Aspects of Quantum Gravity & Strongly Coupled Field Theories

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by

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Declaration of Authorship

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

The work was done under the guidance of Professor Gautam Mandal, at the Tata Institute of Fundamental Research, Mumbai.

Pranjal Nayak

In my capacity as the supervisor of the candidates thesis, I certify that the above statements are true to the best of my knowledge.

Prof. Gautam Mandal

Date: 15.12.17

"Reality is a cloud of possibility, not a point."

-Amos Tversky

Abstract

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Understanding Quantum Gravity and Strongly Coupled Field theories is a challenge that lies at the forefront of theoretical physics. This is fundamental to improving our understanding of various problems in physics: both theoretical and practical. In the various works presented in this thesis, an attempt has been made to improve our current understanding of the above mentioned problems. This has been done through various techniques like AdS/CFT duality, bootstrap of scattering amplitudes and other techniques that enable the computation of arbitrary scattering amplitudes in a quantum field theory.

Acknowledgements

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Lastly, and most importantly, I thank my parents and my brother for being so supportive and for believing in me. To them I dedicate this thesis.

Contents

To my Brother and Parents

&

Infinity and Beyond!

List of Publications

Papers relevant to the thesis work:

- "Revisiting AdS/CFT at a finite radial cut-off"
  G. Mandal and **P. Nayak**
  JHEP 12 (2016) 125, arXiv:1608.00411

- "Coadjoint orbit action of Virasoro group and two-dimensional quantum gravity dual to SYK/tensor models"
  G. Mandal, **P. Nayak** and S. R. Wadia
  JHEP 11 (2017) 046, arXiv:1702.04266

- "A Note on S-Matrix Bootstrap for Amplitudes with Linear Spectrum"
  P. Nayak, R. R. Poojary and R. M. Soni
  Submitted to JHEP, arXiv:1707.08135

- "All tree level scattering amplitudes in Chern-Simons theories with fundamental matter"
  K. Inbasekar, S. Jain, **P. Nayak** and V. Umesh
  Submitted to PRL, arXiv:1710.04227

Other papers:

- "Dual Superconformal Symmetry of N = 2 Chern-Simons theory with Fundamental Matter and Non-Renormalization at Large N"
  K. Inbasekar, S. Jain, S. Majumdar, **P. Nayak**, T. Neogi, T. Sharma and V. Umesh
  arXiv:1711.02672

- "On the Dynamics of Near-Extremal Black Holes"
  P. Nayak, A. Shukla, R. M. Soni, S. P. Trivedi and V. Vishal
  arXiv:1802.09547

Chapter 1

Synopsis

1.1 Motivation

The discovery that the world we live in is inherently *quantum* is one of the biggest discoveries in the history of physics. The most natural mathematical framework that is used to describe the nature around us is in terms of certain classical fields, which when appropriately quantized, give an apt understanding of the many-body physics that surrounds us. While there is little doubt that these quantum fields describe almost everything that we see around us, such a description of nature in terms of the quantum fields comes with its own problems that are both conceptual and technical.

An important conceptual problem that has plagued physicists for a whole century is that of writing down a quantum theory of gravity. General Relativity (GR), which is another big discovery of the twentieth century, is a classical theory of gravity that has withstood the test of time in its own right. While there are well-understood methods that enable us to build a quantum theory given a classical theory, one encounters some impassable problems when trying to do the same for GR. While these problems are well-studied and well-understood, and answered to difficult levels of satisfaction, we will not go into those subtilties and details in this introduction, introducing them as and when needed in later chapters.

A technical problem that plagues most field theories is our inability to solve them in complete generality. Perturbation theory has proved to be a useful and powerful technique to describe the theories that can be written in terms of deformations of a theory for which the exact solutions are known. Such techniques provide a reasonably correct description of the concerned quantum field theories as long as the deformations are 'small', and have been successfully applied to quantum field theories that describe electrodynamics (QED) and weak forces. Perturbative techniques are not applicable to the theories where these deformations are not small: the *strongly coupled quantum field theories*. Such theories form an

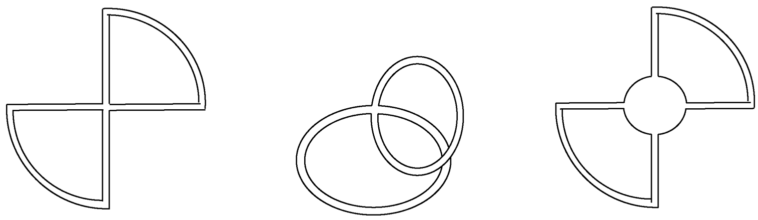


FIGURE 1.1: A representative diagram to demonstrate the Feynman diagrams containing topologies with holes and handles that are suppressed in large N counting. In all the diagrams *the double line notations* are used in which gluons (particles in adjoint representation) of a gauge theory (say $SU(N)$ or $U(N)$ or $SO(N)$) are denoted by double line notations. The single lines denote the quarks in the theory (particles in fundamental representation). The first Feynman diagram is what we call a planar diagram, since it can be drawn on a sheet of paper. The second diagram above is an example of non-planar diagram, no amount of deformations done to the line will make the diagram planar. The loop on the top can be thought of as a *handle* to lift this entire diagram. The third diagram is a typical example of diagrams with holes in them. The quark loop in the center is what we call a *hole*

important part of the description of the nature around us, like the theory of strong nuclear interactions and various condensed matter systems. Over the decades, physicists have come up with various approaches to understand such theories but what we have come to realise is that there is no one-size-fits-all solution. The use of various of these techniques will form an important part of the work that is presented in this synopsis and it would be instructive to review them once in the introduction.[1]

A very important approach in understanding strongly coupled quantum field theories was initiated in the seminal work of 't Hooft in early 1970s [5, 6]. It was shown that increasing the number of degrees of freedom ($\sim N$) in a quantum field theory leads to an unexpected simplification that provides an extremely useful organising principle for the Feynman diagrams: non-planar diagrams with holes and handles are increasingly suppressed in powers of N that depend on the topology of the Feynman diagrams (see Figure 1.1). While in general, this gives a very good qualitative and conceptual understanding of a large class of strongly interacting quantum field theories, quantitative success has been achieved only in a few specific cases, precisely because in most theories it is not possible to sum all the planar Feynman graphs. In some cases where it is possible to sum all the planar Feynman diagrams, it is often possible to invent some new effective fields which provide a classical description of the original theory. As will become clear in the discussion below, AdS/CFT provides one such description of strongly coupled field theories in terms of some new variables which describe the physics classically. Such a classical description is also useful because by studying the fluctuations in such variables one can understand the $1/N$ corrections of the original theory without actually having to sum the non-planar Feynman diagrams.[2]

[1] There are also various numerical techniques like Lattice Gauge theory, which are very important. Since they don't form part of the work presented in this thesis, we will not review them here.

[2] Another useful example of such description for the vector models is provided in [7]; and for SYK-model (discussed in chapter 3) in [8]

Another approach that has been used time and again to solve problems not only in strongly coupled quantum field theories but in general is *Bootstrap* (see [9] and references therein). G. Chew, one of the original proponents of bootstrap, once described it as a collection of techniques to "pull themselves up by their own bootstraps" using some minimal set of principles that underlie the theory of interest. In general, when studying any quantum field theory we work to compute various quantities that are representative of quantities that we measure in experiments. These could be correlation functions of various operators, as is the case in most condensed matter applications; or, these could be scattering amplitudes, as in the case of particle physics. The most common approach to bootstrapping involves identifying the symmetries of our theory, and the behaviour of these observables under application of these symmetries (often these behaviours manifest themselves as Ward identities or some modification of them). Enforcing that the observables respect these symmetries without referring to the underlying theory, in particular, is the common theme of almost all bootstrap techniques known. This makes the bootstrap techniques quite amenable to the study of strongly coupled field theories, providing in-depth insights into the particular constraints imposed by various symmetries of the theory. Historically, the studies in S-matrix bootstrap in 1960s had eventually led to the birth of String theory, which was originally proposed as a theory of strong interactions, but finally became a UV-complete theory of quantum gravity. It is interesting how two apparently different problems in quantum field theories found a solution in a single theory! Over the next two decades, string theory was developed into a tightly constrained structure that unifies all the forces of nature into one single theory.

A very important tool to understand the strongly coupled quantum field theories: the AdS/CFT conjecture, eventually emerged out of the study of String theory [10]. Originally conjectured as a strong-weak coupling duality between string theory on $\mathrm{AdS}_5\times S^5$ and $\mathcal{N}=4$ Supersymmetric Yang-Mills (SYM) theory in 4-dimensions, the conjecture has since been extended to arbitrary dimensions as a duality between a classical gravity theory (GR) on AdS_{d+1} background,

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 - dt^2 + d\vec{x}_{d-1}^2\right) \tag{1.1}$$

and a d-dimensional quantum (conformal) field theory, CFT_d. In the above coordinates, the boundary of the AdS_{d+1} lies at $z = 0$, and is conformally flat.[3] The dual field theory is said to 'live' on this boundary.

The power of this conjecture lies in the particular order of limits: the large N limit in the dual field theory corresponds to a classical theory of strings on AdS_{d+1} background, where the (stringy-)loop corrections are suppressed; a further strong coupling limit in the 't Hooft coupling, $\lambda = g_{YM}^2 N \gg 1$ corresponds to a classical theory of gravity. Thus, using a 'dictionary' that defines the duality one can compute observables in a strongly coupled field

[3] The metric on the boundary is conformally scaled flat metric $\sim \frac{L^2}{z^2}\eta_{\mu\nu}$, where the boundary coordinates are $\{t, \vec{x}_{d-1}\}$.

theory by doing the computations in a classical theory of gravity. The different limits of the AdS/CFT correspondence are summarized in Figure 1.2. Table 1.1 summarizes some of the elements of the dictionary that relate various quantities in the bulk to the quantities in the field theory. Due to the fact that this duality relates a theory of quantum gravity to a non-gravitating quantum field theory, the AdS/CFT correspondence has emerged, over the years, as a tool that enables us to use quantum field theories to understand various aspects of quantum gravity; and also use classical theory of gravity to understand strongly-coupled quantum field theories, which otherwise lie outside the realm of perturbation theory.

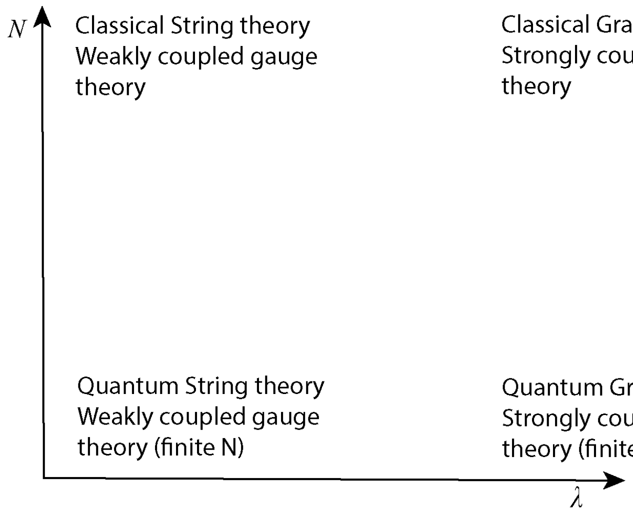


FIGURE 1.2: Behaviour of the Gravity and Field theory description of AdS/CFT correspondence in different regimes of parameters.

	Bulk	Field theory
Generating Functions[4]	$Z_{bulk}[\phi_0] = \lim_{\substack{\epsilon\to 0\\ \phi(\epsilon,x)=\phi_0(x)\sim J(x)\\ g_{\mu\nu}(\epsilon,x)\sim\delta_{\mu\nu}}} \int e^{\frac{1}{16\pi G_N} S_g[g_{\mu\nu}]+S_m[\phi]}$	$Z[J] = \langle e^{\int J(x)\mathcal{O}(x)}\rangle$
Parameters	$\frac{1}{G_N}$	N^2
	$\left(\frac{l}{l_s}\right)^4 = 4\pi g_s N$	$\lambda = g_{YM}^2 N$
Bulk fields dual to field theory operators		
Scalars	$\phi(z,x)$	$\mathcal{O}(x)$
Fermions	$\Psi^\alpha(z,x)$	$\psi^\alpha(x)$
Gauge Fields	$A^\mu(z,x)$	$J^\mu(x)$
Spin-2 fields	$g_{\mu\nu}(z,x)$	$T_{\mu\nu}(x)$

TABLE 1.1: A summary of some elements in AdS/CFT dictionary. The dictionary for the various parameters in this table is for the particular case of AdS_5/CFT_4 correspondence.

[4]Here we are using $\sim$ symbol to denote that the quantities are related upto some scaling in terms of ϵ.

In the work that is summarized in this synopsis, we have used both these aspects of AdS/CFT to learn new lessons in quantum gravity as well as quantum field theories. In section 1.2 we elaborate on how the radial coordinate of the AdS bulk is same as the direction of the RG flow in the field theory. Using the dictionary for the 'double-trace operators' between the bulk and field theory we improve upon certain aspects of the duality and give precise relations between the flow equations of certain couplings when computed using the bulk dual and when computed in the field theory.

Studies of simpler systems instead of directly studying the system of interest has often provided great insights in the past. In section 1.3 we consider a simpler case of a 1-dimensional quantum mechanical model that shows a behaviour that points towards a possible existence of a 2-dimensional bulk dual. We propose a bulk theory of gravity that is able to reproduce some universal features of the 1-dimensional quantum mechanical model. We expect that the study of such simpler systems will prove to be extremely crucial in our understanding of AdS/CFT correspondence and quantum gravity in general.

As emphasised above, the study of quantum field theories revolves around computing the observables that are directly measurable in experiments and one such important quantity is the S-matrix. S-matrices are directly measurable in high-energy scattering experiments, and provide transition probabilities for evolving from some initial state to a final state. They carry crucial information about the interactions of the theory, their symmetries, their spectrum of states and other properties like causality of the theory, among others. So it is hardly surprising that a lot of study is dedicated to understanding the properties of S-matrices of various theories.

In section 1.4 we revisit the Bootstrap approach to understand quantum field theories with a motivation to understand quantum gravity. In a simpler problem which we believe acts as a *proof of concept*, we constrain 4-scalar scattering amplitudes that obey certain symmetry principles, in particular, channel duality. In the process, we discover some constraints on the asymptotic behaviour of such amplitudes and discover some bootstrap equations that any amplitude obeying these assumptions needs to satisfy.

In section 1.5, we discuss a 3-dimensional supersymmetric theory, the study of which is insightful for many reasons: firstly, due to its direct application to condensed matter systems; and secondly, due to the implications it holds for a specific example of AdS/CFT correspondence involving higher-spin fields. We show that in this theory one can compute arbitrary n-point tree-level scattering amplitudes in terms of the 4-point functions of the theory. We further argue about a possibility to generalise such recursion relations to arbitrary loop order.

We conclude in section 1.6 with the discussion of important lessons that we have learnt through the work that is presented here.

1.2 Revisiting AdS/CFT at a Finite Radial Cut-off

1.2.1 Background & Motivation

Utility of any duality in physics lies in the knowledge of relation between the physically relevant quantities between the two theories that are dual; because it is only then one can use the computations in that one theory to draw inference about the other. The introduction to this synopsis briefly introduced the duality between string theory in a AdS background and a gauge theory theory in one lower dimension. Let us elaborate a little more on the duality now. A more precise statement of the duality is presented in terms of the equality between the full string theory partition function evaluated on AdS background and the generating function of the field theory in one lower dimension evaluated at arbitrary value of coupling.[5] However, in the more simplifying probe approximation, where gravitational backreaction can be neglected, this is an equality between the gravity action coupled to matter evaluated on a classical background and a generating function of the lower dimensional conformal field theory, with the source, $J(x)$, for some operator $\mathcal{O}(x)$ turned on:

$$Z_{bulk}[\phi_0] = \lim_{\epsilon\to 0} \int_{\substack{\phi(\epsilon,x)=\phi_0(x)\sim J(x)\\ g_{\mu\nu}(\epsilon,x)\sim\delta_{\mu\nu}}} e^{\frac{1}{16\pi G_N}S_g[g_{\mu\nu}]+S_m[\phi]} = \langle e^{\int J(x)\mathcal{O}(x)}\rangle = Z[J] = e^{W[J]} \tag{1.2}$$

In the AdS/CFT dictionary, the source, $J(x)$ of the operator is related to the boundary value of a bulk field [11, 12]. In Table 1.1 lists the dual bulk fields, the boundary value of which acts as the source for various field theory operators. The duality as stated above can immediately be extended to the computations of correlation functions of the field theory operators using the gravity dual:[6]

$$\begin{aligned}\langle\mathcal{O}(x)\rangle &\sim \frac{1}{Z_{bulk}}\frac{\delta}{\delta\phi_0(x)}Z_{bulk}\Big|_{\phi_0(x)=0}\\ \langle\mathcal{O}(x)\mathcal{O}(y)\rangle &\sim \frac{1}{Z_{bulk}}\frac{\delta^2}{\delta\phi_0(x)\delta\phi_0(y)}Z_{bulk}\Big|_{\phi_0(x)=0}\end{aligned} \tag{1.3}$$

Solving for the onshell solutions of the bulk fields results in two independent solutions (the bulk action is quadratic): one of them dominates the other near the boundary:

$$\phi(z,x) \underset{z\to 0}{\sim} z^{\Delta_+}A(x) + z^{\Delta_-}B(x) \qquad \Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4}+m^2L^2} = \frac{d}{2} \pm \nu \text{ }^{7} \tag{1.4}$$

[5] Such an equality between partition function and generating function holds for all four corners of the duality as presented in Figure 1.2.

[6] For the sake of definiteness, we will be considering the case of scalar field theory operators (and dual scalar bulk fields) in subsequent discussions.

Conventionally, the boundary values of the bulk fields refers to the value of the fields that dominates near the boundary. In the AdS/CFT dictionary, $B(x)$, the coefficient of the dominating mode is related to the source of the dual operator, $J(x)$; while the coefficient $A(x)$ is related to the expectation value of the dual operator, $\langle \mathcal{O}(x) \rangle$. It is well known that the computation of RHS in (1.2) involves renormalization to obtain meaningful observables. The dual bulk computation doesn't come without its share of divergences. It is very easy to see the source of divergences in the bulk: the bulk metric blows up near the boundary of AdS, $z \to 0$. One regulates the field theory computations by introducing a high energy/short-distance cut-off; likewise the bulk computations are regulated by an introduction of a near boundary cut-off at $z = \epsilon$.[8] The treatment of these divergences within the AdS/CFT correspondence has a history similar to the treatment in quantum field theories. In the initial computations of observables, which also proved to be the first checks of the conjecture, the polynomial divergences were simply dropped. In the subsequent works, a more refined understanding was developed through covariant counterterms [13]. In a parallel approach to understanding β-functions of the field theory couplings [14] made the observation that sources for operators in a field theory can be treated as couplings in a long distance expansion:

$$\begin{aligned} S_{pert} = S_{CFT} + \int J(x)\mathcal{O}(x) = S_{CFT} + \int J(k)\mathcal{O}(-k) \\ \approx S_{CFT} + \int (J_0 + k_\mu J_1^\mu + k_\mu k_\nu J_2^{\mu\nu} + \ldots)\mathcal{O}(-k) \\ \approx S_{CFT} + \int J_0 \mathcal{O}(x) + \int J_1^\mu \partial_\mu \mathcal{O}(x) + \int J_2^{\mu\nu} \partial_\mu \partial_\nu \mathcal{O}(-k) + \ldots \end{aligned} \tag{1.5}$$

where, J_i are the couplings constants of the theory. In a Lorentz invariant theory, the terms with odd number of derivatives don't appear, and the perturbations take the form:

$$S_{pert} \approx S_{CFT} + J_0 \int \mathcal{O}(x) + J_2 \int \partial^2 \mathcal{O}(-k) + \ldots \tag{1.6}$$

This is consistent with the solutions of the bulk equations of motion, where (1.4) takes following form in momentum space,

$$\phi(z,k) \underset{z\to 0}{\sim} z^{\Delta_+} A(kz) + z^{\Delta_-} B(kz) \tag{1.7}$$ [9]

where, $A(kz), B(kz)$ are even functions of their argument. The fact that $B(kz)$ is related to the source of the boundary operator led [14] to proposed an interpretation of the coefficients of $B(kz)$ in small momentum expansion (or equivalently a derivative expansion) as boundary

[7] Here, m is the mass of the bulk scalar field $\phi(x)$. When ν takes integer values, one of the solutions behaves logarithmically near the boundary. We will oversee these cases for the time being.

[8] We have been using this cut-off parameters in writing our equations in (1.2) and Table 1.1 without having mentioned the meaning of it.

[9] Here k is the transverse momentum in the boundary directions.

FIGURE 1.3: A cartoon showing the integration of the near boundary degrees of freedom in AdS. These degrees of freedom corresponds to UV degrees of freedom in the dual boundary theory.

coupling constants; and the Hamilton-Jacobi equations for the bulk fields, which give the evolution these coefficients with the radial direction (z) as the β-function equations for these couplings. These works substantiate the relation between the radial direction of AdS bulk dual and the energy scale of the field theory, an idea that was quite apparent in the original D-brane construction of Maldacena, to a generalized AdS/CFT correspondence.

A more 'Wilsonian' understanding of these ideas was further developed in [15–17], where the β-functions of the field theory couplings was computed by an explicit integration of the degrees of freedom between two radial slices, $z = \epsilon_0$ and $z = \epsilon'$ (see Figure 1.3). This process of integration of degrees of freedom gives rise to a 'wavefunctional' on the new boundary at $z = \epsilon'$. AdS/CFT correspondence was generalized to accommodate multi-trace interactions[10] in [18]. The under-lying idea can be (slightly heuristically) explained as follows. Consider a conformal field theory perturbed by an addition of some multi-trace deformation,

$$S_{pert} = S_{CFT} + f[\mathcal{O}] \tag{1.8}$$

where, $f[\mathcal{O}]$ is an arbitrary polynomial function of the operator and its derivatives. Inside a field theory path integral, we can rewrite this as,

$$\begin{aligned}
Z &= \int \mathcal{D}\Phi\mathcal{D}\lambda\ \delta(\lambda(x) - \mathcal{O}(x)) \exp\big(- S_{CFT} - f[\lambda(x)]\big) \\
&= \int \mathcal{D}\Phi\mathcal{D}\lambda\mathcal{D}\sigma\ \ \exp\Big(- S_{CFT} - \int \sigma(x)\big(\lambda(x) - \mathcal{O}(x)\big) - f[\lambda(x)]\Big) \\
&= \int \mathcal{D}\sigma\mathcal{D}\lambda\ \ \exp\Big(- f[\lambda(x)] - \int \sigma(x)\lambda(x)\Big) Z_{bulk}[\sigma(x)] \\
&= \int \mathcal{D}\phi_0\ \ \exp\big(- \tilde{f}[\phi_0(x)]\big) Z_{bulk}[\phi_0(x)]\ ^{11}
\end{aligned} \tag{1.9}$$

[10] In standard AdS/CFT dictionary only the gauge invariant operators are described by the bulk dual. In a gauge theory, such operators are of the kind, $\mathrm{Tr}[\Phi^m(x)],\ m \in \mathbb{Z}$. Double-trace operators, as name suggests, are gauge-invariant operators of the kind $\mathrm{Tr}[\Phi^m(x)]\mathrm{Tr}[\Phi^n(x)],\ m, n \in \mathbb{Z}$. In the subsequent, for any operator $(\mathcal{O})$ that has a dual bulk field (ϕ) under AdS/CFT, by a double trace operator we mean operators of kind: $\mathcal{O}^m$ and derivatives thereof.

where, in the last line we have used AdS/CFT correspondence to write the generating function of the field theory in terms of the bulk action, which has been evaluated with the boundary conditions: $\phi(z,x) \underset{z\to 0}{\sim} z^{\Delta_-}\sigma(x)$. Simply stated, multi-trace deformations in field theory correspond to the addition of a 'boundary wavefunctional' on the boundary ($z=0$) of the AdS to the bulk action:

$$\Psi[\phi_0(x)] = \exp\big(-\tilde{f}[\phi_0(x)]\big) \tag{1.10}$$

The wavefunctionals generated in the integration of the near boundary degrees of freedom in [15, 16] has an interpretation of the generation of double-trace deformations. The bulk Hamilton-Jacobi equations are subsequently used to compute the β-functions of these couplings.

1.2.2 Summary & Results

We have developed these ideas of holographic renormalization into a more robust 'Wilsonian picture'. The two main results in this work are:

1. We started with the question: If the GKPW [11, 12] prescription of using AdS/CFT, coupled with the counterterms computed in [13] is true in the limit cutoff, $\epsilon_0 \to 0$; what is the correct prescription for AdS/CFT at finite cut-off? Generalising in terms of 'wavefunctionals' introduced above, the Dirichlet boundary condition of GKPW prescription can be written as a δ-function wavefunctional on the boundary of AdS:

$$\Psi[\phi_0(x)] = \lim_{\epsilon_0\to 0} \delta\Big(\phi_0(x) - \epsilon_0^{\Delta_-} J(x)\Big) \tag{1.11}$$

We find that the original GKPW δ-function prescription, coupled with the Solodukhin counterterms and applied to a finite radial cut-off $z=\epsilon_0$, corresponds to a wavefunctional which cannot be obtained by the evolution of the known GKPW δ-function boundary condition at $z=0$. We argued for this in multiple ways:

   (a) If one uses the GKPW prescription at finite cut-off, along with the correct counterterms then all the divergences in the correlators are cancelled. However, there is still some non-trivial dependence on the cut-off of the correlators, which should not be the case if we view the theory at finite cut-off as being obtained by integration of degrees of freedom starting with the continuum theory. This is in contradiction with the Wilsonian philosophy that all the physical observables remain unchanged in the process of renormalization.

[11] In this last line, we have just gone back to more familiar notation for the boundary value, $\phi_0(x)$ of the bulk field, $\phi(z,x)$. Moreover, we have also intergrated out the $\lambda(x)$ field.

(b) The β-functions computed using the techniques of [15, 16] for the double-trace operators don't vanish at what one would have expected to the fixed point theory.

We found two specific 'wavefunctionals' that correspond to a fixed point theory (CFT):

$$\Psi_1^0[\phi_0;\epsilon_0] = \exp\left[-\frac{1}{2}\int_{z=\epsilon_0} d^dk\sqrt{\gamma_0}\frac{\left(\phi+\mathscr{A}^*_{ST}\cdot B^*_{ST}(k\epsilon_0)\ \epsilon_0^{d-\Delta_+}J\right)_k\left(\phi+\mathscr{A}^*_{ST}\cdot B^*_{ST}(k\epsilon_0)\ \epsilon_0^{d-\Delta_+}J\right)_{-k}}{\mathscr{A}^*_{ST}(k\epsilon_0)}\right.$$
$$\left.-\frac{1}{2}\int_{z=\epsilon_0}\sqrt{\gamma_0}\phi_k\hat{\mathcal{D}}_{ct}(k\epsilon_0)\phi_{-k}\right] \tag{1.12}$$

$$\Psi_2^0[\phi_0;\epsilon_0] = \exp\left[-\frac{1}{2}\int_{z=\epsilon_0} d^dk\sqrt{\gamma_0}\left(\phi_k\hat{\mathcal{D}}_{ct}(k\epsilon_0)\phi_{-k} + 2\epsilon_0^{d-\Delta_-}B^*_{AQ}(k\epsilon_0)\phi_kJ_{-k}\right.\right.$$
$$\left.\left.+\epsilon_0^{2(d-\Delta_-)}C^*_{AQ}(k\epsilon_0)J_kJ_{-k}\right)\right] \tag{1.13}$$

where, various functions above take the following specific values:

$$\frac{1}{\mathscr{A}^*_{ST}} = 2\nu\left(1-\frac{1}{2\left(1-\nu^2\right)}(k\epsilon_0)^2 - \frac{\left(5+\nu^2\right)}{8\left(4-\nu^2\right)\left(1-\nu^2\right)^2}(k\epsilon_0)^4+\cdots\right) \tag{1.14a}$$

$$\mathscr{A}^*_{ST}\cdot B^*_{ST}(k\epsilon_0) = -\left(1+\frac{1}{4(1-\nu)}(k\epsilon_0)^2+\frac{1}{32(1-\nu)(2-\nu)}(k\epsilon_0)^4+\cdots\right) \tag{1.14b}$$

$$B^*_{AQ}(k\epsilon_0) = 1-\frac{1}{4(1-\nu)}(k\epsilon_0)^2+\frac{(3-\nu)}{32(2-\nu)(1-\nu)^2}(k\epsilon_0)^4+\cdots \tag{1.14c}$$

$$C^*_{AQ}(k\epsilon_0) = -\frac{1}{2\nu}+\frac{1}{4\left(1-\nu^2\right)}(k\epsilon_0)^2-\frac{(5-2\nu)}{32(1-\nu)^2\left(4-\nu^2\right)}(k\epsilon_0)^4+\cdots \tag{1.14d}$$

$$\hat{\mathcal{D}}_{ct}(\epsilon_0k) = \Delta_- - \frac{1}{2(\nu-1)}(k\epsilon_0)^2+\frac{1}{8(\nu-2)(\nu-1)^2}(k\epsilon_0)^4+\cdots \tag{1.14e}$$

They correspond to IR and UV (where it exists) fixed points, respectively, of the field theory; and to standard and alternate (where it exists) quantization [19] in the bulk. More generally, we consider a quadratic wavefunctional:

$$\Psi_0[\phi_0;\epsilon_0] = \exp\left[-\frac{1}{2}\int_{z=\epsilon_0} d^dk\sqrt{\gamma_0}\Big(A(k,\epsilon_0)\phi_k\phi_{-k}+2\epsilon_0^{d-\Delta_+}B(k,\epsilon_0)J_k\phi_{-k}\right.$$
$$\left.+\epsilon_0^{2(d-\Delta_+)}C(k,\epsilon_0)J_kJ_{-k}\Big)\right] \tag{1.15}$$

of which the above cases ($\Psi_1^0[\phi_0;\epsilon_0]$ and $\Psi_2^0[\phi_0;\epsilon_0]$) are two particular cases. For this more general wavefunctional, we give a field theory interpretation to various coefficients when they deviate from the special values in $\Psi_1^0[\phi_0;\epsilon_0]$ and $\Psi_2^0[\phi_0;\epsilon_0]$. These interpretations are summarized in Table 1.2:

$A(k\epsilon)$	$B(k\epsilon)$	$C(k\epsilon)$
Double-trace deformation	Wavefunction renormalization	Contact terms

TABLE 1.2: Interpretation of different coefficients in wavefunctional (1.15) away from the fixed point values, A^*, B^*, C^*.

2. How does the Holographic-scheme of renormalization match up with the Wilsonian-scheme in a field theory computation? Through explicit computations of the β-functions in the bulk as well as field theory[12], we found that the β-functions have the following structure:

$$\begin{aligned}
\beta_0 &= 2\nu \bar{f}_0 - \mathcal{A}_0 \bar{f}_0^2 \\
\beta_1 &= (2\nu - 2)\bar{f}_1 - \mathcal{A}_1 \bar{f}_0^2 - 2\mathcal{A}_0 \bar{f}_0 \bar{f}_1 \\
\beta_2 &= (2\nu - 4)\bar{f}_2 - \mathcal{A}_2 \bar{f}_0^2 - 2\mathcal{A}_1 \bar{f}_0 \bar{f}_1 - \mathcal{A}_0 \left(2\bar{f}_0 \bar{f}_2 + \bar{f}_1^2\right) \\
\beta_3 &= (2\nu - 6)\bar{f}_3 - \mathcal{A}_3 \bar{f}_0^2 - 2\mathcal{A}_2 \bar{f}_0 \bar{f}_1 - \mathcal{A}_1 \left(2\bar{f}_0 \bar{f}_2 + \bar{f}_1^2\right) - \mathcal{A}_0 \left(2\bar{f}_1 \bar{f}_2 + 2\bar{f}_0 \bar{f}_3\right) \\
&\vdots
\end{aligned} \tag{1.16}$$

for some values of $\mathcal{A}_i$ that are different in field theory and bulk. However, we are able to derive a coordinate redefinition in the coupling space that related the two schemes of renormalization.

1.3 2-Dimensional Quantum Gravity Dual to SYK/tensor Models

1.3.1 Background & Motivation

SYK-type [8, 20–24] models have drawn a lot of attention in the literature recently, primarily because of the following features in a large N limit:

1. There is an infrared fixed point with an emergent time reparametrisation symmetry, denoted henceforth as *Diff*.[13] The symmetry is spontaneously broken, at the IR fixed point, to $\mathbb{SL}(2,\mathbb{R})$ by the large N classical solution, leading to Nambu-Goldstone (NG) bosons characterised by the coset *Diff* $/\mathbb{SL}(2,\mathbb{R})$.[14] At the IR fixed point all

[12] The exact computations in the field theory are facilitated by the large N factorization.

[13] We use *Diff* to denote either Diff(R) or Diff(S^1), depending on whether we are at zero temperature or finite temperature. This group is alternatively called the *Virasoro* group.

[14] As explained later in more detail, unlike in higher dimensions where Nambu-Goldstone modes are zero modes of the action promoted to spacetime fields, here they remain zero modes (do not acquire kinetic terms) since they cannot be made dependent on any other dimension.

these are precise zero modes of the action as one might expect from a one-dimensional CFT. Slightly away from the IR fixed point, the *Diff* symmetry is explicitly broken, the 'Nambu-Goldstone' modes cease to be zero modes and their dynamics is described by a Schwarzian term (which is the equivalent of a 'pion mass' term). The particular interest in these modes arises due to the observation that they seem to be responsible for the chaotic behaviour of the 4-point functions in the SYK-like models. Moreover, it has been conjectured that (see, e.g. [25]) that this situation is similar to a bulk model in which the AdS_2 symmetry is slightly broken (this is called a *near* AdS_2 geometry, in the sense of an s-wave reduction from higher dimensions, as in [26]).

2. The possibility of a gravity dual is further reinforced by the fact that the Lyapunov exponent in the SYK model saturates the chaos bound, which is characteristic of a theory of gravity that has black hole solutions [27, 28].

3. The full model has an approximately linearly rising ('Regge-type') spectrum of conformal weights, with $O(1)$ spacing near the IR fixed point. This behaviour is unexpected both from string theory in the limit $\alpha' \to 0$, or from Vasiliev theory. Thus while the dynamics of the soft modes appears to have a simple dual gravity description, it is not clear if it can naturally incorporate the rest of the Regge-type spectrum description. In this chapter we primarily concern ourselves with a bulk gravity dual which describes the soft modes. We leave the larger issue for later work.

In short, a duality between SYK-like models and AdS_2 gravity, if shown explicitly, might be an extremely useful tool to understand the inner working and intricacies of the more general AdS/CFT correspondence. The fact that the correspondence exists in low dimensions might prove to be particularly useful in unravelling some key features of not just AdS/CFT duality but also quantum gravity and strongly-coupled field theories; in a fashion that 't Hooft's 2-dimensional model of QCD was particularly enlightening in understanding the workings of QCD in higher dimensions.

The strategy we pursue for the proposed bulk dual is as follows. As explained in [8, 21], the NG modes of the SYK-type model can be characterised by *Diff* orbits of the classical solution G_0 (at the IR fixed point $J = \infty$) or *Diff* orbits of G_0' which is the deformed value of G_0 after turning on a small value of $1/J$ (see figure 1.4). Any given point on the *Diff* orbit can be obtained from the reference point, G_0 or G_0', by the action of an appropriate one-dimensional diffeomorphism.

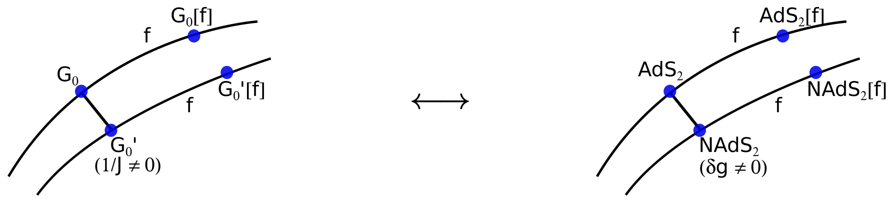


FIGURE 1.4: In the left panel, the top curve represents the Diff(R)-orbit (or a Diff(S^1)-orbit in case of finite temperature), *at the IR fixed point* $J=\infty$, of the classical large N solution for the fermion bilocal $G_0(\tau_1,\tau_2)\sim(\tau_1-\tau_2)^{-2\Delta}$; this represents the Nambu-Goldstones of Diff(R)/$\mathbb{SL}(2,\mathbb{R})$. The lower curve represents the orbit of a deformed solution G_0' slightly away from the IR fixed point, with a small positive $1/J$. In the right panel, the top curve represents the orbit of the AdS$_2$ spacetime (these are asymptotically AdS$_2$ spacetimes, the two-dimensional equivalent of Brown-Henneaux geometries, which we will describe explicitly in Section 3.5). The bottom curve represents the orbit of a slightly deformed AdS$_2$ spacetime NAdS$_2$, with a controlled non-normalizable deformation (see section 3.5).

It is known from the previous work of [29, 30] that the quantization of the coset space $Diff/S^1$ of coadjoint orbits of *Diff* using the natural symplectic form *a la* Kirillov [31], leads to Polyakov's two-dimensional quantum gravity action [32]. Following this observation, one might wonder whether such a two-dimensional quantum gravity action, obtained by the coadjoint orbit method, naturally describes a bulk dual to the SYK model. For this particular scenario we will be interested in the quantization of the orbits corresponding to $Diff/\mathbb{SL}(2,\mathbb{R})$. While it is possible to follow the methods of Kirillov and write down an action, it is not immediately clear what the covariant form of this action is. In this work we conjecture that a generalization of the Polyakov action, which includes a cosmogical constant and boundary terms (the boundary terms are found by requiring the existence of a well-defined variational principle; these are also the terms required by consistency with the Weyl anomaly in a manifold with a boundary):

$$S_{cov}[g]=\frac{1}{16\pi b^2}\int_\Gamma\sqrt{g}\left[R\,\frac{1}{\Box}R-16\pi\mu\right]+\frac{1}{4\pi b^2}\int_{\partial\Gamma}\sqrt{\gamma}\mathcal{K}\,\frac{1}{\Box}R+\frac{1}{4\pi b^2}\int_{\partial\Gamma}\sqrt{\gamma}\mathcal{K}\,\frac{1}{\Box}\mathcal{K}\quad(1.17)$$

is the correct covariant form of this action.

This new action has asymptotically AdS$_2$ geometries as solutions,

$$ds^2=\frac{1}{4\pi\mu\,\zeta^2}\left(d\zeta^2+d\tau^2\left(1-\zeta^2\frac{\{f(\tau),\tau\}}{2}\right)^2\right)\tag{1.18}$$

which are all generated from AdS$_2$, $ds^2=\frac{1}{4\pi\mu\zeta^2}\left(d\tilde{\zeta}^2+d\tilde{\tau}^2\right)$, by the action of following large diffeomorphisms:

$$\tilde{\tau}=f(\tau)-\frac{2\zeta^2f''(\tau)f'(\tau)^2}{4f'(\tau)^2+\zeta^2f''(\tau)^2},\quad\tilde{\zeta}=\frac{4\zeta f'(\tau)^3}{4f'(\tau)^2+\zeta^2f''(\tau)^2}\tag{1.19}$$

Here, $\{f(\tau),\tau\}$ denotes the *Schwarzian derivative*:

$$\{f(\tau),\tau\} = \frac{f'''(\tau)}{f'(\tau)} - \left(\frac{f''(\tau)}{f'(\tau)}\right)^2 \tag{1.20}$$

The schematics of these solutions is described in the right panel of Figure 1.4. We find it particularly useful to analyse the above action and its solutions in conformal gauge ($g_{\alpha\beta} = e^{2\phi}\hat{g}_{\alpha\beta}$), where the above action takes the following form:

$$\begin{aligned} S_{cov}[g] = -\frac{1}{4\pi\, b^2}\left[\int_\Gamma \sqrt{\hat{g}}\left(\hat{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \hat{R}\phi + 4\pi\mu e^{2\phi}\right) + 2\int_{\partial\Gamma}\sqrt{\hat{\gamma}}\hat{\mathcal{K}}\phi + \int_{\partial\Gamma}\sqrt{\hat{\gamma}}\hat{n}^\mu\phi\partial_\mu\phi\right] \\ + \frac{1}{16\pi\, b^2}\int_\Gamma \sqrt{\hat{g}}\hat{R}\,\frac{1}{\hat{\Box}}\hat{R} \end{aligned} \tag{1.21}$$

The most general solution to the equations of motion derived from this action then are:

$$\begin{aligned} &g_{\alpha\beta} = e^{2\phi}\hat{g}^{AAdS}_{\alpha\beta} \\ &\phi = \frac{1}{2}\log\left[(z+\bar{z})^2\frac{\partial g(z)\bar{\partial}\bar{g}(\bar{z})}{(g(z)+\bar{g}(\bar{z}))^2}\right], \qquad \text{with } g(z),\bar{g}(\bar{z}) \text{ satisfying following conditions:} \\ &\{g(z),z\}=0,\ \{\bar{g}(\bar{z}),\bar{z}\}=0 \Rightarrow g(z) = \frac{az+ib}{icz+d}\ ,\quad \bar{g}(\bar{z}) = \frac{\bar{a}\bar{z}-i\bar{b}}{-i\bar{c}\bar{z}+\bar{d}}\ ,\quad a,b,c,d\in\mathbb{C}, \\ &\qquad ad+bc=1,\ ^{15} \end{aligned} \tag{1.22}$$

Here, $z,\bar{z} = \tilde{\zeta} \pm i\tilde{\tau}$ coordinates can be written in terms of ζ,τ coordinates using the coordinate transformations (1.19) and $\hat{g}^{AAdS}_{\alpha\beta}$ geometries appearing the the first line are the AAdS geometries of (1.18). When a,b,c,d are real parameters, then those solutions correspond to global $\mathbb{SL}(2,\mathbb{R})$rotations of the geometry, which is an isometry for all the geometries in (1.18). The remaining 3-parameter set of solutions, which corresponds to the point marked NAdS$_2$ in Figure 1.4 are the solutions of our primary interest. These do not preserve the boundary of AdS. In general, the boundary of the spacetime is given by the curve, $g(z)+\bar{g}(\bar{z})=0$, which for a general function of the kind, (1.22), is not the same as $z+\bar{z}=0$. These solutions will subsequently be referred to as *non-normalizable* solutions following the standard AdS/CFT language. We are interested in *small* non-normalizable deformations near the identity transformation which correspond to the choice,

$$a = 1+\delta a \quad b=\delta b \quad c=\delta c \quad d = 1-\delta a \tag{1.23}$$

Here we assume that $\delta a, \delta b, \delta c$ are complex numbers with non-zero imaginary parts (in fact, we can take the real parts to be zero to separate these solutions from the $\mathbb{SL}(2,\mathbb{R})$ transformations that are the isometry of the AdS$_2$ and AAdS$_2$ geometries). With these parameters,

[15] The difference from the standard $\mathbb{SL}(2,\mathbb{R})$condition $ad-bc=1$ arises because we are working with the right-half complex plane compared to upper-half complex plane as in standard treatments.

the solution for the metric becomes

$$ds^2 = e^{2\phi}\widehat{ds^2} \tag{1.24}$$

with $\widehat{ds^2}$ given by the AdS$_2$ metric (1.18), and ϕ, using (1.22) has the near-boundary form

$$\phi = \frac{\delta g(\tau)}{\zeta} + \mathcal{O}(\delta a^2, \delta b^2, \delta c^2), \qquad \delta g(\tau) = \mathrm{Im}(\delta b) + 2\,\mathrm{Im}(\delta a)\tau + \mathrm{Im}(\delta c)\tau^2 \tag{1.25}$$

For simplicity, we may choose $\delta a = \delta c = 0$, so that $\phi = \mathrm{Im}(\delta b)/\zeta$. For the Liouville factor $e^{2\phi}$ not to destroy the asymptotic AdS$_2$ structure altogether, we will assume here that $\delta g \lesssim \delta$ which ensures that $\delta g < \zeta$.[16] These modes play the role of explicit symmetry breaking in the IR limit of the SYK-like model.

1.3.2 Summary & Results

The main point of our work, [2], is that the *two-dimensional quantum gravity theory*, arrived at in this fashion, *provides a bulk dual to the Nambu-Goldstone sector of the SYK models.* We find a number of strong evidences for this duality:

1. the configuration space of the bulk theory reduces to *Diff* $/\mathbb{SL}(2,\mathbb{R})$, which is the same as the configuration space of the Nambu-Goldstone bosons. In the bulk theory these degrees of freedom emerge as the space of large diffeomorphisms (analogous to Brown-Henneaux diffeomorphisms in AdS$_3$). In addition to these, the bulk metric admits a fixed, non-dynamical conformal factor of a simple functional form. In the SYK theory this parameterizes the departure from strong coupling.

2. The bulk path integral reduces to a path integral over *Diff* $/\mathbb{SL}(2,\mathbb{R})$ with a Schwarzian action:

$$S_{hydro} = \frac{\delta g}{2\pi b^2}\int d\tilde{\tau}\,\left\{\tilde{f}(\tilde{\tau}), \tilde{\tau}\right\} \tag{1.26}$$

characterized by a non-zero overall coefficient coming from the conformal factor, ϕ, (1.25). This is the same Schwarzian action that is also obtained for the pseudo-Goldstone modes from the field theory computations.

[16] There is a natural RG interpretation of this inequality in terms of the boundary theory. We will later identify δg with $\sim 1/J$ (see (3.81)). Together with the natural identification of $1/\zeta$, for small ζ, with a Wilsonian floating cut-off Λ (to be distinguished from the bare cut-off $\Lambda_0 = 1/\delta$, see [15, 16], also [1]), we find $\delta g/\zeta \sim \Lambda/J = 1/\bar{J}$, where $\bar{J} = J/\Lambda$ is the dimensionless coupling. Since $\bar{J}$ grows large near the IR cut-off, it follows that $\delta g/\zeta \ll 1$ near the IR cut-off.

3. the low temperature free energy qualitatively agrees with that of SYK model:

$$\log(Z) = -\beta F = -\frac{2}{b^2}\log(\beta/\delta) + \frac{2\log(4\pi) - 3}{b^2} + \frac{\delta g}{2b^2\,\beta} + \mathcal{O}(\delta g^2) \tag{1.27}$$

The qualitative features that match with that of the SYK-like models is the presence of non-zero zero-temperature entropy and a specific heat that scales linearly with temperature: both of which are the universal features of SYK-like models.

1.4 S-Matrix Bootstrap for Amplitudes with Linear Spectrum

1.4.1 Background & Motivation

This work, in a way, consists of going back to the basics of string theory. In this work we have tried to take an approach very similar to the one taken by original proponents of bootstrap in late 1960s and early 1970s [33–35]. The main motivation of development of the S-matrix techniques then laid in the difficulty in understanding the physics of strong interactions through conventional field theoretic approach, as explained in the introduction to this synopsis. To recall, the main issues as they appeared then were: (1) inability to use perturbation theory in a theory of strong interactions (2) the rich spectrum of hadrons (particles interacting through strong interactions) made it impossible to write down a theory using Lagrangian approach which had a field associated with each of these particles. The interest then shifted from writing down a Lagrangian to directly constraining the S-matrices for the theory of interest. As we discussed above, S-matrices are a very important class of observables in particle-physics and are directly related to the experimental measurements. What's more, they directly shed light on various aspects of the interactions of the theory under study. So the philosophy that was promoted under the S-matrix approach was to try and restrict the space of analytic functions that obeyed the correct properties for them to be considered as an S-matrix of a physical process. The main assumptions required of the S-matrices that we are interested in are listed as follows [9]:

1. Poincare invariance
2. Causality
3. Unitarity
4. Crossing symmetry: it is a statement that when looking at two different scattering process like $a + b \to c + d$ and $a + \bar{c} \to \bar{b} + d$, then the amplitudes of the two processes are related to each other by analytic continuations.

5. Existence of only pole singularities in the amplitude; this basically is a statement that we are restricting to tree-level amplitudes. Often the higher-loop amplitudes are constructed by stitching together the tree-level amplitudes using the Unitarity equation: $S^{\dagger}S = \mathbb{I}$. Demanding that the total amplitude thus constructed be 'physically sensible' again imposes additional constraints. However, in the work presented here we have restricted ourselves to tree-level amplitudes.

6. Linear spectrum of poles that correspond to massive particle exchange at the tree-level.

7. Regge asymptotic behaviour: It is a statement about the asymptotic behaviour of the scattering amplitudes:

$$A(s,t) \underset{s\to\infty, t \text{ fixed}}{\longrightarrow} (-s)^{k(t)} \tag{1.28}$$

Here $k(t)$ is a function such that $k(t) < 0$ for $t < 0$. In our work we work with the case where $k(t) = k\,t$ is a linear function of its argument. What we observe is that since even the linear structure is tightly constrained, it is likely that no consistent amplitude exists where this function is a polynomial.

This approach has received a renewed interest mainly based on the interesting result shown in [36] that any theory of gravity for which the graviton 3-point function deviates from the value given by Einstein-Hilbert action violates causality unless one includes infinitely many higher spin fields. A more technical motivation of revisiting the bootstrap ways is the recent success of similar ideology to conformal field theories, known as Conformal Bootstrap, [37, 38].

A distinct advantage of such an approach lies in its power of generality, and how it helps us develop insights into the precise role of various assumptions and symmetries in governing the behaviour of the physical observables. While the original inventors of S-matrix approach were mainly concerned with understanding a theory of strong interactions, in the work that is presented here we are concerned about a different problem: what is the space of physically consistent theories of quantum gravity that includes infinitely many spinning particles (consistent with the results of [36])? Not many examples are known to us, basically only the examples that arise in String theory. We wish to explore how similar/different is the space of such theories from known examples through our study of S-matrices. While this requires one to study graviton scattering amplitudes, we consider a 'simpler' problem of 4-identical scalar scattering. We have left the study of gravitons for future.

1.4.2 Summary & Results

In [3] we have developed a systematic approach to consider a scattering amplitude as a sum of poles in the s-channel and u-channel processes. We show that under the assumptions that are listed above, especially the assumption on the asymptotic behaviour, one can show that an appropriate analytic continuation of this sum also includes the t-channel poles. Such amplitudes that don't require a sum over all three channels, and in which the sum of poles in only one (or two) of the channels give you the complete amplitude are known as channel-dual amplitudes or simply dual-amplitudes (see Figure 1.5).[17] It can be argued that channel

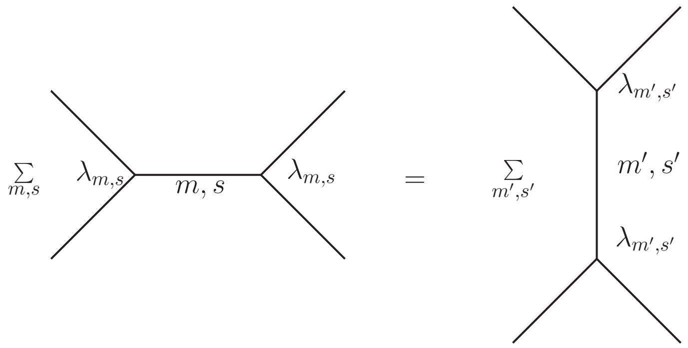


Figure 1.5: Channel dual amplitudes are those in which the sum over the poles in one channel automatically includes the poles in the other channel.

duality follows from crossing symmetry and Regge asymptotics of an amplitude. We have developed a systematic technique to re-sum the amplitude written as a sum over poles in s-complex plane (as a sum over s-channel and u-channel processes):

$$A(a,b) = \sum_{n=0}^{\infty} \frac{f_n(b)}{a+n} + \frac{f_n(b)}{c+n}, \quad \mathrm{Re}\, b > 0,\text{[18]} \tag{1.29}$$

to extract the t-channel poles from it. It can be shown that the Regge behaviour of the full amplitude translates to following behaviour of the residues:

$$f_n(b) = \sum_{j=0}^{\infty} g_j(b) n^{-k(b)-j}. \tag{1.30}$$

In the process of re-summing the amplitude in (1.29) we obtain poles in t-channel which (a) correspond to the physical poles corresponding to the linear spectrum that we started with; (b) some additional 'spurious' poles that are not present in the spectrum. We demand that the residues at the physical poles be consistent with the residues of the corresponding poles in the s-channel,

$$\frac{1}{k} \sum_{J=0}^{kn} g_J(-n) \left\{ (-a)^{kn-J} + (a-n-P)^{kn-J} \right\} = f_n(a) \tag{1.31}$$

[17] Contrast this with the simple examples that one learns in a graduate QFT course, where one necessarily has to consider a sum of diagrams in all three channels.

[18] a, b, c are some rescaled variables defined in terms of Mandelstam variables as: $a = -\alpha' s - \alpha(0)$, $b = -\alpha' t - \alpha(0)$, $c = -\alpha' u - \alpha(0)$, and are constrained by the equation: $a + b + c \equiv P = -4\alpha' M_{ext}^2 - 3\alpha(0)$, where M_{ext} is the mass of the external particle.

At the same time because the amplitude should not contain the above mentioned 'spurious poles' we demand that the residues at such poles vanish identically,

$$\sum_{J=0}^{N} g_J\left(-\frac{N}{k}\right)\left\{(-a)^{N-J}+\left(a-\frac{N}{k}-P\right)^{N-J}\right\}=0 \tag{1.32}$$

where in both the above equations, LHS is the residue at the poles of t-channel that have been obtained by resumming (1.29).
Imposing these constraints on the amplitude we conclude:

1. That the only non-trivial solution to the above equations exists when the asymptotic Regge fall-off is given by $k(t) = kt$ with $k = 2$ in (1.28). This is the same asymptotic behaviour that one finds in String theory.

2. For the case of $k = 2$ any amplitude that is consistent with the above assumption should obey the following *Bootstrap Equations*:

$$\text{Definitions: } f_n(b)=\sum_{j=0}^{\infty} g_j(b) n^{-2b-j}=\sum_{J=0}^{2n} h_J(-n)(-b)^{2n-J}, \tag{1.33}$$

$$\text{Residue-Matching Eqns: } g_j(-n)=h_j(-n), \quad j\le 2n, \tag{1.34}$$

Spurious-Pole Eqns:

$$g_J(b)=-\frac{1}{2}\sum_{j=1}^{J}(-1)^j\frac{\Gamma(-2b-J+j+1)}{\Gamma(j+1)\Gamma(-2b-J+1)}(P-b)^j g_{J-j}(b), \quad J \text{ odd},$$

$$0=\sum_{j=1}^{J}(-1)^j\frac{\Gamma(-2b-J+j+1)}{\Gamma(j+1)\Gamma(-2b-J+1)}(P-b)^j g_{J-j}(b), \qquad J \text{ even}. \tag{1.35}$$

We also found that in general if $A_0(a,b,c)$ is an amplitude that obeys the above requirements then any amplitude that is constructed as follows also obeys it:

$$A(a,b,c)=\sum_{m=0}^{\infty} a_m A_m(a,b,c)\equiv\sum_{m=0}^{\infty} a_m A_0(a+m,b+m,c+m) \tag{1.36}$$

In particular, we can start with $A_0(a,b,c)$ to be Virasoro-Shapiro amplitude and construct a whole family of amplitudes that obey the above requirements.

3. Imposing unitarity doesn't reduce the above space of allowed amplitudes considerably.

1.5 BCFW Recursion Relations in Chern-Simons Theories Coupled to Vector Matter

1.5.1 Background & Motivation

Chern-Simons theories, given by the following action in 3-dimensions,

$$S_{CS} = \frac{i\kappa}{4\pi} \int d^3x \mathrm{Tr}[A \wedge dA - \frac{2i}{3} \epsilon^{\mu\nu\rho} A_\mu A_\nu A_\rho] \tag{1.37}$$

have a very rich history of its application to various different fields in Mathematics and physics. Pure Chern-Simons theory was used in the study of the knot-invariants and Jones polynomials [39]. They are also used in the study of gravity in 3-dimensions [40]. Chern-Simons theories also have applications in Condensed matter theories and provide a coarse-grained effective field theory description of the quantum Hall-effect. Supersymmetric modifications of Chern-Simons theories with (bifundamental-)matter, like ABJM theory [41] have been shown to be dual to M-theory on $\mathrm{AdS}_4 \times S^7$ and is an interesting example of AdS/CFT correspondence. It has also been conjectured that the 3-d Chern-Simons theories coupled to fundamental matter are dual to Vasiliev theory: a theory of higher spins in 4-dimensions [42]. Such varied applications asks for a better understanding of these theories in general. What is more, when the gauge group of the Chern-Simons theories is $SU(N)$ or $U(N)$, then in the $N \to \infty$ limit the theory becomes exactly solvable. Exactly solvable quantum field theories are hard to come by, so it is only natural for one to try and understand such examples. These matter-Chern-Simons theories are also special because they show some interesting dualities: Chern-Simons theory with gauge group $SU(N_F)_{\kappa_F}$ coupled to fundamental fermionic matter,

$$\begin{aligned} S = \int d^3x \, \Big(-\frac{\kappa_F}{4\pi}\Big) \epsilon^{\mu\nu\rho} \, \mathrm{Tr}(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho) \\ + \bar{\psi} \, i \not{D}_\mu \psi + m_F^{\mathrm{reg}} \bar{\psi}\psi \end{aligned} \tag{1.38}$$

at the regular point ($m_F^{reg} = 0$) has been shown to be dual to Chern-Simons theory with gauge group $SU(N_B)_{\kappa_B}$ coupled to fundamental bosons at Wilson-Fisher fixed point ($m_B = 0, \lambda_4 \to \infty$),

$$\begin{aligned} S = \int d^3x \, \Big(-\frac{\kappa_B}{4\pi}\Big) \epsilon^{\mu\nu\rho} \, \mathrm{Tr}(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho) \\ - \mathcal{D}_\mu \bar{\phi} \mathcal{D}^\mu \phi - \lambda_4 (\bar{\phi}\phi)^2 - m_B^2 (\bar{\phi}\phi) \end{aligned} \tag{1.39}$$

The parameters of the two dual theories are related by the following relations:

$$N_B = |\kappa_F| - N_F, \quad \kappa_B = -\kappa_F, \quad \lambda_B = \lambda_F - sgn(\lambda_F) \tag{1.40}$$

This is once again an example of a strong-weak duality[19] and will provide us with a better understanding of the strongly-coupled field theories. This fermion-boson duality is also the only known example of *bosonization* in 3-dimensions.
While the above duality has been verified in the limit, $N_B \to \infty$ or $N_F \to \infty$, a lot of work still needs to be done for theories at finite N. A $\mathcal{N}=1$ supersymmetric generalization of the above duality is described in terms of,

$$\begin{aligned} S^L_{\mathcal{N}1} = \int d^3x \Big[&- \frac{\kappa}{4\pi}\epsilon^{\mu\nu\rho}\mathrm{Tr}\left(A_\mu\partial_\mu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right) + \bar{\psi}(i\not{D} + m_0)\psi - \mathcal{D}^\mu\bar{\phi}\mathcal{D}_\mu\phi - m_0^2\bar{\phi}\phi \\ &- \frac{4\pi m_0}{\kappa}(\bar{\phi}\phi)^2 - \frac{4\pi^2}{\kappa^2}w^2(\bar{\phi}\phi)^3 + \frac{2\pi}{\kappa}(1+w)(\bar{\phi}\phi)(\bar{\psi}\psi) + \frac{2\pi}{\kappa}w(\bar{\psi}\phi)(\bar{\phi}\psi) \\ &- \frac{\pi}{\kappa}(1-w)\big((\bar{\phi}\psi)(\bar{\phi}\psi) + (\bar{\psi}\phi)(\bar{\psi}\phi)\big)\Big] \end{aligned} \quad [20] \tag{1.41}$$

which is self dual under $\kappa \to -\kappa$. The above theory has an enhanced $\mathcal{N}=2$ symmetry for the particular choice $w=1$, and also has a conformal symmetry when $m_0 = 0$. This $\mathcal{N}=2$ theory forms the subject of our study in the work presented in this section. We study arbitrary m-particle to n-particle S-matrices in these theories and comment on the self-duality of the theory in terms of scattering matrices. Such studies of scattering amplitudes are quite important from various points of views. As we have emphasised earlier, exactly solvable quantum field theories are quite rare and even in perturbative field theories it is not possible to compute arbitrary scattering amplitudes even at tree levels. In the known examples where it has been possible to do such computations, often a rich structure of symmetries as been discovered. For example, integrability in $\mathcal{N}$=4 SYM in 4-dimensions and in $\mathcal{N}$=6 ABJM theories have been argued based on the BCFW recursion relations. In these known works it has been anticipated that some minimum amount of supersymmetry is required for BCFW recursion relations to work. Through our work we show that these recursion relations are true for $\mathcal{N}$=2 theories but don't hold for $\mathcal{N}$=1 theories. Quite interestingly, it is also possible to understand arbitrary scattering amplitudes in non-supersymmetric theories using the supersymmetric results. One can immediately note that the fermionic sector of the action Equation 1.41 is same as the non-supersymmetric theory of regular fermions coupled to Chern-Simons theory. Scattering amplitudes involving only fermions are, hence, same in the supersymmetric theory as in the non-supersymmetric theory at tree level. The recursion relations that we derive for the supersymmetric theory also hold for the the non-supersymmetric theory, despite the fact that the non-supersymmetric theory by itself doesn't

[19]Strong-weak because the fermionic theory has no self interaction terms, while the critical bosonic theory is defined at the Wilson-Fisher fixed point, which is a strongly interacting fixed point.

[20]In our notations,

$$\begin{aligned} &\mathcal{D}^\mu\bar{\phi} = \partial^\mu\bar{\phi} + i\bar{\phi}A^\mu \ , \ \mathcal{D}_\mu\phi = \partial_\mu\phi - iA_\mu\phi \ , \\ &\not{D}\bar{\psi} = \gamma^\mu(\partial_\mu\bar{\psi} + i\bar{\psi}A_\mu) \ , \ \not{D}\psi = \gamma^\mu(\partial_\mu\psi - iA_\mu\psi) \end{aligned} \tag{1.42}$$

obey the postulates needed for validity of BCFW relations.
Higher point scattering amplitudes will also shed light on the Aharonov-Bohm effect in multi-particle systems in the non-relativistic limit. While it is common in literature to study Aharonov-Bohm effect when one particle revolves around the other, it would be quite interesting to predict the phases various particles pick up in a multi-particle system.

1.5.2 Summary & Results

In our work [4], we have shown that arbitrary m-particle to n-particle scattering amplitude can be written in terms of 2-particle to 2-particle scattering amplitudes using the BCFW recursion relations. BCFW recursion relations were originally discovered to compute arbitrary tree-level amplitudes in Yang-Mills theories in 4-dimensions. It relies on the following ideas:

1. S-matrices in a theory are characterized by the momenta of the external particles, p_i, and the other quantum numbers of the particles (like helicity, spin, etc.). All the external particles are onshell, and for massless particles we have, $p_i^2 = 0$. S-matrices also obey the momentum conservation condition, $\sum_i p_i = 0$.
2. The momenta of the external particles can be deformed in complex momentum plane, $\hat{p}_i(z_i)$, as a function of complex variables z_i, such that the momentum conservation and on-shell conditions still hold ($\sum_i \hat{p}_i(z_i) = 0$ and $\hat{p}_i^2(z_i) = 0$). In our case (and in the general application of BCFW in the literature) only two of the external particles (say, p_j and p_k) are deformed in terms of one complex variable, z.
3. For such deformations, for different values of the complex parameter, z leads to different internal particles going on-shell, and a tree-level amplitude has corresponding poles. Moreover, in such a case the amplitude factorizes depicted in the following diagram,

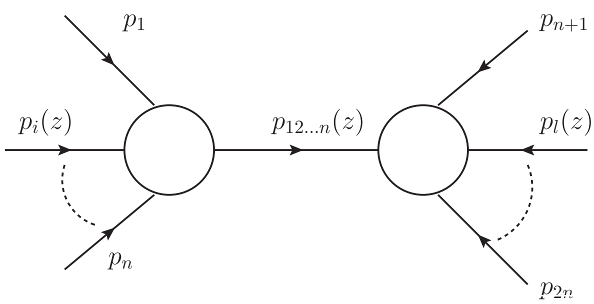


FIGURE 1.6: Factorization of the higher point scattering amplitude under deformation of external momenta $p_i(z), p_l(z)$.

where both the left and the right blobs are individually some on-shell S-matrices with fewer number of particles. When the original amplitude has regular behaviour at $z \to \infty$, it is possible to write the original scattering amplitude of more number of particles in terms of the scattering amplitudes of fewer particles.

In our work we show the following results:

1. It is possible to generalize the above ideas to 3-dimensions and check that the tree-level super-amplitudes for the $\mathcal{N}2$ theory, under appropriate deformations of the external momenta can be written in terms of the smaller-point super-amplitudes:

$$\mathcal{A}_{2n} = \frac{1}{2\pi i} \oint_{\mathbb{C}(1,\epsilon)} dz \frac{\hat{\mathcal{A}}_{2n}(z)}{(z-1)} = -\sum_f \int d\theta_f \frac{(-i)}{p_f^2} \Big[H(z_1, z_2) \mathcal{A}^L(z_1) \mathcal{A}^R(z_1) + \{z_1 \leftrightarrow z_2\} \Big] \tag{1.43}$$

$$H(z_1, z_2) = \begin{cases} z_1 \frac{z_2^2 - 1}{z_1^2 - z_2^2}, & \text{when an external boson and a fermion are deformed} \\ z_1^2 \frac{z_2^2 - 1}{z_1^2 - z_2^2}, & \text{when the deformed particles are either both bosons or both fermions} \end{cases} \tag{1.44}$$

In the above expressions sum over f denotes different channels of factorizations corresponding to different intermediate particles going on-shell; $\mathcal{A}^{L/R}$ are the super-amplitudes corresponding to the left and right factorized amplitudes in Figure 1.6; and, p_f is the momentum of the intermediate particle with undeformed external momenta. The θ_f is a Grassman integral corresponding to super-space coordinate of the intermediate particle. z_i are the values of the complex parameter z for which the given intermediate particle in a given channel goes on-shell.

We have explicitly verified the validity of the BCFW recursion relations by directly computing the 6-point super-amplitude, $\mathcal{A}_6(\bar{\Phi}_1 \Phi_2 \bar{\Phi}_3 \Phi_4 \bar{\Phi}_5 \Phi_6)$ and also in terms of the 4-point amplitudes $\mathcal{A}_4(\bar{\Phi}_1 \Phi_2 \bar{\Phi}_3 \Phi_4)$ using the above equation.[21] We find an exact matching of the results.

2. It can be noted that the fermionic sector of the action in (1.41) is the same as the non-supersymmetric theory. In a theory of only fermions, all the external particles in any S-matrix can only be fermions. Such all-fermion scattering amplitudes appear as component amplitudes in supersymmetric amplitudes. However, at tree level, even in supersymmetric theory there are no internal bosons in the Feynman diagrams contributing to such amplitudes. This is because all interaction vertices that appear in the Lagrangian of the supersymmetric theory have 2 bosons in them, making it impossible to contract them all without forming a loop. Now since the super-amplitudes factorize,

[21] Here Φ are the superfields in the $\mathcal{N}1$ superspace language: $\Phi(\theta, x) = \phi(x) + \theta\psi(x)$

it is not hard to see that one of the components of the super-amplitude, the one that contains all external fermions would also factorize under BCFW deformations. Hence, we argue that a modified BCFW recursion relation holds for the tree-level amplitudes in non-supersymmetric fermionic theory, despite the fact that the amplitudes in that theory doesn't obey the requirements of BCFW.

1.6 Conclusion

Through the work presented in this synopsis we have made an attempt to improve our understanding of quantum theory of gravity as well as strongly coupled field theories through techniques that tie these two important questions in modern day physics together.
We have discovered some interesting facts about the connection between the degrees of freedom in the theory of gravity on AdS background and its dual conformal field theory. We have developed a Wilsonian understanding of these degrees of freedom and developed a precise improvement in AdS/CFT correspondence to account for field theories defined at finite cut-off.
We also studied AdS/CFT correspondence in reference to a toy example of a 1-dimensional field theory and showed how the states relevant to the effective low energy description of the field theory are related to the classical geometries that are related to AdS_2 through large diffeomorphisms. We also proposed a dual theory and provided evidences for the validity of this proposed duality.
Through S-matrix bootstrap techniques we have found some interesting constraints on the space of dual-amplitudes. Using some novel techniques that we have developed to make channel-duality explicit in certain kind of amplitudes, we have also presented a set of equations that every amplitude that obeys channel duality needs to satisfy.
Lastly, we have shown the existence of recursion relations in supersymmetric Chern-Simons theories coupled to vector matter that enables one to compute arbitrary scattering amplitudes in this theory. We have discussed some interesting implications this has on the non-supersymmetric theories. We find, quite surprisingly, that in some non-supersymmetric theories that don't obey the postulates of the BCFW recursion relations, it might still be possible to discover recursion relations that relate the higher point scattering amplitudes in terms of lower point scattering amplitudes.

Chapter 2

AdS/CFT at a finite radial cut-off

2.1 Introduction

In AdS/CFT, conformal field theory partition function at a finite UV cut-off (Λ) is given by an AdS partition function at a finite radial cut-off $z = \epsilon = R^2_{AdS}/\Lambda$. The latter quantity, of course, needs a boundary condition. For example, the original GKPW prescription is a Dirichlet boundary condition. It is well-known, however, that the bulk path integral with this boundary condition leads to correlators with contact terms some of which may diverge in the limit $\epsilon \to 0$. Following de Haro et al [13], it is possible to add bulk counterterms to remove these contact terms (completely or partially). With recent insight from hWRG (holographic Wilsonian RG [15, 16]), boundary conditions at $z = \epsilon$ can be treated as a wavefunction $\Psi_0[\phi_0, \epsilon]$ (e.g. Dirichlet b.c. is a delta-function wavefunction). More generally, it is possible to impose a boundary condition on the bulk fields by introducing a boundary term in the action. In the context of AdS/CFT correspondence it begs for a better understanding of: (i) What are the physically allowed boundary wavefunctionals (equivalently, boundary conditions)? (ii) What does a choice of boundary condition/wavefunction in the bulk path integral correspond to in the CFT?

Since the AdS/CFT dictionary discusses a limited number of boundary conditions that are relevant from the field theory point of view, one way to answer (i) is to study the wavefunctionals that arise by the integration of the near boundary degrees of freedom in the bulk geometry. Equivalently, this corresponds to integration of UV degrees of freedom in the field theory. A boundary wavefunction $\Psi_0[\phi_0, \epsilon]$ is allowed provided its ϵ-dependence follows the radial Schrödinger equation $\partial_\epsilon \Psi_0[\phi_0, \epsilon] = H_{rad}[\phi_0, \partial/\partial\phi_0]\ \Psi_0[\phi_0, \epsilon]$. In the limit of $G_N \to 0$ (implicit in the above equation), the Schrödinger equation reduces to a Hamilton-Jacobi equation for $S[\phi_0, \epsilon] = \log \Psi_0[\phi_0, \epsilon]$:

$$\partial_\epsilon S = H_{rad}[\phi_0, \partial S/\partial \phi_0]$$

For example, for a quadratic bulk action such as (2.4), the space of allowed boundary wavefunctions $\Psi_0 = e^S$ is given by the (2.10), which we reproduce schematically as (here we suppress the ϵ-dependent factors in $\mathbf{B}, \mathbf{C}$)

$$\Psi_0[\phi_0; \epsilon] = \exp\Bigg[-\frac{1}{2}\int \sqrt{\gamma_0}\left(\mathbf{A}(k,\epsilon)\phi_0(k)\phi_0(-k) + 2\mathbf{B}(k,\epsilon)J(k)\phi_0(-k) + \mathbf{C}(k,\epsilon_0)J(k)J(-k)\right)\Bigg] \tag{2.1}$$

We will show below that the wavefunctional corresponding to GKPW [11, 12] boundary conditions, normally taken to represent the CFTs (Dirichlet boundary condition for standard quantization and Neumann for alternative quantization when the latter exists), correspond to a wavefunctional with a wrong ϵ-dependence when taken with the counterterms in [13], as they do not satisfy the radial Schrödinger equation. This wavefunctional also leads to spurious double trace deformations in the dual CFT. The correct wavefunctions which represent the IR and UV CFT's (standard and alternative CFTs) are the wavefunctions Ψ_1^0 and Ψ_2^0 described below (Equation 2.16 and 2.23, respectively).[1]

A partial answer to question (2) appears in [18] where it is shown that a subset of the above wavefunctions represents a CFT with double-trace deformations (see Section 2.4). This chapter provides a detailed and improved interpretation of the A, B, C coefficients[2]. In particular it is shown that various choices of the A, B, C terms correspond to (i) double-trace deformations,

$$S = S_{CFT} + \sum_{n=0}^{\infty} f_n \int \mathbb{O}_n, \quad \mathbb{O}_n = \mathcal{O}(x)(\partial^2)^n\mathcal{O}(x) \tag{2.2}$$

and (ii) contact terms. We have summarized the interpretation of these coefficients in Table 2.1. One of the main observations of the work presented in this chapter is that there exist special wavefunctionals (with special choices of A, B, C) such that both (i) and (ii) are absent and the correlators become pure power laws. Indeed, as mentioned above, there are just two such special choices Ψ_1^0 and Ψ_2^0 in the context studied in this paper: one corresponds to the IR CFT (standard quantization) without any deformations and the other corresponds to the UV CFT (alternative quantization) without any deformation. In section 2.2 these

[1] Ψ_2^0, the wavefunctional corresponding to the UV fixed point has an interpretation of a unitary quantum field theory only inside the Klebanov-Witten window.

[2] J will continue to represent the source for the single trace operator $\mathcal{O}(x)$ dual to the bulk field ϕ.

will correspond to setting a quantity $\chi(k)$ (characterizing A, B, C, and hence the boundary wavefunctional) to zero or ∞.

We have shown (in section 2.3) that the wavefunctions Ψ_1^0 and Ψ_2^0 have a simple geometric interpretation. Each of them corresponds to a specific smearing of the boundary points in Witten diagrams; as mentioned above, the defining property of the above smearing is that even when the cut-off surface is moved inside, the resulting correlators remain a power law. As an application of the above insight, the Wilsonian holographic beta-functions of the double-trace operators are computed and compared with those obtained from direct calculations in field theory. We find that the infinite number of coupled beta-functions can be exactly mapped between field theory and holographic calculations. The existence of such a mapping is nontrivial since both the field theory and holographic beta-functions are exact and strictly quadratic. The correct identification of the double trace deformations with the boundary wavefunctionals plays here an essential role.

This chapter is organized as follows:
Section 2.2 discusses the allowed boundary conditions at finite cut-off and arrive at the two wavefunctionals Ψ_1^0 and Ψ_2^0 which correctly represent the IR and UV CFTs respectively. In section 2.3, a geometric interpretation of these wavefunctions is discussed. It can be shown that these boundary wavefunctionals represent a specific kind of non-locality which smears the boundary points in Witten diagrams in a particular way. Section 2.4 presents the exact identification of the coefficients in a general boundary wavefunctional with coupling constants of double trace deformations in Eq. (2.2) and the contact terms (a generic boundary wavefunction represents both). In section 2.5, we use the above characterization of double trace deformations to compute the infinite series of coupled beta-functions. A detailed dual field theory computation of these infinite series of beta-functions is presented in section 2.6 and a discussion of the matching between the two results in section 2.7. The matching works with a mapping between the field theory and the bulk couplings; such a map is highly constrained because the beta-functions are quadratic and exact on both sides. The details of various calculations have been reserved to appendices B-E.

The details of various calculations in the paper were shared along with the arXiv preprint and are available at arXiv:1608.00411 as a Mathematica notebook named *CalculationsFile.nb*.

2.2 AdS/CFT at a finite radial cut-off: fixed points

The section presents a precise extension of the GKPW prescription ([11, 12]) to a finite cut-off. We present the ideas in the context of correlation functions of a single trace operator $\mathcal{O}(x)$, which is dual to a scalar field $\phi(z, x)$ in $d+1$ dimensional AdS spacetime. The spacetime metric and the scalar action are given by[3]

$$ds^2 \equiv g_{MN} dX^M dX^N = \frac{dz^2 + dx_\mu dx_\mu}{z^2}, \tag{2.3}$$

$$S_b = \frac{1}{2}\int d^d x dz \sqrt{g}\left((\partial\phi)^2 + m^2\phi^2\right) \tag{2.4}$$

Mass of the scalar field, ϕ, is related to the scaling dimension of the dual field theory scalar operator, $\mathcal{O}$, by the following relation [4]

$$\Delta = \Delta_+ \equiv d/2 + \nu, \quad \nu = \sqrt{d^2/4 + m^2 R_{AdS}^2}, \tag{2.5}$$

(we use units where $R_{AdS} = 1$). This relation is commonly referred to as the *mass-dimension relation*.[5] This relation is just one entry of a more elaborate dictionary between the conformal dimension of the field theory operators and mass of the dual bulk field in the AdS/CFT dictionary. The generalization of such a mass-dimension relation for various fields appearing in Table 1.1 can be found in [43]. For our current purposes, the scalar action is the only relevant part of the bulk action since we will work in the "probe approximation" in which the AdS metric g_{MN} remains unaltered (see Appendix E).

2.2.1 Standard quantization

As explained in the introduction to this chapter, the title 'standard quantization' refers to the quantum theory defined by the usual GKPW prescription, characterized by the mass-dimension relation (2.5). Under special circumstances we can define an 'alternative quantization' (see footnote 4 and more detailed discussions below). Recall that in our notation, various quantities associated with the 'standard quantization' are denoted by a subscript + (e.g. Δ_+) (and similarly those associated with 'alternative quantization' by a subscript $-$).

[3] Euclidean metric is considered for simplicity.

[4] As discussed in the introduction to this chapter, in the Klebanov-Witten window $\nu \in (0, 1)$, two distinct unitary CFT duals can be found, corresponding to $\mathcal{O}(x)$ having scaling dimensions $\Delta_\pm = d/2 \pm \nu$. For the new CFT, defined as 'alternative quantization', the conformal dimension is Δ_-.

[5] Sometimes, in the literature, the relation $\Delta(\Delta - d) = m^2 R_{AdS}^2$ is also known as the mass-dimension relation, to which (2.5) is one of the solutions

Let us begin with the following *putative definition* of AdS/CFT for standard quantization (GKPW)

$$Z_+[J_k] = \langle \exp\left[\int d^d k J_k \mathcal{O}_{-k}\right]\rangle_+ = \int \mathcal{D}\phi_0 \Psi_0[\phi_0;\epsilon_0] \int_{z>\epsilon_0} \mathcal{D}\phi e^{-S_b} \tag{2.6}$$

$$\Psi_0[\phi_0;\epsilon_0] = \Psi_{\text{GKPW}} \times \Psi_{ct},\ \ \Psi_{\text{GKPW}} = \delta\left(\phi_0(k) - \epsilon_0^{d-\Delta_+} J(k)\right),$$
$$\Psi_{ct} = \exp\left(-\frac{1}{2}\int_{z=\epsilon_0} \sqrt{\gamma_0}\phi_k \hat{\mathcal{D}}_{ct}(k\epsilon_0)\phi_{-k}\right) \tag{2.7}$$

Here γ_0 is the determinant of the induced metric $\gamma_{\mu\nu}$ at a radial cut-off $z = \epsilon_0$.

The δ-function above is equivalent to imposing the Dirichlet boundary condition on the bulk field at $z = 0$, where the boundary value of the bulk field is related to the source, $J(k)$, of the dual field theory operator $\mathcal{O}(k)$ with some appropriate renormalization. In addition to the original δ-function of GKPW, (2.7) also includes the counter-terms denoted by $\hat{\mathcal{D}}_{ct}(k\epsilon_0)$ conventionally introduced to ensure finiteness of the bulk partition function in the $\epsilon \to 0$ limit [13] (see also [19, 44]; soon we will rediscover these counterterms from the requirement of a well-defined variational principle, cf. (2.35) below). Expanded to several orders in $(k\epsilon_0)^2$, it reads ([13] gives the first two terms; the expansion can be worked out to arbitrary orders with the help of the Mathematica notebook shared as Ancillary files on arXiv:1608.00411, [1])

$$\hat{\mathcal{D}}_{ct}(\epsilon_0 k) = \Delta_- - \frac{1}{2(\nu-1)}(k\epsilon_0)^2 + \frac{1}{8(\nu-2)(\nu-1)^2}(k\epsilon_0)^4 + \cdots \tag{2.8}$$

Quite interestingly, it is also possible to derive these counterterms by demanding exact conformal invariance as explained below.

We now demonstrate the need for improvement of the conventional definition of AdS/CFT (2.6) using the wavefunctional (2.7). It will be noticed that if the correlators of the dual field theory operators are computed using the above prescription, (2.7), for $\epsilon_0 > 0$, then the answer is not consistent with Ward identities of conformal symmetry. Rather, the correlators are of the form (2.89), corresponding to correlators computed in a regulated field theory perturbed by double trace operators. While, in some sense, these correlators do limit to those expected from conformal symmetry, strictly speaking, these can't be interpreted as coming from an exact conformal field theory through Wilsonian philosophy. It could still be argued that the above prescription is valid only at $\epsilon_0 = 0$, however, it is hardly clear how to take this limit in (2.7). The resolution to this inconsistency lies in the modification of the wavefunctional on the boundary. The subsequent analysis will also shed light on how to take the limit $\epsilon_0 \to 0$ in a well defined manner.

The space of allowed wavefunctionals: The general form of the wavefunctional, $\Psi_0[\phi_0, \epsilon_0]$, in particular the dependence on ϵ_0, can be inferred from the fact that it must satisfy the radial Schrödinger equation, which, in the case of a bulk theory with a free massive scalar without gravitational back reaction, takes the form

$$-\partial_{\epsilon_0}\Psi[\psi_0;\epsilon_0] = \hat{H}_{rad}\Psi[\psi_0;\epsilon_0], \text{ where,} \tag{2.9}$$
$$\hat{H}_{rad} = \int d^dx\ \hat{\mathcal{H}}_{rad} = \frac{1}{2}\left(\int d^dk\ \frac{1}{z^{1-d}}\hat{\Pi}_k\hat{\Pi}_{-k} + z^{-1-d}\left(z^2k^2+m^2\right)\hat{\phi}_k\hat{\phi}_{-k}\right) \text{ and, } \hat{\Pi} \equiv i\frac{\delta}{\delta\phi}$$

The general solution for the wavefunctional is of the following quadratic form in the bulk field ϕ_0, of the form [6]

$$\Psi_0[\phi_0;\epsilon_0] = \exp\left[-\frac{1}{2}\int\limits_{z=\epsilon_0} d^dk\sqrt{\gamma_0}\Big(A(k,\epsilon_0)\phi_k\phi_{-k} + 2\epsilon_0^{d-\Delta_+}B(k,\epsilon_0)J_k\phi_{-k} + \epsilon_0^{2(d-\Delta_+)}C(k,\epsilon_0)J_kJ_{-k}\Big)\right] \tag{2.10}$$

Eqn. (2.9), computed in the Hamilton-Jacobi approximation [15, 16] gives [7]

$$\dot{A} = -(A-\Delta_+)(A-\Delta_-) + (k\epsilon)^2, \quad \dot{B} = \Delta_+\ B - A\ B, \quad \dot{C} = (2\Delta_+ - d)\ C - B^2 \tag{2.11}$$

here, $\dot{X}$ denotes, $\epsilon_0\partial_{\epsilon_0}X$. The general closed form solution for $A(k,\epsilon_0)$ is,

$$A(k,\epsilon) = \frac{\chi(k)\ \Big((\frac{d}{2}+\nu)I_{-\nu}(k\epsilon) + k\epsilon_0 I_{-\nu-1}(k\epsilon)\Big) + (-1)^\nu\frac{\Gamma(\nu+1)}{\Gamma(1-\nu)}\Big((\frac{d}{2}-\nu)I_\nu(k\epsilon) + k\epsilon I_{\nu-1}(k\epsilon)\Big)}{\chi(k)\ I_{-\nu}(k\epsilon) + (-1)^\nu\frac{\Gamma(\nu+1)}{\Gamma(1-\nu)}I_\nu(k\epsilon)}$$
$$= \frac{2^\nu\chi(k)\left((d-2\nu) + (k\epsilon)^2\ \frac{(d-2\nu+4)}{4(1-\nu)} + \dots\right) + \left(-\frac{1}{2}\right)^\nu(k\epsilon)^{2\nu}\left((d+2\nu) + (k\epsilon)^2\frac{(d+2\nu+4)}{4(\nu+1)} + \dots\right)}{2^{\nu+1}\chi(k)\left(1 + (k\epsilon)^2\frac{1}{4(1-\nu)} + \dots\right) + 2\left(-\frac{1}{2}\right)^\nu(k\epsilon)^{2\nu}\left(1 + (k\epsilon)^2\frac{1}{2(4(\nu+1))} + \dots\right)} \tag{2.12}$$

Here, $\chi(k)$ is a constant of integration, fixed by solving with a boundary condition at some cut-off $z = \epsilon_0$. Note that the above solution in the series form has two independent series, a series in integer powers of $(k\epsilon_0)$ and another series in powers of $(k\epsilon_0)^{2\nu}$. It will be shown that the series corresponding to $(k\epsilon_0)^{2\nu}$ contains information about the double trace deformations

[6] The explicit ϵ_0-dependent factors in front of B and C are chosen so that the parameters A, B, and C in the wavefunctional are dimensionless (note our choice of units where $R_{AdS} = 1$). The form of the wavefunctional can also be argued based on explicit integration of the near boundary degrees of freedom in the bulk action, as is done in [15–17], and also in Appendix C. Without any interactions, the wavefunctional obtained by integrating out degrees of freedom between $z = 0$ and some $z = \epsilon_0$ can only be quadratic.

[7] In this particular quadratic case, Hamilton-Jacobi approximation is equivalent to exact Schrödinger equations. The second and third equations of (2.11) are slightly different from the corresponding equations in [15, 16] due to the fact that the their B, C are dimensionful.

FIGURE 2.1: The wavefunctional with the coefficients A^*, B^*, C^* gives the correct effective description of the continuum theory, which was obtained by the $\epsilon_0 = 0$ bulk action. This wavefunctional is effectively obtained by the integration of near boundary degrees of freedom in AdS.

around the fixed point. Similar solutions exist for $B(k, \epsilon_0)$ and $C(k, \epsilon_0)$ but have not been reproduced here because they aren't particularly insightful.

Wavefunctional satisfying exact scaling In general, the partition function can be computed by integrating out the bulk fields exactly,

$$Z[J_k] = \exp\left[-\frac{1}{2}\int d^d k\ J_k J_{-k}\epsilon_0^{d-2\Delta_+}\left(C(k,\epsilon_0) - \frac{B^2(k,\epsilon_0)}{k\epsilon_0\frac{K_{\nu-1}(k\epsilon_0)}{K_\nu(k\epsilon_0)} - \Delta_- + A(k,\epsilon_0)}\right)\right] \tag{2.13}$$

where, $K_\nu(k\epsilon_0)$ are the modified Bessel functions of second kind. There are two special choices of $\chi(k)$ above, i.e. $\chi(k) = 0$, or ∞, for which the partition function in (2.13) becomes exactly that of a conformal theory.[8]

To the leading order in $k\epsilon_0$, the solution for these particular choices of the wavefunctionals are $A = \Delta_+$ or Δ_-, as can also be seen from the leading order truncation of (2.12). Let us consider the solution with $A = \Delta_+$. In this case, B-evolution equation is identically satisfied, and the value of B is fixed by the boundary value enforced by (2.7) (to leading order, in continuum limit) to $B = -2\nu$. Finally, this fixes $C = 2\nu$, and the wavefunctional

[8] What we really mean here is that the partition function computed above doesn't explicitly depend on the cut-off ϵ_0, thus obeying the correct scaling laws corresponding to the dual field theory operator $\mathcal{O}$. This is also the reason to claim that such a wavefunctional can be understood as being generated by integrating out the degrees of freedom between $z = 0$ and $z = \epsilon_0$ in the bulk theory that is exactly dual to the conformal field theory, the limiting action given by (2.7). This is consistent with the Wilsonian philosophy of RG.

is given by,

$$\Psi_0[\phi_0;\epsilon_0] \sim \exp\left[-\frac{1}{2}\int\limits_{z=\epsilon_0} d^dk\sqrt{\gamma_0}\Big(\Delta_+\ \phi_k\phi_{-k} - 4\nu\epsilon_0^{d-\Delta_+}J_k\phi_{-k} + 2\nu\epsilon_0^{2(d-\Delta_+)}J_kJ_{-k}\Big)\right]$$
$$\sim \exp\left[-\frac{1}{2}\times(2\nu)\int\limits_{z=\epsilon_0} d^dk\sqrt{\gamma_0}\Big(\phi_k - \epsilon_0^{d-\Delta_+}J_k\Big)_k\Big(\phi_k - \epsilon_0^{d-\Delta_+}J_k\Big)_{-k}\right]$$
$$\times\exp\left[-\frac{1}{2}\Delta_-\int\limits_{z=\epsilon_0}\sqrt{\gamma_0}\phi_k\phi_{-k}\right]\ [9] \tag{2.14}$$

Thus, with $A = \Delta_+, B = -2\nu, C = 2\nu$, we have an appropriately regulated, and correct form of the wavefunctional (2.7). Clearly, (2.14) also reproduced the correct δ-function boundary condition of GKPW in the $\epsilon_0 \to 0$ limit.

The solution with the sub-leading corrections can be found to arbitrary order in $(k\epsilon_0)$ and are given by,

$$A^*_{ST}(k\epsilon_0) = \Delta_+ + \frac{1}{2(1+\nu)}(k\epsilon_0)^2 - \frac{1}{8(2+\nu)(1+\nu)^2}(k\epsilon_0)^4 + \cdots \tag{2.15a}$$
$$= \hat{\mathcal{D}}_{ct}(k\epsilon_0) + 2\nu\left(1 - \frac{1}{2\,(1-\nu^2)}(k\epsilon_0)^2 + \frac{(5+\nu^2)}{8\,(4-\nu^2)\,(1-\nu^2)^2}(k\epsilon_0)^4 + \cdots\right) \tag{2.15b}$$
$$= \hat{\mathcal{D}}_{ct}(k\epsilon_0) + 1/\mathscr{A}^*_{ST} \tag{2.15c}$$
$$\mathscr{A}^*_{ST}\cdot B^*_{ST}(k\epsilon_0) = -\left(1 + \frac{1}{4(1-\nu)}(k\epsilon_0)^2 + \frac{1}{32(1-\nu)(2-\nu)}(k\epsilon_0)^4 + \cdots\right) \tag{2.15d}$$
$$\mathscr{A}^*_{ST}\cdot C^*_{ST}(k\epsilon_0) = 1 + \frac{1}{2-2\nu}(k\epsilon_0)^2 + \frac{(3-2\nu)}{16(2-\nu)(1-\nu)^2}(k\epsilon_0)^4 + \cdots \tag{2.15e}$$

where it can be checked that $(\mathscr{A}^*_{ST}\cdot B^*_{ST}(k\epsilon_0))^2 = \mathscr{A}^*_{ST}\cdot C^*_{ST}(k\epsilon_0)$. So the wavefunctional at the finite cut-off is,

$$\Psi^0_1[\phi_0;\epsilon_0] = \exp\left[-\frac{1}{2}\int\limits_{z=\epsilon_0} d^dk\sqrt{\gamma_0}\frac{\Big(\phi + \mathscr{A}^*_{ST}\cdot B^*_{ST}(k\epsilon_0)\ \epsilon_0^{d-\Delta_+}J\Big)_k\Big(\phi + \mathscr{A}^*_{ST}\cdot B^*_{ST}(k\epsilon_0)\ \epsilon_0^{d-\Delta_+}J\Big)_{-k}}{\mathscr{A}^*_{ST}(k\epsilon_0)}\right.$$
$$\left. - \frac{1}{2}\int_{z=\epsilon_0}\sqrt{\gamma_0}\phi_k\hat{\mathcal{D}}_{ct}(k\epsilon_0)\phi_{-k}\right] \tag{2.16}$$

[9] '$\sim$' signifies that the subleading terms have not been included. Also, we are not careful about the normalizations, since they are inconsequential in a quadratic theory.

and $\mathscr{A}^*_{ST}$ is just the shorthand for the series,

$$\frac{1}{\mathscr{A}^*_{ST}} = 2\nu\left(1 - \frac{1}{2\left(1-\nu^2\right)}(k\epsilon_0)^2 - \frac{\left(5+\nu^2\right)}{8\left(4-\nu^2\right)\left(1-\nu^2\right)^2}(k\epsilon_0)^4 + \cdots\right)$$

Two things can be noted from the above results:

1. δ-function of the GKPW prescription needs to be regularized at a finite radial cut-off.
2. The source, J, for the dual field theory operator, $\mathcal{O}$, gets renormalized. The wavefunction renormalization at the finite cut-off is given by $Z_J^{-1} = \left(-\mathscr{A}^*_{ST}(k\epsilon_0)\cdot B^*_{ST}(k\epsilon_0)\right)^{-1} = Z_{\mathcal{O}}$.[10]

With such a choice of wavefunction, the functional integral, (2.6), is actually *independent* of the cut-off parameter ϵ_0 (see (2.17) below)! This is true even when the holographic computation is done with a finite radial cut-off, $z = \epsilon_0$. Consequently, as we see below, the correlators computed from this prescription exhibit a pure power law behaviour, consistent with the conformal Ward identities.

$\mathcal{O}(k)\mathcal{O}(-k)$ correlator We compute the correlators with the new prescription for AdS/CFT at finite radial cut-off with the inclusion of the boundary wavefunctional (2.16) by integrating out the bulk fields ϕ. The exact partition function becomes,

$$Z_+[J_k] = \exp\left[-\frac{1}{2}\int d^dk\ J_k\left(k^{2\nu}\ \frac{2^{1-2\nu}\Gamma(1-\nu)}{\Gamma(\nu)}\right)J_{-k}\right] \tag{2.17}$$

This is the exact partition function to all orders with the correct solutions of $\mathscr{A}^*_{ST}, B^*, C^*$.[11] Thus the connected two point function for the boundary operator is,

$$\langle\mathcal{O}(k)\mathcal{O}(-k)\rangle_+ = k^{2\nu}\ \frac{2^{1-2\nu}\Gamma(1-\nu)}{\Gamma(\nu)} \tag{2.18}$$

This is the correct 2-point function as governed by conformal symmetry. If one follows the Wilsonian principles of integrating out the degrees of freedom such that all the physical observables remain invariant, then this is the wavefunctional that will be obtained from (2.7). This result is slightly surprising because it tells us that it is possible to define AdS/CFT correspondence with a finite bulk cut-off, such that we still describe the field theory in the continuum limit. Alternatively, from the conventional renormalization point of view, in the field theory this is analogous to finding out all the correct counter-terms and/or

[10] Renormalization factors, $Z_{\mathcal{O}}, Z_J$ are defined by $\mathcal{O}^{(\epsilon)} = Z_{\mathcal{O}}\cdot\mathcal{O}^{(0)}$ and $J^{(\epsilon)} = Z_J\cdot J^{(0)}$. Alternatively, one can identify the source through (2.16), without any mention of wavefunction renormalization.

[11] We have checked it to the sixth order in $k\epsilon$ expansion, but with the inclusion of the exact solutions for $\mathscr{A}^*_{ST}, B^*, C^*$ this will hold true to all orders.

vacuum energy terms that make the partition function at a finite cut-off exactly conformally invariant. This view point is discussed in detail in subsection 2.2.3.

Correlator for a regulated field theory Since we want to find a bulk dual to field theory that is regulated at short distances (section 2.6), we want to introduce an explicit cut-off dependence in our correlator/partition function which replicates the regulation-dependence in the field theory (see subsection 2.6.4). A position space regulated correlator, (2.73), in momentum space is given by (2.84). To include a similar regulation in the bulk calculation, we need to include an extra contact term piece in our bulk action,

$$S_{extra} = \frac{1}{2}\int d^d k \; \epsilon_0^{-d+2(d-\Delta_+)} \delta C(k\epsilon_0) J_k J_{-k} \tag{2.19}$$

which modifies the correlator (2.18) to,

$$\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle_+ = k^{2\nu}\; \frac{2^{1-2\nu}\Gamma(1-\nu)}{\Gamma(\nu)} + \epsilon_0^{-2\nu}\; \delta C(k\epsilon_0) \tag{2.20}$$

One could argue that any perturbation away from the fixed point could ideally be achieved by changing any of A, B or C away from the fixed point values, A^*, B^*, C^*. But as can be seen in section 2.4, each of these coefficients have a different field theory interpretation of double-trace perturbation, wavefunction renormalization and contact terms in the correlators/partition function, respectively. So the change of each one of them contributes in a different manner to the observables like correlators of the theory.

$A(k\epsilon)$	$B(k\epsilon)$	$C(k\epsilon)$
Double-trace deformation	Wavefunction renormalization	Contact terms

Table 2.1: Interpretation of different coefficients in wavefunctional (2.10) away from the fixed point values, A^*, B^*, C^*. This interpretation is slightly heuristic and the exact relations are given in section 2.4.

In what comes next, we study the RG flows of field theories regulated in this fashion through direct as well as holographic computations. But before that we also establish the AdS/CFT duality at a finite cut-off in alternative quantization.

2.2.2 Alternative Quantization

In Klebanov-Witten window $\nu = \Delta_+ - d/2 \in (0,1)$ [19] the bulk gravitational theory is dual to two different quantum field theories in the boundary which are related to each other through Legendre transform. Thus, the generating function of one quantum field theory is the $1PI$ effective action of the other and vice versa, with the distinction that $1PI$ effective action is itself a local action for such theories.

Alternative fixed point can be understood as a UV completion of the standard IR theory within the Klebanov-Witten window by analysing the flow equations (2.11).[12] However, we treat this as a stand-alone prescription to begin with, and will connect them using the flow in double trace couplings in Appendix C.1. The usual AdS/CFT prescription for the alternative quantization is given by,

$$\begin{aligned} Z_-[J_k] &= \langle \exp \int d^d k J_k \mathcal{O}_{-k} \rangle_- \\ &= \int\limits_{z \geq \epsilon_0} \mathcal{D}\phi \exp \left[-S_b - \lim_{\epsilon_0 \to 0} \left(\int\limits_{z=\epsilon_0} d^d k \sqrt{\gamma_0} \epsilon_0^{d-\Delta_-} \phi_k J_{-k} + \frac{1}{2} \int\limits_{z=\epsilon_0} d^d x \sqrt{\gamma_0} \phi_k \hat{\mathcal{D}}_{ct}(\epsilon_0 k) \phi_{-k} \right) \right] \end{aligned} \tag{2.21}$$

The boundary part of the action, which is also the wavefunctional $\Psi[\phi_0]$, in the above equation is such that the variation principle imposes a modified Neumann condition on the boundary $z = \epsilon_0 \to 0$. This relates the normalizable part of the classical solution for π (conjugate momentum to the bulk field ϕ) to the source, J for the dual field theory operator $\mathcal{O}$, which now has the conformal dimension $\Delta_- = d/2 - \nu$, [19, 45]. In this case, the wavefunctional can be generalized to a finite cut-off without any ambiguity. Evolution equations for alternative quantization in terms of A, B, C are (B, C equations are modified due to difference in normalization of the sources with respect to the bulk field ϕ),

$$\dot{A} = -(A - \Delta_+)(A - \Delta_-) + (k\epsilon_0)^2, \quad \dot{B} = \Delta_- \, B - A \, B, \quad \dot{C} = (2\Delta_- - d) \, C - B^2 \tag{2.22}$$

It can be checked immediately that $\hat{\mathcal{D}}_{ct}$ given by [13] is identically a stationary point for A. At the leading order in $k\epsilon_0$, $A = \Delta_-$ and $B = 1$, and B equation is identically satisfied. However, in the limiting prescription of (2.21), we don't have any C, which clearly is not a stationary point. The concluding discussion of the previous section emphasized the interpretation of C-terms, that are quadratic in the sources J_k, as a choice of regulation scheme at a finite cut-off; this term adds contact terms to the bulk action and the $\mathcal{O}$ correlators. We modify the wavefunctional in (2.21) to include such terms and demand that this be at a fixed point as we did for standard quantization. On the upside, inclusion of such a term makes the alternative theory the exact Legendre transform of the standard theory along with all the counter-terms in both the theories. Solving for the stationary point of C to the

[12] It is the solution corresponding to $\chi \to \infty$ in (2.12), with the corresponding solutions for $B(k, \epsilon_0)$ and $C(k, \epsilon_0)$.

leading order, the wavefunctional becomes,

$$\Psi_2^0[\phi_0;\epsilon_0] = \exp\Bigg[-\frac{1}{2}\int\limits_{z=\epsilon_0} d^dk\sqrt{\gamma_0}\,\Big(\phi_k\hat{\mathcal{D}}_{ct}(k\epsilon_0)\phi_{-k} + 2\epsilon_0^{d-\Delta_-}B^*_{AQ}(k\epsilon_0)\phi_k J_{-k} + \epsilon_0^{2(d-\Delta_-)}C^*_{AQ}(k\epsilon_0)J_kJ_{-k}\Big)\Bigg] \tag{2.23}$$

where,

$$B^*_{AQ}(k\epsilon_0) = 1 - \frac{1}{4(1-\nu)}(k\epsilon_0)^2 + \frac{(3-\nu)}{32(2-\nu)(1-\nu)^2}(k\epsilon_0)^4 + \cdots \tag{2.24a}$$

$$C^*_{AQ}(k\epsilon_0) = -\frac{1}{2\nu} + \frac{1}{4(1-\nu^2)}(k\epsilon_0)^2 - \frac{(5-2\nu)}{32(1-\nu)^2\,(4-\nu^2)}(k\epsilon_0)^4 + \cdots \tag{2.24b}$$

It is interesting to note that, $B^*_{AQ}(k\epsilon_0) = -1/(\mathscr{A}^*_{ST}\cdot B^*_{ST}(k\epsilon_0))$ and $C^*_{AQ}(k\epsilon_0) = -1/(\mathscr{A}^*_{ST}\cdot {B^*_{ST}}^2)$. This shows that the alternative theory given by the wavefunctional (2.23) is exactly the Legendre transform of the standard theory defined by the wavefunctional (2.16) at cut-off $z=\epsilon_0$.

$\mathcal{O}(k)\mathcal{O}(-k)$ correlator The partition function and the correlator computation follows similar to that in standard quantization and can be computed exactly by using the wavefunctional (2.23), and integrating out the ϕ fields in the bulk,

$$Z_+[J_k] = \exp\left[-\frac{1}{2}\int d^dk\; J_k\left(-k^{-2\nu}\,\frac{2^{2\nu-1}\Gamma(\nu)}{\Gamma(1-\nu)}\right)J_{-k}\right] \tag{2.25}$$

Again, this is the exact correlator to all orders with the correct solutions of A^*, B^*, C^*. Thus the connected two point function for the boundary operator is,

$$\langle\mathcal{O}(k)\mathcal{O}(-k)\rangle_- = -k^{-2\nu}\,\frac{2^{2\nu-1}\Gamma(\nu)}{\Gamma(1-\nu)} \tag{2.26}$$

This is the correct 2-point function as governed by conformal symmetry for a continuum theory around the UV-fixed point.

Correlator for a regulated field theory Following the discussion in previous subsection, we can study a regulated field theory by including an extra piece in the wavefunctional, (2.23),

$$S_{extra} = \frac{1}{2}\int d^dk\; \epsilon_0^{-d+2(d-\Delta_-)}\delta C(k\epsilon_0)J_kJ_{-k} \tag{2.27}$$

which once again modifies the correlator above to,

$$\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle_- = -k^{-2\nu}\,\frac{2^{2\nu-1}\Gamma(\nu)}{\Gamma(1-\nu)} + \epsilon_0^{2\nu}\delta C(k\epsilon_0) \tag{2.28}$$

2.2.3 Choice of regulation scheme and comparison with field theory

In a dual field theory calculation, Wilsonian principles demand that under integration of degrees of freedom in a field theory, all physical observables remain unchanged. This gives us an effective description of the same theory with reduced degrees of freedom. In particular, if we start with a continuum quantum field theory and integrate out the UV degrees of freedom (either in position or momentum space), then the correlation functions computed using the new effective Lagrangian are the same as that of the continuum theory. In a continuum conformal field theory in which the correlation functions of the primary operators obey the scaling laws, an effective description with integration of certain degrees of freedom will reproduce the same power law correlators. However, a particular choice of regulation scheme in the field theory changes the short-distance/UV behaviour of the correlators (e.g. (2.73)) by an addition of certain counter-terms in the momentum space (Equation 2.83). For example, for the Θ-function regulated theory this choice corresponds to, (see (2.84)),

$$c_0 = \pm\frac{2\pi^{\frac{d-1}{2}}}{\nu\,\Gamma\left(\frac{d-1}{2}\right)},\quad c_1 = \frac{\pi^{\frac{d-1}{2}}}{3(\nu+1)\Gamma\left(\frac{d-1}{2}\right)},\quad c_2 = -\frac{\pi^{\frac{d-1}{2}}}{60(\nu+2)\Gamma\left(\frac{d-1}{2}\right)},\quad \cdots \tag{2.29}$$

where, $\delta C = c_0 + c_1(k\epsilon)^2 + c_2(k\epsilon)^4 + \cdots$. These coefficients depend only on the choice of regulation scheme and not on the cut-off ϵ at which the theory is regulated. Within such a scheme, with the regulated correlator, one needs to modify the effective Lagrangian appropriately to obtain the continuum power-law-obeying correlators. In conventional renormalization this is done by adding appropriate counter-terms in the Lagrangian. It is shown in [46] that for a generic large N theory the conformal invariance is broken by the running of double-trace couplings (which, as emphasized there, is a leading large N behaviour), unless the theory is at a conformal fixed point of all the double-trace couplings. Since we identify the alternative/standard quantizations with the UV/IR fixed points in the double-trace sectors, we are assured that no new counter-terms are generated for double-trace deformations. So, the corrections required in the regulated effective theory with certain UV cut-off can't be obtained by some double-trace counter terms. This argument is further strengthened by an explicit calculation with the inclusion of double-trace counter terms. As shown in various places in this chapter, inclusion of any double-trace interaction in the Lagrangian (away from the fixed point values) necessarily modifies the correlators by addition of terms proportional to $k^{4\nu}, k^{6\nu}, \ldots$ – which is not the same as the momentum space counter-terms that are present in the regulated theory. However, the inclusion of terms quadratic in the

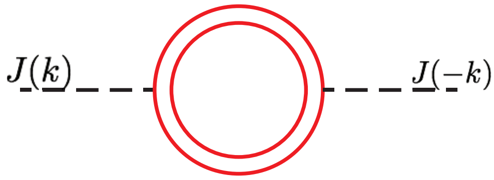


FIGURE 2.2: Diagrams contributing to generation of terms quadratic in source, $J(k)$. As a standard convention throughout the chapter, colored propagators denote 'heavy' modes (see section 2.6 for conventions used in Feynman diagrams).

source, $J(k)$, of the operator, $\mathcal{O}(k)$ in the Lagrangian provides the required correction that makes the correlators same as that of the continuum theory. Normally, in the partition function (which is computed with $J(k) = 0$, as opposed to the generating function), one would think that such terms are inconsequential. However, such terms necessarily correct the generating function, $W[J] = \log Z[J]$, of the theory and hence all the correlators of the theory. Particularly, in the quadratic effective action that we have in the large N theory, we obtain the power-law 2-point functions with the inclusion of appropriate terms. Within Wilson-Polchinski fRG treatment, such terms are necessarily generated as we integrate out the degrees of freedom (Figure 2.2).

The bulk computation at finite radial cut-off, (2.17), automatically corresponds to the regulated field theory with the inclusion of such terms. However, we emphasise the need to differentiate the contribution of the regulation scheme from that of the quadratic J term. In a regulated field theory with a double-trace deformation the regulation of the correlators (contact terms coming due to the regulation scheme) participates dynamically in the computation of the Feynman diagrams that gives rise to the rational fraction form of the correlator, (2.85), in the perturbed theory. The quadratic J term corrects this correlator by an additive term (which cancels the regulation-scheme contact terms in absence of the perturbation). Analogously, in the bulk computation, we treat the two contributions separately. This is done by a deviation of the boundary wavefunctional, Ψ, from C^* by some δC corresponding to the particular choice of scheme in the field theory. Then we use this wavefunctional in our Hubbard-Stratonovic transformation to describe the regulated, double-trace deformed field theory (as in (2.41) and (2.49)). It is hence important to compute the β-functions for the double-trace couplings using this prescription.

2.3 Geometric interpretation: smeared Witten diagram

The above improvement of the AdS/CFT prescription at finite radial cut-off has a natural generalization in the limit of massive, $mR_{AdS} \gg 1$, bulk fields. It is known that in this limit, the field theory correlators are approximated by geodesics between the points of operator insertions in the boundary, [47, 48] . Geodesic length between the points (ϵ, x_1) and (ϵ, x_2) in AdS is given by

$$L_\epsilon(x_1-x_2) = \cosh^{-1}\left(1+\frac{1}{2}(|x_1-x_2|/\epsilon)^2\right) = 2\log[|x_1-x_2|/\epsilon]+2(\epsilon/|x_1-x_2|)^2+O(\epsilon/|x_1-x_2|)^4$$

This is related to the correlator $\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_\epsilon$ for large $\Delta \approx m$ (with $R_{AdS} = 1$) as (Δ is the operator dimension of $\mathcal{O}$)

$$G_\epsilon(x_1-x_2) = \text{constant } \exp[-\Delta L_\epsilon(x_1-x_2)] = (1/|x_1-x_2|)^{2\Delta} e^{\left(1+2\Delta(\epsilon/|x_1-x_2|)^2+O(\epsilon/|x_1-x_2|)^4\right)} \tag{2.30}$$

where the 'constant'$= \epsilon^{-2\Delta}$ (in accordance with the dimension $[O(x)] = \Delta$, and Zamolodchikov's convention $G(0,1) = 1$). The corrections that appear in the exponential of the correlator above can be thought of as a regulation scheme for the correlator. It can be easily checked that this scheme obeys all the general discussion of subsection 2.6.4 and has the momentum space counter-terms as discussed there.

Like the conventional GKPW prescription, this should also be understood as a limiting prescription which is well defined only in $\epsilon \to 0$ limit. Our finite radial cut-off modification to the GKPW prescription suggests that we need to modify the geodesic prescription too. The source corresponding to the insertion of boundary operator $\mathcal{O}$ at x_1, x_2 is $J(\vec{x}) = \delta(\vec{x}-\vec{x}_1)+\delta(\vec{x}-\vec{x}_2)$. Using the boundary condition, (2.37) (with $\mathfrak{f} = 0$), we find that the bulk field, ϕ, at finite radial cut-off in the momentum space is,

$$\phi(k,\epsilon_0) = \frac{2^{1-\nu}\epsilon_0^{d/2}}{\Gamma(\nu)}\left(e^{i\vec{k}\cdot\vec{x}_1}+e^{i\vec{k}\cdot\vec{x}_2}\right)k^\nu K_\nu(k\epsilon_0) \tag{2.31}$$

where we have used $J(k) = \left(e^{i\vec{k}\cdot\vec{x}_1}+e^{i\vec{k}\cdot\vec{x}_2}\right)$. Similar to the law of superposition, we simply add the field due to the presence of one source at $\vec{x} = \vec{x}_1$ to that due to source at $\vec{x} = \vec{x}_2$. In position space, the field due to an individual source is given by,

$$\begin{aligned}\phi(k,\epsilon_0) = \frac{2^{1-\nu}\epsilon_0^{d/2}}{\Gamma(\nu)}e^{i\vec{k}\cdot\vec{x}_1}k^\nu K_\nu(k\epsilon_0) \xrightarrow[\text{transform}]{\text{Fourier}} \\ \frac{2^{d-1}\pi^{\frac{d-2}{2}}\epsilon_0^{-\frac{d}{2}-\nu}}{(d+2\nu-1)}\left(\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d}{2}+\nu\right)}{\Gamma\left(\frac{d+1}{2}\right)\Gamma(\nu)}\right)\left(\left(1+\frac{\epsilon_0^2}{\rho^2}\right){}_2F_1\left(\frac{d}{2},\frac{d}{2}+\nu;-\frac{1}{2};-\frac{\rho^2}{\epsilon_0^2}\right)\right. \\ \left. -\left(2(d+\nu)+\frac{\epsilon_0^2}{\rho^2}\right){}_2F_1\left(\frac{d}{2},\frac{d}{2}+\nu;\frac{1}{2};-\frac{\rho^2}{\epsilon_0^2}\right)\right)\end{aligned} \tag{2.32}$$

This function is peaked around $\rho = 0$, where $\vec{\rho} = \vec{x}-\vec{x}_1$, with a half-width of the order of $\mathcal{O}(\epsilon_0)$.

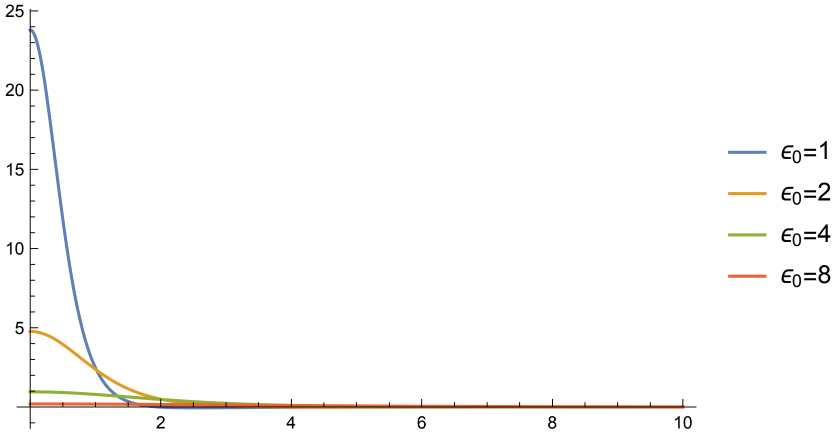


FIGURE 2.3: Plots for the boundary fields at finite radial cut-off with double-centered delta function source.

This solution for ϕ_0 corresponds to a distribution for ϕ_0 smeared around $J(x) = \delta(\vec{x}-\vec{x}_1)+\delta(\vec{x}-\vec{x}_2)$. This is schematically represented by the right panel of the diagram, Figure 2.4. Note that since the correlator at any cut-off surface is a pure power law by this device, *the motion of the cut-off surface into the AdS bulk does not change the correlator.*

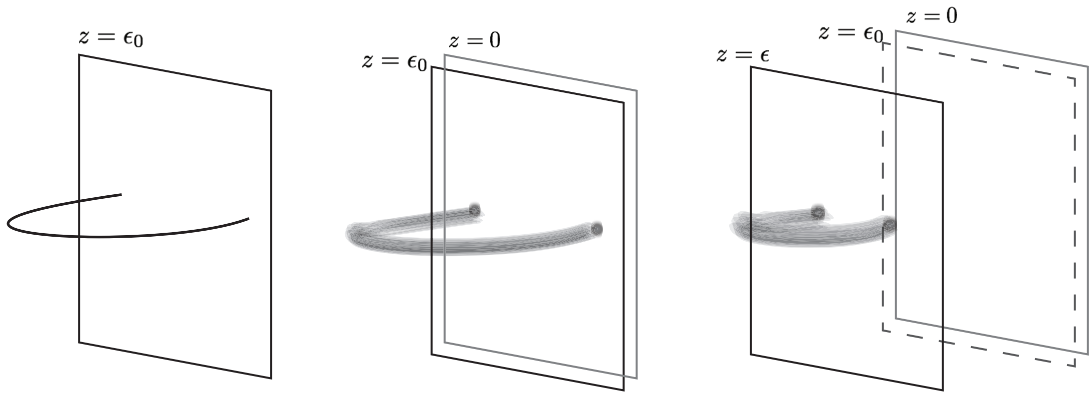


FIGURE 2.4: (Left) Witten diagram for a delta-function boundary term corresponds to a scaling violation, as in (2.30). (Center) Witten diagram with our smearing over the delta-function boundary condition gives the pure power law. (Right) Smearing increases as one moves deeper in the radial direction. However, the exact correlator in both the centre and the right diagram are equal.

2.4 Double trace perturbations

Having defined our fixed point theories with a finite cut-off and before we move on to computation of β-function in dual bulk theory, we review ([18]) and extend the AdS/CFT dictionary for the derivative double-trace operators. The same bulk field which is dual to a scalar primary operator $\mathcal{O}$ of scaling dimension Δ also describes the physics of derivative multi-trace operators with an appropriately modified boundary condition that is discussed in this section.

The action with a double-trace perturbation and inclusion of a source term is given by (2.72)

$$S = S_0 + \frac{1}{2}\int d^d k\ \mathcal{O}\mathfrak{f}(\partial^2)\mathcal{O}(x) - \int d^d x\ J(x)\mathcal{O}(x)$$

Since the bulk computations will give us different β-functions, to differentiate between the two sets of couplings we have denoted the couplings used in the bulk calculations by $\mathfrak{f}$ instead of f for the dimensionful couplings, and $\bar{\mathfrak{f}}$ instead $\bar{f}$ for the dimensionless couplings. Same notation is used for $\mathfrak{f}(\partial^2)$ as in (2.71). In the subsequent discussions, we work in momentum space,

$$\mathfrak{f}(k^2) \ = \ \mathfrak{f}_0 + \mathfrak{f}_2 k^2 + \mathfrak{f}_4 k^4 + \dots$$

We use Hubbard-Stratonovich trick to write the perturbation terms above as,

$$\begin{aligned} &\exp\left[\int J(k)\mathcal{O}(-k) - \int \frac{\mathfrak{f}(k^2)}{2}\mathcal{O}(k)\mathcal{O}(-k)\right] \\ &\qquad = \int \mathcal{D}\tilde{\phi}\ \exp\left[\int \frac{\left(\tilde{\phi}-J\right)_k \left(\tilde{\phi}-J\right)_{-k}}{2\mathfrak{f}(k^2)} + \int \tilde{\phi}(k)\mathcal{O}(-k)\right] \end{aligned} \tag{2.33}$$

Standard Quantization Using (2.33), and the statement of duality for standard quantization at finite radial cut-off given by the wavefunctional (2.16), a bulk partition function dual to the double-trace perturbed field theory can be obtained,

$$\begin{aligned} Z_+[J, \mathfrak{f}(k^2)] \ = \ &\int \mathcal{D}\phi\ \exp\left[-S_b - \int\limits_{z=\epsilon_0} d^d k\ \sqrt{\gamma_0}\ \frac{\left(\phi + \mathscr{A}^*_{ST}\cdot B^*_{ST}(k\epsilon_0)\ \epsilon_0^{d-\Delta_+} J\right)^2_k}{2\mathscr{A}^*_{ST}(k\epsilon_0)\left(1 - {B^*_{ST}}^2\ \mathscr{A}^*_{ST}\frac{\mathfrak{f}(k^2)}{\epsilon_0^{2\nu}}\right)}\right. \\ &\left. - \int d^d k \frac{\sqrt{\gamma_0}}{2}\phi_k \hat{\mathcal{D}}_{ct}(\epsilon_0 k)\phi_{-k}\right] \end{aligned} \tag{2.34}$$

Variational principle imposes following condition at the boundary $z = \epsilon_0$,

$$\pi(k,\epsilon_0) - \sqrt{\gamma_0}\ \frac{\left(\phi + \mathscr{A}^*_{ST}\cdot B^*_{ST}(k\epsilon_0)\ \epsilon_0^{d-\Delta_+} J\right)_k}{\mathscr{A}^*_{ST}(k\epsilon_0)\left(1 - {B^*_{ST}}^2\ \mathscr{A}^*_{ST}\frac{\mathfrak{f}(k^2)}{\epsilon_0^{2\nu}}\right)} - \sqrt{\gamma_0}\ \hat{\mathcal{D}}_{ct}(\epsilon_0 k)\phi(k,\epsilon_0) = 0 \tag{2.35}$$

where, $\pi(k,z) = \sqrt{g}\ \partial^z\phi(k,z)$ is the conjugate momentum of the bulk field.
Using the near boundary expansion of the bulk field $\phi(k,z)$,

$$\phi(k,z) = z^{d-\Delta_+}\ a(k)\left(1 - \frac{(kz)^2}{2^2(\nu-1)} + \cdots\right) + z^{\Delta_+}\ b(k)\left(1 + \frac{(kz)^2}{2^2(\nu+1)} + \cdots\right) \tag{2.36}$$

the boundary condition becomes,

$$J(k) = 2\nu \mathfrak{f}(k^2) b(k) + \ a(k) \text{ [13]} \tag{2.37}$$

In the above expression, in the $\epsilon_0 \to 0$ limit, $b(k)$ is the expectation value of the operator $\mathcal{O}$, and $a(k)$ is the source. The above expression can be rewritten as,

$$\begin{aligned} a(k) &= J(k) - 2\nu \mathfrak{f}(k^2)\ b(k) \ = \ J(k) - 2\nu \mathfrak{f}(k^2) \langle \mathcal{O}(k) \rangle \\ \equiv a(x) &= J(x) - 2\nu \Big(\mathfrak{f}_0 \langle \mathcal{O}(x) \rangle + \mathfrak{f}_2 \langle \partial^2 \mathcal{O}(x) \rangle + \mathfrak{f}_4 \langle \partial^4 \mathcal{O}(x) \rangle + \dots \Big) \end{aligned} \tag{2.38}$$

IR boundary condition in the bulk at $z = \infty$ imposes an additional condition on the on-shell field $\phi(k, z)$. In the pure AdS geometry, demanding the regularity of the field at IR determines $b(k)$ in terms of $a(k)$,

$$b(k) = 2^{-2\nu} k^{2\nu} \frac{\Gamma(-\nu)}{\Gamma(\nu)} \cdot a(k)$$

So the improved relationship between the boundary value of the bulk field, $\phi(k, \epsilon_0)$, and the field theory source for the dual operator $\mathcal{O}$, in the absence of the double-trace deformation, $\mathfrak{f}(k^2)$, is

$$\phi(k, \epsilon_0) = \epsilon_0^{d-\Delta_+} J(k) \left[\left(1 - \frac{(k\epsilon_0)^2}{2^2(\nu - 1)} + \cdots \right) + \left(\frac{k\epsilon_0}{2} \right)^{2\nu} \frac{\Gamma(-\nu)}{\Gamma(\nu)} \left(1 + \frac{(k\epsilon_0)^2}{2^2(\nu + 1)} + \cdots \right) \right] \tag{2.39}$$

In the limit, $\epsilon_0 \to 0$, this gives back the well known GKPW prescription between the field and the source, $\lim_{\epsilon_0 \to 0} \epsilon_0^{\Delta_+ - d} \phi(k, \epsilon_0) = J(k)$. This is a reaffirmation of the limiting δ-function prescription, (2.7), originally known in the correspondence.
In the presence of the double-trace deformation this relation gets modified to,

$$\phi(k, \epsilon_0) = \epsilon_0^{d-\Delta_+} J(k) \frac{\left[\left(1 - \frac{(k\epsilon_0)^2}{2^2(\nu-1)} + \cdots \right) + \left(\frac{k\epsilon_0}{2} \right)^{2\nu} \frac{\Gamma(-\nu)}{\Gamma(\nu)} \left(1 + \frac{(k\epsilon_0)^2}{2^2(\nu+1)} + \cdots \right) \right]}{1 + 2^{1-2\nu}\ \bar{\mathfrak{f}}(k^2\epsilon_0^2)\ \ (k\epsilon_0)^{2\nu}\ \ \frac{\nu\Gamma(-\nu)}{\Gamma(\nu)}} \tag{2.40}$$

With the regulator counter-terms Since we are particularly interested in field theories that are regulated at short distances in position space (or equivalently, have certain counter-terms in the momentum space) it is also important that we establish our duality for the

[13]This equation is correct to all orders with the inclusion of all the correct counterterms that we have derived at finite cut-off, viz., the values of $B^*_{ST}, \mathscr{A}^*_{ST}, \hat{\mathcal{D}}_{ct}$.

double-trace perturbations with the inclusion of such regulators, (2.19).

$$Z_+[J,\mathfrak{f}(k^2)] \;=\; \int \mathcal{D}\phi\ \exp\left[-S_b - \frac{1}{2}\int_{z=\epsilon_0} d^dk\ \sqrt{\gamma_0}\ \left(\frac{1-\delta C\cdot\frac{\mathfrak{f}(k^2)}{\epsilon_0^{2\nu}}}{\mathscr{A}^*_{ST}\left(1-\left(\delta C + {B^*_{ST}}^2\ \mathscr{A}^*_{ST}\right)\frac{\mathfrak{f}(k^2)}{\epsilon_0^{2\nu}}\right)}\phi_k\phi_{-k}\right.\right.$$

$$\left.+2\frac{B^*_{ST}\ \epsilon_0^{d-\Delta_+}}{\left(1-\left(\delta C + {B^*_{ST}}^2\ \mathscr{A}^*_{ST}\right)\frac{\mathfrak{f}(k^2)}{\epsilon_0^{2\nu}}\right)}J_k\phi_{-k} + \frac{\left(\delta C + \mathscr{A}^*_{ST}\cdot {B^*_{ST}}^2\right)\epsilon_0^{2(d-\Delta_+)}}{\left(1-\left(\delta C + {B^*_{ST}}^2\ \mathscr{A}^*_{ST}\right)\frac{\mathfrak{f}(k^2)}{\epsilon_0^{2\nu}}\right)}\ J_kJ_{-k}\right)$$

$$\left. - \int_{z=\epsilon_0} d^dk\frac{\sqrt{\gamma_0}}{2}\phi_k\hat{\mathcal{D}}_{ct}(\epsilon_0 k)\phi_{-k}\right] \tag{2.41}$$

Variational principle imposes following condition at the boundary $z=\epsilon_0$,

$$\pi(k,\epsilon_0) - \frac{\sqrt{\gamma_0}}{\mathscr{A}^*_{ST}}\left(\frac{1-\delta C\cdot\frac{\mathfrak{f}(k^2)}{\epsilon_0^{2\nu}}}{1-\left(\delta C + {B^*_{ST}}^2\ \mathscr{A}^*_{ST}\right)\frac{\mathfrak{f}(k^2)}{\epsilon_0^{2\nu}}}\right)\phi_k - \sqrt{\gamma_0}\left(\frac{\epsilon_0^{d-\Delta_+}B^*_{ST}}{1-\left(\delta C + {B^*_{ST}}^2\ \mathscr{A}^*_{ST}\right)\frac{\mathfrak{f}(k^2)}{\epsilon_0^{2\nu}}}\right)J_k$$

$$-\sqrt{\gamma_0}\ \hat{\mathcal{D}}_{ct}(\epsilon_0 k)\phi(k,\epsilon_0) = 0 \tag{2.42}$$

the boundary condition becomes,

$$J(k) = 2\nu\mathfrak{f}(k^2)b(k) + \left(1-\frac{\mathfrak{f}(k^2)}{\epsilon_0^{2\nu}}\delta C(k\epsilon_0)\right)\ a(k) \tag{2.43}$$

In the double-trace perturbed theory the exact two point function, $\langle\mathcal{O}(k)\mathcal{O}(-k)\rangle_{\mathfrak{f}}$ is given by the summing over all the connected diagrams. Since the bulk partition function of the perturbed theory, (2.34) or (2.41), is quadratic in bulk fields ϕ_k, we can perform the gaussian integral exactly and compute the 2-point function from the resulting generating function,

$$\langle\mathcal{O}(k)\mathcal{O}(-k)\rangle^{(+)}_{\mathfrak{f}} = \frac{G^{\epsilon_0}_{(+)}(k)}{1+\mathfrak{f}(k^2)G^{\epsilon_0}_{(+)}(k)} \tag{2.44}$$

for any value of the coupling $\mathfrak{f}(k^2)$. Here $G^{(\epsilon_0)}_{+}$ is given by either (2.18) or (2.20).[14]

Alternative Quantization From the duality for alternative quantization without double-trace perturbation (2.23) and (2.33), the bulk dual to double-trace deformed alternative

[14]Note that we have dropped the contribution coming from the quadratic J explained in subsection 2.2.3 as we won't need them for the β-function calculation, but we should remember their presence.

quantized theory is,

$$\begin{aligned}
Z_-[J,\mathfrak{f}(k^2)] &= \int \mathcal{D}\Phi \exp\left(-S_0^{(-)} + \int d^dk\; J(k)\mathcal{O}(-k) - \int d^dk\; \frac{\mathfrak{f}(k^2)}{2}\mathcal{O}(k)\mathcal{O}(-k)\right) \\
&= \int\limits_{z\geq\epsilon_0} \mathcal{D}\phi \exp\Bigg[-S_b - \int\limits_{z=\epsilon_0} d^dk \frac{\sqrt{\gamma_0}}{2}\left(\frac{{B^*_{AQ}}^2\; \mathfrak{f}(k^2)\epsilon_0^{2\nu}}{1 - C^*_{AQ}\; \mathfrak{f}(k^2)\epsilon_0^{2\nu}} + \hat{\mathcal{D}}_{ct}(\epsilon_0 k)\right)\phi_k\phi_{-k} \\
&\qquad - \int\limits_{z=\epsilon_0} d^dk\sqrt{\gamma_0}\left(\frac{B^*_{AQ}}{1 - C^*_{AQ}\; \mathfrak{f}(k^2)\epsilon_0^{2\nu}}\right)\epsilon_0^{d-\Delta_-}\phi_k J_{-k} \\
&\qquad -\frac{1}{2}\int\limits_{z=\epsilon_0} d^dk\sqrt{\gamma_0}\left(\frac{C^*_{AQ}}{1 - C^*_{AQ}\; \mathfrak{f}(k^2)\epsilon_0^{2\nu}}\right)\epsilon_0^{2(d-\Delta_-)}J_kJ_{-k}\Bigg]
\end{aligned} \tag{2.45}$$

Variation of the fields on the boundary $z = \epsilon_0$ imposes the condition,

$$\pi(k,\epsilon_0) - \sqrt{\gamma_0}\,\phi(k,\epsilon_0)\left(\frac{{B^*_{AQ}}^2\; \mathfrak{f}(k^2)\epsilon_0^{2\nu}}{1 - C^*_{AQ}\; \mathfrak{f}(k^2)\epsilon_0^{2\nu}} + \hat{\mathcal{D}}_{ct}(\epsilon_0 k)\right) = \sqrt{\gamma_0}\left(\frac{B^*_{AQ}}{1 - C^*_{AQ}\; \mathfrak{f}(k^2)\epsilon_0^{2\nu}}\right)\epsilon_0^{d-\Delta_-}J(k)$$

Using the near boundary expansion of the bulk field $\phi(k, z)$ in the boundary condition we get, [15]

$$J(k) = 2\nu\; a(k) - \mathfrak{f}(k^2)b(k) \tag{2.46}$$

which can be rewritten as,

$$\begin{aligned}
a(k) &= \frac{1}{2\nu}\left(J(k) + \mathfrak{f}(k^2)b(k)\right) = \frac{1}{2\nu}\left(J(k) + \mathfrak{f}(k^2)\langle\mathcal{O}(k)\rangle\right) \\
\equiv\;\; a(x) &= \frac{1}{2\nu}\left(J(x) + \mathfrak{f}_0\langle\mathcal{O}(x)\rangle + \mathfrak{f}_2\langle\partial^2\mathcal{O}(x)\rangle + \mathfrak{f}_4\langle\partial^4\mathcal{O}(x)\rangle + \ldots\right)
\end{aligned} \tag{2.47}$$

As in the standard quantization, demanding regular IR boundary condition in the pure AdS bulk geometry, at $z = \infty$, determines $b(k)$ in terms of $a(k)$,

$$b(k) = 2^{2\nu}k^{-2\nu}\frac{\Gamma(\nu)}{\Gamma(-\nu)}\cdot a(k)$$

[15] $\phi(k,z) = z^{d-\Delta_-}\; a(k)\left(1 + \frac{(kz)^2}{2^2(\nu+1)} + \cdots\right) + z^{\Delta_-}\; b(k)\left(1 - \frac{(kz)^2}{2^2(\nu-1)} + \cdots\right)$ where $a(k)$ is the coefficient of normalizable part and hence the source for alternative quantization. Also the expression in (2.46) is exact to all orders.

So the improved relationship between the boundary value of the bulk field, $\phi(k, \epsilon_0)$, and the field theory source for the dual operator $\mathcal{O}$, now of dimension Δ_-, is

$$\phi(k,\epsilon_0) = \epsilon_0^{\Delta_-} J(k) \left(\frac{k}{2}\right)^{-2\nu} \frac{\left[\left(\frac{k\epsilon_0}{2}\right)^{2\nu}\left(1+\frac{(k\epsilon_0)^2}{2^2(\nu+1)}+\cdots\right)+\frac{\Gamma(\nu)}{\Gamma(-\nu)}\left(1-\frac{(k\epsilon_0)^2}{2^2(\nu-1)}+\cdots\right)\right]}{2\nu - 2^{2\nu}\,\bar{\mathfrak{f}}(k^2\epsilon_0^2)\,\,(k\epsilon_0)^{-2\nu}\,\,\frac{\Gamma(\nu)}{\Gamma(-\nu)}} \tag{2.48}$$

which, again limits to the known relationship between the source and the normalizable part of the bulk field, $J(k) = 2\nu a(k)$ in the $\epsilon_0 \to 0$ limit in the absence of the double-trace deformations.

With the regulator counter-terms If we however start with (2.27), then,

$$\begin{aligned} Z_-[J,\mathfrak{f}(k^2)] &= \int \mathcal{D}\Phi \exp\left(-S_0^{(-)} + \int d^dk\ J(k)\mathcal{O}(-k) - \int d^dk\ \frac{\mathfrak{f}(k^2)}{2}\mathcal{O}(k)\mathcal{O}(-k)\right) \\ &= \int_{z\geq\epsilon_0} \mathcal{D}\phi \exp\Bigg[-S_b - \frac{1}{2}\int_{z=\epsilon_0} d^dk\sqrt{\gamma_0}\left(\frac{{B^*_{AQ}}^2\ \epsilon_0^{2\nu}\ \mathfrak{f}(k^2)}{1-\epsilon_0^{2\nu}\ \mathfrak{f}(k^2)(C^*_{AQ}+\delta C)} + \hat{\mathcal{D}}_{ct}(\epsilon_0 k)\right)\phi_k\phi_{-k} \\ &\qquad - \int_{z=\epsilon_0} d^dk\sqrt{\gamma_0}\ \frac{B^*_{AQ}}{1-\epsilon_0^{2\nu}\ \mathfrak{f}(k^2)(C^*_{AQ}+\delta C)}\epsilon_0^{d-\Delta_-}\ \phi_k J_{-k} \\ &\qquad -\frac{1}{2}\int_{z=\epsilon_0} d^dk\sqrt{\gamma_0}\ \frac{C^*_{AQ}+\delta C}{1-\epsilon_0^{2\nu}\ \mathfrak{f}(k^2)(C^*_{AQ}+\delta C)}\ \epsilon_0^{2(d-\Delta_-)} J_k J_{-k}\Bigg] \end{aligned} \tag{2.49}$$

which leads to boundary condition,

$$\begin{aligned} \pi(k,\epsilon_0) - \sqrt{\gamma_0}\ \phi(k,\epsilon_0)\left(\frac{{B^*_{AQ}}^2\ \epsilon_0^{2\nu}\ \mathfrak{f}(k^2)}{1-\epsilon_0^{2\nu}\ \mathfrak{f}(k^2)(C^*_{AQ}+\delta C)} + \hat{\mathcal{D}}_{ct}(\epsilon_0 k)\right) = \\ \sqrt{\gamma_0}\ \frac{B^*_{AQ}}{1-\epsilon_0^{2\nu}\ \mathfrak{f}(k^2)(C^*_{AQ}+\delta C)}\ \epsilon_0^{d-\Delta_-} J(k) \end{aligned} \tag{2.50}$$

$$J(k) = 2\nu\left(1-\epsilon_0^{2\nu}\mathfrak{f}(k^2)\ \delta C(k\epsilon_0)\right)\ a(k) - \mathfrak{f}(k^2)b(k) \tag{2.51}$$

As in standard quantization, the 2-point function is evaluated exactly by integrating out (2.45) or (2.49),

$$\langle\mathcal{O}(k)\mathcal{O}(-k)\rangle_{\mathfrak{f}}^{(-)} = \frac{G^{\epsilon_0}_{(-)}(k)}{1+\mathfrak{f}(k^2)G^{\epsilon_0}_{(-)}(k)} \tag{2.52}$$

Equations (2.38), (2.43), (2.47) and (2.51) are our proposed generalisation of the boundary prescription originally given by [18] for the derivative multi-trace deformations around a conformal field theory in standard and alternative quantization, respectively. These have the same structure as we had found for the field theory correlators in subsection 2.6.4.
For even more general higher-derivative multi-trace operators, we expect that the above formulae generalises as long as we include all the derivative terms inside the expectation values. Corresponding computation for triple-trace operators without derivatives is done in [45], and we think the generalisation shouldn't be difficult.

2.5 Holographic computation of β-functions

Having established the duality for the double-trace operators in previous section, we know that the couplings of the field theory double-trace operators are contained in the coefficient of the $\phi_k\phi_{-k}$ in the boundary part of the bulk action (2.34),(2.45). AdS/CFT naturally incorporates a holographic version of RG flow, because of the correspondence between the radial coordinate in the bulk and the energy scale in the boundary field theory, see, e.g., [14, 49–52]. Holographic Wilsonian RG flow of double-trace operators without derivatives was considered in [15, 16], which was generalised in [17] to double trace operators with derivatives. In the following we essentially build up on the treatment in [17]. For other relevant work on renormalization of multi-trace operators from holographic and field theoretic viewpoints, see, e.g. [45, 46, 53–55]).

An essential feature of the AdS/CFT correspondence is the connection between the energy scale of the conformal field theory (CFT) and the radial coordinate of the AdS dual. More precisely, AdS/CFT states that the bulk partition function in Euclidean AdS, defined with a radial cut-off $r = r_0$, equals the dual field theory partition function with a UV momentum cut-off Λ given in terms of r_0 (for large Λ, $\Lambda = r_0/R^2_{AdS}$ [56]). A corollary of this statement, in the semi-classical limit, is that the running of field theory couplings is identified with the radial dependence of classical field configurations in the dual gravitational theory (see, e.g., [14, 49–51]). Motivated by this feature, in [15, 16], the near-boundary degrees of freedom in the bulk are identified with the *heavy*/short-distance modes of the dual field theory. They work in *probe approximation* with a fluctuating field $\phi(x, z)$ on a fixed AdS background given by,

$$ds^2 = \frac{1}{z^2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right) \tag{2.53}$$

Integration of the near boundary modes in the bulk gives a new holographic version of Wilsonian effective action in the field theory. Stated mathematically,

$$
\begin{aligned}
Z_{bulk,\epsilon_0} &= \int_{z\geq\epsilon_0} \mathcal{D}[\phi] e^{-\mathcal{S}[\phi]} \\
&= \int \mathcal{D}\phi|_{z>\epsilon}\ \mathcal{D}\tilde{\phi}\ \mathcal{D}\phi|_{\epsilon_0\leq z<\epsilon} \quad e^{-\mathcal{S}[\phi]|_{z>\epsilon}} e^{-\mathcal{S}[\phi]|_{z<\epsilon}} \\
&= \int \mathcal{D}\tilde{\phi}\ Z_{bulk,\epsilon}(\epsilon,\tilde{\phi}) Z_{UV}(\epsilon,\tilde{\phi}) \qquad (2.54)
\end{aligned}
$$

The role of Z_{UV} is an addition of a boundary wavefunctional, $\Psi[\phi_0;\epsilon_0]$ to the bulk action at the new cutoff $z=\epsilon$, $Z_{bulk,\epsilon}$. This, in the AdS/CFT dictionary has the interpretation of addition of higher-trace terms in the field theory, as discussed in section 2.4. Following Wilsonian principles, same as in the field theory computations, we demand,

$$
\begin{aligned}
&\frac{d}{d\epsilon} Z_{bulk,\epsilon_0} = 0 \\
\Rightarrow &\int \mathcal{D}\tilde{\phi} \left(\frac{\partial Z_{bulk,\epsilon}}{\partial\epsilon}\ Z_{UV} + Z_{bulk,\epsilon}\ \frac{\partial Z_{UV}}{\partial\epsilon} \right) = 0 \qquad (2.55)
\end{aligned}
$$

here, the evolution of Z_{UV} can be computed using the Hamiltonian corresponding to radial slicing,

$$
\frac{\partial Z_{UV}}{\partial\epsilon}(\tilde{\phi},\epsilon) = -H(\tilde{\phi},\tilde{\pi}) Z_{UV}(\epsilon,\tilde{\phi}) \qquad (2.56)
$$

which will be henceforth refered to as radial Schrödinger evolution equations. Here $\tilde{\pi} = -i\kappa^2\ \delta/\delta\tilde{\phi}$. In general, Z_{UV} contains the details of the various field theory couplings which enables us to compute the β-functions of these couplings using (2.56). These ideas have been worked out for the bulk duals of double-traced deformed field theories (2.34), (2.45) in Appendix C. Only the final β-functions are quoted here.

Standard Quantization: Working with the bulk action, (2.41), which is dual to the regulated field theory and keeping in mind the subtleties that we remarked upon in the subsection 2.2.3, we get the β-function equation,

$$\begin{aligned}
\epsilon\partial_\epsilon\bar{\mathfrak{f}} = \bar{\mathfrak{f}}^2 \times \Bigg(& {B^*_{ST}}^2\ {\mathscr{A}^*_{ST}}^2 \left(k^2\epsilon^2 + m^2 + \hat{\mathcal{D}}_{ct}(d-\hat{\mathcal{D}}_{ct}) - \epsilon\partial_\epsilon\hat{\mathcal{D}}_{ct}\right) - \epsilon\partial_\epsilon\delta C \\
& + \frac{(\delta C)^2}{{B^*_{ST}}^2{\mathscr{A}^*_{ST}}^2}\left(\epsilon\partial_\epsilon\bar{\mathfrak{f}}^* - 1 + \mathscr{A}^*_{ST}(d-2\hat{\mathcal{D}}_{ct}) + {\mathscr{A}^*_{ST}}^2\left(k^2\epsilon^2+m^2+\hat{\mathcal{D}}_{ct}(d-\hat{\mathcal{D}}_{ct}) - \epsilon\partial_\epsilon\hat{\mathcal{D}}_{ct}\right)\right) \\
& - \delta C\left(-\frac{2\epsilon\partial_\epsilon B^*_{ST}}{B^*_{ST}} - \frac{2\ \epsilon\partial_\epsilon\mathscr{A}^*_{ST}}{\mathscr{A}^*_{ST}} - 2\mathscr{A}^*_{ST}\left(k^2\epsilon^2+m^2+\hat{\mathcal{D}}_{ct}(d-\hat{\mathcal{D}}_{ct}) - \epsilon\partial_\epsilon\hat{\mathcal{D}}_{ct}\right) - d + 2\hat{\mathcal{D}}_{ct}\right)\Bigg) \\
& + \bar{\mathfrak{f}}\Bigg(- \frac{2\ \epsilon\partial_\epsilon B^*_{ST}}{B^*_{ST}} - \frac{2\ \epsilon\partial_\epsilon\mathscr{A}^*_{ST}}{\mathscr{A}^*_{ST}} - 2\mathscr{A}^*_{ST}\left(k^2\epsilon^2+m^2+\hat{\mathcal{D}}_{ct}(d-\hat{\mathcal{D}}_{ct}) - \epsilon\partial_\epsilon\hat{\mathcal{D}}_{ct}\right) - d + 2\hat{\mathcal{D}}_{ct}(k\epsilon) \\
& - 2\frac{\delta C}{{B^*_{ST}}^2{\mathscr{A}^*_{ST}}^2}\left(\epsilon\partial_\epsilon\bar{\mathfrak{f}}^* - 1 + \mathscr{A}^*_{ST}(d-2\hat{\mathcal{D}}_{ct}) + {\mathscr{A}^*_{ST}}^2\left(k^2\epsilon^2+m^2+\hat{\mathcal{D}}_{ct}(d-\hat{\mathcal{D}}_{ct}) - \epsilon\partial_\epsilon\hat{\mathcal{D}}_{ct}\right)\right)\Bigg) \\
& + \frac{\epsilon\partial_\epsilon\mathscr{A}^*_{ST} + \mathscr{A}^*_{ST}\ (d-2\hat{\mathcal{D}}_{ct}) + {\mathscr{A}^*_{ST}}^2\left(k^2\epsilon^2+m^2+\hat{\mathcal{D}}_{ct}(d-\hat{\mathcal{D}}_{ct}) - \epsilon\partial_\epsilon\hat{\mathcal{D}}_{ct}\right) - 1}{{B^*_{ST}}^2\ {\mathscr{A}^*_{ST}}^2}
\end{aligned} \tag{2.57}$$

Alternative Quantization: Bulk action, (2.49), corresponds to the regulated theory,

$$\begin{aligned}
\epsilon\partial_\epsilon\bar{\mathfrak{f}} = & \frac{1}{{B^*_{AQ}}^2}\Bigg[\bar{\mathfrak{f}}^2\Big(2B^*_{AQ}\ \epsilon\partial_\epsilon B^*_{AQ}\left(C^*_{AQ}+\delta C\right) - {B^*_{AQ}}^2\left(\epsilon\partial_\epsilon C^*_{AQ} + \epsilon\partial_\epsilon\delta C + \left(C^*_{AQ}+\delta C\right)(d-2\hat{\mathcal{D}}_{ct})\right) \\
& - {B^*_{AQ}}^4 + \left(C^*_{AQ}+\delta C\right)^2\left(\hat{\mathcal{D}}_{ct}(d-\hat{\mathcal{D}}_{ct}) - \epsilon\partial_\epsilon\hat{\mathcal{D}}_{ct} + k^2\epsilon^2 + m^2\right)\Big) \\
& + \bar{\mathfrak{f}}\left(-2B^*_{AQ}\ \epsilon\partial_\epsilon B^*_{AQ} + {B^*_{AQ}}^2(d-2\hat{\mathcal{D}}_{ct}) - 2\left(C^*_{AQ}+\delta C\right)\left(\hat{\mathcal{D}}_{ct}(d-\hat{\mathcal{D}}_{ct}) - \epsilon\partial_\epsilon\hat{\mathcal{D}}_{ct} + k^2\epsilon^2 + m^2\right)\right) \\
& + \hat{\mathcal{D}}_{ct}(d-\hat{\mathcal{D}}_{ct}) - \epsilon\partial_\epsilon\hat{\mathcal{D}}_{ct} + k^2\epsilon^2 + m^2\Bigg]
\end{aligned} \tag{2.58}$$

The β-function equations for individual couplings $\bar{\mathfrak{f}}_i$ are listed in (C.7) and (C.8). One can note that they follow the same general structure as the β-functions computed from the field theory. Although, even for the same choice of the regulator (or equivalently, δC) at a given cut-off, the β-functions are different. We associate this additional 'scheme-dependence' of the β-functions with reparametrization in the space of couplings as explained in section 2.7.

2.6 β-function for double-trace operators from field theory

2.6.1 Warming up: β-function of f_0

Before we get into a full-fledged calculations of β-function for general double trace couplings mentioned above, let us first describe, following [18, 57], the Wilsonian computation of the β-function in the space of the single coupling f_0. Double-trace perturbations without derivatives, i.e. (2.2) with only $f_0 \neq 0$ and their renormalizations have been discussed extensively in the literature see, e.g. [15–18, 45, 46, 52–55, 57, 58].

Let us consider a double-trace perturbation given by,

$$S = S_{CFT} + \frac{f_0}{2}\int d^dx\ \mathcal{O}^2(x) \tag{2.59}$$

The single-trace operator $\mathcal{O}(x)$ is a primary of conformal dimension $\Delta_- = \dfrac{d}{2} - \nu$ at the fixed point given by $f_i = 0$. The double-trace operator will then be a relevant operator with dimension (at leading large N). [16]

$$\Delta_{\mathcal{O}^2} = 2\Delta \equiv d - \nu,\ \nu > 0 \tag{2.60}$$

In [18] β-function for f_0 was computed for a marginal double-trace deformation. This was generalised in [57] to arbitrary $\Delta_{\mathcal{O}^2}$, where a Wilsonian RG using real space integration shells was used. See also [59] and [45] for a general perspective. Partition function of the deformed theory is given by,

$$Z = \int \mathcal{D}\Phi\ e^{-S[\Phi]} = \int \mathcal{D}\Phi\ e^{-S_{CFT}[\Phi]}\left(1 - \frac{f_0}{2}\int d^dx\ \mathcal{O}^2(x) + \frac{f_0^2}{4\cdot 2!}\int d^dx\, d^dy\ \mathcal{O}^2(x)\mathcal{O}^2(y) - \ldots\right) \tag{2.61}$$

Here, Φ are the 'fundamental fields' in the theory. The omitted terms in (2.61) organise in themselves in form of a Dyson-Schwinger sum in the final answer. If the theory is regulated at some cut-off a, such that the correlator $\langle \mathcal{O}(x)\mathcal{O}(y)\rangle$ vanishes for $|x-y| \le a$, one can write (for more general treatment see (2.73) and the discussion in Section 2.6.2)

$$G_a(w) = \langle \mathcal{O}(x)\mathcal{O}(x+w)\rangle_a = \frac{\Theta(|w|/a - 1)}{|w|^{2\Delta}} \tag{2.62}$$

this regulator is also used in [57] (see Section 2.6.2, especially (2.74) for other choices). As explained in detail in following subsection (Figure 2.6), the third term in parenthesis in

[16] This makes the theory at $f_0 = 0$ a UV CFT. In earlier sections discussing the holographic setup, this CFT was identified with the so-called 'alternative quantization'. However, we keep our subsequent analysis more general and won't use any specific value of Δ. Only in (2.69a) is the specific value in (2.60) used.

(2.61) can be rewritten as,

$$\begin{aligned}\frac{f_0^2}{4\cdot 2!}\int d^dx\, d^dw\ \mathcal{O}^2(x)\mathcal{O}^2(x+w) &= \frac{f_0^2}{2!}\int d^dx\, d^dw\ \mathcal{O}(x)\ G_a(w)\ \mathcal{O}(x+w)\\ &= \frac{f_0^2}{2!}\int d^dx\, d^dw\ \mathcal{O}(x)\ \Big(G_{a'}(w)+(a-a')G'_{a'}(w)\\ &\quad + \frac{(a-a')^2}{2}G''_{a'}(w)+\cdots\Big)\ \mathcal{O}(x+w) \qquad (2.63)\end{aligned}$$

In (2.63) the terms that are suppressed in the large N limit (see subsection 2.6.3) have been omitted. In the simple case of Θ-function cut-off as in (2.62), it can be written more simply as,

$$\begin{aligned}\frac{f_0^2}{4\cdot 2!}\int_a d^dx\, d^dw\ \mathcal{O}^2(x)\mathcal{O}^2(x+w) &= \frac{f_0^2}{4\cdot 2!}\Bigg(\int_{a'} d^dx\, d^dw\ \mathcal{O}^2(x)\mathcal{O}^2(x+w)\\ &\quad + 4\int_a^{a'} d^dx\, d^dw\ \mathcal{O}^2(x)\mathcal{O}^2(x+w)\Bigg)\\ &= \frac{f_0^2}{4\cdot 2!}\Bigg(\int_{a'} d^dx\, d^dw\ \mathcal{O}^2(x)\mathcal{O}^2(x+w)\\ &\quad + 4\int_a^{a'} d^dx\, d^dw\ \mathcal{O}(x)\ \frac{1}{|w|^{2\Delta}}\ \mathcal{O}(x+w)\Bigg) \qquad (2.64)\end{aligned}$$

The factors of 4 in both (2.63) and (2.64) are due to 4 possible combinations of contractions between $\mathcal{O}(x)$ and $\mathcal{O}(x+y)$. While the first term in (2.64) is the standard contribution for a new theory defined at cut-off a', the second term corrects the value of f_0 in (2.61). In second term on RHS of (2.64), expanding $\mathcal{O}(x+w)$ in a Taylor series

$$\begin{aligned}\frac{f_0^2}{2}\int d^dx\ \Bigg(\mathcal{O}^2(x)\int_a^{a'} d^dw\frac{1}{|w|^{2\Delta}} + \mathcal{O}(x)\ \partial_\mu\mathcal{O}(x)\int_a^{a'} d^dw\frac{w^\mu}{|w|^{2\Delta}}\\ +\frac{1}{2!}\mathcal{O}(x)\ \partial_\mu\partial_\nu\mathcal{O}(x)\int_a^{a'} d^dw\frac{w^\mu w^\nu}{|w|^{2\Delta}}+...\Bigg) \qquad (2.65)\end{aligned}$$

and using the result (B.1) in Appendix B,

$$\begin{aligned}=&\frac{f_0^2}{2}\left(\frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}\right)\left(\frac{a'^{d-2\Delta}-a^{d-2\Delta}}{d-2\Delta}\right)\left(\int d^dx\ \mathcal{O}^2(x)\right)\\ &+\frac{f_0^2}{2}\left(\frac{\pi^{d/2}}{2\,\Gamma\left(\frac{d}{2}+1\right)}\right)\left(\frac{a'^{d-2\Delta+2}-a^{d-2\Delta+2}}{d-2\Delta+2}\right)\left(\int d^dx\ \mathcal{O}\,(\partial^2)\mathcal{O}(x)\right)+\ldots \qquad (2.66)\end{aligned}$$

we see that derivative double-trace couplings are automatically generated. The couplings at the new cut-off a' are then,

$$\begin{aligned}
f_0' &= f_0 - f_0^2\left(\frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}\right)\left(\frac{a'^{d-2\Delta}-a^{d-2\Delta}}{d-2\Delta}\right)+\ldots \\
f_1' &= -f_0^2\left(\frac{\pi^{d/2}}{2\;\Gamma\left(\frac{d}{2}+1\right)}\right)\left(\frac{a'^{d-2\Delta+2}-a^{d-2\Delta+2}}{d-2\Delta+2}\right)+\ldots \\
f_2' &= -f_0^2\left(\frac{\pi^{d/2}}{16\;\Gamma\left(\frac{d}{2}+2\right)}\right)\left(\frac{a'^{d-2\Delta+4}-a^{d-2\Delta+4}}{d-2\Delta+4}\right)+\ldots\text{[17]}
\end{aligned} \tag{2.67}$$

The ellipsis in the above equations denotes higher order terms coming from ellipsis in (2.61). (2.67) can be used to compute β-functions. The contributions coming from terms in ellipsis above are $\sim(\delta a)^2$ and hence don't contribute to β-function computations.

$$\beta_0^{(d)} = \lim_{a\to a'}\left(a\cdot\frac{f_0'-f_0}{a'-a}\right) = -f_0^2\;a^{d-2\Delta}\left(\frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}\right) \tag{2.68a}$$

$$\beta_1^{(d)} = \lim_{a\to a'}\left(a\cdot\frac{f_1'}{a'-a}\right) = -f_0^2\;a^{d-2\Delta+2}\left(\frac{\pi^{d/2}}{2\;\Gamma\left(\frac{d}{2}+1\right)}\right) \tag{2.68b}$$

$$\beta_2^{(d)} = \lim_{a\to a'}\left(a\cdot\frac{f_2'}{a'-a}\right) = -f_0^2\;a^{d-2\Delta+4}\left(\frac{\pi^{d/2}}{16\;\Gamma\left(\frac{d}{2}+2\right)}\right) \tag{2.68c}$$

$$\vdots$$

where, $\beta^{(d)}$ are the β-functions for the dimensionful couplings. In terms of the dimensionless couplings, for the operators with dimension given by (2.60), these become,

$$\beta_0 = 2\nu\bar{f}_0 - \bar{f}_0^2\left(\frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}\right) \tag{2.69a}$$

$$\beta_1 = (2\nu-2)\bar{f}_1 - \bar{f}_0^2\left(\frac{\pi^{d/2}}{2\;\Gamma\left(\frac{d}{2}+1\right)}\right) \tag{2.69b}$$

$$\beta_2 = (2\nu-4)\bar{f}_2 - \bar{f}_0^2\left(\frac{\pi^{d/2}}{16\;\Gamma\left(\frac{d}{2}+2\right)}\right) \tag{2.69c}$$

$$\vdots$$

More generally, we can start with double-trace couplings with arbitrary number of derivatives as in (2.2). By a simple generalisation of the above method, we get a closed set of beta-functions. This is described in what follows.

[17] Recall, we had started with only $f_0 \neq 0$, rest all $f_i = 0\,\forall i > 0$.

2.6.2 β-function of a general coupling with arbitrary cut-off regulator

In this section we generalise the above computations of the β-functions to couplings constants of the double-trace operators with derivatives. The fixed point Lagrangian is perturbed by a term as follows,

$$\frac{1}{2}\int d^dx \left(f_0\mathcal{O}^2(x) + f_1\mathcal{O}\partial^2\mathcal{O}(x) + f_2\mathcal{O}\partial^4\mathcal{O}(x) + \cdots\right) \tag{2.70}$$

where f_i are the dimensionful coupling constants for the operators of the type $\mathcal{O}(\partial^2)^i\mathcal{O}(x)$, same as in (2.2), but written in a concise notation. These are the same class of operators for which β-functions were computed in bulk in [15, 16]. In a large N theory, the anomalous dimension of the double-trace operators are suppressed by $1/N$, and so the conformal dimension of any of the above double-trace operators is $\Delta_i = \left[\mathcal{O}(\partial^2)^i\mathcal{O}(x)\right] = d - 2\nu + 2i$. [18] We are considering appropriately orthogonalized single-trace operators at the fixed point such that under RG only the multi-traces and their derivatives are generated. We package the above couplings into a single function of ∂^2 (or equivalently k^2 in momentum space),

$$f(\partial^2) = f_0 + f_1(\partial^2) + f_2(\partial^2)^2 + \cdots \tag{2.71}$$

and hence the double-trace perturbations become,

$$\mathcal{L}_{DT} = \frac{1}{2}\int d^dx \; \mathcal{O}\; f(\partial^2)\mathcal{O}(x) \tag{2.72}$$

In a large-N theory, all the $O(1)$ connected diagrams factorise through the double-trace vertices into chain-like diagrams,

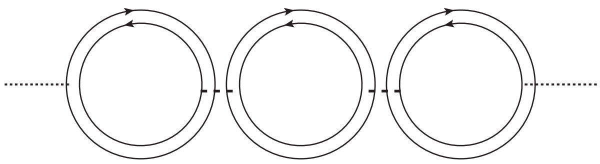


FIGURE 2.5: Factorisation through double-trace vertices in Large-N limit. Each circle is representative of $\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle$ contractions, or of their derivatives.

In the Figure 2.5, each circle is representative of $\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle$ contractions, or of their derivatives coming from the double-trace vertices (although here it looks like $\mathcal{O} = Tr[\Phi^2]$, it is representative of any arbitrary single-trace operator). In a regulated theory a UV cut-off modifies the short-distance behaviour of any correlator. We capture the effect of such regulations in our correlators by introducing a regulating-function, $\mathcal{K}(|x_1 - x_2|/a)$, such that

[18] We only require $\Delta_i = \Delta_{\mathcal{O}^2} + 2i$ in most of our analysis, using the specific value only in β-function computations.

the new regulated correlator becomes,

$$G_a(x_1 - x_2) = \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_a = \frac{\mathcal{K}(|x_1 - x_2|/a)}{|x_1 - x_2|^{2\Delta}}. \tag{2.73}$$

Here 'a' parametrises the length-scale of regulation, and the correlator shows deviation from polynomial law only near length-scales $\lesssim a$, while long distance behaviour remains power-law, as governed by conformal symmetry. Thus, $\mathcal{K}(|x_1 - x_2|/a) \to 1$, when $|x_1 - x_2| \gg a$, but falls off faster than $|x_1 - x_2|^{2\Delta}$, when $|x_1 - x_2| \lesssim a$. In our study, we assume that the short-distance fall-off is fast enough to regulate all the correlators $\langle(\partial^2)^i\mathcal{O}(x_1)\ (\partial^2)^j\mathcal{O}(x_2)\rangle$ at short distances. An example of such a regulator is $\mathcal{K}(r/a) = \Theta(r - a)$, where Θ is the Heaviside-theta function, which was used in [18, 57, 58]. We also use a regulated form of Θ-function,

$$\mathcal{K}(\rho) = \frac{\sqrt{\pi}e^{1/\omega^2}\left(\omega^2+2\right)\left(\operatorname{erf}\left(\frac{\rho-1}{\omega}\right)+\operatorname{erf}\left(\frac{1}{\omega}\right)\right)+2\omega-2(\rho+1)\omega e^{-\left(\rho^2-2\rho\right)/\omega^2}}{\sqrt{\pi}e^{1/\omega^2}\left(\omega^2+2\right)\left(\operatorname{erf}\left(\frac{1}{\omega}\right)+1\right)+2\omega} \tag{2.74}$$

The corresponding regulated δ-function that is

$$\delta_r(\rho-1) = \frac{4\rho^2 e^{-\frac{(\rho-2)\rho}{\omega^2}}}{\omega\left(\sqrt{\pi}e^{\frac{1}{\omega^2}}\left(\omega^2+2\right)\left(\operatorname{erf}\left(\frac{1}{\omega}\right)+1\right)+2\omega\right)}$$

here, ω is the width of the regulated δ-function and regulated Θ-function.
Hence, computation of any physical observable involves evaluation of chain-diagrams with regulated correlators.

Evaluation of β-functions involves studying the change of the coupling constants f_i under the change of the cut-off scale $a \to a'$. All the physical observables in this new theory are required to remain unchanged and the chain diagrams involve the correlators, $G_{a'}(|x_1 - x_2|/a')$. One can relate the diagrams in the original theory at a to those in the new theory at cut-off a' by relating the correlators.

$$G_a(x_1 - x_2) = G_{a'}(x_1 - x_2) + \partial_{a'}G_{a'}(x_1 - x_2)\ (-\delta a) + \ldots \tag{2.75}$$

Note that the second term above involves derivative of $\mathcal{K}(|x-y|/a)$ and is supported only in the region $|x-y| \sim a'$. The first term on the RHS of (2.75) contributes to the chain-diagrams at the new cut-off a' and subsequent terms correct the coupling constant. Integration involving second and subsequent terms can be seen as coming from integration of *heavy modes*, as they contribute only at short distances. They are denoted by coloured contractions in the diagrammatic representations, as in Figure 2.6.

We compute the contribution of the second diagram on the RHS of Figure 2.6 with the

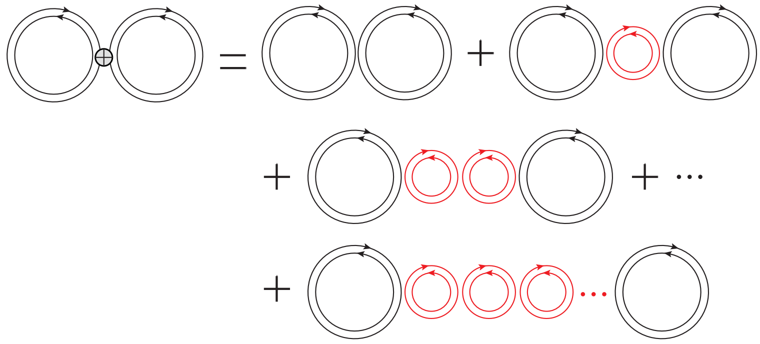


FIGURE 2.6: Corrections to a vertex at new cut-off a'. The crossed vertex on LHS denotes the vertex at new cut-off. Vertices on RHS are original vertices at a. Coloured contractions denote integration of heavy modes coming from higher order corrections in (2.75).

vertices $\frac{1}{2}f_n \int d^d z_1 \mathcal{O}\ (\partial^2)^n \mathcal{O}(z_1)$ and $\frac{1}{2}f_m \int d^d z_2 \mathcal{O}\ (\partial^2)^m \mathcal{O}(z_2)$. There are 4 ways to choose the *heavy contractions* between single-trace operators,

$$
\begin{aligned}
&\frac{f_n f_m}{4}(-\delta a')\Bigg(\int d^d z_1 d^d z_2\ \mathcal{O}(z_1)\ \partial_{a'}\left[(\partial^2)^n G_{a'}(z_1-z_2)\right](\partial^2)^m \mathcal{O}(z_2) \\
&+\int d^d z_1 d^d z_2\ \mathcal{O}(z_1)\ \partial_{a'}\left[(\partial^2)^{m+n} G_{a'}(z_1-z_2)\right]\mathcal{O}(z_2) \\
&+\int d^d z_1 d^d z_2\ (\partial^2)^n \mathcal{O}(z_1)\ \partial_{a'}\left[G_{a'}(z_1-z_2)\right](\partial^2)^m \mathcal{O}(z_2) \\
&+\int d^d z_1 d^d z_2\ (\partial^2)^n \mathcal{O}(z_1)\ \partial_{a'}\left[(\partial^2)^m G_{a'}(z_1-z_2)\right]\mathcal{O}(z_2)\Bigg)
\end{aligned}
\tag{2.76}
$$

Here we have kept only the linear variation in $(\delta a')$, since only that is required in the β-function computations. All the subsequent terms in (2.75) (which are higher order in $(\delta a')$) don't contribute to the β-functions, even though they need to be considered in computation of the exact vertex at the new cut-off. For the same reason second and following rows in Figure 2.6 don't contribute to the β-function computation. As in any differential equation, their contribution is exactly captured in the solution. At this point the β-functions in large-N limit are quadratic, whose exactness will be established in subsection 2.6.3. This is consistent with the holographic computations of the β-functions.
In (2.76), the operator at $\mathcal{O}(z_2)$ is written in a Taylor series expansion around z_1.

$$
\begin{aligned}
(\partial^2)^m \mathcal{O}(z_2) = (\partial^2)^m \mathcal{O}(z_1) + (z_2-z_1)^\mu \partial_\mu\Big((\partial^2)^m \mathcal{O}(z_1)\Big) \\
+\frac{1}{2!}(z_2-z_1)^\mu (z_2-z_1)^\nu \partial_\mu \partial_\nu \Big((\partial^2)^m \mathcal{O}(z_1)\Big)+\cdots
\end{aligned}
\tag{2.77}
$$

From the conformal field theory point of few, this is same as translating the operator at z_1 to z_2. Furthermore, rotational invariance of the theory implies that only the vector-singlets constructed at any level of Taylor series contribute, and hence odd-terms in the Taylor series don't contribute. Thus a general term appearing in the Taylor series can be written as,

$$
\begin{aligned}
&\int d^d z_1 d^d z_2 \mathcal{O}(z_1)\partial_{a'}\left[(\partial^2)^n G_{a'}(|z_1-z_2|)\right]\left(\frac{1}{(2k)!}z_{21}^{\mu_1}\dots z_{21}^{\mu_{2k}}\partial_{\mu_1}\dots\partial_{\mu_{2k}}\Big((\partial^2)^m\mathcal{O}(z_1)\Big)\right)\\
&\qquad = (a')^{2k}\left(\frac{2^{1-2k}\pi^{d/2}}{\Gamma(k+1)\Gamma(k+\frac{d}{2})}\right)\times\left(\int d\rho\ \rho^{d-1+2k}\ \partial_{a'}\left[(a')^{-2n}(\partial_\rho^2)^n G_{a'}(a'\rho)\right]\right)\\
&\qquad\qquad\times\int d^d z_1 \mathcal{O}(z_1)(\partial^2)^{(m+k)}\mathcal{O}(z_1)
\end{aligned}
\tag{2.78}
$$

where, we have used the notation $\vec{\rho} = \dfrac{\vec{z}_{21}}{a'}$, $z_{ij} = z_i - z_j$, $\rho = |\vec{\rho}|$; and the first numerical factor is coming from the angular integrations (see Appendix B).

Clearly, β-function of every coupling constant in the double-trace perturbation, f_i, is quadratic in every other coupling constant, f_j. It is instructive to note that the contribution of some coupling f_n to the β_i, where $n > i$ comes only from those terms in (2.76) in which the operator $(\partial^2)^n\mathcal{O}$ is involved in a contraction.

We show here only first few β-functions, while the details of calculations have been relegated to Appendix B:

$$
\begin{aligned}
\beta_0 = 2\nu\bar{f}_0 &+ \bar{f}_0^2\left(\alpha_0\mathbb{G}_\Delta^{\mathcal{K}'}\right) + \bar{f}_0\bar{f}_1\ \alpha_0\left[\rho^{d-2\Delta-1}\Big(\rho\ \mathcal{K}^{(2)}(\rho) - (2\Delta-1)\mathcal{K}^{(1)}(\rho)\Big)\right]_0^\infty\\
&+ \bar{f}_1^2\ \alpha_0\left[\rho^{d-2\Delta-3}\Big(\rho^3\mathcal{K}^{(4)}(\rho) - \rho^2(6\Delta-d-2)\mathcal{K}^{(3)}(\rho)\right.\\
&\qquad + \rho\left(12\Delta^2-(4d+2)\Delta+d-1\right)\mathcal{K}^{(2)}(\rho)\\
&\qquad \left.-\ \left(4\Delta^2-1\right)(2\Delta-d+1)\mathcal{K}^{(1)}(\rho)\Big)\right]_0^\infty
\end{aligned}
\tag{2.79a}
$$

$$
\begin{aligned}
\beta_1 = (2\nu-2)\bar{f}_1 &+ \bar{f}_0^2\left(\alpha_1\mathbb{G}_{\Delta-1}^{\mathcal{K}'}\right)\\
&+ \bar{f}_0\bar{f}_1\left((\alpha_0+2d\ \alpha_1)\ \mathbb{G}_\Delta^{\mathcal{K}'} + \alpha_1\ \left[\rho^{d-2\Delta+1}\Big(\rho\mathcal{K}^{(2)}(\rho) - (2\Delta+1)\mathcal{K}^{(1)}(\rho)\Big)\right]_0^\infty\right)\\
&+ \bar{f}_1^2\Bigg(\alpha_1\ \left[\rho^{d-2\Delta-1}\Big(\rho^3\mathcal{K}^{(4)}(\rho) - \rho^2(6\Delta-d)\mathcal{K}^{(3)}(\rho)\right.\\
&\qquad + \rho\left(12\Delta^2-(4d+6)\Delta+d-3\right)\mathcal{K}^{(2)}(\rho)\\
&\qquad \left.-\ \left(4\Delta^2-1\right)(2\Delta-d+3)\mathcal{K}^{(1)}(\rho)\Big)\right]_0^\infty\\
&\quad + 2\ \alpha_0\ \left[\rho^{d-2\Delta-1}\Big(\rho\ \mathcal{K}^{(2)}(\rho) - (2\Delta-1)\mathcal{K}^{(1)}(\rho)\Big)\right]_0^\infty\Bigg)
\end{aligned}
\tag{2.79b}
$$

where, the following short-hand notations have been used to avoid clutter

$$\begin{aligned} \alpha_i &= \frac{2^{1-2i}\ \pi^{d/2}}{\Gamma(i+1)\Gamma(i+\frac{1}{2}d)} \\ \mathbb{G}_{\Delta}^{\mathcal{K}^{(n)}} &= \int d\rho\ \rho^{d-2\Delta}\ \mathcal{K}^{(n)}(\rho) \end{aligned} \tag{2.80}$$

$\bar{f}_i = a^{2\nu-2i} f_i$ are the dimensionless coupling constants for the operator $\mathcal{O}\,(\partial^2)^i\mathcal{O}(x)$ (corresponding to choice (2.60)), and the β-functions are computed for these dimensionless couplings.

It is apparent that some of the coefficients in the above β-functions are simply boundary terms. With our assumption that the regulation scheme, $\mathcal{K}$ falls off fast enough at the origin to regulate all the correlators, these coefficients vanish. Thus the β-functions become,

$$\beta_0 = 2\nu\bar{f}_0 - \bar{f}_0^2\left(\alpha_0\mathbb{G}_{\Delta}^{\mathcal{K}'}\right) \tag{2.81a}$$

$$\beta_1 = (2\nu-2)\bar{f}_1 - \bar{f}_0^2\left(\alpha_1\mathbb{G}_{\Delta-1}^{\mathcal{K}'}\right) - \bar{f}_0\bar{f}_1\left((\alpha_0 + 2d\ \alpha_1)\ \mathbb{G}_{\Delta}^{\mathcal{K}'}\right) \tag{2.81b}$$

We find that the β-functions follow a pattern in which the coefficient of $\bar{f}_i\bar{f}_j$ in β_k is only a boundary term when $i+j>k$, and hence vanish. While we have checked it explicitly for first four β-functions listed below but we could easily see it generalise to any arbitrary order,

$$\begin{aligned}
\beta_0 &= 2\nu\bar{f}_0 - \bar{f}_0^2\left(\alpha_0\mathbb{G}_{\Delta}^{\mathcal{K}'}\right) \\
\beta_1 &= (2\nu-2)\bar{f}_1 - \bar{f}_0^2\left(\alpha_1\mathbb{G}_{\Delta-1}^{\mathcal{K}'}\right) - \bar{f}_0\bar{f}_1\left((\alpha_0 + 2d\ \alpha_1)\ \mathbb{G}_{\Delta}^{\mathcal{K}'}\right) \\
&= (2\nu-2)\bar{f}_1 - \bar{f}_0^2\left(\alpha_1\mathbb{G}_{\Delta-1}^{\mathcal{K}'}\right) - 2\bar{f}_0\bar{f}_1\left(\alpha_0\ \mathbb{G}_{\Delta}^{\mathcal{K}'}\right) \\
\beta_2 &= (2\nu-4)\bar{f}_2 - \bar{f}_0^2\left(\alpha_2\mathbb{G}_{\Delta-2}^{\mathcal{K}'}\right) - \bar{f}_0\bar{f}_1\left((\alpha_1 + 4(d+2)\alpha_2)\mathbb{G}_{\Delta-1}^{\mathcal{K}'}\right) \\
&\qquad - \bar{f}_0\bar{f}_2\left((\alpha_0 + 8d(d+2)\alpha_2)\mathbb{G}_{\Delta}^{\mathcal{K}'}\right) - \bar{f}_1^2\ \frac{1}{4}\left((\alpha_0 + 4d\alpha_1 + 8d(d+2)\alpha_2)\mathbb{G}_{\Delta}^{\mathcal{K}'}\right) \\
&= (2\nu-4)\bar{f}_2 - \bar{f}_0^2\left(\alpha_2\mathbb{G}_{\Delta-2}^{\mathcal{K}'}\right) - 2\bar{f}_0\bar{f}_1\left(\alpha_1\mathbb{G}_{\Delta-1}^{\mathcal{K}'}\right) - \left(2\bar{f}_0\bar{f}_2 + \bar{f}_1^2\right)\left(\alpha_0\mathbb{G}_{\Delta}^{\mathcal{K}'}\right) \\
\beta_3 &= (2\nu-6)\bar{f}_1 - \bar{f}_0^2\left(\alpha_3\mathbb{G}_{\Delta-3}^{\mathcal{K}'}\right) - 2\bar{f}_0\bar{f}_1\left(\alpha_2\mathbb{G}_{\Delta-2}^{\mathcal{K}'}\right) - \left(2\bar{f}_0\bar{f}_2 + \bar{f}_1^2\right)\left(\alpha_1\mathbb{G}_{\Delta-1}^{\mathcal{K}'}\right) \\
&\qquad - 2\left(\bar{f}_0\bar{f}_3 + \bar{f}_1\bar{f}_2\right)\left(\alpha_0\mathbb{G}_{\Delta}^{\mathcal{K}'}\right) \\
&\ \vdots
\end{aligned} \tag{2.82}$$

We have used the identity $\alpha_i = (2i+2)(d+2i)\alpha_{i+1}$ to simplify coefficients, and α_i and $\mathbb{G}_{\Delta}^{\mathcal{K}'}$ are given by (2.80). Table 2.2 summarises the values of the coefficients above for $\mathcal{K} = \Theta$

	$\Theta(\rho)$	Regulated-$\Theta(\rho)$
$\alpha_0\mathbb{G}^{\mathcal{K}'}_{\Delta}$	$\dfrac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}$	$\dfrac{4\pi^{d/2}\omega^{2\nu+1}\Big[2\Gamma(\nu+2)\,{}_1F_1\Big(\nu+2;\frac{3}{2};\frac{1}{\omega^2}\Big)+\omega\Gamma\big(\nu+\frac{3}{2}\big)\,{}_1F_1\Big(\nu+\frac{3}{2};\frac{1}{2};\frac{1}{\omega^2}\Big)\Big]}{\Gamma\big(\frac{d}{2}\big)\Big[\sqrt{\pi}e^{\frac{1}{\omega^2}}(\omega^2+2)\times\big(\mathrm{erf}\big(\frac{1}{\omega}\big)+1\big)+2\omega\Big]}$
$\alpha_1\mathbb{G}^{\mathcal{K}'}_{\Delta-1}$	$\dfrac{\pi^{d/2}}{2\Gamma(\frac{d}{2}+1)}$	$\dfrac{\pi^{d/2}\omega^{2\nu+3}\Big[2\Gamma(\nu+3)\,{}_1F_1\Big(\nu+3;\frac{3}{2};\frac{1}{\omega^2}\Big)+\omega\Gamma\big(\nu+\frac{5}{2}\big)\,{}_1F_1\Big(\nu+\frac{5}{2};\frac{1}{2};\frac{1}{\omega^2}\Big)\Big]}{\Gamma\big(\frac{d}{2}+1\big)\Big[\sqrt{\pi}e^{\frac{1}{\omega^2}}(\omega^2+2)\times\big(\mathrm{erf}\big(\frac{1}{\omega}\big)+1\big)+2\omega\Big]}$
$\alpha_2\mathbb{G}^{\mathcal{K}'}_{\Delta-2}$	$\dfrac{\pi^{d/2}}{16\Gamma(\frac{d}{2}+2)}$	$\dfrac{\pi^{d/2}\omega^{2\nu+5}\Big[2\Gamma(\nu+4)\,{}_1F_1\Big(\nu+4;\frac{3}{2};\frac{1}{\omega^2}\Big)+\omega\Gamma\big(\nu+\frac{7}{2}\big)\,{}_1F_1\Big(\nu+\frac{7}{2};\frac{1}{2};\frac{1}{\omega^2}\Big)\Big]}{8\Gamma\big(\frac{d}{2}+2\big)\Big[\sqrt{\pi}e^{\frac{1}{\omega^2}}(\omega^2+2)\times\big(\mathrm{erf}\big(\frac{1}{\omega}\big)+1\big)+2\omega\Big]}$
$\alpha_3\mathbb{G}^{\mathcal{K}'}_{\Delta-3}$	$\dfrac{\pi^{d/2}}{192\Gamma(\frac{d}{2}+3)}$	$\dfrac{\pi^{d/2}\omega^{2\nu+7}\Big[2\Gamma(\nu+5)\,{}_1F_1\Big(\nu+5;\frac{3}{2};\frac{1}{\omega^2}\Big)+\omega\Gamma\big(\nu+\frac{9}{2}\big)\,{}_1F_1\Big(\nu+\frac{9}{2};\frac{1}{2};\frac{1}{\omega^2}\Big)\Big]}{96\Gamma\big(\frac{d}{2}+3\big)\Big[\sqrt{\pi}e^{\frac{1}{\omega^2}}(\omega^2+2)\times\big(\mathrm{erf}\big(\frac{1}{\omega}\big)+1\big)+2\omega\Big]}$

Table 2.2: List of the coefficients appearing in (2.82) for choice of two different regulators discussed in the text.

and $\mathcal{K}$ =(2.74), the regulated Θ-function.

2.6.3 Exactness of β-function

The usual Wilsonian or Polchinski-Wilsonian renormalization procedure involves integration of UV/short-distance-degrees of freedom. In a continuum field theory defined around Gaussian fixed point, momentum eigenvalues serve as adequate label to differentiate between UV and IR degrees of freedom, and *heavy modes* are defined as those modes with momentum greater than some arbitrary cut-off value. When we change the value of the cut-off, those modes that lie between the old and new cut-offs are integrated over. Diagrammatically these are denoted by bold lines, and in this chapter they are represented by coloured lines (see Figure 2.7). In this chapter, we perform an integration of *heavy modes* in position space, as demonstrated above and we justify our approach in this subsection.

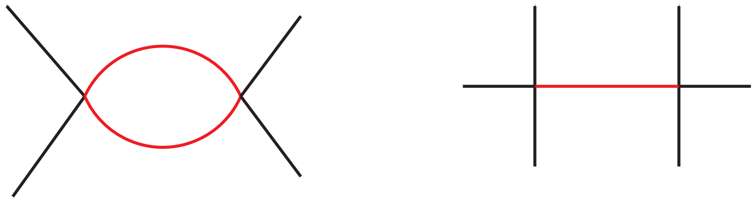


FIGURE 2.7: Types of diagrams that originate in Wilsonian RG due to integration of heavy modes. Coloured lines represent heavy modes that are being integrated out. Above diagrams show the origin of corrections to ϕ^4 and ϕ^6 vertices.

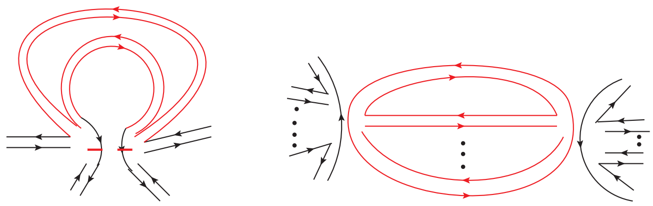


FIGURE 2.8: Diagrams that arise in contraction of *heavy modes* in a matrix theory from the double-trace vertices. The first kind of diagrams correct the single-trace coupling constants at sub-leading order of N counting. Only the second kind of diagrams correct the double-trace coupling constants, at the leading order.

In a large-N matrix theory like the one that we are considering, integration of *heavy modes* generates diagrams shown in Figure 2.8. With our normalisation of operators, it is clear that the leading contribution comes from contracting all the *heavy 'legs'* between two double-trace vertices, so one effectively has $\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle$. Fewer contractions of legs leaves us with one more propagator (with a contribution of $1/N$) than number of loops (which contribute a factor of N each), and hence the contribution is suppressed. Moreover, such a diagram with fewer *heavy contractions* contribute to a triple trace term, which even though comes with the correct normalisation (of $1/N$) in our N counting, doesn't contribute to $O(1)$ part of the effective action.

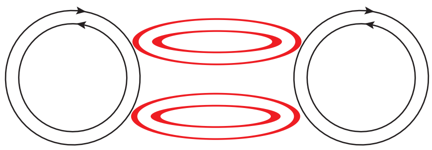


FIGURE 2.9: An example of a 2-loop diagram that is suppressed in large-N counting. Suppression of similar diagrams is also discussed in [46].

There is a class of diagrams as shown in Figure 2.9, which are suppressed by appearance of internal propagators. In general, any diagram that involves internal propagators are suppressed. A similar reasoning appears in [46] in terms of certain auxiliary fields that are used to write the double-trace operators in terms of the single-trace operators. Thus, it is clear that the only diagrams that can possibly contribute at the leading order are the chain-type diagrams discussed previously in this section, and hence the β-functions computed using such diagrams are exact.

2.6.4 Field theory correlators in momentum space

Most of our computations in bulk are in momentum space. For sake of completeness and to be able to compare the results, we will summarize some of the field theory results in momentum space. The momentum space expression for the field theory correlator along with the inclusion of the regulating function, (2.73), in general is of the form,

$$\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle_\epsilon = k^{2\Delta-d} + \epsilon^{d-2\Delta}\left(a_0 + a_1(k\epsilon)^2 + a_2(k\epsilon)^4 + \ldots\right) \tag{2.83}$$

where, a_i are some coefficients that are given by the choice of the regulating function $\mathcal{K}$. For example, for the θ-function regulation, we have following correlator in momentum space (to keep in line with the bulk notations, we are using $\Delta = d/2 \pm \nu$),

$$\begin{aligned}\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle_\epsilon &= k^{\pm 2\nu}\left(-4\pi^{\frac{d-1}{2}}\cos(\pi\nu)\frac{\Gamma(\mp 2\nu - 1)}{\Gamma\left(\dfrac{d-1}{2}\right)}\right) \pm 2\epsilon^{\mp 2\nu}\,\pi^{\frac{d-1}{2}} \\ &\qquad\qquad \times \frac{{}_1F_2\left(\mp\nu; \frac{3}{2}, \mp\nu + 1; -\frac{1}{4}(k\epsilon)^2\right)}{\nu\,\Gamma\left(\frac{d-1}{2}\right)} \\ &= k^{\pm 2\nu}\left(-4\pi^{\frac{d-1}{2}}\cos(\pi\nu)\frac{\Gamma(\mp 2\nu - 1)}{\Gamma\left(\dfrac{d-1}{2}\right)}\right) \\ &+ \epsilon^{\mp 2\nu}\left(\pm\frac{2\pi^{\frac{d-1}{2}}}{\nu\,\Gamma\left(\dfrac{d-1}{2}\right)}\right)\left[1 \pm \nu\frac{(k\epsilon)^2}{6(\nu+1)} \mp \nu\frac{(k\epsilon)^4}{120(\nu+2)} \pm \nu\frac{(k\epsilon)^6}{5040(\nu+3)} + \ldots\right]\end{aligned} \tag{2.84}$$

and the coefficients a_i can be read from the above equation. Strictly speaking, in the correctly regulated IR theory, we don't get the diverging counter terms in the above correlators. That is to say, for example, if $0 < \nu < 1$, then around the IR fixed point, when $\Delta = d/2 + \nu$, in a correctly regulated theory, the first counter term above, $a_0 = 0$. (i.e. we need to add a counter-term with $-a_0$).

In a more general case, it might happen that the kinematic term (the term proportional to $k^{2\Delta-d}$ in the above equation) also has a multiplicative integer power series in $k\epsilon$. We attribute such a series to a multiplicative wavefunction renormalization of the operator $\mathcal{O}$.

Thus, for any choice of a regulator the 2-point function in momentum space can be brought to the above form. For reference, we have presented the correlator computations in a large N bosonic vector model in Appendix D. There the correlator for the ϕ^2 operator in the regulated UV theory is given by, (D.1) which has the same form as presented above.

In a double-trace deformed field theory around a fixed point, the correlator of the $\mathcal{O}$ operator in the large N limit is given by the Schwinger-Dyson series,

$$\begin{aligned}\langle\mathcal{O}(k)\mathcal{O}(-k)\rangle_\epsilon^f &= \langle\mathcal{O}(k)\mathcal{O}(-k)\rangle_\epsilon - f(k^2)\langle\mathcal{O}(k)\mathcal{O}(-k)\rangle_\epsilon^2 + f^2(k^2)\langle\mathcal{O}(k)\mathcal{O}(-k)\rangle_\epsilon^3 + \cdots \\ &= \frac{\langle\mathcal{O}(k)\mathcal{O}(-k)\rangle_\epsilon}{1+f(k^2)\langle\mathcal{O}(k)\mathcal{O}(-k)\rangle_\epsilon}\end{aligned} \tag{2.85}$$

IR fixed point from UV theory Now we analyse the UV and IR limit of the perturbed correlators around the fixed points of the theory. Around the UV fixed point $\Delta = d/2-\nu$, and the dimensionless coupling constants are $\bar{f}_-(k\epsilon) = \epsilon^{2\nu} f_-(k^2)$, so the perturbed correlator is given by,

$$\langle\mathcal{O}(k)\mathcal{O}(-k)\rangle_\epsilon^{f_-} = \frac{k^{-2\nu}+\epsilon^{2\nu}\delta C(k\epsilon)}{1+\epsilon^{-2\nu}\bar{f}_-\left(k^{-2\nu}+\epsilon^{2\nu}\delta C(k\epsilon)\right)} \tag{2.86}$$

Taking the IR limit of this correlator, $k\epsilon \to 0$, we get the following limit of the correlator,

$$\lim_{k\epsilon\to 0}\langle\mathcal{O}(k)\mathcal{O}(-k)\rangle_\epsilon^{f_-} \to \left(\frac{\epsilon^{2\nu}}{\bar{f}_-} - k^{2\nu}\frac{\epsilon^{4\nu}}{\bar{f}_-^2} + k^{4\nu}\frac{\epsilon^{6\nu}}{\bar{f}_-^3}\left(1+\bar{f}_-\delta C\right)+\cdots\right) \tag{2.87}$$

Thus in the strict IR limit, only the second term survives, and in that case we get the correlator of the IR theory upto some wavefunction renormalization, $\epsilon^{4\nu}\bar{f}^2$, and the first contact term, after the inclusion of this wavefunction renormalization becomes, $\bar{f}\cdot\epsilon^{-2\nu}$,

$$\lim_{k\epsilon\to 0}\langle\tilde{\mathcal{O}}(k)\tilde{\mathcal{O}}(-k)\rangle_\epsilon^{f_-^*} \to \left(\bar{f}_-^*\cdot\epsilon^{-2\nu} - p^{2\nu}\right) \tag{2.88}$$

In this limit, even the coupling constants approach their respective IR fixed point value, $\bar{f} \to \bar{f}_-^*$. So the first term is precisely the type of contact term that one expects for the regulated theory with the scaling dimension, $\Delta = d/2+\nu$.

UV fixed point from IR theory Let us analyse the correlator for a double-trace deformed theory around the IR fixed point, and take the UV limit of such a correlator. The correlator given by the exact summation of the Schwinger-Dyson sum in this case is also (2.85), but now with the correlators at the IR fixed point, and also the perturbation,

$\bar{f}_+(k\epsilon) = \epsilon^{-2\nu} f_+(k^2)$, around this fixed point,

$$\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle_\epsilon^{f_+} = \frac{k^{2\nu} + \epsilon^{-2\nu}\delta C(k\epsilon)}{1 + \epsilon^{2\nu}\bar{f}_+\left(k^{2\nu} + \epsilon^{-2\nu}\delta C(k\epsilon)\right)} \tag{2.89}$$

The UV limit in this case is, $k\epsilon \to \infty$,

$$\lim_{k\epsilon\to 0}\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle_\epsilon^{f_+} \to \left(\frac{\epsilon^{-2\nu}}{\bar{f}_+} - k^{-2\nu}\frac{\epsilon^{-4\nu}}{\bar{f}_+^2} + k^{-4\nu}\frac{\epsilon^{-6\nu}}{\bar{f}_+^3}\left(1 + \bar{f}_+\delta C\right) + \cdots\right)$$
$$\xrightarrow[\text{renormalization}]{\text{on wavefunction}} \left(\bar{f}_+^* \cdot \epsilon^{2\nu} - p^{-2\nu}\right) \tag{2.90}$$

Thus, it can be seen that starting with either of the fixed points, in correct limits, one can flow to the other fixed point. It is clear that the properties of the correlators and the β-functions that are discussed in this section are also true for the holographic computations. A few subtleties that are involved in the duality between the field theory and the gravitational theory are discussed next.

2.7 Scheme-dependence and coupling constant redefinition

This section discusses *(a)* the relationship between the choice of regulator $\mathcal{K}$ in the field theory and radial cut-off in the holographic computations, and *(b)* how are different choices of regulators $\mathcal{K}$ related to diffeomorphisms in the space of couplings (or equivalently, in the space of field theories).

In the derivation of β-functions for a general regulator $\mathcal{K}$, (2.82), it is clear that all the independent coefficients appearing there are of the form

$$\mathbb{G}^{\mathcal{K}'}_{\Delta-j} = \int d\rho\ \rho^{d-2\Delta+2j}\ \mathcal{K}'(\rho), \qquad j \in \{\mathbb{Z}^+ \cup 0\} \tag{2.91}$$

These are almost like moments of derivative of the regulating function, $\mathcal{K}$.[19] Thus knowledge of all these coefficients, along with the behaviour of $\mathcal{K}$ at 0 and ∞, is, in principle, enough to reconstruct $\mathcal{K}$. From the holographic computation we only have a knowledge of the coefficients and one would like to understand what choice of $\mathcal{K}$ it corresponds to?We next discuss the class of diffeomorphisms in the space of couplings, $\bar{f}_i$, that correspond to different choices of regulating function in the Wilsonian computation. The general structure of the

[19] we say almost, because $d - 2\Delta = 2\nu$ is not an integer

β-functions either in bulk (C.7) and (C.8) or field theory (2.82) is:

$$\begin{aligned}
&\beta_0 = 2\nu\bar{f}_0 - \mathcal{A}_0\bar{f}_0^2\\
&\beta_1 = (2\nu-2)\bar{f}_1 - \mathcal{A}_1\bar{f}_0^2 - 2\mathcal{A}_0\bar{f}_0\bar{f}_1\\
&\beta_2 = (2\nu-4)\bar{f}_2 - \mathcal{A}_2\bar{f}_0^2 - 2\mathcal{A}_1\bar{f}_0\bar{f}_1 - \mathcal{A}_0\left(2\bar{f}_0\bar{f}_2 + \bar{f}_1^2\right)\\
&\beta_3 = (2\nu-6)\bar{f}_3 - \mathcal{A}_3\bar{f}_0^2 - 2\mathcal{A}_2\bar{f}_0\bar{f}_1 - \mathcal{A}_1\left(2\bar{f}_0\bar{f}_2 + \bar{f}_1^2\right) - \mathcal{A}_0\left(2\bar{f}_1\bar{f}_2 + 2\bar{f}_0\bar{f}_3\right)\\
&\quad\vdots
\end{aligned} \tag{2.92}$$

for some values of $\mathcal{A}_i$.

Above β-functions, β_i and couplings, f_i can be packaged into generating functions defined as

$$\beta(\kappa) = \beta_0 + \kappa^2\beta_1 + \kappa^4\beta_2 + \kappa^6\beta_3 + \cdots \tag{2.93a}$$

$$\bar{f}(\kappa) = \bar{f}_0 + \kappa^2\bar{f}_1 + \kappa^4\bar{f}_2 + \kappa^6\bar{f}_3 + \cdots \tag{2.93b}$$

and then (2.92) is re-packaged into a single equation,

$$\beta(\kappa) = 2\nu\bar{f}(\kappa) - \mathcal{A}(\kappa)\bar{f}^2(\kappa) - \kappa\partial_\kappa\bar{f}(\kappa) \tag{2.94}$$

where,

$$\mathcal{A}(\kappa) = \mathcal{A}_0 + \kappa^2\mathcal{A}_1 + \kappa^4\mathcal{A}_2 + \kappa^6\mathcal{A}_3 + \cdots \tag{2.95}$$

Note that, with the identification $\kappa = \epsilon k$ in (2.93b), we have the dimensionless version of $\mathfrak{f}(k) = \sum_{n=0}^{\infty}\mathfrak{f}_n(k^2)^n$ in (2.2). Then, (2.94) becomes,

$$\dot{\bar{f}}(\kappa) = \epsilon\partial_\epsilon\bar{f}(\kappa)|_k = 2\nu\bar{f}(\kappa) - \mathcal{A}(\kappa)\bar{f}^2(\kappa)$$

Such a packaged form of β-functions appears naturally in the bulk computations (see (2.57) and (2.58)).

The above differential equation can be rewritten as,

$$\left(\frac{\epsilon^{2\nu}}{\bar{f}(\kappa)}\right)^{\bullet} = \epsilon^{2\nu}\mathcal{A}(\kappa)$$

From the field theory computations, we see that different choices of regulating functions, $\mathcal{K}$, correspond to different $\mathcal{A}_i$. Now, consider another set of β-function differential equations with different coefficients, packaged into $\mathbb{A}(\kappa)$, which denotes a different *scheme* of renormalization. Then for the two different set of β-functions, the couplings in these two different schemes, $\bar{f}(\kappa)$ and $\bar{\mathfrak{f}}(\kappa)$, can be related by,[20]

$$\left(\epsilon^{2\nu}\left(\frac{1}{\bar{f}(\kappa)}-\frac{\mathsf{d}}{\bar{\mathfrak{f}}(\kappa)}\right)\right)^{\bullet}=\epsilon^{2\nu}\left(\mathcal{A}(\kappa)-\mathsf{d}\mathbb{A}(\kappa)\right)$$

here, we have allowed for a relative scaling by d, which is a consistent rescaling within a scheme: the coefficients and the couplings need to be simultaneously scaled by d and $1/\mathsf{d}$, respectively, which leaves the β-function equations invariant. Defining, $c(\kappa)=\dfrac{1}{\bar{f}(\kappa)}-\dfrac{\mathsf{d}}{\bar{\mathfrak{f}}(\kappa)}$, which can be viewed as an expansion by itself, $c(\kappa)=c_0+\kappa^2c_2+\kappa^4c_4+\cdots$, we can solve for $c(\kappa)$,[21]

$$e^{2\nu t}c(e^tk)-\lim_{t\to-\infty}\left(e^{2\nu t}c(e^tk)\right)=\int_{-\infty}^{t}dt\; e^{2\nu t}\left[\mathcal{A}(e^tk)-\mathsf{d}\mathbb{A}(e^tk)\right] \tag{2.96}$$

here we have used, $\kappa=\epsilon k$ and the redefinition $\epsilon=e^t$. Solving the above equation (2.96) term by term as a series in κ, we get,

$$\begin{aligned} c(e^tk)=\sum_{j=0}\left[\left(e^{2t}\right)^j k^{2j}c_j\right]&=\sum_{j=0}\left(e^{2t}\right)^j k^{2j}\,\frac{\mathcal{A}_j-\mathsf{d}\mathbb{A}_j}{2\nu+2j}\\ c_i&=\frac{\mathcal{A}_i-\mathsf{d}\mathbb{A}_i}{2\nu+2i},\qquad i\geq 0 \end{aligned} \tag{2.97}$$

The relation $c(\kappa)=\dfrac{1}{\bar{f}(\kappa)}-\dfrac{\mathsf{d}}{\bar{\mathfrak{f}}(\kappa)}$ gives us a transformation in the coupling-space which relates the two RG-schemes at an arbitrary cut-off.

2.8 Discussions

In this chapter we have determined all possible boundary conditions for a single bulk scalar field in AdS/CFT. The principle is that these boundary conditions can be regarded as wavefunctionals whose z-dependence is determined by a radial Schrödinger equation. It was found that the original GKPW prescription [11, 12], coupled with the counterterms discovered in [13] and applied to a finite radial cut-off $z=\epsilon_0$, corresponds to a wavefunctional which cannot be obtained by the evolution of the known GKPW δ-function boundary condition at $z=0$. In addition, it contains some spurious double trace deformations. A precise field theory correspondence for all allowed boundary conditions was found in the discussion above. Moreover, two specific wavefunctionals: Ψ_1^0 and Ψ_2^0 (Equation 2.16 and

[20] For our interest, the Wilsonian/Polchinski-Wilsonian scheme and Holographic scheme are the ones that we want to relate, and hence we use the same notations for couplings as those we have used previously in this chapter, $\bar{f}$ for dimensionless field theory couplings, and $\bar{\mathfrak{f}}$ for dimensionless bulk couplings

[21] this expansion is motivated by RHS of the equation, and has some non-trivial implication. Since the relation between $\bar{\mathfrak{f}}$ and $\bar{f}$ doesn't depend explicitly on t, this can be directly understood as a diffeomorphism in the space of couplings.

2.23), were computed, which represent the pure CFTs (respectively, IR and UV CFT, corresponding to standard and alternative quantizations). This enabled us to isolate the real double trace deformations from spurious ones and find that the holographic beta-functions can be matched to the ones computed from field theory. A geometric interpretation of the specific wavefunctionals in terms of a specific form of non-locality of the boundary 'points' in Witten diagrams was also presented in section 2.3.

As mentioned above, we have discussed the field theory equivalent of the above boundary wavefunctional in terms of properties of the generating functional $Z[J]$. In field theory, it is in principle possible, though difficult in practice, to reproduce the continuum result (power law scaling) at a finite cut-off scale, in terms of effective Wilsonian vertices plus a J^2 term in $\log Z[J]$ However, holography gives such an 'RG scheme' in a rather straightforward fashion. This is one of the important aspect of the results presented in this chapter.

Throughout this chapter, we considered a probe approximation; it was sufficient for our purposes to consider a quadratic bulk scalar action. We expect that for an interacting bulk action, with possibly multiple fields, it should again be possible to discover boundary wavefunctionals defining AdS/CFT at a finite cut-off, such that the pure CFT correlators are reproduced at a finite cut-off. The argument for the existence of such boundary conditions follows from the abstract argument, presented in subsection 2.6.3 for existence of such RG schemes in field theory.

Chapter 3

AdS_2/CFT_1 correspondence

3.1 Introduction

AdS/CFT conjecture has been subject to scrutinity of past 20 years. It has withstood various tests of verification during this time and has simultaneously been used to perform various computations in the strongly coupled field theories. However, it still holds various mysteries from us: emergence of bulk locality from boundary field theory; resolution of the information-loss paradox using the dual unitary field theory; description of the black-hole interior and the resolution of the spacetime singularity in a blackhole background; to name a few. It would be quite instructive if one could develop some simpler models of AdS/CFT correspondence where these questions could be understood more unambiguously.

SYK model is a Quantum mechanical model of fermions which was originally proposed to explain strange metal behaviour in the Condensed matter literature around 25 years back, [20]. However, recent observations suggest that it might provide us with one of the simplest models of holography between the theories which are moderately difficult (or moderately easy, depending on one's outlook), [8, 21]. The Sachdev-Ye-Kitaev (SYK) model and other tensor models that have universal IR properties [8, 20–24, 60, 61], are described by a Hamiltonian which, for Euclidean time $\tau = it$, can be viewed alternatively as a one-dimensional statistical model of fermions. The SYK model has random couplings $J_{i_1 i_2 \ldots i_q}$, representing disorder, and does not correspond to a unitary quantum mechanics. A different version without the random disorder, but with the same leading large N behaviour, has been proposed by Gurau [22, 62], Witten [23], and Klebanov and Tarnopolsky [24]. Here we are interested only in the large N behaviour and will call the set of models SYK-type models. More recently, higher dimensional generalizations of such models have also been a subject of study with the expectation that various interesting properties that make such models a good playground to study black hole physics can be carried over to the higher dimensions, [63–65].

The interest in these models as simpler models of holography is primarily because of the following features in a large N limit:

(1) There is an infrared fixed point with an emergent time reparametrisation symmetry, denoted henceforth as *Diff*.[1] The symmetry is spontaneously broken, at the IR fixed point, to $\mathbb{SL}(2,\mathbb{R})$ by the large N classical solution, leading to Nambu-Goldstone (NG) bosons characterized by the coset *Diff* $/\mathbb{SL}(2,\mathbb{R})$.[2] At the IR fixed point all these are precise zero modes of the action as one might expect from a one-dimensional CFT. Slightly away from the IR fixed point, the *Diff* symmetry is explicitly broken, the 'Nambu-Goldstone' modes cease to be zero modes and their dynamics is described by a Schwarzian term (which is the equivalent of a 'pion mass' term). It has been conjectured that (see, e.g. [25, 66]) that this situation is similar to a bulk model in which the AdS_2 symmetry is slightly broken (this is called a *near* AdS_2 geometry, in the sense of an s-wave reduction from higher dimensions, as in [26]).

(2) The possibility of a gravity dual is further reinforced by the fact that the Lyapunov exponent in the SYK model saturates the chaos bound, which is characteristic of a theory of gravity that has black hole solutions [27, 28, 67].

(3) The full model has an approximately linearly rising ('Regge-type') spectrum of conformal weights near the IR fixed point, with $O(1)$ anomalous dimension even for operators with spin higher than two. This behaviour is unexpected both from string theory in the limit $\alpha' \to 0$, or from Vasiliev theory (see, for example, [8]). Thus while the dynamics of the soft modes appears to have a simple dual gravity description, it is not clear if it can naturally incorporate the rest of the Regge-type spectrum description. In this chapter we primarily concern ourselves with a bulk gravity dual which describes the soft modes.

A partial list of various works exploring SYK-type models and other related developments is [8, 21–24, 63–65, 67–86].

In this regard, the work presented in this chapter has concentrated on trying to understand the duality between SYK model and a two-dimensional theory of gravity. We have made our own proposal of such a dual gravity theory using the geometrical techniques of co-adjoint orbits and verified that certain features of the SYK model can be reproduced using this proposed dual: (1) it reproduces correctly the effective low energy effective action of the

[1] We use *Diff* to denote either $\mathrm{Diff}(R)$ or $\mathrm{Diff}(S^1)$, depending on whether we are at zero temperature or finite temperature. This group is alternatively called the *Virasoro* group.

[2] As explained later in more detail, unlike in higher dimensions where Nambu-Goldstone modes are zero modes of the action promoted to spacetime fields, here they remain zero modes (do not acquire kinetic terms) since they cannot be made dependent on any other dimension.

SYK model, the Schwarzian action; (2) it reproduces the correct Thermodynamic behaviour of the SYK model.

The strategy pursued in this chapter to build a bulk-dual of the SYK-type models can be summarized as follows:

As explained in [8, 21], the NG modes of the SYK-type model can be characterized by *Diff* orbits of the classical solution G_0 (at the IR fixed point $J = \infty$) or *Diff* orbits of G_0' which is the deformed value of G_0 after turning on a small value of $1/J$ (see figure 3.1). Any given point on the *Diff* orbit can be obtained from the reference point, G_0 or G_0', by the action of an appropriate one-dimensional diffeomorphism.

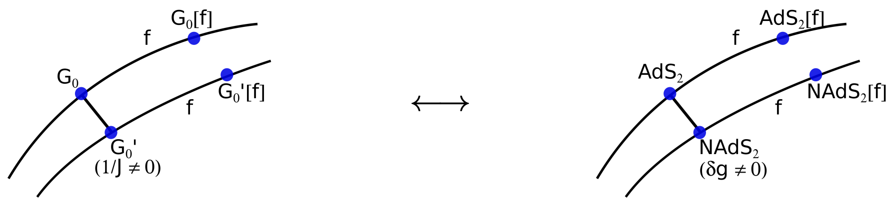


FIGURE 3.1: In the left panel, the top curve represents the Diff(R)-orbit (or a Diff(S^1)-orbit at finite temperature), *at the IR fixed point* $J = \infty$, of the classical large N solution for the fermion bilocal $G_0(\tau_1, \tau_2) \sim (\tau_1 - \tau_2)^{-2\Delta}$; this represents the Nambu-Goldstones of Diff(R)/$\mathbb{SL}(2, \mathbb{R})$. The lower curve represents the orbit of a deformed solution G_0' slightly away from the IR fixed point, with a small positive $1/J$. In the right panel, the top curve represents the orbit of the AdS_2 spacetime (these are asymptotically AdS_2 spacetimes, the two-dimensional equivalent of Brown-Henneaux geometries, which we will describe explicitly in Section 3.5). The bottom curve represents the orbit of a slightly deformed AdS_2 spacetime $NAdS_2$, with a controlled non-normalizable deformation (see section 3.5).

It is shown in [29, 30] that the space of coadjoint orbits of *Diff* can be quantized using a natural symplectic form *a la* Kirillov [31], leading to Polyakov's two-dimensional quantum gravity action [32]. This observation is reminiscent of the emergent two-dimensional bulk description from the $c = 1$ model, which is a matrix quantum mechanics. It was found in [87, 88] that the semiclassical (large N) singlet configurations of the matrix quantum mechanics, described by fermion droplets on a two-dimensional phase plane, could be understood as coadjoint orbits of W_∞ algebra generated by bi-local boson operators made out of fermions. A representation of this algebra in $c = 1$ was found in [89]. The coadjoint orbit action *a la* Kirillov [31] in the space of these configurations gave rise to a two-dimensional action whose low energy sector reproduced the (massless) tachyons of two-dimensional string theory.[3] A similar approach was taken in [91] to arrive at a moduli space action of LLM geometries [92] describing half-BPS giant gravitons.

Following the above examples, one might wonder whether such a two-dimensional quantum gravity action, obtained by the coadjoint orbit method, naturally describes a bulk dual to

[3]The precise correspondence required some additional structure ('leg-poles'); see [90] for some recent insight.

the SYK model. It turns out that *a priori* it is not possible since the gravity action does not have a cosmological constant and it describes asymptotically flat spaces. This prompts one to consider a generalization of the Polyakov action, which includes a cosmological constant and boundary terms (the boundary terms are found by requiring the existence of a well-defined variational principle; these are also the terms required by consistency with the Weyl anomaly in a manifold with a boundary, see Appendix K for details). The new action, described in Section 3.3, has asymptotically AdS_2 geometries as solutions (see Section 3.5 and 3.4), which are all generated from AdS_2 by the action of *Diff*. The schematics of these solutions is described in the right panel of Figure 3.1.

The main point of this chapter is that the *two-dimensional quantum gravity theory*, arrived at in this fashion, *provides a bulk dual to the Nambu-Goldstone sector of the SYK models.* We find a number of strong evidences for this duality:

(a) the space on which path integral of the bulk theory is performed reduces to $Diff/\mathbb{SL}(2,\mathbb{R})$, which is the same as that of the Nambu-Goldstone bosons in the SYK model. In the bulk theory these degrees of freedom emerge as the space of large diffeomorphisms (analogous to Brown-Henneaux diffeomorphisms in AdS_3). In addition to these, the bulk metric admits a fixed, non-dynamical conformal factor of a simple functional form. In the SYK theory this parameterizes the departure from strong coupling.

(b) The bulk path integral reduces to a path integral over $Diff/\mathbb{SL}(2,\mathbb{R})$ with a Schwarzian action section 3.6, characterized by a non-zero overall coefficient coming from the conformal factor.

(c) the low temperature free energy qualitatively agrees with that of SYK model, section 3.6. In the Discussion section, we show how to go beyond the low energy sector, and describe the higher mass modes of the SYK model, by introducing bulk matter fields. We show, up to quadratic order, how to couple an infinite series of bulk scalars to the Polyakov model and show that it reproduces the coupling of the higher modes of the SYK model with the NG bosons.

Subsequently in this chapter, Section 3.2 briefly reviews the SYK model and summarizes the key features. Section 3.3 motivates our proposed bulk action (3.28) from the viewpoint of coadjoint orbits of $Diff/\mathbb{SL}(2,\mathbb{R})$. The two subsequent sections analyze the theory in the conformal gauge $ds^2 = e^{2\phi}\widehat{ds^2}$. In Section 3.4, solutions of the equation of motion where $\widehat{ds^2}$ represents pure AdS_2 geometry are described; it turns out that the 'Liouville mode' ϕ gets completely fixed by the equations of motion (in fact, by just the Virasoro constraints, as shown in Appendix H), up to three real parameters which define boundary conditions for the metric. In Section 3.5 we find a larger class of solutions, which represent large diffeomorphisms of AdS_2 (similar to Brown-Henneaux geometries in asymptotically AdS_3 spacetimes). These are normalizable modes of the metric ('boundary gravitons') and represent dynamical variables of the path integral, which is described in Section 3.5.1. In Section 3.6 the effective action of these boundary gravitons is obtained by an on-shell evaluation of the path-integral; it is found to be given by a Schwarzian (3.67). Thus, the boundary gravitons are found to represent the pseudo-Nambu-Goldstone modes of the SYK model. Section 3.7 focuses on a large diffeomorphism which leads to a Euclidean black hole geometry (this turns the boundary direction into a circle). On-shell action for this geometry reproduces the qualitative features of the free energy of the SYK models. Detailed comparison with the SYK model is carried out in Section 3.8. Finally, in Section 3.9, we discuss how to describe the 'hard' modes of the SYK model in terms of external probe scalars coupled to the metric. The Appendices contain detailed derivations of some formulae and supplementary arguments.

3.2 Review of SYK model

SYK-model

As mentioned above, SYK-model is a model of N-interacting fermions in which q of the N fermions interact via a Hamiltonian given by,

$$H = \frac{(i)^{q/2}}{q!} \sum_{1 \le i_1 < i_2 \ldots i_q \le N} J_{i_1 i_2 \ldots i_q} \; \psi^{i_1} \psi^{i_2} \cdots \psi^{i_q} \tag{3.1}$$

The corresponding Lagrangian can be written as,

$$\mathcal{L} = \frac{i}{2} \sum_{1 \le i \le N} \psi^i \partial_t \psi^i - \frac{1}{q!} (i)^{q/2} \sum_{1 \le i_1 < i_2 \ldots i_q \le N} J_{i_1 i_2 \ldots i_q} \; \psi^{i_1} \psi^{i_2} \cdots \psi^{i_q} \tag{3.2}$$

The original model proposed in [20] was in terms of the Dirac fermions, however, the key features of the theory can be captured by the Majorana fermions.

In general *a well defined* large N limit is one where the contribution to partition function coming from the action and entropy is comparable in large N counting. In a system with N fermions the entropy contributes at $\mathcal{O}(N)$. Thus we require that the above Lagrangian also shows similar behaviour under large N scaling. The interaction term, naively behaves as $\mathcal{O}(N^q)$ and hence for a well-defined large N behaviour, we require that the coupling scale as $\mathcal{O}(N^{1-q})$. Moreover, in this model the coupling $J_{i_1 \ldots i_q}$ is chosen from a Gaussian ensemble of zero mean and variance given by,

$$\langle J^2_{i_1 \ldots i_q} \rangle = \frac{J^2\,(q-1)!}{N^{q-1}} \tag{3.3}$$

The physical observables in this theory are given by,

$$\overline{\langle O_1(t_1)\cdots O_i(t_i)\rangle} = \frac{\int \mathcal{D}J_{i_1\ldots i_q} \exp\left[-\frac{N^{q-1}\;J^2_{i_1\ldots i_q}}{2J^2}\right]\langle O_1(t_1)\cdots O_i(t_i)\rangle}{\int \mathcal{D}J_{i_1\ldots i_q} \exp\left[-\frac{N^{q-1}\;J^2_{i_1\ldots i_q}}{2J^2}\right]} \tag{3.4}$$

$$\text{where,}\quad \langle O_1(t_1)\cdots O_i(t_i)\rangle = \frac{\int \mathcal{D}\psi^i e^{i\int dt\mathcal{L}}\; O_1(t_1)\cdots O_i(t_i)}{\int \mathcal{D}\psi^i e^{i\int dt\mathcal{L}}}$$

However, even though this is technically what we are interested in computing, this isn't what we can compute! But following observation comes to our rescue here: treat Js as a field of the theory and define a new path integral as follows:

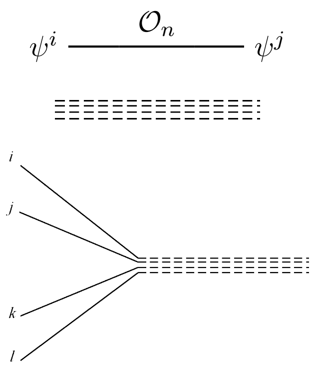
$$\overline{\langle O_1(t_1)\cdots O_i(t_i)\rangle} = \frac{\int \mathcal{D}J_{i_1\ldots i_q}\;\mathcal{D}\psi^i\;\exp\left[-\frac{N^{q-1}\;J^2_{i_1\ldots i_q}}{2J^2}\right]e^{i\int dt\mathcal{L}}\;O_1(t_1)\cdots O_i(t_i)}{\int \mathcal{D}J_{i_1\ldots i_q}\;\mathcal{D}\psi^i\;\exp\left[-\frac{N^{q-1}\;J^2_{i_1\ldots i_q}}{2J^2}\right]e^{i\int dt\mathcal{L}}} \tag{3.5}$$

The integration in (3.4) is called *quenching* while the one in (3.5) is called *annealing*. It can be argued that the two theories given by (3.4) and (3.5) differ only at subleading order of large N counting, [93, 94].

One way to understand the equivalence between the quenching and annealing in the large N limit due to [93]. Note that the equation(3.4) can be interpreted as starting with some m copies of the system, each with different values of the couplings. The observables of interest are computed in each of the system and then averaged over using the Gaussian weights to compute the average value of the observables. Think of this as a *quadratic theory of* $J_{i_1\ldots i_q}$ *with no interactions*. However, when we consider the expression in (3.5), then we are essentially considering a theory of $J_{i_1\ldots i_q}$ that has interactions with the fermions. So to show that the two theories are equivalent, it suffices to show that the interactions generate corrections that are suppressed in large N.

3.2.1 Physical Quantities

Now let us look into the computation of some physical quantities in the annealed model. The Feynman rules are listed as below:



$-\delta^{ij}/(i\mathcal{O}_n)$

J^2/N^{q-1}

$-i^{q/2+1}$

FIGURE 3.2: Feynman Rules. Here for demonstration we have chosen $q = 4$. The first diagram denotes the fermion propagator. The second diagram is the value of propagator of $J_{i_1 i_2 i_3 i_4}$ fields in the annealed model. The third diagram is the value of the interaction vertex $J_{i_1 i_2 i_3 i_4}\ \psi^{i_1}\psi^{i_2}\psi^{i_3}\psi^{i_4}$.

Partition function Diagrams contributing to the Partition function are:

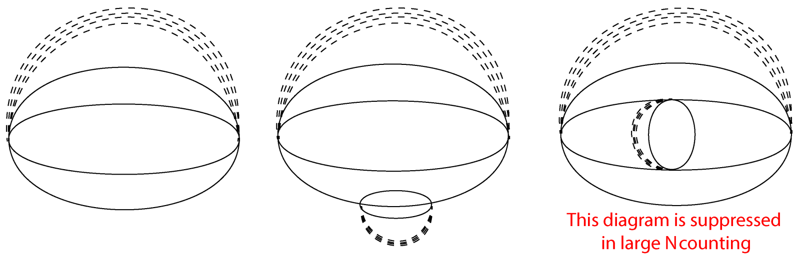


FIGURE 3.3: Melonic diagram contributing to the computation of the partition function. The rightmost diagram demonstrates an example of non-melonic diagram that doesn't contribute at the leading large N order.

2-pt function Diagrams contributing to the 2-pt function are:

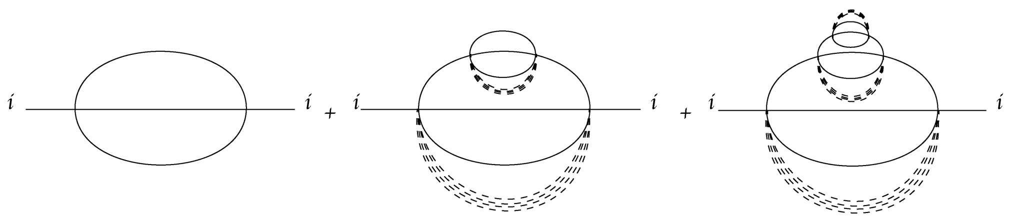


FIGURE 3.4: Melonic diagram contributions to the 2-point function.

4-pt function Diagrams contributing to the 4-pt function are:

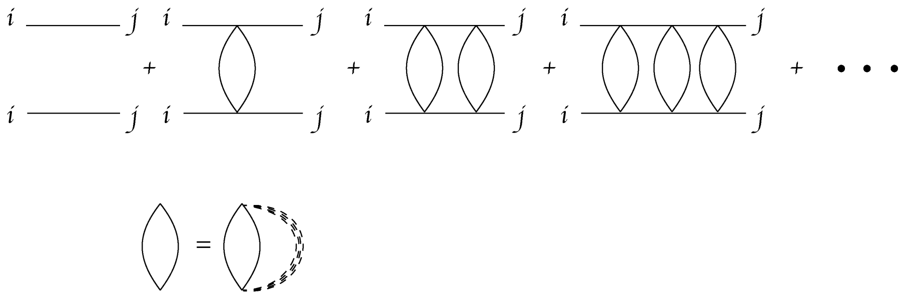


FIGURE 3.5: Diagrams contributing to the 4-point function to the leading large N counting.

The dominating diagrams that we have introduced above are known as *Melonic diagrams*. SYK-model is not the only theory where these diagrams dominate. In fact, the dominance of such diagrams had been known in a different type of theories that we explore next.

Tensor models

It was noticed by Witten that there is a class of theories which shows similar behaviour in terms of perturbative diagrammatic computations as the SYK model, called the *Tensor Models*.[4] Moreover, it has an inherent advantage over SYK model. In SYK model the couplings are quenched, i.e, they are chosen arbitrarily from an ensemble. Thus the computation of the observables involves averaging over measurements across various instances of the system with different values of the couplings. Such a *system* is inherently non-unitary. Moreover, in the annealed model instead, where we treated the couplings as some slowly varying fields, it is important to note that the thermodynamic entropy of these $J_{i_1 \cdots i_q}$ fields is $\sim N^q/q!$ far exceeds the entropy of the fermions. Thus all the computations done using the above action are subleading. Alternatively, since the tensor models are inherently unitary but with same physics, the observations of chaotic behaviour is more significant in such models. It is clear that such unitary models will do more justice in our attempt to explain the problems surrounding balck hole thermodynamics that the SYK model. Also, this model is closer to the conventional models of bulk-boundary dualities than SYK model. For one, in SYK model, as we will see later, there are certain fields bilinear in ψ^i fermions, which have been conjectured to have dual bulk fields; however the same is not true with the fermions themselves. At least from our knowledge of the dualities in higher dimensions we know that such fields don't have a bulk dual. Gauged tensor models provide a natural set-up of gauge-singlet operators that, in line with our understanding of the higher dimensional dualities have a bulk dual. In this subsection we will discuss some of these nuances and in

[4]These models were originally proposed to generalize the construction of 2-dimensional geometries from matrix models in higher dimensions.

the rest of the lectures, we will try to make our analysis as independent of specific details of the models as possible.

Following the initial suggestion [23] there have been various proposed tensor models that are different in ways that doesn't affect the physics governed by the melonic diagrams, see [24, 78–86]. This section should serve as a brief collection of the references where such models are discussed in literature with a short discussion of their properties.

Witten's model We replace the SYK Lagrangian with the following Lagrangian,

$$\mathcal{L} = \frac{i}{2}\sum_{i=1\ldots q}\psi_i\partial_t\psi_i - i^{q/2}j\psi_1\ldots\psi_q \tag{3.6}$$

The contraction of indices in the above vertices is as follows:

$$\psi_1\psi_2\psi_3\psi_4 = \psi_1^{l_{14}l_{13}l_{12}}\psi_2^{l_{12}l_{24}l_{23}}\psi_3^{l_{23}l_{13}l_{34}}\psi_4^{l_{34}l_{24}l_{14}} \tag{3.7}$$

In this model the fermions are coloured under $q-1$ copies of gauge group ($SO(n)$ or $SU(n)$, let's call them G_{ij}, labelled by the two particles it 'runs in between'), hence there are a total of $q(q-1)/2$ copies of the gauge group. The total fermionic degrees of freedom are: qn^{q-1}; while the degrees of freedom in the gauge group is: $\sim q(q-1)n^2$. Thus, the effective 'large N' in this theory is: $N = qn^{q-1}$. Again for a good large N limit, we should scale the coupling constant as,

$$j = \frac{J}{n^{(q-1)(q-2)/4}}$$

Klebanov-Tarnopolsky model This is a simplification of the Witten's model by taking 'same' fermions on the diagonal elements of the q-simplex that is formed in the Witten's vertex. Thus the vertex contraction now becomes,

$$\psi_1\psi_2\psi_3\psi_4 \equiv \psi\psi\psi\psi = \psi^{b_1c_1a_1}\psi^{a_1c_2b_1}\psi^{b_1c_1a_2}\psi^{a_2c_2b_2} \tag{3.8}$$

While in Witten's model l_{ij} represented the index of the gauge group between ψ_i and ψ_j (G_{ij}), in this model the gauge groups are labelled by indices a, b, c. So while each fermion is coloured under the same gauge groups (lets call them G_a, G_b, G_c), they are charged under different copies of the groups and hence are technically distinct fermions and can have coincident insertions.

Klebanov's Bosonic model We replace the SYK Lagrangian with the following Lagrangian,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_{abc}\partial^\mu \phi^{abc} + \frac{1}{4} g \phi^{b_1 c_1 a_1}\phi^{a_1 c_2 b_1}\phi^{b_1 c_1 a_2}\phi^{a_2 c_2 b_2} \tag{3.9}$$

Such models also have a Wilson-Fisher fixed points and the analysis of the same is provided in [24].

Computing the Physical Quantitites

Now let us go back to the computation of the 2-point function. From now on we will concentrate on the diagrams, which are not model specific. We can write down the Schwinger-Dyson equation for the computation of the 2-point function as follows:

$$\begin{aligned}
G(\tau_1,\tau_2) &= G_0(\tau_1,\tau_2) + J^2 \int d\tau d\tau' G_0(\tau_1,\tau)\left[G_0(\tau,\tau')\right]^{q-1} G_0(\tau',\tau_2) + \cdots \\
&= G_0 + G_0 * \Sigma * G_0 + G_0 * \Sigma * G_0 * \Sigma * G_0 + \cdots \\
\text{where,} \qquad & \Sigma(\tau_1,\tau_2) = J^2 G(\tau_1,\tau_2)^{q-1} \qquad\qquad (3.10)\\
& G_0(\tau_1,\tau_2) = \frac{1}{2}\, sgn(\tau_1-\tau_2), \quad \text{is the free propagator}
\end{aligned}$$

$$\Rightarrow G(\tau_1,\tau_2) = \left[G_0 * \left[1 - \Sigma * G_0\right]^{-1}\right](\tau_1,\tau_2) \tag{3.11}$$

The above equation is easier written in momentum space as,

$$\begin{aligned}
G^{-1}(p) &= G_0^{-1}(p) - \Sigma(p) \\
\Rightarrow G^{-1}(p) &= -ip - \Sigma(p)
\end{aligned} \tag{3.12}$$

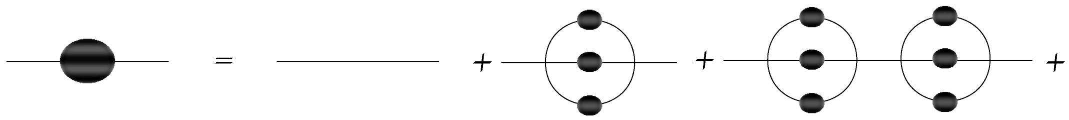
Writing (3.12) back in position space it reads:

$$-\delta(\tau_1-\tau_2) = \partial_{\tau_1} G(\tau_1,\tau_2) + \int d\tau\, G(\tau_1,\tau)\Sigma(\tau,\tau_2) \tag{3.13}$$

Now we want to solve this equation. It is not an easy problem in general, and therefore one can take a simplifying assumption: $J \to \infty$. In this case we can drop the first term in (3.13),

$$-\delta(\tau_1-\tau_2) = \int d\tau\, G(\tau_1,\tau)\Sigma(\tau,\tau_2) \tag{3.14}$$

To interpret this assumption, recall that around the UV fixed point, $[J] = 1$, and hence it is a relevant coupling. In absence of an IR fixed point (which can be argued for given that there is no other length scale in the problem), the coupling flows to infinity. Hence, our simplifying assumption is really the computation at the IR fixed point. Other way to arrive at the same conclusion is through working it out in position space: the condition under



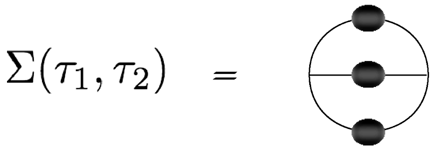


FIGURE 3.6: Diagramatic representation of the Schwinger-Dyson equations in (3.10) and (3.11)

which the first term becomes unimportant is when the length scales at which the correlator starts varying is significantly bigger than the natural length scale in the problem: $1/J$.

At this stage we conclude that:

(1) The scaling above is the same as what one expects from a scaling field of scaling dimension Δ and $\Delta(q-1)$ respectively in a conformal field theory. Since $G(\tau,\tau')$ is related to the fermion 2-point function, the fermions appear to obtain an anomalous dimension $1/q$ at the IR fixed point.
(2) If this is true, then we can guess a solution to the above integral equation:

$$G(\tau,\tau') = \frac{b\, sgn(\tau-\tau')}{|\tau-\tau'|^{2\Delta}} \tag{3.15}$$

What is the conformal symmetry in one dimension? Every redefinition of the coordinate, $f(\tau)$. Hence what above symmetry tells us is that there is an *emergent* reparametrization invariance in the IR limit.[5]

The above invariance of the theory under reparametrizations can be used to compute the 2-point function on a circle. Idea being that when we are working with a Euclidean theory, a theory on a circle is like a theory with finite temperature. Using the transformation, $\tau = \tan(\pi\theta/\beta)$, where $\theta \in [-\beta/2, \beta/2]$ we find that,

$$G_\beta(\tau,\tau') = b\left[\frac{\pi}{\beta\sin\left(\frac{\pi(\tau-\tau')}{\beta}\right)}\right]^{2\Delta} sgn(\tau-\tau') \tag{3.16}$$

[5]The word *emergent* has been emphasized because there is no strict IR fixed point here, unlike, say, Wilson-Fisher fixed point; it is only a limiting statement. Any measurement of finitely separated correlators will necessarily deviate from the ansatz that we have computed above.

What is more important, one can write an effective low energy theory in terms of the path integral variables $\tilde{G}(\tau_1,\tau_2), \tilde{\Sigma}(\tau_1,\tau_2)$ which reproduces the correct large N dynamics of all the above theories. The effective action is given by,

$$\frac{S}{N} = -\frac{1}{2}\log\det\left(\partial_t - \tilde{\Sigma}\right) + \frac{1}{2}\int d\tau_1 d\tau_2 \left[\tilde{\Sigma}(\tau_1,\tau_2)\tilde{G}(\tau_1,\tau_2) - \frac{J^2}{q}\tilde{G}(\tau_1,\tau_2)^q\right] \tag{3.17}$$

If one defines the fluctuations as, $\tilde{G} = \bar{G} + |G|^{\frac{2-q}{2}} g$ and $\tilde{\Sigma} = \bar{\Sigma} + |G|^{\frac{2-q}{2}}\sigma$,[6] and integrates out the σ modes in the resulting quadratic fluctuation action then one obtains an action for the g-modes:

$$\frac{S}{N} = -\frac{J^2(q-1)}{4} g * \left(\tilde{K}^{-1} - 1\right) * g \tag{3.18}$$

where $\tilde{K}$ is the same kernel that is used to define the 4-point function, [25]:

$$\tilde{K}(\tau_1,\tau_2;\tau_3,\tau_4) = -J^2(q-1)|G(\tau_{12})|^{\frac{q-2}{2}} G(\tau_{13})G(\tau_{24})|G(\tau_{34})|^{\frac{q-2}{2}} \tag{3.19}$$

Specifically, the action for the reparametrization modes of the theory corresponding to

$$g(\tau_1,\tau_2) = \delta_\epsilon G(\tau_1,\tau_2) = \left[\Delta\left(\partial_{\tau_1}\epsilon(\tau_1) + \partial_{\tau_2}\epsilon(\tau_2)\right) + \epsilon(\tau_1)\partial_{\tau_1} + \epsilon(\tau_2)\partial_{\tau_2}\right]\bar{G}(\tau_1,\tau_2) \tag{3.20}$$

is given by the *Schwarzian action*:

$$\frac{S}{N} \sim \int d\tau \{f(\tau),\tau\} \tag{3.21}$$

where, $f(\tau) = \tau + \epsilon(\tau)$ has been used. These modes are essentially responsible for the chaotic behaviour of this theory and will also form the subject of study from the holographic point of view in the remaining part of this chapter.

3.3 2D quantum gravity action

This section briefly reviews some of the material on coadjoint orbits of *Diff* in [29, 30, 95], focussing on the emergence of 2D quantum gravity represented by the Polyakov action [32].

As explained in [8, 21], and briefly mentioned in the Introduction, the zero modes of the SYK model at the IR fixed point (these modes are suggestively called the Nambu-Goldstone (NG) modes, although they differ somewhat from their higher dimensional counterpart, as explained below) are given by *Diff* transforms of the large N condensate of the bilocal

[6] $\bar{G}, \bar{\Sigma}$ are the saddle point solutions of the above action.

'meson' variable $G(\tau_1,\tau_2) = \psi_I(\tau_1)\psi_I(\tau_2)$[7]

$$
\begin{aligned}
G_0(\tau_1,\tau_2) &\sim \frac{1}{(\tau_1-\tau_2)^{2\Delta}} \xrightarrow{f\in Diff(R^1)} G_0[f](\tau_1,\tau_2) \\
G_0[f](f(\tau_1),f(\tau_2)) &\equiv G_0(\tau_1,\tau_2)\left(\frac{\partial f(\tau_1)}{\partial\tau_1}\frac{\partial f(\tau_2)}{\partial\tau_2}\right)^{-\Delta}
\end{aligned}
\tag{3.22}
$$

Here $f:\tau\to f(\tau)$ represents an element of Diff(R^1). This orbit is represented pictorially by the top curve in the left panel of Fig 3.1. In case of finite temperature, the time direction is considered Euclidean and compactified into a circle of size $\beta = 1/T$: in that case the appropriate group of transformations is Diff(S^1).

The second line of the above equation essentially says that G transforms as a bilocal tensor of weight 2Δ under the diffeomorphism f. For later reference, the infinitesimal version of this transformation as represented in the space of bilocal variables is given by (for $f(\tau) = \tau+\epsilon(\tau)$),

$$
\delta_\epsilon G(\tau_1,\tau_2) = \left[\Delta\left(\partial_{\tau_1}\epsilon(\tau_1)+\partial_{\tau_2}\epsilon(\tau_2)\right)+\epsilon(\tau_1)\partial_{\tau_1}+\epsilon(\tau_2)\partial_{\tau_2}\right]G(\tau_1,\tau_2) \tag{3.23}
$$

Note that G_0, as defined in the first line, is invariant under $\mathbb{SL}(2,\mathbb{R})$, i.e. under *Diff* elements of the form $h(\tau) = (a\tau+b)/(c\tau+d)$, with $ad-bc=1$. This implies that the orbit described above parameterizes a coset *Diff* $/\mathbb{SL}(2,\mathbb{R})$, namely the set of *Diff* elements quotiented by the identification $f(\tau)\sim f(h(\tau))$.

An important issue in the context of the SYK model is the quantum mechanical realization of the *Diff* algebra; in particular, it is an important question what the central charge of the corresponding Virasoro algebra is. We will find below, in terms of the bulk dual described by (3.28), that the central charge of the two-dimensional realization is proportional to N.[8]

In higher dimensions, such as in pion physics, the elements of the coset represent Nambu-Goldstone bosons, with kinetic terms given by a nonlinear sigma model (see, e.g. the discussion of pions in [96], Chapter 19). The Nambu-Goldstone bosons are zero-modes promoted to spacetime-dependent fields. In the SYK model, the zero-modes are described by $f(\tau)$, or in the infinitesimal form $\epsilon(\tau)$, (3.23). Their definition already uses up the only dimension available in the model, and hence they cannot be made dependent on any other coordinate and remain zero modes (do not pick any kinetic terms). As explained above, following [8, 21], when one moves away from the strict IR limit (i.e. a small value of $1/J$ is turned on), these modes cease to be zero modes and pick up a non-zero action, given in

[7] We are using a generalized notation here, in which 'I' denotes the appropriate indices of a given SYK/tensor model. For example, in SYK model it denotes the 'flavour indices' of fermions ψ_i, while in Witten-Gurau model it denotes the tri-fundamental index the fermions carry.

[8] More precisely, the *Diff* group is realized here as a subgroup of a two-dimensional conformal algebra which is unbroken by the presence of the boundary.

terms of the Schwarzian derivative

$$S_{\text{eff}} \sim \frac{N}{J} \int d\tau \{f,\tau\}, \text{ where } \{f,\tau\} \equiv \frac{f'''(\tau)}{f'(\tau)} - \frac{3}{2}\left(\frac{f''(\tau)}{f'(\tau)}\right)^2 \tag{3.24}$$

In spite of the appearance of the derivatives, the above is a ‘potential’ term for the zero modes, similar to a pion mass term.[9]

3.3.1 Coadjoint orbits

The above discussion shows that the degrees of freedom of the low energy (NG) sector of the SYK theory are characterized by elements of $M = \mathit{Diff}/\mathbb{SL}(2,\mathbb{R})$. In particular, the free energy is given by a path integral over M with the above Schwarzian action.

In this subsection, the question of possible quantization of this configuration space is addressed. This question has a natural interpretation in terms of AdS/CFT correspondence, since the bulk path integral can, in a sense, be regarded as a radial quantum evolution of boundary data [15, 16, 56]. [15, 16, 56][10]

The quantum theory envisaged above has a configuration space given by the group of paths in M (the group of closed paths in M is called loop(M)). An action functional on this space was formulated in [29, 30], using the formalism of coadjoint orbits and the resulting symplectic form in M [31, 95]. Let's consider a path $\mathcal{P}(\sigma)$ in the space of *Diff* elements, with $\mathcal{P}(0) = P_0$, $\mathcal{P}(1) = P_1$. Since each point of the path is represented by a diffeomorphism, we can label the path as $f(\tau,\sigma)$ where the initial point P_0 corresponds to some diffeomorphism $f_0(\tau)$ and the final point P_1 to another diffeomorphism $f_1(\tau)$. The above mentioned action functional for such a path, also called the coadjoint orbit action or the Kirillov action, is given by [29, 30] (where the symplectic form is $\Omega = d\Theta$)

$$\begin{aligned} S_{\text{Kirillov}} &= \int d\sigma \Theta(\sigma, \{f(\tau,\sigma)\}) \\ &= \int d\sigma d\tau \left[-b_0(\{f(\tau)\})\, f'\dot{f} + \frac{c}{48\pi} \frac{f'}{\dot{f}} \left(\frac{\dddot{f}}{\dot{f}} - 2\frac{\ddot{f}^2}{\dot{f}^2} \right) \right] \end{aligned} \tag{3.25}$$

where $\dot{f} = \partial_\tau f, f' = \partial_\sigma f$ etc. Here c represents a possible central term in the coadjoint representation of *Diff* [29, 30, 95]; b_0 is an arbitrary functional, representing the choice of a reference point on the orbit (different inequivalent orbits correspond to different inequivalent choices of b_0.)

[9] One way to appreciate this is to regard the Euclidean time as a discrete lattice and think of the ‘time’ derivatives in terms of discrete differences $f'(\tau) \sim f_{i+1} - f_i$ where $f(\tau)$ is regarded as a collection of constant zero modes f_i.

[10] See [1] for a detailed treatment of the boundary wavefunction which represents the CFT data accurately.

It was observed in [29, 30] that, with the choice $b_0 = 0$ (we discuss this more later), the Kirillov action becomes the same as the two-dimensional quantum gravity action of Polyakov [32]

$$S[g] = \frac{c}{24\pi} \int_\Gamma \sqrt{g} R \, \frac{1}{\Box} R \tag{3.26}$$

where the metric is [29][11]

$$ds^2 = \partial_\sigma f \; d\tau d\sigma \tag{3.27}$$

Here, R is the Ricci scalar of the geometry, $\frac{1}{\Box}$ is a notation used for the inverse of the scalar Laplacian in the geometry.

3.3.2 Two-dimensional quantum gravity action

It is rather remarkable that the two-dimensional quantum gravity action of Polyakov emerges from the quantization of the *Diff* configuration space.[12] Identifying such a quantization with the holographic path integral, as mentioned in the previous subsection, one would tend to identify (3.26) with a possible bulk dual for the Nambu-Goldstone sector of the SYK model. This does not work, however, since the action (3.26) does not have a cosmological constant and therefore pertains to asymptotically flat spaces without a boundary. To qualify as the bulk dual, the classical action must admit asymptotically AdS_2 spaces as solutions. Is there a natural generalization of the Polyakov action (3.26) which admits such solutions?

It turns out that there is such an action, given by[13]

$$S_{cov}[g] = \frac{1}{16\pi b^2} \int_\Gamma \sqrt{g} \left[R \, \frac{1}{\Box} R - 16\pi\mu \right] + \frac{1}{4\pi b^2} \int_{\partial\Gamma} \sqrt{\gamma} \mathcal{K} \, \frac{1}{\Box} R + \frac{1}{4\pi b^2} \int_{\partial\Gamma} \sqrt{\gamma} \mathcal{K} \, \frac{1}{\Box} \mathcal{K} \tag{3.28}$$

Here $\mathcal{K}$ is the extrinsic curvature of the boundary. The constant $b^2 = \frac{3}{2c}$ is the dimensionless Newton's constant in two dimensions; we are interested in the classical limit $b \to 0$. A bulk cosmological constant, $(-\mu) < 0$[14], is also included (to accommodate asymptotically AdS_2 spaces). The boundary terms are dictated by the requirement of a well-defined variational principle (see Appendix G for derivation); these terms can also be independently derived from the considerations of Weyl anomaly on manifolds with a boundary, see Appendix K.

[11] The function $f(\tau, \sigma)$ here should be compared with $F(x, t)$ of [29]

[12] In the foregoing discussion, the fact that the *Diff* symmetry is slightly broken does not appear to be taken into account. Shortly we discuss how the broken *Diff* symmetry gets incorporated from the 2D gravity perspective.

[13] One might wonder whether other non-local terms like $\left(\frac{1}{\Box} R\right)^n$, $n \in \mathbb{Z}^+$ are allowed in the action. It can be shown that including such higher order terms in general leads to equations of motion that do not admit an asymptotically AdS_2 spacetime.

[14] We have already incorporated a negative sign while writing the action, thus leaving $\mu > 0$

We have presented a discussion of the quantum corrections contributing to the action in Appendix J.

The main proposal in the work [2], on which this chapter is based, is that the modified quantum gravity action (3.28) describes a bulk dual of the low energy sector of the SYK model. The remaining chapter is a presentation of the evidence in support of this conjectured duality.

The next section discusses the above action in more detail. We will discuss in the subsequent section the *Diff* orbit of AdS_2 (asymptotically AdS_2 metrics) in detail, and show that they are solutions of the equations of motion. We should note that the specific realization of this *Diff* orbit will differ somewhat from that of the above discussion. The most important difference is that in the above discussion (which assumes spacetime without a boundary) various points of the *Diff* orbit are actually diffeomorphic in 2D; in our construction below, the *Diff* orbits involve *large diffeomorphisms in 2D* which are nontrivial near the boundary, and hence constitute physically distinguished configurations.

Before we proceed we would like to emphasize following points:

1. The action (3.26) involves the dynamical variables $f(\tau, \sigma)$ representing the loop space $L(\mathit{Diff})$ (more precisely, $L(M)$, $M = \mathit{Diff}/\mathbb{SL}(2,\mathbb{R})$). It describes a quantization of M, which is different from simply integrating over M. The latter emerges in the description of the pseudo-Nambu-Goldstone modes of the SYK model. It is possible to identify the quantization of M as the two-dimensional boundary dual to gravity on AdS_3 (see, e.g., [97]).[15]

2. In this work, however, we consider a different variant of the model, namely (3.28), which, in addition to the term in (3.26) includes a negative cosmological constant and boundary terms, and consequently defines a theory of gravity in asymptotically AdS_2 spaces.

3. As we will find, the only physical degrees of freedom of (3.28), reduce to M, parametrized by $f(\tau)$ (see, e.g. (3.48)) which lives on the boundary. The bulk-boundary correspondence in this case essentially follows from two-dimensional diffeomorphism (this is somewhat reminiscent of Chern-Simon theories on a manifold with boundaries, or of $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality). We will also find that the action describing the modes $f(\tau)$ is the Schwarzian action of SYK-type model and that the low temperature thermodynamics also have qualitative agreement with that of SYK.

4. We would like to emphasize that while (3.26), in the gauge (3.27), arises from a coadjoint orbit action of *Diff*, we do not yet have an explicit proof that our proposed

[15] We thank D. Stanford and E. Witten for illuminating correspondences on these points.

bulk dual, described by (3.28), is also a coadjoint orbit action of *Diff* for asymptotically AdS_2 geometries in some gauge. While this may eventually turn out to be true, the verification of our proposed duality in the rest of chapter is independent of such a connection.

3.4 Solutions of equations of motion and the Liouville action

This section starts with a discussion of the equations of motion arising from the action (3.28). The solutions describe spacetimes of constant negative curvature, which include AdS_2 as well as a three-parameter 'non-normalizable' deformation, which correspond to geometries whose boundary is displaced with respect to the original boundary of AdS_2. We will subsequently discuss the on-shell action.

Equations of motion

We now discuss the solutions of the above action, (3.28). The details of the computations of the equations of motion have been relegated to Appendix G and only the important results are summarized here. The equations of motion are,

$$\begin{aligned} 0 = \frac{1}{16\pi b^2}\Bigg(& g_{\mu\nu}(w)\Big(2R(w)+8\pi\mu\Big) + \int_\Gamma^x \Big[-2\nabla_\mu^{(w)}\nabla_\nu^{(w)} G(w,x)R(x)\Big] \\ & + \int_\Gamma^x\int_\Gamma^y \left[\frac{\partial G(w,x)}{\partial w^\mu}\frac{\partial G(w,y)}{\partial w^\mu} - \frac{1}{2}g_{\mu\nu}(w)g^{\alpha\beta}(w)\frac{\partial G(w,x)}{\partial w^\alpha}\frac{\partial G(w,y)}{\partial w^\beta}\right]R(x)R(y)\Bigg) \end{aligned} \tag{3.29}$$

It is more instructive to study the trace and traceless part of the equations separately,[16]

$$\text{Trace part:} \quad R(x) = -8\pi\mu \tag{3.30}$$

$$\begin{aligned} \text{Traceless part:} \quad 0 = & \int_\Gamma^x \left[-2\left(\nabla_\mu^{(w)}\nabla_\nu^{(w)} G(w,x) - \frac{1}{2}g_{\mu\nu}(w)\Box^{(w)}G(w,x)\right)R(x)\right] \\ & + \int_\Gamma^x\int_\Gamma^y \left[\frac{\partial G(w,x)}{\partial w^\mu}\frac{\partial G(w,y)}{\partial w^\mu} - \frac{1}{2}g_{\mu\nu}(w)g^{\alpha\beta}(w)\frac{\partial G(w,x)}{\partial w^\alpha}\frac{\partial G(w,y)}{\partial w^\beta}\right]R(x)R(y) \end{aligned} \tag{3.31}$$

Note that since $\mu > 0$, the first equation, (3.30), signifies that the metric must have a constant negative curvature, which of course includes AdS_2. However, one still needs to check if AdS_2 still satisfies (3.31). Moreover, one should also ask what is the most general solutions to these equations?

[16]We will subsequently write the action, (3.28) itself as sum over the trace and traceless part.

In the following paragraphs we summarize the mail results, while the details are discussed in Appendix G. Let us write the metric in conformal gauge around an AdS_2 background, $g_{\alpha\beta} = e^{2\phi}\hat{g}_{\alpha\beta}$, where

$$\widehat{ds^2} \equiv \hat{g}_{\alpha\beta}dx^\mu dx^\nu = \frac{1}{\pi\mu(z+\bar{z})^2}\,dz\,d\bar{z} = \frac{1}{4\pi\mu\zeta^2}\left(d\zeta^2 + d\tau^2\right) \tag{3.32}$$

Eq. (3.30) then becomes the same as Liouville equation of motion (see below for detail),

$$2\hat{\Box}\phi = \hat{R} + 8\pi\mu e^{2\phi} \tag{3.33}$$

which has the general solution [98, 99],

$$\phi = \frac{1}{2}\log\left[(z+\bar{z})^2\frac{\partial g(z)\bar{\partial}\bar{g}(\bar{z})}{(g(z)+\bar{g}(\bar{z}))^2}\right] \tag{3.34}$$

where $g, \bar{g}$ are arbitrary complex functions, conjugate of each other.[17] In the same gauge and background, (3.31) gives us the following *Virasoro constraints* ,

$$\partial^2\phi(z,\bar{z}) - (\partial\phi(z,\bar{z}))^2 + 2\frac{\partial\phi(z,\bar{z})}{z+\bar{z}} = 0, \qquad \bar{\partial}^2\phi(z,\bar{z}) - \left(\bar{\partial}\phi(z,\bar{z})\right)^2 + 2\frac{\bar{\partial}\phi(z,\bar{z})}{z+\bar{z}} = 0 \tag{3.35}$$

Solving (3.30) and (3.31) (or, equivalently (3.35)) simultaneously, one gets solutions (3.34) with following conditions on $g, \bar{g}$,

$$\{g(z), z\} = 0, \quad \{\bar{g}(\bar{z}), \bar{z}\} = 0 \Rightarrow g(z) = \frac{az+ib}{icz+d}, \quad \bar{g}(\bar{z}) = \frac{\bar{a}\bar{z}-i\bar{b}}{-i\bar{c}\bar{z}+\bar{d}}, \quad a,b,c,d \in \mathbb{C} \;\; {}^{18} \tag{3.36}$$

Here, and subsequently in this chapter we denote the Schwarzian derivative of a function, $f(\tau)$, by $\{f(\tau),\tau\} = \frac{f'''(\tau)}{f'(\tau)} - \frac{3}{2}\left(\frac{f''(\tau)}{f'(\tau)}\right)^2$. Of these solutions, the choice $a,b,c,d \in \mathbb{R}$ corresponds to $\mathbb{SL}(2,\mathbb{R})$transformations of AdS_2 coordinates, and are the exact isometries of the geometry.

The remaining 3-parameter set of solutions, which corresponds to the point marked NAdS_2 in Figure 3.1 are the solutions of our primary interest. These do not preserve the boundary of AdS_2. In general, the boundary of the spacetime is given by the curve, $g(z)+\bar{g}(\bar{z}) = 0$, which for a general function of the kind, (3.36), is not the same as $z+\bar{z} = 0$. These solutions will subsequently be referred to as *non-normalizable* solutions following the standard AdS/CFT language.

The set of non-normalizable solutions obtained above is parameterized by $(a,b,c,d) \in \mathbb{SL}(2,\mathbb{C})/\mathbb{SL}(2,\mathbb{R})$, which can be identified with a hyperboloid (see Appendix H, especially

[17] In Lorentzian signature, these functions can be chosen to be two independent real functions.

[18] Here the independent set of parameters are constrained by $ad + bc = 1$, which is the same as $\mathbb{SL}(2,\mathbb{C})$.

(H.7) for more details). The point $(a, b, c, d) = (1, 0, 0, 1)$ corresponds to the identity transformation $g(z) = z$ in (3.36). We are interested in *small* non-normalizable deformations near the identity transformation. It is possible to choose a set of coordinates of $\mathbb{SL}(2, \mathbb{C})/\mathbb{SL}(2, \mathbb{R})$, in which such deformations are given by

$$a = 1 + i\,\delta a^I \quad b = i\,\delta b^I \quad c = i\,\delta c^I \quad d = 1 - i\,\delta a^I \quad , \tag{3.37}$$

where $\delta a^I, \delta b^I, \delta c^I$ are real numbers. With these parameters, the solution for the metric becomes

$$ds^2 = e^{2\phi}\widehat{ds^2} \tag{3.38}$$

with $\widehat{ds^2}$ given by the AdS_2 metric (3.32), and ϕ, using (3.34) and (3.36) has the near-boundary form

$$\phi = -\frac{\delta g(i\tau)}{\zeta} + \mathcal{O}(\delta a^2, \delta b^2, \delta c^2), \qquad -\delta g(i\tau) = \delta b^I + 2\delta a^I \tau + \delta c^I \tau^2 \tag{3.39}$$

Eventually, $\delta a^I = \delta c^I = 0$ is chosen, so that $\delta g = -\delta b^I$, and $\phi = \delta b^I/\zeta$. We will find that the δb^I deformation (more precisely, $-\delta b^I$) corresponds to the irrelevant coupling $1/J$ of the SYK model. The other parameters δa^I and δc^I are physically distinct; it would be interesting to explore their significance, which we leave for future work.

For the Liouville factor $e^{2\phi}$ not to destroy the asymptotic AdS_2 structure altogether, we will assume here that $\delta g \lesssim \delta$;[19] this ensures that $\delta g < \zeta$.[20] Note that the expression for the Liouville field in (3.39) is similar to that of the dilaton in [25], and plays a somewhat similar role as we will see later. In the next section, more solutions are generated from the above three-parameter solutions by using large diffeomorphisms, which we cannot capture staying within the conformal gauge.

Liouville action

We now show that the above analysis of equations of motion with separation into trace and traceless parts also works for the classical action. Writing the induced gravity action in a

[19] δ is a radial cut-off that we have introduced to regulate the UV divergences in the bulk computations, akin to ϵ_0 of previous chapter.

[20] There is a natural RG interpretation of this inequality in terms of the boundary theory. We will later identify δg with $\sim 1/J$ (see (3.81)). Together with the natural identification of $1/\zeta$, for small ζ, with a Wilsonian floating cut-off Λ (to be distinguished from the bare cut-off $\Lambda_0 = 1/\delta$, see [15, 16], also [1]), we find $\delta g/\zeta \sim \Lambda/J = 1/\bar{J}$, where $\bar{J} = J/\Lambda$ is the dimensionless coupling. Since $\bar{J}$ grows large near the IR cut-off, it follows that $\delta g/\zeta \ll 1$ near the IR cut-off.

conformal gauge around an arbitrary fiducial metric, $\hat{g}_{\alpha\beta}$, one gets the action,[21]

$$
\begin{aligned}
S_{cov}[g] = -\frac{1}{4\pi\, b^2}\Bigg[&\int_\Gamma \sqrt{\hat{g}}\Big(\hat{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \hat{R}\phi + 4\pi\mu e^{2\phi}\Big) + 2\int_{\partial\Gamma}\sqrt{\hat{\gamma}}\hat{\mathcal{K}}\phi + \int_{\partial\Gamma}\sqrt{\hat{\gamma}}\hat{n}^\mu\phi\partial_\mu\phi \\
&- \frac{1}{2}\int_{\partial\Gamma}\sqrt{\hat{\gamma}}\hat{n}^\mu\,\hat{\nabla}_\mu\Big(\phi\frac{1}{\hat{\Box}}\hat{R}\Big)\Bigg] + \frac{1}{16\pi\, b^2}\int_\Gamma\sqrt{\hat{g}}\hat{R}\,\frac{1}{\hat{\Box}}\hat{R} + \frac{1}{4\pi\, b^2}\int_{\partial\Gamma}\sqrt{\gamma}\hat{\mathcal{K}}\,\frac{1}{\hat{\Box}}\hat{R} \\
= -\frac{1}{4\pi\, b^2}\Bigg[&\int_\Gamma \sqrt{\hat{g}}\Big(\hat{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \hat{R}\phi + 4\pi\mu e^{2\phi}\Big) + 2\int_{\partial\Gamma}\sqrt{\hat{\gamma}}\hat{\mathcal{K}}\phi + \int_{\partial\Gamma}\sqrt{\hat{\gamma}}\hat{n}^\mu\phi\partial_\mu\phi\Bigg] \\
&+ \frac{1}{16\pi\, b^2}\int_\Gamma\sqrt{\hat{g}}\hat{R}\,\frac{1}{\hat{\Box}}\hat{R}
\end{aligned}
\tag{3.40}
$$

In all the above equations, the coordinate dependence of the functions is understood. In the second line above, the boundary terms containing the Green's function, $\frac{1}{\hat{\Box}}$, have been dropped, given the fall-off properties of the Green's function. The part of the action in (3.40) which depends on ϕ field can be identified with Liouville action on a background with metric $\hat{g}$.

$$
S_L[\phi,\hat{g}] = -\frac{1}{4\pi\, b^2}\Bigg[\int_\Gamma \sqrt{\hat{g}}\Big(\hat{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \hat{R}\phi + 4\pi\mu e^{2\phi}\Big) + 2\int_{\partial\Gamma}\sqrt{\hat{\gamma}}\hat{\mathcal{K}}\phi + \int_{\partial\Gamma}\sqrt{\hat{\gamma}}\hat{n}^\mu\phi\partial_\mu\phi\Bigg]
\tag{3.41}
$$

We are interested in computing the above action in the classical limit, $b \to 0$. The classical equation of motion for the ϕ field turns out to be exactly the same as (3.33), the trace part of the equations of motion coming from the Polyakov action, as expected. We emphasize the fact that if one chooses to study (3.40) as a theory of quantum gravity, then the trace of background metric appearing there should not be treated as independent degree of freedom.

One-dimensional Liouville equation of motion has appeared in [68, 73, 100] in the context of boundary dynamics. However, their connection to the induced gravity action that we have discussed here is not clear.

No dynamical Liouville mode: It is important to note that in the above system there are no dynamical Liouville modes at all. The Liouville mode is entirely fixed in terms of three parameters which, furthermore, correspond to non-normalizable modes. These are specified as boundary conditions of the path integral and are not dynamical variables. This point is elaborated further in Appendix H where it is shown that the form of the Liouville mode, with three real constants, is completely fixed by the Virasoro constraints alone.

[21]Later in this chapter we will choose the fiducial metric from a class of Asymptotic AdS_2 ($AAdS_2$) geometries. Although none of the analysis depends on the choice of this fiducial metric, it is only economical for a classical analysis that we choose it to be one of the saddle point solutions.

3.5 Asymptotically AdS_2 geometries

In this section, we will construct asymptotically AdS_2 geometries as a *Diff* orbit of the solutions constructed in (3.38) (*see the orbits in the right panel of Figure 3.1*). To begin with, we will construct these asymptotic geometries purely kinematically, from an analysis of asymptotic Killing vectors (AKV) of AdS_2 geometry (also see Appendix I for some details). Later, we argue that they solve the equations of motion and evaluate the on-shell action for these configurations. AKV's of AdS_2 have been studied earlier in [101, 102] in the near-boundary region, inspired by earlier work of Brown and Henneaux in one higher dimension [103]. We show below that it is possible to integrate the infinitesimal diffeomorphisms exactly to find the full nonlinear solution. This will lead to a class of $AAdS_2$ geometries that are related to each other by diffeomorphisms that become tangential at the boundary. These geometries are dual to the conformally transformed states in the 1-D field theory.[22] We mainly consider Euclidean metrics below.

Euclidean AdS_2 metric in Poincare coordinates is defined by (3.32). The $AAdS_2$ geometries are defined by the fall-off conditions [101–103],

$$g_{\zeta\zeta} = \frac{1}{4\pi\mu\,\zeta^2} + \mathcal{O}(\zeta^0), \quad g_{\zeta\tau} = \mathcal{O}(\zeta^0), \quad g_{\tau\tau} = \frac{1}{4\pi\mu\,\zeta^2} + \mathcal{O}(\zeta^0) \tag{3.42}$$

Variation of the metric under most general diffeomorphism is,

$$\delta g_{\alpha\beta} = \nabla_\alpha \epsilon_\beta + \nabla_\beta \epsilon_\alpha = \begin{pmatrix} -\dfrac{\epsilon^\zeta(\zeta,\tau) - \zeta\partial_\zeta\epsilon^\zeta(\zeta,\tau)}{2\pi\mu\zeta^3} & \dfrac{\partial_\tau\epsilon^\zeta(\zeta,\tau) + \partial_\zeta\epsilon^\tau(\zeta,\tau)}{4\pi\mu\zeta^2} \\ \dfrac{\partial_\tau\epsilon^\zeta(\zeta,\tau) + \partial_\zeta\epsilon^\tau(\zeta,\tau)}{4\pi\mu\zeta^2} & -\dfrac{\epsilon^\zeta(\zeta,\tau) - \zeta\partial_\tau\epsilon^\tau(\zeta,\tau)}{2\pi\mu\zeta^3} \end{pmatrix} \tag{3.43}$$

The asymptotic Killing vectors can be solved for by imposing on (3.43) the fall-off conditions in (3.42), [101, 102]. However, we choose to work in Fefferman-Graham gauge which is defined by,

$$\delta g_{\zeta\zeta} = 0, \quad \delta g_{\zeta\tau} = 0 \tag{3.44}$$

The solution for the asymptotic Killing vectors is given in terms of an arbitrary function, $\delta f(\tau)$,

$$\epsilon^\zeta(\zeta,\tau) = \zeta\delta f'(\tau), \quad \epsilon^\tau(\zeta,\tau) = \delta f(\tau) - \frac{1}{2}\zeta^2\delta f''(\tau) \tag{3.45}$$

[22] As indicated before, precisely at the conformal point, the stress tensor vanishes trivially; hence all states are ground states. However, slightly away from the conformal point, the (broken) conformal transformations lead to nontrivial states.

It is clear from the above solution, that the diffeomorphism is tangential at the boundary of AdS_2, $\zeta = 0$. The integrated form of the coordinate transformations is,

$$\tilde{\tau} = f(\tau) - \frac{2\zeta^2 f''(\tau) f'(\tau)^2}{4f'(\tau)^2 + \zeta^2 f''(\tau)^2}, \quad \tilde{\zeta} = \frac{4\zeta f'(\tau)^3}{4f'(\tau)^2 + \zeta^2 f''(\tau)^2} \tag{3.46}$$

Although we think that this choice of gauge should not be necessary and it should be possible to integrate the diffeomorphisms more generally, we found it easier to do so with this gauge choice. This was largely motivated by [104, 105] who performed similar integrations of diffeomorphisms in AdS_3 case. The details of this computation are presented in Appendix I.

The result of this diffeomorphism can be stated as follows. If we start with the AdS_2 metric in the $\tilde{\zeta}$-$\tilde{\tau}$ coordinates

$$\widehat{ds^2} = \frac{1}{4\pi\mu\,\tilde{\zeta}^2}\Big(d\tilde{\zeta}^2 + d\tilde{\tau}^2\Big),$$

in the original ζ-τ coordinates it becomes

$$\widehat{ds^2} = \frac{1}{4\pi\mu\,\zeta^2}\left(d\zeta^2 + d\tau^2\left(1 - \zeta^2\frac{\{f(\tau),\tau\}}{2}\right)^2\right) \tag{3.47}$$

Recall that $\{f(\tau),\tau\} = \frac{f'''(\tau)}{f'(\tau)} - \frac{3}{2}\left(\frac{f''(\tau)}{f'(\tau)}\right)^2$ is the standard notation for Schwarzian derivative that we use throughout our discussion. We emphasize that the class of geometries given by (3.47) also have constant negative curvature, $\hat{R} = -8\pi\mu$. As in AdS_3, it should be possible to identify these geometries as different sections of the global AdS_2 geometry. Some discussion of how various AdS_2 geometries are related is provided in [102].

One can carry out the above diffeomorphism in the presence of the non-normalizable solutions described in the previous section. To do this, begin with the metric (3.38) in the $\tilde{\zeta}$-$\tilde{\tau}$ coordinates:

$$ds^2 = e^{2\tilde{\phi}(\tilde{x}^\mu)}\widehat{ds^2}, \quad \tilde{\phi}(\tilde{x}^\mu) = \frac{\tilde{\delta g}}{\tilde{\zeta}} + \mathcal{O}(\delta a^2, \delta b^2, \delta c^2),\ \tilde{\delta g} = \mathrm{Im}(\delta b) + 2\,\mathrm{Im}(\delta a)\tilde{\tau} + \mathrm{Im}(\delta c)\tilde{\tau}^2$$

and transform to ζ-τ coordinates, yielding the metric

$$\begin{aligned} ds^2 &= e^{2\phi}\widehat{ds^2},\ \widehat{ds^2} = \frac{1}{4\pi\mu\,\zeta^2}\left(d\zeta^2 + d\tau^2\left(1 - \zeta^2\frac{\{f(\tau),\tau\}}{2}\right)^2\right), \\ \phi &= -\frac{\delta g(i\tilde{\tau})}{\tilde{\zeta}(\zeta,\tau)} + \mathcal{O}(\delta a^2, \delta b^2, \delta c^2),\ -\delta g(i\tilde{\tau}) = \delta b^I + 2\delta a^I\tilde{\tau} + \delta c^I\tilde{\tau}^2,\ \tilde{\tau} = f(\tau) \end{aligned} \tag{3.48}$$

In terms of the Figure 3.1, the above solutions (3.47), (3.48) represent the Diff orbit of AdS_2 and NAdS_2 on the right panel.

As remarked below (3.39), only the one-parameter deformation parameterized by δb^I is eventually chosen, which will turn out to correspond to the $1/J$ deformation of the strong coupling fixed point of the SYK theory. However, for the sake of generality, we will for now continue with the more general form of δg.

3.5.1 Proper treatment of the bulk path integral

To this point we have not discussed the issue of gauge fixing inside the quantum mechanical path integral. While we are largely interested in a classical computation in the bulk, where the path integral measure due to gauge fixing is not important, we now shed some light on this issue. The computation of the ghost action is discussed in detail in Appendix J. The gauge fixing δ-function and the corresponding Faddeev-Popov determinant is given by,

$$\begin{aligned} 1 = \Delta_{FP}\Big[\hat{g}[f(\tau)],\phi\Big] \times \int [\mathcal{D}\epsilon^{(s)}][\mathcal{D}\phi][\mathcal{D}f(\tau)]\ \delta\Big(g^{\epsilon^{(s)}} - e^{2\phi}\hat{g}[f(\tau)]\Big) \\ \times\, \delta\Big(\epsilon^{(s)}(z_1)\Big)\,\delta\Big(\epsilon^{(s)}(z_2)\Big)\,\delta\Big(\epsilon^{(s)}(z_3)\Big) \end{aligned} \tag{3.49}$$

In line with the discussion of the previous sections, we gauge fix an arbitrary metric to be conformally related to the AAdS$_2$ metrics. In the choice of this gauge, there is an additional $\mathbb{SL}(2,\mathbb{R})$residual gauge freedom that has been fixed using the δ-functions that anchor three arbitrary points in the geometry.[23] Going through the standard procedure of introducing the fermionic ghosts, we obtain a ghost action (J.3). This procedure should not only capture the correct Jacobian required for the gauge fixing, but also for defining an invariant measure on the space of $f(\tau)$ integrations.

With the above ingredients, the path integral is given by

$$Z = \int \frac{\mathcal{D}f(\tau)'}{f'(\tau)}\exp[-S_{hydro} + ...] \tag{3.50}$$

where S_{hydro} is the effective action (3.67), describing the hydrodynamic modes (see the next section). The terms in the ellipsis denote subleading terms which get contribution from the Faddeev-Popov determinant mentioned above and discussed in detail in Appendix J. The integration measure is the invariant integration measure in the space of $f(\tau)$ functions. The *prime* on the measure denotes the exclusion of the integration over $\mathbb{SL}(2,\mathbb{R})$ degrees of freedom due to the treatment of $\mathbb{SL}(2,\mathbb{R})$ modes discussed above.[24]

[23] This is the standard prescription followed in open-string path integral computations. See also the relevant discussion in [8].

[24] This measure should appear from a proper treatment of the Faddeev-Popov procedure which is sketched in Appendix J. We leave details of this to subsequent work.

3.6 Action of hydrodynamics modes

In this section, the on-shell action of the above geometries is computed to determine the contribution of the large diffeomorphisms to the partition function of the system.

3.6.1 Boundary action

It is clear from the analysis of equations of motion in Appendix G that all of $AAdS_2$ geometries satisfy the bulk equations of motion. Thus we can safely anticipate that the major contribution to the action of hydrodynamics modes will come from the boundary terms in (3.40). The boundary terms of the action are given by,

$$S_L^{bdy}[\phi, \hat{g}] = -\frac{1}{4\pi\, b^2}\left[2\int_{\partial\Gamma}\sqrt{\hat{\gamma}}\hat{\mathcal{K}}\phi + \int_{\partial\Gamma}\sqrt{\hat{\gamma}}\hat{n}^{\mu}\phi\partial_{\mu}\phi\right] \tag{3.51}$$

The second term above doesn't contribute at the leading order. The contribution of this term starts at $\mathcal{O}(\delta g)^2$ and hence won't contribute to the leading order answers that are subsequently computed.

We also emphasize on the correct way to regulate the geometries for the subsequent computations. To keep the notations unambiguous, the coordinates of AdS_2 are denoted by $\tilde{\zeta}, \tilde{\tau}$ and that of $AAdS_2$ geometries by ζ, τ. We know that $AAdS_2$ geometries are related to AdS_2 geometry by large diffeomorphisms. Hence, the application of these large diffeomorphisms on a radial cut-off in AdS_2 at $\tilde{\zeta} = \delta$ maps the boundary at constant $\tilde{\zeta}$ to some *wiggly-curves* in ζ-τ coordinates,[25]

$$\begin{aligned}
\delta &= \frac{4\zeta f'(\tau)^3}{4f'(\tau)^2 + \zeta^2 f''(\tau)^2} \\
\Rightarrow \zeta &= \frac{2\, f'(\tau)}{\delta\, f''(\tau)^2}\left[f'(\tau)^2 - \sqrt{f'(\tau)^4 - \delta^2\, f''(\tau)^2}\right]
\end{aligned} \tag{3.52}$$

These are the same wiggles as discussed in [25]. To consider physically distinct geometries in ζ-τ coordinates, we put a cut-off at $\zeta = \delta$ and compare the action with that of geometry corresponding to $\tilde{\zeta} = \delta$.

[25] There are two solutions for ζ satisfying $\tilde{\zeta} = \delta$ (because the second equation in (3.46) is a quadratic in ζ), one of which doesn't satisfy the boundary condition, $\zeta \to 0$ as $\delta \to 0$, and hence is unphysical.

In AdS$_2$ On the boundary $\tilde{\zeta} = \delta$, $\sqrt{\hat{\gamma}}\hat{\mathcal{K}} = \frac{1}{\delta}$

$$S_L^{bdy}[\tilde{\phi}, \tilde{g}_{\alpha\beta}] = -\frac{1}{2\pi\, b^2}\int_{\partial\Gamma} \sqrt{\hat{\gamma}}\hat{\mathcal{K}}\tilde{\phi} = -\frac{1}{2\pi\, b^2}\int d\tilde{\tau}\Bigg[\left(\frac{\delta g(i\tilde{\tau})}{\delta^2} - \frac{1}{2}\,\delta g''(i\tilde{\tau}) + \mathcal{O}(\delta^2)\right) + \mathcal{O}\big[\delta g(i\tilde{\tau})^2\big]\Bigg] \tag{3.53}$$

where, $\delta g(\tilde{\tau})$ was defined in (3.39). To be able to compare with the AAdS$_2$ answer later, we do the coordinate transformation from $\tilde{\tau} \to \tau$ coordinates,

$$\begin{aligned} S_L^{bdy}[\tilde{\phi}, \tilde{g}_{\alpha\beta}] &= -\frac{1}{2\pi\, b^2}\int_{\partial\Gamma} \sqrt{\hat{\gamma}}\hat{\mathcal{K}}\tilde{\phi} \\ &= -\frac{1}{2\pi\, b^2\,\delta}\int d\tau \frac{\partial\tilde{\tau}(\tau)}{\partial\tau}\Bigg[\left(\frac{\delta g(i\tilde{\tau}(\tau))}{\delta} - \frac{\delta}{2}\,\delta g''(i\tilde{\tau}(\tau)) + \mathcal{O}(\delta^2)\right) + \mathcal{O}\big[\delta g(i\tilde{\tau}(\tau))^2\big]\Bigg] \end{aligned} \tag{3.54}$$

$$\tilde{\tau} = f(\tau) - \frac{f'(\tau)^2}{f''(\tau)}\left(1 - \sqrt{1 - \delta^2\left(\frac{f''(\tau)}{f'(\tau)^2}\right)^2}\right) \tag{3.55}$$

Here, it is important to note that we need to implement the coordinate transformation at the $\tilde{\zeta} = \delta$ slice. To this effect, we need to solve for ζ at $\tilde{\zeta} = \delta$ using the second equation in (3.46) and substitute it back in the first equation there.

In AAdS$_2$ On the boundary $\zeta = \delta$, $\sqrt{\hat{\gamma}}\hat{\mathcal{K}} = \frac{1}{\delta} + \delta\frac{\{f(\tau),\tau\}}{2}$

$$\begin{aligned} S_L^{bdy}[\phi, g_{\alpha\beta}] = -\frac{1}{2\pi\, b^2}\int_{\partial\Gamma} \sqrt{\hat{\gamma}}\hat{\mathcal{K}}\phi = -\frac{1}{2\pi\, b^2}\int d\tau\left[\frac{1}{\delta} + \delta\frac{\{f(\tau),\tau\}}{2}\right] \times \Bigg[\Bigg(\frac{1}{\delta}\,\frac{\delta g(if(\tau))}{f'(\tau)} \\ - \delta\,\frac{\Big(-\delta \mathrm{g}(if(\tau))f''(\tau)^2 + 2f'(\tau)^4\delta \mathrm{g}''(if(\tau)) + 2if'(\tau)^2 f''(\tau)\delta \mathrm{g}'(if(\tau))\Big)}{4f'(\tau)^3} \\ + \mathcal{O}(\delta^2)\Bigg) + \mathcal{O}\big[\delta g(if(\tau))^2\big]\Bigg] \end{aligned} \tag{3.56}$$

Hence,

$$\begin{aligned} \delta S_L^{bdy} = S_L^{bdy}[\phi, g_{\alpha\beta}] - S_L^{bdy}[\tilde{\phi}, \tilde{g}_{\alpha\beta}] = \frac{1}{2\pi\, b^2}\int d\tau\Bigg[\Bigg(\frac{\delta g(if(\tau))}{\delta^2}\left(f'(\tau) - \frac{1}{f'(\tau)}\right) \\ - \frac{\delta g(if(\tau))}{f'(\tau)}\{f(\tau),\tau\} + \mathcal{O}(\delta^2)\Bigg) + \mathcal{O}\big[\delta g(if(\tau))^2\big]\Bigg] \end{aligned} \tag{3.57}$$

The $O(1/\delta^2)$ divergent term can be subtracted by introducing following counterterm in the action (3.41),

$$S_{ct} = \frac{4\sqrt{\pi\mu}}{4\pi\, b^2}\int_{\partial\Gamma}\sqrt{\hat{\gamma}}\phi \tag{3.58}$$

which essentially replaces $\sqrt{\hat{\gamma}}\hat{\mathcal{K}} \to \sqrt{\hat{\gamma}}(\hat{\mathcal{K}}-1)$.[26] To the linear order in δg under consideration here, this is the same as the fully covariant counterterm $-4\sqrt{\pi\mu}\frac{1}{4\pi b^2}\int_{\partial\Gamma}\sqrt{\gamma}\,\frac{1}{\Box}R$. The finite part of the answer is,

$$\delta S_L^{bdy} = -\frac{1}{2\pi b^2}\int d\tau \frac{\delta g(if(\tau))}{f'(\tau)}\{f(\tau),\tau\} = \frac{1}{2\pi b^2}\int d\tilde{\tau}\;\delta g(i\tilde{\tau})\,\left\{\tilde{f}(\tilde{\tau}),\tilde{\tau}\right\} \tag{3.59}$$

Here the third term is written in terms of the $\tilde{\tau}$ coordinate, the boundary coordinate of the unperturbed AdS_2[27] Also, note that we have defined $\tilde{f}(\tilde{\tau}) = \tau$ as the reparametrized coordinate starting with the unperturbed AdS_2coordinate $\tilde{\tau}$.[28]

The function $\delta g(\tau)$ is given by (3.39). As indicated below that equation, henceforth δg = constant is chosen. One might wonder if one can absorb the τ and τ^2 deformations in δg, parameterized by δa^I and δc^I, by a possible reparameterization of the boundary coordinate τ; this, however, turns out impossible for any value of these parameters since the corresponding transformation turns out to be singular. Thus, the $\delta b^I, \delta a^I, \delta c^I$ represent different physics, and we will find that it is only the δb^I deformation, that is, a constant δg, which will correspond to the SYK model. It will be seen that the non-normalizable mode corresponding to constant δg, corresponds to the irrelevant coupling $1/J$ of the SYK model.

Section 3.8 details the matching of the above results with the boundary field theory. Note that the $\mathbb{SL}(2,\mathbb{R})$transformations that correspond to the 'global conformal transformations' of one dimensional space remain the symmetry of this action. Moreover, a discussion of the correct measure of integration over the $\tilde{f}(\tau)$ modes is presented in subsection 3.5.1 and Appendix J.

[26]A similar counterterm is also implied in [8] in removing a quadratic divergence from their computation of the Schwarzian term.

[27]In going from the second expression to the third term, we have first transformed to the time coordinate $\tilde{\tau} = f(\tau)$, with $\tau = \tilde{f}(\tilde{\tau})$, $\tilde{f} \equiv f^{-1}$, and used the Schwarzian composition rule

$$\{\tilde{f}(f(\tau)),\tau\} = \{\tilde{f}(f(\tau)),f(\tau)\}f'(\tau)^2 + \{f(\tau),\tau\} = \{\tilde{f}(\tilde{\tau}),\tilde{\tau}\}f'(\tau)^2 + \{f(\tau),\tau\},$$

The LHS equals $\{\tau,\tau\}$ and vanishes.

[28]It is important to note that the large diffeomorphism $\tilde{f}$ is what corresponds to the pseudo-Nambu-Goldstone mode f of the SYK model.

3.6.2 Bulk action

The bulk part of the Liouville action is,

$$\begin{aligned}
S_L^{bulk}[\phi,\hat{g}] &= -\frac{1}{4\pi\, b^2}\int_\Gamma \sqrt{\hat{g}}\Big(\hat{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \hat{R}\phi + 4\pi\mu e^{2\phi}\Big)\\
&= -\frac{1}{4\pi b^2}\int_{\partial\Gamma}\sqrt{\hat{\gamma}}\hat{n}^\alpha\phi\partial_\alpha\phi - \frac{1}{4\pi b^2}\int_\Gamma\sqrt{\hat{g}}\Big(-\phi\hat{\Box}\phi + \hat{R}\phi + 4\pi\mu e^{2\phi}\Big)\\
&= -\frac{1}{4\pi b^2}\int_{\partial\Gamma}\sqrt{\hat{\gamma}}\hat{n}^\alpha\phi\partial_\alpha\phi - \frac{1}{4\pi b^2}\int_\Gamma\sqrt{\hat{g}}\Big(\frac{1}{2}\hat{R}\phi + 4\pi\mu e^{2\phi}(1-\phi)\Big)\\
&= -\frac{1}{4\pi b^2}\int_{\partial\Gamma}\sqrt{\hat{\gamma}}\hat{n}^\alpha\phi\partial_\alpha\phi - \frac{\mu}{b^2}\int_\Gamma\sqrt{\hat{g}}\Big(-\phi + e^{2\phi}(1-\phi)\Big)
\end{aligned} \tag{3.60}$$

here in the second line we have shifted the derivatives, while in the second term we have used the equation of motion of the ϕ field, $\hat{\Box}\phi = \frac{1}{2}\hat{R} + 4\pi\mu e^{2\phi}$. In the last line, the value of $\hat{R} = -8\pi\mu$ has been substituted. The first term in the above equation contributes at subleading order, as argued under (3.51). Using the on-shell value of the ϕ we evaluate the above action in AdS_2 and $AAdS_2$ backgrounds.

In AdS_2 For AdS_2 background metric, the action is given by,

$$\begin{aligned}
S_L^{bulk}[\tilde{\phi},\tilde{g}_{\alpha\beta}] &= -\frac{\mu}{b^2}\int\limits_{-\infty}^{\infty} d\tilde{\tau}\int\limits_{\tilde{\zeta}=\delta}^{\infty} d\tilde{\zeta}\sqrt{\tilde{g}}\Big(-\tilde{\phi} + e^{2\tilde{\phi}}\Big(1-\tilde{\phi}\Big)\Big)\\
&= -\frac{\mu}{b^2}\int\limits_{-\infty}^{\infty} d\tau\int\limits_{\zeta>\partial\Gamma}^{\infty} d\zeta\sqrt{g}\Big(-\phi + e^{2\phi}(1-\phi)\Big)
\end{aligned} \tag{3.61}$$

Here, in the second line we have used the coordinate transformations, (3.46) and the boundary in ζ coordinates is now given by the wiggly curve, (3.52),

$$\partial\Gamma \equiv \zeta = \frac{2}{\delta\ f''(\tau)^2}\Big[f'(\tau)^3 - \sqrt{f'(\tau)^6 - \delta^2\, f'(\tau)^2\, f''(\tau)^2}\Big]$$

in $AAdS_2$ Similarly for the $AAdS_2$ background we have the action,

$$S_L^{bulk}[\phi, g_{\alpha\beta}] = -\frac{\mu}{b^2}\int\limits_{-\infty}^{\infty} d\tau\int\limits_{\zeta=\delta}^{\infty} d\zeta\sqrt{g}\Big(-\phi + e^{2\phi}(1-\phi)\Big) \tag{3.62}$$

Thus we have,

$$\delta S_L^{bulk} = S_L^{bulk}[\phi, g_{\alpha\beta}] - S_L^{bulk}[\tilde{\phi},\tilde{g}_{\alpha\beta}] = -\frac{\mu}{b^2}\int\limits_{-\infty}^{\infty} d\tau\int\limits_{\delta}^{\zeta=\partial\Gamma} d\zeta\sqrt{g}\Big(-\phi + e^{2\phi}(1-\phi)\Big) \tag{3.63}$$

It is easy to approximate this expression close to the boundary of the geometry, *i.e.* when $\delta \to 0$. In that case the difference between $\zeta = \partial\Gamma$ and $\zeta = \delta$ reduces to a small strip as shown in Figure 3.7.

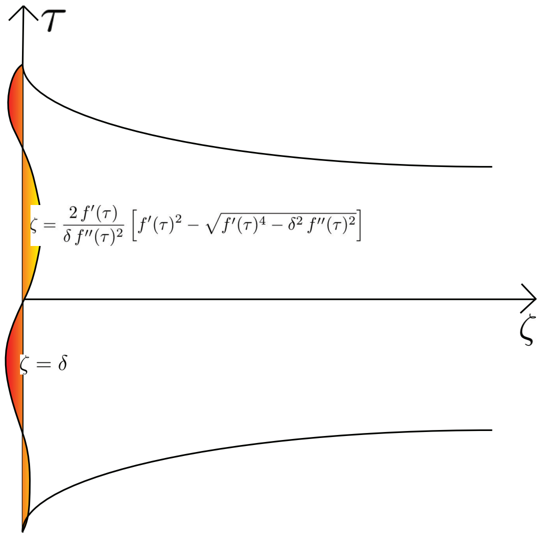


FIGURE 3.7: The difference in bulk action between AdS_2 and $AAdS_2$ geometries gets contribution only from the shaded region

Moreover $(-\phi + e^{2\phi}(1-\phi)) \sim 1 + \mathcal{O}[\delta^3, \delta g(i\tau)^2]$, and hence, that part of the integrand becomes trivial. The integrand can be approximated by,

$$\delta S_L^{bulk} = \frac{1}{4\pi b^2}\int_{-\infty}^{\infty} d\tilde{\tau}\left[\frac{1}{\delta}\left(\tilde{f}'(\tilde{\tau}) - 1\right) + \frac{\delta}{4}\left(\frac{\left(2\tilde{f}'(\tilde{\tau}) - 3\right)\tilde{f}''(\tilde{\tau})^2 - 2\tilde{f}'''(\tilde{\tau})\left(\tilde{f}'(\tilde{\tau}) - 1\right)\tilde{f}'(\tilde{\tau})}{\tilde{f}'(\tau)^3}\right)\right] \tag{3.64}$$

The first term is linearly divergent,[29] however while considering the coordinate transformations which approach identity transformations asymptotically this term integrates to zero. In other words if we consider a transformation, $\tilde{f}(\tilde{\tau}) = \tilde{\tau} + \epsilon(\tilde{\tau})$[30], then $\tilde{f}'(\tilde{\tau}) = 1 + \epsilon'(\tilde{\tau})$. In this case the first term becomes,

$$\frac{1}{4\pi b^2}\int_{-\infty}^{\infty} d\tilde{\tau}\,\frac{1}{\delta}\epsilon'(\tilde{\tau}) = \frac{1}{4\pi b^2\,\delta}[\epsilon(\infty) - \epsilon(-\infty)] \tag{3.65}$$

Clearly a *good* coordinate transformation has to be monotonically increasing. Additionally we require, for the transformation to remain invertible, that $\epsilon(\infty) = 0 = \epsilon(-\infty)$. In fact, the transformation that is used to map the theory on a line to a theory on a thermal circle is not of this kind and the regulation scheme adopted in that case is explained in section 3.7.

[29] We thank Shiraz Minwalla for a crucial discussion on this point.
[30] Here $\epsilon(\tilde{\tau})$ is not necessarily small, but just a rewriting of the coordinate transformations.

Leaving aside the issue of the regulation, the reparametrization of a thermal quantum mechanical theory can be achieved starting from a quantum mechanical theory in two steps: firstly, the straight line is mapped to a thermal circle using the map $\tilde{\tau} = \tan\Big(\pi\tilde{\theta}/\beta\Big)$; then, one reparametrizes the thermal circle with appropriate boundary conditions, ensuring reparametrization doesn't change the winding around the circle and is invertible. In this case, the Schwarzian action becomes [8, 25]:

$$S^{\beta}_{hydro} = \frac{\delta g}{2\pi b^2}\int d\tilde{\theta}\left\{\frac{\beta}{2}\tan\left(\pi\frac{f(\tilde{\theta})}{\beta}\right), \tilde{\theta}\right\} \tag{3.66}$$

3.6.3 Summary

From the preceding discussion, the following low energy effective action (in the leading large $1/b$ limit) for the 'hydrodynamic modes' can be deduced,

$$S_{hydro} = \frac{\delta g}{2\pi b^2}\int d\tilde{\tau}\ \left\{\tilde{f}(\tilde{\tau}), \tilde{\tau}\right\} \tag{3.67}$$

In section 3.8 we will compare this with the Schwarzian term which appears in the SYK-type models.

It is important to mention that the bulk dual discussed in [25, 26], leads to a similar Schwarzian term starting from a dilaton gravity model, while the bulk dual presented here has only the metric field described by the Polyakov action. The source of the hydrodynamic modes in both cases involves the *large diffeomorphisms* which are nontrivial at the boundary. In a very recent paper [106], another proposal for a bulk dual has appeared which has a Liouville field and the Almheiri-Polchinski action, [26]. They also appear to get a Schwarzian term rather differently, from the Liouville fluctuations similar to our functions $g(z), \bar{g}(\bar{z})$ in (3.34). However, as explained in detail above, except for an $\mathbb{SL}(2,\mathbb{R})$worth of degrees of freedom (see (3.36), (3.37)), these Liouville fluctuations are frozen by the Virasoro gauge conditions (3.35). It is also pertinent here to mention the theorems due to Schwarz and Pick [107]; these restrict the class of conformal transformations that map the boundary of Poincare half-plane to itself to only $\mathbb{SL}(2,\mathbb{R})$transformations.

3.7 Thermodynamic partition function from bulk dual

This section discusses the computation of the Euclidean bulk partition function in the classical limit for a black hole geometry. The standard prescription of [108] is used to renormalize the bulk partition function by subtracting the partition function of thermal AdS_2 geometry from the Euclidean black-hole geometries that we describe below.[31] Following [8, 25], we can do a reparametrization of the Euclidean time to study a field theory defined on a thermal circle of length β,

$$\tilde{\tau} = \tan\left(\frac{\pi\tilde{\theta}}{\beta}\right) \tag{3.68}$$

Using (3.47), one can compute the Euclidean geometry that is dual to the thermal field theory,

$$ds^2 = \frac{1}{4\pi\mu\,\tilde{\zeta}^2}\left[d\tilde{\zeta}^2 + \left(1-\pi^2\frac{\tilde{\zeta}^2}{\beta^2}\right)^2 d\tilde{\tau}^2\right], \quad \tilde{\tau} \in \left(-\frac{\beta}{2},\frac{\beta}{2}\right) \text{ and } \tilde{\zeta} \in \left(0,\frac{\beta}{\pi}\right) \tag{3.69}$$

This geometry is a *capped AdS_2 geometry* in two dimensions. There is no deficit angle near the horizon of the geometry, which can be easily checked by doing a near horizon expansion, $\tilde{\zeta} = \beta/\pi - \rho$,

$$ds^2 \sim \frac{\pi}{4\mu\,\beta^2}\left[d\rho^2 + 4\frac{\pi^2}{\beta^2}\rho^2 d\tilde{\theta}^2\right]$$

Analytically continuing this geometry to Lorentzian space we get,

$$ds^2 = \frac{1}{4\pi\mu\,\tilde{\zeta}^2}\left[d\tilde{\zeta}^2 - \left(1-\pi^2\frac{\tilde{\zeta}^2}{\beta^2}\right)^2 dt^2\right] \tag{3.70}$$

which is a geometry with a horizon at $\tilde{\zeta} = \beta/\pi$.

To get the free energy of the theory, we compute the on-shell bulk action for this geometry, but with a small non-normalizable deformation turned on (*smallness* is understood as explained in the previous section).

Bulk action We first compute the bulk part of the action given in (3.41). The bulk part of the Liouville action is,

$$\begin{aligned} S_L^{bulk}[\phi,\hat{g}] &= -\frac{1}{4\pi\,b^2}\int_\Gamma \sqrt{\hat{g}}\Big(\hat{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \hat{R}\phi + 4\pi\mu e^{2\phi}\Big) \\ &= -\frac{1}{4\pi b^2}\int_{\partial\Gamma}\sqrt{\hat{\gamma}}\hat{n}^\alpha\phi\partial_\alpha\phi - \frac{1}{4\pi b^2}\int_\Gamma\sqrt{\hat{g}}\Big(-\phi\hat{\Box}\phi + \hat{R}\phi + 4\pi\mu e^{2\phi}\Big) \\ &= -\frac{1}{4\pi b^2}\int_{\partial\Gamma}\sqrt{\hat{\gamma}}\hat{n}^\alpha\phi\partial_\alpha\phi - \frac{1}{4\pi b^2}\int_\Gamma\sqrt{\hat{g}}\Big(\frac{1}{2}\hat{R}\phi + 4\pi\mu e^{2\phi}(1-\phi)\Big) \\ &= -\frac{1}{4\pi b^2}\int_{\partial\Gamma}\sqrt{\hat{\gamma}}\hat{n}^\alpha\phi\partial_\alpha\phi - \frac{\mu}{b^2}\int_\Gamma\sqrt{\hat{g}}\Big(-\phi + e^{2\phi}(1-\phi)\Big) \end{aligned} \tag{3.71}$$

here in the second line we have shifted the derivatives, while in the third line we have used the equation of motion of ϕ field, $\hat{\Box}\phi = \frac{1}{2}\hat{R} + 4\pi\mu e^{2\phi}$. In the last line we have substituted the

[31] Thermal AdS_2 geometry is obtained simply by identifying the boundary time coordinate in (3.32) over a period β.

value of $\hat{R} = -8\pi\mu$. The first boundary term in the last line combines with the boundary term already present in (3.41). However, these terms are not important for our analysis because they only contribute at $\mathcal{O}(\delta g^3)$. We don't have any leading contribution coming from the δg modes from the bulk action,

$$S_L^{bulk}[\tilde{\phi}, \tilde{g}_{\alpha\beta}] = \frac{1}{2b^2} - \frac{\beta}{4\pi b^2\,\delta} - \frac{\pi\,\delta}{4b^2\,\beta} + \mathcal{O}(\delta g^3) \tag{3.72}$$

As was the case with the previous Hydrodynamics calculation, all the divergent as well as finite terms above are cancelled by subtraction of the thermal AdS_2 partition function. This is the standard prescription to regulate the partition function of the black hole geometries (see [108]). Thus the bulk contribution starts only at $\mathcal{O}(\delta g^3)$.

Boundary action Computing the boundary terms of the action (3.41). Again, as argued above, the last term in (3.41) doesn't contribute at leading order. The term containing extrinsic curvature when evaluated on the boundary gives,

$$S_L^{bdy}[\tilde{\phi}, \tilde{g}_{\alpha\beta}] = \frac{\delta g}{2b^2\,\beta} + \frac{\beta\,\delta g}{4\pi^2 b^2\,\delta^2} + \mathcal{O}(\delta g^2) \tag{3.73}$$

In both the above expressions we have taken the boundary value of the $\delta g(i\tilde{\tau})$ field to be constant, as explained earlier, and have denoted it by δg. Again, the quadratically divergent term is cancelled by inclusion of the counterterm discussed in (3.58).

One last piece that needs to be evaluated is the bulk term $\int \sqrt{\hat{g}}\hat{R}\frac{1}{\Box}\hat{R}$ that depends only on the background geometry. The Green's function in hyperbolic spaces is a well studied subject. In Green's function can be evaluated by taking a limit of the 'resolvent' of the Laplacian.[32] The resolvent of the Laplacian on right half Poincare-plane, $\mathbb{H}$, is given by,

$$\Big(-\hat{\Box}_z + 4\pi\mu s(s-1)\Big) R_{\mathbb{H}}(s;z,w) = 4\pi\mu\delta^{(2)}(z-w)$$
$$R_{\mathbb{H}}(s;z,w) = \frac{1}{4\pi}\frac{\Gamma(s)^2}{\Gamma(2s)}\left(1+\frac{|z-w|^2}{4\,\mathrm{Re}(z)\,\mathrm{Re}(w)}\right)^{-s} {}_2F_1\left[s,s;2s;\frac{1}{1+\frac{|z-w|^2}{4\,\mathrm{Re}(z)\,\mathrm{Re}(w)}}\right] \tag{3.74}$$

Here, z, w are the complexified coordinates, $z = \zeta_1 + i\tau_1$ and $w = \zeta_2 + i\tau_2$. The $s \to 1$ limit of this function is,

$$G(\{\zeta_1,\tau_1\};\{\zeta_2,\tau_2\}) = -\frac{1}{4\pi}\log\left(1 - \frac{4\zeta_1\zeta_2}{(\zeta_1+\zeta_2)^2 + (\tau_1-\tau_2)^2}\right) \tag{3.75}$$

However, the above results are in $\mathbb{H}$, while we are interested in solving the Green's function for the geometry in (3.69). The Green's function can be obtained easily using the coordinate

[32] A resolvent in defined as the classical Green's function of the operator $-\Box + 4\pi\mu s(s-1)$. Thus the required Green's function is the $s \to 1$ limit of the resolvent.

transformations in (3.46) with the choice of function in (3.68). We get,

$$G = -\frac{1}{4\pi}\log\left[1 - \frac{8\pi^2\beta^2\zeta_1\zeta_2}{\left(\begin{array}{l}\beta^4 + \pi^2\beta^2\left(\zeta_1^2 + 4\zeta_1\zeta_2 + \zeta_2^2\right) + \pi^4\zeta_1^2\zeta_2^2 \\ \qquad\qquad -(\beta^2 - \pi^2\zeta_1^2)(\beta^2 - \pi^2\zeta_2^2)\cos\left(\frac{2\pi(\theta_1 - \theta_2)}{\beta}\right)\end{array}\right)}\right] \tag{3.76}$$

With this Green's function we solve the

$$\int \sqrt{\hat{g}}\hat{R}\frac{1}{\hat{\Box}}\hat{R} = \int \sqrt{\hat{g}(\zeta_1,\tau_1)}\int \sqrt{\hat{g}(\zeta_2,\tau_2)}R_1 G(\{\zeta_1,\tau_1\};\{\zeta_2,\tau_2\})R_2$$

term for the geometry, (3.69). We get,

$$\int \sqrt{\hat{g}}\hat{R}\frac{1}{\hat{\Box}}\hat{R} = \frac{\beta}{\pi b^2\delta} - \frac{2}{b^2}\log(\beta/\delta) + \frac{2\log(4\pi) - 3}{b^2} - \frac{\pi\delta}{3b^2\beta} \tag{3.77}$$

Again, the linearly divergent piece that appears above is cancelled by the contribution coming from the thermal AdS_2 partition function.[33]

Thus, the total action is

$$\log(Z) = -\beta F = -\frac{2}{b^2}\log(\beta/\delta) + \frac{2\log(4\pi) - 3}{b^2} + \frac{\delta g}{2b^2\,\beta} + \mathcal{O}(\delta g^2) \tag{3.78}$$

3.8 Comparison with field theory

Finally, it is important that the results obtained from the above bulk dual are compared with the field theory results directly obtained from the SYK-model. This section is a summary of such comparisons.

3.8.1 Hydrodynamics and a double scaling

The gravity dual leads to the following low energy effective action

$$S_{hydro} = \frac{\delta g}{2\pi b^2}\int d\tilde{\tau}\,\left\{\tilde{f}(\tilde{\tau}),\tilde{\tau}\right\} \tag{3.79}$$

while the SYK model has the following expression for the same quantity [8, 21]

$$S_{hydro} = N\frac{\alpha(q)}{\mathcal{J}}\int d\tau\,\left\{\tilde{f}(\tau),\tau\right\} \tag{3.80}$$

[33] It can be seen easily by doing a similar computation using the Green's function, (F.12), on the thermal AdS_2 geometry as discussed in Appendix F.

As argued above, δg plays the role of the explicit symmetry breaking parameter $1/J$ in the SYK model. Further, the classical limit in the bulk model corresponds to $b \to 0$, which, therefore corresponds to the limit $N \to \infty$. Therefore, these quantities can be identified up to constants, thus:

$$\frac{1}{b^2} = c_1 N, \ \delta g = c_2 \frac{1}{\mathcal{J}} \tag{3.81}$$

For the two hydrodynamic expressions above to match, we need to have $c_1 c_2 = \alpha(q)$. A q-dependence in the coefficients c_1, c_2 may appear strange; however, it may indicate the existence of a double scaling in the theory. Note that at large q, $\alpha(q) = a_0/q^2$ (a_0=constant). A possible choice of the coefficients is $c_1 = \alpha(q)$, $c_2 = 1$. In this case, we are essentially identifying

$$\frac{1}{b^2} = a_0 N/q^2, \ \delta g = c_2 \frac{1}{\mathcal{J}} \tag{3.82}$$

Thus, if we take the limit $N \to \infty$, *and* q^2/N fixed (*cf.* [73] appendix B), the corresponding scaled quantity corresponds to the bulk Newton's constant:

$$q^2/N = a_0 b^2$$

3.8.2 Thermodynamics

At low temperatures, the bulk partition function is given by (3.78), with a divergence of the form $\log(\beta/\delta)$. With the logarithmically divergent term we might typically be left with finite parts, say P_0, after cancellation of the divergence. The low temperature partition function will then be given by, ignoring subleading order terms in $\delta g/\delta$,

$$\log(Z) = -\beta F = \frac{1}{b^2}\left[\left(-2P_0 + \frac{4\log(4\pi) - 5}{2}\right) + \frac{\delta g}{2\beta} + \mathcal{O}(\delta g^2)\right] \tag{3.83}$$

The corresponding expression in the SYK model is [8, 73, 74]

$$\log(Z) = -\beta F = N\left[\beta\mathcal{J}\frac{1}{q^2} + \frac{1}{2}\log 2 - \frac{\pi^2}{4q^2} + \frac{1}{\beta\mathcal{J}}\frac{\pi^2}{2q^2} + O(\frac{1}{q^4})\right] \tag{3.84}$$

It is then possible that by suitably adjusting the finite part P_0 and the constant c_2 introduced above, one can match the zero-temperature entropy and the low temperature specific heat. The SYK zero-temperature entropy here does not seem to be universal; however, in the double scaling limit mentioned above, the N/q^2 term *is* universal.

3.9 Discussion

In this chapter, we arrive at a proposal for a gravity dual of the low energy sector of SYK-type models from symmetry considerations, more precisely from the fact that the coadjoint orbit action of the *Diff* group is the Polyakov action (3.28). Subsequently, the classical equations of motion were solved and it was found that the solutions are parametrized by a large diffeomorphism together with a specific conformal factor (value of the Liouville mode) representing a non-normalizable deformation. We compute the on-shell action which evaluates the classical contribution to $\log Z$. The computation leads to a Schwarzian action for the low energy hydrodynamic modes and a specific heat which is linear at low temperatures. Thus, the low energy behaviour of our proposed gravity dual reproduces that of SYK-type models.

Let's end this chapter with some remarks about possible UV properties of the bulk dual. Recall that in usual AdS/CFT, such as in the example of $\mathcal{N} = 4$ SYM theory on $S^3 \times R$, states with spin > 2 acquire very large anomalous dimensions $\gamma \sim (g_{YM}^2 N)^{1/4}$ at strong coupling [34] The energy grows as $E \sim \gamma / R_{AdS}$ and the corresponding bulk state is identified as a string state with mass $m_s = (g_{YM}^2 N)^{1/4}/R_{AdS}$. This corresponds to the fact that the UV completion of the gravity theory is string theory in AdS. In case of SYK-type models, the anomalous dimensions of operators with spin higher than two, which form an approximate Regge trajectory, remain $O(1)$ even at strong coupling. From the point of the bulk dual, the usual mass-dimension formula (which follows by using the relation between the AdS Laplacian and Casimir of $\mathbb{SL}(2,\mathbb{R})$) implies $E \sim \Delta/R_{AdS}$ (in our model, $R_{AdS} \sim 1/\sqrt{\mu}$, see (3.30)). If we wish to identify the 'Reggeons' with possible string states, this would imply that the 'string length' is of the same order as the AdS radius. It is not clear what such a dual string theory of light strings is. On the other hand, the spectrum of these massive modes suggests that it may be possible to incorporate these states in our bulk dual by adding to the Polyakov action (3.28) an infinite number of matter fields η_r minimally coupled to the metric (see [25] for related ideas), with masses m_r given in terms of the conformal dimensions Δ_r. In such a scenario, the Polyakov action (3.28) would still continue to represent the physics of the 'Nambu-Goldstone' modes. The full action will have the structure

$$S = S_{cov}[g] + S_{matter}[g, \{\eta_r\}] \tag{3.85}$$

[34] Primary operators with spin ≤ 2 retain $O(1)$ anomalous dimensions. These correspond to spherical harmonics of gravitons with $E \sim O(1)/R_{AdS}$.

where $S_{cov}[g]$ is the Polyakov action, given by (3.28). The matter action

$$\begin{aligned} S_{matter}[g, \{\eta_r\}] &= \frac{1}{2}\int_\Gamma \sqrt{g}\left[\sum_r \left(g^{\alpha\beta}\partial_\alpha\eta_r\partial_\beta\eta_r + m_r^2\eta_r^2\right) + ...\right] \\ &= \frac{1}{2}\int_\Gamma \sqrt{\hat{g}}\left[\sum_r \left(\hat{g}^{\alpha\beta}\partial_\alpha\eta_r\partial_\beta\eta_r + m_r^2 e^{2\phi}\eta_r^2\right) + ...\right] \end{aligned} \tag{3.86}$$

where in the second step, we have used (3.48). Note that since the metric $\hat{g}$ contains the Nambu-Goldstone modes f (see (3.47)), the above action automatically incorporates a coupling between these modes and the higher mass modes η_r; this fact plays an important role in computing the chaotic growth of the out-of-time correlator. Using the action (8.1) we can derive the exponentially growing behaviour of the out-of-time ordered 4- point functions, $\langle\mathcal{O}(\tau)\mathcal{O}(0)\mathcal{O}(\tau)\mathcal{O}(0)\rangle$ (where $\tau > 0$), which gives the Lyapunov exponent, $2\pi/\beta$, consistent with the bound on chaos derived in [27]. Note also the appearance of the Liouville factor in the mass term (this is to be contrasted with proposed bulk duals based on Jackiw-Teitelboim models, e.g. in [25]). This implies subleading correction to the mass term proportional to $1/J$ (see (3.48)). However, as shown in [8, 66] one doesn't need to break the conformal symmetry explicitly to study the physics of these excited states. In fact, the $1/J$ corrections for these states are truly subleading. The terms in the ellipsis above denote interaction terms, which are suppressed in large N counting. Whether the procedure of incorporating bulk fields outlined above can be consistently extended to an interacting level with local interactions in the bulk, of course, remains an interesting question.

Chapter 4

Interlude

The chapter 2 and chapter 3 in this thesis were a study of AdS/CFT correspondence. The chapter 2 was a study of renormalization group flows of a particular kind of operators in a general conformal field theory through its gravity dual. It was shown that the integration of near boundary degrees of freedom in the AdS space correspond to a particular choice of Polchinski-Wilsonian RG scheme in which the short-distance degrees of freedom in the dual field theory are integrated over. This observation was used to improve the AdS/CFT dictionary to propose bulk duals to regulated conformal field theories. These results are important not only because they make certain aspects of the duality more robust, but also because they can be used to predict the behavior of certain class of irrelevant operators in a field theory: something that is in general extremely difficult to achieve.

While AdS/CFT conjecture has been verified quite robustly over the past two decades, it has still not been able to answer some very important questions in physics. One of them is the problem of apparent non-unitarity in gravitating systems. Creation of blackholes is an 'apparently' non-unitary phenomenon within semi-classical analysis. This is because the final blackhole, or the Hawking radiation that fills the universe after the blackhole evaporation doesn't seem to contain the information of the constituents that created the black-hole. This is in contrast with quantum mechanics and this problem is even more apparent within AdS/CFT correspondence, where a gravity theory allowing blackhole solutions is dual to a good old unitary quantum field theory; an apparent non-unitary theory can't be dual to a unitary theory. This puzzle stays unresolved even now. In general, it is quite difficult to resolve this issue in the context of the duaity between (super-)gravity on $\mathrm{AdS}_5 \times S^5$ background and $\mathcal{N}$=4 SYM theory. The need of the hour is to understand a resolution in simpler systems. SYK model provides one such model where we can hope to be able to resolve such issues. Chapter 3 forms the discussion of AdS/CFT correspondence in this light.

While AdS/CFT correspondence tells us a lot about both the gravitational theories and the strongly coupled quantum field theories, it doesn't tell us: (1) about the gravity in flat space, precisely because there is no known example of Flat-space holography; (2) some interesting details specific to some particular strongly coupled field theories.

In the subsequent chapters, we attempt to answer such questions using the tools of S-matrices. S-matrices, as we have discussed above, are powerful observables in a theory that contain the information about the interactions of the theory, its symmetries, its states, among other things. It is often possible to restrict the space of theories by imposing the symmetries of the theory directly on the S-matrices. This procedure is referred to as Bootstrap. In chapter 5 we use such ideas to restrict S-matrices in a certain class of theories that have been motivated by known String theories; with an aim to generalize these techniques to theories that necessarily have a massless spin-2 gauge boson (a graviton).

Chapter 6 is a study of S-matrices in strongly interacting $\mathcal{N}$=2 Chern-Simons theories with fundamental matter. While in a general quantum field theory, computation of the S-matrices is, in general, an arduous task; in these class of theories we observe that the computation of an arbitrary tree-level n-point function is facilitated by some recursion relations: BCFW relations. We compute the arbitrary n-point function in the above mentioned theory and make some comments about possible implications on non-supersymmetric theories. We also comment on possibilities of some additional symmetries in these theories that were not known to this point.

Chapter 5

S-Matrix Bootstrap for Amplitudes with Linear Spectrum

5.1 Introduction

String theory arose from an attempt to write down scattering amplitudes for strong interactions from consistency conditions rather than from a Lagrangian, also known as the S-Matrix bootstrap program. This program, however, was eventually abandoned in favour of $SU(3)$ Yang-Mills theory. The amplitudes written by Veneziano [33] and generalised by Virasoro [34, 35], while they didn't prove very useful for understanding strong interactions, eventually gave rise to string theory, which then shed its roots in this program to become a field in its own right.

This program has, in some sense, seen a revival in recent years, both indirectly through applications in conformal bootstrap [37, 38, 109] and more directly through a striking result about the three-point functions between two gravitons and higher-spin fields: [36] proved that the three-point functions must either match Einstein gravity, or there must exist an infinite tower of higher-spin fields in the theory.

This last result raises a very interesting question. This question is predicated on the fact that there are only a few classical, tree-level, amplitudes known that have an infinite tower of higher spins: as many as there are different string theories. While, quantum-mechanically, string theory is plagued by a large number of possible compactifications – the so-called "landscape" problem –, each string theory gives a unique tree-level amplitude, for the simple reason that without the moduli space integrations required at loop level the different dimensions of space-time correspond to decoupled CFTs on the worldsheet. The question is this: given that there are so few examples known of tree-level graviton amplitudes, could it be that these are the only examples? In other words, could it be that the string theories are the only consistent extensions of classical gravity?

The obvious question is: consistent with what? A minimal list of conditions would include: Lorentz-invariance, causality, unitarity and crossing symmetry. So, we could ask the question of what the most general four-point graviton scattering amplitude consistent with this minimal set of consistency conditions is.

This chapter heads towards this problem via a simpler, more restricted problem. We consider the scattering of four identical scalars (instead of gravitons), and further assume a linear spectrum of exchanged particles and Regge behaviour at large energies; and arrive at the conclusion that crossing symmetry restricts the Regge asymptotic behaviour to be $A(s,t) \xrightarrow{s\to\infty} (-s)^{2t}$ but still allows for an infinite-dimensional parameter space of amplitudes, and argue numerically that unitarity doesn't significantly reduce the dimensionality of this allowed space.

This problem has been addressed recently, using very different methods, in [110–113]. The results of [110, 111] are more or less assumed in our work, in the assumption of linear spectrum. Those of [112] are not relevant for this work, since the case of linear spectrum is a very degenerate one and that work goes beyond this case. Finally, the authors of [113] are able to show that string theory amplitudes can be derived from monodromy and BCFW recursion relations. It would be interesting to use the recursion relations that they have developed for the string theory amplitudes to constraint the higher-point scattering amplitudes using our techniques.

More precisely, for tree-level four-point amplitudes of four identical scalars, we impose the conditions that its behaviour at large s is[1]

$$A(s,t) \xrightarrow{s\to\infty} (-s)^{-k(-t)}, \tag{5.1}$$

and, that the mass-squared of exchanged particles are evenly spaced,

$$m_n^2 = \frac{n-\alpha_0}{\alpha'}, \tag{5.2}$$

along with some other conditions listed in section 5.2. The restrictions imposed first by crossing symmetry and then by unitarity are investigated.

It should be mentioned here that, throughout this chapter, we don't work with the standard Mandelstam variables s, t, u, but with their shifted and rescaled versions a, b, c (see (5.7)) such that the poles from the intermediate particles going on-shell are at $a = 0, -1, -2, \cdots$ for the s-channel, $b = 0, -1, -2\cdots$ for the t-channel and $c = 0, -1, -2\cdots$ for the u-channel.

[1] This equation is missing some factors. See (5.10) for a more precise statement.

For clarity, however, the remainder of this introduction is written in terms of the standard Mandelstam variables while being cavalier about factors.

With the asymptotic Regge behaviour, (5.1), it is known that crossing symmetry must be implemented not by an independent sum of s-channel, t-channel and u-channel diagrams but by "channel duality": the t-channel poles have to be hidden in an infinite sum over s-channel poles, as illustrated schematically in figure 5.1. This is because a t-channel diagram with an exchange of a spin l particle behaves as s^l at large s, which necessarily overpowers the exponential falloff that is the Regge behaviour. Therefore, it must not be possible to write the entire amplitude in the region $s > 0$ without summing over t-channel diagrams. See [110, 111] for further discussion.

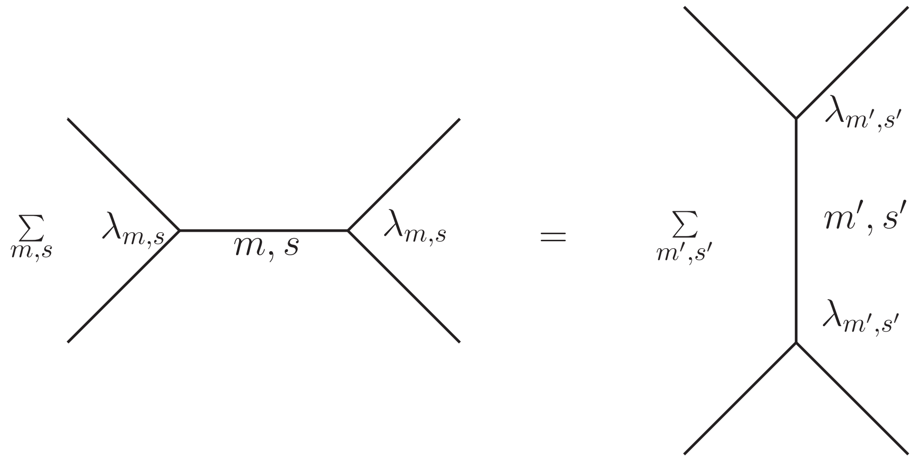


FIGURE 5.1: Crossing symmetry in theories with Regge asymptotic behvaiour is imposed by "channel duality," the requirement that the sum over s-channel poles be equal to the sum over t-channel poles.

In section 5.3, the condition of channel duality are reduced to a countably infinite set of equations (5.27) in terms of the values at a discrete set of points of the coefficients (5.19) of the Laurent expansion of the amplitude about $s = \infty$ (they are still functions of t). We also obtain a physical interpretation of the function $k(-t)$ that appears in the exponent in the Regge behaviour, (5.1): $-k(-n)$ is the maximum spin exchanged at level n (where the lightest exchanged particle corresponds to $n = 0$).

Furthermore, the condition that the function k is linear in its argument is imposed in section 5.4. Although, it's taken as an external condition to facilitate the solutions of the equations, one can see that a more general function is unlikely to have a consistent set of solutions. First, using various complex analysis techniques (mostly Carlson's theorem), we're able to show that the function k can't be just any linear function but has to be $k(-t) = 2(-t) + l$, which is exactly the sort of behaviour shown by the amplitudes in string theory! This is one of the main results presented in this chapter.[2] The final conclusion of section 5.4, the channel duality equations (5.39), (5.40) and (5.41) are among the other major results discussed in this chapter; they are necessary and sufficient conditions for an amplitude that satisfies our assumptions to be channel-dual. In the discussion subsection, we try to relate each of these equations to some physical meaning.

Section 5.5 discusses the solutions to these equations. While these equations are rather hard to solve, it is shown that starting with a channel-dual amplitude $A_{a,b,c}$ one can construct an infinite-dimensional parameter space of amplitudes $\sum_{m=0}^{\infty} a_m A(a+m, b+m, c+m)$ that are all channel dual with the same poles. That these class of 4-point amplitudes satisfy all the assumptions of dual amplitudes is known from the works of [114–117].

Finally, the section 5.6 is a discussion of unitarity of these amplitudes. It is shown that for the Virasoro-Shapiro amplitude and the dilaton amplitude in closed bosonic string theory, many perturbations of these base amplitudes seem to be consistent with unitarity. This section consists only of numerical arguments.

Appendix L summarises some useful facts about the kinematics of the amplitudes we consider here. Appendix M shows explicitly how our analytic continuation and other techniques and results apply to the Euler beta function, which is the building block of the Veneziano amplitude.

Mathematica files containing parts of relevant computations of are available as ancillary files at arXiv:1707.08135.

5.2 Postulates

We begin by laying out the properties that we require the amplitude to satisfy. The physical situation under consideration is the 2-2 tree-level scattering of four identical scalars of mass M_{ext} in a D-dimensional Minkowski spacetime. The incoming particles are labelled 1 through 4, with k_i labelling their respective momenta. All four momenta are taken to be ingoing and the momentum conservation with this convention reads, $k_1 + k_2 + k_3 + k_4 = 0$. We take the metric to have mostly positive signature, $\{-1, 1, 1, \ldots, 1\}$. The Mandelstam variables are

$$\begin{aligned} s &= -(k_1 + k_2)^2 \\ t &= -(k_1 + k_3)^2 \\ u &= -(k_1 + k_4)^2. \end{aligned} \tag{5.3}$$

[2]This result was indicated by, though not quite proved, by previous work [114].

Having set the stage, let us list the assumptions. For a more detailed review, see [9].

1. **Lorentz-Invariance**: The amplitude A is only a function of the Mandelstam variables s, t, u, and momentum is conserved at every vertex.

2. **Causality, or analyticity**: In the two-dimensional complex s, t plane the singularities occur only when one or more intermediate particles go on-shell. This allows both tree-level and loop diagrams (in which case the singularities are branch cuts). In particular, the singularities in the amplitude appear only for real values of s, t or $u = 4M_{ext}^2 - s - t$.

3. **Restriction to Tree-Level Amplitudes**: The amplitude is a sum of only tree-level diagrams. Combined with the previous assumption, this means that the only singularities of the amplitude are poles, at values of s, t, u equal to the mass-squared of a particle in the spectrum of the theory.

4. **Unitarity, or Cutting Rules**: Unitarity of scattering amplitudes is generally ensured by cutting rules. Suppose there are particles of spin 0 through L at some mass m. Then, the residue of the amplitude at the pole $s = m^2$ must be

$$Res_{s=m^2} A(s,t) = \sum_{l=0}^{L} \lambda_{m,l}^2 C_l^{\left(\frac{D-3}{2}\right)} \left(1 + \frac{2t}{s - M_{ext}^2}\right), \quad \lambda_{m,l}^2 \geq 0. \tag{5.4}$$

   Here, each $\lambda_{m,l}^2 = \sum_i \lambda_{m,l,i}^2$ is the sum of squares of cubic couplings $\lambda_{m,l,i}$ of two external scalars and all the particles of mass m and spin l (which we have labelled by i), and the functions $C_l^{(\alpha)}$ are Gegenbauer polynomials (which reduce to Legendre polynomials for $D = 4$); the argument of the Gegenbauers is $\cos\theta$, where θ is the scattering angle in the centre-of-momentum frame, see appendix L for details.

   It should be noted that this requirement is only a necessary and not a sufficient condition for unitarity, because of the possibility of many particles with the same mass and spin. In the case when there are multiple such particles, it doesn't restrict all the $\lambda_{m,l,i}^2$s to be positive but only their sum, $\sum_i \lambda_{m,l,i}^2$.

5. **Crossing Symmetry**: The amplitude should be invariant under exchange of any pair of external particles, since all four particles are identical. In terms of the Mandelstam variables, this means that the amplitude should, as a function, satisfy the relations

$$A(s,t) = A(t,s) = A(s, u = -4M_{ext}^2 - s - t). \tag{5.5}$$

6. **Linear Spectrum**: The mass-squareds of the exchanged particles are spaced linearly, that is

$$m_n^2 = \frac{n - \alpha(0)}{\alpha'}, \quad /n \in \{0, 1, 2 \cdots\}. \tag{5.6}$$

This is the first really non-trivial assumption here, it gives us a lot of control over the problem. The ones earlier were all properties that we must require of all tree-level amplitudes.

To take advantage of this simple behaviour, we define proxy variables a, b, c for s, t, u as, [3]

$$a = -\alpha' s - \alpha(0), \tag{5.7}$$

and similarly for b and c, so that the poles in the amplitude are at $a = -n$,

$$s = m_n^2 \quad \Leftrightarrow \quad a = -n \tag{5.8}$$

and similarly for t, u and b, c. Also note that this α' isn't the constant that appears in the string action, but just the inverse of the level spacing; in particular for closed bosonic string theory our α' is related to that one as $\alpha'_{us} = \frac{1}{4}\alpha'_{closed\ bosonic}$.

Because $s + t + u = 4M_{ext}^2$ is a constant, so is $a + b + c$. We call this constant P for the remainder of this note,

$$a + b + c \equiv P = -4\alpha' M_{ext}^2 - 3\alpha(0). \tag{5.9}$$

For later convenience, we note that in these variables, the physical s-channel scattering region is given by $a < 0, -\alpha(0) < b < -a - \alpha(0)$.

7. **Regge Asymptotic Behaviour, or "Analyticity of the Second Kind"**: At large s (or t or u), the amplitude behaves as

$$A(a, b, c) \xrightarrow{a \to -\infty} a^{-k(b)} \sim (-s)^{-k(-t)}. \tag{5.10}$$

where $k(b) > 0$ in the physical s-channel region, $b > 0$.

This appears to be non-analytic at $a = -\infty$ because $k(b)$ need not be an integer. However, this is not a true non-analyticity of the amplitude, but the result of the fact that we are restricting ourselves to physical scattering wedge in writing the above asymptotic behaviour.

In particular, we will heavily use the fact that the amplitude admits a Laurent expansion around infinity once we factor out this apparent non-analyticity. Before we go ahead to discuss the implications of this assumption, we wish to mention that "analyticity of the second kind" often refers to a class of assumptions on the asymptotic behaviour of the amplitudes. These assumptions are used as additional postulates

[3] These were called $-\alpha(s), -\alpha(t), -\alpha(u)$ in the old bootstrap literature, but we use this notation to avoid clutter, and also because these are the natural variables that turn up in the Virasoro-Shapiro amplitude, as named in [118].

that differentiate the amplitudes of weakly interacting theories like QED and the weak force from those of strongly interacting theories, [9]. Moreover, these assumptions are independent of the bounds that are implied on the asymptotic behaviour of any general amplitude that obeys all the previous assumptions, like the Froissart and Martic bounds.

The above assumption, (5.10), is inconsistent with the amplitude being a sum of separate diagrams in the s, t and u channels, because a t-channel diagram of spin l behaves as s^l at large s, and the fact that $l \geq 0$ means that this will overpower the Regge falloff; see [110, 111] for more details. So, the assumption of Regge behaviour means that crossing symmetry is implemented by **channel duality**; the sum over all s-channel diagrams has all the t-channel poles hidden in it. The main thrust of this work is to understand how they're hidden in it.

Before going ahead, we note that these assumptions are all true for the Virasoro-Shapiro amplitude. The Virasoro-Shapiro amplitude, which is the scattering amplitude of four tachyons in bosonic string theory, is

$$\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b)\Gamma(b+c)\Gamma(c+a)}, \tag{5.11}$$

with

$$a = -\frac{\alpha'}{4}s - 1, \tag{5.12}$$

where α' is not the Regge slope from (5.7) but is related to the inverse of string tension. In this case, the external mass-squared is $M_{ext}^2 = -\frac{4}{\alpha'}$ and $a + b + c = 1$. The asymptotic behaviour is

$$A(a, b, c) \xrightarrow{a\to\infty} a^{-2b}, \tag{5.13}$$

as can be easily shown using Stirling's approximation.

5.3 Channel Duality Equations

Having set up the problem, we now try to understand the constraints imposed by channel duality. In this section, we show that the constraints can be reduced to a countably infinite set of equations that have to be simultaneously satisfied. The strategy will be to take the pole-sum form in the a and c channels,

$$A(a, b) = \sum_{n=0}^{\infty} \frac{f_n(b)}{a+n} + \frac{f_n(b)}{c+n}, \quad \mathrm{Re}\, b > 0, \tag{5.14}$$

and recreate the poles in the b-channel by a suitable analytic continuation. Since there are no explicit poles in b,[4] it must be the case that these poles come from the infinite sum. To see that this is the case, we can ask how the above function can diverge for a particular value of b; the answer is clearly that the infinite sum may diverge for particular values of b. In particular, that means that this sum must converge for all positive values of b, where there are no poles (the c-channel poles are being ignored here because they are already explicit in the above expression). Appendix M demonstrates the ideas discussed in this and the next section by implementing them on Euler beta functions, which are the building blocks of Veneziano amplitudes.

Our first step is to realise that the assumptions about analyticity allow us to say something about the residues in the pole-sum form (5.14). First of all, by the fact that the sum converges for $a \neq -n$ (since otherwise it would be a non-analyticity away from the physical poles, which is not allowed), it can be deduced that for large enough n for fixed b, $f_n(b) \leq n^{-K}$, where $K > 0$; a good way to see this is to take $a = 5.5$ and demand convergence of the sum from $n = 6$ to ∞. By the Euler-Maclaurin formula,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{f_n(b)}{a+n} &\lesssim \sum_{n=0}^{\lfloor |a| \rfloor} \frac{f_n(b)}{a+n} + \sum_{\lceil |a| \rceil} \left(n^{-K-1} + \text{subleading}\right) \\ &\lesssim \sum_{n=0}^{\lfloor |a| \rfloor} \frac{f_n(b)}{a+n} + |a|^{-K} + \text{subleading}. \end{aligned} \tag{5.15}$$

This tells us that $K = k(b)$ is an allowed bound, since otherwise the $|a|^{-K}$ term would swamp out the Regge behaviour. By taking $a = -n + \varepsilon$, we find that the amplitude is

$$A(-n+\varepsilon, b) \sim \frac{1}{\varepsilon} f_{-n+\varepsilon}(b) \leq \frac{1}{\varepsilon} n^{-k(b)}. \tag{5.16}$$

The $\frac{1}{\varepsilon}$ part is just the oscillatory envelope one expects to find close to the negative real line in the a complex plane, and so we see that

$$f_n(b) = g_0(b) n^{-k(b)} + \text{subleading at large } n. \tag{5.17}$$

Now, consider the auxiliary function

$$\tilde{f}_n(b) = n^{k(b)} f_n(b). \tag{5.18}$$

This function can be analytically continued to the complex n-plane. Because it doesn't diverge exponentially at large n and is defined on all the positive integers, there is a unique analytic continuation that doesn't diverge exponentially (roughly speaking) by Calrson's

[4] Technically, the $1/(P-a-b+n)$ part has singularities in the b complex plane, but we're only looking for b-channel poles, whose positions are independent of a

theorem, described in further detail in section 5.4, specifically in the paragraphs following (5.36). In particular, we expect it to have a Laurent expansion about $n = \infty$, since it is analytic there. Thus, the residue can be written as

$$f_n(b) = \sum_{j=0}^{\infty} g_j(b) n^{-k(b)-j}. \tag{5.19}$$

We assume that this expansion converges for $n > 1$; we expect that the details will not be substanitally different if we take a smaller radius of convergence, as long as it is finite. Also note that we chose to do the expansion in n instead of $-n \sim a$; this is a matter of convenience and will have implications later.

We proceed by subsituting the $1/n$ expansion of the residue (5.19) into the pole-sum form of the amplitude (5.14),

$$\begin{aligned} A(a,b) = &\sum_{n=2}^{\infty}\left(\sum_{j=0}^{\infty} g_j(b) n^{-k(b)-j}\right)\left(\frac{1}{a+n}+\frac{1}{P-a-b+n}\right) \\ &+\sum_{n=0}^{1} f_n(b)\left(\frac{1}{a+n}+\frac{1}{P-a-b+n}\right), \end{aligned} \tag{5.20}$$

where the $n \leq 1$ terms have been split off because they're clearly outside the radius of convergence of the $1/n$ expansion. However, since the part in the second line above is clearly regular in b – it's just a polynomial, we may safely ignore it; to put this another way, the divergence must come from the tail end of the sum so dropping a finite sum in the beginning should not be a problem.

For all $n > |a|$, we can also expand

$$\frac{1}{a+n} = \sum_{r=0}^{\infty} \frac{(-a)^r}{n^{r+1}}, \tag{5.21}$$

and similarly for the $1/(c+n)$ term.

Plugging this into the amplitude (5.20) gives[5]

$$
\begin{aligned}
A(a,b) &= \sum_{n=[a]}^{\infty} \left\{ \left(\sum_{j=0}^{\infty} g_j(b) n^{-k(b)-j} \right) \sum_{r=0}^{\infty} \frac{(-a)^r}{n^{r+1}} \right\} \\
&\quad + \sum_{n=[P-a-b]}^{\infty} \left\{ \left(\sum_{j=0}^{\infty} g_j(b) n^{-k(b)-j} \right) \sum_{r=0}^{\infty} \frac{(a+b-P)^r}{n^{r+1}} \right\} + reg. \\
&= \sum_{n=1}^{\infty} \left(\sum_{j=0}^{\infty} g_j(b) n^{-k(b)-j} \right) \left(\sum_{r=0}^{\infty} \frac{(-a)^r + (a+b-P)^r}{n^{r+1}} \right) + reg. \qquad (5.22)
\end{aligned}
$$

where we have again ignored all finite sums in n, which are denoted by reg. The sum in the last expression can be rearranged as follows:

$$
A(a,b) = \sum_{j,r=0}^{\infty} g_j(b) \left\{ (-a)^r + (a+b-P)^r \right\} \sum_{n=1}^{\infty} n^{-k(b)-j-r-1} + reg. \qquad (5.23)
$$

So far, all the manipulations are valid only for $\mathrm{Re}\, b > 0$. To find the poles in b, which reside at $b = -n$, we need to analytically continue the expression (5.23) to the left half plane of b. The innermost sum in (5.23) diverges for all b whenever $k(b) \leq 0$; this is an artefact of the fact that the expression isn't valid in that region. To analytically continue, the sum over n is replaced by Riemann zeta functions – that is, only the logarithmic divergences in the sum over n are kept,[6]

$$
\begin{aligned}
A(a,b) &= \sum_{j,r=0}^{\infty} g_j(b) \left\{ (-a)^r + (a+b-P)^r \right\} \zeta(k(b)+j+r+1) + reg. \\
&= \sum_{N=0}^{\infty} \sum_{J=0}^{N} g_J(b) \left\{ (-a)^{N-J} + (a+b-P)^{N-J} \right\} \zeta(k(b)+N+1) + reg. \qquad (5.24)
\end{aligned}
$$

This expression now makes sense for $\mathrm{Re}\, b < 0$ as well. While this analytic continuation can be justified merely on the grounds of being an analytic continuation and therefore unique, we also show in appendix M in more detail how it works.

The Riemann zeta function $\zeta(z)$ has a pole at $z = 1$ with residue 1. Thus, the expression (5.24) has poles at $k(b) = -N$ with residues

$$
Res_{b=k^{-1}(-N)} A(a,b) = \frac{\sum_{J=0}^{N} g_J(k^{-1}(-N)) \left\{ (-a)^{N-J} + (a + k^{-1}(-N) - P)^{N-J} \right\}}{k'(k^{-1}(-N))} \qquad (5.25)
$$

[5]Here $[a]$ denotes smallest integer greater than a.

[6]This is the well-known Zeta function regularization.

Consistency with crossing symmetry therefore gives us a condition on the function $k(b)$,

$$k(-n) = -N, \text{so that there are poles at } b = -n, \tag{5.26}$$

and two countably infinite sets of conditions on the functions g_J,

$$\begin{aligned} k^{-1}(-N) \notin \{0, -1, \cdots\} &\Rightarrow \frac{\sum_{J=0}^{N} g_J(k^{-1}(-N)) \left\{(-a)^{N-J} + (a + k^{-1}(-N) - P)^{N-J}\right\}}{k'(k^{-1}(-N))} \\ &= 0 \\ k^{-1}(-N) = -n &\Rightarrow \frac{1}{k'(-n)} \sum_{J=0}^{N} g_J(-n) \left\{(-a)^{N-J} + (a - n - P)^{N-J}\right\} = f_n(a). \end{aligned} \tag{5.27}$$

The first equation ensures that the spurious poles occuring at non-integer values of b vanish, and the second ensures that the real poles have the correct residues, those required by channel duality. Eqns (5.26) and (5.27) are the general conditions for duality.

5.3.1 Discussion

Before going ahead, we note that the linearity of the spectrum has not been substantially used anywhere yet. In the case of a nonlinear spectrum, we can take a to be a non-linear function of s such that the poles are at $a = -n$ and similarly for b and c. The major difference in this case is that there's no expression of the form $c = P - a - b$; however, it is still true that c can be determined given a and b, since a must still be an invertible function of s. Then, eqns (5.26) and (5.27) will be valid with the modification that $a + k^{-1}(N) - P$ must be replaced by $c|_{b=k^{-1}(-N)}$.

Another point worth noting is that the function $k(b)$ has a physical meaning – the residue $f_n(b)$ is a polynomial of degree $-k(-n)$, which means that $-k(-n)$ is the spin of the maximum spin particle exchanged at level n. This might seem odd at first, since the function was defined in terms of not the physical variables s, t, u but the made-up variables a, b, c. The point is that the normalisation for a – that the poles are at $a = -n$ – is important. If we redefined a, then the degree of the n^{th} polynomial wouldn't match the function in the exponent of the asymptotic behaviour. They will still be related by a simple algebraic relation.

Having noted that, it should also be noted that these general conditions aren't very easy to solve. Hence, in the following discussion the asymptotic function $k(b)$ will be restricted to be linear. Although this might look like an ad-hoc simplification, it will be noted after solving the case of linear $k(b)$ that it is unlikely to find solutions for the case of non-linear functions. This is because the (5.27) will provide more constraints than variables (both in the sense of counting infinities).

5.4 The Case of a Linear Asymptotic Function

The necessary conditions that a channel-dual amplitude needs to satisfy, equations (5.26) and (5.27), that were computed in the previous section, aren't very easy to solve. To facilitate control over the equations, the following discussion is restricted to the case of linear $k(b)$,

$$k(b) = kb. \tag{5.28}$$

With this restriction, we show that the only consistent value of k is 2. The main reason $k = 2$ is special is that the duality equations simplify greatly at this value; we end this section with the simplest form of these equations. The details for the case $k(b) = kb - l$ are not substantially different, so we drop it to aid clarity.

When $k(b)$ is a linear function, the condition (5.26) is automatically satisfied, and the countably infinite set of constraints (5.27) on the g_Js become

$$\begin{aligned} N \neq kn &\Rightarrow \sum_{J=0}^{N} g_J\left(-\frac{N}{k}\right)\left\{(-a)^{N-J} + \left(a - \frac{N}{k} - P\right)^{N-J}\right\} = 0 \\ N = kn &\Rightarrow \frac{1}{k}\sum_{J=0}^{kn} g_J(-n)\left\{(-a)^{kn-J} + (a-n-P)^{kn-J}\right\} = f_n(a). \end{aligned} \tag{5.29}$$

We expand the residue $f_n(a)$ in powers of $-a$ as

$$f_n(a) = \sum_{J=0}^{kn} h_J(-n)(-a)^{kn-J}. \tag{5.30}$$

The choice of $-a$ instead of a in this equation is parallel to the choice of expanding in n instead of $-n$ in (5.19). Using this and the binomial expansion of $\left(a - \frac{N}{k} - P\right)^{N-J}$, we find

$$\begin{aligned} h_J\left(-\frac{N}{k}\right) &= \frac{1}{k}\left\{g_J\left(-\frac{N}{k}\right) + (-1)^{N-J}\sum_{j=0}^{J}(-1)^j\binom{N-J+j}{j}\left(\frac{N}{k}+P\right)^j g_{J-j}\left(-\frac{N}{k}\right)\right\} \\ &= \frac{1}{k}\left\{[1+(-1)^{N-J}]g_J\left(-\frac{N}{k}\right) + (-1)^{N-J}\sum_{j=1}^{J}(-1)^j\binom{N-J+j}{j}\left(\frac{N}{k}+P\right)^j g_{J-j}\left(-\frac{N}{k}\right)\right\}, \end{aligned} \tag{5.31}$$

with the definition

$$h_J(\text{non-integer}) = 0. \tag{5.32}$$

The first line in (5.31) is written in a form such that the a-channel and c-channel contibutions are separate from each other, while the second line collects the g_js together.

The notation that is used for the h_Js is (almost falsely) suggestive – we've covertly treated them as functions of the level n. The reason for this is that in the equations (5.31) the right hand side are in fact functions, and therefore we can promote h to a function of the level by requiring equality with the right hand side. This has to be consistent with the analytic continuation that allowed us to expand the residues $f_n(b)$ in a $1/n$ expansion. But, notice that the right hand side has the factor $(-1)^N$, which has an essential singularity at ∞; this contradicts our assumption about analyticity at ∞, as used in (5.19).

For odd values of k, this poses an insurmountable problem. Consider $k = 1$ for example, so that the left hand side of (5.31) is non-zero for all values of N. Then, the residue at level n is

$$f_n(a) = \sum_{J=0}^{n} g_J(-n)(-a)^{n-J} + (-1)^n \sum_{J=0}^{n}\sum_{j=0}^{J}(-1)^{j+J}\binom{n-J+j}{j}(n+P)^j g_{J-j}(-n)(-a)^{n-J} \tag{5.33}$$

Because of the $(-1)^n$, this can't be analytic at ∞, assuming the g_js are analytic at ∞. However, we know that the g_js are, in fact, analytic at ∞, which can be seen by taking the limit $b \to \infty$ of their definition (5.19), where it must exhibit Regge behaviour.[7] Thus, the channel duality equations for $k = 1$ are inconsistent with our assumptions about analyticity at ∞. It is easy enough to see that this extends to all odd values of k, and therefore that our assumptions aren't compatible with k being odd.

What saves the case of even k is that the physical poles are all at even values of N in (5.31); that means we can promote the h_Js to *two* sets of functions, h_J^e obtained from analytically continuing off even N and h_J^o obtained from analytically continuing off odd N; note here that the h_J^os are identically 0, because only then can the residue vanish for these spurious poles.

So we consider the equations for odd N and even N separately. For odd N, the equations are

$$\left\{1-(-1)^J\right\} g_J\left(-\frac{N}{k}\right) = \sum_{j=1}^{J}(-1)^{J+j}\binom{N-J+j}{j}\left(\frac{N}{k}+P\right)^j g_{J-j}\left(-\frac{N}{k}\right), \quad N \text{ odd.} \tag{5.34}$$

[7] One may object that (5.33) is a polynomial not in a but in $-a$; pulling out those signs merely shifts the problematic $(-1)^n$ to the other term and the non-analyticity remains. It is also useful to remember to note that, while $\Gamma(z)$ has an essential singularity at ∞, the binomial coefficients diverge only polynomially.

It is useful to again split these into two sets of equations, those in which J is odd and those in which J is even. For even J, the left hand side is 0 and we find

$$0 = \sum_{j=1}^{J} (-1)^j \binom{N-J+j}{j} \left(\frac{N}{k}+P\right)^j g_{J-j}\left(-\frac{N}{k}\right), \quad N \text{ odd}, \ J \text{ even}. \tag{5.35}$$

And for odd J we find

$$g_J\left(-\frac{N}{k}\right) = -\frac{1}{2} \sum_{j=1}^{J} (-1)^j \binom{N-J+j}{j} \left(\frac{N}{k}+P\right)^j g_{J-j}\left(-\frac{N}{k}\right), \quad N \text{ odd}, \ J \text{ odd}. \tag{5.36}$$

As can be seen, these are all constraints that relate various g_Js evaluated at the same point, for a countably infinite set of points. Moreover, (5.36) can be used recursively to evaluate g_Js.

As it happens, this is enough to argue that equations (5.35) and (5.36) are valid everywhere – that is, that they are functional relations among the g_Js. This is possible because of a theorem in complex analysis called Carlson's theorem:

> **Carlson's theorem** If $f(z)$ is an entire function[8] that satisfies following properties:
>
> - $|f(z)| \leq Ce^{\tau|z|}$, $\forall\ z \in \mathbb{C}$, for some $C, \tau \in \mathbb{R}$,
> - $\exists\ \mu < \pi$, such that $|f(iy)| \leq Ce^{\mu|y|}$, $\forall\ y \in \mathbb{R}$,
> - $f(n) = 0$, $\forall\ n \in \mathbb{Z}^+ \cup)0$
>
> then, f is identically 0.

Given that the g_Js satisfy the conditions of this theorem, we can see that equations (5.35) and (5.36) must be valid everywhere very easily. Take, for example, eqn (5.36) and separately analytically both the left and the right hand sides. The analytic continuation of the left hand side is clearrly g_J. The analytic continuation of the right hand side is also unique, since it behave roughly as $\sum N^{2j} g_{J-j}$ and if the g_Js satisfy the conditions so do these. Thus, eqn (5.36) is a functional relation. One can similarly argue for eqn (5.35).

What remains is to show that the g_Js in fact satisfy the conditions of Carlson's theorem. Since we are analytically continuing off the negative integers, we need to bound the growth of $g_J(b)$ in the left-half plane. Suppose it grows exponentially on the left half plane, $g_J(b) \sim$

[8] An entire function is a complex-valued function that is holomorphic at all finite points over the whole complex plane.

$e^{-\nu b}$. Then, because of the duality equations (5.31) and the definition (5.30) of the h_Js,

$$f_n(a) \sim e^{\nu n} \sum_J (-a)^{kn-J}. \tag{5.37}$$

This blow-up at large n contradicts Regge behaviour. This means that the g_Js can't grow exponentially in the left-half plane, and by continuity can't grow expnonentially on the (upper or lower) imaginary axis either. It is worth noting that this doesn't constrain the behaviour on the right-half plane; in particular, for the Virasoro-Shapiro amplitude, it behaves like b^b, which is super-exponential on the right-half plane and goes to 0 on the left-half plane while being oscillatory on the imaginary axis. Thus, we have proved that the relations (5.36) and (5.35) are functional relations valid for all values of N (but, remember, not J).

Having dealt with the odd N equations, we can now deal with those for even N. The major difference is that the h^e_Js that appear in these equations are not 0. The equations are

$$\begin{aligned} h_J\left(-\frac{N}{k}\right) &= \frac{1}{k}\left\{\left[1+(-1)^J\right] g_J\left(-\frac{N}{k}\right) + (-1)^J \sum_{j=1}^{J} (-1)^j \binom{N-J+j}{j} \left(\frac{N}{k}+P\right)^j g_{J-j}\left(-\frac{N}{k}\right)\right\} \\ &= \frac{2}{k} g_J\left(-\frac{N}{k}\right), \end{aligned} \tag{5.38}$$

where we have used the functional relations (5.35) and (5.36) – for odd J the first term in the first line vanished and the second term becomes the final answer, and for even J the second term vanishes.

These are nearly the final forms of the duality equations. To see where the final simplification comes from, consider the case $k = 4$. In this case, $N = 4n + 2$ are all spurious poles; thus, $h_J(-n-1/2) = 0$ and therefore $g_J(-n-1/2) = 0$. Since the g_Js are 0 at an infinite number of evenly spaced points and g_J can't grow exponentially, the only solution is $g_J(b) = 0$. This argument clearly generalises to all values of k except 2, since for all $k > 2$ we can find an infinite set of evenly spaced points where N is even and N/k isn't an integer.

Thus, the final forms of the duality equations are

$$\text{Definitions: } f_n(b) = \sum_{j=0}^{\infty} g_j(b) n^{-2b-j} = \sum_{J=0}^{2n} h_J(-n)(-b)^{2n-J}, \tag{5.39}$$

$$\text{Residue-Matching Eqns: } g_j(-n) = h_j(-n), \quad j \leq 2n, \tag{5.40}$$

$$\begin{aligned} \text{Spurious-Pole Eqns: } g_J(b) &= -\frac{1}{2}\sum_{j=1}^{J}(-1)^j \frac{\Gamma(-2b-J+j+1)}{\Gamma(j+1)\Gamma(-2b-J+1)}(P-b)^j g_{J-j}(b), \quad J \text{ odd}, \\ 0 &= \sum_{j=1}^{J}(-1)^j \frac{\Gamma(-2b-J+j+1)}{\Gamma(j+1)\Gamma(-2b-J+1)}(P-b)^j g_{J-j}(b), \qquad J \text{ even}. \end{aligned} \tag{5.41}$$

The reason for naming the equations such is that, when $k = 2$, all the equations from even N involve matching the residues of poles that in fact exist in the amplitude and all those from odd N are those that involve demanding that the residues of the spurious poles vanish.

These equations are among the main results of this chapter.

5.4.1 Discussion

The first order of business for this duiscussion section is to dispel an obvious objection, which is that the Veneziano amplitude

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \xrightarrow{a\to\infty} a^{-b}, \tag{5.42}$$

which flatly contradicts our assertion that the asymptotic behaviour has to be a^{-2b}. The answer, of course, is that this is not actually the Veneziano amplitude; the full Veneziano amplitude is the sum of three terms

$$A_{Ven}(a,b) = B(a,b) + B(a,c) + B(b,c), \xrightarrow{a\to\infty} a^{-2b}, \tag{5.43}$$

In fact, the full Veneziano amplitude can be written in the functional form of the Virasoro-Shapiro amplitude, see [34] for details.

Turning to other things, one might wonder if these duality equations (5.39), (5.40), (5.41) mean anything physically. While it is hard to give an overarching narrative to these equations, various aspects of them reflect various physical facts.

Most obviously, the value of k is related to the particle with highest spin at every level, as seen in (5.29) for example – the highest spin particle propagating at level n has spin kn. This throws some light on why $k = 2$ is special. If k is odd, then levels at odd values of n

have a highest spin particle with odd spin, but odd spins can't propagate in the scattering of identical scalar bosons; thus, k being odd would require an infinite number of intricate cancellations, and our analyticity arguments are essentially that they can't actually happen. It's not as clear why $k = 2$ is preferred over a generic even k.

First, note that the RMEs (5.40) can be obtained from the definitions of the h_Js and g_Js (5.39) by a very simple prescription: interchanging b and $-n$ interchanges the roles of the h_js and g_js in (5.39) (up to problems with the summation limits). It seems that a very simple prescription ensures channel duality.

The SPEs are equations that express the coefficients of odd powers of b in terms of the coefficients of the even powers of b, as can clearly be seen by their explicit solutions[9]

$$g_{2j+1}(-n) = \sum_{p=0}^{j}(-1)^p(n+P)^{2p+1}\frac{2n+1-2j+2p}{2n-2j}Z(p+1)g_{2j-2p}(-n), \tag{5.44}$$

where

$$Z(x) = \frac{4}{\pi^{2x}}\sum_{k=-\infty}^{\infty}(4k+1)^{-2x} = \frac{4}{(4\pi)^{2x}}\left\{\zeta\left(2x,\frac{1}{4}\right)+(-1)^{2x}\zeta\left(2x,\frac{3}{4}\right)\right\}. \tag{5.45}$$

This fact suggests that the content of the SPEs is that there are no odd spin particles propagating in the amplitude. And this is in fact the case; an arbitrary sum of even-spin Gegenbauer polynomials satsifies these equations, assuming the RMEs.

Thus, the value of k encodes the spin spectrum of the exchanged paricles, the RMEs encode the actual non-trivial s-t crossing symmetry, and the SPEs (given the RMEs) encode the much simpler t-u crossing symmetry (which is equivalent to there not being any odd-spin particles).

5.5 Solving the Bootstrap Equations

These equations can be solved for the scattering of four identical scalars. In this section, we show that in this case there is an infinite-dimensional parameter space of solutions. In particular, it is shown using the bootstrap equations that given a proposed amplitude $A_0(a,b,c)$ that is symmetric in its arguments and has the correct poles and asymptotics, any amplitude of the form

$$A(a,b,c) = \sum_{m=0}^{\infty} a_m A_m(a,b,c) \equiv \sum_{m=0}^{\infty} a_m A_0(a+m,b+m,c+m) \tag{5.46}$$

[9] This is shown numerically in the Mathematica notebook available as ancillary file at arXiv:1707.08135.

also satisfies these same conditions [114–117].

One can start with comparing the poles of the new amplitude with the required poles. The term A_m has poles at $a = -m, -m-1, -m-2, \cdots$ and similarly for b and c, all of which are the poles of A_0, and therefore the entire sum has the same set of poles as the first term. Similarly, the m^{th} term has asymptotic behaviour n^{-2b-2m}, which is dominated by the asymptotic behaviour of A_0. Finally, it is manifestly crossing symmetric. Thus, we expect these class amplitudes to satisfy the crossing equations, (5.39)-(5.41). This is verified in the subsequent discussion of this section.

To usefully solve the bootstrap equations, we need to parametrise the residues in some fashion, so that we work with the minimum amount of independent data. The natural way to parametrise residues in a scattering amplitude is as sums of Gegenbauer polynomials with different spins. For our purpose, however, this is an inconvenient basis, because it is very hard to find the combination of Gegenbauers that has Regge asymptotic behaviour; even in the case of the simplest example of the Virasoro-Shapiro amplitude, the general decomposition of the residues into Gegenbauers is not known, and thus even in that case we can't ascertain exactly how the sum of Gegenbauers attains this behaviour. Much more convenient would be a basis which has the correct asymptotic behaviour. Luckily, there is one right at hand, given by the A_ms in (5.46). Using this decomposition, we'll see that there are no constraints on the coefficients a_m coming from crossing symmetry.

We write the residue as

$$f_n(b) = \sum_{m=0}^{n} a_m F_{n,m}(b), \quad / F_{n.m}(b) = F_{n-m,0}(b+m), \tag{5.47}$$

where $F_{n,m}(b)$ is the residue of A_m at $a = -n$. The expansion of $F_{n,m}(b)$ about $n = \infty$ is

$$F_{n,m}(b) = \sum_{j=0}^{\infty} G_j^m(b) n^{-2b-j} \quad / G_{j<2m}^m(b) = 0. \tag{5.48}$$

The condition on the G_j^ms comes from the fact that $A_m \sim n^{-2b-2m}$.

Because the A_ms satisfy the bootstrap equations themselves, we can apply the RMEs (5.40) to rerwrite the full residue as

$$\begin{aligned} f_n(b) &= \sum_{m=0}^{n} a_m F_{n,m}(b) \\ &= \sum_{m=0}^{n} a_m \sum_{j=0}^{2n} G_j^m(-n)(-b)^{2n-j} \\ &= \sum_{j=0}^{2n} \left\{ \sum_{m=0}^{n} a_m G_j^m(-n) \right\} (-b)^{2n-j}. \end{aligned} \tag{5.49}$$

From here, we can again apply the RMEs to the full amplitude to find

$$g_j(-n) = \sum_{m=0}^{n} a_m G_j^m(-n). \tag{5.50}$$

Now,we can plug this form for the g_j to find

$$\begin{aligned} \sum_{m=0}^{n} a_m G_{2J+1}^m(-n) &= \sum_{m=0}^{n} a_m \left\{ -\frac{1}{2} \sum_{j=1}^{2J+1} (-1)^j \binom{2n-2J+j}{j} (P+n)^j G_{2J+1-j}^m(-n) \right\} \\ 0 &= \sum_{m=0}^{n} a_m \sum_{j=1}^{2J} (-1)^j \binom{2n-2J+j+1}{j} (P+n)^j G_{2J-j}^m(-n). \end{aligned} \tag{5.51}$$

We see that the expression multiplying each a_m exactly vanishes by the fact that the A_ms satisfy crossing, and thus it has been established, using the bootstrap equations, that (5.46) is crossing symmetric.

5.5.1 Discussion

An important question the above analysis leaves unanswered is whether the form (5.46) is the most general allowed form of the amplitude. For tachyons, it is, but not for particles of positive mass-squared.

One way we could have gone about solving the bootstrap equations might have been to parametrise the residues by an arbitrary sum of even-spin polynomials (polynomials that can be written as a sum of even-spin Gegenbauers). Given that the bootstrap equations guarantee us that the residue at level n is a polynomial of degree $2n$, and that the Gegenbauer with spin s is a polynomial of degree s, the n^{th} residue is a sum of $n+1$ even-spin Gegenbauers (of spins $0, 2, \cdots 2n$); in other words, the residue at level n is given by $n+1$ real numbers.

We already used such a paramterisation, (5.47), in which the residue is given by $a_m, m = 0, 1 \cdots n$. Each polynomial used in this parametrisation, further, is guaranteed to be a positive sum of even-spin Gegenbauers, because of the $b-c$ $(t-u)$ symmetry of the different components in (5.46). Thus, it naively seems that this is the most general parametrisation and thus (5.46) is the most general amplitude allowed by the bootstrap equations.

There is a problem, however, because some of these polynomials are in general either 0 or a constant. The problem stems due to the special kinematics of the three-point function of two particle at mass m and one particle at mass $2m$. By going to the rest frame of the heavy particle, one realises that both light particles have 0 spatial momentum. Because the incoming state has no orbital momentum, the heavy particle has to be a scalar. Another way to see the same thing is that if the heavy particle weren't a scalar the interaction vertex would necessarily have some derivatives acting on the light particles, and because of the lack of momentum these derivatives owuld be 0. Further, in the case of three massless particles, the three-point function has to be 0 on-shell.

Because of this, in the case of massless or massive particles, when the particle of mass $2m$ is necessarily in the spectrum, one of the residues of the original amplitude A_0 has a residue which isn't a polynomial of degree $2n$ but a constant. And because of the structure of (5.47), this means that every subsequent residue is short a polynomial. Thus, it is not obviously true that (5.46) is the most general amplitude satisfying our conditions when the external particles aren't tachyons. We have not been able to usefully add a polynomial at each level that corrects this problem to check whether it is the most general form, however.

5.6 Unitarity

Having shown that crossing symmetry allows for an infinite-dimensional parameter space of amplitudes, we may still hope that unitarity constrains it more. While unitarity is hard to analyse in general, we numerically argue in this section that unitarity still alows for an open set in this infinite-dimensional paramter space – that is, we argue that arbitrary small perturbations don't violate unitarity. It is difficult to make an argument in the general case, so the class of amplitudes that can be constructed with A_0 as the Virasoro-Shapiro amplitude is considered in 4 spacetime dimensions.

First, ampltiudes of the form $A_0 + a_m A_m$ are considered for a particular value of m. The coupling constant of the l^{th} Regge trajectory (set of particles with spin $2(n-l)$ for all levels n) takes the form

$$\lambda^2_{n,2n-2l} = f_l(n) + a_m g_l(n). \tag{5.52}$$

The unitarity constraint, that this be positive, provides a lower bound for a_m for values of n such that $g_l(n)$ is positive; and an upper bound when $g_l(n)$ is negative. Maximising the lower bound and minimising the lower bound over all n in the respective regions gives the unitarity constraint on a_m coming from the l^{th} Regge trajectory.

We calculated it for $m = 1, 2 \cdots 6$ using 8 Regge trajectories. The bounds obtained on the various coefficients are as follows,[10]

$$\begin{aligned}
-8. <&a_1 < 246\\
-1230. <&a_2 < 1744\\
-753408. <&a_3 < 545260\\
-6.19451 \times 10^8 <&a_4 < 5.61988 \times 10^8\\
-1.04535 \times 10^{12} <&a_5 < 1.30753 \times 10^{12}\\
-3.5675 \times 10^{15} <&a_6 < 5.70538 \times 10^{15}.
\end{aligned} \tag{5.53}$$

Second, we consider perturbations with two subleading Virasoro-Shapriros at a time, $A_0 + a_{m_1} A_{m_1} + a_{m_2} A_{m_2}$. The cases where (m_1, m_2) takes the values $(1, 2)$, $(1, 3)$ and $(2, 3)$ are considered. Then, we consider 25×25 points in the region allowed by eqn (5.53) and check (again, a Regge trajectory at a time) whether each of these points is allowed by unitarity or not. For this, we used 5 Regge trajectories. The allowed values of the parameters are coloured in yellow in fig Figure 5.2.

5.6.1 Discussion

It can be seen from the above numerical results that the unitarity of the Virasoro-Shapiro ampltude is stable to perturbations. The cases of three or more subleading Virasoro-Shapiros was not subsequently considered because it seems pretty apparent from the above numbers and graphs that unitarity doesn't strongly constrain the amplitude.

One explanation of this observed stability of the Virasoro-Shapiro amplitude to perturbations is that it doesn't seem to have any couplings that are 0, [119], and so there's always a perturbation small enough that it doesn't cause a problem. Further, even if one of the couplings were 0, it would only put a hard constraint on perturbation in one direction. Thus, to get a non-trivial unitarity constraint, it must be that the same perturbation moves two couplings that are 0 in oppposite directions.

[10] Numerical computations of these bounds are available in the Mathematica files at arXiv:1707.08135.

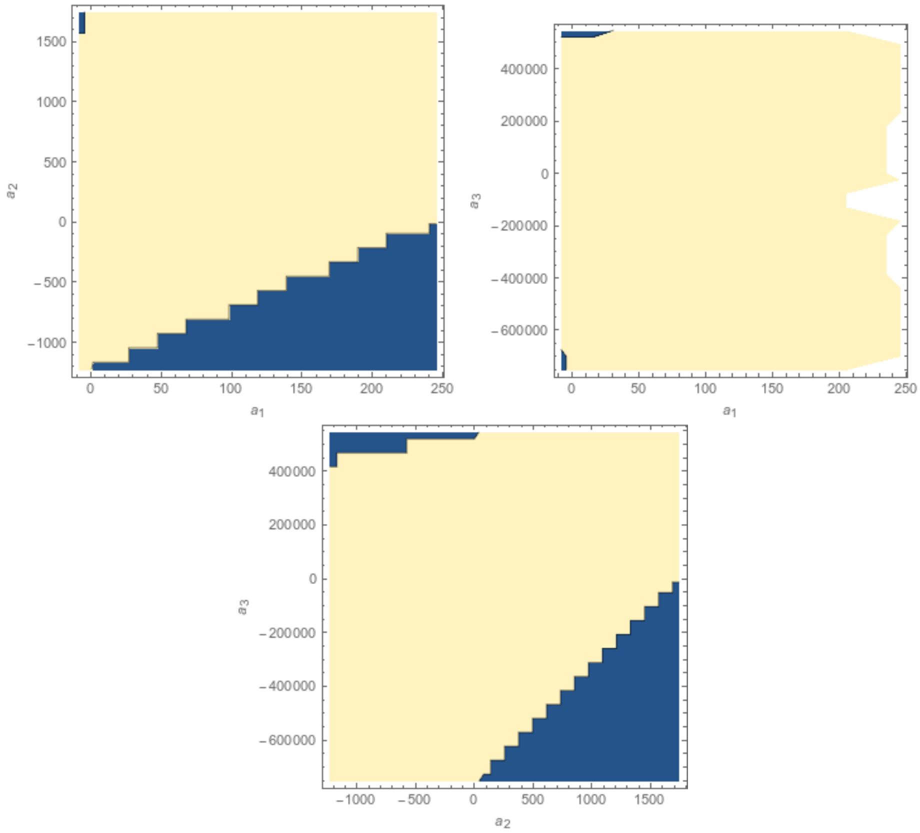


FIGURE 5.2: We checked unitarity for 25×25 points for amplitudes of the form $A_0 + a_{m_1}A_{m_1} + a_{m_2}A_{m_2}$, using couplings from 5 Regge trajectories. The yellow regions are the ones allowed by unitarity, The three cases are (a) $m_1 = 1, m_2 = 2$ (b) $m_1 = 1, m_2 = 3$ (c) $m_1 = 2, m_2 = 3$. The ranges scanned were decided by the bounds shown in eqn (5.53).

5.7 Conclusions

In this work, the conditions imposed by Regge asymptotic behaviour, crossing symmetry and unitarity on the scattering amplitude of four identical scalars was studied in the case when the exchanged particles have a linear spectrum. In the general case, a countably infinite set of bootstrap equations required for channel duality, (5.27), were found. Moreover, the exponent in the Regge behaviour is restricted to be $k(b) = 2b$, and in this case the bootstrap equations are reduced to a simpler form given in (5.39), (5.40) and (5.41). We've shown that these equations allow an infinite-dimensional parameter space of solutions and that unitarity doesn't seem to impose strong enough constraints to reduce the dimensionality of the parameter space.

The above results can be improved by calculating many different amplitudes featuring overlapping sets of cubic couplings; and to truly understand the conditions posed by unitarity, one must grapple with this whole morass of amplitudes. Not only is it a much tougher problem, it is also apparent that the study of unitarity of the Virasoro-Shapiro amplitude is extremely cumbersome without resorting to the worldsheet (in fact, the worldsheet was discovered in the process of investigating the unitarity of these amplitudes). However, a problem of immediate interest that begs further investigation is to try and understand what happens if we demand that the spectrum of intermediate particles be only asymptotically linear (the spectrum becomes linear only for very massive exchanged particles). Besides this, the original problem of understanding graviton scattering amplitudes still remains an important open problem.

Chapter 6

Tree-Level Scattering Amplitudes in Chern-Simons Theories with Fundamental Matter

6.1 Introduction

Chern Simons gauge theories (with gauge group $SU(N)$ or $U(N)$) coupled to matter fields have a wide variety of applications in areas as diverse as quantum hall physics, anyonic physics, topology of three manifolds, quantum gravity via the AdS/CFT correspondence, etc. These theories are conjectured to enjoy a strong-weak duality that has been tested in several intense computations at large N and κ, while keeping the 't Hooft coupling $\lambda = \frac{N}{\kappa}$ fixed [42, 120–147]. Recently, a finite N, κ form of the duality was proposed [148–158]. An example of the strong-weak duality is the duality between Chern-Simons gauge theory coupled to fundamental fermions and Chern-Simons gauge theory coupled to fundamental critical bosons. Other examples include self dual theories, such as $\mathcal{N} = 1$, $\mathcal{N} = 2$ supersymmetric CS matter theories. Very recently, at large N it was demonstrated that the S matrix for the $2 \to 2$ scattering computed exactly to all orders in the 't Hooft coupling displays an unusual modified crossing relation [133, 142, 159]. Moreover, for $\mathcal{N} = 2$ theory, the result is tree level exact [159] except in the anyonic channel, where it gets renormalized by a simple function of the 't Hooft coupling.

A natural question to ask would be, is it possible to compute arbitrary $m \to n$ scattering amplitudes at all values of the 't Hooft coupling at large N, κ? Given the simplicity of the results at least in the supersymmetric case, it is also interesting to ask if the computability of scattering amplitudes extends to finite N, κ. However, the usual method of computing these amplitudes via the Feynman diagrams, though very useful, is restricted only to low orders in perturbation theory and small number of scattering particles as the computational difficulty rapidly increases with the number of loops and particles involved (and especially with the number of particles involved, [160]). And despite this computational difficulty, it has been observed in various cases that the final result of the scattering amplitudes is rather simple [161–165]. This leads to the following conclusions:

1. The dramatic cancelation between the Feynman diagrams that leads to simpler amplitudes implies not all Feynman diagrams are independent and are probably highly redundant.

2. Secondly, most probably the use of Feynman diagrams is not the most efficient way to compute the scattering amplitudes.

A case for the first point above is made in various papers (see [166, 167] for original work and [168, 169] and references therein for recent reviews) in context of Yang-Mills theory where they show that of the $\sim n!$ color-ordered Feynman diagrams that one would draw, only $(n-3)!$ are independent. However, this isn't yet drastically simplifying. Over the past decade or so, significant progress has been made by approaching this problem mainly from two ways. Firstly, methods have been developed in which an arbitrary higher n-point function can be recursively computed in terms of the lower point functions, which are comparatively easy to compute, [170–172]. This provides an excellent tool for both analytical as well as numerical computations. Secondly, a lot of progress has been made in developing alternative ways to compute amplitudes that utilize hidden symmetry structures not visible directly from the Lagrangian [173–175], both for tree amplitudes as well as for efficient loop computations (see [169] and reference therein for more details). In these works, these novel tools have been used in context of both supersymmetric and non-supersymmetric Yang-Mills theories and $\mathcal{N}=6$ ABJM theory, which is a Chern-Simons gauge theory in 3-dimensions coupled to bi-fundamental matter.

In this chapter, these techniques are to both supersymmetric and non-supersymmetric Chern-Simons theories coupled to vector matter to compute the scattering amplitudes in these theories. As a first step towards these questions, we compute all tree level amplitudes for the $\mathcal{N}=2$ theory and the regular fermionic theory. The self-dual $\mathcal{N}=2$ supersymmetric theory is particularly interesting and important since via RG flow, we can obtain non supersymmetric dual pairs such as critical bosons coupled to CS and regular fermions coupled to CS [131, 138].

The introduction to the special 3-dimensional kinematics that are used in this chapter are available in Chapter 11 of [169]. The remaining chapter is organized as follows. In section 6.2 the four point scattering amplitude in the fermionic and $\mathcal{N}=2$ theory is reviewed.

In section 6.3 general criteria for the BCFW recursions to hold for the $\mathcal{N}=2$ theory is presented. In section 6.4, we give a formal argument using background field method to show that BCFW works for the $\mathcal{N}=2$ theory. In section 6.5 we present a recursion relation for all tree level amplitudes for the $\mathcal{N}=2$ theory. Furthermore in section 6.6, we discuss how to use the $\mathcal{N}=2$ results to obtain recursion relations for all tree level amplitudes in fermionic theory. We end the letter with a discussion and possible future directions.

6.2 Four point scattering amplitude

In this section, the scattering amplitudes are computed in following theories:

1. Fermion coupled to $SU(N)$ Chern-Simons theory (FCS)

$$\int d^3x\left[-\frac{\kappa}{4\pi}\epsilon^{\mu\nu\rho}\mathrm{Tr}\left(A_\mu\partial_\mu A_\rho-\frac{2i}{3}A_\mu A_\nu A_\rho\right)+\bar{\psi}i\not{D}\psi\right], \tag{6.1}$$

2. $\mathcal{N}=2$ Chern-Simons matter theory coupled to a Chiral multiplet given by

$$\begin{aligned}\mathcal{S}^L_{\mathcal{N}=2}=\int d^3x&\left[-\frac{\kappa}{4\pi}\epsilon^{\mu\nu\rho}\mathrm{Tr}\left(A_\mu\partial_\mu A_\rho-\frac{2i}{3}A_\mu A_\nu A_\rho\right)\right.\\&+\bar{\psi}i\not{D}\psi-\mathcal{D}^\mu\bar{\phi}\mathcal{D}_\mu\phi+\frac{4\pi^2}{\kappa^2}(\bar{\phi}\phi)^3+\frac{4\pi}{\kappa}(\bar{\phi}\phi)(\bar{\psi}\psi)\\&\left.+\frac{2\pi}{\kappa}(\bar{\psi}\phi)(\bar{\phi}\psi)\right].\end{aligned} \tag{6.2}$$

For our purposes, it is convenient to introduce the spinor helicity basis [169] defined by

$$p_i^{\alpha\beta}=p_i^\mu\sigma_\mu^{\alpha\beta}=\lambda_i^\alpha\lambda_i^\beta,\quad (p_i+p_j)^2=2p_i.p_j=\langle\lambda_i^\alpha\lambda_{i,\alpha}\rangle^2\ . \tag{6.3}$$

A shorthand notation is used $\langle\lambda_i^\alpha\lambda_{j,\alpha}\rangle=\langle ij\rangle$ in the following discussion. For a supersymmetric amplitude, the standard procedure involves introduction of on-shell Grassmann variables θ such that the super-creation and super-annihilation operators are given by

$$A_i=a_i+\theta_i\alpha_i,\quad A_i^\dagger=\theta_i a_i^\dagger+\alpha_i^\dagger, \tag{6.4}$$

where $\left(a_i^\dagger,a_i\right)/\left(\alpha_i^\dagger,\alpha_i\right)$ create and annihilate a boson/fermion with momenta p_i respectively. The two on-shell supercharges for n point scattering amplitudes are given by

$$Q=\sum_{i=1}^n q_i=\sum_{i=1}^n\lambda_i\theta_i,\quad \bar{Q}=\sum_{i=1}^n\bar{q}_i=\sum_{i=1}^n\lambda_i\partial_{\theta_i}\ . \tag{6.5}$$

For FCS theory in (6.1), the tree level $2 \to 2$ scattering amplitude is given by [133]

$$A_4^F = \langle \bar{\psi}(p_1)\psi(p_2)\bar{\psi}(p_3)\psi(p_4)\rangle = \frac{\langle 12\rangle\langle 24\rangle}{\langle 23\rangle}\delta(\sum_{i=1}^{4} p_i) \ . \tag{6.6}$$

For $\mathcal{N} = 2$ theory in (6.2), the tree level $2 \to 2$ super amplitude is given by

$$A_4^S = \frac{\langle 12\rangle}{\langle 23\rangle}Q^2 = \frac{\langle 12\rangle}{\langle 23\rangle}\delta(\sum_{i=1}^{4} p_i) \sum_{1=i<j}^{4} \langle ij\rangle\theta_i\theta_j \ . \tag{6.7}$$

Here A_4^S is the super-amplitude computed using the super-creation/annihilation operators defined in (6.4). Any component amplitude can be obtained from (6.7) by picking up the coefficient of products of two θ's. For example, the four fermion amplitude is given by the coefficient of $\theta_2\theta_4$ that coincides precisely with (6.6).

6.3 Higher point scattering amplitude

BCFW recursion relations are an efficient method to compute and express arbitrary higher point scattering amplitudes in terms of product of lower point amplitudes. Standard procedure for BCFW involves the deformation of two external momenta of the particles by a complex parameter z. The deformation is such that the particles continue to remain 'on shell' and the total momentum conservation of the process continues to hold. In 3-dimensions, BCFW deformations are a little more involved than in 4-dimensions and were first discussed in [176] (the discussion in section 2 of this reference is followed closely here). BCFW recursion relations are applicable in 3-dimensions provided that the higher point amplitudes are regular functions at $z \to \infty$ and $z \to 0$. In the following section the $z \to \infty$ (and $z \to 0$) behavior of the amplitudes in the theories described earlier is studied. We find it convenient to deform color contracted (we label them as '1' and '2') external legs. In 3-dimensions, momentum deformation of particles 1 and 2 can be written in terms of the spinor-helicity variables as

$$\begin{pmatrix} \hat{\lambda}_1 \\ \hat{\lambda}_2 \end{pmatrix} = R \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \quad \text{where} \quad \mathrm{R} = \begin{pmatrix} \frac{z+z^{-1}}{2} & -\frac{z-z^{-1}}{2\,i} \\ \frac{z-z^{-1}}{2\,i} & \frac{z+z^{-1}}{2} \end{pmatrix} \ . \tag{6.8}$$

In the theories (6.1),(6.2), all 3-point vertices involve gauge fields and since the Chern-Simons gauge field does not have an on shell propagating degree of freedom, it follows that only even-point functions are non-zero. This also implies that the 4-point functions are fundamental building blocks for higher point functions.

Under the deformation (6.8), any tree-level scattering amplitude for FCS in (6.1) is not well behaved at large z and hence doesn't obey the requirements of BCFW. However this situation is cured for the $\mathcal{N} = 2$ theory defined in (6.2). Additionally, conservation of the super-charges in (6.5) require that the on-shell spinor variables θ be deformed as

$$\begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{pmatrix} = R \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} , \tag{6.9}$$

where the R matrix is defined by (6.8).

Here, the $2n$-point super-amplitude is denoted by $A_{2n}(\lambda_1, \lambda_2, \cdots \lambda_{2n}, \theta_1, \theta_2, \cdots \theta_{2n})$ and the deformed amplitude by $A_{2n}(\hat{\lambda}_1, \hat{\lambda}_2, \cdots \lambda_{2n}, \hat{\theta}_1, \hat{\theta}_2, \cdots \theta_{2n}, z)$. The deformed super-amplitude can be explicitly written as an expansion in the θ variables as follows

$$\begin{aligned} A_{2n}(z) &= A^0(z) + A^1(z)\hat{\theta}_1(z) + A^2(z)\hat{\theta}_2(z) \\ &\qquad\qquad\qquad + A^{12}(z)\hat{\theta}_1(z)\hat{\theta}_2(z) \\ &= A^0(z) + \tilde{A}^1(z)\theta_1 + \tilde{A}^2(z)\theta_2 + A^{12}(z)\theta_1\theta_2, \end{aligned} \tag{6.10}$$

where in the last line of (6.10) we have used (6.8) and the fact that $\hat{\theta}_1(z)\hat{\theta}_2(z) = \theta_1\theta_2$. We have also defined

$$\begin{pmatrix} \tilde{A}^1(z) \\ \tilde{A}^2(z) \end{pmatrix} = R^T \begin{pmatrix} A^1(z) \\ A^2(z) \end{pmatrix} , \tag{6.11}$$

where R^T is the transpose of the R matrix defined in (6.8), with $RR^T = 1$. The super-momentum conservation implies that the large z behavior of the super-amplitude $A_{2n}(z)$ is identical to that of the components A^0, A^{12}. Hence it is sufficient to show that either of A^0 or A^{12} are well behaved since supersymmetric Ward identity guarantees the required behavior for the rest of the amplitudes. It is convenient to write the fields in pair wise contractions since they transform in the fundamental representation of the gauge group. For instance we are interested in the large z behavior of amplitudes such as $(\bar{\psi}_1^i \phi_{2i})(\bar{\phi}_3^j \psi_{4j}) \ldots$ and $(\bar{\phi}_1^i \psi_{2i})(\bar{\psi}_3^j \phi_{4j}) \ldots$, where $\ldots$ represent color contracted bosonic or fermionic particles allowed by interactions in (6.2). These amplitudes appear in A^0, A^{12} in (6.10) respectively.

The explicit Feynman diagram computations of the amplitude $A^0 = A_6(\bar{\psi}_1\phi_2\bar{\phi}_3\psi_4\bar{\phi}_5\phi_6)$ and the verification that its $z \to \infty$ behaviour is well behaved, is presented in Appendix N. We discuss the large z behavior of the general $2n$ point amplitude using the background field method [165] in the next section.

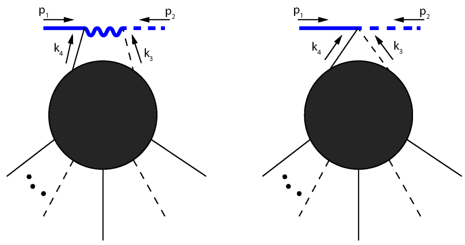


FIGURE 6.1: The diagrams that have a non-regular $z \to \infty$ behavior. $\mathcal{O}(z)$ part of these two diagrams cancel against each other to give a regular $z \to \infty$ behavior of the total amplitude. In the above diagram, the solid lines correspond to fermions and the dashed lines correspond to bosons. This amplitude appears in A^0 in (6.10). The blue color lines corresponds to deformed hard particle.

6.4 Asymptotic behavior of amplitudes

To understand the large z behavior of various scattering amplitudes, it is extremely useful to think from the background field method point of view introduced in [165]. Here z-deformed particles are considered as hard particles propagating in a background of soft particles. The amplitude is modified due to (a) modified propagator of intermediate hard particle; (b) the modified contribution of various vertices; and, (c) modified fermion wave function, in case an external deformed particle is a fermion. Detailed analysis shows (see Appendix O) that the non-trivial $z \to \infty$ behavior of the amplitude is due to diagrams of the kind depicted in fig. 6.1. The values of these diagrams are:

$$\text{Gauge-field exchange: } \frac{4\pi i}{\kappa} \langle k_4|\gamma^\mu|1\rangle \frac{k_3^\nu p_2^\rho}{(k_3+p_2)^2} \epsilon_{\mu\nu\rho} \tag{6.12}$$

$$\text{Contact vertex: } -\frac{2\pi}{\kappa}\langle k_4|1\rangle \tag{6.13}$$

Under the 1-2 z-deformations, (6.8), in the $z \to \infty$ limit the $\mathcal{O}(z)$ part of the amplitude cancels and the amplitude behaves as $\mathcal{O}(1/z)$. Hence this amplitude has a regular $z \to \infty$ behavior for $\mathcal{N} = 2$ theory. This cancellation works even for the 4-point function $\langle \bar{\psi}_1 \phi_2 \bar{\phi}_3 \psi_4 \rangle$, which receives contributions from the diagrams in fig. 6.1 with the blob removed and $k_3 \to p_3, k_4 \to p_4$ are taken to be on-shell momenta. It is important to emphasize that we need minimum $\mathcal{N} = 2$ amount of supersymmetry for this to work. [1]

6.5 Recursion relations in $\mathcal{N} = 2$ theory

In the last section, it was demonstrated that A^0 is well behaved in large z. Hence we can apply the BCFW recursion relation directly to the super amplitude in the left hand side of

[1] see Appendix O

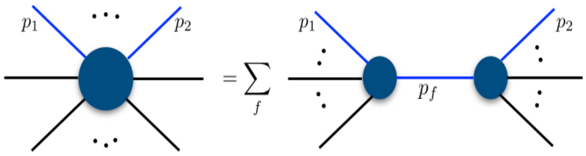


FIGURE 6.2: Recursion formula for a 2n point amplitude: The black lines denote the undeformed legs, the external blue lines represent the deformed legs and p_f represents the momentum in the factorization channel.

(6.10). The recursion formula for a $2n$ point superamplitude can be expressed in terms of lower point superamplitudes as follows (see fig 6.2)

$$A_{2n}(z=1) = \sum_f \int \frac{d\theta}{p_f^2}\Bigg(z_{a;f}\frac{z_{b;f}^2 - 1}{z_{a;f}^2 - z_{b;f}^2} A_L(z_{a;f},\theta)A_R(z_{a;f}, i\theta) + (z_{a;f} \leftrightarrow z_{b;f})\Bigg) , \tag{6.14}$$

where the integration is over the intermediate Grassmann variable θ and $A_{2n}(z=1)$ is the undeformed $2n$-point amplitude that we are interested in computing. In the above, p_f is the undeformed momentum that runs in the factorization channel f and the summation in (6.14) runs over all the factorization channels corresponding to different intermediate particles going on-shell. Here, $z_{a;f}$ and $z_{b;f}$ are given by

$$\left(z_{a;f}^2\ ,\ z_{b;f}^2\right) = \frac{-(p_f - p_2).(p_f + p_1) \pm \sqrt{(p_f - p_2)^2(p_f + p_1)^2}}{4q.(p_f - p_2)} , \tag{6.15}$$

where the null momenta q are defined in terms of the spinor helicity variables as

$$q^{\alpha\beta} = \frac{1}{4}(\lambda_2 + i\lambda_1)^\alpha(\lambda_2 + i\lambda_1)^\beta\ , \tilde{q}^{\alpha\beta} = \frac{1}{4}(\lambda_2 - i\lambda_1)^\alpha(\lambda_2 - i\lambda_1)^\beta\ . \tag{6.16}$$

Note that the formula (6.14) has a very similar form (but not quite the same as discussed below) to the one obtained in [176] for the ABJM theory[2] that enjoys $\mathcal{N} = 6$ supersymmetry. It is remarkable that such recursion formulae exist in a theory with much lesser supersymmetry such as the one in discussion.

As an explicit demonstration, consider the six point[3] amplitude $A_6(\lambda_1 \dots \lambda_6) \equiv (\bar{\phi}\psi)(\bar{\psi}\phi)(\bar{\phi}\phi)$

[2] Although, formula (6.14) looks very similar to ABJM case, the details are different since the external matter particles are in fundamental representation. For example, in general there will be more factorization channels in this case as compared to the ABJM case. For example, in the six point function, as will be clear below, there are two factorized channels, where as for the corresponding deformation in ABJM, there is only one factorized channel.

[3] A general six point super amplitude in $\mathcal{N} = 2$ theory can be written in terms of two independent functions as

$$A_6 = Q^2\left(f_1(p)\sum_{i=1}^{3}\epsilon^{ijk}\lambda(p_j)\lambda(p_k)\theta_i + f_2(p)\sum_{i=4}^{6}\epsilon^{ilm}\lambda(p_l)\lambda(p_m)\theta_i\right) \tag{6.17}$$

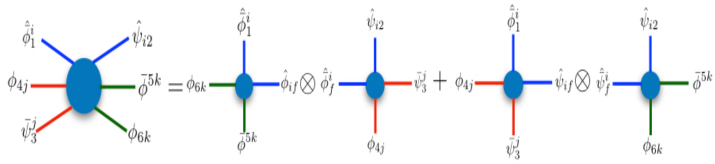


FIGURE 6.3: BCFW recursion for the six point amplitude: Factorization into two channels: Each four point amplitude on the RHS is on shell. Two adjacent lines with the same color are color contracted. Note that the blue lines in particular represent the BCFW deformed legs.

in the $\mathcal{N}=2$ SCS theory. This is the same as the A^0 component amplitude in (6.10) for $\bar\Phi_1\Phi_2\bar\Phi_3\Phi_4\bar\Phi_5\Phi_6$ scattering. This amplitude factorizes into two channels as shown in fig 6.3 (and discussed in detail in Appendix N). The recursion formula can be explicitly written as

$$
\begin{aligned}
\langle\bar\phi_1\psi_2\bar\psi_3\phi_4\bar\phi_5\phi_6\rangle =&\Bigg(z_{a;f}\frac{z_{b;f}^2-1}{z_{a;f}^2-z_{b;f}^2}\langle\hat{\bar\phi}_1\hat\phi_f\bar\phi_5\phi_6\rangle_{z_{a;f}}\langle\hat{\bar\phi}_{(-f)}\hat\psi_2\bar\psi_3\phi_4\rangle_{z_{a;f}}\\
&\qquad\qquad\qquad\qquad\qquad+(z_{a;f}\leftrightarrow z_{b;f})\Bigg)\frac{i}{p_f^2}\Bigg|_{p_f=p_{234}}\\
&+\Bigg(z_{a;f}\frac{z_{b;f}^2-1}{z_{a;f}^2-z_{b;f}^2}\langle\hat{\bar\phi}_1\hat\psi_f\bar\psi_3\phi_4\rangle_{z_{a;f}}\langle\hat{\bar\psi}_{(-f)}\hat\psi_2\bar\phi_5\phi_6\rangle_{z_{a;f}}\\
&\qquad\qquad\qquad\qquad\qquad+(z_{a;f}\leftrightarrow z_{b;f})\Bigg)\frac{i}{p_f^2}\Bigg|_{p_f=p_{256}}, \quad (6.18)
\end{aligned}
$$

where $z_{a;f}, z_{b;f}$ are defined in (6.15). Fields with hats corresponds to deformed momenta. We have checked (6.18) explicitly by computing the relevant Feynman diagrams. It is a curious fact that, the total number of Feynman graphs that contribute to A_6 is 15. Of these, eleven are reproduced by the channel $p_f = p_{234}$ and the remaining four in the channel $p_f = p_{256}$.

6.6 Recursion Relations in the Fermionic Theory

In this section, it is shown that the BCFW recursion relations can be used to compute $2n$−point amplitude $A_{2n} = (\bar\psi_1\psi_2)\cdots(\bar\psi_{2n-1}\psi_{2n})$ for the regular fermionic theory coupled to CS gauge field (6.1). If the deformations (6.8) are applied to this amplitude, it is easy to show that, it does not have a good large z (as well as $z\to 0$) behavior, hence we cannot readily apply the BCFW recursion relation[4] to determine all higher point fermionic amplitudes. However, we show below that we can use the recursion relation of the $\mathcal{N}=2$ to write a recursion relation for the fermionic theory.

where $Q=\sum_{i=1}^6\lambda_i\theta_i$ as defined in (6.5).

[4]There will be some non-trivial boundary terms that do not vanish and in general there are no good prescriptions to compute them systematically.

As a first step towards this, let us note that the Feynman diagrams for any tree-level all-fermion scattering amplitude in the $\mathcal{N}=2$ theory (6.2) is identical to that of the tree-level scattering amplitude in the fermionic theory (6.1). In the previous section it was proved for the $\mathcal{N}=2$ theory that an arbitrary higher-point super-amplitude can be written only in terms of the 4-point super-amplitude. Same can be said for the component amplitudes including the purely fermionic component amplitude.[5] Let us note that for the four point super-amplitude, supersymmetry relates all the component 4-point amplitudes to one component amplitude, which for instance can be taken to be 4-fermion scattering amplitude (see (6.7)). Thus an arbitrary higher-point component amplitude can be written only in terms of 4-fermion amplitude. This can be recursively done for an arbitrary $2n$ point amplitude, however for simplicity we write the recursion relation for the six point amplitude below

$$\begin{aligned}
\langle\bar{\psi}_1\psi_2\bar{\psi}_3\psi_4\bar{\psi}_5\psi_6\rangle = &\Bigg(z_{a;f}\frac{z_{b;f}^2-1}{z_{a;f}^2-z_{b;f}^2}\left[-\frac{z_{a;f}^2+1}{2z_{a;f}}+i\,\frac{z_{a;f}^2-1}{2z_{a;f}}\,\frac{\langle\hat{1}4\rangle}{i\langle\hat{f}4\rangle}\frac{\langle\hat{f}6\rangle}{\langle\hat{2}6\rangle}\right] \\
&\qquad\qquad\times\langle\hat{\bar{\psi}}_1\hat{\psi}_f\bar{\psi}_3\psi_4\rangle\langle\hat{\bar{\psi}}_{(-f)}\hat{\psi}_2\bar{\psi}_5\psi_6\rangle_{z_{a;f}} \\
&+(z_{a;f}\leftrightarrow z_{b;f})\Bigg)\frac{i}{p_f^2}\Bigg|_{p_f=p_{234}} \\
&-\Bigg(z_{a;f}\frac{z_{b;f}^2-1}{z_{a;f}^2-z_{b;f}^2}\left[-\frac{z_{a;f}^2+1}{2z_{a;f}}+i\,\frac{z_{a;f}^2-1}{2z_{a;f}}\,\frac{\langle\hat{1}6\rangle}{i\langle\hat{f}6\rangle}\frac{\langle\hat{f}4\rangle}{\langle\hat{2}4\rangle}\right] \\
&\qquad\qquad\times\langle\hat{\bar{\psi}}_1\hat{\psi}_f\bar{\psi}_5\psi_6\rangle\langle\hat{\bar{\psi}}_{(-f)}\hat{\psi}_2\bar{\psi}_3\psi_4\rangle_{z_{a;f}} \\
&+(z_{a;f}\leftrightarrow z_{b;f})\Bigg)\frac{i}{p_f^2}\Bigg|_{p_f=p_{256}}
\end{aligned} \tag{6.19}$$

Hence, while the amplitudes in the fermionic theory by themselves don't obey the requirements for BCFW relations, using $\mathcal{N}=2$ theory we can find out the recursion relations for the fermionic theory too.

6.7 Discussion

This chapter established the validity of BCFW recursion relations for all tree level amplitudes in $\mathcal{N}=2$ CS matter theory and CS theory coupled to regular fermions. The knowledge of arbitrary scattering amplitudes to all loop-orders is important to verify the duality statements that were made in the starting of this chapter. Computation of these

[5]Note that the recursion relation in the $\mathcal{N}=2$ theory (6.14) does not directly give $(\bar{\psi}_1\psi_2)\cdots(\bar{\psi}_{2n-1}\psi_{2n})$ in terms of the lower point fermionic amplitude. However, we can use BCFW relations recursively to write down any higher point amplitude in terms of four point amplitudes such as $(\bar{\psi}\psi)(\bar{\psi}\psi)$, $(\bar{\phi}\phi)(\bar{\phi}\phi)$, $(\bar{\phi}\phi)(\bar{\psi}\psi)$, $(\bar{\phi}\psi)(\bar{\psi}\phi)$ etc. Moreover, at the level of the four point amplitude, one can rewrite this in terms of $(\bar{\psi}\psi)(\bar{\psi}\psi)$. For example, $(\bar{\psi}\psi)(\bar{\phi}\phi)=\frac{\langle 23\rangle}{\langle 24\rangle}(\bar{\psi}\psi)(\bar{\psi}\psi)$. This implies that we get a recursion relation for $(\bar{\psi}_1\psi_2)\cdots(\bar{\psi}_{2n-1}\psi_{2n})$ in terms of lower point fermionic amplitudes only. Hence this can be interpreted as a BCFW recursion relation in the regular fermionic theory coupled to CS gauge field (6.1).

scattering amplitudes is a first step in this direction. Moreover, it was shown in [159], that the $2 \to 2$ scattering amplitude in the $\mathcal{N} = 2$ theory does not get renormalized except in the anyonic channel, where it gets renormalized by a simple function of the 't Hooft coupling. A natural question is, why in the $\mathcal{N} = 2$ theory the scattering amplitude has such a simple form, whereas the corresponding amplitudes in the fermionic [133] and other less susy $\mathcal{N} = 1$ [159] theories are quite complicated. A possible explanation is that there exists some symmetry such as dual conformal invariance that appears in the $\mathcal{N} = 2$ theory and it protects the amplitude from loop corrections [177]. It is natural to ask, if the simplicity of the amplitudes continues to persist with higher point amplitudes. It is also interesting to explore an analog of the Aharonov-Bohm phase for higher point amplitudes. It may very well turn out that the Aharanov-Bohm phases of higher point amplitudes are products of the Aharonov-Bohm phases of the $2 \to 2$ amplitude. BCFW recursion relations provide a strong indication towards this result.

To answer the above questions, one needs to compute higher scattering amplitudes to all orders in λ. A possible way is to investigate the Schwinger-Dyson equation. However, the Schwinger-Dyson equation approach is quite complicated even at the 6-point level. A refined approach might be to look for a larger class of symmetries such as Yangian symmetry [177] and use the powerful formulation of [178] to obtain results. Given the fact that, these theories are exactly solvable at large-N as well as the fact that $\mathcal{N} = 2$ theory is self-dual, it could turn out that the $\mathcal{N} = 2$ theory may be one of the simplest play grounds to develop new techniques in computing S-matrices to all orders [178]. Furthermore exact solvability at large N indicates that these models might even be integrable. One possible way to investigate integrability is to show the existence of an infinite dimensional Yangian symmetry. Since these theories relate to various physical situations, any of the above exercises may provide insight into finite N, κ computations.

Appendix A

Notations & Identities

A.1 For Chapter 2

We will generally use the notation f_n for the double-trace couplings introduced in (2.2), and $\bar{f}_n$ for their dimensionless counterparts. In addition, depending on the context we will denote these couplings by the following specialized notations:

Field Theory	Dimensionful: f	Dimensionless: $\bar{f}$
Bulk	Dimensionful: $\mathfrak{f}$	Dimensionless: $\bar{\mathfrak{f}}$

A.2 For Chapter 3

For $g_{\alpha\beta} = e^{2\phi}\hat{g}_{\alpha\beta}$

$$R = e^{-2\phi}\Big(\hat{R} - 2\hat{\Box}\phi\Big) \tag{A.1a}$$

$$\Box = e^{-2\phi}\hat{\Box} \tag{A.1b}$$

$$g := \det(g_{\alpha\beta}) = e^{4\phi}\hat{g} \tag{A.1c}$$

$$n_\mu = e^{\phi}\hat{n}_\mu \tag{A.1d}$$

$$n^\mu = e^{-\phi}n^\mu \tag{A.1e}$$

$$\gamma := \det\{\gamma_{\alpha\beta}\} = e^{2\phi}\hat{\gamma} \tag{A.1f}$$

$$\sqrt{\gamma}\mathcal{K} = \sqrt{\hat{\gamma}}\Big(\hat{\mathcal{K}} + \hat{n}^\mu\partial_\mu\phi\Big) \tag{A.1g}$$

A.3 For Chapter 5

A.3.1 Notations and Conventions for $2+1$D $\mathcal{N}=1,2$ SUSY

In this section, we present some notations and conventions that we will use throughout the article. These are taken from [179] and [159].

A.3.2 Spinors

We work with the metric signature $(-,+,+)$. The Lorentz group in $2+1$D is $SL(2,\mathbb{R})$. The fundamental representation of the group acts on two-component real spinors $\psi^\alpha \equiv (\psi^+,\psi^-)$. These are also referred to as *Majorana* spinors. The spinor indices are raised and lowered using the antisymmetric symbol $C_{\alpha\beta}$, defined by

$$C_{\alpha\beta} = -\,C_{\beta\alpha} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -\,C^{\alpha\beta}. \tag{A.2}$$

Notice that $C_{\alpha\beta}$ is same as the Pauli matrix σ_2. Spinor index raising and lowering follows the NW-SE convention, which implies

$$\begin{aligned} \psi^\alpha &= C^{\alpha\beta}\psi_\beta, \\ \psi_\alpha &= \psi^\beta C_{\beta\alpha}. \end{aligned} \tag{A.3}$$

Thus, ψ_α is purely imaginary. A crucial property satisfied by the antisymmetric symbol $C_{\alpha\beta}$ is

$$C_{\alpha\beta}C^{\sigma\tau} = \delta^\sigma_\alpha\delta^\tau_\beta - \delta^\sigma_\beta\delta^\tau_\alpha. \tag{A.4}$$

As a consequence of eq.(A.4), we get

$$\begin{aligned} C_{\alpha\gamma}C^{\gamma\beta} &= -\,\delta^\beta_\alpha, \\ C_{\alpha\beta}C^{\alpha\beta} &= 2. \end{aligned} \tag{A.5}$$

Note that the expression for the square of spinorial quantities has an additional factor of half in front; for instance,

$$\psi^2 \equiv \frac{1}{2}\,\psi^\alpha\psi_\alpha = i\psi^+\psi^-. \tag{A.6}$$

An interesting property following from the antisymmetry of $C_{\alpha\beta}$ is

$$A^\alpha B_\alpha = -A_\alpha B^\alpha. \tag{A.7}$$

A.3.3 γ-matrices

We choose the γ-matrices (with both spinor indices lowered) to be real and symmetric, given by

$$(\gamma^\mu)_{\alpha\beta} = \{\mathbb{I}, \sigma_3, \sigma_1\}. \tag{A.8}$$

Here, the spacetime index μ runs over the set $(0, 1, 3)$.[1] From eq.(A.8), we can write

$$(\gamma^\mu)_\alpha{}^\beta = \{\sigma_2, -i\sigma_1, i\sigma_3\}, \tag{A.9}$$

all of which are purely imaginary. The Clifford algebra satisfied by the γ-matrices is given by

$$(\gamma^\mu)_\alpha{}^\tau(\gamma^\nu)_\tau{}^\beta + (\gamma^\nu)_\alpha{}^\tau(\gamma^\mu)_\tau{}^\beta = -2\eta^{\mu\nu}\delta^\beta_\alpha. \tag{A.10}$$

As a consequence, we have

$$(\gamma^\mu)_\alpha{}^\beta(\gamma^\nu)_\beta{}^\alpha = -\,2\eta^{\mu\nu}. \tag{A.11}$$

Note that

$$(\gamma^0)^2 = \mathbb{I}, (\gamma^1)^2 = -\mathbb{I}, (\gamma^3)^2 = -\mathbb{I}. \tag{A.12}$$

The γ-matrices with both upper indices are given by

$$(\gamma^\mu)^{\alpha\beta} = \{-\mathbb{I}, \sigma_3, \sigma_1\}. \tag{A.13}$$

An important property satisfied by the γ-matrices is

$$[\gamma^\mu, \gamma^\nu] = -2i\epsilon^{\mu\nu\rho}\gamma_\rho. \tag{A.14}$$

Another useful identity is

$$(\gamma^\mu)_\alpha{}^\beta(\gamma_\mu)_\gamma{}^\delta = C_{\alpha\gamma}C^{\beta\delta} - \delta_\alpha{}^\delta\delta_\gamma{}^\beta. \tag{A.15}$$

We also have

$$\begin{aligned}
(\gamma^\mu)_\alpha{}^\tau(\gamma^\nu)_\tau{}^\beta &= -\,\eta^{\mu\nu}\delta_\alpha{}^\beta - i\epsilon^{\mu\nu\rho}(\gamma_\rho)_\alpha{}^\beta\,,\\
(\gamma^\mu)^{\alpha\tau}(\gamma^\nu)_\tau{}^\beta &= -\,\eta^{\mu\nu}C^{\alpha\beta} - i\epsilon^{\mu\nu\rho}(\gamma_\rho)^{\alpha\beta}\,,\\
(\gamma^\mu)_\alpha{}^\tau(\gamma^\nu)_{\tau\beta} &= -\,\eta^{\mu\nu}C_{\alpha\beta} - i\epsilon^{\mu\nu\rho}(\gamma_\rho)_{\alpha\beta}\,,\\
(\gamma^\mu)^{\alpha\tau}(\gamma^\nu)_{\tau\beta} &= \eta^{\mu\nu}\delta^\alpha{}_\beta - i\epsilon^{\mu\nu\rho}(\gamma_\rho)^\alpha{}_\beta\,.
\end{aligned} \tag{A.16}$$

[1] This seemingly arbitrary choice has been made to facilitate the process of Wick rotation later. Under a Wick rotation, the index 0 goes over to 2. Our convention for the completely antisymmetric Levi-Civita symbol is $\epsilon^{013} = -1$; therefore, after Wick rotation it becomes $\epsilon^{213} = -1$. This matches with the standard Euclidean convention $\epsilon^{123} = 1$.

The Euclidean γ-matrices can be obtained from the Lorentzian definition eq.(A.9) by using the rule $\gamma^0 \to -i\gamma_E^2$,

$$(\gamma_E^\mu)_\alpha{}^\beta = \{i\sigma_2, -i\sigma_1, i\sigma_3\}, \quad \mu = (2,1,3). \tag{A.17}$$

The Clifford algebra satisfied by the Euclidean γ-matrices is

$$(\gamma_E^\mu)_\alpha{}^\tau (\gamma_E^\nu)_\tau{}^\beta + (\gamma_E^\nu)_\alpha{}^\tau (\gamma_E^\mu)_\tau{}^\beta = -2\delta^{\mu\nu}\delta_\alpha^\beta. \tag{A.18}$$

Spacetime three-vectors are represented by symmetric second rank spinors. For instance, for a three-vector V_μ we have the spinor representation $V_{\alpha\beta}$, defined via

$$V_{\alpha\beta} = V_\mu (\gamma^\mu)_{\alpha\beta}. \tag{A.19}$$

Using the Clifford algebra eq.(A.10), this gives

$$k_\alpha{}^\beta k_\beta{}^\alpha = -2k^2, \tag{A.20}$$

where $k^2 = \eta^{\mu\nu} k_\mu k_\nu$.

A.3.4 Superspace

The superspace consists of three spacetime coordinates $x^{\alpha\beta}$, and two anticommuting spinor coordinates θ^α. The ordinary derivatives are defined by[2]

$$\begin{aligned} \frac{\partial \theta^\alpha}{\partial \theta^\beta} &= \delta_\beta^\alpha, \\ \frac{\partial x^{\alpha\beta}}{\partial x^{\sigma\tau}} &= \frac{1}{2}(\delta_\sigma^\alpha\, \delta_\tau^\beta + \delta_\tau^\alpha\, \delta_\sigma^\beta). \end{aligned} \tag{A.21}$$

The momentum operators have the Hermiticity property

$$\begin{aligned} \left(i\,\frac{\partial}{\partial \theta^\alpha}\right)^\dagger &= -\,i\,\frac{\partial}{\partial \theta^\alpha}, \\ \left(i\,\frac{\partial}{\partial x^{\alpha\beta}}\right)^\dagger &= i\,\frac{\partial}{\partial x^{\alpha\beta}}. \end{aligned} \tag{A.22}$$

[2]For notational convenience, we will sometimes use the shorthand notation

$$\frac{\partial}{\partial \theta^\alpha} = \partial_\alpha, \quad \frac{\partial}{\partial x^{\alpha\beta}} = \partial_{\alpha\beta}.$$

The defining properties for Grassmannian integration on the superspace are[3]

$$\int d\theta = 0, \quad \int d\theta\, \theta = 1. \tag{A.23}$$

As a consequence, we get[4]

$$\int d^2\theta\, \theta^2 = -1, \tag{A.24}$$

$$\int d^2\theta\, \theta^\alpha \theta^\beta = C^{\alpha\beta}. \tag{A.25}$$

Eq.(A.24) is used to give the definition of the δ-function on the superspace,

$$\delta^2(\theta) = -\,\theta^2. \tag{A.26}$$

This implies

$$\delta^2(\theta_1 - \theta_2) = -(\,\theta_1^2 + \,\theta_2^2 - \theta_1\theta_2), \tag{A.27}$$

where $\theta_1\theta_2 = \theta_1^\alpha\theta_{2\alpha}$. Note that $\delta^2(\theta_1 - \theta_2) = \delta^2(\theta_2 - \theta_1)$. We also have

$$\delta^2(\theta_1 - \theta_2)\, \delta^2(\theta_2 - \theta_1) = 0. \tag{A.28}$$

Some more useful properties are

$$C^{\alpha\beta} \frac{\partial}{\partial\theta^\beta} \frac{\partial}{\partial\theta^\alpha}\, \theta^2 \;=\; -2. \tag{A.29}$$

and

$$\begin{aligned} \theta^\alpha\theta^\beta &= -\,C^{\alpha\beta}\theta^2, \\ \theta_\alpha\theta_\beta &= -\,C_{\alpha\beta}\theta^2. \end{aligned} \tag{A.30}$$

Also, the D'Alembertian operator is given by

$$\Box = \frac{1}{2}\,\partial^{\alpha\beta}\partial_{\alpha\beta}. \tag{A.31}$$

[3] Note that Grassmann integration is equivalent to differentiation,

$$\int d\theta \equiv \frac{\partial}{\partial\theta}.$$

[4] As already mentioned before eq.(A.6), the square of spinorial quantities is defined with an extra factor of half. Thus,

$$\theta^2 = \frac{1}{2}\,\theta^\alpha\theta_\alpha, \quad d^2\theta = \frac{1}{2}\,d\theta^\alpha d\theta_\alpha.$$

Superspace covariant derivatives are defined by

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\theta^\beta \frac{\partial}{\partial x^{\alpha\beta}}. \tag{A.32}$$

with $D^\alpha = C^{\alpha\beta}D_\beta$. Also

$$D_{\alpha\beta} = \partial_{\alpha\beta} = \frac{\partial}{\partial x^{\alpha\beta}}. \tag{A.33}$$

They satisfy the algebra

$$\begin{aligned} \{D_\alpha, D_\beta\} &= 2iD_{\alpha\beta}, \\ [D_\alpha, D_{\beta\gamma}] &= 0, \\ [D_{\alpha\beta}, D_{\gamma\delta}] &= 0. \end{aligned} \tag{A.34}$$

We often work in momentum space. To convert the above equations into momentum space expressions, we use the replacement $i\partial_{\alpha\beta} \to k_{\alpha\beta}$. This gives

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \theta^\beta k_{\alpha\beta}, \tag{A.35}$$

and the algebra becomes

$$\{D_\alpha, D_\beta\} = 2k_{\alpha\beta}. \tag{A.36}$$

Using the tracelessness of $(\gamma^\mu)_\alpha{}^\beta$ in eq.(A.36), we get

$$\{D^\alpha, D_\alpha\} = 0. \tag{A.37}$$

From the expression for D_α, we find

$$\begin{aligned} D^2 &\equiv \frac{1}{2} D^\alpha D_\alpha \\ &= \frac{1}{2}\left(C^{\alpha\beta}\frac{\partial}{\partial\theta^\beta}\frac{\partial}{\partial\theta^\alpha} + 2\theta^\alpha k_\alpha{}^\beta \frac{\partial}{\partial\theta^\beta} + 2\theta^2 k^2\right). \end{aligned} \tag{A.38}$$

This leads to the following identities

$$\begin{aligned} D_\alpha D_\beta &= k_{\alpha\beta} + C_{\beta\alpha}D^2, \\ (D^2)^2 &= \Box = -k^2, \\ D^\alpha D_\beta D_\alpha &= 0, \\ D^2 D_\alpha &= -D_\alpha D^2 = k_{\alpha\beta}D^\beta. \end{aligned} \tag{A.39}$$

The derivatives of the δ-function satisfy the identities

$$D_\alpha^{\theta_1,k}\delta^2(\theta_1-\theta_2) = -\,\theta_{1\alpha} + \theta_{2\alpha} - k_{\alpha\beta}\,\theta_1^\beta\theta_2^2 + k_{\alpha\beta}\,\theta_1^\beta(\theta_1\theta_2), \tag{A.40}$$

and

$$D^2_{\theta_1,k}\delta^2(\theta_1-\theta_2) = 1 - k_{\alpha\beta}\,\theta_1^\alpha\theta_2^\beta - \theta_1^2\theta_2^2k^2 = \exp(-k_{\alpha\beta}\,\theta_1^\alpha\theta_2^\beta). \tag{A.41}$$

Using these identities, we get

$$\begin{aligned}
&\delta^2(\theta_1-\theta_2)D^\alpha_{\theta_2,k}\delta^2(\theta_2-\theta_1) = 0,\\
&\delta^2(\theta_1-\theta_2)D^2_{\theta_2,k}\delta^2(\theta_2-\theta_1) = \delta^2(\theta_1-\theta_2),\\
&D^{\theta_1,k}_\alpha\delta^2(\theta_1-\theta_2) = -\,D^{\theta_2,-k}_\alpha\delta^2(\theta_2-\theta_1).
\end{aligned} \tag{A.42}$$

The third identity in eq.(A.42) above is also referred to as the "transfer rule."

The supersymmetry generators Q_α have the superspace representation

$$Q_\alpha = i\left(\frac{\partial}{\partial\theta^\alpha} - \theta^\beta k_{\alpha\beta}\right), \tag{A.43}$$

and satisfy the anti-commutation relations

$$\begin{aligned}
&\{Q_\alpha, Q_\beta\} = 2k_{\alpha\beta},\\
&\{Q_\alpha, D_\beta\} = 0.
\end{aligned} \tag{A.44}$$

A.3.5 Superfields

Superfields are functions of the superspace coordinates (x,θ). For $\mathcal{N}=1$ SUSY, we work with a scalar and a vector superfield. The scalar superfield $\Phi(x,\theta)$ consists of a complex scalar field $\phi(x)$, a complex spinor field $\psi^\alpha(x)$ and an auxiliary complex scalar $F(x)$. The θ-expansion for the scalar superfield is given by

$$\Phi = \phi + \theta\psi - \theta^2 F. \tag{A.45}$$

This implies that[5]

$$\bar\Phi = \bar\phi + \theta\bar\psi - \theta^2\bar F. \tag{A.46}$$

The component fields can be extracted from the scalar superfield by using

$$\begin{aligned}
&\phi(x) = \Phi(x,\theta)|_{\theta=0},\\
&\psi_\alpha(x) = D_\alpha\Phi(x,\theta)|_{\theta=0},\\
&F(x) = D^2\Phi(x,\theta)|_{\theta=0}.
\end{aligned} \tag{A.47}$$

From eq.(A.45) and eq.(A.46), it is easy to compute that

$$\bar\Phi\Phi = \bar\phi\phi + \theta^\alpha(\bar\phi\psi_\alpha + \bar\psi_\alpha\phi) - \theta^2(\bar F\phi + \bar\phi F + \bar\psi\psi). \tag{A.48}$$

[5]Note that θ^α are real valued Grassmann variables.

We also have

$$
\begin{aligned}
D_\alpha \Phi &= \psi_\alpha - \theta_\alpha F + i\theta^\beta \partial_{\alpha\beta}\phi + i\theta^2 \partial_\alpha{}^\beta \psi_\beta, \\
D_\alpha \bar{\Phi} &= \bar{\psi}_\alpha - \theta_\alpha \bar{F} + i\theta^\beta \partial_{\alpha\beta}\bar{\phi} + i\theta^2 \partial_\alpha{}^\beta \bar{\psi}_\beta.
\end{aligned}
\tag{A.49}
$$

These equations can be combined to give

$$
D^\alpha \bar{\Phi} D_\alpha \Phi|_{\theta^2} = \theta^2 \big(2\bar{F}F - 2\partial\bar{\phi}\,\partial\phi + 2i\bar{\psi}^\alpha \partial_\alpha{}^\beta \psi_\beta \big) + i\theta^2 \partial_{\alpha\beta} \big(\bar{\psi}^\alpha \psi^\beta \big). \tag{A.50}
$$

The last term in eq.(A.50) is a total derivative term; it can therefore be dropped out of integrals over the superspace. Note that $\partial\bar{\phi}\,\partial\phi \equiv \eta^{\mu\nu}\partial_\mu\bar{\phi}\,\partial_\nu\phi$. Another result of interest is

$$
D^2_{k,\theta}(\bar{\Phi}\Phi) = (\bar{F}\phi + \bar{\phi}F + \bar{\psi}\psi) + \theta^\alpha k_\alpha{}^\beta (\bar{\phi}\psi_\beta + \bar{\psi}_\beta \phi) + 2\theta^2 k^2 (\bar{\phi}\phi). \tag{A.51}
$$

The vector superfield $\Gamma^\alpha(x,\theta)$ is composed of a gauge field $A_{\alpha\beta}(x)$, the gaugino $\lambda^\alpha(x)$, an auxiliary scalar $B(x)$, and an auxiliary fermion $\chi^\alpha(x)$. The θ-expansion for the vector superfield is given by

$$
\Gamma^\alpha = \chi^\alpha - \theta^\alpha B + i\theta^\beta A_\beta{}^\alpha - \theta^2 \big(2\lambda^\alpha - i\partial^{\alpha\beta}\chi_\beta \big). \tag{A.52}
$$

Appendix B

Some mathematical results

Integrals

Integrals of the following type appear in the calculation of β-functions,

$$\int\limits_{a'>|w|>a} d^d w \, \frac{w^{\mu_1} w^{\mu_2} \cdots w^{\mu_{2n}}}{|w|^p} = \int\limits_{a'>w>a} dw \frac{1}{|w|^{p-d+1-2n}} \int d\Omega_{d-1} \hat{w}^{\mu_1} \hat{w}^{\mu_2} \cdots \hat{w}^{\mu_{2n}}$$
$$= \frac{1}{d+2n-p} \left(a'^{d+2n-p} - a^{d+2n-p} \right)$$
$$\times \left(\frac{2^{1-2n} \pi^{d/2}}{\Gamma\left(\frac{d}{2}+n\right) \Gamma(n+1)} \right) \sum_{\mathcal{P} \in S_{2n}} \left(\delta^{\mu_{\mathcal{P}(1)} \mu_{\mathcal{P}(2)}} \; \delta^{\mu_{\mathcal{P}(3)} \mu_{\mathcal{P}(4)}} \cdots \delta^{\mu_{\mathcal{P}(2n-1)} \mu_{\mathcal{P}(2n)}} \right) \tag{B.1}$$

here, $\mathcal{P}$ runs over all permutations of $2n$ numbers, and hence a lot of terms in the parenthesis in the last line are equivalent. The pre-factor has been accordingly calculated to account for these redundancies. This convention is useful because contractions of the $(2n)!$ different permutations of Kronecker-δ above with $\partial_{\mu_1} \ldots \partial_{\mu_{2n}}$ generates $(2n)!(\partial^2)^n$, and the $(2n)!$ here exactly cancels with $1/(2n)!$ coming from the Taylor series. We have used the following short-hand notation for the pre-factor in the paper,

$$\alpha_n = \left(\frac{2^{1-2n} \pi^{d/2}}{\Gamma\left(\frac{d}{2}+n\right) \Gamma(n+1)} \right) \tag{B.2}$$

This factor also obeys an identity,

$$\frac{\alpha_n}{\alpha_{n+1}} = 2(n+1)(d+2n) \tag{B.3}$$

which is useful in simplifying the coefficients of the β-functions.

Variation of derivatives of propagators

Coefficients of all the terms in the β-function equations are of the form (see (2.78)),

$$\left(\int d\rho\ \rho^{d-1+2k}\ \partial_{a'}\left[(a')^{-2n}(\partial_\rho^2)^n G_{a'}(a'\rho)\right]\right)$$

In this appendix, we list first few expressions for $(\partial^2)^n G_{a'}(w)$ and $\partial_{a'}\left[(\partial^2)^n G_{a'}(w)\right]$, in terms of various derivatives of the regulating function, $\mathcal{K}^{(n)}$.

n	$\boldsymbol{(\partial^2)^n G_{a'}(w)}$
0	$\dfrac{\mathcal{K}(w/a)}{w^{2\Delta}}$
1	$\dfrac{\mathcal{K}''(w/a)}{a^2\ w^{2\Delta}} - (4\Delta-d+1)\dfrac{\mathcal{K}'(w/a)}{a\ w^{2\Delta+1}} + 2\Delta(2\Delta-d+2)\dfrac{\mathcal{K}(w/a)}{w^{2(\Delta+1)}}$
2	$\dfrac{\mathcal{K}^{(4)}(w/a)}{a^4\ w^{2\Delta}} - 2(4\Delta-d+1)\dfrac{\mathcal{K}^{(3)}(w/a)}{a^3\ w^{2\Delta+1}} + \Big(d^2-4d(3\Delta+1)+3\left(8\Delta^2+8\Delta+1\right)\Big)\times \dfrac{\mathcal{K}''(w/a)}{a^2\ w^{2(\Delta+1)}} - (4\Delta-d+3)(4\Delta(2\Delta+3)-4d\Delta-d+1)\dfrac{\mathcal{K}'(w/a)}{a\ w^{2\Delta+3}} +4\Delta(\Delta+1)(2\Delta-d+4)(2\Delta-d+2)\dfrac{\mathcal{K}(w/a)}{w^{2(\Delta+2)}}$
$\vdots$	$\vdots$

TABLE B.1: List of various powers of Laplacian acting on propagator $G_{a'}(w)$, which are needed in the computation of β-functions.

n	$\boldsymbol{\partial_{a'}\left[(\partial^2)^n G_{a'}(w)\right]}$
0	$-\dfrac{\mathcal{K}'(w/a)}{a^2\ w^{2\Delta-1}}$
1	$-\dfrac{\mathcal{K}^{(3)}(w/a)}{a^4\ w^{2\Delta-1}} + (4\Delta-d-1)\dfrac{\mathcal{K}''(w/a)}{a^3\ w^{-2\Delta}} - \left(4\Delta^2-2d\Delta+d-1\right)\dfrac{\mathcal{K}'(w/a)}{a^2\ w^{2\Delta+1}}$
2	$-\dfrac{\mathcal{K}^{(5)}\,(w/a)}{a^6\ w^{2\Delta-1}} + 2(4\Delta-d-1)\dfrac{\mathcal{K}^{(4)}\,(w/a)}{a^5\ w^{2\Delta}} - \left(24\Delta^2-12d\Delta+d(d+2)-3\right)\dfrac{\mathcal{K}^{(3)}\,(w/a)}{a^4\ w^{2\Delta+1}} +(4\Delta-d+1)\left(8\Delta^2-4d\Delta+4\Delta+d-3\right)\dfrac{\mathcal{K}''\,(w/a)}{a^3\ w^{2(\Delta+1)}} - \left(4\Delta^2-1\right)(2\Delta-d+3)(2\Delta-d+1)\dfrac{\mathcal{K}'\,(w/a)}{a^2\ w^{-2\Delta-3}}$
$\vdots$	$\vdots$

TABLE B.2: List of variation of $(\partial^2)^n G_{a'}(w)$ with respect to a'.

These expressions listed in Table B.2 are part of the integrands that appear in (2.78). For a general coefficient, we use following notation for these integrals (2.80),

$$\mathbb{G}_{\Delta}^{\mathcal{K}^{(n)}} = \int d\rho \; \rho^{d-2\Delta} \; \mathcal{K}^{(n)}(\rho) \tag{B.4}$$

and corresponding values from Table B.2 have been used to exactly compute the coefficients in (2.79), (2.82) and Table 2.2 for the choices $\mathcal{K}(\rho) = \Theta(\rho-1)$ and $\mathcal{K}(\rho) = \text{regulated-}\Theta(\rho-1)$ for the regulating function.

Appendix C

Holographic Wilsonian Renormalization: Explicit solution

In this appendix we compute the β-functions using Holographic Wilsonian RG techniques. Calculations are based on the work that appears in [15–17] with modifications called for by introducing finite cut-off as discussed in section 2.2.

As explained in section 2.5 we separate the bulk degrees of freedom into UV and IR degrees of freedom and integrate out the near boundary (UV) degrees of freedom, as we change the radial cut-off surface from $z = \epsilon_0$ to $z = \epsilon$. In the process we generate a modified wavefunctional $\Psi[\phi_0; \epsilon] = Z_{UV}$ at the new boundary $z = \epsilon$, whose coefficients contain that information about the couplings of double-trace operators in the field theory at the new cut-off as given by (2.34) and (2.45).

The bulk evolution equation in radial direction can be determined by computing the *radial Hamiltonian*.

$$\mathcal{H} = \frac{1}{2}\left(\frac{\pi^2}{z^{1-d}} + \frac{z^{-1-d}}{2}\left(\partial^\mu\phi\partial_\mu\phi + m^2\phi^2\right)\right) \tag{C.1}$$

in operator language, the evolution Hamiltonian in the radial direction is,

$$\hat{H} = \int d^dx\ \hat{\mathcal{H}} = \frac{1}{2}\left(\int d^dk\ \frac{1}{z^{1-d}}\hat{\Pi}_k\hat{\Pi}_{-k} + z^{-1-d}\left(z^2k^2 + m^2\right)\hat{\phi}_k\hat{\phi}_{-k}\right) \tag{C.2}$$

here $\hat{\Pi} \equiv i\frac{\delta}{\delta\phi}$ in the 'field basis', where $\hat{\phi}(x)\,|\phi\rangle = \phi(x)\,|\phi\rangle$. The radial Schrödinger equation for the radial wavefunctional Z_{UV} is given by (2.56), [1]

$$-\partial_\epsilon Z_{UV} = \hat{H} Z_{UV}$$

Since we are working with a quadratic theory and the boundary wavefunctional at $z = \epsilon_0$ is also quadratic, the wavefunctional generated at any other cut-off $z = \epsilon$, $Z_{UV} = \Psi[\phi_\epsilon; \epsilon]$ is also quadratic. So let us consider a general form of the wavefunctional,

$$Z_{UV} = \exp\left[-\frac{1}{2}\int d^dk\sqrt{\gamma}\left(A(k\epsilon; k\epsilon_0)\phi^\epsilon_k\phi^\epsilon_{-k} + 2\epsilon^{d-\Delta}B(k\epsilon; k\epsilon_0)J^0_k\phi^\epsilon_{-k} \right.\right.$$
$$\left.\left. + \epsilon^{2(d-\Delta)}C(k\epsilon; k\epsilon_0)J^0_kJ^0_{-k}\right)\right] \tag{C.3}$$

to keep the calculation more general, we don't specify Δ here. In subsequent computations $\Delta = \Delta_+$ for standard quantization and $\Delta = \Delta_-$ for alternative quantization. We now derive the general evolution equations for the coefficients $A(k,\epsilon,\epsilon_0), B(k,\epsilon,\epsilon_0), C(k,\epsilon,\epsilon_0)$. The exact form of these coefficients can be obtained by starting with the appropriate wavefunctionals (2.34) or (2.45) at $z = \epsilon_0$ but since the evolution equation doesn't depend on the initial wavefunctional it is not required here. When substituted in the radial Schrödinger equation we get,

$$-\partial_\epsilon Z_{UV} = \left(\frac{1}{2}\int d^dk\left[\partial_\epsilon\left(\epsilon^{-d}A(k\epsilon; k\epsilon_0)\right)\phi^\epsilon_k\phi^\epsilon_{-k} + 2\partial_\epsilon\left(\epsilon^{-\Delta}B(k\epsilon; k\epsilon_0)\right)\phi^\epsilon_kJ^0_{-k}\right.\right.$$
$$\left.\left. + \partial_\epsilon\left(\epsilon^{d-2\Delta}C(k\epsilon; k\epsilon_0)J^0_kJ^0_{-k}\right)\right]\right) \times Z_{UV}$$
$$\hat{H}Z_{UV} = \left(\frac{1}{2}\int d^dk\epsilon^{-d-1}\Big[\left(\epsilon^2k^2 + m^2 - A^2(k\epsilon; k\epsilon_0)\right)\phi^\epsilon_k\phi^\epsilon_{-k}\right.$$
$$\left. - 2A(k\epsilon; k\epsilon_0)B(k\epsilon; k\epsilon_0)\epsilon^{d-\Delta}\phi^\epsilon_kJ^0_{-k} - \epsilon^{2(d-\Delta)}B^2(k\epsilon; k\epsilon_0)J^0_kJ^0_{-k}\Big] + \cdots\right) \times Z_{UV}$$
$$\tag{C.4}$$

the terms in the ellipsis in the above equation are not important and don't arise when we keep track of the overall normalisation of Z_{UV}. J_0 above is the source for the operator $\mathcal{O}$ at $z = \epsilon_0$. This implies following evolution equations for the coefficients,

$$\epsilon\partial_\epsilon A = -A^2 + dA + (\epsilon^2k^2 + m^2) \tag{C.5a}$$

$$\epsilon\partial_\epsilon B = \Delta\ B - A\ B \tag{C.5b}$$

$$\epsilon\partial_\epsilon C = (2\Delta - d)\ C - B^2 \tag{C.5c}$$

[1] In the particular case of quadratic bulk action, the case we are demonstrating here, the Schrödinger equation and the semi-classical Hamiltonian-Jacobi equations are equivalent.

the field theory double-trace couplings are related to $A(k,\epsilon,\epsilon_0)$ by (2.34) and (2.45) (recall, $\bar{\mathfrak{f}}$ denotes dimensionless coupling),

$$\bar{\mathfrak{f}}^{ST}(k^2\epsilon^2) = \frac{\left(A(k\epsilon;k\epsilon_0) - \hat{\mathcal{D}}_{ct}(k\epsilon)\right)\,\mathscr{A}^*_{ST} - 1}{{B^*_{ST}}^2 {\bar{\mathfrak{f}}^*_{ST}}{}^2 \left(A(k\epsilon;k\epsilon_0) - \hat{\mathcal{D}}_{ct}(k\epsilon)\right)} \tag{C.6a}$$

while, with the inclusion of the counter-term,

$$\bar{\mathfrak{f}}^{ST}(k^2\epsilon^2) = \frac{\left(A(k\epsilon;k\epsilon_0) - \hat{\mathcal{D}}_{ct}(k\epsilon)\right)\,\mathscr{A}^*_{ST} - 1}{\left({B^*_{ST}}^2 {\bar{\mathfrak{f}}^*_{ST}}{}^2 + \mathscr{A}^*_{ST}\cdot\delta C\right)\left(A(k\epsilon;k\epsilon_0) - \hat{\mathcal{D}}_{ct}(k\epsilon)\right) - \delta C} \tag{C.6b}$$

$$\bar{\mathfrak{f}}^{AQ}(k^2\epsilon^2) = \frac{A(k\epsilon;k\epsilon_0) - \hat{\mathcal{D}}_{ct}(k\epsilon)}{C^*_{AQ}\left(A(k\epsilon;k\epsilon_0) - \hat{\mathcal{D}}_{ct}(k\epsilon;k\epsilon_0)\right) + {B^*_{AQ}}^2} \tag{C.6c}$$

and with the inclusion of the counter-terms,

$$\bar{\mathfrak{f}}^{AQ}(k^2\epsilon^2) = \frac{A(k\epsilon;k\epsilon_0) - \hat{\mathcal{D}}_{ct}(k\epsilon)}{\left(C^*_{AQ} + \delta C\right)\left(A(k\epsilon;k\epsilon_0) - \hat{\mathcal{D}}_{ct}(k\epsilon;k\epsilon_0)\right) + {B^*_{AQ}}^2} \tag{C.6d}$$

Equations (C.5) can be used to compute the β-function equations for these couplings, For standard quantization (see (2.57)),

$$\begin{aligned}
\epsilon\partial_\epsilon\bar{\mathfrak{f}} = \bar{\mathfrak{f}}^2 \times \Bigg(& {B^*_{ST}}^2\,{\mathscr{A}^*_{ST}}^2\left(k^2\epsilon^2 + m^2 + \hat{\mathcal{D}}_{ct}(d-\hat{\mathcal{D}}_{ct}) - \epsilon\partial_\epsilon\hat{\mathcal{D}}_{ct}\right) - \epsilon\partial_\epsilon\delta C \\
& + \frac{(\delta C)^2}{{B^*_{ST}}^2{\mathscr{A}^*_{ST}}^2}\left(\epsilon\partial_\epsilon\bar{\mathfrak{f}}^* - 1 + \mathscr{A}^*_{ST}(d-2\hat{\mathcal{D}}_{ct}) + {\mathscr{A}^*_{ST}}^2\left(k^2\epsilon^2 + m^2 + \hat{\mathcal{D}}_{ct}(d-\hat{\mathcal{D}}_{ct}) - \epsilon\partial_\epsilon\hat{\mathcal{D}}_{ct}\right)\right) \\
& - \delta C\left(-\frac{2\epsilon\partial_\epsilon B^*_{ST}}{B^*_{ST}} - \frac{2\,\epsilon\partial_\epsilon\mathscr{A}^*_{ST}}{\mathscr{A}^*_{ST}} - 2\mathscr{A}^*_{ST}\left(k^2\epsilon^2 + m^2 + \hat{\mathcal{D}}_{ct}(d-\hat{\mathcal{D}}_{ct}) - \epsilon\partial_\epsilon\hat{\mathcal{D}}_{ct}\right) - d + 2\hat{\mathcal{D}}_{ct}\right)\Bigg) \\
& + \bar{\mathfrak{f}}\Bigg(- \frac{2\,\epsilon\partial_\epsilon B^*_{ST}}{B^*_{ST}} - \frac{2\,\epsilon\partial_\epsilon\mathscr{A}^*_{ST}}{\mathscr{A}^*_{ST}} - 2\mathscr{A}^*_{ST}\left(k^2\epsilon^2 + m^2 + \hat{\mathcal{D}}_{ct}(d-\hat{\mathcal{D}}_{ct}) - \epsilon\partial_\epsilon\hat{\mathcal{D}}_{ct}\right) - d + 2\hat{\mathcal{D}}_{ct}(k\epsilon) \\
& - 2\frac{\delta C}{{B^*_{ST}}^2{\mathscr{A}^*_{ST}}^2}\left(\epsilon\partial_\epsilon\bar{\mathfrak{f}}^* - 1 + \mathscr{A}^*_{ST}(d-2\hat{\mathcal{D}}_{ct}) + {\mathscr{A}^*_{ST}}^2\left(k^2\epsilon^2 + m^2 + \hat{\mathcal{D}}_{ct}(d-\hat{\mathcal{D}}_{ct}) - \epsilon\partial_\epsilon\hat{\mathcal{D}}_{ct}\right)\right)\Bigg) \\
& \qquad + \frac{\epsilon\partial_\epsilon\mathscr{A}^*_{ST} + \mathscr{A}^*_{ST}\,(d-2\hat{\mathcal{D}}_{ct}) + {\mathscr{A}^*_{ST}}^2\left(k^2\epsilon^2 + m^2 + \hat{\mathcal{D}}_{ct}(d-\hat{\mathcal{D}}_{ct}) - \epsilon\partial_\epsilon\hat{\mathcal{D}}_{ct}\right) - 1}{{B^*_{ST}}^2\,{\mathscr{A}^*_{ST}}^2}
\end{aligned}$$

And for alternative quantization (2.58),

$$\begin{aligned}\epsilon\partial_\epsilon\bar{\mathfrak{f}} =& \frac{1}{{B^*_{AQ}}^2}\Bigg[\bar{\mathfrak{f}}^2\Big(2B^*_{AQ}\ \epsilon\partial_\epsilon B^*_{AQ}\left(C^*_{AQ}+\delta C\right) - {B^*_{AQ}}^2\left(\epsilon\partial_\epsilon C^*_{AQ} + \epsilon\partial_\epsilon\delta C + \left(C^*_{AQ}+\delta C\right)(d-2\hat{\mathcal{D}}_{ct})\right) \\ &- {B^*_{AQ}}^4 + \left(C^*_{AQ}+\delta C\right)^2\left(\hat{\mathcal{D}}_{ct}(d-\hat{\mathcal{D}}_{ct}) - \epsilon\partial_\epsilon\hat{\mathcal{D}}_{ct} + k^2\epsilon^2 + m^2\right)\Big) \\ &+\bar{\mathfrak{f}}\left(-2B^*_{AQ}\ \epsilon\partial_\epsilon B^*_{AQ} + {B^*_{AQ}}^2(d-2\hat{\mathcal{D}}_{ct}) - 2\left(C^*_{AQ}+\delta C\right)\left(\hat{\mathcal{D}}_{ct}(d-\hat{\mathcal{D}}_{ct}) - \epsilon\partial_\epsilon\hat{\mathcal{D}}_{ct} + k^2\epsilon^2 + m^2\right)\right) \\ &+\hat{\mathcal{D}}_{ct}(d-\hat{\mathcal{D}}_{ct}) - \epsilon\partial_\epsilon\hat{\mathcal{D}}_{ct} + k^2\epsilon^2 + m^2\Bigg]\end{aligned}$$

In the above equations, we have suppressed the functional dependence of $\hat{\mathcal{D}}_{ct}(k\epsilon), B^*_{ST}(k\epsilon)$ and $\mathscr{A}^*_{ST}(k\epsilon)$ to avoid clutter. Although the above equations look horrendous, when resolved in components of the coupling $\bar{\mathfrak{f}} = \bar{\mathfrak{f}}_0 + \bar{\mathfrak{f}}_1(k\epsilon)^2 + \bar{\mathfrak{f}}_2(k\epsilon)^4 + \cdots$, and on substituting the values of $\hat{\mathcal{D}}_{ct}(k\epsilon), B^*_{ST}(k\epsilon)$ and $\mathscr{A}^*_{ST}(k\epsilon)$ given by (2.8), (2.15), (2.24), the β-functions for individual couplings become quite simple,

Standard Quantization:

$$\begin{aligned}\dot{\bar{\mathfrak{f}}}_0 &= -2\nu\ \bar{\mathfrak{f}}_0 + 2\nu\ c_0\ \bar{\mathfrak{f}}_0^2 \\ \dot{\bar{\mathfrak{f}}}_1 &= -(2\nu+2)\ \bar{\mathfrak{f}}_1 - 2(1-\nu)\ c_1\ \bar{\mathfrak{f}}_0^2 + 4\nu\ c_0\ \bar{\mathfrak{f}}_0\bar{\mathfrak{f}}_1 \\ \dot{\bar{\mathfrak{f}}}_2 &= -(2\nu+4)\ \bar{\mathfrak{f}}_2 - 2(2-\nu)\ c_2\ \bar{\mathfrak{f}}_0^2 - 4(1-\nu)\ c_1\ \bar{\mathfrak{f}}_0\bar{\mathfrak{f}}_1 + 4\nu\ c_0\ \bar{\mathfrak{f}}_0\bar{\mathfrak{f}}_2 + 2\nu\ c_0\ \bar{\mathfrak{f}}_1^2 \\ &\ \ \vdots\end{aligned} \tag{C.7}$$

Alternative Quantization:

$$\begin{aligned}\dot{\bar{\mathfrak{f}}}_0 &= 2\nu\bar{\mathfrak{f}}_0 - 2\nu\ c_0\ \bar{\mathfrak{f}}_0^2 \\ \dot{\bar{\mathfrak{f}}}_1 &= (2\nu-2)\bar{\mathfrak{f}}_1 - 2(1+\nu)\ c_1\ \bar{\mathfrak{f}}_0^2 - 4\nu\ c_0\ \bar{\mathfrak{f}}_0\bar{\mathfrak{f}}_1 \\ \dot{\bar{\mathfrak{f}}}_2 &= (2\nu-4)\bar{\mathfrak{f}}_2 - 2(2+\nu)\ c_2\ \bar{\mathfrak{f}}_0^2 - 4(1+\nu)\ c_1\ \bar{\mathfrak{f}}_0\bar{\mathfrak{f}}_1 - 4\nu\ c_0\ \bar{\mathfrak{f}}_2\bar{\mathfrak{f}}_0 - 2\nu\ c_0\ \bar{\mathfrak{f}}_1^2 \\ &\ \ \vdots\end{aligned} \tag{C.8}$$

The fixed point values for the coupling constants given by solving the stationary points of the above equations are (for both standard and alternative quantization),

$$\begin{aligned}\text{Trivial Fixed Point: } &\bar{\mathfrak{f}}_i = 0\ \forall\ \{i \in \mathbb{Z}^+ \cup 0\} \\ \text{Non-Trivial Fixed Point: } &\bar{\mathfrak{f}}_0 \to \frac{1}{c_0},\ \bar{\mathfrak{f}}_1 \to -\frac{c_1}{c_0^2},\ \bar{\mathfrak{f}}_2 \to \frac{c_1^2 - c_0c_2}{c_0^3}\ldots\end{aligned} \tag{C.9}$$

It might look strange that the fixed point for both standard and alternative quantization in (C.9) is the same. This happens because the counter-terms, δC in one theory aren't the same as those in the other theory. Here we have only used them as a notational device and so they should not be confused to be equivalent. We discuss the relation between the non-trivial fixed points of one theory with the trivial fixed point of the other theory in the next subsection.

C.1 Relation between Standard and Alternative Quantizations

We had remarked in subsection 2.2.2 how the undeformed alternative and standard quantized theories are Legendre transform of each other. This relationship doesn't hold exactly anymore for the regulated theories given by the inclusion of (2.19) and (2.27). However, as one would expect, the UV fixed point of the regulated standard quantized theory is the alternative theory and vice versa. In the following discussion we show this relationship explicitly.

From (C.9) we see that the non-trivial fixed point corresponds to couplings $\bar{\mathfrak{f}}(k^2\epsilon^2) = \dfrac{1}{\delta C(k\epsilon)}$. So the correlators at the non-trivial fixed points are given by, (2.44) and (2.52),

$$\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle_+^{fp} = \frac{k^{2\nu}\ \dfrac{2^{1-2\nu}\Gamma(1-\nu)}{\Gamma(\nu)} + \epsilon^{-2\nu}\ \delta C_{ST}(k\epsilon)}{2 + \dfrac{(k\epsilon)^{2\nu}}{\delta C_{ST}(k\epsilon)}\ \dfrac{2^{1-2\nu}\Gamma(1-\nu)}{\Gamma(\nu)}} \tag{C.10a}$$

$$\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle_-^{fp} = \frac{-k^{-2\nu}\ \dfrac{2^{2\nu-1}\Gamma(\nu)}{\Gamma(1-\nu)} + \epsilon^{2\nu}\ \delta C_{AQ}(k\epsilon)}{2 - \dfrac{(k\epsilon)^{-2\nu}}{\delta C_{AQ}(k\epsilon)}\ \dfrac{2^{2\nu-1}\Gamma(\nu)}{\Gamma(1-\nu)}} \tag{C.10b}$$

here, the superscript fp signifies that we are computing the correlator at the non-trivial fixed point of the theory. The flow towards UV starting from the standard quantization is defined by taking the limit $k\epsilon \to \infty$ in (C.10a). In this limit the correlation function becomes,

$$\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle_+^{fp}\Big|_{k\epsilon\to\infty} = \left(\epsilon^{-2\nu}\delta C_{ST}\right)^2 \left[\frac{\epsilon^{2\nu}}{\delta C_{ST}} - k^{-2\nu}\ \frac{2^{2\nu-1}\Gamma(\nu)}{\Gamma(1-\nu)}\right] \tag{C.11}$$

which is the same as the correlator of the regulated alternative theory if we identify $\delta C_{AQ} = 1/\delta C_{ST}$, upto some overall multiplicative wavefunctional renormalization, $\mathcal{O}_-(k) = (\epsilon^{-2\nu}\delta C_{ST})^{-1}$.

$\mathcal{O}_+(k) = \mathfrak{f}_+^* \mathcal{O}_+(k)$.[2] Similarly for the flow towards IR fixed point from the alternative fixed point, we take the IR limit, $k\epsilon \to 0$ in (C.10b),

$$\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle_-^{fp}\Big|_{k\epsilon\to 0} = \left(\epsilon^{2\nu}\delta C_{AQ}\right)^2 \left[\frac{\epsilon^{-2\nu}}{\delta C_{AQ}} + k^{2\nu}\ \frac{2^{1-2\nu}\Gamma(1-\nu)}{\Gamma(\nu)}\right] \tag{C.12}$$

which again is the same as the correlator of the regulated standard theory with the identification $\delta C_{ST} = 1/\delta C_{AQ}$, and $\mathcal{O}_+(k) = (\epsilon^{2\nu}\delta C_{AQ})^{-1}\cdot\mathcal{O}_-(k) = \mathfrak{f}_-^*\mathcal{O}_-(k)$. Thus clearly, the standard quantized theory and alternative quantized theory are connected to each other with RG flow as IR and UV fixed points.

All the results discussed here are parallel to the field theory calculations that were presented in subsection 2.6.4.

[2]This wavefunctional renormalization is well known in the literature and provides for the correct scaling dimension of the operators at the non-trivial fixed point.
Also, for clarification of notation, $\mathfrak{f}_\pm^*$ are the non-trivial fixed points for the standard and alternative theories. $\mathcal{O}_+$ and $\mathcal{O}_-$ are the operators dual to the bulk field ϕ at the standard and alternative fixed points respectively.

Appendix D

Large N limit of $O(N)$ Wilson-Fisher model

Let us consider the following Euclidean action in $d = 4 - \epsilon$ dimensions (see, e.g. [7])

$$S = \int d^d x \left\{ \frac{1}{2} \left(\partial_\mu \phi_i\right)^2 + \frac{1}{2} m_0^2 \mathcal{O}(x) + \frac{1}{4!} \frac{g_0}{N} \Lambda^\epsilon \mathcal{O}(x)^2 \right\}, \quad \mathcal{O}(x) = \phi_i \phi_i(x)$$

The phase diagram and fixed points of this model are shown in Fig. D.1. The model possesses a critical surface (where the correlation length diverges) given by

$$m_0^2 = -g_0 \frac{1}{6} \Lambda^\epsilon \Omega_d(0), \; \Omega_d(0) \equiv \frac{1}{(2\pi)^d} \int^\Lambda \frac{d^d k}{k^2} \propto \Lambda^{d-2}$$

The β-function is given by

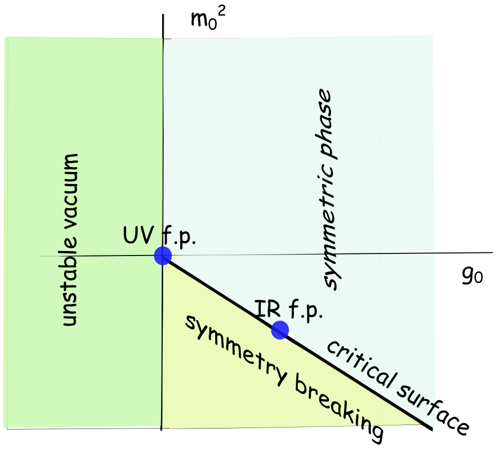


FIGURE D.1: Large-N Wilson-Fisher: fixed points and phase diagram.

$$\Lambda \partial_\Lambda g_0 = \beta(g_0) = -\epsilon g_0 + \frac{N+8}{N} \frac{g_0^2}{48\pi^2} + O(g_0^3)$$

which shows a UV fixed point at $g_0 = 0$ and an IR fixed point at

$$g_0^* = \epsilon \frac{48\pi^2 N}{N+8} + O(\epsilon^2)$$

The two-point function of $\mathcal{O}(x)$ can be obtained in the large N limit by saddle point methods, and is given by (see Sections 2.3 and 2.4 of [7], especially Eqs. (2.57) and (2.59))

$$\begin{aligned} &\langle \mathcal{O}(p)\mathcal{O}(q)\rangle = G(p)\delta(p+q), \; G(p) = -\Lambda^{-\epsilon}\frac{\frac{12}{g_0}}{1+\Lambda^{\epsilon}\frac{g_0}{6}B_\Lambda(p)} \\ &B_\Lambda(p) = \int^\Lambda \frac{d^dk}{k^2(k-p)^2} = p^{-\epsilon}\left(b_0 + b_1(p/\Lambda)^2 + \cdots\right) + \Lambda^{-\epsilon}\left(a_0 + a_1(p/\Lambda)^2 + \cdots\right) \end{aligned} \tag{D.1}$$

where b, a are some constants.

The IR behaviour: IR limit is given by $p/\Lambda \to 0$,

$$\begin{aligned} G_{IR}(p) &= -\frac{72\Lambda^{-2\epsilon}}{g_0^2 Z^2} p^\epsilon \left[1 + \left(\frac{p}{\Lambda}\right)^\epsilon \left(\delta C + \frac{6}{g_0 Z^2}\right)\right]^{-1} \\ &\xrightarrow{p/\Lambda\to\infty} -\frac{72\Lambda^{-2\epsilon}}{g_0^2 Z^2} p^\epsilon \end{aligned} \tag{D.2}$$

where, we have used the notation, $Z^2 = \left(b_0 + b_1(p/\Lambda)^2 + \cdots\right)$, $Z^2 \cdot \delta C = \left(a_0 + a_1(p/\Lambda)^2 + \cdots\right)$. The renormalized IR operators are given by $\mathcal{O}_{IR} = \left(\frac{g_0\,\Lambda^\epsilon}{12}\right)^2 \mathcal{O}_{UV}$, which is well known for the Wilson-Fisher fixed point.[1]

The UV behaviour: In the limit $p/\Lambda \to \infty$, we get[2]

$$\begin{aligned} G_{UV}(p) &= -\frac{\frac{12\Lambda^{-\epsilon}}{g_0}}{1+\frac{g_0}{6}Z^2\cdot\delta C}\left[1 + \frac{\frac{g_0}{6}Z^2}{1+\frac{g_0}{6}Z^2\cdot\delta C}\left(\frac{\Lambda}{p}\right)^\epsilon\right]^{-1} \\ &\xrightarrow{p/\Lambda\to\infty} -\frac{\frac{12\Lambda^{-\epsilon}}{g_0}}{1+\frac{g_0}{6}Z^2\cdot\delta C} + \frac{2Z^2}{\left(1+\frac{g_0}{6}Z^2\cdot\delta C\right)^2}\, p^{-\epsilon} \end{aligned} \tag{D.3}$$

which again agrees with the general analysis presented in subsection 2.6.4, upto some normalization and contact terms which can be attributed different regulation used in [7].

[1] Note that there is a slight difference in the correlator here compared to subsection 2.6.4 because the correlator in (D.1) is not of the form $\frac{G}{1+fG}$, and the conventions in [7] are such that the IR correlator appears without the contact-terms.

[2] Note that the normalization of the two-point function differs from the main text, due to a different normalization of the operator $\mathcal{O}(x)$. We can identify correctly normalized UV operator as, $\mathcal{O}_{UV} = \sqrt{2}\, Z\, \mathcal{O}$

Appendix E

Large N, Probe approximation and Hamilton-Jacobi

Probe approximation: Let us consider a free massive scalar field described by (2.4) but coupled to a perturbed metric of the form $g_{MN} = \bar{g}_{MN} + \sqrt{\kappa}\, h_{MN}$ where $\bar{g}_{MN}$ is now the AdS metric (2.3). In this case the bulk action is of the schematic form (where we focus on the κ-dependence)

$$\begin{aligned} &S \sim S_b + S_{grav} + S_{int}, \\ &S_b \sim \int (\partial\phi)^2 + m^2\phi^2, \quad S_{grav} \sim \int (\partial h)^2, \quad S_{int} \sim \int \sqrt{\kappa}(h\partial\phi\partial\phi + h\partial h\partial h) + \kappa\ hh\partial h\partial h \end{aligned} \tag{E.1}$$

The bulk partition function, computed from the above, clearly matches (in large N counting) a field theory partition of the form $\langle \exp\big[\int \phi_0(x)\mathcal{O}(x)\big]$ where the connected two-point function is normalized as $\langle \mathcal{O}\mathcal{O}\rangle \sim O(1)$. The connected 3-point function $\langle \mathcal{O}\tilde{T}\tilde{T}$ (where $\tilde{T}$ is the normalized stress tensor satisfying $\langle \tilde{T}\tilde{T}\rangle \sim O(1)$) from the AdS computation is now $\sim \sqrt{\kappa}$ which matches with the field theory result $O(1/N)$.[1] In the above we have assumed that the scaling dimension of $\mathcal{O}(x)$ is $O(1)$ (compared with N, or more generally, with the central charge c of the CFT). The back-reaction on the metric is then given by the equation of motion for the graviton $\partial^2 h \sim \sqrt{\kappa}\langle\partial\phi\partial\phi\rangle$. Now $\langle\phi\phi\rangle \sim O(1)$ since ϕ is canonically normalized. (Alternatively, $\langle\phi\phi\rangle$ is related to $\langle\mathcal{O}\mathcal{O}\rangle$ by bulk-boundary correspondence and the latter is, by convention, $O(1)$. We could also arrive at this result by noting that $\delta g \sim G_N T_{\text{bulk},\mu\nu}$ which is $\sim G_N\langle O|T_{\mu\nu}|O\rangle \sim G_N \sim 1/N^2$ (which matches $h \sim 1/N$. From the last point of view, it is clear that we need the single trace operator to have scaling dimension $\Delta \sim O(1)$.

[1] We made these arguments for a large N gauge theory such as $\mathcal{N} = 4$ SYM, but for vector models and other examples, this counting can be appropriately modified.

The above argument about probe approximation can be easily extended to the case when the CFT is deformed by both single trace and double trace operators. The zero-th order bulk scalar action, S_b remains quadratic.

We should make a remark here about self-interaction of the bulk scalar. Typically the connected 3-point function $\langle \mathcal{O}\mathcal{O}\mathcal{O} \rangle$ will be non-vanishing. But this will also be $O(1/N)$. Hence S_{int} will have a term $\sim \int \sqrt{\kappa}\phi^3$.

Justification of Hamilton-Jacobi: We argued above that in the large N approximation, it suffices to consider a quadratic action, making Hamilton-Jacobi approximation to the Schrödinger equation is exact (up to a pre-factor which is not important for our purpose).

Appendix F

Green's function of Laplacian in AdS$_2$

Green's functions in hyperbolic spaces are well studied. Therefore, in this appendix, following [180], we only provide a quick review of some results that are important for this paper. In the Poincare half plane, $\mathbb{H}$, the Laplacian is given by,

$$\hat{\Box} = \zeta^2\left(\partial_\zeta^2 + \partial_\tau^2\right) \tag{F.1}$$

We are interested in solving the Green's function equation,

$$\hat{\Box} G(\vec{x}, \vec{x}') = \zeta^2 \delta^{(2)}(\vec{x} - \vec{x}') \tag{F.2}$$

It is convenient to work with the coordinates, $z = \zeta + i\tau, \bar{z} = \zeta - i\tau$. Geodesic distances between two points, z, z', on $\mathbb{H}$ are given by,

$$d(z, z') = \frac{1}{\sqrt{4\pi\mu}} \operatorname{arccos}\left(1 + \frac{|z - z'|^2}{2\,\mathrm{Re}[z]\,\mathrm{Re}[z']}\right) \tag{F.3}$$

Hyperbolic symmetry implies that the Green's function depends only on the geodesic distance, $G(z, z') = f(d)$. Switching to geodesic polar coordinates centered around z',

$$ds^2 = dr^2 + \sinh^2(2\sqrt{\pi\mu}\, r) d\theta^2 \tag{F.4}$$

In these coordinates, (F.2) becomes,

$$\left[\frac{1}{\sinh\left(2\sqrt{\pi\mu}\, r\right)} \partial_r (\sinh(2\sqrt{\pi\mu}\, r)\partial_r)\right] f(r) = \frac{\delta(r)}{\sinh\left(2\sqrt{\pi\mu}\, r\right)} \tag{F.5}$$

We regulate the above equation by first solving the resolvent for the operator $-\Box+4\pi\mu\, s(s-1)$, and then taking the limit, $s \to 1$. Moreover, we first solve the homogeneous condition and then impose an appropriate condition on the discontinuity of the resolvent at origin to solve for the Green's function. The solution to the regulated homogeneous equation,

$$\left[\frac{1}{\sinh(2\sqrt{\pi\mu}\, r)}\,\partial_r(\sinh(2\sqrt{\pi\mu}\, r)\partial_r) + s(s-1)\right] f_s(r) = 0 \tag{F.6}$$

is given by,

$$f_s(r) = a_1 Q_{s-1}(\cosh(2\sqrt{\pi\mu}\, r)) + a_2 P_{s-1}(\cosh(2\sqrt{\pi\mu}\, r)) \tag{F.7}$$

where, P_s, Q_s are Legendre functions of first and second kind respectively, and a_1, a_2 are some constant of integrations. To fix the normalization and the discontinuity at the origin, we substitute (F.7) into (F.5) and integrate on both sides. This fixes the solution for the resolvent to be,

$$\begin{aligned} f_s(r) &= -\frac{1}{2\pi} Q_{s-1}(\cosh(2\sqrt{\pi\mu} r)) \\ G_s(z,z') &= -\frac{1}{2\pi} Q_{s-1}\left(1 + \frac{|z-z'|^2}{2\,\mathrm{Re}[z]\,\mathrm{Re}[z']}\right) \\ &= -\frac{\Gamma(s)^2}{4\pi\Gamma(2s)}\left(1+\frac{|z-z'|^2}{4\,\mathrm{Re}[z]\,\mathrm{Re}[z']}\right)^{-s} {}_2F_1\left(s,s;2s;\left(1+\frac{|z-z'|^2}{4\,\mathrm{Re}[z]\,\mathrm{Re}[z']}\right)^{-1}\right) \end{aligned} \tag{F.8}$$

Taking $s \to 1$, the Green's function is given by,

$$G(z,z') = \frac{1}{4\pi}\log\left[1-\left(1+\frac{|z-z'|^2}{4\,\mathrm{Re}[z]\,\mathrm{Re}[z']}\right)^{-1}\right] \tag{F.9}$$

In terms of the $\zeta - \tau$ coordinates, this is,

$$G(\{\zeta_1,\tau_1\},\{\zeta_2,\tau_2\}) = \frac{1}{4\pi}\log\left[\frac{(\zeta_1-\zeta_2)^2+(\tau_1-\tau_2)^2}{(\zeta_1+\zeta_2)^2+(\tau_1-\tau_2)^2}\right] \tag{F.10}$$

The Green's function is quite instructive in this form. It is same as the flat space Green's function in 2-dimensions with an additional contribution coming from the 'mirror charge' at $\{-\zeta_2, \tau_2\}$. This is not surprising because AdS_2 is Weyl scaled flat metric and hence has the same Green's function up to imposition of boundary conditions.

F.1 Green's function for thermal AdS_2

Thermal AdS_2 is defined by periodic identification of τ coordinate over a length β. Thus the metric remains same as pure AdS_2 and so does the Laplacian given in (F.1). However, now the Green's function should be invariant under the shift $\Delta\tau = \tau_1 - \tau_2 \to \Delta\tau + n\beta$, with

$n \in \mathbb{Z}$. This can be achieved by taking using the method of images,

$$G_{thermal}(\{\zeta_1, \tau_1\}, \{\zeta_2, \tau_2\}) = \frac{1}{4\pi} \sum_{n=-\infty}^{\infty} \log \left[\frac{(\zeta_1 - \zeta_2)^2 + (\Delta\tau + n\beta)^2}{(\zeta_1 + \zeta_2)^2 + (\Delta\tau + n\beta)^2} \right] \tag{F.11}$$

This sum can be computed explicitly,

$$G_{thermal}(\{\zeta_1, \tau_1\}, \{\zeta_2, \tau_2\}) = \frac{1}{4\pi} \log \left[\frac{\cosh\left(\frac{2\pi(\zeta_1 - \zeta_2)}{\beta}\right) - \cos\left(\frac{2\pi\Delta\tau}{\beta}\right)}{\cosh\left(\frac{2\pi(\zeta_1 + \zeta_2)}{\beta}\right) - \cos\left(\frac{2\pi\Delta\tau}{\beta}\right)} \right] \tag{F.12}$$

This Green's function was used in the computations of the partition function in section 3.7 which was then subtracted from the partition function in black hole geometries discussed in that section.

Appendix G

Variation of the induced gravity (Polyakov) action

In this appendix we will study the exact variation of the Polyakov action, (3.28). We haven't found a discussion of these covariant equations of motion anywhere in literature, we think that it might have been worked out personally, they haven't been presented in published form. Since the action is non-local, so are the equations of motion.[1] While we won't be solving the equations in full generality, we show,

1. That the diagonal part of the equations of motion are the same as the one we obtain for the *Liouville mode*, ϕ, in conformal gauge. These is the equation of motion that one obtains for Liouville field theory with a background metric, $\hat{g}$.

2. AdS_2 and AAdS_2 geometries that we have discussed in the paper satisfy the equations of motion.

3. The most general solutions ([98, 99]) of the Liouville mode, ϕ, in AdS_2 background,

$$\phi = \frac{1}{2}\log\left[(z+\bar{z})^2 \frac{\partial g(z)\bar{\partial}\bar{g}(\bar{z})}{(g(z)+\bar{g}(\bar{z}))^2}\right]$$

   obtain further constraints from the equations of motion. That is not surprising because the above solutions were obtained from solving only the Liouville equation. This also bodes well with the degree of freedom counting in 2d theory of gravity. These constraints force the solutions of $g(z), \bar{g}(\bar{z})$ to be $\mathbb{SL}(2,\mathbb{C})$ transformations of complex plane, (3.36). However, the boundary conditions reduce it to $\mathbb{SL}(2,\mathbb{R})$ transformations, which are the isometries of the geometries that we are interested in. The remaining

[1] This also makes this section pretty ugly in terms of the equations.

solutions that don't satisfy the boundary conditions are what we call *non-normalizable* solutions.

4. This exercise also justifies the boundary terms that we have introduced in (3.28) that we have argued are required for a well defined variational principle.

We use following notations to avoid clutter in the forthcoming equations:

$$\int_\Gamma^x \equiv \int_\Gamma d^2x\sqrt{g(x)} \tag{G.1a}$$

$$\int_{\partial\Gamma}^s \equiv \int_{\partial\Gamma} ds\sqrt{\gamma(s)} \text{ where } s \text{ is the boundary coordinate} \tag{G.1b}$$

$$G(x,y) \text{ is the Green's function of the Laplacian satisfying, } \Box^{(x)}G(x,y) = \frac{\delta^2(x-y)}{\sqrt{g(x)}} \tag{G.1c}$$

$$\nabla_\mu^{(x)} \text{ denotes the covariant derivative with respect to variable } x \tag{G.1d}$$

Bulk Term We start with varying the bulk term in (3.28).

$$\begin{aligned}
\delta S_{cov}^{bulk}[g] &= \frac{1}{16\pi b^2}\int_\Gamma \delta\left(\sqrt{g}\left[R\frac{1}{\Box}R - 16\pi\mu\right]\right) \\
&= \frac{1}{16\pi b^2}\int_\Gamma d^2x\int_\Gamma d^2y\; \delta\left[\sqrt{g(x)}\sqrt{g(y)}R(x)G(x,y)R(y)\right] + \frac{1}{16\pi b^2}\int_\Gamma d^2x\; \delta\left[\sqrt{g(x)}\right](-16\pi\mu) \\
&= \frac{1}{16\pi b^2}\int_\Gamma d^2x\int_\Gamma d^2y\left(2\,\delta\left[\sqrt{g(x)}R(x)\right]\sqrt{g(y)}R(y)G(x,y) + \sqrt{g(x)}R(x)\sqrt{g(y)}R(y)\;\delta[G(x,y)]\right) \\
&\quad + \frac{1}{16\pi b^2}\int_\Gamma d^2x\;\delta\left[\sqrt{g(x)}\right](-16\pi\mu)
\end{aligned}$$

Here, in the last equation on RHS we have used the symmetry of Green's function in $x-y$ coordinates to multiply the first term by 2. In the above equation, the first and last term are very easy to compute while the second term is slightly more non-trivial. The variations of the Ricci scalar and metric determinant are:

$$\begin{aligned}
\delta\left[\sqrt{g(x)}\right] &= -\frac{1}{2}\sqrt{g(x)}g_{\mu\nu}(x)\delta g^{\mu\nu}(x) \\
\delta[R(x)] &= R_{\mu\nu}\delta g^{\mu\nu} + \nabla_\mu v^\mu \\
&\qquad \text{where, } v^\sigma = g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu}) - \nabla_\alpha(\delta g^{\alpha\sigma})
\end{aligned}$$

Henceforth, we are dropping the overall factor of $1/16\pi b^2$ and will reinstate it at the end.

$$\begin{aligned}
\delta S^{bulk}_{cov}[g] &= 2\int_\Gamma^x \int_\Gamma^y \left(R_{\mu\nu}(x) - \frac{1}{2}g_{\mu\nu}(x)R(x)\right)\delta g^{\mu\nu}(x)\, R(y)G(x,y) + 2\int_\Gamma^x \int_\Gamma^y \nabla^{(x)}_\mu v^\mu R(y)G(x,y) \\
&\quad + \int_\Gamma^x \int_\Gamma^y R(x)R(y)\, \delta[G(x,y)] + \int_\Gamma^x 8\pi\mu\, g_{\mu\nu}(x)\delta g^{\mu\nu}(x) \\
&= 2\int_\Gamma^x \int_\Gamma^y \nabla^{(x)}_\sigma \Big[g_{\mu\nu}(x)\nabla^\sigma_{(x)}(\delta g^{\mu\nu}(x)) - \nabla^{(x)}_\alpha(\delta g^{\alpha\sigma}(x))\Big] R(y)G(x,y) \\
&\quad + \int_\Gamma^x \int_\Gamma^y R(x)R(y)\, \delta[G(x,y)] + \int_\Gamma^x 8\pi\mu\, g_{\mu\nu}(x)\delta g^{\mu\nu}(x)
\end{aligned}$$

where, in the second line we have dropped the term containing Einstein tensor which is identically zero in 2 dimensions. Subsequently, we integrate by parts, keeping track of all the boundary terms that we pick in the process.

$$\begin{aligned}
\delta S^{bulk}_{cov}[g] &= -2\int_\Gamma^x \int_\Gamma^y \Big[\nabla^\sigma_{(x)}(g_{\mu\nu}(x)\delta g^{\mu\nu}(x)) - \nabla^{(x)}_\alpha(\delta g^{\alpha\sigma}(x))\Big] R(y)\nabla^{(x)}_\sigma G(x,y) \\
&\quad + 2\int_\Gamma^y \int_\Gamma^x \nabla^{(x)}_\sigma [v^\sigma(x)\, R(y)G(x,y)] + \int_\Gamma^x \int_\Gamma^y R(x)R(y)\, \delta[G(x,y)] + \int_\Gamma^x 8\pi\mu\, g_{\mu\nu}(x)\delta g^{\mu\nu}(x) \\
&= -2\int_\Gamma^x \int_\Gamma^y \Big[\nabla^\sigma_{(x)}(g_{\mu\nu}(x)\delta g^{\mu\nu}(x)) - \nabla^{(x)}_\alpha(\delta g^{\alpha\sigma}(x))\Big] R(y)\nabla^{(x)}_\sigma G(x,y) \\
&\quad + 2\int_\Gamma^y \int_{\partial\Gamma}^s \hat{n}_\sigma(s)v^\sigma(s)\, R(y)G(x,y) + \int_\Gamma^x \int_\Gamma^y R(x)R(y)\, \delta[G(x,y)] + \int_\Gamma^x 8\pi\mu\, g_{\mu\nu}(x)\delta g^{\mu\nu}(x) \\
&= -4\int_\Gamma^y \int_{\partial\Gamma}^s \delta\mathcal{K}\, R(y)G(x,y) - 2\int_\Gamma^x \int_\Gamma^y \nabla^\sigma_{(x)}\Big(g_{\mu\nu}(x)\delta g^{\mu\nu}(x)\, R(y)\nabla^{(x)}_\sigma G(x,y)\Big) \\
&\quad + \int_\Gamma^x g_{\mu\nu}(x)\delta g^{\mu\nu}(x)\Big(2R(x) + 8\pi\mu\Big) + 2\int_\Gamma^x \int_\Gamma^y \nabla^{(x)}_\alpha\Big(\delta g^{\alpha\sigma}(x)\, R(y)\nabla^{(x)}_\sigma G(x,y)\Big) \\
&\quad - 2\int_\Gamma^x \int_\Gamma^y \delta g^{\alpha\sigma}(x)\nabla^{(x)}_\alpha\nabla^{(x)}_\sigma G(x,y)\, R(y) + \int_\Gamma^x \int_\Gamma^y R(x)R(y)\, \delta[G(x,y)]
\end{aligned}$$

in the second line on RHS, we have used the Gauss's law to make the bulk integral into a surface integral. The first term in the second line is also the term that needs to be cancelled because it involves derivatives of variation of metric. Using $\hat{n}^\sigma v_\sigma = -2\delta\mathcal{K}$, one can clearly see that this term is cancelled by the variation of second term in (3.28). In the third line, we have used integration by parts in the second term of the second line. We have obtained two boundary terms in the process (2^{nd} and 4^{th} term in the third line), both of which involve variation of the metric on the boundary, and under Dirichlet boundary condition, are zero. They will be dropped from here onwards.

$$\begin{aligned}
\delta S^{bulk}_{cov}[g] &= -4\int_\Gamma^y \int_{\partial\Gamma}^s \delta\mathcal{K}\, R(y)G(x,y) + \int_\Gamma^x g_{\mu\nu}(x)\delta g^{\mu\nu}(x)\Big(2R(x) + 8\pi\mu\Big) \\
&\quad - 2\int_\Gamma^x \int_\Gamma^y \delta g^{\alpha\sigma}(x)\nabla^{(x)}_\alpha\nabla^{(x)}_\sigma G(x,y)\, R(y) + \int_\Gamma^x \int_\Gamma^y R(x)R(y)\, \delta[G(x,y)]
\end{aligned} \tag{G.2}$$

Now we embark upon the computation of $\delta[G(x,y)]$. The Green's function in a curved background is defined in a covariant manner by (G.1c). Varying this equation with respect to metric,

$$\begin{aligned}\delta\Box^{(x)}\,G(x,y)+\Box^{(x)}\delta G(x,y)&=\frac{1}{2\sqrt{g(x)}}g_{\mu\nu}(x)\delta g^{\mu\nu}(x)\;\delta^2(x-y)\\ \delta G(x,y)&=\frac{1}{2}g_{\mu\nu}(y)\delta g^{\mu\nu}(y)\;G(x,y)-\int_\Gamma^w G(x,w)\delta\Box^{(w)}\,G(w,y)\end{aligned}\tag{G.3}$$

Here in the second line we have integrated both sides with a Green's function. The action of Laplacian on a scalar is also given by, $\Box^{(x)}f(x)=\frac{1}{\sqrt{g(x)}}\partial_\mu\Big(\sqrt{g(x)}g^{\mu\nu}(x)\partial_\nu f(x)\Big)$. Thus the variation of the $\Box$ operator is,

$$\begin{aligned}\delta\Box^{(x)}\;f(x)&=\frac{1}{2}g_{\mu\nu}(x)\delta g^{\mu\nu}(x)\;\Box^{(x)}f(x)-\frac{1}{2\sqrt{g(x)}}\partial_\mu\Big(\sqrt{g(x)}g_{\rho\sigma}(x)\delta g^{\rho\sigma}(x)g^{\mu\nu}(x)\partial_\nu f(x)\Big)\\ &\quad+\frac{1}{\sqrt{g(x)}}\partial_\mu\Big(\sqrt{g(x)}\delta g^{\mu\nu}(x)\partial_\nu f(x)\Big)\\ &=-\frac{1}{2}g^{\mu\nu}(x)\partial_\nu f(x)\;\nabla_\mu\Big(g_{\rho\sigma}(x)\delta g^{\rho\sigma}(x)\Big)+\frac{1}{\sqrt{g(x)}}\partial_\mu\Big(\sqrt{g(x)}\delta g^{\mu\nu}(x)\partial_\nu f(x)\Big)\\ &=-\frac{1}{2}g^{\mu\nu}(x)\partial_\nu f(x)\;\nabla_\mu\Big(g_{\rho\sigma}(x)\delta g^{\rho\sigma}(x)\Big)+\partial_\mu(\delta g^{\mu\nu}(x))\;\partial_\nu f(x)+\delta g^{\mu\nu}(x)\;\partial_\mu\partial_\nu f(x)\\ &\quad+\frac{1}{2g(x)}\partial_\mu(g(x))\;\delta g^{\mu\nu}(x)\partial_\nu f(x)\\ &=-\frac{1}{2}g^{\mu\nu}(x)\partial_\nu f(x)\;\nabla_\mu\Big(g_{\rho\sigma}(x)\delta g^{\rho\sigma}(x)\Big)+\partial_\mu(\delta g^{\mu\nu}(x))\;\partial_\nu f(x)+\delta g^{\mu\nu}(x)\;\partial_\mu\partial_\nu f(x)\\ &\quad+\Gamma^\sigma{}_{\mu\sigma}\;\delta g^{\mu\nu}(x)\partial_\nu f(x)\\ \Rightarrow\delta\Box^{(x)}\;f(x)&=-\frac{1}{2}g^{\mu\nu}(x)\partial_\nu f(x)\;\nabla_\mu\Big(g_{\rho\sigma}(x)\delta g^{\rho\sigma}(x)\Big)+\nabla_\mu(\delta g^{\mu\nu}(x))\;\partial_\nu f(x)+\delta g^{\mu\nu}(x)\;\nabla_\mu\partial_\nu f(x)\end{aligned}\tag{G.4}$$

We have used chain rule of differentiation to come from the first line on RHS to the second line. We have also changed the normal derivative acting on $g_{\rho\sigma}(x)\delta g^{\rho\sigma}(x)$ into a covariant derivative because it is a scalar. In the fourth line we have used the identity, $\partial_\mu g(x)=2g(x)\Gamma^\nu{}_{\mu\nu}$. We have also converted some of the differentiations into covariant derivatives in last line. For our computations, the role of f in the above computations is played by,

$\int_\Gamma^y \sqrt{g(y)} R(y) G(w,y)$. Using (G.4) in (G.3) and substituting back into last term of (G.2),

$$\begin{aligned}
\int_\Gamma^x \int_\Gamma^y R(x)R(y)\ \delta[G(x,y)] &= \int_\Gamma^x \int_\Gamma^y R(x)R(y)\Big[\frac{1}{2} g_{\mu\nu}(y)\delta g^{\mu\nu}(y)\ G(x,y) - \int_\Gamma^w G(x,w)\delta\Box^{(w)}\ G(w,y)\Big] \\
&= \frac{1}{2}\int_\Gamma^x \int_\Gamma^y R(x)R(y) g_{\mu\nu}(y)\delta g^{\mu\nu}(y)\ G(x,y) \\
&+ \frac{1}{2}\int_\Gamma^x \int_\Gamma^y \int_\Gamma^w R(x)R(y)G(x,w)\Big[g^{\mu\nu}(w)\frac{\partial}{\partial w^\nu} G(w,y)\ \nabla^w_\mu\Big(g_{\rho\sigma}(w)\delta g^{\rho\sigma}(w)\Big) \\
&\quad - 2\nabla^w_\mu(\delta g^{\mu\nu}(w))\ \frac{\partial}{\partial w^\nu} G(w,y) - 2\delta g^{\mu\nu}(w)\ \nabla^w_\mu \frac{\partial}{\partial w^\nu} G(w,y)\Big] \\
&= \frac{1}{2}\int_\Gamma^x \int_\Gamma^y R(x)R(y) g_{\mu\nu}(y)\delta g^{\mu\nu}(y)\ G(x,y) \\
&- \frac{1}{2}\int_\Gamma^x \int_\Gamma^y \int_\Gamma^w R(x)R(y)\Big[G(x,w)\ \Box^{(w)} G(w,y)\ g_{\rho\sigma}(w)\delta g^{\rho\sigma}(w)\Big] \\
&- \frac{1}{2}\int_\Gamma^x \int_\Gamma^y \int_\Gamma^w R(x)R(y)\Big[g_{\rho\sigma}(w)\ g^{\mu\nu}(w)\frac{\partial}{\partial w^\mu} G(x,w)\frac{\partial}{\partial w^\nu} G(w,y)\Big]\delta g^{\rho\sigma}(w) \\
&+ \frac{1}{2}\int_\Gamma^x \int_\Gamma^y \int_\Gamma^w R(x)R(y)\nabla^w_\mu\Big[G(x,w) g^{\mu\nu}(w)\frac{\partial}{\partial w^\nu} G(w,y)\ g_{\rho\sigma}(w)\delta g^{\rho\sigma}(w)\Big] \\
&\quad - \int_\Gamma^x \int_\Gamma^y \int_\Gamma^w R(x)R(y)\nabla^w_\mu\Big[G(x,w)\delta g^{\mu\nu}(w)\ \frac{\partial}{\partial w^\nu} G(w,y)\Big] \\
&\quad + \int_\Gamma^x \int_\Gamma^y \int_\Gamma^w R(x)R(y)\Big[\nabla^w_\mu G(x,w)\delta g^{\mu\nu}(w)\ \frac{\partial}{\partial w^\nu} G(w,y)\Big] \\
&\quad + \int_\Gamma^x \int_\Gamma^y \int_\Gamma^w R(x)R(y)G(x,w)\ \delta g^{\mu\nu}(w)\ \nabla^w_\mu \frac{\partial}{\partial w^\nu} G(w,y) \\
&\quad - \int_\Gamma^x \int_\Gamma^y \int_\Gamma^w R(x)R(y)G(x,w)\delta g^{\mu\nu}(w)\ \nabla^w_\mu \frac{\partial}{\partial w^\nu} G(w,y)
\end{aligned}$$

in the third line on RHS, the first two terms cancel between themselves, while the last two terms also cancel between themselves. The fourth and the fifth terms are total derivative terms that are essentially some boundary terms. These terms vanish since we are working with Dirichlet boundary conditions such that the Green's function vanishes on the boundary.

$$\begin{aligned}
&\int_\Gamma^x \int_\Gamma^y R(x)R(y)\ \delta[G(x,y)] = \\
&\quad \int_\Gamma^x \int_\Gamma^y \int_\Gamma^w R(x)R(y)\Big[\frac{\partial G(w,x)}{\partial w^\mu}\frac{\partial G(w,y)}{\partial w^\mu} - \frac{1}{2} g_{\mu\nu}(w) g^{\alpha\beta}(w)\frac{\partial G(w,x)}{\partial w^\alpha}\frac{\partial G(w,y)}{\partial w^\beta}\Big]\delta g^{\mu\nu}(w)
\end{aligned} \tag{G.5}$$

Thus, the final expression of the variation of the bulk action is (with the reinstating of the overall $\frac{1}{16\pi b^2}$ factor),

$$\begin{aligned}\delta S^{bulk}_{cov}[g] = &-\frac{1}{4\pi b^2}\int_\Gamma^y \int_{\partial\Gamma}^s \delta\mathcal{K}\; R(y)G(x,y) + \frac{1}{16\pi b^2}\int_\Gamma^x g_{\mu\nu}(w)\delta g^{\mu\nu}(w)\Big(2R(w)+8\pi\mu\Big)\\ &-\frac{1}{8\pi b^2}\int_\Gamma^x\int_\Gamma^y \delta g^{\alpha\sigma}(w)\nabla^{(w)}_\alpha\nabla^{(w)}_\sigma G(w,y)\;R(y)\\ &+\frac{1}{16\pi b^2}\int_\Gamma^x\int_\Gamma^y\int_\Gamma^w R(x)R(y)\left[\frac{\partial G(w,x)}{\partial w^\mu}\frac{\partial G(w,y)}{\partial w^\mu}-\frac{1}{2}g_{\mu\nu}(w)g^{\alpha\beta}(w)\frac{\partial G(w,x)}{\partial w^\alpha}\frac{\partial G(w,y)}{\partial w^\beta}\right]\delta g^{\mu\nu}(w)\end{aligned} \tag{G.6}$$

The bulk equations of motion are non local and given by:

$$\begin{aligned}0 = \frac{1}{16\pi b^2}\Bigg(&g_{\mu\nu}(w)\Big(2R(w)+8\pi\mu\Big)+\int_\Gamma^x\left[-2\nabla^{(w)}_\mu\nabla^{(w)}_\nu G(w,x)R(x)\right]\\ &+\int_\Gamma^x\int_\Gamma^y\left[\frac{\partial G(w,x)}{\partial w^\mu}\frac{\partial G(w,y)}{\partial w^\mu}-\frac{1}{2}g_{\mu\nu}(w)g^{\alpha\beta}(w)\frac{\partial G(w,x)}{\partial w^\alpha}\frac{\partial G(w,y)}{\partial w^\beta}\right]R(x)R(y)\Bigg)\end{aligned} \tag{G.7}$$

Now let us look at the trace part of the equations of motion. The last term in the above equation doesn't contribute in that case.

$$\begin{aligned}0 &= \left(2[2R(w)+8\pi\mu]+\int_\Gamma^x\left[-2\Box^{(w)}G(w,x)R(x)\right]\right)\\ &= R(w)+8\pi\mu\end{aligned} \tag{G.8}$$

In conformal gauge, where $g_{\mu\nu}(x) = e^{2\phi(x)}\hat{g}_{\mu\nu}(x)$, this is same as, (3.33),

$$\hat{R}(x) - 2\hat{\Box}\phi(x) = -8\pi\mu e^{2\phi(x)} \tag{G.9}$$

which is also the equation of motion for the Liouville mode ϕ with background metric $\hat{g}$. In AdS$_2$ background ($d\hat{s}^2 = \big(1/\pi\mu(z+\bar{z})^2\big)dz\,d\bar{z}$), the most general solution of this equation is, [98, 99],

$$\phi = \frac{1}{2}\log\left[(z+\bar{z})^2\frac{\partial g(z)\bar{\partial}\bar{g}(\bar{z})}{(g(z)+\bar{g}(\bar{z}))^2}\right] \tag{G.10}$$

where, in Euclidean space, $g(z), \bar{g}(\bar{z})$ are complex function which are complex conjugate of each other. Equivalently, in Lorentzian space, they can be chosen to be two independent real functions.

Solving (G.7) in full generality is a daunting task that we don't undertake. We show that

AdS_2 satisfies these equations of motion, and also provide an argument that $AAdS_2$ geometries satisfy them too. Traceless part of (G.7) is,

$$
\begin{aligned}
0 = &\int_\Gamma^x \left[-2\left(\nabla_\mu^{(w)}\nabla_\nu^{(w)}G(w,x) - \frac{1}{2}g_{\mu\nu}(w)\Box^{(w)}G(w,x)\right)R(x)\right] \\
&+ \int_\Gamma^x\int_\Gamma^y \left[\frac{\partial G(w,x)}{\partial w^\mu}\frac{\partial G(w,y)}{\partial w^\mu} - \frac{1}{2}g_{\mu\nu}(w)g^{\alpha\beta}(w)\frac{\partial G(w,x)}{\partial w^\alpha}\frac{\partial G(w,y)}{\partial w^\beta}\right]R(x)R(y)
\end{aligned}
\tag{G.11}
$$

One way to check that AdS_2 satisfies the on-shell equations of motion is to directly use the (F.9) in the above expression and do the exact computation. However, it is much easier if we think of AdS_2 as Weyl scaling of flat space, $g^{AdS}_{\alpha\beta} = e^{2\Omega}\eta_{\alpha\beta}$, where for $\mathbb{H}$, $\Omega = -\log\big(\sqrt{\pi\mu}(z+\bar{z})\big) = -\log\big(\sqrt{4\pi\mu}\zeta\big)$. We use the formula for Ricci scalar, $R(x) = -2e^{-2\Omega}\Box_{flat}\Omega(x)$ to write (G.11) as,

$$
\begin{aligned}
0 = &\,4\int_\Gamma d^2x e^{2\Omega}\left[\left(\nabla_\mu^{(w)}\nabla_\nu^{(w)}G(w,x) - \frac{1}{2}g_{\mu\nu}(w)\Box^{(w)}G(w,x)\right)\left(e^{-2\Omega}\Box^{(x)}_{flat}\Omega(x)\right)\right] \\
&+ 4\int_\Gamma d^2x\int_\Gamma d^2y e^{2\Omega(x)}e^{2\Omega(y)}\left[\frac{\partial G(w,x)}{\partial w^\mu}\frac{\partial G(w,y)}{\partial w^\mu} - \frac{1}{2}g_{\mu\nu}(w)g^{\alpha\beta}(w)\frac{\partial G(w,x)}{\partial w^\alpha}\frac{\partial G(w,y)}{\partial w^\beta}\right] \\
&\qquad\times\left(e^{-2\Omega(x)}\Box^{(x)}_{flat}\Omega(x)\right)\left(e^{-2\Omega(y)}\Box^{(y)}_{flat}\Omega(y)\right) \\
= &\,4\int_\Gamma d^2x\left[\left(\nabla_\mu^{(w)}\nabla_\nu^{(w)}\Box^{(x)}_{flat}G(w,x) - \frac{1}{2}g_{\mu\nu}(w)\Box^{(w)}\Box^{(x)}_{flat}G(w,x)\right)\Omega(x)\right] \\
&+ 4\int_\Gamma d^2x\int_\Gamma d^2y\left[\frac{\partial\left(\Box^{(x)}_{flat}G(w,x)\right)}{\partial w^\mu}\frac{\partial\left(\Box^{(y)}_{flat}G(w,y)\right)}{\partial w^\mu}\right. \\
&\qquad\left. - \frac{1}{2}g_{\mu\nu}(w)g^{\alpha\beta}(w)\frac{\partial\left(\Box^{(x)}_{flat}G(w,x)\right)}{\partial w^\alpha}\frac{\partial\left(\Box^{(y)}_{flat}G(w,y)\right)}{\partial w^\beta}\right]\times\Omega(x)\Omega(y) \\
= &\,4\left[\nabla_\mu^{(x)}\nabla_\nu^{(x)}\Omega(x) - \frac{1}{2}g_{\mu\nu}(w)\Box^{(w)}\Omega(x)\right] + 4\left[\frac{\partial\Omega(x)}{\partial w^\mu}\frac{\partial\Omega(y)}{\partial w^\mu} - \frac{1}{2}g_{\mu\nu}(x)g^{\alpha\beta}(x)\frac{\partial\Omega(x)}{\partial x^\alpha}\frac{\partial\Omega(y)}{\partial x^\beta}\right] \\
= &\,0
\end{aligned}
\tag{G.12}
$$

In the second line we have used integration by parts to shift $\Box_{flat}$ on the corresponding Green's functions; we have dropped the vanishing boundary terms on our way. We also use the fact discussed at the end of Appendix F, that the Green's function remain unchanged for the Weyl scaled metrics, upto impositions of boundary condition. In this case the boundary condition, $G(\{\zeta,\tau\},\{0,\tau_2\}) = 0$, is imposed by adding a contribution of a 'mirror charge' at a point reflected across the boundary. Thus the flat space Laplacian acting on this Green's function gives two δ-functions, one each for the 'original charge' and 'mirror charge'.[2] The δ-function of the mirror charge lies outside the region of integration and hence doesn't contribute.

[2]The δ-function is a flat space δ-function.

The equations of motion (G.9),(G.11) are covariant equations under diffeomorphisms. Thus they will also be satisfied for the class of geometries that we constructed in section 3.5.
We can still do slightly better and solve the equations of motion for Weyl scaled metrics around a given background. Around AdS_2 background, from (G.11) we get following *Virasoro constraints* for ϕ,

$$4\begin{pmatrix} \partial^2\phi(z,\bar{z}) - (\partial\phi(z,\bar{z}))^2 + 2\dfrac{\partial\phi(z,\bar{z})}{z+\bar{z}} & 0 \\ 0 & \bar{\partial}^2\phi(z,\bar{z}) - \left(\bar{\partial}\phi(z,\bar{z})\right)^2 + 2\dfrac{\bar{\partial}\phi(z,\bar{z})}{z+\bar{z}} \end{pmatrix} = 0 \tag{G.13}$$

Solving (G.9) and (G.13) simultaneously, we get solutions of the type (G.10), but with $g, \bar{g}$ additionally restricted by the conditions,

$$0 = \begin{pmatrix} 2\left(\dfrac{g^{(3)}(z)}{g'(z)} - \dfrac{3}{2}\dfrac{g''(z)^2}{g'(z)^2}\right) & 0 \\ 0 & 2\left(\dfrac{\bar{g}^{(3)}(z)}{\bar{g}'(z)} - \dfrac{3}{2}\dfrac{\bar{g}''(z)^2}{\bar{g}'(z)^2}\right) \end{pmatrix} \tag{G.14}$$

which is basically the Schwarzian derivatives of $g(z)$ and $\bar{g}(\bar{z})$. This restricts $g(z)$ to be of the form,

$$g(z) = \frac{az+ib}{icz+d} \tag{G.15}$$

for $a, b, c, d \in \mathbb{C}$, and $\bar{g}(\bar{z})$ is its complex conjugate. Imposing the boundary condition, $g(z) + \bar{g}(\bar{z})|_{z+\bar{z}=0} = 0$ further restricts $a, b, c, d \in \mathbb{R}$. These precisely corresponds to the isometries of the geometries that we are considering. However, more general choice of these parameters gives us solutions that we call *non-normalizable* in this paper. These solutions diverge as $1/\zeta$ for small deviations around identity,

$$\begin{aligned} a &= 1 + \delta a \\ b &= \delta b \\ c &= \delta c \\ d &= 1 - \delta a \end{aligned} \tag{G.16}$$

Boundary Term A similar analysis for the variation of boundary terms in (3.28) gives,

$$
\begin{aligned}
\delta S^{bdy}_{cov}[g] &= \frac{1}{16\pi b^2}\int_\Gamma \delta\left(4\sqrt{\gamma}\mathcal{K}\frac{1}{\Box}R\right) \\
&= \frac{1}{4\pi b^2}\int_\Gamma^x\int_{\partial\Gamma}^s \delta\mathcal{K}(s)\ G(x,s)R(x) - \frac{1}{4\pi b^2}\int_\Gamma^x\int_{\partial\Gamma}^s \delta g^{\mu\nu}(x)\left[\nabla^{(x)}_\mu\nabla^{(x)}_\nu G(x,s)\mathcal{K}(s)\right] \\
&+ \frac{1}{4\pi b^2}\int_{\partial\Gamma}^s\int_\Gamma^x\int_\Gamma^w \delta g^{\mu\nu}(w)\left[\frac{\partial G(w,x)}{\partial w^\mu}\frac{\partial G(w,s)}{\partial w^\mu} - \frac{1}{2}g_{\mu\nu}(w)g^{\alpha\beta}(w)\frac{\partial G(w,x)}{\partial w^\alpha}\frac{\partial G(w,s)}{\partial w^\beta}\right]\mathcal{K}(s)R(x) \\
&- \frac{1}{2\pi b^2}\int_{\partial\Gamma}^s\int_{\partial\Gamma}^{s'} \delta\mathcal{K}(s)G(s,s')\mathcal{K}(s')
\end{aligned}
\tag{G.17}
$$

Note that the first term in RHS of (G.17) exactly cancels the last term in RHS of (G). Moreover, the last term in (G.17) exactly cancels the variation arising from the last term in (3.28). Also in writing the above expressions we have made use of the fact that we are imposing Dirichlet boundary conditions on the metric, $\delta g_{\mu\nu}|_{\partial\Gamma} = 0$

Appendix H

Analysing off-shell constraints

In this section we demonstrate that the constraints coming from the traceless part of the equations of motion in the conformal gauge, *viz.* the 'Virasoro constraints' (3.35), do not permit any off-shell degrees of freedom apart from those representing large diffeomorphisms of AdS_2 geometry.

It is enough to carry out this analysis in absence of the large diffeomorphisms, with $\widehat{ds}^2$ as in (3.32). The generalization to (3.47) is obtained by applying the large diffeomorphism (3.46), in the manner explained in Section 3.5.

Simplifying the holomorphic part of the constraints,

$$
\begin{aligned}
&\partial^2\phi(z,\bar{z}) - (\partial\phi(z,\bar{z}))^2 + 2\frac{\partial\phi(z,\bar{z})}{z+\bar{z}} = 0 \\
\Rightarrow\ &\partial\Big((z+\bar{z})^2\,\partial\Big(e^{-\phi(z,\bar{z})}\Big)\Big) = 0 \qquad\qquad \text{(H.1)} \\
\Rightarrow\ &\partial\Big(e^{-\phi(z,\bar{z})}\Big) = \frac{A(\bar{z})}{(z+\bar{z})^2} \\
\Rightarrow\ &e^{-\phi(z,\bar{z})} = -\frac{A(\bar{z})}{(z+\bar{z})} + B(\bar{z}) \qquad\qquad \text{(H.2)}
\end{aligned}
$$

Similarly, solving the anti-holomorphic part gives,

$$
e^{-\phi(z,\bar{z})} = -\frac{C(z)}{(z+\bar{z})} + D(z) \tag{H.3}
$$

In the above equations, the functions A, B, C, D are arbitrary and independent, to begin with, as they appear as "constants' of integration. However, they must satisfy the requirement that the two expressions (H.2) and (H.3) for the same quantity $e^{-\phi(z,\bar{z})}$ must be (i) equal to each other and (ii) real. Assuming a general power series form of each of the functions, we find that these two requirements can only be met if A, C are quadratic and B, D

are linear, and, in particular, of the form

$$\begin{aligned} A(\bar{z}) &= \mathbf{a}\bar{z}^2 + 2i\mathbf{b}\bar{z} + \mathbf{c}, \quad B(\bar{z}) = \mathbf{a}\bar{z} + \mathbf{d} + i\mathbf{b} \\ C(z) &= \mathbf{a}z^2 - 2i\mathbf{b}z + \mathbf{c}, \quad D(z) = \mathbf{a}z + \mathbf{d} - i\mathbf{b}, \end{aligned} \tag{H.4}$$

leading to the following solution for the Liouville field,

$$\begin{aligned} e^{-\phi(z,\bar{z})} &= \frac{\mathbf{a}z\bar{z} + (\mathbf{d}+i\mathbf{b})z + (\mathbf{d}-i\mathbf{b})\bar{z} - \mathbf{c}}{z+\bar{z}} \\ \Rightarrow \phi(z,\bar{z}) &= \frac{1}{2}\log\left[\frac{(z+\bar{z})^2}{(\mathbf{a}z\bar{z} + (\mathbf{d}+i\mathbf{b})z + (\mathbf{d}-i\mathbf{b})\bar{z} - \mathbf{c})^2}\right] \end{aligned} \tag{H.5}$$

Here the constants $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are real. Out of these four, only three are physical. The reason is that the Virasoro constraints, expressed as in (H.1) (and the similar, antiholomorphic equation) are homogeneous linear equations in the variable $e^{-\phi(z,\bar{z})}$, which implies that $e^{-\phi(z,\bar{z})} \to \text{constant} \times e^{-\phi(z,\bar{z})}$ is a symmetry of the equations. Hence, the constants $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are only determined up to a (real) scale factor.

It is important to check that the solution (H.5) of the Virasoro constraints satisfies the equation of motion (3.33). This can be done in two ways:
(i) By direct substitution of (H.5) into (3.33), we find that (3.33) is satisfied up to a term proportional to $\mathbf{ac} + (\mathbf{d}+i\mathbf{b})(\mathbf{d}-i\mathbf{b}) - 1$. By using the scale symmetry mentioned above, we can clearly make this vanish, e.g. by treating $\mathbf{a}, \mathbf{b}$ and $\mathbf{d}$ as independent variables and fixing $\mathbf{c} = (1 - (\mathbf{d}+i\mathbf{b})(\mathbf{d}-i\mathbf{b}))/\mathbf{a}$ (this is equivalent to choosing a gauge).
(ii) Alternatively, one can match (H.5) with the solution (3.34). We find that the parameters of the two solutions are related as follows

$$\bar{a}c - a\bar{c} = -i\mathbf{a},\ \bar{b}d - b\bar{d} = -i\mathbf{c},\ \bar{b}c + a\bar{d} = \mathbf{d} - i\mathbf{b} \tag{H.6}$$

The $\mathbb{SL}(2,\mathbb{C})$ conditions $ad + bc = 1$ translate to the condition

$$\mathbf{ac} + (\mathbf{d}+i\mathbf{b})(\mathbf{d}-i\mathbf{b}) = 1, \tag{H.7}$$

As mentioned before, on this surface (H.5) solves the equation of motion (3.33). Furthermore, in the analysis of (3.34), we found that the $\mathbb{SL}(2,\mathbb{R})$subgroup, parameterized by real values of a, b, c, d, correspond to trivial isometries of AdS_2, and did not generate a new solution; there is a natural interpretation of this fact according to (H.6): real a, b, c, d translate to $\mathbf{a} = \mathbf{b} = \mathbf{c} = 0$, $\mathbf{d} = 1$, leading to the trivial solution $\phi = 0$. Thus the variables $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, given by (H.6) actually parameterize the nontrivial coset $\mathbb{SL}(2,\mathbb{C})/\mathbb{SL}(2,\mathbb{R})$. In fact, the hyperboloid (H.7) parameterizes this coset.

As mentioned before, the above analysis can be generalized to the case of the reference metric (3.47) by applying the large diffeomorphism (3.46) to the solution (H.5).

Conclusion: The Virasoro constraints completely fix the Liouville field ϕ (up to three real constants). As explained in the text (see Section 3.5), the three constants need to be fixed as boundary conditions for the path integral (since they correspond to non-normalizable deformations). Thus, there are no off-shell variables (*i.e.* variables appearing in the path integration) that come from the Liouville field ϕ. The only off-shell variables are represented by the large diffeomorphisms as in (3.50).

Appendix I

Exact computation of asymptotic AdS_2 geometries

We use the knowledge of exact asymptotically AdS_3 geometries to construct $AAdS_2$ geometries. In AdS_3 the space of solutions of spacetimes with constant negative curvature is is given by, [104, 105],

$$ds^2 = L^2_{(AdS_3)}\left(\frac{d\zeta^2 + 2dxd\bar{x}}{\zeta^2} + L(x)dx^2 + \bar{L}(\bar{x})d\bar{x}^2 - \frac{\zeta^2}{2}L(x)\bar{L}(\bar{x})dxd\bar{x}\right) \tag{I.1}$$

where, $L(x)$, $\bar{L}(\bar{x})$ are holomorphic and anti-holomorphic functions, and related to the holographic stress tensor, [44]. In the above references it is discussed how following large diffeomorphisms generate the above class of geometries from the Poincare AdS_3 geometry $(ds^2 = L^2_{(AdS_3)}(du^2 + 2dyd\bar{y})/u^2)$,

$$\begin{aligned} y &= f(x) + \frac{2\zeta^2 f'(x)^2 \bar{f}''(\bar{x})}{8f'(x)\bar{f}'(\bar{x}) - \zeta^2 f''(x)\bar{f}''(\bar{x})} \\ y &= \bar{f}(\bar{x}) + \frac{2\zeta^2 \bar{f}'(\bar{x})^2 f''(x)}{8f'(x)\bar{f}'(\bar{x}) - \zeta^2 f''(x)\bar{f}''(\bar{x})} \\ u &= \zeta\frac{\left(4f'(x)\bar{f}'(\bar{x})\right)^{3/2}}{8f'(x)\bar{f}'(\bar{x}) - \zeta^2 f''(x)\bar{f}''(\bar{x})} \end{aligned} \tag{I.2}$$

In general, in any arbitrary dimensions, it is not difficult to solve for the asymptotic Killing vectors for any spacetime. The special feature of AdS_3 is the fact that these infinitesimal diffeomorphisms can be integrated to non-linear order. AdS_2 being a more constrained geometry, also enjoys this same feature. Here, we use the known results of the exact non-linear diffeomorphisms in AdS_3 to construct the class of asymptotic AdS_2 solutions. In Cartesian coordinates, $y = (x + i\tau)/\sqrt{2}, \bar{y} = (x - i\tau)/\sqrt{2}$, one can obtain AdS_2 as a reduction of AdS_3 b y restricting to $x = 0$ slice. Restricting ourselves to those transformations that keeps this

AdS_2 slice invariant, i.e., for $f(x)+\bar{f}(\bar{x})|_{x+\bar{x}=0}=0$, we find that the following coordinate transformations are precisely those which generate large diffeomorphisms in AdS_2, (3.46), while keeping us within Fefferman-Graham gauge,

$$\tilde{\tau}=f(\tau)-\frac{2\zeta^2 f''(\tau)f'(\tau)^2}{4f'(\tau)^2+\zeta^2 f''(\tau)^2},\quad \tilde{\zeta}=\frac{4\;\zeta f'(\tau)^3}{4f'(\tau)^2+\zeta^2 f''(\tau)^2}$$

These transformations map the AdS_2 metrics, $ds^2=(d\tilde{\zeta}^2+d\tilde{\tau}^2)/(4\pi\mu\,\tilde{\zeta}^2)$ to AAdS_2 geometries, $ds^2=\left(d\zeta^2+\left(1-\frac{\zeta^2}{2}\{f(\tau),\tau\}\right)^2 d\tau^2\right)/(4\pi\mu\,\zeta^2)$.

Appendix J

Quantum corrections to the classical action

In this section we discuss the issue of gauge fixing the action (3.28). The idea and Faddeev-Popov procedure to arrive at the same: We introduce a functional delta-function in our path integral using the Faddeev-Popov prescription. The corresponding determinant is then written in terms of fermionic ghosts, which gives rise to new ghost-graviton interaction vertices.[1]

The Faddeev-Popov determinant is defined in terms of the gauge-fixing δ-function as follows,

$$1 = \Delta_{FP}\Big[\hat{g}[f(\tau)], \phi\Big] \times \int [\mathcal{D}\epsilon^{(s)}][\mathcal{D}\phi][\mathcal{D}f(\tau)]\ \delta\Big(g^{\epsilon^{(s)}} - e^{2\phi}\hat{g}[f(\tau)]\Big) \times \delta\Big(\epsilon^{(s)}(z_1)\Big)\ \delta\Big(\epsilon^{(s)}(z_2)\Big)\ \delta\Big(\epsilon^{(s)}(z_3)\Big) \tag{J.1}$$

Here, we are denoting the small diffeomorphisms (these are the gauge-symmetry of the theory) by $\epsilon^{(s)}$. In the subsequent discussion we will drop the (s) superscript to conciseness. ϕ denotes the Weyl degree of freedom and will eventually become the Liouville mode. Since our theory is not Weyl-invariant, unlike in String theory, we don't factor out these degrees of freedom. Finally, $f(\tau)$ denotes the degree of freedom due to large diffeomorphisms that is discussed in section 3.5. We are gauge fixing (using only small diffeomorphisms) an arbitrary metric to a metric that is Weyl equivalent to metrics (3.47). This procedure will give us the Jacobian corresponding to change of integration 'variable' from $[\mathcal{D}g]$ to $[\mathcal{D}\phi][\mathcal{D}f(\tau)]$. The δ-functions have been included in the above expression to fix the residual gauge symmetry corresponding to our gauge choice. This is precisely the $\mathbb{SL}(2,\mathbb{R})$ isometry of the geometries (3.47), and hence we choose to fix three arbitrary points in the interior of AAdS_2 geometries.

[1] In the subsequent discussion in this particular appendix, we call the f degree of freedom of section 3.5 corresponding to large diffeomorphisms of AdS_2 as 'gravitons'.

The path integral that we are interested in computing is formally written as,

$$Z = \int \frac{[\mathcal{D}g]}{V_\epsilon} e^{-S[g]}, \quad V_\epsilon \text{ is the volume of the symmetry group}$$

and inserting (J.1) into this path integral, we get,

$$\begin{aligned}
Z &= \int \frac{[\mathcal{D}g][\mathcal{D}\epsilon][\mathcal{D}\phi][\mathcal{D}f(\tau)]}{V_\epsilon} \times \Delta_{FP}\Big[\hat{g}[f(\tau)],\phi\Big]\ \delta\Big(g^\epsilon - e^{2\phi}\hat{g}[f(\tau)]\Big) \times e^{-S[g]} \times (\delta\text{-functions}) \\
&= \int \frac{[\mathcal{D}\tilde{g}][\mathcal{D}\epsilon][\mathcal{D}\phi][\mathcal{D}f(\tau)]}{V_\epsilon} \times \Delta_{FP}\Big[\hat{g}[f(\tau)],\phi\Big]\ \delta\Big(\tilde{g} - e^{2\phi}\hat{g}[f(\tau)]\Big) \times e^{-S[\tilde{g}]} \times (\delta\text{-functions}) \\
&= \int \frac{[\mathcal{D}\epsilon][\mathcal{D}\phi][\mathcal{D}f(\tau)]}{V_\epsilon} \times \Delta_{FP}\Big[\hat{g}[f(\tau)],\phi\Big] \times e^{-S[e^{2\phi}\hat{g}[f(\tau)]]} \times (\delta\text{-functions}) \\
&= \int [\mathcal{D}\phi][\mathcal{D}f(\tau)] \times \Delta_{FP}\Big[\hat{g}[f(\tau)],\phi\Big] \times e^{-S[e^{2\phi}\hat{g}[f(\tau)]]} \times (\delta\text{-functions})
\end{aligned} \tag{J.2}$$

In the second line on RHS, we have changed integration 'variables' from $\mathcal{D}g$ to $\mathcal{D}\tilde{g}$, where, $g =: \tilde{g}^{\epsilon^{-1}}$ and used the fact that action and measure are both gauge invariant. In the third line we have integrated over the metric degrees of freedom using the δ-function. In the last line we have used the fact that the integrand of third line doesn't depend on ϵ anymore, integration over which simply gives us the volume of the symmetry group.
Faddeev Popov determinant can be easily written in terms of the b and c ghosts as,

$$\Delta_{FP}\Big[\hat{g}[f(\tau)],\phi\Big] = \int \mathcal{D}c_\alpha\ \mathcal{D}b^{\alpha\beta}\ \mathcal{D}\mathfrak{f}_\alpha \exp\Big[-\Big(b^{\alpha\beta}(\hat{P}c)_{\alpha\beta} - b^{\alpha\beta}(\hat{P}\mathfrak{f})_{\alpha\beta}\Big)\Big] \times \left[\frac{c(z_1)c(z_2)c(z_3)}{(z_1-z_2)(z_2-z_3)(z_3-z_1)}\right] \tag{J.3}$$

here, c-insertions are equivalent to the δ-functions appearing in the previous expressions. $b^{\alpha\beta}$ is a symmetric-traceless tensor, and thus has only 2 degrees of freedom. We have defined operator $\hat{P}$ such that,

$$(\hat{P}x)_{\alpha\beta} := {}^{(f)}\nabla_{(\alpha}x_{\beta)} - ({}^{(f)}\nabla \cdot x)\ \hat{g}[f(\tau)]_{\alpha\beta}$$

${}^{(f)}\nabla$ is the covariant derivative w.r.t geometries in (3.47). $\mathfrak{f}$ is defined in terms of the fermionized large diffeomorphisms (3.45) as,

$$\mathfrak{f} = \begin{pmatrix} \zeta\ \mathfrak{k}'(\tau) \\ \mathfrak{k}(\tau) - \frac{\zeta^2}{2}\mathfrak{k}''(\tau) \end{pmatrix}$$

where again, $\mathfrak{k}$ are the fermionized ghost counter-part of the field appearing in (3.45). The above action can be expanded and written in terms of the components.

$$\hat{P}\mathfrak{f} = \begin{pmatrix} \begin{array}{c} \frac{1}{(\{f(\tau),\tau\}\zeta^2-2)^3}\Big[-2\zeta^4\,\partial_\tau\Big(\{f(\tau),\tau\}\Big)\mathfrak{k}''(\tau) \\ +4\zeta^2\,\partial_\tau\Big(\{f(\tau),\tau\}\Big)\mathfrak{k}(\tau)+2\zeta^2\,\Big(\{f(\tau),\tau\}\zeta^2-2\Big)\mathfrak{k}^{(3)}(\tau) \\ +\Big(\{f(\tau),\tau\}\zeta^2-2\Big)\Big(\{f(\tau),\tau\}\Big(\{f(\tau),\tau\}\zeta^2-8\Big)\zeta^2+8\Big)\mathfrak{k}'(\tau)\Big] \end{array} & \frac{\left(\zeta^2\{f(\tau),\tau\}+2\right)\left(\zeta^2\mathfrak{k}''(\tau)-2\mathfrak{k}(\tau)\right)}{\zeta(\zeta^2\{f(\tau),\tau\}-2)} \\ \frac{\left(\zeta^2\{f(\tau),\tau\}+2\right)\left(\zeta^2\mathfrak{k}''(\tau)-2\mathfrak{k}(\tau)\right)}{\zeta(\zeta^2\{f(\tau),\tau\}-2)} & \begin{array}{c} \frac{1}{4\{f(\tau),\tau\}\zeta^2-8}\Big[16\mathfrak{k}'(\tau)+2\zeta^4\partial_\tau\Big(\{f(\tau),\tau\}\Big)\mathfrak{k}''(\tau) \\ -\zeta^2\{f(\tau),\tau\}\Big(\{f(\tau),\tau\}\zeta^2-6\Big)\Big(\{f(\tau),\tau\}\zeta^2-4\Big)\mathfrak{k}'(\tau) \\ -4\zeta^2\partial_\tau\Big(\{f(\tau),\tau\}\Big)\mathfrak{k}(\tau)-2\zeta^2\Big(\{f(\tau),\tau\}\zeta^2-2\Big)\mathfrak{k}^{(3)}(\tau)\Big] \end{array} \end{pmatrix} \tag{J.4}$$

$$\hat{P}c = \begin{pmatrix} \begin{array}{c} \frac{1}{\zeta(\{f(\tau),\tau\}\zeta^2-2)^3}\Big[4\zeta^3\,\partial_\tau\Big(\{f(\tau),\tau\}\Big)c_\tau(\zeta,\tau) \\ -4\zeta\,\Big(\{f(\tau),\tau\}\zeta^2-2\Big)c_\tau^{(0,1)}(\zeta,\tau) \\ +\zeta\,\Big(\{f(\tau),\tau\}\zeta^2-2\Big)^3 c_\zeta^{(1,0)}(\zeta,\tau) \\ -4\Big(\{f(\tau),\tau\}\zeta^2-2\Big)^2 c_\zeta(\zeta,\tau)\Big] \end{array} & \begin{array}{c} \Big(c_\zeta^{(0,1)}(\zeta,\tau)+c_\tau^{(1,0)}(\zeta,\tau)\Big) \\ -2\frac{\left(\zeta^2\{f(\tau),\tau\}+2\right)c_\tau(\zeta,\tau)}{\zeta\left(\zeta^2\{f(\tau),\tau\}-2\right)} \end{array} \\ \begin{array}{c} \Big(c_\zeta^{(0,1)}(\zeta,\tau)+c_\tau^{(1,0)}(\zeta,\tau)\Big) \\ -2\frac{\left(\zeta^2\{f(\tau),\tau\}+2\right)c_\tau(\zeta,\tau)}{\zeta\left(\zeta^2\{f(\tau),\tau\}-2\right)} \end{array} & \begin{array}{c} \frac{1}{4\zeta(\{f(\tau),\tau\}\zeta^2-2)}\Big[-4\zeta^3\,\partial_\tau\Big(\{f(\tau),\tau\}\Big)c_\tau(\zeta,\tau) \\ +4\zeta\,\Big(\{f(\tau),\tau\}\zeta^2-2\Big)c_\tau^{(0,1)}(\zeta,\tau) \\ +4\Big(\{f(\tau),\tau\}\zeta^2-2\Big)^2 c_\zeta(\zeta,\tau) \\ -\zeta\,\Big(\{f(\tau),\tau\}\zeta^2-2\Big)^3 c_\zeta^{(1,0)}(\zeta,\tau)\Big] \end{array} \end{pmatrix} \tag{J.5}$$

Appendix K

Weyl anomaly in manifolds with boundary

In this section we compute the most general boundary term for Weyl anomaly in 2-dimensions on a manifold with a boundary allowed by the Wess-Zumino consistency condition. Let us start with the variation of (3.28) under a Weyl transformation,

$$
\begin{aligned}
\delta_W S_{cov} = & -\frac{1}{4\pi b^2}\int_{\Gamma}\sqrt{g}(R+8\pi\mu)\delta\omega + \frac{1}{4\pi b^2}\int_{\partial\Gamma}\sqrt{\gamma(s)}\int_{\Gamma}\sqrt{g(x)}\ \delta\omega(s)R(x)\ \hat{n}^{\mu}(s)\frac{\partial}{\partial y^{\mu}}G(x,y)|_{y=s} \\
& + \frac{1}{2\pi b^2}\int_{\partial\Gamma}\sqrt{\gamma(s_1)}\int_{\partial\Gamma}\sqrt{\gamma(s_2)}\ \delta\omega(s_2)\mathcal{K}(s_1)\ \hat{n}^{\mu}(s_2)\frac{\partial}{\partial y^{\mu}}G(s_1,y)|_{y=s_2} \\
& - \frac{1}{2\pi b^2}\int_{\partial\Gamma}\sqrt{\gamma(s)}\ \mathcal{K}(s)\delta\omega(s)
\end{aligned}
\tag{K.1}
$$

Under a second Weyl transformation

$$
\begin{aligned}
\delta_{W_2}(\delta_{W_1}S_{cov}) = & -\frac{1}{2\pi b^2}\int_{\partial\Gamma}\sqrt{\gamma(s)}\ \partial^{\mu}\delta\omega(s)\ \partial_{\mu}\delta\omega_2(s) - \frac{4\mu}{b^2}\int_{\Gamma}\sqrt{g}\ \delta\omega_2(x)\ \delta\omega_1(x) \\
& + \frac{1}{2\pi b^2}\int_{\partial\Gamma}\sqrt{\gamma(s_1)}\int_{\partial\Gamma}\sqrt{\gamma(s_2)}\ \delta\omega_1(s_1)\delta\omega_2(s_2)\ \hat{n}^{\mu}(s_1)\hat{n}^{\nu}(s_2)\ \partial_{\mu}\partial_{\nu}G(s_1,s_2)
\end{aligned}
\tag{K.2}
$$

therefore, since all the terms in the above equation are symmetric in $\delta\omega_1$ and $\delta\omega_2$, we have,

$$
\delta_{W_2}(\delta_{W_1}S_{cov}) - \delta_{W_1}(\delta_{W_2}S_{cov}) = 0 \tag{K.3}
$$

Thus the boundary terms that we have introduced are consistent with the Wess-Zumino conditions. All the boundary terms that we have introduced are consistent with the general analysis in [181].

Appendix L

Kinematics

In this appendix, we collect some useful facts about the kinematics of the scattering we're considering. The basic setup is the tree-level scattering of four identical scalars. We ignore any four-point coupling between these scalars (since it's channel-dual already), and focus on the particles the scalars have a three-point coupling with. These intermediate particles are exchanged in the s, t and u channels.

L.1 Spinning intermediate particles

Most general scalar-scalar-spin(l) interaction The most general 3-point interaction between 2 scalars and a spin-l particle is given by:

$$S_{int} = \lambda \int d^D x \sigma^{\mu_1 \cdots \mu_l}(x) \Big(\phi(x) \big(\overleftrightarrow{\partial}_{(\mu_1} \overleftrightarrow{\partial}_{\mu_2} \dots \overleftrightarrow{\partial}_{\mu_l)} \big) \phi(x) \Big), \qquad \overleftrightarrow{\partial} = i(\overleftarrow{\partial} - \overrightarrow{\partial}) \tag{L.1}$$

In writing the above interaction term we have taken into account the symmetric-transverse-traceless representation of an arbitrary spin-l particle.[1]

Note that the vertex identically vanishes when l is odd. This happens because the above vertex picks up a sign $(-1)^l$ under the exchange of the two ϕ fields, which is basically a symmetry. Another way to see this is to consider a 3-point interaction as shown below. The amplitude should not change under the exchange of particle $1-2$. However, this corresponds to a rotation by an angle π in the center of mass frame and the odd-spin particle picks up a phase, (-1). Thus for consistency, this 3-point interaction vanishes identically.

[1] In symmetric-transverse-traceless representation, the polarization of the spinning particle can be expanded in a basis like: $\epsilon^{(\mu_1}\epsilon^{\mu_2} \cdots \epsilon^{\mu_l)}$, where $\epsilon \cdot \epsilon = 0$, $\epsilon \cdot p = 0$. Here p is the momentum of the spinning particle and ϵ is its polarization.

Propagator of a spin-l particle Finding the propagator of a general spin-l particle is a matter of projecting out a l-tensor in the correct symmetric-transverse-traceless representation and has been worked out in [182, 183]. We quote the momentum space propagator here:

$$\frac{-i\Theta^{(l)}_{\mu_1\ldots\mu_s,\nu_1\ldots\nu_l}|_{p^2=m^2}}{-p^2+m^2} \tag{L.2}$$

where $\Theta^{(l)}_{(\mu),(\nu)}$ is the spin-(l) analogue of the projection operator given by:

$$\begin{aligned}\Theta^{(s)}_{\mu_1\ldots\mu_s,\nu_1\ldots\nu_l} \quad = \quad & \Bigg\{\sum_{p=0}^{[l/2]}\frac{(-1)^p l!(2l+D-2p-5)!!}{2^p p!(l-p)!(2l+D-5)!!}\Theta_{\mu_1\mu_2}\Theta_{\nu_1\nu_2}\ldots\Theta_{\mu_{2p-1}\mu_{2p}}\Theta_{\nu_{2p-1}\nu_{2p}} \\ & \times\Theta_{\mu_{2p+1}\nu_{2p+1}}\ldots\Theta_{\mu_l,\nu_l}\Bigg\}_{sym(\mu),sym(\nu)} \quad ,\end{aligned} \tag{L.3}$$

where $\Theta_{\mu\nu} = \eta_{\mu\nu} - p_\mu p_\nu/p^2$ is the spin one projection operator and $[l/2]$ gives the largest integer lesser than $l/2$. In the above expression $\{\cdot\}_{sym(\mu),sym(\nu)}$ denotes that the expression needs to be symmetrized in all the μ,ν indices.

4-scalar scattering with spin(l) exchange Using the expressions for the interaction between scalar and spin-l in (L.1) and the propagator, (L.3), one can easily write down the 4-scalar scattering amplitude with spin-l intermediate particle.

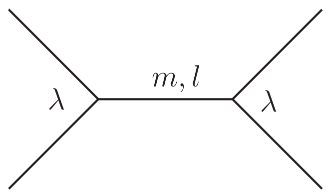


FIGURE L.1: 4-scalar scattering with an exchange of particle of spin (l) and mass (m)

$$\langle\phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\rangle_l = \lambda^2\left(\frac{l!\,\Gamma\left(\frac{D-3}{2}\right)\left(s-4M^2\right)^l}{2^l\Gamma\left(\frac{D-3}{2}+l\right)}\times C_l^{\frac{D-3}{2}}\left[1+\frac{2t}{s-4M^2}\right]\right)\times\frac{1}{s-m^2} \tag{L.4}$$

Here, $C_l^{\frac{D-3}{2}}$ are the Gegenbauer polynomials, which originate in the above expression due the particular structure of contractions that appears in (L.3). The Gegenbauer polynomials obey the following orthogonality condition:

$$\int_0^\pi d\theta(\sin\theta)^{D-3}\,C_l^{\frac{D-3}{2}}[cos\theta]C_{l'}^{\frac{D-3}{2}}[cos\theta] = 2^{4-D}\pi\frac{\Gamma(l+D-3)}{(l+\frac{D-3}{2})\Gamma(l+1)\Gamma\left(\frac{D-3}{2}\right)^2}\,\delta_{ll'} \tag{L.5}$$

The residue of the pole at $s = m^2$ is given by,

$$\lambda^2 \left(\frac{l!\, \Gamma\left(\frac{D-3}{2}\right) \left(m^2 - 4M^2\right)^l}{2^l \Gamma\left(\frac{D-3}{2} + l\right)} \times C_l^{\frac{D-3}{2}}[\cos(\theta)] \right) \tag{L.6}$$

here, $\cos(\theta)$ is the angle of scattering in the center of mass coordinates and is related to the Mandelstam variables by following relations,

$$\cos(\theta) = 1 + \frac{2t}{s - 4M^2} = \frac{u - t}{u + t}$$

L.2 Threshold kinematics

We show that for the scattering of massive scalar particles of mass M, there exists a threshold at mass $2M$, at which the kinematics becomes trivial. This is clear from the expression (L.6) for the residue of a physical pole. If the pole occurs at $s = m^2 = 4M^2$, corresponding to a particle of mass $2M$, then the residue vanishes identically, for $l \neq 0$. Thus, at such a threshold mass only scalar particles are allowed. Consequently, the residue, instead of being a polynomial of appropriate degree in t or $\cos(\theta)$ is a constant. For the case of massless external particles the threshold particle is also massless and hence the residue vanishes identically even for $l = 0$.

For the class of amplitudes that interest us: those one with a linear spectrum, the threshold condition is always met for particles with positive or zero mass. For such particles, the residue of the amplitude becomes a constant at some excited level. However, this doesn't happen for tachyonic particles, for which the threshold mass is a particle with an even more negative mass.

Appendix M

Explicit Demonstration of Channel Duality

In this appendix, we explicitly show how channel duality works in the case of the Euler Beta function

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \tag{M.1}$$

which is the building block of the Veneziano amplitude. This serves both as an intuition-building exercise, and as a demonstration of the vailidity of our techniques in the main text. We choose this function for simplicity; it doesn't have any c-poles, which makes the equations significantly shorter. However, precisely because it doesn't have these poles, it also lacks full crossing symmetry under arbitrary permutations of a, b, c; it is, however, still invariant under the exchange of a and b.

First, we go to the region $a < 0, b > 0$, which is the physical s-channel scattering regime. There, we may write

$$B(a,b) = \sum_{n=0}^{\infty} \frac{1}{a+n} \frac{(1-b)(2-b)\cdots(n-b)}{n!}. \tag{M.2}$$

This sum converges for $\mathrm{Re}\, b > 0$, since the summand behaves for large n as n^{-b-1}.

At the edge of this region of convergence, $b = 0$, all the residues in the expansion are 1. We can recreate this result from the t-channel expansion, valid for $a > 0, b < 0$,

$$B(a,b) = \sum_{n=0}^{\infty} \frac{1}{b+n} \frac{(1-a)(2-a)\cdots(n-a)}{n!} \tag{M.3}$$

by the following (strictly invalid) trick. First, we take $b = 0$ and $a = -m$ (which, notice, is outside the region of convergence); the expression (M.3) then becomes

$$\begin{aligned} B(a,0) &\xrightarrow{a\to 0} \sum_n \frac{1}{n} \\ &\xrightarrow{a\to -1} \sum_n 1 + \frac{1}{n} \\ &\xrightarrow{a\to -2} \sum_n \frac{n}{2} + \frac{3}{2} + \frac{1}{n} \\ &\xrightarrow{a\to -m} \sum_n \cdots + \frac{1}{n}. \end{aligned} \tag{M.4}$$

It seems that the coefficient of the $\sum \frac{1}{n}$ term gives the correct residue for this value of b. While recreating a factor of 1 is neither useful nor kosher, it is still valid for the reason that we can recreate it using a lot less arbitrary prescription: replacing $\sum n^{-s}$ by the Riemann zeta function $\zeta(s)$, which has a simple pole at $s = -1$! This prescription is less arbitrary for the simple reason that it provides an analytic continuation off the right-half-plane, and it is therefore the unique prescription.

It is useful to do this in a more systematic manner. We rewrite the s-channel expansion as

$$B(a,b) = \sum_{n=0}^{\infty} \frac{f_n(b)}{a+n} \quad \Big/ f_n(b) = \frac{\Gamma(n+1-b)}{\Gamma(n+1)\Gamma(1-b)}. \tag{M.5}$$

We can expand the residues $f_n(b)$ in a $1/n$-series,

$$f_n(b) = \sum_{j=0}^{\infty} g_j(b) n^{-b-j}, \tag{M.6}$$

where, for reference, the first few g_js are

$$\begin{aligned} g_0(b) &= \frac{1}{\Gamma(1-b)} \\ g_1(b) &= \frac{b(b-1)}{2\Gamma(1-b)} \\ g_2(b) &= \frac{b(2-3b-2b^2+3b^3)}{24\Gamma(1-b)} \\ g_3(b) &= \frac{(-1+b)^2 b^2 (2+3b+b^2)}{48\Gamma(1-b)} \\ g_4(b) &= \frac{b(-48+20b+180b^2-25b^3-192b^4-10b^5+60b^6+15b^7)}{5760\Gamma(1-b)}. \end{aligned} \tag{M.7}$$

The manipulations from eqn (5.20) to eqn (5.27) go through essentially as in the main text, except without the terms coming from the poles in c, with $k(b) = b$. Eqn. (5.27) then becomes

$$f_n(b) = \sum_{j=0}^{n} g_j(-n)(-b)^{n-j}, \tag{M.8}$$

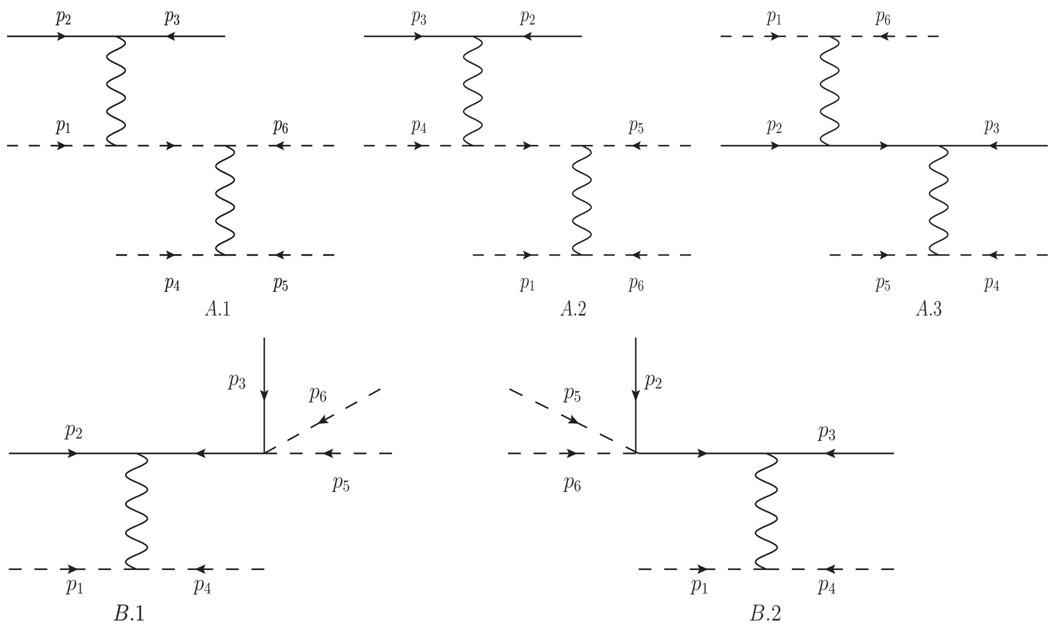
which are exactly the residue-matching equations (5.40). As an aside, we note the lack of any spurious-pole equations (5.41); this is because of the lack of poles in c, giving further credence to their interpretation as arising from $b - c$ symmetry.

The reader may readily check using the g_js listed here that eqn (M.8) is indeed correct. A much simpler way to check is to note the eqn (M.8) is the same as eqn (M.6) with b and $-n$ interchanged. The exact expression for the residue, eqn (M.5), is manifestly invariant under that interchange, and therefore we must have arrived at the right answer.

Appendix N

Large z behavior of the component 6-point scattering amplitude

In this section, we compute the six point scattering amplitude $\left(\bar{\phi}(p_1)\psi(p_2)\right)\left(\bar{\psi}(p_3)\phi(p_4)\right)\left(\bar{\phi}(p_5)\phi(p_6)\right)$ and demonstrate that it is well behaved under the BCFW deformations. The Feynman diagrams that contribute to the six point function under consideration are displayed in fig N.1.



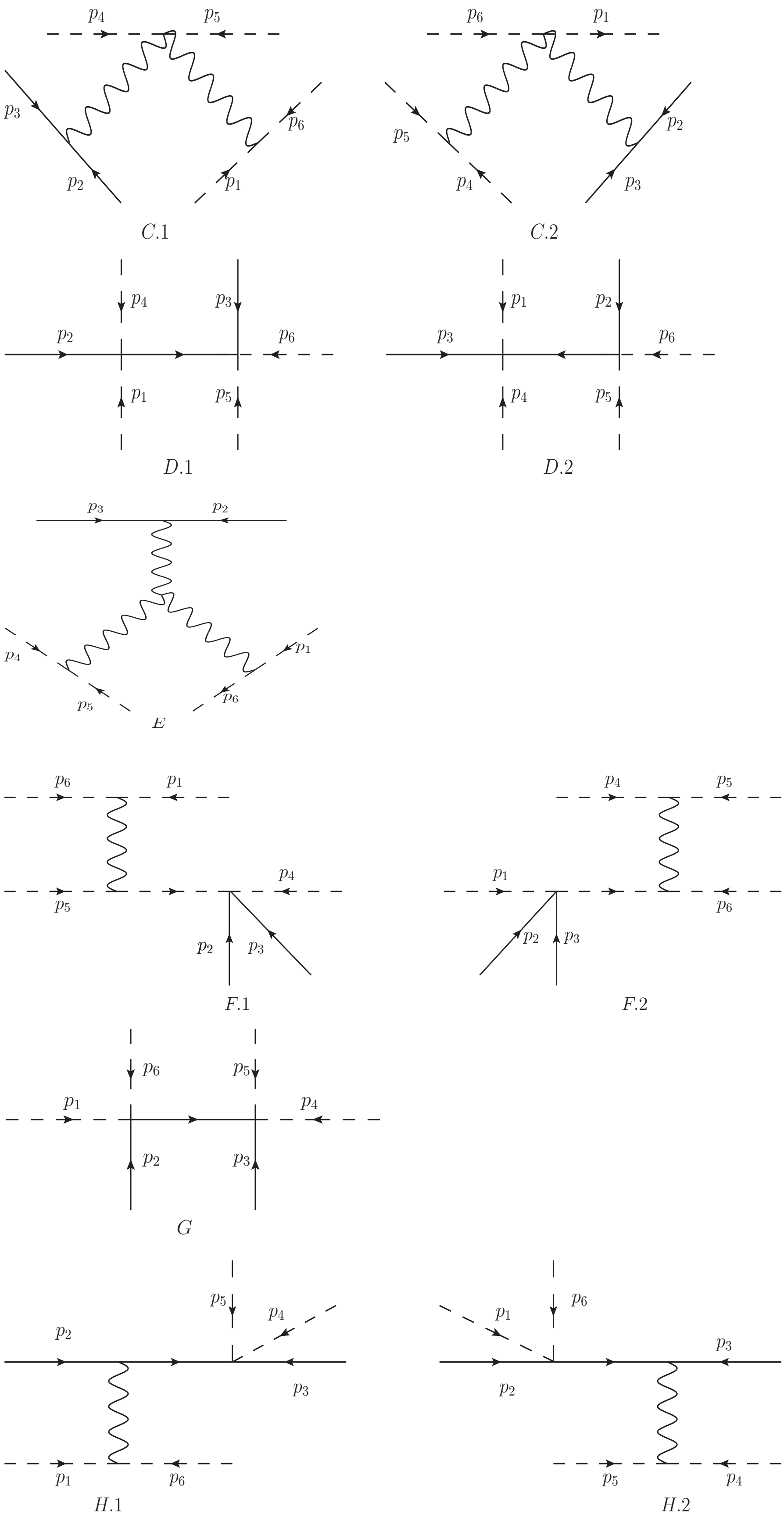


FIGURE N.1: Feynman diagrams for the amplitude $(\bar{\phi}(p_1)\psi(p_2))(\bar{\psi}(p_3)\phi(p_4))(\bar{\phi}(p_5)\phi(p_6))$

We give below explicit expression for each diagram appearing in Fig.N.1

$$A_1 = -\frac{16\pi^2 i}{\kappa^2}\frac{p_1.(p_2-p_3)}{p_{23}^2 p_{45}^2 p_{123}^2}\langle 23\rangle\langle 45\rangle\langle 56\rangle\langle 46\rangle \tag{N.1}$$

$$A_2 = \frac{16\pi^2 i}{\kappa^2}\frac{p_4.(p_2-p_3)}{p_{23}^2 p_{16}^2 p_{234}^2}\langle 23\rangle\langle 16\rangle\langle 65\rangle\langle 15\rangle \tag{N.2}$$

$$A_3 = \frac{4i\pi^2}{\kappa^2}\frac{\langle 16\rangle\langle 45\rangle}{p_{16}^2 p_{45}^2 p_{126}^2}\left(\langle 21\rangle(\langle 34\rangle\langle 5|p_{12}|6\rangle + \langle 35\rangle\langle 6|p_{12}|4\rangle) + \langle 26\rangle(\langle 34\rangle\langle 1|p_{26}|5\rangle + \langle 35\rangle\langle 1|p_{26}|4\rangle)\right) \tag{N.3}$$

$$B_1 = \frac{8\pi^2 i}{\kappa^2}\langle 14\rangle\left(\frac{\langle 1|p_{56}|3\rangle\langle 24\rangle + \langle 3|p_{56}|4\rangle\langle 21\rangle}{p_{14}^2 p_{356}^2}\right) \tag{N.4}$$

$$B_2 = \frac{8\pi^2 i}{\kappa^2}\langle 14\rangle\left(\frac{\langle 1|p_{56}|2\rangle\langle 34\rangle + \langle 2|p_{56}|4\rangle\langle 31\rangle}{p_{14}^2 p_{256}^2}\right) \tag{N.5}$$

$$C_1 = -\frac{8\pi^2 i}{\kappa^2}\left(\frac{\langle 2|p_1|3\rangle(p_{23}.p_6) - \langle 2|p_6|3\rangle(p_{23}.p_1)}{p_{23}^2 p_{16}^2}\right) \tag{N.6}$$

$$C_2 = -\frac{8\pi^2 i}{\kappa^2}\left(\frac{\langle 2|p_5|3\rangle(p_{23}.p_4) - \langle 2|p_4|3\rangle(p_{23}.p_5)}{p_{23}^2 p_{45}^2}\right) \tag{N.7}$$

$$D_1 = \frac{8\pi^2 i}{\kappa^2}\frac{\langle 2|p_{14}|3\rangle}{p_{124}^2} \tag{N.8}$$

$$D_2 = \frac{8\pi^2 i}{\kappa^2}\frac{\langle 2|p_{56}|3\rangle}{p_{256}^2} \tag{N.9}$$

$$E = -\frac{4\pi^2 i}{\kappa^2}\langle 45\rangle\langle 23\rangle\left(\frac{(\langle 12\rangle^2 - \langle 13\rangle^2)\langle 46\rangle\langle 56\rangle - (\langle 26\rangle^2 - \langle 36\rangle^2)\langle 14\rangle\langle 15\rangle}{p_{23}^2 p_{45}^2 p_{16}^2}\right) \tag{N.10}$$

$$F_1 = \frac{8\pi^2 i}{\kappa^2}\frac{\langle 23\rangle\langle 16\rangle\langle 65\rangle\langle 15\rangle}{p_{16}^2 p_{234}^2} \tag{N.11}$$

$$F_2 = \frac{8\pi^2 i}{\kappa^2}\frac{\langle 23\rangle\langle 45\rangle\langle 56\rangle\langle 46\rangle}{p_{45}^2 p_{231}^2} \tag{N.12}$$

$$G = \frac{4\pi^2 i}{\kappa^2}\frac{\langle 2|p_{16}|3\rangle}{p_{126}^2} \tag{N.13}$$

$$H_1 = \frac{4\pi^2 i}{\kappa^2}\langle 16\rangle\left(\frac{\langle 12\rangle\langle 6|p_{12}|3\rangle - \langle 26\rangle\langle 1|p_{26}|3\rangle}{p_{16}^2 p_{126}^2}\right) \tag{N.14}$$

$$H_2 = -\frac{4\pi^2 i}{\kappa^2}\langle 45\rangle\left(\frac{\langle 35\rangle\langle 2|p_{16}|4\rangle + \langle 34\rangle\langle 2|p_{16}|5\rangle}{p_{45}^2 p_{126}^2}\right) \tag{N.15}$$

It is easy to verify that the the asymptotic behavior of the full set of diagram is well behaved by deforming the momentum p_1 and p_2, as discussed in section 6.3 and 6.4. We apply the BCFW deformations in the large z limit using

$$\begin{aligned} p_2 &\to q z^2 \\ p_1 &\to -q z^2 \end{aligned} \tag{N.16}$$ [1]

and obtain the asymptotic behavior of the amplitudes to leading order in z as follows. The diagrams A_2, C_1, D_2, F_1 in (N.1) go as $O(1/z)$ to the leading order in the large z limit.

$$A_2 \sim \frac{2\pi^2 i}{\kappa^2 z} \frac{(q.p_4)\langle q3\rangle\langle q6\rangle\langle 65\rangle\langle q5\rangle}{(q.p_3)(q.p_{34})(q.p_6)} + \mathcal{O}\left(\frac{1}{z^3}\right) \tag{N.17}$$

$$B_2 \sim \frac{2\pi^2 i}{\kappa^2 z} \frac{\langle q4\rangle(\langle q|p_{56}|q\rangle\langle 34\rangle + \langle q|p_{56}|4\rangle\langle 3q\rangle)}{(q.p_4)(q.p_{56})} + \mathcal{O}\left(\frac{1}{z^3}\right) \tag{N.18}$$

$$C_1 \sim \frac{2\pi^2 i}{\kappa^2 z} \frac{\langle q|p_6|3\rangle}{(q.p_6)} + \mathcal{O}\left(\frac{1}{z^3}\right) \tag{N.19}$$

$$D_2 \sim \frac{4\pi^2 i}{\kappa^2 z} \frac{\langle q|p_{56}|3\rangle}{q.p_{56}} + \mathcal{O}\left(\frac{1}{z^3}\right) \tag{N.20}$$

$$F_1 \sim \frac{2\pi^2 i}{\kappa^2 z} \frac{\langle q3\rangle\langle q6\rangle\langle 65\rangle\langle q5\rangle}{(q.p_6)(q.p_{34})} + \mathcal{O}\left(\frac{1}{z^3}\right) \tag{N.21}$$

(N.22)

For the remaining diagrams we just display the leading large z behavior. They are given by

$$A_1 \sim -\frac{8\pi^2 iz}{\kappa^2} \frac{\langle q3\rangle\langle 45\rangle\langle 56\rangle\langle 46\rangle}{p_{45}^2 p_{123}^2} + \mathcal{O}\left(\frac{1}{z}\right) \qquad F_2 \sim \frac{8\pi^2 iz}{\kappa^2} \frac{\langle q3\rangle\langle 45\rangle\langle 56\rangle\langle 46\rangle}{p_{45}^2 p_{123}^2} + \mathcal{O}\left(\frac{1}{z}\right) \tag{N.23}$$

$$B_1 \sim \frac{8\pi^2 iz}{\kappa^2} \frac{\langle q|p_{56}|3\rangle}{p_{356}^2} + \mathcal{O}\left(\frac{1}{z}\right) \qquad D_1 \sim -\frac{8\pi^2 iz}{\kappa^2} \frac{\langle q|p_{56}|3\rangle}{p_{356}^2} + \mathcal{O}\left(\frac{1}{z}\right) \tag{N.24}$$

$$\begin{aligned} A_3 &\sim -\frac{4\pi^2 iz}{\kappa^2} \frac{\langle 45\rangle(\langle 34\rangle\langle q|p_{34}|5\rangle + \langle 35\rangle\langle q|p_{35}|4\rangle)}{p_{45}^2 p_{126}^2} + \mathcal{O}\left(\frac{1}{z}\right) \\ H_2 &\sim \frac{4\pi^2 iz}{\kappa^2} \frac{\langle 45\rangle(\langle 34\rangle\langle q|p_{34}|5\rangle + \langle 35\rangle\langle q|p_{35}|4\rangle)}{p_{45}^2 p_{126}^2} + \mathcal{O}\left(\frac{1}{z}\right) \end{aligned} \tag{N.25}$$

$$C_2 \sim \frac{2\pi^2 iz}{\kappa^2} \frac{\langle q4\rangle\langle q5\rangle\langle 3q\rangle\langle 45\rangle}{p_{45}^2 (q.p_3)} + \mathcal{O}\left(\frac{1}{z}\right) \qquad E \sim -\frac{2\pi^2 iz}{\kappa^2} \frac{\langle q4\rangle\langle q5\rangle\langle 3q\rangle\langle 45\rangle}{p_{45}^2 (q.p_3)} + \mathcal{O}\left(\frac{1}{z}\right) \tag{N.26}$$

$$G \sim -\frac{4\pi^2 iz}{\kappa^2} \frac{\langle q|p_{45}|3\rangle}{p_{126}^2} + \mathcal{O}\left(\frac{1}{z}\right) \qquad H_1 \sim \frac{4\pi^2 iz}{\kappa^2} \frac{\langle q|p_{45}|3\rangle}{p_{126}^2} + \mathcal{O}\left(\frac{1}{z}\right) \tag{N.27}$$

Even though some of the individual diagrams are divergent linearly in z, the divergences in the total amplitude cancel pair wise in the large z limit as is evident from the way we have written the results. For example linear in z behavior cancelling pair wise in (A_1, F_2), (B_1, D_1) etc. Thus the total amplitude is well behaved as $z \to \infty$. A straightforward computation yields the analogous result for the $z \to 0$ limit. Thus the amplitude $A_6\big((\bar{\phi}(p_1)\psi(p_2))(\bar{\psi}(p_3)\phi(p_4))(\bar{\phi}(p_5)\phi(p_6))\big)$ is well behaved under the BCFW deformations both at $z \to \infty$ and $z \to 0$.

[1] q is defined in (6.16)

Towards the end of section 6.5 we had mentioned that four of the diagrams are reproduced in the factorization channel $p_f = p_{256}$, these diagrams are B_1, B_2, D_1, D_2 in fig N.1. The remaining eleven diagrams in fig N.1 are reproduced in the factorization channel $p_f = p_{234}$.

N.1 A Dyson-Schwinger equation for all loop six point correlator

As we saw earlier in section 6.2, the basic building block of higher point amplitudes in the Chern-Simons matter theories at the tree level is the four point amplitude. In this section we describe the Dyson-Schwinger construction of the all loop six point correlator[2]

$$\langle \bar{\hat{P}}^i(p+q,\theta_1)\hat{P}_i(-p,\theta_2)\bar{\hat{P}}^j(k+q',\theta_3)\hat{P}_j(-k-q,\theta_6)\bar{\hat{P}}^k(r,\theta_5)\hat{P}_k(-r-q',\theta_4)\rangle \tag{N.28}$$

using the superspace Schwinger-Dyson construction developed in [159]. In the above Φ^i is a complex scalar superfield in $\mathcal{N}=1$ superspace defined by

$$\Phi^i = \phi^i + \theta\psi^i - \theta^2 F^i \tag{N.29}$$

where ϕ^i is a complex scalar, ψ^i is a complex fermion and F^i is a complex auxiliary field. The $\mathcal{N}=2$ theory can be written in $\mathcal{N}=1$ superspace in terms of Φ^i. For more details see [159]. Before presenting the central idea it is informative to understand the color structure of the tree level and one loop amplitudes in the theory. In the supersymmetric Light cone gauge these are described succinctly in fig N.2 and in fig N.3. It turns out that, there are

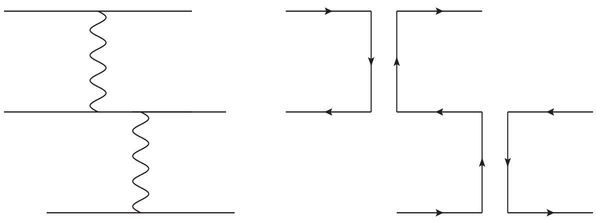


FIGURE N.2: Six point correlator: We display tree diagrams in supersymmetric light cone gauge. For simplicity we have only displayed the ladder diagrams.The tree diagrams are of order $\mathcal{O}(\frac{1}{\kappa^2})$ since the gauge field propagator contributes a factor of $\mathcal{O}(\frac{1}{\kappa})$.

six different diagrams for a given color contracted correlator. We have displayed only one in Fig.N.2 for brevity. The situation is a little bit more complicated at one loop as three different type of diagrams can appear as displayed in fig N.3. Note that diagrams like fig N.3 b) are suppressed in the large N, κ limit (keeping $\lambda = \frac{N}{\kappa}$ fixed). So they don't contribute to

[2]A similar discussion can be carried out for higher point function.

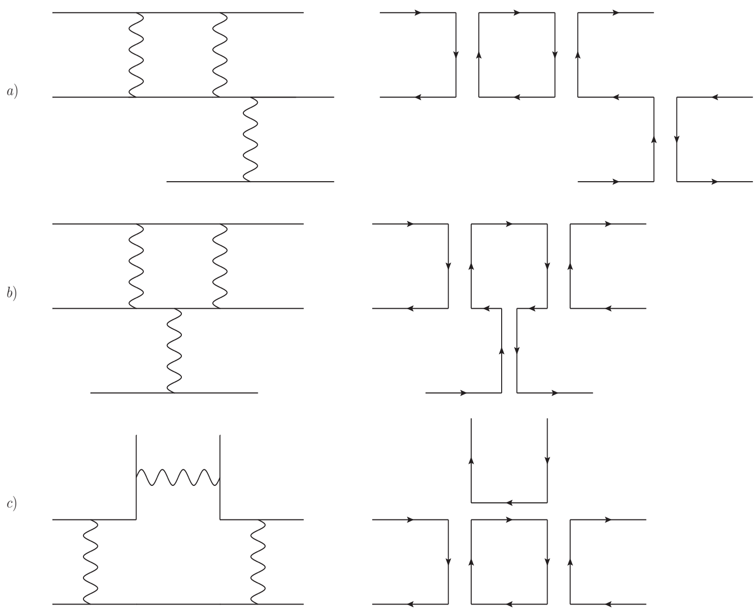


Figure N.3: Six point correlator: We have listed the various contributions to the one loop correlator in supersymmetric light cone gauge. For simplicity we have displayed only the ladder diagrams. In fig a) and c) the three gauge field propagators contribute a factor of $\mathcal{O}(\frac{1}{\kappa^3})$ and the single color loop gives a factor of N, leading to a contribution of the order $\frac{\lambda}{\kappa^2}$. Note that this is of the same order in κ as the tree level diagram displayed in fig N.2. On the other hand fig b) has three gauge fields and no color loops, rendering it to be $O(\frac{1}{\kappa^3})$.

the Schwinger-Dyson equation at this order. It can be checked that these type of diagrams continue to remain suppressed at higher loops.

This paves way for the construction of all loop higher point correlators entirely in terms of all-loop four point correlators at least in the planar approximation. The case for the six point correlator is displayed in (see fig N.4). It is straightforward to write down the correlator for

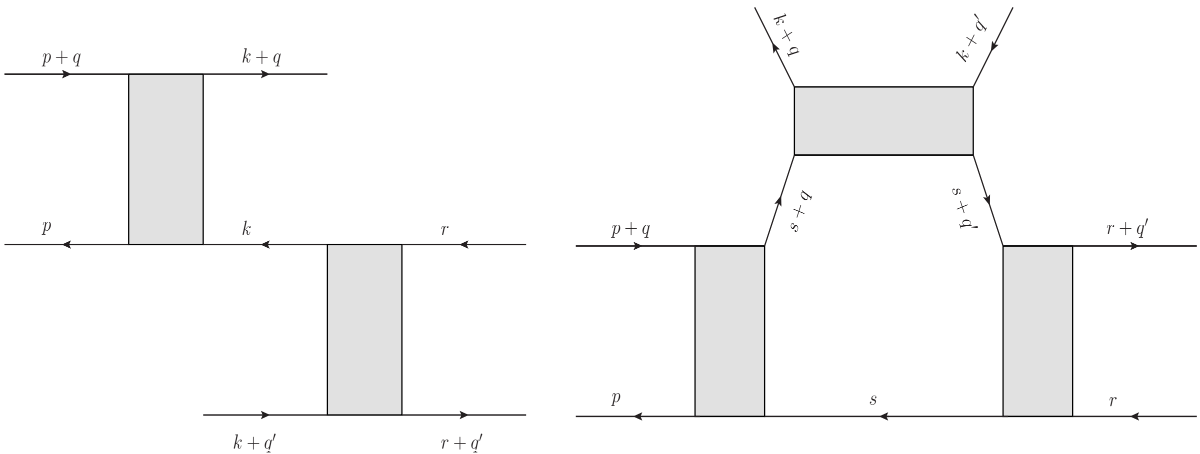


Figure N.4: Six point correlator in superspace: The grey boxes represent the $\mathcal{N}=2$ all-loop four point correlator computed in [159] the first diagram in N.4, the second contribution however requires a loop integration over both intermediate grassmann and momentum variables and is quite complicated, we defer a detailed treatment to future works.

Appendix O

Large z behaviour of arbitrary $2n$-point amplitudes

This appendix provides the arguments for a good large z behaviour of arbitrary tree-level amplitudes in $\mathcal{N} = 2$ theory. In section O.1 we argue why same is not true for $\mathcal{N} = 1$ theory.

Backgound Field Method We use the backgound field method of [165] to understand the large z behaviour of the amplitudes in $\mathcal{N}$=2 theory. All the fields in the theory are expanded into a background part plus a quantum fluctuation,

$$\begin{aligned} &\Phi \to \tilde{\Phi} + \hat{P},\ \Gamma_\alpha{}^\beta \to \tilde{\Gamma}_\alpha{}^\beta + \mathbf{g}_\alpha{}^\beta \\ \text{for on-shell component fields:}\quad &\phi \to \tilde{\phi} + \phi,\ \psi \to \tilde{\psi} + \psi,\ A_\alpha{}^\beta \to \tilde{A}_\alpha{}^\beta + a_\alpha{}^\beta \end{aligned} \tag{O.1}$$

In the first line we have expanded the superfields around a background field while the expression in the second line is the background field expansion for the component fields.[1] In this section we will need to use only the expansion in the component fields. In the above expansion the background fields, $\tilde{\cdot}$, are solutions to the classical equations of motion.
In the $z \to \infty$ limit, the two external particles with shifted momenta (which we have labeled '1' and '2' in chapter 6) are extremely hard light-like particles scattering through a background of soft-particles. Such interactions can be captured by diagrams represented in (Figure O.1). Since in the problems that we are considering only two external momenta are shifted, which corresponds to only two fluctuation fields, we only need to expand out Lagrangian to quadratic order in the fluctuations. Let us now discuss the $z \to \infty$ behaviour of various terms that contribute to Lagrangian at quadratic order. The kinetic terms in the

[1] Here and in all subsequent discussion of this section, the Hermitian conjugate fields won't be mentioned separately. In fact we will use a schematic notation where ϕ, ψ denote both the fields and their Hermitian conjugates.

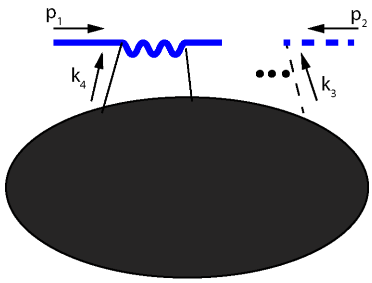


FIGURE O.1: Diagram depicting the interaction of hard particle (denoted by blue) with the soft *background* particles. The blob in the above diagram denotes the remaining Feynman diagram which contains no hard particle.

Lagrangian are:[2]

$$\epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho, \quad \bar{\phi}\big(-\Box + m_0^2\big)\phi, \quad -\bar{\psi}\big(\partial\!\!\!/ + m_0\big)\psi \tag{O.2}$$

The momentum space propagators and their $z \to \infty$ behaviour is given by (for gauge field, boson and fermions, in that order),

$$G_a(\hat{p}) = i\epsilon_{\mu\nu\rho} \frac{(\frac{p_1+p_l}{2} + z^2 q + z^{-2}\tilde{q})^\rho}{\big(\frac{p_1+p_l}{2} + z^2 q + z^{-2}\tilde{q}\big)^2} \quad \xrightarrow{z \to \infty} \quad i\epsilon_{\mu\nu\rho} \frac{q^\rho}{(p_1+p_l)\cdot q} \sim \mathcal{O}(1) \tag{O.3}$$

$$G_\phi(\hat{p}) = \frac{i}{\big(\frac{p_1+p_l}{2} + z^2 q + z^{-2}\tilde{q}\big)^2 + m_0^2} \quad \xrightarrow{z \to \infty} \quad i\frac{1}{z^2 (p_1+p_l)\cdot q} \sim \frac{1}{z^2} \tag{O.4}$$

$$G_\psi(\hat{p}) = -i \frac{(\frac{p\!\!\!/_1+p\!\!\!/_l}{2} + z^2 q\!\!\!/ + z^{-2}\tilde{q\!\!\!/}) + m_0}{\big(\frac{p_1+p_l}{2} + z^2 q + z^{-2}\tilde{q}\big)^2 + m_0^2} \quad \xrightarrow{z \to \infty} \quad -i\frac{q\!\!\!/}{(p_1+p_l)\cdot q} \sim \mathcal{O}(1) \tag{O.5}$$

Recall that the sources of the z-dependence in the amplitudes are:

1. modified propagator of intermediate hard particle
2. the modified contribution of various vertices; and,
3. modified fermion wave function, in case an external deformed particle is a fermion

Vertices in the theory are modified only then there is momentum factor arising due to derivatives acting on the vertices. Hence, following interaction vertices without derivatives don't have any z dependence in them and hence behave as $\mathcal{O}(1)$ at $z \to \infty$.

$$\phi^2\tilde{\phi}^4,\ \tilde{\phi}^2\psi^2,\ \tilde{\psi}^2\phi^2,\ \tilde{\psi}\tilde{\phi}\psi\phi,\ \tilde{A}^\mu\psi^2,\ \tilde{\psi}a^\mu\psi,\ \tilde{A}^\mu\tilde{A}_\mu\phi^2,\ \tilde{A}^\mu\tilde{\phi}a_\mu\phi,\ \tilde{\phi}^2a^\mu a_\mu,\ \epsilon^{\mu\nu\rho}\tilde{A}_\mu a_\nu a_\rho \tag{O.6}$$

[2]see Equation (2.11) of [159]. In the massive case, $p_1^2 = -m_0^2 = p_l^2$. Moreover, of all the conditions that are imposed in massless case on $q, \tilde{q}$, viz., $q^2 = 0 = \tilde{q}^2$, $(p_1+p_l)\cdot q = 0 = (p_1+p_l)\cdot\tilde{q}$, $q\cdot\tilde{q} = -p_1\cdot p_l/4$, the last one gets modified to $2q\cdot\tilde{q} = (p_1-p_l)^2/4$. With respect to $q, \tilde{q}$ satisfying this new criterion, we again define $\hat{p}_{1/l} = (p_1+p_l)/2 \pm z^2 q \pm \tilde{q}/z^2$

Additional vertices coming from gauge-boson interaction terms are,

$$a^\mu \phi \partial_\mu \tilde{\phi}, \quad a^\mu \tilde{\phi} \partial_\mu \phi, \quad \tilde{A}^\mu \phi \partial_\mu \phi \tag{O.7}$$

The first term above contains a derivative of the background field and is independent of the momentum of the fluctuation, hence it has $\mathcal{O}(1)$ behaviour. Recall that the gauge fields are not dynamical and any gauge field appearing in the vertices are essentially internal fields in a diagram. Thus the contribution of vertices like the second one above always has an accompanying gauge propagator with it,

$$\epsilon^{\mu\nu\rho} \frac{(z^2 q)_\rho (k + z^2 q)_\mu}{(k + z^2 q)^2} \sim 1 \tag{O.8}$$

The last term in (O.7) can be set to 0 by the choice of background gauge condition, $q \cdot \tilde{A} = 0$. Thus all the vertices and propagators behave at worst as $\mathcal{O}(1)$. However, since external fermion wavefunction itself behaves as $\mathcal{O}(z)$, the n-point function can possibly behave as $\mathcal{O}(z)$ in $z \to \infty$ limit. To understand how this large z behaviour is tamed we study how the external fermion is connected to the rest of the diagram though various vertices:

$$\tilde{\psi} \tilde{\phi} \psi \phi, \ \tilde{\phi}^2 \psi^2, \ \tilde{A}^\mu \psi^2, \ \tilde{\psi} a^\mu \psi \tag{O.9}$$

Recall that the boson propagators are suppressed as $1/z^2$, and hence all diagrams with at least one internal boson will have a good $z \to \infty$ behaviour and subsequently we will assume that the Feynman diagrams that we are studying this section don't have internal boson propagators. For the same reason, if the external fermion is connected through the first vertex in (O.9) then the only diagram of concern is where the boson will be hard external particle Figure O.2.

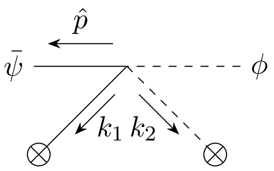


FIGURE O.2: The only diagram involving quartic $(\bar{\psi}\phi)(\bar{\phi}\psi)$ which contributes in $z \to \infty$ limit is where the hard boson is an external perticle.

The contribution of this diagram is cancelled against other vertices that are discuss below. Essentially, this cancellation happens because of the particular vertex factors that appear in $\mathcal{N}$=2 Lagrangian. In fact even for $\mathcal{N}$=1 theory, where the interactions are less constrained, this doesn't happen and we don't get a good large z behaviour. If the fermion is connected through second or third vertex above,[3] then the contribution coming from these terms is,

$$\lim_{z\to\infty} \langle\psi(\hat{p})|\, \bar{\tilde{\bar{\phi}}}\tilde{\phi}\bar{\psi}^{\beta}\psi_{\beta}\ldots, \qquad \lim_{z\to\infty} \langle\psi(\hat{p})|\, \bar{\psi}^{\alpha}\tilde{\bar{A}}_{\alpha}{}^{\beta}\psi_{\beta}\ldots \tag{O.10}$$

where the ellipsis denotes the subsequent vertices in the Feynman diagram. Each of the above vertices contains another fermion propagator that connects it to the rest of the diagram. So the contribution becomes,

$$\lim_{z\to\infty} \Big(\langle\psi(\hat{p})|\, \bar{\psi}^{\beta}\Big)\frac{\big(\not{k}+z^2\not{q}\big)_{\beta\gamma}}{(k+z^2q)^2}\ldots = \frac{2z\lambda_q^{\beta}\not{k}_{\beta\gamma}}{2z^2k\cdot q}\ldots \sim \frac{1}{z} \tag{O.11a}$$

$$\lim_{z\to\infty} \big(\langle\psi(\hat{p})|\, \bar{\psi}^{\alpha}\big)\tilde{\bar{A}}_{\alpha}{}^{\beta}\frac{\big(\not{k}+z^2\not{q}\big)_{\beta\gamma}}{(k+z^2q)^2}\ldots = \frac{2z\lambda_q^{\beta}\not{k}_{\beta\gamma}}{2z^2k\cdot q}\ldots \sim \frac{1}{z} \tag{O.11b}$$

Here, we have used $\big(\langle\psi(\hat{p})|\, \bar{\psi}^{\beta}\big) \to z\lambda_q^{\beta}$. Now let us look at the last term in (O.9). In this case, a_μ will propagate to another internal vertex in the diagram.[4] We will next consider all possible vertices that this gauge field can connect to:

$$\tilde{\psi}a^{\mu}\psi,\ \tilde{A}^{\mu}\tilde{\phi}a_{\mu}\phi,\ \tilde{\phi}^2a^{\mu}a_{\mu},\ \epsilon^{\mu\nu\rho}\tilde{A}_{\mu}a_{\nu}a_{\rho},\ a^{\mu}\phi\partial_{\mu}\tilde{\phi},\ a^{\mu}\tilde{\phi}\partial_{\mu}\phi \tag{O.12}$$

Connection though $\tilde{\psi}\mathbf{a}^{\mu}\psi$

When the gauge field in $\tilde{\psi}a^{\mu}\psi$ connects to another fermion vertex, then we typically have a diagram as in Figure O.3,

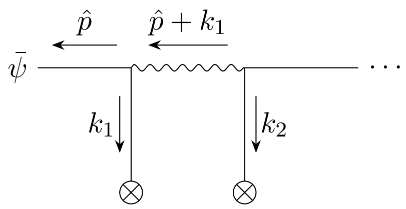


FIGURE O.3: Gluon propagator connecting through fermion vertex.

This subgraph contributes,

$$\begin{aligned}
&\lim_{z\to\infty} \big(\langle\psi(\hat{p})|\, \bar{\psi}^{\alpha}\big)a^{\mu}\gamma_{\mu_{\alpha}}{}^{\beta}\tilde{\psi}_{\beta}(k_1)\ a^{\nu}\gamma_{\nu_{\delta}}{}^{\gamma}\bar{\tilde{\bar{\psi}}}^{\delta}(k_2)\psi_{\gamma}(k_1+k_2+\phi)\ldots \\
&= z\lambda_q^{\alpha}\gamma_{\mu_{\alpha}}{}^{\beta}\tilde{\psi}_{\beta}\ \epsilon^{\mu\nu\rho}\frac{\gamma_{\rho_{\epsilon}}{}^{\kappa}q_{\kappa}{}^{\epsilon}}{2k_1\cdot q}\ \gamma_{\nu_{\gamma}}{}^{\delta}\bar{\tilde{\bar{\psi}}}^{\gamma}(k_2)\ \frac{\not{q}_{\delta\sigma}}{2(k_1+k_2)\cdot q}\ldots
\end{aligned} \tag{O.13}$$

[3]The origin of such terms in the Lagrangian are: $\bar{\phi}\psi\,\bar{\psi}\phi$ and $\bar{\phi}\phi\,\bar{\psi}\psi$ for $\tilde{\phi}^2\bar{\psi}\psi$; and, $\bar{\psi}\tilde{\bar{A}}\psi$ for $\tilde{A}^{\mu}\bar{\psi}\psi$.

[4]Unlike the discussion of (O.11) where the fermion on the right hand side could directly be connected to an external leg, a_μ necessarily has to connect to an internal vertex since gauge fields are not dynamical degrees of freedom.

Here in the second line, we have taken into account the subsequent fermion propagator that connects this sub-diagram to the rest of the diagram on the RHS. Now using the identity,

$$\epsilon^{\mu\nu\rho}\gamma_{\mu_\alpha}{}^{\beta}\,\gamma_{\nu_\gamma}{}^{\delta}\,\gamma_{\rho_\epsilon}{}^{\kappa} = i\Big(-\delta^{\beta}_{\epsilon}C_{\gamma\alpha}C^{\delta\kappa} - \delta^{\beta}_{\gamma}C^{\delta\kappa}C_{\epsilon\alpha} - \delta^{\delta}_{\epsilon}C^{\beta\kappa}C_{\gamma\alpha} - \delta^{\delta}_{\alpha}\delta^{\beta}_{\epsilon}\delta^{\kappa}_{\gamma} + \delta^{\kappa}_{\gamma}C_{\epsilon\alpha}C^{\beta\delta} + \delta^{\delta}_{\alpha}C^{\beta\kappa}C_{\epsilon\gamma} + \delta^{\kappa}_{\alpha}C_{\epsilon\gamma}C^{\beta\delta} + \delta^{\beta}_{\gamma}\delta^{\kappa}_{\alpha}\delta^{\delta}_{\epsilon}\Big)\,[5] \tag{O.14}$$

the above expression vanishes as long as q is light-like (see footnote 2).

Connection though $\tilde{\mathbf{A}}^{\mu}\tilde{\phi}\mathbf{a}_{\mu}\phi$

The corresponding diagram is given by Figure O.4,

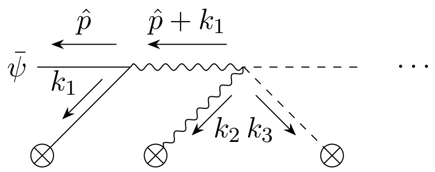


FIGURE O.4: Gluon propagator connecting through boson vertex.

We have already argued above that internal boson propagators are well behaved as $z \to \infty$. Thus this diagram contributes only when the fluctuation boson on RHS is an external particle. Then the value of this diagram is,

$$z\lambda^{\alpha}_{q}(\gamma_\mu)_\alpha{}^{\beta}\tilde{\psi}_\beta(k_1)\;\epsilon^{\mu\nu\rho}\frac{q_\rho}{2q\cdot k_1}\tilde{A}_\nu\tilde{\phi} = \frac{z}{2q\cdot k_1}\Big(-i\tilde{A}\cdot q\Big)\;\lambda^{\beta}_{q}\tilde{\psi}_\beta \tag{O.15}$$

This is zero by our gauge choice for background gauge field.

Connection though $\tilde{\phi}^{\mathbf{2}}\mathbf{a}^{\mu}\mathbf{a}_{\mu}$

The Feynman diagram corresponding the interaction with $\tilde{\phi}^2 a^\mu a_\mu$ is (Figure O.5),

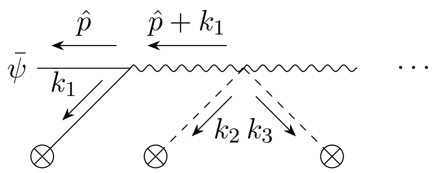


FIGURE O.5: Gluon propagator connecting through background boson vertex.

[5]Here $C^{\alpha\beta}, C_{\alpha\beta}$ are as defined in [159].

and the corresponding value of the diagram is,

$$
\begin{aligned}
&\lim_{z\to\infty} \left(\langle\psi(\hat{p})|\, \bar{\psi}^{\alpha}\right)\ a^{\mu}(\gamma_{\mu})_{\alpha}{}^{\beta}\tilde{\psi}_{\beta}(k_1)\ a^{\nu}\ a_{\nu}\ldots \\
&\sim z\lambda_q^{\alpha}\ (\gamma_{\mu})_{\alpha}{}^{\beta}\tilde{\psi}_{\beta}(k_1)\ \epsilon^{\mu\nu\rho}\frac{q_{\rho}}{2q\cdot k_1}\ \epsilon_{\nu\kappa\omega}\frac{q^{\omega}}{2q\cdot(k_1+k_2+k_3)}\cdots \\
&\sim \frac{z}{8}\lambda_q^{\alpha}\ (\gamma_{\mu})_{\alpha}{}^{\beta}\tilde{\psi}_{\beta}(k_1)\ \frac{1}{q\cdot k_1}\ \frac{1}{q\cdot(k_1+k_2+k_3)}\left[q_{\kappa}q^{\mu}-\delta^{\mu}_{\kappa}q^2\right]\ldots \\
&\sim \frac{z}{8}\frac{1}{q\cdot k_1}\ \frac{1}{q\cdot(k_1+k_2+k_3)}\left[\lambda_q^{\alpha}\ \not{q}_{\alpha}{}^{\beta}\tilde{\psi}_{\beta}q_{\kappa}-\lambda_q^{\alpha}\ (\gamma_{\kappa})_{\alpha}{}^{\beta}\tilde{\psi}_{\beta}q^2\right]\ldots=0
\end{aligned}
\tag{O.16}
$$

above in the second line, we have also included the propagator of the next gauge field. We have used the identity, $\epsilon^{\mu\nu\rho}\epsilon_{\nu\kappa\omega}=\frac{1}{2}\delta^{\mu}_{[\omega}\delta^{\rho}_{\kappa]}$ and $\lambda_q\cdot\not{q}=0, q^2=0$.

Connection though $\epsilon^{\mu\nu\rho}\tilde{\mathbf{A}}_{\mu}\mathbf{a}_{\nu}\mathbf{a}_{\rho}$

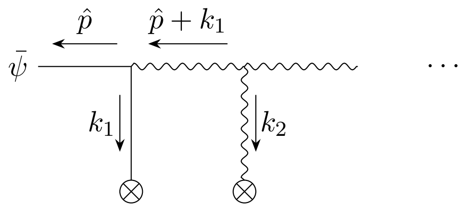


FIGURE O.6: Gluon propagator connecting through gluon vertex.

The gauge field in vertex, $\tilde{\psi}a^{\mu}\psi$, to which the external field is connected has two choices of contractions in the vertex $\epsilon^{\mu\nu\rho}\tilde{A}_{\mu}a_{\nu}a_{\rho}$, see Figure O.6. But this vertex is anti-symmetric with respect to these two choices and hence vanishes. Mathematically,

$$
\begin{aligned}
&\lim_{z\to\infty} \left(\langle\psi(\hat{p})|\, \bar{\psi}^{\alpha}\right)\ a_{\mu}(\gamma^{\mu})_{\alpha}{}^{\beta}\tilde{\psi}_{\beta}(k_1)\ \epsilon^{\sigma\kappa\rho}\tilde{A}_{\sigma}a_{\kappa}a_{\rho}\ldots \\
&\lim_{z\to\infty} \left(\langle\psi(\hat{p})|\, \bar{\psi}^{\alpha}\right)\ (\gamma^{\mu})_{\alpha}{}^{\beta}\tilde{\psi}_{\beta}(k_1)\ \tilde{A}_{\sigma}\left(\epsilon^{\sigma\kappa\rho}\epsilon_{\mu\kappa\omega}\frac{q^{\omega}}{2k_1\cdot q}a_{\rho}-\epsilon^{\sigma\rho\kappa}\epsilon_{\mu\rho\omega}\frac{q^{\omega}}{2k_1\cdot q}a_{\kappa}\right)\ldots=0
\end{aligned}
\tag{O.17}
$$

Where the term in the parenthesis is exactly zero.

Connection though $\mathbf{a}^{\mu}\phi\partial_{\mu}\tilde{\phi}$ and $\mathbf{a}^{\mu}\tilde{\phi}\partial_{\mu}\phi$

These vertices come as a part of current-gauge field interaction in the Lagrangian. So the actual vertex that appears in the Lagrangian is,

$$
ia^{\mu}\left(\bar{\tilde{\phi}}\partial_{\mu}\phi-\partial_{\mu}\bar{\tilde{\phi}}\,\phi\right) \tag{O.18}
$$

We need to consider only those diagrams in which the bosonic propagator is external. Therefore the diagram that has non-trivial large z contribution is,

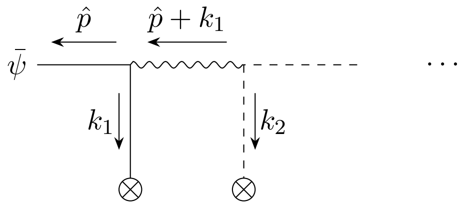


FIGURE O.7: Gluon propagator connecting through fermion vertex.

The contribution coming from this Feynman graph is cancelled against the contribution coming from the Feynman graph Figure O.2. Thus total contribution coming from the contact interaction term is,

$$\begin{aligned}&\lim_{z\to\infty}\left(\langle\psi(\hat{p})|\,\bar{\psi}^{\alpha}\right)\,(\gamma^{\mu})_{\alpha}{}^{\beta}\tilde{\psi}_{\beta}(k_1)\left(-\frac{2\pi i}{\kappa}\right)\tilde{\phi}(k_2)\\ &\sim z\left(-\frac{2\pi i}{\kappa}\right)\lambda_q^{\alpha}(\gamma^{\mu})_{\alpha}{}^{\beta}\tilde{\psi}_{\beta}(k_1)\;\tilde{\phi}(k_2)\end{aligned} \tag{O.19}$$

and the contribution coming from Figure O.7 is,

$$\lim_{z\to\infty}\left(\langle\psi(\hat{p})|\,\bar{\psi}^{\alpha}\right)\,(\gamma^{\mu})_{\alpha}{}^{\beta}\tilde{\psi}_{\beta}(k_1)\left(-\frac{2\pi i}{\kappa}\,\epsilon^{\mu\nu\sigma}\frac{(\hat{p}+k_1)_{\sigma}}{(\hat{p}+k_1)^2}\right) \tag{O.20}$$

O.1 Un-improved $z\to\infty$ behaviour of $\mathcal{N}=1$ theory

In the previous section it was shown that certain diagrams that don't have a good $z\to\infty$ behaviour in $\mathcal{N}$=2 theory cancel against each other, giving us a good overall behaviour of the amplitude in large z limit. In the less restricting $\mathcal{N}$=1 theory, not only does this cancellation not happen, there are additional contributions coming from the new vertices that are present in the =1 Lagrangian but not in $\mathcal{N}$=2 Lagrangian. These vertices are of the form $\big(\bar{\phi}\psi\big)\big(\bar{\phi}\psi\big)+\big(\bar{\psi}\phi\big)\big(\bar{\psi}\phi\big)$ and contribute to diagrams similar to Figure O.2, as well as (O.11a). Thus clearly, one can't use the BCFW recursion relations in $\mathcal{N}$=1 SCS theory to compute the higher point scattering matrices.

O.2 Fermioninc CS theory

Based on the above discussion we can easily argue that any tree-level diagram in pure fermion coupled to Chern-Simons theory is not well behaved in the $z\to\infty$ limit, and hence, a priori, there is no reason to assume that one can use the BCFW recursion relations. The Lagrangian of this theory is simply,

$$\mathcal{L}_F=\int d^3x\bigg[\Big(-\frac{\kappa}{2\pi}\Big)\epsilon^{\mu\nu\rho}\mathrm{Tr}[A_{\mu}\partial_{\nu}A\rho-\frac{2i}{3}A_{\mu}A_{\nu}A_{\rho}]-\bar{\psi}(i\not{D}+m_0)\psi\bigg] \tag{O.21}$$

The large z behaviour of the propagator and the vertices is the same as discussed in the previous section (every thing behaves at $\mathcal{O}(1)$). The external hard fermions can be attached to the rest of the diagram only through vertices: $\tilde{A}^{\mu}\psi^{2}$, $\tilde{\psi}a^{\mu}\psi$, and subsequently, the gauge propagator can connect only through vertices: $\tilde{\psi}a^{\mu}\psi$, $\epsilon^{\mu\nu\rho}\tilde{A}_{\mu}a_{\nu}a_{\rho}$.

However, there is an alternate way to argue that the BCFW relations should be applicable to this theory despite it not obeying the postulates of BCFW recursion relations. In a theory of only fermions, all the external particles in any S-matrix can only be fermions. Such all-fermion scattering amplitudes appear as component amplitudes in supersymmetric amplitudes. However, at tree level, even in supersymmetric theory there are no internal bosons in the Feynman diagrams contributing to such amplitudes. This is because all interaction vertices that appear in the Lagrangian of the supersymmetric theory have 2 bosons in them, making it impossible contract them all without forming a loop.

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