$$\mathcal{T}_F^{\mathcal{N}=2} = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2} + \frac{8\pi m}{\kappa} \ . \qquad (287)$$

The S matrices above are simply those for tree level scattering. It follows that the tree level S matrices in the three non-anyonic channels are not renormalized, at any order in the coupling constant, in the $\mathcal{N}=2$ theory.

[67] This is because the J functions reported in (252) and (253) have an extremely simple form at $w=1$ (see (492)).

There is an immediate (but rather trivial) check of this result. Recall that according to §4.7.3 the four boson and four fermion scattering amplitudes are not independent in the $\mathcal{N} = 2$ theory; supersymmetry determines the former in terms of the latter. The precise relation is derived in 4.7.3 and is given by (404) for particle-antiparticle scattering and (409) for particle-particle scattering. It is easy to verify that (286) and (287) trivially satisfy (404) (or (409)) using (155),(156) and appropriate momentum assignments for the channels of scattering discussed in section §4.3.9. [68]

For completeness we now present explicit formulae for the S matrices of the $\mathcal{N} = 2$ theory in the three non-anyonic channels.

For the U_d channel

$$\begin{aligned} \mathcal{T}_B^{U_d;\mathcal{N}=2} &= E(q, p-k, p+k)\frac{4\pi i}{k}\sqrt{\frac{ts}{u}} - \frac{8\pi m}{\kappa} \ , \\ \mathcal{T}_F^{U_d;\mathcal{N}=2} &= E(q, p-k, p+k)\frac{4\pi i}{k}\sqrt{\frac{ts}{u}} + \frac{8\pi m}{\kappa} \ . \end{aligned} \tag{289}$$

For the U_e channel

$$\begin{aligned} \mathcal{T}_B^{U_e;\mathcal{N}=2} &= E(q, p-k, p+k)\frac{4\pi i}{k}\sqrt{\frac{us}{t}} - \frac{8\pi m}{\kappa} \ , \\ \mathcal{T}_F^{U_e;\mathcal{N}=2} &= E(q, p-k, p+k)\frac{4\pi i}{k}\sqrt{\frac{us}{t}} + \frac{8\pi m}{\kappa} \ . \end{aligned} \tag{290}$$

[68] As an example, in the T channel (see (266)) we substitute the coefficients (155), (156) into (404) and evaluate it to get

$$\mathcal{S}_B = \mathcal{S}_F \frac{-2m(k-p)_- + iq_3(k+p)_-}{2m(k-p)_- + iq_3(k+p)_-} \ . \tag{288}$$

It is clear that the covariant form of the S matrices given in (286) and (287) trivially satisfy (288). Similarly it can be easily checked that the result (288) follows from (409) for particle-particle scattering.

For the T channel

$$
\begin{aligned}
\mathcal{T}_B^{T;\mathcal{N}=2} &= E(q,p-k,p+k)\frac{4\pi i}{k}\sqrt{\frac{tu}{s}} - \frac{8\pi m}{\kappa} \ , \\
\mathcal{T}_F^{T;\mathcal{N}=2} &= E(q,p-k,p+k)\frac{4\pi i}{k}\sqrt{\frac{tu}{s}} + \frac{8\pi m}{\kappa} \ .
\end{aligned} \tag{291}
$$

Let us now turn to the singlet channel. As described in §4.3.10, we cannot compute the S channel S matrix directly because of our choice of the kinematic regime $q_\pm = 0$. The naive analytic continuation of (286) and (287) to the S channel gives

$$
\begin{aligned}
\mathcal{T}_B^{S;\text{naive};\mathcal{N}=2} &= E(q,p-k,p+k)4\pi i\lambda\sqrt{\frac{su}{t}} - 8\pi m\lambda \ , \\
\mathcal{T}_F^{S;\text{naive};\mathcal{N}=2} &= E(q,p-k,p+k)4\pi i\lambda\sqrt{\frac{su}{t}} + 8\pi m\lambda \ .
\end{aligned} \tag{292}
$$

Thus the naive S channel S matrix for the $\mathcal{N}=2$ theory is

$$
\begin{aligned}
\mathcal{S}_B^{S;\text{naive};\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) =& I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \\
&+ i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_B^{S;\text{naive};\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \ , \\
\mathcal{S}_F^{S;\text{naive};\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) =& I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \\
&+ i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_F^{S;\text{naive};\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \ .
\end{aligned} \tag{293}
$$

As explained in the introduction §4.1, this result is obviously non-unitary. Applying the modified crossing symmetry transformation rules (279) we obtain our conjecture for the $\mathcal{N}=2$ S matrix in the singlet channel

$$
\begin{aligned}
\mathcal{S}_B^{S;\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_B^{S;\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \ , \\
\mathcal{S}_F^{S;\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_F^{S;\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \ ,
\end{aligned} \tag{294}
$$

where

$$\mathcal{T}_B^{S;\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) = -i(\cos(\pi\lambda)-1)I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + \frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}_B^{S;\text{naive};\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \ ,$$
$$\mathcal{T}_F^{S;\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) = -i(\cos(\pi\lambda)-1)I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + \frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}_F^{S;\text{naive};\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \ . \tag{295}$$

In the center of mass frame the conjectured S channel S matrix in the $\mathcal{N}=2$ theory takes the form

$$\mathcal{S}_B^{S;\mathcal{N}=2}(s,\theta) = 8\pi\sqrt{s}\delta(\theta) + i\mathcal{T}_B^{S}(s,\theta) \ ,$$
$$\mathcal{S}_F^{S;\mathcal{N}=2}(s,\theta) = 8\pi\sqrt{s}\delta(\theta) + i\mathcal{T}_F^{S}(s.\theta) \ , \tag{296}$$

where

$$\mathcal{T}_B^{S;\mathcal{N}=2}(s,\theta) = -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + \sin(\pi\lambda)(4i\sqrt{s}\cot(\theta/2) - 8m) \ ,$$
$$\mathcal{T}_F^{S;\mathcal{N}=2}(s,\theta) = -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + \sin(\pi\lambda)(4i\sqrt{s}\cot(\theta/2) + 8m) \ . \tag{297}$$

Note that as $\sqrt{s} \to 2m$ (297) reproduces the straightforward non-relativistic limit of the $\mathcal{N}=1$ theory (285).

In other words the conjectured S channel S matrix for the $\mathcal{N}=2$ theory takes the following form in the center of mass frame

$$\mathcal{S}_B^{S;\mathcal{N}=2}(s,\theta) = 8\pi\sqrt{s}\cos(\pi\lambda)\delta(\theta) + i\sin(\pi\lambda)\left(4i\sqrt{s}\cot(\theta/2) - 8m\right) \ ,$$
$$\mathcal{S}_F^{S;\mathcal{N}=2}(s,\theta) = 8\pi\sqrt{s}\cos(\pi\lambda)\delta(\theta) + i\sin(\pi\lambda)\left(4i\sqrt{s}\cot(\theta/2) + 8m\right) \ . \tag{298}$$

We explicitly show that the conjectured S channel S matrix is unitary in the following section.

4.4 Unitarity

In this section, we first show that the S matrices in the T and U channel obey the unitarity conditions (190) and (191) at leading order in the large N limit. As the relevant unitarity equations are linear, the unitarity equation is a relatively weak consistency check of the S matrices computed in this chapter.

We then proceed to demonstrate that the S matrix (279) also obeys the constraints of unitarity. As the unitarity equation is nonlinear in the S channel, this constraint is highly nontrivial, we believe it provides an impressive consistency check of the conjecture (279).

4.4.1 Unitarity in the T and U channel

We begin by discussing the unitarity condition for the T (adjoint) and U (particle - particle) channels. Firstly we note that the S matrices in these channels are $O(1/N)$. Therefore the LHS of (190) and (191) are $O(1/N^2)$. It follows that the unitarity equations (190) and (191) are obeyed at leading order in the large N limit provided

$$\begin{aligned}
\mathcal{T}_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) &= \mathcal{T}_B^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_1, \mathbf{p}_2) \ , \\
\mathcal{T}_F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) &= \mathcal{T}_F^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_1, \mathbf{p}_2) \ .
\end{aligned} \tag{299}$$

The four boson and four fermion S matrices in the T channel are given in terms of the universal functions in (252) and (253) after applying the momentum assignments (266). It follows that (299) holds in the T channel provided

$$\begin{aligned}
\mathcal{T}_B^T(p+q, -k-q, p, -k) &= \mathcal{T}_B^{T*}(p, -k, p+q, -k-q) \ , \\
\mathcal{T}_F^T(p+q, -k-q, p, -k) &= \mathcal{T}_F^{T*}(p, -k, p+q, -k-q) \ .
\end{aligned} \tag{300}$$

This equation may be verified to be true (see below for some details).

Similarly the U_d channel S matrix is obtained via the momentum assignments (260); It follows that (299) is obeyed provided

$$\begin{aligned}\mathcal{T}_B^{U_d}(p+q,k,p,k+q) &= \mathcal{T}_B^{U_d*}(p,k+q,p+q,k)\ , \\ \mathcal{T}_F^{U_d}(p+q,k,p,k+q) &= \mathcal{T}_F^{U_d*}(p,k+q,p+q,k)\ , \end{aligned} \tag{301}$$

which can also be checked to be true.

Finally in the U_e channel it follows from the momentum assignments (263) that (299) holds provided

$$\begin{aligned}\mathcal{T}_B^{U_e}(k,p+q,p,k+q) &= \mathcal{T}_B^{U_e*}(p,k+q,k,p+q)\ , \\ \mathcal{T}_F^{U_e}(k,p+q,p,k+q) &= \mathcal{T}_F^{U_e*}(p,k+q,k,p+q)\ , \end{aligned} \tag{302}$$

which we have also verified.

The T matrices for all the above channels of scattering are reported in §4.3.9. Note that the starring of the T matrices in (299) also involves a momentum exchange $p_1 \Leftrightarrow p_3$ and $p_2 \Leftrightarrow p_4$. It follows that under this exchange $q \to -q$. [69]

In verifying (300), (301) and (302) we have used the fact that the functions J_B and J_F are both invariant under the combined operation of complex conjugation accompanied by the flip $q \to -q$ (see (488)). We also use the fact that in each case (T, U_d and U_e) the factor $E(q, p-k, p+k)$ flips sign under the momentum exchange $p_1 \Leftrightarrow p_3$ and $p_2 \Leftrightarrow p_4$; the sign obtained from this process compensates the minus sign from complex conjugating

[69] For instance in the T channel, we get the equations

$$p'+q' = p\ ,\ p' = p+q\ ,\ -k'-q' = -k\ ,\ -k' = -k-q\ . \tag{303}$$

It follows that $q' = -q$.

the explicit factor of i. [70]

4.4.2 Unitarity in S channel

The S matrix in the S channel is of $O(1)$ and one has to use the full non-linear unitarity conditions (195) and (196) . We reproduce them here for convenience.

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(- Y(s)(\mathcal{T}_B^S(s,\theta)+4Y(s)\mathcal{T}_f^S(s,\theta))(\mathcal{T}_B^{S*}(s,-(\alpha-\theta))+4Y(s)\mathcal{T}_f^{S*}(s,-(\alpha-\theta))) + \mathcal{T}_B^S(s,\theta)\mathcal{T}_B^{S*}(s,-(\alpha-\theta))\Big) = i(\mathcal{T}_B^{S*}(s,-\alpha)-\mathcal{T}_B^S(s,\alpha)) \ , \tag{304}$$

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(Y(s)(\mathcal{T}_B^S(s,\theta)+4Y(s)\mathcal{T}_f^S(s,\theta))(\mathcal{T}_B^{S*}(s,-(\alpha-\theta))+4Y(s)\mathcal{T}_f^{S*}(s,-(\alpha-\theta))) - 16Y(s)^2\mathcal{T}_f^S(s,\theta)\mathcal{T}_f^{S*}(s,-(\alpha-\theta))\Big) = i4Y(s)(-\mathcal{T}_f^S(s,\alpha)+\mathcal{T}_f^{S*}(s,-\alpha)) \ , \tag{305}$$

where

$$Y(s) = \frac{-s+4m^2}{16m^2} \tag{306}$$

is as defined in (183), and $\mathcal{T}_B^S$ corresponds to the bosonic T matrix while $\mathcal{T}_f^S$ corresponds to the phase unambiguous part of the fermionic T matrix in the Singlet (S) channel given in (280) (also see (275)). In center of mass coordinates it takes the form

$$\mathcal{T}_f^S(s,\theta) = -\frac{\mathcal{T}_F^S(s,\theta)}{4Y(s)} \ . \tag{307}$$

[70] The unitarity conditions in these channels are simply the statement that the S matrices are real. The reality of S matrices is tightly connected to the absence of two particle branch cuts in the S matrices in these channels at leading order in large N.

Substituting the above into (304) and (305), the conditions for unitarity may be rewritten as

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Bigg(- Y(s)(\mathcal{T}_B^S(s,\theta) - \mathcal{T}_F^S(s,\theta))(\mathcal{T}_B^{S*}(s,-(\alpha-\theta)) - \mathcal{T}_F^{S*}(s,-(\alpha-\theta)))$$
$$+\mathcal{T}_B^S(s,\theta)\mathcal{T}_B^{S*}(s,-(\alpha-\theta))\Bigg) = i(\mathcal{T}_B^{S*}(s,-\alpha) - \mathcal{T}_B^S(s,\alpha)) \ , \quad (308)$$

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Bigg(Y(s)(\mathcal{T}_B^S(s,\theta) - \mathcal{T}_F^S(s,\theta))(\mathcal{T}_B^{S*}(s,-(\alpha-\theta)) - \mathcal{T}_F^{S*}(s,-(\alpha-\theta)))$$
$$-\mathcal{T}_F^S(s,\theta)\mathcal{T}_F^{S*}(s,-(\alpha-\theta))\Bigg) = i(\mathcal{T}_F^S(s,\alpha) - \mathcal{T}_F^{S*}(s,-\alpha)) \ . \quad (309)$$

Let us pause to note that under duality $\mathcal{T}_B \to \mathcal{T}_F$ and vice versa; it follows then (308) and (309) map to each other under duality. In other words the unitarity conditions are compatible with duality.

We will now verify that our S channel S matrix is indeed compatible with unitarity. Let us recall that the angular dependence of the S matrix, in the center of mass frame is given by

$$\begin{aligned} \mathcal{T}_B^S &= H_B T(\theta) + W_B - iW_2\delta(\theta) \ , \\ \mathcal{T}_F^S &= H_F T(\theta) + W_F - iW_2\delta(\theta) \ , \end{aligned} \quad (310)$$

where

$$T(\theta) = i\cot(\theta/2).$$

We will list the particular values of the coefficient functions $H_B(s)$ etc below; we will be able to proceed for a while leaving these functions unspecified.

Substituting (310) in (308) and doing the angle integrations[71] we find that (308) is obeyed if and only if

$$
\begin{aligned}
&H_B - H_B^* = \frac{1}{8\pi\sqrt{s}}(W_2 H_B^* - H_B W_2^*) \ , \\
&W_2 + W_2^* = -\frac{1}{8\pi\sqrt{s}}(W_2 W_2^* + 4\pi^2 H_B H_B^*) \ , \\
&W_B - W_B^* = \frac{1}{8\pi\sqrt{s}}(W_2 W_B^* - W_2^* W_B) - \frac{i}{4\sqrt{s}}(H_B H_B^* - W_B W_B^*) - \frac{iY}{4\sqrt{s}}(W_B - W_F)(W_B^* - W_F^*) \ .
\end{aligned}
\tag{312}
$$

Similarly (309) is obeyed if and only if

$$
\begin{aligned}
&H_F - H_F^* = \frac{1}{8\pi\sqrt{s}}(W_2 H_F^* - H_F W_2^*) \ , \\
&W_2 + W_2^* = -\frac{1}{8\pi\sqrt{s}}(W_2 W_2^* + 4\pi^2 H_F H_F^*) \ , \\
&W_F - W_F^* = \frac{1}{8\pi\sqrt{s}}(W_2 W_F^* - W_2^* W_F) - \frac{i}{4\sqrt{s}}(H_F H_F^* - W_F W_F^*) - \frac{iY}{4\sqrt{s}}(W_B - W_F)(W_B^* - W_F^*) \ .
\end{aligned}
\tag{313}
$$

The first two equations of (312) and (313) are entirely identical to the first two equations of equation 2.66 in [61] for the non-supersymmetric case. The third equation has an additional contribution due to supersymmetry. Note that (312) and (313) are compatible with duality under $H_B \to H_F$ and $W_B \to W_F$ and vice versa.

Let us now proceed to verify that the equations (312) and (313) are indeed obeyed; for this purpose we need to use the specific values of the coefficient functions in (310). These functions are easily read off from the formulae (282) (that we reproduce here for

[71] The angle integrations in (308) can be done by using the formula

$$
\int d\theta \mathrm{Pv}\cot\left(\frac{\theta}{2}\right) \mathrm{Pv}\cot\left(\frac{\alpha-\theta}{2}\right) = 2\pi - 4\pi^2\delta(\alpha), \tag{311}
$$

where Pv stands for principal value. See (478) for a simple check of this formula.

convenience)

$$
\begin{aligned}
\mathcal{T}_B^S &= -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + \frac{\sin(\pi\lambda)}{\pi\lambda}\left(4\pi i\lambda\sqrt{s}\cot(\theta/2) + J_B(\sqrt{s},\lambda)\right) \ , \\
\mathcal{T}_F^S &= -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + \frac{\sin(\pi\lambda)}{\pi\lambda}\left(4\pi i\lambda\sqrt{s}\cot(\theta/2) + J_F(\sqrt{s},\lambda)\right) \ ,
\end{aligned} \tag{314}
$$

from which we find

$$
\begin{aligned}
W_B &= J_B(\sqrt{s},\lambda)\frac{\sin(\pi\lambda)}{\pi\lambda} \ , \\
W_F &= J_F(\sqrt{s},\lambda)\frac{\sin(\pi\lambda)}{\pi\lambda} \ ,
\end{aligned} \tag{315}
$$

where the explicit form of the J functions are given in (272). While we also identify

$$
H_B = H_F = 4\sqrt{s}\sin(\pi\lambda), \ W_2 = 8\pi\sqrt{s}(\cos(\pi\lambda)-1), \ T(\theta) = i\cot(\theta/2) \ . \tag{316}
$$

Using the above relations it is very easy to see that the first two equations in each of (312) and (313) are satisfied. The first equation in each of (312) and (313) holds because H_B, H_F and W_2 are all real. The second equation in each case boils down to a true trigonometric identity.

The functions W_B and W_F occur only in the third equation in (312) and (313). These equations assert two nonlinear identities relating the (rather complicated) J_B and J_F functions. We have verified by explicit computation that these identities are indeed obeyed. It follows that the conjectured S matrix (279) is indeed unitary.

At the algebraic level, the satisfaction of the unitarity equation appears to be a minor miracle. A small mistake of any sort (a factor or two or an incorrect sign) causes this test to fail badly. In particular, unitarity is a very sensitive test of the conjectured form (279) of the S matrix. Let us recall again that this conjecture was first made in [61], where it was shown that it leads to a unitary $2 \to 2$ S matrix. The supersymmetric S matrices of this chapter are more complicated than the S matrices of the purely bosonic or purely fermionic theories of [61]. In particular the unitarity equation for four boson and four fermion S matrices is different in this chapter from the corresponding equations in [61] (the difference stems from the fact that two bosons can scatter not just to two bosons but also to two fermions, and this second process also contributes to the quadratic part of the unitarity equations). Nonetheless the prescription (279) adopted from [61] turns out to give results that obey the modified unitarity equation of this chapter. In our opinion this constitutes a very nontrivial check of the crossing symmetry relation (279) proposed in [61].

The unitarity equation is satisfied for the arbitrary $\mathcal{N}=1$ susy theory, and so is, in particular obeyed for the $\mathcal{N}=2$ theory. Recall that the $\mathcal{N}=2$ theory has a particularly simple S matrix (297). In fact in the T and U channels the $\mathcal{N}=2$ S matrix is tree level exact at leading order in large N. According to the rules of naive crossing symmetry the S channel S matrix would also have been tree level exact. This result is in obvious conflict with the unitarity equation: in the equation $-i(T-T^\dagger)=TT^\dagger$ the LHS vanishes at tree level while the RHS is obviously nonzero. The modified crossing symmetry rules (279) resolve this paradox in a very beautiful way. According to the rules (279), the T matrix is not Hermitian even if T^{naive} is; as the term in (279) proportional to identity is imaginary. It follows from (279) that both LHS and the RHS of the unitarity equation are nonzero; they are infact equal, as we now pause to explicitly demonstrate. In the $\mathcal{N}=2$ limit (see (297)) we have

$$\begin{aligned}
&H_B = H_F = 4\sqrt{s}\sin(\pi\lambda)\ ,\\
&W_B = -8m\sin(\pi\lambda)\ ,\\
&W_F = 8m\sin(\pi\lambda)\ ,\\
&W_2 = 8\pi\sqrt{s}(\cos(\pi\lambda)-1)\ .
\end{aligned} \tag{317}$$

The first equation in (312) is satisfied because everything is real. We have checked that the second equation is satisfied using a trigonometric identity. [72] The third equation works because we have

$$(H_B H_B^* - W_B W_B^*) = -16\sin^2(\pi\lambda)(-s+4m^2) \tag{318}$$

and

$$Y(W_B - W_F)(W_B^* - W_F^*) = 16\sin^2(\pi\lambda)(-s+4m^2) \tag{319}$$

the other terms don't matter because everything else is real. The same thing is true for (313) since

$$(H_F H_F^* - W_F W_F^*) = -16\sin^2(\pi\lambda)(-s+4m^2) \tag{320}$$

and thus the unitarity conditions are satisfied by the conjectured S matrix (280) in the $\mathcal{N}=2$ theory as well.

4.5 Pole structure of S matrix in the S channel

The S channel S matrix studied in the last two sections turns out to have an interesting analytic structure. In this section we will demonstrate that the S matrix has a pole whenever $w < -1$. As we demonstrate below the pole is at threshold at $w = -1$, migrates to lower masses as w is further reduced until it actually occurs at zero mass at a critical value $w = w_c(\lambda) < -1$. As w is further reduced, the squared mass of the pole increases again, until the pole mass returns to threshold at $w = -\infty$.

In order to establish all these facts let us recall the structure of four boson and four fermion S matrix in the S channel. The S matrices take the form (see (272))

$$\mathcal{T}_B^S = \frac{n_b}{d_1 d_2}, \quad \mathcal{T}_F^S = \frac{n_f}{d_1 d_2}\ , \tag{321}$$

[72] This is the only equation in which the LHS and RHS are both nonzero. The LHS is the imaginary part of the coefficient of identity.

where

$$
\begin{aligned}
d_1 =& -4|m|^2\left(\operatorname{sgn}(\lambda)(w-1)\left(\left(\frac{1+y}{1-y}\right)^\lambda-1\right)+y\left(-w\left(\frac{1+y}{1-y}\right)^\lambda+w+\left(\frac{1+y}{1-y}\right)^\lambda+3\right)\right)\ , \\
d_2 =& \operatorname{sgn}(\lambda)(w-1)\left(\left(\frac{1+y}{1-y}\right)^\lambda-1\right)+y\left(w\left(\left(\frac{1+y}{1-y}\right)^\lambda-1\right)+3\left(\frac{1+y}{1-y}\right)^\lambda+1\right)\ ,
\end{aligned}
\tag{322}
$$

$$
\begin{aligned}
n_b =& -32|m|^3 y\sin(\pi\lambda)\Bigg(8\ \operatorname{sgn}(\lambda)(w+1)y\left(\frac{1+y}{1-y}\right)^\lambda \\
&+(w-1)(\operatorname{sgn}(\lambda)-y)\left(\frac{1+y}{1-y}\right)^{2\lambda}(\operatorname{sgn}(\lambda)(w-1)+(w+3)y) \\
&-(w-1)(\operatorname{sgn}(\lambda)+y)(\operatorname{sgn}(\lambda)(w-1)-(w+3)y)\Bigg)\ , \\
n_f =& 32|m|^3 y\sin(\pi\lambda)\Bigg(8\ \operatorname{sgn}(\lambda)(w+1)y\left(\frac{1+y}{1-y}\right)^\lambda \\
&-(w-1)(\operatorname{sgn}(\lambda)-y)\left(\frac{1+y}{1-y}\right)^{2\lambda}(\operatorname{sgn}(\lambda)(w-1)+(w+3)y) \\
&+(w-1)(\operatorname{sgn}(\lambda)+y)(\operatorname{sgn}(\lambda)(w-1)-(w+3)y)\Bigg)\ ,
\end{aligned}
\tag{323}
$$

where $y=\sqrt{s}/2|m|$. Through this discussion we assume that $\lambda m>0$ (recall this condition was needed for duality invariance).

The denominators d_1, d_2 and the numerators are all polynomials of y and the quantity

$$
X=\left(\frac{1+y}{1-y}\right)^\lambda\ .
$$

Most of the interesting scaling behaviors we will encounter below are a consequence of the dependence of all quantities on X. Note that d_1 and d_2 are linear functions of X while n_b and n_f are quadratic functions of X. It is consequently possible to recast n_b and n_f in the form

$$
\begin{aligned}
n_b &= a_b d_1 d_2 + b_b d_1 + c_b d_2 \ , \\
n_f &= a_f d_1 d_2 + b_f d_1 + c_f d_2 \ .
\end{aligned}
$$

Here a_b, b_b, c_b, a_f, b_f and c_f are polynomials of y (but are independent of X) and are given by

$$
\begin{aligned}
a_b &= y \ , \\
b_b &= (w-1)(\mathrm{sgn}(\lambda)+y)^2 \ , \\
c_b &= -4|m|^2(\mathrm{sgn}(\lambda)-y)(\mathrm{sgn}(\lambda)(w-1)-(w+3)y) \ , \\
a_f &= y \ , \\
b_f &= -(w-1)\left(1-y^2\right) \ , \\
c_f &= 4|m|^2(\mathrm{sgn}(\lambda)+y)(\mathrm{sgn}(\lambda)(w-1)-(w+3)y) \ .
\end{aligned}
\tag{324}
$$

In order to study the poles of the S matrix we need to investigate the zeroes of the functions d_1 and d_2. Let us first consider the case $\lambda > 0$. In this case it turns out that $d_1(y)$ has a zero for $w \in (-\infty, w_c]$, while $d_2(y)$ has a zero in the range $w \in [w_c, -1]$ where

$$
w_c(\lambda) = 1 - \frac{2}{|\lambda|} \ . \tag{325}
$$

At $w = -\infty$ the zero of d_1 occurs at $y = 1$. As w is increased the y value of the zero decreases, until it reaches $y = 0$ at $w = w_c$. At larger values of w, d_1 no longer has a zero. However $d_2(y)$ develops a zero. The zero of $d_2(y)$ starts out at $y = 0$ when $w = w_c$, and then increases, reaching $y = 1$ at $w = -1$. At larger values of w neither d_1 nor d_2 have a zero.

When $\lambda < 0$ we have an identical situation except that the roles of d_1 and d_2 are reversed. $d_2(y)$ has a zero for $w \in (-\infty, w_c]$, while $d_1(y)$ has a zero in the range

$w \in [w_c, -1]$. At $w = -\infty$ the zero of d_2 occurs at $y = 1$. As w is increased the y value of the zero decreases, until it reaches $y = 0$ at $w = w_c$. At larger values of w, d_2 no longer has a zero. However $d_1(y)$ develops a zero. The zero of $d_1(y)$ starts out at $y = 0$ when $w = w_c$, and then increases, reaching $y = 1$ at $w = -1$. At larger values of w neither d_1 nor d_2 have a zero.

In summary our S matrix has a pole for $w \in (-\infty, -1]$. The pole lies at threshold at the end points of this range, and becomes massless at $w = w_c$. There are clearly three special values of w in this range: $w = -1$, $w = w_c$ and $w = -\infty$. In the rest of this section we examine the neighborhood of three special points in turn.

4.5.1 Behavior near $w = -1 - \delta w$

In this subsection we study the pole in the neighborhood of $w = -1$. When $w \to -1 - \delta w$ with $0 < \delta w << 1$, we also expand $y \to 1 - \delta y$ (where $0 < \delta y << 1$) and find that

$$\begin{aligned}
d_1 \sim& 4|m|^2 \left((\mathrm{sgn}(\lambda) - 1) \left(\delta w - 2 \left(\frac{2}{\delta y} \right)^{\lambda} \right) + 2(\mathrm{sgn}(\lambda) + 1) \right) \ , \\
d_2 \sim& (\mathrm{sgn}(\lambda) + 1) \left(2 - \left(\frac{2}{\delta y} \right)^{\lambda} \delta w \right) - 2 \left(\frac{2}{\delta y} \right)^{\lambda} (\mathrm{sgn}(\lambda) - 1) \ , \\
a_b \sim& 1 - \delta y \ , \\
b_b \sim& -(2 + \delta w)(\mathrm{sgn}(\lambda) + 1 - \delta y)^2 \ , \\
c_b \sim& 4|m|^2 (\mathrm{sgn}(\lambda) - 1 + \delta y)(\mathrm{sgn}(\lambda)(2 + \delta w) + (2 - \delta w)(1 - \delta y)) \ , \\
a_f \sim& 1 - \delta y \ , \\
b_f \sim& 2\delta y (2 + \delta w)(2 - \delta y) \ , \\
c_f \sim& -4|m|^2 (\mathrm{sgn}(\lambda) + 1 - \delta y)(\mathrm{sgn}(\lambda)(\delta w + 2) + (2 - \delta w)(1 - \delta y)) \ .
\end{aligned} \tag{326}$$

Let us first consider the case $\lambda > 0$. In this case d_1 equals $16m^2$ at leading order and so does not have a zero for δw and δy small. On the other hand

$$d_2 \propto \left(2 - \left(\frac{2}{\delta y}\right)^{\lambda} \delta w\right)$$

and so vanishes when

$$\frac{\delta w}{2} = \left(\frac{\delta y}{2}\right)^{|\lambda|}, \quad \frac{\delta y}{2} = \left(\frac{\delta w}{2}\right)^{\frac{1}{|\lambda|}} . \tag{327}$$

When $\lambda < 0$, d_2 is a monotonic function that never vanishes. However d_1 vanishes provided the condition (327) is met. It follows that the S matrix has a pole when (327) is satisfied for both signs of λ.

The pole in the S matrix occurs due to the vanishing of the denominator $d_1 d_2$. As this denominator is the same for both the boson boson $\to$ boson boson and the fermion fermion $\to$ fermion fermion S matrices, both these scattering processes have a pole at the value of y listed in (327). The residue of this pole is, however, significantly different in the four boson and four fermion scattering processes. Let us first consider the four boson scattering term. The residue of the pole is determined by b_b evaluated at (327) (in the case $\lambda > 0$) and c_b evaluated at the same pole (in the case $\lambda < 0$). In either case we find the structure of the pole for four boson scattering to be

$$\mathcal{T}_B \sim \frac{\left(\frac{\delta y}{2}\right)^{|\lambda|}}{\delta w - 2\left(\frac{\delta y}{2}\right)^{|\lambda|}} . \tag{328}$$

In a similar manner the residue of the pole for four fermion scattering is determined by b_f evaluated at (327) (in the case $\lambda > 0$) and c_f evaluated at the same pole (in the case $\lambda < 0$). In either case we find that

$$\mathcal{T}_F \sim \frac{\left(\frac{\delta y}{2}\right)^{1+|\lambda|}}{\delta w - 2\left(\frac{\delta y}{2}\right)^{|\lambda|}} . \tag{329}$$

Notice that while the residue of the pole for four fermion scattering is suppressed compared to the residue of the same pole for four boson scattering by a factor of $(\delta w)^{\frac{1}{|\lambda|}}$.

4.5.2 Pole near $y = 0$

There exists a critical value, $w = w_c(\lambda)$, at which both d_1 and d_2 have zeroes at $y = 0$. In order to locate w_c we expand d_1 and d_2 about $y = 0$. To linear order we find

$$d_1 = d_2 \sim y(\lambda \mathrm{sgn}(\lambda)(w-1)+2) \ . \tag{330}$$

Clearly d_1 and d_2 have a common zero at $y = 0$ provided

$$w = w_c(\lambda) = 1 - \frac{2}{|\lambda|} \ . \tag{331}$$

In order to study this pole in the neighborhood of $w = w_c$ we set $w = w_c + \delta w$ (with $|\delta w| < 1$) near $y = \delta y$ (with $0 < \delta y << 1$); expanding in δw and δy we find

$$\begin{aligned} d_1 \sim & \frac{8|m|^2 \delta y\, (\delta w \lambda + 2\delta y(1-|\lambda|))}{\mathrm{sgn}(\lambda)} \ , \\ d_2 \sim & \frac{\delta y\, (\delta w \lambda - 2\delta y(1-|\lambda|))}{\mathrm{sgn}(\lambda)} \ , \\ n_b \sim & -\frac{512|m|^3 \sin(\pi\lambda)\delta y^2(-1+|\lambda|)}{\lambda} \ , \\ n_f \sim & \frac{512|m|^3 \sin(\pi\lambda)\delta y^2(-1+|\lambda|)}{\lambda} \ . \end{aligned} \tag{332}$$

The product $d_1 d_2$ vanishes when

$$\delta y = \frac{|\lambda \delta w|}{2(1-|\lambda|)}, \quad \text{i.e.} \quad \delta y^2 = \frac{\lambda^2 \delta w^2}{4(1-|\lambda|)^2} \ . \tag{333}$$

[73] The residue of the pole at $y=0$ for any sign of λ is given by substituting (333) into the functions n_b and n_f in (332). We find the pole structure of the bosonic S matrix near $y=0$ to be

$$\mathcal{T}_B \sim -\frac{64|m|\sin(\pi\lambda)(-1+|\lambda|)}{|\lambda|\left(\delta w^2\lambda^2 - 4\delta y^2(1-|\lambda|)^2\right)} \ . \tag{334}$$

In a similar manner we find the pole structure of the fermion S matrix near $y=0$ to be

$$\mathcal{T}_F \sim \frac{64|m|\sin(\pi\lambda)(-1+|\lambda|)}{|\lambda|\left(\delta w^2\lambda^2 - 4\delta y^2(1-|\lambda|)^2\right)} \ . \tag{335}$$

4.5.3 Behavior at $w \to -\infty$

We now turn to the analysis of the pole structure at $w \to -\infty$. This is easily achieved by setting $w = -\frac{1}{\delta w}$ with $0 < \delta w << 1$ and $y \to 1-\delta y$ with $0 < \delta y << 1$. The various functions (322) in the S matrix (321) have the behavior

$$\begin{aligned}
d_1 \sim& \frac{4|m|^2}{\delta w}\left((\delta y + \mathrm{sgn}(\lambda) - 1)\left(1 - \left(\frac{2}{\delta y}\right)^{\lambda}\right) + (\mathrm{sgn}(\lambda)+3)\delta w\right) \ , \\
d_2 \sim& \frac{1}{\delta w}\left((-\delta y + \mathrm{sgn}(\lambda) + 1) - \left(\frac{2}{\delta y}\right)^{\lambda}((\mathrm{sgn}(\lambda)-3)\delta w + \mathrm{sgn}(\lambda)+1)\right) \ , \\
a_b \sim& 1 - \delta y \ , \\
b_b \sim& -\frac{1}{\delta w}(\mathrm{sgn}(\lambda) + 1 - \delta y)^2 \ , \\
c_b \sim& -4|m|^2(\mathrm{sgn}(\lambda) - 1 + \delta y)(-\mathrm{sgn}(\lambda)(1+\frac{1}{\delta w}) - (3 - \frac{1}{\delta w})(1-\delta y)) \ , \\
a_f \sim& 1 - \delta y \ , \\
b_f \sim& \frac{\delta y}{\delta w}(2 - \delta y) \ , \\
c_f \sim& 4|m|^2(\mathrm{sgn}(\lambda) + 1 - \delta y)(-\mathrm{sgn}(\lambda)(1+\frac{1}{\delta w}) - (3 - \frac{1}{\delta w})(1-\delta y)) \ .
\end{aligned} \tag{336}$$

[73] $d_1 d_2$ also vanishes quadratically at $\delta y = 0$. Note however that both n_b and n_f are proportional to δy^2. Consequently the factors of δy^2 cancel between the numerator and denominator.

Let us first consider the case $\lambda > 0$. In this case d_2 is a monotonic function that never vanishes and so does not have a zero for δw and δy small. On the other hand

$$d_1 \propto \left(\delta w - \frac{1}{2}\left(\frac{\delta y}{2}\right)^{1-|\lambda|}\right)$$

and so vanishes when

$$\delta w = \frac{1}{2}\left(\frac{\delta y}{2}\right)^{1-|\lambda|}, \quad \delta y = \left(\frac{4\delta w}{2^{|\lambda|}}\right)^{\frac{1}{1-|\lambda|}} . \tag{337}$$

When $\lambda < 0$, d_1 is a constant $-8m^2$. However d_2 vanishes provided the condition (337) is met. It follows that the S matrix has a pole when (337) is satisfied for both signs of λ.

The pole in the S matrix occurs due to the vanishing of the denominator $d_1 d_2$. As this denominator is the same for both the boson boson $\rightarrow$ boson boson and the fermion fermion $\rightarrow$ fermion fermion S matrices, both these scattering processes have a pole at the value of y listed in (337). The residue of this pole is different in the four boson and four fermion scattering processes as before. Let us first consider the four boson scattering term. The residue of the pole is determined by c_b evaluated at (337) (in the case $\lambda > 0$) and b_b evaluated at the same pole (in the case $\lambda < 0$). In either case we find the structure of the pole for four boson scattering to be

$$\mathcal{T}_B \sim \frac{\left(\frac{\delta y}{2}\right)^{2-|\lambda|}}{\delta w - \frac{1}{2}\left(\frac{\delta y}{2}\right)^{1-|\lambda|}} . \tag{338}$$

In a similar manner the residue of the pole for four fermion scattering is determined by c_f evaluated at (327) (in the case $\lambda > 0$) and b_f evaluated at the same pole (in the case $\lambda < 0$). In either case we find that

$$\mathcal{T}_F \sim \frac{\left(\frac{\delta y}{2}\right)^{1-|\lambda|}}{\delta w - \frac{1}{2}\left(\frac{\delta y}{2}\right)^{1-|\lambda|}} . \tag{339}$$

Notice that the residue of the pole for four boson scattering is suppressed by a factor of $(\delta w)^{\frac{1}{1-|\lambda|}}$ compared to the residue for four fermion scattering.

4.5.4 Duality invariance

It is most interesting to note that the statements and results obtained in the above sections ((327), (331) and (337)) are all duality invariant. This is most transparent from the observation that under the duality transformation (130)[74]

$$\begin{aligned} d_1 &\leftrightarrow d_1 \ , \\ d_2 &\leftrightarrow d_2 \ . \end{aligned} \tag{340}$$

Hence the zeroes of d_1 and d_2 ((327) and (337)) should map to themselves, and w_c (331) should be duality invariant. Also recollect that under duality the bosonic and fermionic S matrices map to one another. Thus it is natural to expect that the pole in the bosonic S matrix at $w = -1$ (327) should map to the pole of the fermionic S matrix at $w = -\infty$ (337) and vice versa. Since both the bosonic and fermionic S matrices have a pole at $w = w_c$ (331) at $y = 0$, this pole should be self dual.

Upon using (130) on (331) it is straightforward to see that it is duality invariant. The slightly non-trivial part is the mapping of the two scaling regimes (327) and (337). It is straightforward to obtain the identification from $w = -\infty$ to $w = -1$ from (130)

$$-\frac{1}{\delta w_\infty} = \frac{3 - (-1 - \delta w_{-1})}{1 + (-1 - \delta w_{-1})} \sim -\frac{4}{\delta w_{-1}} \tag{341}$$

Using the above result in (337) and applying (130) for λ it is easy to check that (327) follows (and vice versa).

[74]Under duality transformation d_1 and d_2 transform into one another upto an overall non-zero factor. This overall factor is cancelled by an identical contribution from the duality transform of the numerator.

4.5.5 Scaling limit of the S matrix

In this subsection we discuss a particularly interesting near-threshold limit of the S-matrix. It was shown in [64] that in this limit the S matrices for the boson-boson and fermion-fermion reduce to the ones that are obtained by solving the Schrodinger equation with Amelino-Camelia-Bak boundary conditions [81, 82]. In this subsection we illustrate that the analysis of [64] applies for our results as well. We consider the near threshold region

$$y = 1 + \frac{k^2}{2m^2} \tag{342}$$

with $k << 1$ and

$$w = -1 - \delta w \tag{343}$$

where $0 < \delta w << 1$. In the limit

$$k \to 0, \quad \delta w \to 0, \quad , \frac{k^2}{4m^2}\left(\frac{\delta w}{2}\right)^{-\frac{1}{|\lambda|}} = \text{fixed} \tag{344}$$

the J function in the bosonic S matrix ((321)) reduces to [75]

$$J_B = 8|m \sin(\pi\lambda)| \frac{1 + e^{i\pi|\lambda|}\frac{A_R}{k^{2|\lambda|}}}{1 - e^{i\pi|\lambda|}\frac{A_R}{k^{2|\lambda|}}}. \tag{345}$$

where

$$A_R = \frac{4^{|\lambda|}}{2}|m|^{2|\lambda|}\delta w \ . \tag{346}$$

Comparing our Lagrangian (129) with that of eq 1.1 of [64] we make the parameter identifications

$$\delta w = \frac{\delta b_4}{8|m|\pi\lambda} \ .$$

Substituting δw in (346) we see that (345) matches exactly with eq 1.12 of [64].

[75] Here we work in the regime $\sqrt{s} > 2m$ i.e $y > 1$ and hence the appearance of the factors of $e^{i\pi\lambda}$.

4.5.6 Effective theory near $w = w_c$?

As we have explained above, our theory develops a massless bound state at $w = w_c$; the mass of this bound state scales like $w - w_c$ in units of the mass of the scattering particles. [76] When $w - w_c \ll 1$ there is a separation of scales between the new bound state and all other excitations in our theory. In this regime the effective dynamics of the nearly massless particles should be governed by an autonomous quantum field theory that makes no reference to UV degrees of freedom. It seems likely that the superfield that creates the bound states is a real $\mathcal{N} = 1$ superfield. The fixed point that governs the dynamics of this field presumably has a single relevant deformation; as it was possible to approach this theory with a single fine tuning (setting $w = w_c$). These considerations suggest that the dynamics of the light bound state is governed by an $\mathcal{N} = 1$ Wilson-Fisher theory built out of a single real superfield. If this suggestion is correct it would imply that the long distance dynamics of the light bound states is independent of λ. Given that the bound states are gauge neutral this possibility does not seem absurd to us. It would be interesting to study this suggestion in future work.

4.6 Discussion

In this chapter we have presented computations and conjectures for the all orders S matrix in the most general renormalizable $\mathcal{N} = 1$ Chern-Simons matter theory with a single fundamental matter multiplet. Our results are consistent with unitarity if we assume that the usual results of crossing symmetry are modified in precisely the manner proposed in [61], whereas the usual crossing symmetry rules are inconsistent with unitarity. We view this fact as a nontrivial consistency check of the crossing symmetry rules proposed in [61].

The 'particle - antiparticle' S matrix in the singlet channel conjectured in this

[76] We expect all of these results to continue to hold at finite N at least when N is large; in the rest of the discussion we assume that N is finite, and so the interactions between two bound state particles is not parametrically suppressed.

chapter has an interesting analytic structure. In a certain range of superpotential parameters the S matrix has a bound state pole; a one parameter tuning of superpotential parameters can be used to set the pole mass to zero. We find the existence of a massless bound state in a theory whose elementary excitations are all massive fascinating. It would be interesting to further investigate the low energy dynamics of these massless bound states. It would also be interesting to investigate if these bound states are 'visible' in the explicit results for the partition functions of Chern-Simons matter theories.

As we have explained in the previous section, our singlet sector particle - antiparticle S matrix has a simple non-relativistic limit. It would be useful to reproduce this scattering amplitude from the solution of a manifestly supersymmetric Schrodinger equation.

The results of this chapter suggest many natural extensions and questions. First it would be useful to generalize the computations of this chapter to the mass deformed $\mathcal{N} = 3$ and especially to the mass deformed $\mathcal{N} = 6$ susy gauge theories (the later is necessarily a $U(N) \times U(M)$ theory; the methods of this chapter are likely to be useful in the limit $N \to \infty$ with M held fixed). This generalization should allow us to make contact with earlier studies of scattering in ABJ theory [39–45] that were performed arbitrary values of M and N but perturbatively (to given loop order) in λ.

At the $\mathcal{N} = 2$ point the S matrices presented in this chapter are tree level exact in the three non anyonic channels, and depend on λ in a very simple way in the singlet channel. It is possible that this very simple result can be deduced in a more structural manner using only general principles and $\mathcal{N} = 2$ supersymmetry. It would be interesting if this were the case.

As an intermediate step in the computation of the S matrix we evaluated the off shell four point function of four superfields. This four point correlator was rather complicated in the general $\mathcal{N} = 1$ theory, but extremely simple at the $\mathcal{N} = 2$ point. The four point correlator (or sum of ladder diagrams) is a useful intermediate piece in the evaluation of two, three and four point functions of gauge invariant operators [51, 54, 65, 67]. The simplicity of the $\mathcal{N} = 2$ results suggest that it would be rather easy to explicitly evaluate such correlators, at least in special kinematic limits. Such computations could be used as independent checks of duality as well as well as inputs into $\mathcal{N} = 2$ generalizations of the Maldacena-Zhiboedov solutions of Chern-Simons fundamental matter theories [48, 49].

All of the computations in this chapter have been performed under the assumption $\lambda m \geq 0$. Atleast naively all of the checks of duality (including earlier checks involving the partition function) fail when $\lambda m < 0$. It would be interesting to understand why this is the case. It is possible that our theory undergoes a phase transition as λm changes sign (see [55, 58] for related discussion). It would be interesting to understand this better.

We believe that the results of this chapter put the crossing symmetry relations conjectured in [61] on a firm footing. It would be interesting to find a rigorous proof of these crossing relations, and even more interesting to hit upon a plausible generalization of these relations to finite N and k. From a traditional perturbative point of view the modified crossing symmetry rules are presumably related to infrared divergences. It thus seems possible that one route to a proof and generalization of these relations lies in a detailed study of the infrared divergences of the relevant Feynman graphs. We hope to return to several of these questions in the future.

4.7 Appendices for Chapter 3

4.7.1 Notations and conventions

Gamma matrices In this section, we list the various notations and conventions used in this chapter. We follow those of [83]. We list them here for convenience.

The metric signature is $\eta_{\mu\nu} = \{-, +, +\}$. In three dimensions the Lorentz group is $SL(2,\mathbb{R})$ and it acts on two component real spinors ψ^α, where α are the spinor indices. A vector is represented by either a real and symmetric spinor $V_{\alpha\beta}$ or a symmetric traceless spinor V_α^β, where $V_{\alpha\beta} = V_\mu \gamma^\mu_{\alpha\beta}$. We will choose our gamma matrices in the real and symmetric form [84]

$$\gamma^\mu_{\alpha\beta} = \{\mathbb{I}, \sigma^3, \sigma^1\} \ . \tag{347}$$

The charge conjugation matrix $C_{\alpha\beta}$ is used to raise and lower the spinor indices

$$C_{\alpha\beta} = -C_{\beta\alpha} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = -C^{\alpha\beta} \ . \tag{348}$$

In the above, note that $C_{\beta\alpha} = C^T$ and $C^{\alpha\beta} = (C^T)^{-1}$. It follows that

$$C_{\alpha\gamma} C^{\gamma\beta} = -\delta_\alpha^{\ \beta} \ , \tag{349}$$

where $\delta_\alpha^{\ \beta}$ is the usual identity matrix. The spinor indices are raised and lowered using the NW-SE convention

$$\psi^\alpha = C^{\alpha\beta}\psi_\beta \ , \psi_\alpha = \psi^\beta C_{\beta\alpha} \ . \tag{350}$$

We also use the notation $\psi^2 = \frac{1}{2}\psi^\alpha\psi_\alpha = i\psi^+\psi^-$. Note that ψ^2 is Hermitian. Since ψ^α is real, it is clear that ψ_α is imaginary since the charge conjugate matrix is imaginary.

The Clifford algebra is defined using the matrices $(\gamma^\mu)_\alpha^{\ \beta}$ and these can be obtained by raising the indices using $C^{\alpha\beta}$ as illustrated above

$$(\gamma^\mu)_\alpha^{\ \beta} = \{\sigma^2, -i\sigma^1, i\sigma^3\} \ . \tag{351}$$

Note that these matrices are purely imaginary. Choosing the $\gamma^\mu_{\alpha\beta}$ as real and symmetric always yields this and vice versa. Our $\mu = 0, 1, 3$, since at some point we will do an euclidean rotation from the $\mu = 0$ direction to $\mu = 2$. It is clear that $(\gamma^0)^2 = 1, (\gamma^1)^2 =$

$-1, (\gamma^3)^2 = -1$, therefore with our metric conventions the Clifford algebra is satisfied by

$$(\gamma^\mu)_\alpha^{\ \tau}(\gamma^\nu)_\tau^{\ \beta} + (\gamma^\nu)_\alpha^{\ \tau}(\gamma^\mu)_\tau^{\ \beta} = -2\eta^{\mu\nu}\delta_\alpha^{\ \beta} \ . \tag{352}$$

Another very useful relation is

$$[\gamma^\mu, \gamma^\nu] = -2i\epsilon^{\mu\nu\rho}\gamma_\rho \ , \epsilon^{013} = -1 \ . \tag{353}$$

For completion we also note that

$$(\gamma^\mu)^{\alpha\beta} = \{-\mathbb{I}, \sigma^3, \sigma^1\} \ . \tag{354}$$

As a consequence of the Clifford algebra (352), we get a minus sign in the trace

$$k_\alpha^{\ \beta} k_\beta^{\ \alpha} = -2k^2 \ . \tag{355}$$

The Euclidean counterpart of (352) is obtained by the standard Euclidean rotation $\gamma^0 \to i\gamma^2$

$$(\gamma^\mu)_\alpha^{\ \beta} = \{i\sigma^2, -i\sigma^1, i\sigma^3\} \ , \ \mu = 2, 1, 3, \tag{356}$$

and they satisfy the Euclidean Clifford algebra

$$(\gamma^\mu)_\alpha^{\ \tau}(\gamma^\nu)_\tau^{\ \beta} + (\gamma^\nu)_\alpha^{\ \tau}(\gamma^\mu)_\tau^{\ \beta} = -2\delta^{\mu\nu}\delta_\alpha^{\ \beta} \ . \tag{357}$$

where $\delta_{\mu\nu} = (+, +, +)$. Note that the euclidean continuation of the formula (353) is

$$[\gamma^\mu, \gamma^\nu] = 2\epsilon^{\mu\nu\rho}\gamma_\rho \ , \epsilon^{123} = 1 \ . \tag{358}$$

Superspace The two component Grassmann parameters θ that appear in various places in superspace have the properties

$$\int d\theta = 0 \ , \int d\theta\theta = 1 \ , \int d^2\theta\theta^2 = -1 \ , \int d^2\theta\theta^\alpha\theta^\beta = C^{\alpha\beta} \ ,$$
$$\frac{\partial\theta^\alpha}{\partial\theta^\beta} = \delta_\beta{}^\alpha \ , C^{\alpha\beta}\frac{\partial}{\partial\theta^\beta}\frac{\partial}{\partial\theta^\alpha}\theta^2 = -2 \ , \theta^\alpha\theta^\beta = -C^{\alpha\beta}\theta^2 \ , \theta_\alpha\theta_\beta = -C_{\alpha\beta}\theta^2 \ . \tag{359}$$

The definition of the delta function in superspace follows from the relation

$$\int d^2\theta\theta^2 = -1 \implies \delta^2(\theta) = -\theta^2 \ . \tag{360}$$

Formally we write

$$\delta^2(\theta_1 - \theta_2) = -(\theta_1 - \theta_2)^2 = -(\theta_1^2 + \theta_2^2 - \theta_1\theta_2). \tag{361}$$

The superspace derivatives are defined as

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\theta^\beta\partial_{\alpha\beta} \ , D^\alpha = C^{\alpha\beta}D_\beta \ . \tag{362}$$

We will mostly use the momentum space version of the above in which we replace $i\partial_{\alpha\beta} \to k_{\alpha\beta}$

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \theta^\beta k_{\alpha\beta} \ . \tag{363}$$

Note that the choice of the real and symmetric basis in 347 makes the momentum operator Hermitian. The superspace derivatives satisfy the algebra

$$\{D_\alpha, D_\beta\} = 2k_{\alpha\beta} \ . \tag{364}$$

The tracelessness of $(\gamma^\mu)_\alpha^{\ \beta}$ implies that

$$\{D^\alpha, D_\alpha\} = 0 \ . \tag{365}$$

Care has to be taken when integrating by parts with superderivatives due to their anti-commuting nature. From the expression for D_α we can construct

$$D^2 = \frac{1}{2} D^\alpha D_\alpha = \frac{1}{2}\left(C^{\beta\alpha} \frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta} + 2\theta^\alpha k_\alpha^{\ \beta} \frac{\partial}{\partial\theta^\beta} + 2\theta^2 k^2 \right) \ . \tag{366}$$

From the above it is easy to verify

$$\begin{aligned}
&(D^2)^2 = -k^2, \\
&D^2 D_\alpha = -D_\alpha D^2 = k_{\alpha\beta} D^\beta \\
&D^\alpha D_\beta D_\alpha = 0 \ .
\end{aligned} \tag{367}$$

using the properties given in (359). Yet another extremely useful relation is the action of the superderivative square (366) on the delta function (361)

$$D^2_{\theta_1,k}\delta^2(\theta_1 - \theta_2) = 1 - \theta_1^\alpha \theta_2^\beta k_{\alpha\beta} - \theta_1^2\theta_2^2 k^2 = \exp(-\theta_1^\alpha\theta_2^\beta k_{\alpha\beta}) \ . \tag{368}$$

We will often suppress the spinor indices in the exponential with the understanding that the spinor indices are contracted as indicated above. Some useful formulae are

$$\begin{aligned}
&\delta^2(\theta_1 - \theta_2)\delta^2(\theta_2 - \theta_1) = 0 \ , \\
&\delta^2(\theta_1 - \theta_2) D^\alpha_{\theta_2,k}\delta^2(\theta_2 - \theta_1) = 0 \ , \\
&\delta^2(\theta_1 - \theta_2) D^2_{\theta_2,k}\delta^2(\theta_2 - \theta_1) = \delta^2(\theta_1 - \theta_2) \ ,
\end{aligned} \tag{369}$$

and the transfer rule

$$D_\alpha^{\theta_1,p}\delta^2(\theta_1 - \theta_2) = -D_\alpha^{\theta_2,-p}\delta^2(\theta_2 - \theta_1) \ . \tag{370}$$

The supersymmetry generators

$$Q_\alpha^{\theta,k} = i\left(\frac{\partial}{\partial\theta^\alpha} - \theta^\beta k_{\alpha\beta}\right) \ , \tag{371}$$

satisfy the anticommutation relations

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= 2k_{\alpha\beta} \ , \\ \{Q_\alpha, D_\beta\} &= 0 \ . \end{aligned} \tag{372}$$

It is also clear that the transfer rule (370) is the statement that the delta function of θ is a supersymmetric invariant.

Superfields The scalar superfield $\Phi(x,\theta)$ contains a complex scalar ϕ, a complex fermion ψ^α, and a complex auxiliary field F. The vector superfield $\Gamma^\alpha(x,\theta)$ consists of the gauge field $V_{\alpha\beta}$, the gaugino λ^α, an auxiliary scalar B and an auxiliary fermion χ^α. The following superfield expansions are used repeatedly in several places. We list them here for easy reference.

$$\begin{aligned} &\Phi = \phi + \theta\psi - \theta^2 F \ , \\ &\bar\Phi = \bar\phi + \theta\bar\psi - \theta^2 \bar F \ , \\ &\bar\Phi\Phi = \bar\phi\phi + \theta^\alpha(\bar\phi\psi_\alpha + \bar\psi_\alpha\phi) - \theta^2(\bar F\phi + \bar\phi F + \bar\psi\psi) \ , \\ &D_\alpha\Phi = \psi_\alpha - \theta_\alpha F + i\theta^2\partial_\alpha{}^\beta\psi_\beta + i\theta^\beta\partial_{\alpha\beta}\phi \ , \\ &D^\alpha\bar\Phi D_\alpha\Phi\Big|_{\theta^2} = \theta^2(2\bar F F + 2i\bar\psi^\alpha\partial_\alpha{}^\beta\psi_\beta - 2\partial\bar\phi\partial\phi) \ , \\ &D^2_{q,\theta}(\bar\Phi\Phi) = (\bar\phi F + \bar F\phi + \bar\psi\psi) + \theta^\alpha q_\alpha{}^\beta(\bar\phi\psi + \bar\psi\phi) + \theta^2 q^2(\bar\phi\phi)^2 \ , \\ &\Gamma^\alpha = \chi^\alpha - \theta^\alpha B + i\theta^\beta A_\beta{}^\alpha - \theta^2(2\lambda^\alpha - i\partial^{\alpha\beta}\chi_\beta) \ . \end{aligned} \tag{373}$$

4.7.2 A check on the constraints of supersymmetry on S matrices

In §4.2.4 we demonstrated that the manifestly supersymmetric scattering of any $\mathcal{N} = 1$ theory in three dimensions is described by two independent functions. In this section, we directly verify this result in theories whose offshell effective action takes the form (223) with the function V that takes the particular supersymmetric form (227) (and so is determined by four unspecified functions A, B C and D).

We wish to use (223) to study scattering. In order to do this we evaluate (223) with the fields Φ and $\bar{\Phi}$ in that action chosen to be the most general linearized onshell solutions to the equations of motion. In this appendix we focus on a particular scattering process - scattering in the adjoint channel. At leading order in the large N limit we can focus on this channel by choosing the solution for Φ_m and $\bar{\Phi}^m$ in (223) to be positive energy solutions (representing initial states), while $\bar{\Phi}^m$ and Φ_n are expanded in negative energy solutions (representing final states). The negative and positive energy solutions are both allowed to be an arbitrary linear combination of bosonic and fermionic solutions. Plugging these solutions into (223) yields a functional of the coefficients of the bosonic and fermionic solutions in the four superfields in (223). The coefficients of various terms in this functional are simply the S matrices. For instance the coefficient of the term proportional to the product of four bosonic modes is the four boson scattering amplitude, etc.

Let us schematically represent the scattering process we study by

$$\begin{pmatrix} \Phi(\theta_1, p_1) \\ \bar{\Phi}(\theta_2, p_2) \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\Phi}(\theta_3, p_3) \\ \Phi(\theta_4, p_4) \end{pmatrix}$$

where the LHS represents the in-states and the RHS represents the out-states. The momentum assignments in (223) are

$$p_1 = p + q \ , \ p_2 = -k - q \ , \ p_3 = p \ , \ p_4 = -k \ . \tag{374}$$

In component form (374) encodes the following S matrices

$$\mathcal{S}_B : \begin{pmatrix} \phi(p_1) \\ \bar{\phi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\phi}(p_3) \\ \phi(p_4) \end{pmatrix} \ , \ \mathcal{S}_F : \begin{pmatrix} \psi(p_1) \\ \bar{\psi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\psi}(p_3) \\ \psi(p_4) \end{pmatrix}$$

$$H_1 : \begin{pmatrix} \phi(p_1) \\ \bar{\phi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\psi}(p_3) \\ \psi(p_4) \end{pmatrix} \ , \ H_2 : \begin{pmatrix} \psi(p_1) \\ \bar{\psi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\phi}(p_3) \\ \phi(p_4) \end{pmatrix}$$

$$H_3 : \begin{pmatrix} \phi(p_1) \\ \bar{\psi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\phi}(p_3) \\ \psi(p_4) \end{pmatrix} \ , \ H_4 : \begin{pmatrix} \psi(p_1) \\ \bar{\phi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\psi}(p_3) \\ \phi(p_4) \end{pmatrix}$$

$$H_5 : \begin{pmatrix} \phi(p_1) \\ \bar{\psi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\psi}(p_3) \\ \phi(p_4) \end{pmatrix} \ , \ H_6 : \begin{pmatrix} \psi(p_1) \\ \bar{\phi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\phi}(p_3) \\ \psi(p_4) \end{pmatrix} \tag{375}$$

These S matrix elements are all obtained by the process spelt out above in terms of the four unknown functions A, B, C, D (which we will take to be arbitrary and unrelated). The functions A,B,C,D are to be evaluated at the onshell conditions that follow from taking the momenta onshell, but that will play no role in what follows.

It is not difficult to demonstrate that the boson-boson $\to$ boson boson and the fermion-fermion $\to$ fermion fermion S matrices are given in terms of the functions A, B,

C and D by [77]

$$\mathcal{S}_B = (-4iAm + 4Bm^2 - Bq_3^2 - q_3(Ck_- + Dp_-)) \ ,$$
$$\mathcal{S}_F = (BC^{\beta\alpha}C^{\delta\gamma} - iC\ C^{\beta\alpha}C^{+\gamma}C^{+\delta} + iDC^{\delta\gamma}C^{+\alpha}C^{+\beta})\bar{u}_\alpha(p_3)u_\beta(p_1)v_\gamma(p_2)\bar{v}_\delta(p_4)$$
$$= -B(4m^2 + q_3^2) + Ck_-(2im - q_3) - Dp_-(q_3 + 2im) \ . \tag{377}$$

The S matrices for the remaining processes in (375) are also easily obtained: we find

$$H_i = a_i\mathcal{S}_B + b_i\mathcal{S}_F \tag{378}$$

where the coefficients are given by

$$\begin{aligned}
a_1 &= \frac{(4m^2+q_3^2)\,(q_3(p-k)_- + 2im(k+p)_-)}{32mk_-p_-\sqrt{k_+p_+}} &, b_1 &= \frac{(4m^2+q_3^2)\,(q_3(k-p)_- + 2im(k+p)_-)}{32mk_-p_-\sqrt{k_+p_+}}\\
a_2 &= \frac{(4m^2+q_3^2)\,(q_3(p-k)_- + 2im(k+p)_-)}{32mk_-p_-\sqrt{k_+p_+}} &, b_2 &= \frac{(4m^2+q_3^2)\,(q_3(k-p)_- + 2im(k+p)_-)}{32mk_-p_-\sqrt{k_+p_+}}\\
a_3 &= -\frac{2m+iq_3}{4m} &, b_3 &= \frac{2m+iq_3}{4m}\\
a_4 &= \frac{2m-iq_3}{4m} &, b_4 &= -\frac{2m-iq_3}{4m}\\
a_5 &= \frac{(4m^2+q_3^2)\,(q_3(k+p)_- - 2im(k-p)_-)}{32mk_-p_-\sqrt{k_+p_+}} &, b_5 &= -\frac{i\,(4m^2+q_3^2)\,(2m(k-p)_- - iq_3(k+p)_-)}{32mk_-p_-\sqrt{k_+p_+}}\\
a_6 &= \frac{i\,(4m^2+q_3^2)\,(2m(k-p)_- + iq_3(k+p)_-)}{32mk_-p_-\sqrt{k_+p_+}} &, b_6 &= \frac{(4m^2+q_3^2)\,(q_3(k+p)_- + 2im(k-p)_-)}{32mk_-p_-\sqrt{k_+p_+}}
\end{aligned} \tag{379}$$

The above set of coefficients match with the coefficients directly evaluated from (155) and

[77]For the T channel we have used

$$v_\alpha(-k) = \begin{pmatrix} -\sqrt{k_+} \\ \frac{(q_3-2im)}{2\sqrt{k_+}} \end{pmatrix} \ , \ \bar{v}^\alpha(-k-q) = \begin{pmatrix} -\frac{2m+iq_3}{2\sqrt{k_+}} & i\sqrt{k_+} \end{pmatrix}$$
$$u_\alpha(p+q) = \begin{pmatrix} -i\sqrt{p_+} \\ \frac{2m-iq_3}{2\sqrt{p_+}} \end{pmatrix} \ , \ \bar{u}^\alpha(p) = \begin{pmatrix} -\frac{(2im+q_3)}{2\sqrt{p_+}} & -\sqrt{p_+} \end{pmatrix} \tag{376}$$

(156). This is a consistency check of the results of §4.2.4.

For the $\mathcal{N}=2$ theory the S matrix (154) should also obey an additional constraint (see §4.7.3) that relates $\mathcal{S}_B$ and $\mathcal{S}_F$ through (404). For the T channel this relation was evaluated in (288), substituting this in (378) it is easy to verify that the $\theta_2\theta_3$ and $\theta_1\theta_4$ terms in (154)

$$H_5 : \begin{pmatrix} \phi(p_1) \\ \bar{\psi}(p_2) \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\psi}(p_3) \\ \phi(p_4) \end{pmatrix} \ , \ H_6 : \begin{pmatrix} \psi(p_1) \\ \bar{\phi}(p_2) \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\phi}(p_3) \\ \psi(p_4) \end{pmatrix} \tag{380}$$

vanish for the $\mathcal{N}=2$ theory. This is consistent with the fact that the corresponding terms in the tree level component Lagrangian (129) vanish at the $\mathcal{N}=2$ point $w=1$.

4.7.3 Manifest $\mathcal{N}=2$ supersymmetry invariance

In this appendix we discuss the general constraints on the S matrix obtained by imposing $\mathcal{N}=2$ supersymmetry. In subsection 4.2.4 we have already solved the constraints coming from $\mathcal{N}=1$ supersymmetry. As an $\mathcal{N}=2$ theory is in particular also $\mathcal{N}=1$ supersymmetric, the results of this appendix will be a specialization of those of subsection 4.2.4.

In the case of $\mathcal{N}=2$, we have to recall the notion of chirality. A 'chiral' (antichiral) $\mathcal{N}=2$ superfield Φ is defined as

$$\bar{D}_\alpha \Phi = 0, \qquad D_\alpha \bar{\Phi} = 0. \tag{381}$$

We define the following:

$$\theta_\alpha = \frac{1}{\sqrt{2}}(\theta^{(1)}_\alpha - i\theta^{(2)}_\alpha), \qquad \bar{\theta}_\alpha = \frac{1}{\sqrt{2}}(\theta^{(1)}_\alpha + i\theta^{(2)}_\alpha). \tag{382}$$

Where the superscripts (1) and (2) indicate the two (real) copies of the $\mathcal{N}=1$ superspace.

With these definitions, we can define the supercharges

$$Q_\alpha = \frac{1}{\sqrt{2}}(Q^{(1)}_\alpha + iQ^{(2)}_\alpha) = i\left(\frac{\partial}{\partial\theta^\alpha} - i\bar{\theta}^\beta \partial_{\beta\alpha}\right), \tag{383}$$

$$\bar{Q}_\alpha = \frac{1}{\sqrt{2}}(Q^{(1)}_\alpha - iQ^{(2)}_\alpha) = i\left(\frac{\partial}{\partial\bar{\theta}^\alpha} - i\theta^\beta \partial_{\beta\alpha}\right). \tag{384}$$

Likewise, we can define the supercovariant derivative operators

$$D_\alpha = \frac{1}{\sqrt{2}}(D^{(1)}_\alpha + iD^{(2)}_\alpha) = \left(\frac{\partial}{\partial\theta^\alpha} + i\bar{\theta}^\beta \partial_{\beta\alpha}\right), \tag{385}$$

$$\bar{D}_\alpha = \frac{1}{\sqrt{2}}(D^{(1)}_\alpha - iD^{(2)}_\alpha) = \left(\frac{\partial}{\partial\bar{\theta}^\alpha} + i\theta^\beta \partial_{\beta\alpha}\right). \tag{386}$$

The solutions to the constraints (381) for (off-shell) chiral and anti-chiral fields are

$$\Phi = \phi + \sqrt{2}\theta\psi - \theta^2 F + i\theta\bar{\theta}\partial\phi - i\sqrt{2}\theta^2(\bar{\theta}\partial\!\!\!/\psi) + \theta^2\bar{\theta}^2\partial^2\phi, \tag{387}$$

$$\bar{\Phi} = \bar{\phi} + \sqrt{2}\bar{\theta}\bar{\psi} - \bar{\theta}^2\bar{F} - i\theta\bar{\theta}\partial\bar{\phi} - i\sqrt{2}\bar{\theta}^2(\theta\partial\!\!\!/\bar{\psi}) + \theta^2\bar{\theta}^2\partial^2\bar{\phi}. \tag{388}$$

Here $\theta\bar{\theta}\partial\phi = \theta^\alpha\bar{\theta}^\beta\partial_{\alpha\beta}$ and $\bar{\theta}\partial\!\!\!/\psi = \bar{\theta}^\alpha\partial_\alpha{}^\beta\psi_\beta$ and so on.

In the context of the current chapter the chiral matter superfield transforms in the fundamental representation of the gauge group while the antichiral matter superfield transforms in the antifundamental representation of the gauge group. It follows that it is impossible to add a gauge invariant quadratic superpotential to our action (recall that an $\mathcal{N}=2$ superpotential can only depend on chiral multiplets) in order to endow our fields with mass. However it is possible to make the matter fields massive while preserving $\mathcal{N}=2$ supersymmetry; the fields can be made massive using a D term.

As our theory has no superpotential, it follows that $F = \bar{F} = 0$ on shell. We are interested in the action of supersymmetry on the on-shell component fields ϕ ($\bar{\phi}$) which are defined as

$$\phi(x) = \int \frac{d^2p}{(2\pi)^2\sqrt{2p^0}} \left[a(\mathbf{p})e^{ip\cdot x} + a^{c\dagger}(\mathbf{p})e^{-ip\cdot x}\right], \tag{389}$$

$$\bar{\phi}(x) = \int \frac{d^2p}{(2\pi)^2\sqrt{2p^0}} \left[a^c(\mathbf{p})e^{ip\cdot x} + a^{\dagger}(\mathbf{p})e^{-ip\cdot x}\right]. \tag{390}$$

Likewise, for ψ ($\psi^\dagger$) we have

$$\psi(x) = \int \frac{d^2p}{(2\pi)^2\sqrt{2p^0}} \left[u_\alpha(\mathbf{p})\alpha(\mathbf{p})e^{ip\cdot x} + v_\alpha(\mathbf{p})\alpha^{c\dagger}(\mathbf{p})e^{-ip\cdot x}\right], \tag{391}$$

$$\psi^\dagger(x) = \int \frac{d^2p}{(2\pi)^2\sqrt{2p^0}} \left[u_\alpha(\mathbf{p})\alpha^c(\mathbf{p})e^{ip\cdot x} + v_\alpha(\mathbf{p})\alpha^{\dagger}(\mathbf{p})e^{-ip\cdot x}\right]. \tag{392}$$

In order to obtain this action we used the transformation properties listed in equations F.16-F.20 of [52] and then specialized to the onshell limit. [78] The results may be summarized as follows. As before, we define the (super) creation and annihilation operators

$$A(\mathbf{p}) = a(\mathbf{p}) + \alpha(\mathbf{p})\theta, \qquad A^c(\mathbf{p}) = a^c(\mathbf{p}) + \alpha^c(\mathbf{p})\theta, \tag{393}$$

$$A^\dagger(\mathbf{p}) = a^\dagger(\mathbf{p}) + \theta\alpha^\dagger(\mathbf{p}), \qquad A^{c\dagger}(\mathbf{p}) = a^{c\dagger}(\mathbf{p}) + \theta\alpha^{c\dagger}(\mathbf{p}). \tag{394}$$

The action of Q_α (and $\bar{Q}_\alpha$) on A and $A^\dagger$ is

$$\begin{aligned} [Q_\alpha, A(\mathbf{p})] &= -i\sqrt{2}u_\alpha(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta}, \qquad [\bar{Q}_\alpha, A(\mathbf{p})] = i\sqrt{2}u^*_\alpha(\mathbf{p})\theta, \\ [Q_\alpha, A^\dagger(\mathbf{p})] &= i\sqrt{2}v^*_\alpha(\mathbf{p})\theta, \qquad [\bar{Q}_\alpha, A^\dagger(\mathbf{p})] = i\sqrt{2}v_\alpha(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta}. \end{aligned} \tag{395}$$

[78] Note that the action of Q_α on the chiral field Φ is different from the action on the anti-chiral field $\bar{\Phi}$. Similar remarks apply for $\bar{Q}_\alpha$.

Similarly, the action of Q_α (and $\bar{Q}_\alpha$) on A^c and $A^{c\dagger}$ is

$$
\begin{aligned}
[Q_\alpha, A^c(\mathbf{p})] &= i\sqrt{2}u^*_\alpha(\mathbf{p})\theta, \quad [\bar{Q}_\alpha, A^c(\mathbf{p})] = -i\sqrt{2}u_\alpha(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta}, \\
[Q_\alpha, A^{c\dagger}(\mathbf{p})] &= i\sqrt{2}v_\alpha(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta}, \quad [\bar{Q}_\alpha, A^{c\dagger}(\mathbf{p})] = i\sqrt{2}v^*_\alpha(\mathbf{p})\theta.
\end{aligned} \tag{396}
$$

It is clear from (395) that $(Q_\alpha + \bar{Q}_\alpha)/\sqrt{2}$ produces the action of the first supercharge $Q^{(1)}_\alpha$, which we have seen earlier. That this action produces the correct differential operator given earlier is obvious as well. Therefore, in order to obtain the second supercharge $Q^{(2)}_\alpha$, we simply operate with the other linear combination $(Q_\alpha - \bar{Q}_\alpha)/i\sqrt{2}$.

Note that for the $\mathcal{N} = 1$ case, it doesn't matter if we used $A^\dagger$ or $A^{c\dagger}$ for the initial states (A or A^c for the final states), as is clear from (396). This agrees with the fact that the linear combination $(Q_\alpha + \bar{Q}_\alpha)/\sqrt{2}$ produces the same equation on all S matrix elements. However other linear combinations of the two $\mathcal{N} = 2$ supersymmetries act differently on A and A^c, and so the constraints of $\mathcal{N} = 2$ supersymmetry are different depending on which scattering processes we consider.

Particle - antiparticle scattering Let us first study the invariance of the following S matrix element

$$
S(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) = \langle 0|A_4(\mathbf{p}_4, \theta_4)A^c_3(\mathbf{p}_3, \theta_3)A^\dagger_2(\mathbf{p}_2, \theta_2)A^{c\dagger}_1(\mathbf{p}_1, \theta_1)|0\rangle. \tag{397}
$$

In the context of our chapter, this is the S matrix for particle - antiparticle scattering. The full $\mathcal{N} = 2$ invariance of the S matrix is expressed as

$$
\left(\sum_{i=1}^{4} Q^i_\alpha(\mathbf{p}_i, \theta_i)\right) S(\mathbf{p}_i, \theta_i) = 0, \text{ and } \left(\sum_{i=1}^{4} \bar{Q}^i_\alpha(\mathbf{p}_i, \theta_i)\right) S(\mathbf{p}_i, \theta_i) = 0. \tag{398}
$$

The above conditions (398) produce the following constraints for the S matrix element (397)

$$\begin{aligned}
&\left(\sum_{i=1}^{4} Q^{i}_{\alpha}(\mathbf{p}_i,\theta_i)\right) S(\mathbf{p}_i,\theta_i) = 0 \Rightarrow \\
&\qquad \left(iv_{\alpha}(\mathbf{p}_1)\frac{\overrightarrow{\partial}}{\partial\theta_1} + iv^{*}(\mathbf{p}_2)\theta_2 + iu^{*}_{\alpha}(\mathbf{p}_3)\theta_3 - iu_{\alpha}(\mathbf{p}_4)\frac{\overrightarrow{\partial}}{\partial\theta_4}\right) S(\mathbf{p}_i,\theta_i) = 0, \\
&\left(\sum_{i=1}^{4} \bar{Q}^{i}_{\alpha}(\mathbf{p}_i,\theta_i)\right) S(\mathbf{p}_i,\theta_i) = 0 \Rightarrow \\
&\qquad \left(iv^{*}_{\alpha}(\mathbf{p}_1)\theta_1 + iv_{\alpha}(\mathbf{p}_2)\frac{\overrightarrow{\partial}}{\partial\theta_2} - iu_{\alpha}(\mathbf{p}_3)\frac{\overrightarrow{\partial}}{\partial\theta_3} + iu^{*}_{\alpha}(\mathbf{p}_4)\theta_4\right) S(\mathbf{p}_i,\theta_i) = 0.
\end{aligned} \tag{399}$$

We check in what follows that the combination

$$\left(\frac{1}{\sqrt{2}}\sum_{i=1}^{4} Q^{i}_{\alpha}(\mathbf{p}_i,\theta_i) + \bar{Q}^{i}_{\alpha}(\mathbf{p}_i,\theta_i)\right) S(\mathbf{p}_i,\theta_i) = 0 \tag{400}$$

produces the same equation (and therefore solution) of $\mathcal{N} = 1$ which we have already found. We easily find that this gives

$$\begin{aligned}
&\Bigg(iv_{\alpha}(\mathbf{p}_1)\frac{\overrightarrow{\partial}}{\partial\theta_1} + iv_{\alpha}(\mathbf{p}_2)\frac{\overrightarrow{\partial}}{\partial\theta_2} - iu_{\alpha}(\mathbf{p}_3)\frac{\overrightarrow{\partial}}{\partial\theta_3} - iu_{\alpha}(\mathbf{p}_4)\frac{\overrightarrow{\partial}}{\partial\theta_4} \\
&\qquad + iv^{*}_{\alpha}(\mathbf{p}_1)\theta_1 + iv^{*}_{\alpha}(\mathbf{p}_2)\theta_2 + iu^{*}_{\alpha}(\mathbf{p}_3)\theta_3 + iu^{*}_{\alpha}(\mathbf{p}_4)\theta_4\Bigg) S(\mathbf{p}_i,\theta_i) = 0.
\end{aligned} \tag{401}$$

Now, we turn to the other linear combination, which is

$$\left(\frac{1}{i\sqrt{2}}\sum_{i=1}^{4} Q^{i}_{\alpha}(\mathbf{p}_i,\theta_i) - \bar{Q}^{i}_{\alpha}(\mathbf{p}_i,\theta_i)\right) S(\mathbf{p}_i,\theta_i) = 0. \tag{402}$$

This readily gives the differential equation

$$\left(iv_\alpha(\mathbf{p}_1)\frac{\overrightarrow{\partial}}{\partial\theta_1} - iv_\alpha(\mathbf{p}_2)\frac{\overrightarrow{\partial}}{\partial\theta_2} + iu_\alpha(\mathbf{p}_3)\frac{\overrightarrow{\partial}}{\partial\theta_3} - iu_\alpha(\mathbf{p}_4)\frac{\overrightarrow{\partial}}{\partial\theta_4} \right.$$
$$\left. - iv^*_\alpha(\mathbf{p}_1)\theta_1 + iv^*_\alpha(\mathbf{p}_2)\theta_2 + iu^*_\alpha(\mathbf{p}_3)\theta_3 - iu^*_\alpha(\mathbf{p}_4)\theta_4 \right) S(\mathbf{p}_i, \theta_i) = 0. \qquad (403)$$

The equation (401) is the same as it was for the $\mathcal{N} = 1$ theory, whereas the second equation (403) must be obeyed by the same S matrix in the $\mathcal{N} = 2$ point. Thus (408) is an additional constraint obeyed by the $\mathcal{N} = 2$ S matrix (154). It follows that (403) gives a relation between $\mathcal{S}_B$ and $\mathcal{S}_F$

$$\mathcal{S}_B\left(C_{12}v_\alpha(\mathbf{p}_1) - C_{23}u_\alpha(\mathbf{p}_3) + C_{24}u_\alpha(\mathbf{p}_4) + v^*_\alpha(\mathbf{p}_2)\right) = \mathcal{S}_F(C^*_{13}u_\alpha(\mathbf{p}_4) + C^*_{14}u_\alpha(\mathbf{p}_3) + C^*_{34}v_\alpha(\mathbf{p}_1)) \qquad (404)$$

Thus, the $\mathcal{N} = 2$ S matrix for particle-antiparticle scattering consists of only one independent function, with the other related by (404).

Particle - particle scattering Now, consider the other S matrix element (which was considered in the previous $\mathcal{N} = 1$ computation)

$$S(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) = \langle 0|A_4(\mathbf{p}_4, \theta_4)A_3(\mathbf{p}_3, \theta_3)A^\dagger_2(\mathbf{p}_2, \theta_2)A^\dagger_1(\mathbf{p}_1, \theta_1)|0\rangle. \qquad (405)$$

The conditions (398) produce the following for the S matrix element (405)

$$\left(\sum_{i=1}^{4} Q_{\alpha}^{i}(\mathbf{p}_i,\theta_i)\right) S(\mathbf{p}_i,\theta_i) = 0 \Rightarrow$$
$$\left(iv_{\alpha}^{*}(\mathbf{p}_1)\theta_1 + iv^{*}(\mathbf{p}_2)\theta_2 - iu_{\alpha}(\mathbf{p}_3)\frac{\overrightarrow{\partial}}{\partial\theta_3} - iu_{\alpha}(\mathbf{p}_4)\frac{\overrightarrow{\partial}}{\partial\theta_4}\right) S(\mathbf{p}_i,\theta_i) = 0,$$
$$\left(\sum_{i=1}^{4} \bar{Q}_{\alpha}^{i}(\mathbf{p}_i,\theta_i)\right) S(\mathbf{p}_i,\theta_i) = 0 \Rightarrow$$
$$\left(iv_{\alpha}(\mathbf{p}_1)\frac{\overrightarrow{\partial}}{\partial\theta_1} + iv_{\alpha}(\mathbf{p}_2)\frac{\overrightarrow{\partial}}{\partial\theta_2} + iu_{\alpha}^{*}(\mathbf{p}_3)\theta_3 + iu_{\alpha}^{*}(\mathbf{p}_4)\theta_4\right) S(\mathbf{p}_i,\theta_i) = 0. \tag{406}$$

For the combination (400) we get

$$\begin{aligned}\Bigg(iv_{\alpha}(\mathbf{p}_1)\frac{\overrightarrow{\partial}}{\partial\theta_1} &+iv_{\alpha}(\mathbf{p}_2)\frac{\overrightarrow{\partial}}{\partial\theta_2} - iu_{\alpha}(\mathbf{p}_3)\frac{\overrightarrow{\partial}}{\partial\theta_3} - iu_{\alpha}(\mathbf{p}_4)\frac{\overrightarrow{\partial}}{\partial\theta_4} \\ &+ iv_{\alpha}^{*}(\mathbf{p}_1)\theta_1 + iv_{\alpha}^{*}(\mathbf{p}_2)\theta_2 + iu_{\alpha}^{*}(\mathbf{p}_3)\theta_3 + iu_{\alpha}^{*}(\mathbf{p}_4)\theta_4\Bigg) S(\mathbf{p}_i,\theta_i) = 0,\end{aligned} \tag{407}$$

and for the combination (402) we have

$$\begin{aligned}\Bigg(-iv_{\alpha}(\mathbf{p}_1)\frac{\overrightarrow{\partial}}{\partial\theta_1} &-iv_{\alpha}(\mathbf{p}_2)\frac{\overrightarrow{\partial}}{\partial\theta_2} - iu_{\alpha}(\mathbf{p}_3)\frac{\overrightarrow{\partial}}{\partial\theta_3} - iu_{\alpha}(\mathbf{p}_4)\frac{\overrightarrow{\partial}}{\partial\theta_4} \\ &+ iv_{\alpha}^{*}(\mathbf{p}_1)\theta_1 + iv_{\alpha}^{*}(\mathbf{p}_2)\theta_2 - iu_{\alpha}^{*}(\mathbf{p}_3)\theta_3 - iu_{\alpha}^{*}(\mathbf{p}_4)\theta_4\Bigg) S(\mathbf{p}_i,\theta_i) = 0.\end{aligned} \tag{408}$$

Similar to the particle-anti particle case discussed in the previous section. The equation (407) is the same as it was for the $\mathcal{N}=1$ theory, whereas the second equation (408) must be obeyed by the same S matrix in the $\mathcal{N}=2$ point. It follows that (408) gives a relation between $\mathcal{S}_B$ and $\mathcal{S}_F$

$$\mathcal{S}_B\left(C_{13}u_{\alpha}(\mathbf{p}_3) + C_{14}u_{\alpha}(\mathbf{p}_4) + C_{12}v_{\alpha}(\mathbf{p}_2) + v_{\alpha}^{*}(\mathbf{p}_1)\right) = \mathcal{S}_F(C_{24}^{*}u_{\alpha}(\mathbf{p}_3) - C_{23}^{*}u_{\alpha}(\mathbf{p}_4) + C_{34}^{*}v_{\alpha}(\mathbf{p}_2)) \tag{409}$$

The $\mathcal{N} = 2$ S matrix for particle-particle scattering consists of only one independent function, with the other related by (409).

Thus in the $\mathcal{N} = 2$ theory the S matrix is only made of one independent function. Note that the results of this section are true for *any* three dimensional $\mathcal{N} = 2$ theory. It simply follows from the supersymmetric ward identity (398) and is independent of the details of the theory.

4.7.4 Identities for S matrices in onshell superspace

In this subsection we demonstrate that the product of two supersymmetric S matrices is supersymmetric. In other words we demonstrate that

$$\left(\sum_{i=1}^{4} Q^i_\alpha(\mathbf{p}_i, \theta_i)\right) S_1 \star S_2 = 0. \tag{410}$$

provided S_1 and S_2 independently obey the same equation.

This can be analyzed as follows. We have the invariance (differential) equation for S_1 and S_2

$$\left(\overrightarrow{Q}_{\tilde{v}(\mathbf{p}_1)} + \overrightarrow{Q}_{\tilde{v}(\mathbf{p}_2)} + \overrightarrow{Q}_{u(\mathbf{p}_3)} + \overrightarrow{Q}_{u(\mathbf{p}_4)}\right) S_i(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) = 0$$
$$\text{with } p_1 + p_2 = p_3 + p_4. \tag{411}$$

where the left-acting supercharges $\overrightarrow{Q}_{\tilde{v}(\mathbf{p})}$ are defined as

$$\overrightarrow{Q}_{\tilde{v}(\mathbf{p})} = i\left(v_\alpha(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta} + v^*_\alpha(\mathbf{p})\theta\right) \tag{412}$$

in contrast to (150), because we're acting from the left. It may be easily checked that this indeed produces the correct action of Q on $A^\dagger$. The reader is reminded that the (left-acting) supercharges $\overrightarrow{Q}_{u(\mathbf{p})}$ are defined as

$$\overrightarrow{Q}_{u(\mathbf{p})} = i\left(-u_\alpha(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta} + u^*_\alpha(\mathbf{p})\theta\right) . \qquad (413)$$

Note that

$$\begin{aligned}(\overrightarrow{Q}_{\tilde{v}(\mathbf{p})})^* &= \overrightarrow{Q}_{u(\mathbf{p})} \ , \\ (\overrightarrow{Q}_{u(\mathbf{p})})^* &= \overrightarrow{Q}_{\tilde{v}(\mathbf{p})} \ . \end{aligned} \qquad (414)$$

We have used the fact that while complex conjugating, the grassmannian derivatives acting from the left act from the right (and vice-versa) and to bring any such right acting derivative to the left involves introducing an extra minus sign. Armed with the definitions above, we can rewrite (411) as (all differential operators henceforth, unless noted otherwise, are taken to act from the left)

$$\left(Q^*_{u(\mathbf{p}_1)} + Q^*_{u(\mathbf{p}_2)} + Q_{u(\mathbf{p}_3)} + Q_{u(\mathbf{p}_4)}\right) S_i(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) = 0 \ . \qquad (415)$$

The next step is to observe that

$$\begin{aligned}\left(Q^*_{u(\mathbf{p}_1)} + Q^*_{u(\mathbf{p}_2)} + Q_{u(\mathbf{p}_3)} + Q_{u(\mathbf{p}_4)}\right) \exp(\theta_1\theta_3 + \theta_2\theta_4) 2p^0_3(2\pi)^2\delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_3) \\ 2p^0_4(2\pi)^2\delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_4) = 0 \end{aligned} \qquad (416)$$

after we set $\mathbf{p}_1 = \mathbf{p}_3$ and $\mathbf{p}_2 = \mathbf{p}_4$. We now act on (172) with

$$\begin{aligned}\left(Q^*_{u(\mathbf{p}_1)} + Q^*_{u(\mathbf{p}_2)} + Q_{u(\mathbf{p}_3)} + Q_{u(\mathbf{p}_4)}\right) \int d\Gamma \Big[S_1(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{k}_3, \phi_1, \mathbf{k}_4, \phi_2) \\ \exp(\phi_1\phi_3 + \phi_2\phi_4) 2k^0_1(2\pi)^2\delta^{(2)}(\mathbf{k}_3 - \mathbf{k}_1) 2k^0_2(2\pi)^2\delta^{(2)}(\mathbf{k}_4 - \mathbf{k}_2) \\ S_2(\mathbf{k}_1, \phi_3, \mathbf{k}_2, \phi_4, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) \Big] \ . \end{aligned} \qquad (417)$$

Proceeding with (417), one finds

$$
\begin{aligned}
-\int d\Gamma \Big[& \left(Q_{u(\mathbf{k}_3)} + Q_{u(\mathbf{k}_4)}\right) S_1(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{k}_3, \phi_1, \mathbf{k}_4, \phi_2) \exp(\phi_1\phi_3 + \phi_2\phi_4) \\
& 2k_1^0(2\pi)^2\delta^{(2)}(\mathbf{k}_3 - \mathbf{k}_1) 2k_2^0(2\pi)^2\delta^{(2)}(\mathbf{k}_4 - \mathbf{k}_2) S_2(\mathbf{k}_1, \phi_3, \mathbf{k}_2, \phi_4, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) \\
& + S_1(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{k}_3, \phi_1, \mathbf{k}_4, \phi_2) \exp(\phi_1\phi_3 + \phi_2\phi_4) 2k_1^0(2\pi)^2\delta^{(2)}(\mathbf{k}_3 - \mathbf{k}_1) \\
& 2k_2^0(2\pi)^2\delta^{(2)}(\mathbf{k}_4 - \mathbf{k}_2) \left(Q^*_{u(\mathbf{k}_1)} + Q^*_{u(\mathbf{k}_2)}\right) S_2(\mathbf{k}_1, \phi_3, \mathbf{k}_2, \phi_4, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) \Big]. \quad (418)
\end{aligned}
$$

We next integrate by parts keeping in mind that only the derivative parts of the Q change sign (as a consequence of the integration by parts). This gives

$$
\begin{aligned}
\int d\Gamma \Big[& S_1(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{k}_3, \phi_1, \mathbf{k}_4, \phi_2) \\
& \left(\tilde{Q}_{u(\mathbf{k}_3)} + \tilde{Q}_{u(\mathbf{k}_4)} + \tilde{Q}^*_{u(\mathbf{k}_1)} + \tilde{Q}^*_{u(\mathbf{k}_2)}\right) \exp(\phi_1\phi_3 + \phi_2\phi_4) 2k_1^0(2\pi)^2\delta^{(2)}(\mathbf{k}_3 - \mathbf{k}_1) \\
& 2k_2^0(2\pi)^2\delta^{(2)}(\mathbf{k}_4 - \mathbf{k}_2) S_2(\mathbf{k}_1, \phi_3, \mathbf{k}_2, \phi_4, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) \Big]. \quad (419)
\end{aligned}
$$

Here, by $\tilde{Q}_{u(p)}$ and $\tilde{Q}^*_{u(p)}$ we mean

$$
\tilde{Q}_{u(p)} = i\left(u_\alpha(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta} + u^*_\alpha(\mathbf{p})\theta\right) , \quad (420)
$$

$$
\tilde{Q}^*_{u(p)} = i\left(u^*_\alpha(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta} - u_\alpha(\mathbf{p})\theta\right) . \quad (421)
$$

It can be easily checked (just like (416)) that (on setting $\mathbf{k}_3 = \mathbf{k}_1$ and $\mathbf{k}_4 = \mathbf{k}_2$)

$$
\begin{aligned}
& \left(\tilde{Q}_{u(\mathbf{k}_3)} + \tilde{Q}_{u(\mathbf{k}_4)} + \tilde{Q}^*_{u(\mathbf{k}_1)} + \tilde{Q}^*_{u(\mathbf{k}_2)}\right) \exp(\phi_1\phi_3 + \phi_2\phi_4) 2k_1^0(2\pi)^2\delta^{(2)}(\mathbf{k}_3 - \mathbf{k}_1) \\
& \qquad 2k_2^0(2\pi)^2\delta^{(2)}(\mathbf{k}_4 - \mathbf{k}_2) = 0, \quad (422)
\end{aligned}
$$

completing the proof.

4.7.5 Details of the unitarity equation

In this section, we simplify the unitarity equations (180) and (181). We define

$$Z(\mathbf{p}_i) = \frac{1}{4m^2} v^*(\mathbf{p}_1) v^*(\mathbf{p}_2)\ v(\mathbf{p}_3) v(\mathbf{p}_4)$$

and rewrite (180) and (181) as

$$\begin{aligned}
&\int d\Gamma' \,[\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\\
&-Y(\mathbf{p}_3,\mathbf{p}_4)\left(\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\right.\\
&+4Y(\mathbf{p}_3,\mathbf{p}_4)\left(\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)+\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\right)\\
&\left.+16Y^2(\mathbf{p}_3,\mathbf{p}_4)\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\right)\Big] = 2p_3^0(2\pi)^2\delta^{(2)}(\mathbf{p}_1-\mathbf{p}_3)2p_4^0(2\pi)^2\delta^{(2)}(\mathbf{p}_2-\mathbf{p}_4)
\end{aligned} \tag{423}$$

and

$$\begin{aligned}
&Z(\mathbf{p}_i)\int d\Gamma' \,[-4Y(\mathbf{p}_3,\mathbf{p}_4)\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\\
&+\big(4Y^2(\mathbf{p}_3,\mathbf{p}_4)\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\\
&+Y(\mathbf{p}_3,\mathbf{p}_4)\left(\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)+\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\right)\\
&\left.+\frac{1}{4}\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\Big)\right] = -2p_3^0(2\pi)^2\delta^{(2)}(\mathbf{p}_1-\mathbf{p}_3)2p_4^0(2\pi)^2\delta^{(2)}(\mathbf{p}_2-\mathbf{p}_4).
\end{aligned} \tag{424}$$

Since the factor $Z(\mathbf{p}_i)$ depends only on the external momenta $\mathbf{p}_i$, we may evaluate it on the delta functions and this simply yields $Z(\mathbf{p}_i) = 4Y(\mathbf{p}_3, \mathbf{p}_4)$. We finally arrive at

$$\begin{aligned}
&\int d\Gamma' \Big[\mathcal{S}_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{S}_B^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) \\
&- Y(\mathbf{p}_3, \mathbf{p}_4) \Big(\mathcal{S}_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{S}_B^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) \\
&+ 4Y(\mathbf{p}_3, \mathbf{p}_4) \big(\mathcal{S}_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{S}_F^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) + \mathcal{S}_F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{S}_B^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) \big) \\
&+ 16Y^2(\mathbf{p}_3, \mathbf{p}_4) \mathcal{S}_F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{S}_F^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) \Big) \Big] = 2p_3^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_3) 2p_4^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_4)
\end{aligned} \tag{425}$$

and

$$\begin{aligned}
&\int d\Gamma' \Big[-16Y^2(\mathbf{p}_3, \mathbf{p}_4) \mathcal{S}_F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{S}_F^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) \\
&+ Y(\mathbf{p}_3, \mathbf{p}_4) \Big(\mathcal{S}_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{S}_B^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) \\
&+ 4Y(\mathbf{p}_3, \mathbf{p}_4) \big(\mathcal{S}_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{S}_F^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) + \mathcal{S}_F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{S}_B^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) \big) \\
&+ 16Y^2(\mathbf{p}_3, \mathbf{p}_4) \mathcal{S}_F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{S}_F^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) \Big) \Big] = -2p_3^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_3) 2p_4^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_4).
\end{aligned} \tag{426}$$

The above equations can be more compactly written as (184) and (185) respectively (since $p_3 \cdot p_4 = p_1 \cdot p_2$).

4.7.6 Going to supersymmetric Light cone gauge

In this brief appendix we will demonstrate that (upto the usual problem with zero modes) it is always possible to find a super gauge transformation that takes us to the supersymmetric lightcone gauge $\Gamma_- = 0$

Let us start with a gauge configuration that obeys our gauge condition $\Gamma_- = 0$. Starting with this gauge configuration, we will now demonstrate that we can perform a gauge transformation that will take Γ_- to any desired value, say $\tilde{\Gamma}_-$.

Performing the gauge transformation (122) we find that the new value of Γ_- is simply $D_- K$. Let

$$K = M + \theta\zeta - \theta^2 P, \tag{427}$$

where M, ζ^α, P are gauge parameters. It follows that

$$D_- K = \zeta_- - \theta_-(\partial_{-+}M + P) + \theta_+\partial_{--}M - i\theta_+\theta_-(\partial_{-+}\zeta_- - \partial_{--}\zeta_+) \tag{428}$$

Now let us suppose that

$$-\tilde{\Gamma}_- = \chi_- - \theta_-(B + A_{+-}) + \theta_+ A_{--} + i\theta_+\theta_-(2\lambda_- + \partial_{--}\chi_+ - \partial_{-+}\chi_-)$$

We need to find K so that

$$D_- K = \tilde{\Gamma}_-$$

Equating coefficients on the two sides of this equation we find

$$\begin{aligned} \chi_- + \zeta_- &= 0 \ , \\ B + A_{+-} + P + \partial_{-+}M &= 0 \ , \\ A_{--} + \partial_{--}M &= 0 \ , \\ 2\lambda_- + \partial_{--}(\chi_+ + \zeta_+) - \partial_{-+}(\chi_- + \zeta_-) &= 0 \ , \end{aligned} \tag{429}$$

which are then solved to get,

$$\begin{aligned} \zeta_- &= -\chi_- \ , \\ \zeta_+ &= -2\partial_{--}^{-1}\lambda_- - \chi_+ \ , \\ M &= -\partial_{--}^{-1}A_{--} \ , \\ P &= -B - A_{+-} + \partial_{-+}(\partial_{--}^{-1}A_{--}) \ . \end{aligned} \tag{430}$$

Substituting the above expressions in the expansion for K, we can write

$$K = -\partial_{--}^{-1} A_{--} - i\theta_- (2\partial_{--}^{-1}\lambda_- + \chi_+) + i\theta_+\chi_- + i\theta_+\theta_-(\partial_{-+}\partial_{--}^{-1} A_{--} - B - A_{+-}) \ . \quad (431)$$

It can be checked that the form of K obtained above follows from

$$K = i\partial_{--}^{-1} D_- \Gamma_- \ , \quad (432)$$

which is a supersymmetric version of the gauge transformation used to generate an arbitrary A_- starting from usual lightcone gauge.

4.7.7 Details of the self energy computation

In this subsection, we will demonstrate that the self energy $\Sigma(p, \theta_1, \theta_2)$ is a constant independent of the momenta p. As discussed in §4.3.3 $\Sigma(p, \theta_1, \theta_2)$ obeys the integral equation

$$\begin{aligned}\Sigma(p, \theta_1, \theta_2) = {} & 2\pi\lambda w \int \frac{d^3 r}{(2\pi)^3} \delta^2(\theta_1 - \theta_2) P(r, \theta_1, \theta_2) \\ & - 2\pi\lambda \int \frac{d^3 r}{(2\pi)^3} D_-^{\theta_2, -p} D_-^{\theta_1, p} \left(\frac{\delta^2(\theta_1 - \theta_2)}{(p-r)_{--}} P(r, \theta_1, \theta_2) \right) \\ & + 2\pi\lambda \int \frac{d^3 r}{(2\pi)^3} \frac{\delta^2(\theta_1 - \theta_2)}{(p-r)_{--}} D_-^{\theta_1, r} D_-^{\theta_2, -r} P(r, \theta_1, \theta_2,) \ . \end{aligned} \quad (433)$$

We will now simplify the second and third terms in (433). In §4.3.3 we already observed that the general form of the propagator is of the form given by (206). Using the formulae (368) and (369) we can write (206) as

$$P(p, \theta_1, \theta_2) = \big(C_1(p) D_{\theta_1, p}^2 + C_2(p)\big)\, \delta^2(\theta_1 - \theta_2) \quad (434)$$

In the second term of (433) we have to evaluate

$$C_1(p)D_-^{\theta_2,-p}D_-^{\theta_1,p}\left(\delta^2(\theta_1-\theta_2)D^2_{\theta_1,p}\delta^2(\theta_1-\theta_2)\right) \ , \tag{435}$$

since the product of $\delta^2(\theta_1-\theta_2)$ vanishes. We further use the formulae (369) and then the transfer rule (370) to get

$$\begin{aligned}-C_1(p)D_-^{\theta_2,-p}D_-^{\theta_2,-p}\delta^2(\theta_1-\theta_2) &= p_{--}C_1(p)\delta^2(\theta_1-\theta_2)\\ &= p_{--}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2) \ ,\end{aligned} \tag{436}$$

where we have used the algebra (364) in the first line and (369) in the second.

Let us now proceed to simplify the third term in (433). We need to evaluate

$$\begin{aligned}&\delta^2(\theta_1-\theta_2)D_-^{\theta_1,r}D_-^{\theta_2,-r}\left(C_1(p)D^2_{\theta_1,r}\delta^2(\theta_1-\theta_2)+C_2(p)\delta^2(\theta_1-\theta_2)\right)\\ &\qquad = C_1(p)\delta^2(\theta_1-\theta_2)D_-^{\theta_1,r}D_-^{\theta_2,-r}D^2_{\theta_1,r}\delta^2(\theta_1-\theta_2) \ ,\end{aligned} \tag{437}$$

where we have used the transfer rule (370) and the fact that the product of $\delta^2(\theta_1-\theta_2)$ vanishes. We further simplify

$$\begin{aligned}C_1(p)\delta^2(\theta_1-\theta_2)D_-^{\theta_1,r}D_-^{\theta_2,-r}D^2_{\theta_1,r}\delta^2(\theta_1-\theta_2) &= -C_1(p)\delta^2(\theta_1-\theta_2)r_-^{\beta}D_-^{\theta_1,r}D_\beta^{\theta_2,-r}\delta^2(\theta_1-\theta_2)\\ &= C_1(p)\delta^2(\theta_1-\theta_2)r_-^{+}D_-^{\theta_1,r}D_+^{\theta_1,r}\delta^2(\theta_1-\theta_2)\\ &= C_1(p)\delta^2(\theta_1-\theta_2)(-ir_-^{+})D^2_{\theta_1,r}\delta^2(\theta_1-\theta_2)\\ &= r_{--}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2) \ ,\end{aligned} \tag{438}$$

where in the first line we have used (367), in the second line the expression is nonzero for $\beta=-$ and we have used the transfer rule (370), while the third line follows from the identity $-iD^2=D_-D_+$ and the last line follows from the arguments used before.

Thus, using the results (438) and (436) in (433) we get the final form as given in

(213)

$$\Sigma(p,\theta_1,\theta_2) = 2\pi\lambda w \int \frac{d^3r}{(2\pi)^3}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2)$$
$$- 2\pi\lambda \int \frac{d^3r}{(2\pi)^3}\frac{p_{--}}{(p-r)_{--}}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2)$$
$$+ 2\pi\lambda \int \frac{d^3r}{(2\pi)^3}\frac{r_{--}}{(p-r)_{--}}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2)\ . \qquad (439)$$

From the above it is clear that the momentum dependence cancels between the second and third terms and we get

$$\Sigma(p,\theta_1,\theta_2) = 2\pi\lambda(w-1) \int \frac{d^3r}{(2\pi)^3}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2)\ . \qquad (440)$$

4.7.8 Details relating to the evaluation of the offshell four point function

Supersymmetry constraints on the offshell four point function In this section we will constrain the most general form of the four point function using supersymmetry (see fig 5). Supersymmetric invariance of the four point function in superspace (219) implies

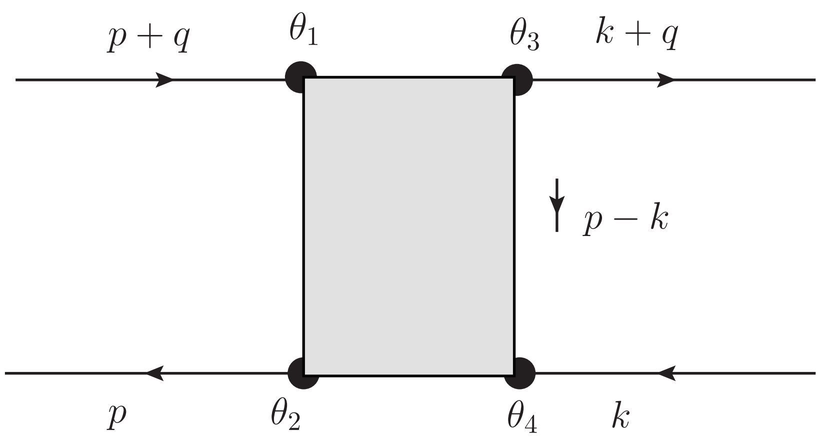

Figure 5: Four point function in superspace

that

$$(Q_{\theta_1,p+q}+Q_{\theta_2,-p}+Q_{\theta_3,-k-q}+Q_{\theta_4,k})V(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k)=0\ . \tag{441}$$

This can be simplified using (371) and written as

$$\sum_{i=1}^{4}\left(\frac{\partial}{\partial\theta_i^\alpha}-p_{\alpha\beta}(\theta_1-\theta_2)^\beta-q_{\alpha\beta}(\theta_1-\theta_3)^\beta-k_{\alpha\beta}(\theta_4-\theta_3)^\beta\right)V(\theta_1,\theta_2,\theta_3,p,q,k)=0\ . \tag{442}$$

We can make the following variable changes to simplify the equation (we suppress spinor indices for simplicity in notation)

$$\begin{aligned}
X&=\sum_{i=1}^{4}\theta_i\ ,\\
X_{12}&=\theta_1-\theta_2\ ,\\
X_{13}&=\theta_1-\theta_3\ ,\\
X_{43}&=\theta_4-\theta_3\ .
\end{aligned}\tag{443}$$

The inverse coordinates are

$$\begin{aligned}
\theta_1&=\frac{1}{4}(X+X_{12}+2X_{13}-X_{43})\ ,\\
\theta_2&=\frac{1}{4}(X-3X_{12}+2X_{13}-X_{43})\ ,\\
\theta_3&=\frac{1}{4}(X+X_{12}-2X_{13}-X_{43})\ ,\\
\theta_4&=\frac{1}{4}(X+X_{12}-2X_{13}+3X_{43})\ .
\end{aligned}\tag{444}$$

In terms of the new coordinates, the derivatives are then expressed as

$$\begin{aligned}
\frac{\partial}{\partial \theta_1} &= \frac{\partial}{\partial X} + \frac{\partial}{\partial X_{12}} + \frac{\partial}{\partial X_{13}} \ , \\
\frac{\partial}{\partial \theta_2} &= \frac{\partial}{\partial X} - \frac{\partial}{\partial X_{12}}, \\
\frac{\partial}{\partial \theta_3} &= \frac{\partial}{\partial X} - \frac{\partial}{\partial X_{13}} - \frac{\partial}{\partial X_{43}} \ , \\
\sum_{i=1}^{4} \frac{\partial}{\partial \theta_i} &= 4\frac{\partial}{\partial X} \ .
\end{aligned} \tag{445}$$

Using the above, one can rewrite (442) as

$$(4\frac{\partial}{\partial X} - p.X_{12} - q.X_{13} - k.X_{43})V(X, X_{12}, X_{13}, X_{43}, p, q, k) = 0, \tag{446}$$

where $p.X_{12} = p_{\alpha\beta}X_{12}^{\beta}$. The above equation can be thought of as a differential equation in the variables X_{ij} and is solved by

$$V(\theta_1, \theta_2, \theta_3, \theta_4, p, q, k) = \exp\left(\frac{1}{4}X.(p.X_{12} + q.X_{13} + k.X_{43})\right)F(X_{12}, X_{13}, X_{43}, p, q, k) \ . \tag{447}$$

This is the most general form of a four point function in superspace that is invariant under supersymmetry.

Explicitly evaluating V_0 In this subsection, we will compute the tree level diagram for the four point function due to the gauge superfield interaction. (see fig 6). In fig 6 the two diagrams are equivalent ways to represent the same process.

$$V_0(\theta_1, \theta_2, \theta_3, \theta_4, p, q, k)^{gauge} = \frac{-2\pi}{\kappa(p-k)_{--}}(D_-^{\theta_2,-p} - D_-^{\theta_4,k})(D_-^{\theta_1,p+q} - D_-^{\theta_3,-(k+q)})(\delta_{13}^2\delta_{24}^2\delta_{12}^2) \ , \tag{448}$$

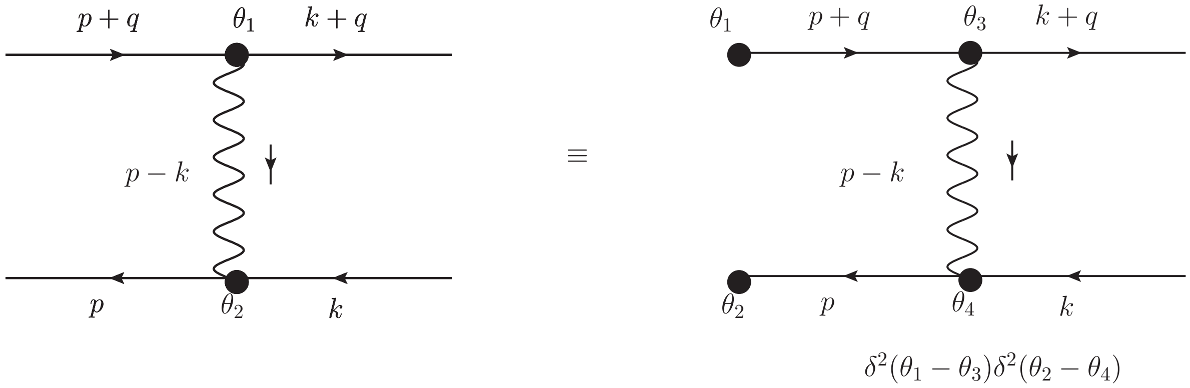

Figure 6: Four point function for gauge interaction: Tree diagram

where $\delta_{ij}^2 = \delta^2(\theta_i - \theta_j)$. [79]

It can be explicitly checked that (see (221) for definition of X_{ij})

$$(D_-^{\theta_2,-p} - D_-^{\theta_4,k})(D_-^{\theta_1,p+q} - D_-^{\theta_3,-(k+q)})(\delta_{13}^2\delta_{24}^2\delta_{12}^2) = \exp\left(\frac{1}{4}X.(p.X_{12} + q.X_{13} + k.X_{43})\right)$$
$$F_{tree}(X_{12}, X_{13}, X_{43}) \ , \tag{449}$$

where

$$F_{tree} = 2iX_{12}^+X_{13}^+X_{43}^+(X_{12}^- + X_{34}^-) \ . \tag{450}$$

Thus the final result for the tree level diagram is given by

$$V_0(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k)^{gauge} = -\frac{4\pi i}{\kappa(p-k)_{--}} \exp\big(\frac{1}{4}X_{1234}.(p.X_{12} + q.X_{13} + k.X_{43})\big)$$
$$X_{12}^+X_{13}^+X_{43}^+(X_{12}^- + X_{34}^-) \ . \tag{451}$$

It is clear that the shift invariant function (450) has the general structure of (227), with

[79] Note that each vertex factor in Fig 6 has a factor of D, resulting in two powers of D in (448).

the appropriate identification

$$A(p,q,k) = -\frac{4\pi i}{\kappa}\frac{1}{(p-k)_{--}} \ , \ B(p,q,k) = -\frac{4\pi i}{\kappa}\frac{1}{(p-k)_{--}} \tag{452}$$

Note that the Fig 6 has the $\mathbb{Z}_2$ symmetry (224). It is straightforward to check that (451) is invariant under (224).

Closure of the ansatz (227) In this section, we establish the consistency of the ansatz (227) as a solution of the integral equation (225). Consistency is established by plugging the ansatz (227) into the RHS of this integral equation, and verifying that the resultant θ structure is once again of the form given in (227). In other words we will show that the dependence of

$$\int \frac{d^3r}{(2\pi)^3} d^2\theta_a d^2\theta_b d^2\theta_A d^2\theta_B \Big(N V_0(\theta_1,\theta_2,\theta_a,\theta_b,p,q,r) P(r+q,\theta_a,\theta_A) \\ P(r,\theta_B,\theta_b) V(\theta_A,\theta_B,\theta_3,\theta_4,r,q,k)\Big) \tag{453}$$

on θ_1, θ_2, θ_3 and θ_4 is given by the form (227) with appropriately identified functions A, B, C D.

The algebraic closure described above actually follows from a more general closure property that we now explain. Note that the tree level four point function V_0 (226) is itself of the form (227). The more general closure property (which we will explain below) is that the expression

$$V_{12} = V_1 \star V_2 \equiv \int \frac{d^3r}{(2\pi)^3} d^2\theta_a d^2\theta_b d^2\theta_A d^2\theta_B \Big(V_1(\theta_1,\theta_2,\theta_a,\theta_b,p,q,r) P(r+q,\theta_a,\theta_A) \\ P(r,\theta_B,\theta_b) V_2(\theta_A,\theta_B,\theta_3,\theta_4,r,q,k)\Big) \tag{454}$$

takes the form (227) whenever V_1 and V_2 are both also of the form (227). In other words (454) defines a closed multiplication rule on expressions of the form (227).

The explicit verification of the closure described the last paragraph follows from straightforward algebra. Let [80]

$$V_1(\theta_1,\theta_2,\theta_a,\theta_b,p,q,r) = \exp\big(\frac{1}{4}X_{12ab}.(p.X_{12}+q.X_{1a}+r.X_{ba})\big)F_1(X_{12},X_{1a},X_{ba},p,q,r) \tag{455}$$

where

$$F_1(X_{12},X_{1a},X_{ba},p,q,r) = X^+_{AB}X^+_{43}\Big(A_1(p,r,q)X^-_{12}X^-_{ba}X^+_{1a}X^-_{1a}+B_1(p,r,q)X^-_{12}X^-_{ba} \\ +C_1(p,r,q)X^-_{12}X^+_{1a}+D_1(p,r,q)X^+_{1a}X^-_{ba}\Big)\ . \tag{456}$$

and

$$V_2(\theta_A,\theta_B,\theta_3,\theta_4,r,q,k) = \exp\big(\frac{1}{4}X_{AB34}.(r.X_{AB}+q.X_{A3}+k.X_{43})\big)F_2(X_{AB},X_{A3},X_{43},r,q,k)\ , \tag{457}$$

where

$$F_2(X_{AB},X_{A3},X_{43},r,q,k) = X^+_{AB}X^+_{43}\Big(A_2(r,k,q)X^-_{AB}X^-_{43}X^+_{A3}X^-_{A3}+B_2(r,k,q)X^-_{AB}X^-_{43} \\ +C_2(r,k,q)X^-_{AB}X^+_{A3}+D_2(r,k,q)X^+_{A3}X^-_{43}\Big)\ . \tag{458}$$

Evaluating the integrals over $\theta_a,\theta_b,\theta_A,\theta_B$, we find that V_{12} in (454) is of the form (227) with

$$A_{12} = -\frac{1}{4}q_3\int d^3\mathcal{R}\Big((C_1C_2k_- - D_1D_2p_- + 2B_2C_1q_3 - 2B_1D_2q_3)r_- \\ +2A_2(D_1p_- + 2B_1q_3 + 2C_1r_-) + 2A_1(C_2k_- + 2B_2q_3 + 2D_2r_-)\Big),$$

[80] We have used the notations $X_{12ab}=\theta_1+\theta_2+\theta_a+\theta_b$ and $X_{AB34}=\theta_A+\theta_B+\theta_3+\theta_4$.

$$B_{12} = -\frac{1}{4}\int d^3\mathcal{R}\Big((2A_2 - C_2k_-)(2A_1 + D_1p_-) + 4B_1B_2q_3^2 + 3C_1D_2r_-^2 \\ + (2A_2C_1 - 2A_1D_2 - C_1C_2k_- - D_1D_2p_- + 4B_2C_1q_3 + 4B_1D_2q_3)r_-\Big),$$

$$\begin{aligned} C_{12} =& -\frac{1}{2}\int d^3\mathcal{R}C_2q_3(2A_1 + D_1p_- + 2B_1q_3 + 3C_1r_-) \ , \\ D_{12} =& -\frac{1}{2}\int d^3\mathcal{R}D_1q_3(-2A_2 + C_2k_- + 2B_2q_3 + 3D_2r_-) \ . \end{aligned} \tag{459}$$

where

$$d^3\mathcal{R} = \frac{d^3r}{(2\pi)^3}\frac{1}{(r^2+m^2)((r+q)^2+m^2)}$$

It follows from (454) that

$$(V_1 \star V_2) \star V_3 = V_1 \star (V_2 \star V_3) \tag{460}$$

as both expressions in (460) are given by the same integral (the expressions differ only in the order in which the θ and internal momentum integrals are performed). In other words the product defined above is associative. We have directly checked that the explicit multiplication formula (459) defines an associative product rule.

Consistency check of the integral equation In this section, we demonstrate that the integral equations (228)-(231) are consistent with the $\mathbb{Z}_2$ symmetry (224). First we note that the $\mathbb{Z}_2$ invariance (224) of (227) imposes the following conditions on the unknown functions of momenta

$$\begin{aligned} &A(p,k,q) = A(k,p,-q) \ , B(p,k,q) = B(k,p,-q) \ , \\ &C(p,k,q) = -D(k,p,-q) \ , D(p,k,q) = -C(k,p,-q) \ . \end{aligned} \tag{461}$$

These conditions can be written in the form of a matrix given by

$$E(p,k,q) = TE(k,p,-q) \ , \tag{462}$$

where

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} , E(p,k,q) = \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} \tag{463}$$

The integral equations (228)-(231) can be written in differential form by taking derivatives of p_+ and using the formulae in Appendix §4.7.8 [81]

$$\partial_{p_+} E(p,k,q) = S(p,k,q) + H(p,k_-,q)E(p,k,q) \tag{464}$$

where $S(p,k,q)$ is a source term. The equation for k_+ can be obtained from the above equation as follows

$$\begin{aligned} \partial_{k_+} E(p,k,q) &= T\partial_{k_+} E(k,p,-q) \ , \\ &= TS(k,p,-q) + TH(k,p_-,-q)E(k,p,-q) \ , \\ &= TS(k,p,-q) + TH(k,p_-,-q)TE(p,k,q) \ , \end{aligned} \tag{465}$$

where we have used (462). Applying k_+, p_+ derivative on (464) and (465) respectively and

[81]Taking derivatives with respect to p_+ eliminates the $r_\pm$ integrals because of the delta functions. The remaining r_3 integrals can be easily performed (see Appendix §4.7.8).

taking the difference we get

$$\partial_{k_+} S(p,k,q) + H(p,k_-,q)\Big(TS(k,p,-q) + TH(k,p_-,-q)TE(p,k,q) \Big)$$
$$= T\partial_{p_+} S(k,p,-q) + TH(k,p_-,-q)T\Big(S(p,k,q) + H(p,k_-,-q)E(p,k,q) \Big) \ . \tag{466}$$

Comparing coefficients of $E(p,k,q)$ in the above equation we get the condition

$$[H(p,k_-,q), TH(k,p_-,-q)T] = 0 \ . \tag{467}$$

For the integral equations (228)-(231), the $H(p,k_-,q)$ are given by

$$H(p,k_-,q_3) = \frac{1}{a(p_s,q_3)} \begin{pmatrix} (6q_3-4im)p_- & 2q_3(2im+q_3)p_- & (2im+q_3)k_-p_- & -(2im+q_3)p_-^2 \\ 4p_- & 4q_3p_- & -2k_-p_- & 2p_-^2 \\ 0 & 0 & 8q_3p_- & 0 \\ 8im-4q_3 & 4q_3(q_3-2im) & 2(q_3-2im)k_- & (4im+6q_3)p_- \end{pmatrix} \tag{468}$$

where

$$a(p_s,q_3) = \frac{\sqrt{m^2+p_s^2}\,(4m^2+q_3^2+4p_s^2)}{2\pi} \ . \tag{469}$$

The matrix $TH(k,p,-q_3)T$ is

$$TH(k,p,-q_3)T = \frac{1}{a(k_s,q_3)} \begin{pmatrix} -(4im+6q_3)k_- & 2q_3(q_3-2im)k_- & -(q_3-2im)k_-^2 & (q_3-2im)k_-p_- \\ 4k_- & -4q_3k_- & -2k_-^2 & 2k_-p_- \\ -8im-4q_3 & 4(-2im-q_3)q_3 & (4im-6q_3)k_- & -(4im+2q_3)p_- \\ 0 & 0 & 0 & -8q_3k_- \end{pmatrix} \tag{470}$$

It is straightforward to check that (468) and (470) commute. Thus the system of differential equations (464) obey the integrability conditions (467). Thus the differential equations (464) will have solutions that respect the $\mathbb{Z}_2$ symmetry.

Useful formulae The Euclidean measure for the basic integrals are

$$\int \frac{(d^3r)_E}{(2\pi)^3} = \frac{1}{(2\pi)^3}\int r_s dr_s dr_3 d\theta \ , \tag{471}$$

where $r_s^2 = r_+ r_- = r_1^2 + r_2^2$ and $r^2 = r_s^2 + r_3^2$. Here the integration limits are $-\infty \leq r_3 \leq \infty$, $0 \leq r_s \leq \infty$. Most often we encounter integrals of the type,

$$H(q) = \int \frac{d^3r}{(2\pi)^3}\frac{1}{(r^2+m^2)((r+q)^2+m^2)} = \frac{1}{4\pi|q_3|}\tan^{-1}\left(\left|\frac{q_3}{2m}\right|\right) \tag{472}$$

where we have set $q_\pm = 0$. Another frequently appearing integral is

$$\int \frac{d^3r}{(2\pi)^3}\frac{1}{r^2+m^2} = -\frac{|m|}{4\pi} \tag{473}$$

where we have regulated the divergence using dimensional regularization.

In the integral equations (228)-(231), there are no explicit functions of r_3 appearing in the integral equations and the r_3 integral can be exactly done

$$\int_{-\infty}^{\infty} \frac{dr_3}{(r_s^2+r_3^2+m^2)(r_s^2+(r_3+q_3)^2+m^2)} = \frac{2\pi}{\sqrt{r_s^2+m^2}(4m^2+q_3^2+4r_s^2)} \ . \tag{474}$$

The results for the angle integrals are

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{(r-p)_-\ (k-r)_-} &= \frac{2\pi}{(k-p)_-}\left(\frac{k_+}{k_s^2}\theta[k_s-r_s] - \frac{p_+}{p_s^2}\theta[p_s-r_s]\right) \ , \\
\int_0^{2\pi} \frac{d\theta\ r_-}{(r-p)_-\ (k-r)_-} &= \frac{2\pi}{(k-p)_-}\left(\theta[k_s-r_s] - \theta[p_s-r_s]\right) \ , \\
\int_0^{2\pi} \frac{d\theta\ r_-^2}{(r-p)_-\ (k-r)_-} &= -\frac{2\pi}{(k-p)_-}\left(k_-(1-\theta[k_s-r_s]) - p_-(1-\theta[p_s-r_s])\right) \ .
\end{aligned} \tag{475}$$

while the r_s integrals are done with the limits from 0 to ∞. We will also make use of the formula

$$\partial_{\bar{z}}\left(\frac{1}{z}\right) = 2\pi\delta^2(z,\bar{z}) \tag{476}$$

to derive the differential form of the integral equations.

For doing the angle integrations in (308) we used the formula (311)

$$\int d\theta \mathrm{Pv}\cot\left(\frac{\theta}{2}\right)\mathrm{Pv}\cot\left(\frac{\alpha-\theta}{2}\right)=2\pi-4\pi^2\delta(\alpha), \tag{477}$$

where Pv stands for principal value. This formula is easily verified by calculating the Fourier coefficients as follows

$$\begin{aligned}\int\frac{d\alpha}{2\pi}e^{-i\alpha}\int d\theta \mathrm{Pv}\cot\left(\frac{\theta}{2}\right)\mathrm{Pv}\cot\left(\frac{\alpha-\theta}{2}\right) &= \oint\frac{d\omega}{2\pi\omega}\omega^{-m}\oint\frac{dz}{z}\mathrm{Pv}\left(\frac{z+1}{z-1}\right)\mathrm{Pv}\left(\frac{z+\omega}{\omega-z}\right)\\ &=\begin{cases}-i\oint dz\,\mathrm{Pv}\left(\frac{z+1}{z-1}\right)z^{-m-1}=-2\pi & (m>0)\\ 0 & (m=0)\\ i\oint dz\,\mathrm{Pv}\left(\frac{z+1}{z-1}\right)z^{-m-1}=-2\pi & (m<0)\end{cases}\end{aligned} \tag{478}$$

where $z=e^{i\theta}$ and $\omega=e^{i\alpha}$. By comparing (478) with Fourier coefficients of delta function,

$$\delta(\alpha)=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}e^{im\alpha}, \tag{479}$$

we can immediately check (311).

4.7.9 Properties of the J functions

The J functions are given by

$$\begin{aligned}J_B(q_3,\lambda)&=\frac{4\pi q_3}{\kappa}\frac{n_1+n_2+n_3}{d_1+d_2+d_3}\ ,\\ J_F(q_3,\lambda)&=\frac{4\pi q_3}{\kappa}\frac{-n_1+n_2+n_3}{d_1+d_2+d_3}\ ,\end{aligned} \tag{480}$$

where the parameters are

$$
\begin{aligned}
n_1 =&16mq_3(w+1)e^{i\lambda\left(2\tan^{-1}\frac{2|m|}{q_3}+\pi\mathrm{sgn}(q_3)\right)} \ , \\
n_2 =&(w-1)(q_3+2im)(2m(w-1)+iq_3(w+3))\left(-e^{2i\pi\lambda\mathrm{sgn}(q_3)}\right) \ , \\
n_3 =&(w-1)(2m+iq_3)(q_3(w+3)+2im(w-1))e^{4i\lambda\tan^{-1}\frac{2|m|}{q_3}}) \ , \\
d_1 =&(w-1)\left(4m^2(w-1)-8imq_3+q_3^2(w+3)\right)e^{4i\lambda\tan^{-1}\frac{2|m|}{q_3}} \ , \\
d_2 =&(w-1)\left(4m^2(w-1)+8imq_3+q_3^2(w+3)\right)e^{2i\pi\lambda\mathrm{sgn}(q_3)} \ , \\
d_3 =&-2\left(4m^2(w-1)^2+q_3^2(w(w+2)+5)\right)e^{i\lambda\left(2\tan^{-1}\frac{2|m|}{q_3}+\pi\mathrm{sgn}(q_3)\right)} \ .
\end{aligned}
\tag{481}
$$

Both the J functions (480) are even functions of q_3

$$
J_B(q_3,\lambda)=J_B(-q_3,\lambda) \ , \ J_F(q_3,\lambda)=J_F(-q_3,\lambda) \ . \tag{482}
$$

Therefore in (480) we can replace q_3 with $|q_3|$ and rewrite them as

$$
\begin{aligned}
J_B(|q_3|,\lambda) &= \frac{4\pi|q_3|}{\kappa}\frac{(\tilde{n}_1+\tilde{n}_2+\tilde{n}_3)}{(\tilde{d}_1+\tilde{d}_2+\tilde{d}_3)} \ , \\
J_F(|q_3|,\lambda) &= \frac{4\pi|q_3|}{\kappa}\frac{(-\tilde{n}_1+\tilde{n}_2+\tilde{n}_3)}{(\tilde{d}_1+\tilde{d}_2+\tilde{d}_3)} \ ,
\end{aligned}
\tag{483}
$$

where

$$
\begin{aligned}
\tilde{n}_1 =&16m|q_3|(w+1)e^{i\lambda\left(2\tan^{-1}\frac{2|m|}{|q_3|}+\pi\right)} \ , \\
\tilde{n}_2 =&(w-1)(|q_3|+2im)(2m(w-1)+i|q_3|(w+3))\left(-e^{2i\pi\lambda}\right) \ , \\
\tilde{n}_3 =&(w-1)(2m+i|q_3|)(|q_3|(w+3)+2im(w-1))e^{4i\lambda\tan^{-1}\frac{2|m|}{|q_3|}}) \ , \\
\tilde{d}_1 =&(w-1)\left(4m^2(w-1)-8im|q_3|+|q_3|^2(w+3)\right)e^{4i\lambda\tan^{-1}\frac{2|m|}{|q_3|}} \ , \\
\tilde{d}_2 =&(w-1)\left(4m^2(w-1)+8im|q_3|+|q_3|^2(w+3)\right)e^{2i\pi\lambda} \ , \\
\tilde{d}_3 =&-2\left(4m^2(w-1)^2+|q_3|^2(w(w+2)+5)\right)e^{i\lambda\left(2\tan^{-1}\frac{2|m|}{|q_3|}+\pi\right)} \ .
\end{aligned}
\tag{484}
$$

Another useful way to write the J function is to use the following identities

$$\begin{aligned}\tan^{-1}\frac{2m}{q} &= \frac{\pi}{2} - \tan^{-1}\frac{q}{2m}\\ \tan^{-1}\frac{q}{2m} &= \frac{1}{2i}\log\left(\frac{1+\frac{iq}{2m}}{1-\frac{iq}{2m}}\right)\end{aligned} \tag{485}$$

Using this relations, it is easy to write the J functions in a factorized form as given in (254)

$$\begin{aligned}J_B(q,\lambda) =& \frac{4\pi q}{\kappa}\frac{N_1N_2+M_1}{D_1D_2} \ ,\\ J_F(q,\lambda) =& \frac{4\pi q}{\kappa}\frac{N_1N_2+M_2}{D_1D_2} \ ,\end{aligned} \tag{486}$$

where

$$\begin{aligned}N_1 =& \left(\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}(w-1)(2m+iq)+(w-1)(2m-iq)\right) \ ,\\ N_2 =& \left(\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}(q(w+3)+2im(w-1))+(q(w+3)-2im(w-1))\right) \ ,\\ M_1 =& -8mq((w+3)(w-1)-4w)\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda} \ ,\\ M_2 =& -8mq(1+w)^2\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda} \ ,\\ D_1 =& \left(i\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}(w-1)(2m+iq)-2im(w-1)+q(w+3)\right) \ ,\\ D_2 =& \left(\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}(-q(w+3)-2im(w-1))+(w-1)(q+2im)\right) \ .\end{aligned} \tag{487}$$

Another useful property of the J function is manifest in the above form is its reality under complex conjugation

$$J_B(q,\lambda) = J_B^*(-q,\lambda) \ , \ J_F(q,\lambda) = J_F^*(-q,\lambda) \ . \tag{488}$$

Yet another useful way to write the J function is to note that the basic integral which appears in the four point function of scalars in an ungauged theory has the form

$$H(q) = \int \frac{d^3r}{(2\pi)^3} \frac{1}{(r^2+m^2)((r+q)^2+m^2)} = \frac{1}{4\pi|q_3|} \tan^{-1}\left(\left|\frac{q_3}{2m}\right|\right) \tag{489}$$

for $q_\pm = 0$. Thus we can also write

$$\begin{aligned} J_B(|q|,\lambda) =& \frac{4\pi|q|}{\kappa} \frac{N_1N_2+M_1}{D_1D_2} \ , \\ J_F(|q|,\lambda) =& \frac{4\pi|q|}{\kappa} \frac{N_1N_2+M_2}{D_1D_2} \ , \end{aligned} \tag{490}$$

where

$$\begin{aligned} N_1 =& \left(e^{-8\pi i\lambda|q|H(q)}(w-1)(2m+i|q|)+(w-1)(2m-i|q|)\right) \ , \\ N_2 =& \left(e^{-8\pi i\lambda|q|H(q)}(|q|(w+3)+2im(w-1))+(|q|(w+3)-2im(w-1))\right) \ , \\ M_1 =& -8m|q|((w+3)(w-1)-4w)e^{-8\pi i\lambda|q|H(q)} \ , \\ M_2 =& -8m|q|(1+w)^2e^{-8\pi i\lambda|q|H(q)} \ , \\ D_1 =& \left(ie^{-8\pi i\lambda|q|H(q)}(w-1)(2m+i|q|)-2im(w-1)+|q|(w+3)\right) \ , \\ D_2 =& \left(e^{-8\pi i\lambda|q|H(q)}(-|q|(w+3)-2im(w-1))+(w-1)(|q|+2im)\right) \ . \end{aligned} \tag{491}$$

Limits of the J function

$\mathcal{N}=2$ point The $\mathcal{N}=1$ theory studied in this chapter enjoys an enhanced $\mathcal{N}=2$ supersymmetry when $w=1$. Naturally in this limit we expect the J functions to have a simplification. In particular we get

$$\begin{aligned} J_B^{w=1} =& -\frac{8\pi m}{\kappa} \ , \\ J_F^{w=1} =& \frac{8\pi m}{\kappa} \ . \end{aligned} \tag{492}$$

Massless limit There exists a consistent massless limit for the J functions

$$J_B^{m=0} = J_F^{m=0} = \frac{4\pi|q_3|}{\kappa} \frac{(w-1)(w+3)\sin(\pi\lambda)}{(w-1)(w+3)\cos(\pi\lambda) - w(w+2) - 5} \ . \tag{493}$$

This expression is self dual under the duality map (130). Note that when $w = 1$ this vanishes and is consistent with the $m \to 0$ limit of (492).

Non relativistic limit in the singlet channel The J functions for the S channel are given in (272). The non-relativistic limit of the J functions is obtained by taking $\sqrt{s} \to 2m$ with all the other parameters held fixed. In this limit, remarkably we recover the $\mathcal{N} = 2$ result.

$$\begin{aligned} J_B^{\sqrt{s}\to 2m} &= -\frac{8\pi m}{\kappa} \ , \\ J_F^{\sqrt{s}\to 2m} &= \frac{8\pi m}{\kappa} \ . \end{aligned} \tag{494}$$

5 Conclusion

I conclude my thesis by summarizing the results in the topics presented in the thesis. In the first part of the thesis is build upon the idea that in large spacetime dimensions the nonlinear black hole dynamics is dual to equations of motion of a codimension one non-gravitational "Membrane" moving in flat space-time.

In chapter 1 we have shown that membrane equations admit a simple static solution with shape $S^{D-p-2} \times R^{p,1}$. We studied the equations for small fluctuations about this solution in a limit in which the amplitude and length scale of the fluctuations are simultaneously scaled to zero as D is taken to infinity. We have demonstrated that the resultant nonlinear equations, which capture the Gregory-Laflamme instability and its end point, exactly agree with the effective dynamical 'black brane' equations due to Emparan, Suzuki and Tanabe. Our results thus identify the 'black brane' equations as a special limit of the membrane EOMs and so unify these approaches to large D black hole dynamics.

In chapter 2 we have demonstrated that this duality extends to all orders in a $\frac{1}{D}$ expansion and outlined a systematic method for deriving the corrected membrane equation in a power series of $\frac{1}{D}$. Through this method we determined the first subleading corrections to the membrane equations of motion. We found a qualitatively new effect and showed that the divergence of the membrane velocity is nonzero and proportional to the square of the shear tensor in this order of perturbation theory; this is reminiscent of the entropy current of hydrodynamics. We calculated the frequencies of light quasinormal modes from our second order EOM about the Schwarzschild black hole. We noticed a perfect match with earlier computations performed directly in the gravitational bulk.

It would be interesting in future if we could simulate some of the membrane EOM's and see how much we can capture about the dynamics of the real black holes. In future I would also like to address interesting unanswered structural questions about gravity, viz. to understand about the second law (Wald Entropy) of thermodynamics in higher derivative gravity.

In chapter 3 we studied the most general renormalizable $\mathcal{N} = 1$ U(N) Chern-Simons gauge theory coupled to a single (generically massive) fundamental matter multiplet. We presented computations and conjectures for the 2×2 S-matrix in these theories at leading order in the ‘t Hooft large N limit but can be applied at all orders in the ‘t Hooft coupling and the matter self interaction. We have shown that our results are consistent with unitarity if and only if we assume that the textbook results of channel crossing symmetry are modified in precisely the manner proposed in [61]. We view this fact as a nontrivial consistency check of the crossing symmetry rules proposed in [61]. The ‘particle-antiparticle’ S-matrix in the singlet channel conjectured in this chapter has an interesting analytic structure. In a specific range of superpotential parameters, the S-matrix has a bound state pole. We have seen that one can tune the only arbitrary parameter superpotential to set the pole mass to zero. We found the existence of a massless bound state in a theory whose elementary excitations are all massive very much interesting.

Let me finish by mentioning that the Chern-Simons Matter theories, which I have worked on, are also connected to many interesting areas like Bosonization and Quantum Hall Physics and dualities (generalization of Particle/Vortex dualities) in $3D$ [85–87], Trace Identities, modified crossing symmetry rules, relationship with ABJM theories, non-supersymmetric modified Giveon-Kutasov dualities with two bosons and two fermions(one fundamental and one antifundamental) and so on. However, the things we know today might be the tip of the iceberg, and in the future, I would like to understand them better and would try to contribute along these lines of research.

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