Aspects of 2-Dimensional Conformal Field Theory

Thesis submitted to the
Tata Institute of Fundamental Research
for the degree of
Master of Science, in Physics

by
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December, 2008

Dedicated to my family

Acknowledgements

I would like to thank my supervisor, Prof. Sunil Mukhi for his constant guidance and inspiring discussions on various topics in Physics. I am also thankful to my friends Anindya, Rudra, Sashideep, Suresh and Suvrat for several interesting and insightful sessions on different ideas. My special thanks to the department staff for all their support and help.

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Chapter 1

Introduction

Symmetry principles play an important role with respect to the laws of nature. They summarize the regularities of the laws that are independent of the specific dynamics. Thus invariance principles provide a structure and coherence to the laws of nature just as the laws of nature provide a structure and coherence to the set of events. With the development of quantum mechanics in the 1920s, symmetry principles came to play a fundamental role. In the latter half of the 20th century symmetry has been the most dominant concept in the exploration and formulation of the fundamental laws of physics. Today it serves as a guiding principle in the search for further unification and progress.

Compared to ordinary quantum field theories in four dimensions, conformal field theories in two dimensions can be defined in a rather abstract way via operator algebras and their representation theory. In fact, there are many examples of CFTs where the usual description in terms of a Lagrangian action with resulting perturbative expansion is not even known. Instead, following a so-called boot-strap approach, one can define these theories without making reference to an action and sometimes one can even solve them exactly. Such a procedure is possible because the algebra of infinitesimal conformal transformations in two dimensions is very special in contrast to its higher dimensional counterparts - it is infinite dimensional and therefore highly constraining.

The main feature of a conformal field theory is the invariance under conformal transformations. Roughly speaking, these are transformations leaving angles invariant and a particular example is the scaling $x \to ax$ of a distance x by some constant a. A field theory exhibiting such symmetry has no preferred scale and one can only expect a physical system to have this property, if there are no dimensionful scales involved. Polyakov [1] conjectured that systems (with assumptions such as isotropy and possibly locality of interactions) exhibiting scale invariance in 2D possess a symmetry larger than simple scaling. This symmetry group is called the conformal group.

As an example, the field theory of a free boson encounters a conformal symmetry for the case of vanishing mass. And even for interacting theories it is known that at the fixed point of a renormalisation group flow, there are only long-range correlations. Therefore, the natural mass scale at this point, that is, the inverse of the correlation length, vanishes and a conformal field theory description might be available. Physical systems with a conformal symmetry are thus more common than one would have naively expected.

Another important instance featuring conformal symmetry is string theory, which is a candidate theory for the unification of all interactions including gravity. Here, the CFT arises as a two-dimensional field theory living on the world-volume of a string which moves in some background spacetime. The dynamics of this string is governed by a non-linear sigma model whose condition for conformal invariance, that is, the vanishing of the β function, gives the string equations of motion. The sigma model perturbation theory is governed by an expansion in s/R, where s is the natural string length and R a typical length scale of the background geometry. With the help of CFT techniques, one can sometimes solve such theories exactly to all orders in perturbation theory and one can sum all contributions of so-called world-sheet instantons. Therefore, conformal field theory is a very powerful tool for string theory, not only in the perturbative regime but also at small length scales where genuine string effects become important and geometric intuition often fails.

Chapter 2

Conformal transformation in general d dimensions

In this chapter we discuss the conformal symmetry in arbitrary dimension. We investigate the generators and their commutation relations and further, the conformal group is identified with the non-compact group $SO(d+1,1)$. We study the action of a conformal transformation on fields at the classical level and relate the scale invariance to tracelessness of the energy momentum tensor. We then look at the consequences of conformal invariance at the quantum level on the structure of correlation functions and derive the form of two- and three-point functions and the Ward identities.

2.1 Conformal Group

Given a metric tensor $g_{\mu\nu}$, a conformal transformation of the coordinates is an invertible mapping $x \to x'$, such that the metric tensor is invariant up to a scale:

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x) \tag{2.1}$$

The set of conformal transformations manifestly form a group. It has the Poincare group as a subgroup which corresponds to $\Lambda(x) = 1$. Conformal field theories care only about angles and not about distances and the physics of the theory looks the same at all length scales.

We begin with the consequence of the definition 2.1 on an infinitesimal transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$. Under $x \to x'$, we have $g_{\mu\nu} \to g'_{\mu\nu}(x') =$

$\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}g_{\alpha\beta}$. So to first order in ϵ, the metric will transform in following manner:

$$g_{\mu\nu} \to g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) \tag{2.2}$$

For this transformation to be conformal, it requires that $(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu)$ is proportional to the metric. Hence

$$(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = f(x) g_{\mu\nu} \tag{2.3}$$

The proportionality factor can be derived by contracting both the sides with the metric:

$$f(x) = \frac{2}{d}\partial_\rho \epsilon^\rho \tag{2.4}$$

For the sake of simplicity, we assume that the metric is the Cartesian metric $g_{\mu\nu} = \delta_{\mu\nu}$, where $\delta_{\mu\nu} = \mathrm{diag}(1, 1, ..., 1)$. By applying an extra derivative ∂_ρ on 2.3, permuting the indices and taking a linear combination, one gets

$$2\partial_\mu \partial_\nu \epsilon_\rho = \delta_{\mu\rho}\partial_\nu f + \delta_{\nu\rho}\partial_\mu f - \delta_{\mu\nu}\partial_\rho f \tag{2.5}$$

and upon contracting with $\delta^{\mu\nu}$, this becomes

$$2\partial^2 \epsilon_\mu = (2 - d)\partial_\mu f \tag{2.6}$$

Using this and 2.3 we finally get

$$(d - 1)\partial^2 f = 0 \tag{2.7}$$

We can get a good idea of the form of the function f using the above equation for different dimensions. When $d = 1$, no constraint is imposed on f and therefore any smooth transformation is a conformal transformation in one dimension. It is quite obvious as the notion of angle doesn't exist in one dimension. For the case $d > 2$, above equations imply that $\partial_\mu \partial_\nu f = 0$. This is true if f is a linear function of coordinates:

$$f(x) = A + B_\mu x^\mu \tag{2.8}$$

A and B_μ are constants. Now we insert this expression into 2.5 to get the general form of ϵ. As f is linear in x, ϵ can be at most quadratic in x. We can write the general expression

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho \tag{2.9}$$

Here $c_{\mu\nu\rho}$ is symmetric in ν and ρ. Now we can consider each of the terms in the expansion separately. $\epsilon_\mu = a_\mu$ is ordinary translation independent of x. To get the properties of $b_{\mu\nu}$, we insert $\epsilon_\mu = b_{\mu\nu}x^\nu$ into the equation 2.3 which yields

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d} b^\mu{}_\mu \delta_{\mu\nu} \tag{2.10}$$

which implies that $b_{\mu\nu}$ is a sum of antisymmetric part and a pure trace:

$$b_{\mu\nu} = \alpha\delta_{\mu\nu} + m_{\mu\nu} \tag{2.11}$$

The trace part signifies a scale transformation while the antisymmetric part is an infinitesimal rigid rotation. Substitution of the quadratic term into eq 2.5 yields

$$c_{\mu\nu\rho} = \delta_{\mu\rho}b_\nu + \delta_{\mu\nu}b_\rho - \delta_{\nu\rho}b_\mu, \qquad b_\mu = \frac{1}{d}c^\kappa{}_{\kappa\mu} \tag{2.12}$$

This gives $\epsilon^\mu = 2x^\mu x.b - b^\mu x^2$ which is called the special conformal transformation (SCT). This may also be viewed as $x'^\mu/x'^2 = x^\mu/x^2 + b^\mu$, a combination of inversion and translation. Note that the points on the surface $1 + 2b.x + b^2x^2 = 1$ have their distances from the origin preserved while the points on the exterior of this surface are sent to the interior and vice versa. Now the generators of the conformal group (discussed in appendix A), take the following form

$$\begin{aligned} Translation \quad & P_\mu = -i\partial_\mu \\ Dilation \quad & D = -ix^\mu\partial_\mu \\ Rotation \quad & L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) \\ SCT \quad & K_\mu = -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu) \end{aligned}$$

This admits a total of $d+1+\frac{1}{2}d(d-1)+d = \frac{1}{2}(d+1)(d+2)$ generators which as we demonstrate now, is isomorphic to $SO(d+1,1)$. These generators obey the following commutation rules:

$$\begin{aligned}
[D,P_\mu] &= iP_\mu \\
[D,K_\mu] &= -iK_\mu \\
[K_\mu,P_\nu] &= 2i(\eta_{\mu\nu}D - L_{\mu\nu}) \\
[K_\rho,L_{\mu\nu}] &= i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) \\
[P_\rho,L_{\mu\nu}] &= i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \\
[L_\mu,L_\rho] &= i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}l_{\mu\rho})
\end{aligned}$$

To make the above commutation rules more familiar, we define the following generators:

$$\begin{aligned}
J_{\mu\nu} &= L_{\mu\nu} \qquad & J_{-1,\mu} &= \frac{1}{2}(P_\mu - K_\mu) \\
J_{-1,0} &= D \qquad & J_{0,\mu} &= \frac{1}{2}(P_\mu + K_\mu)
\end{aligned}$$

where $J_{ab} = -J_{ba}$ and a,b take values $-1,0,1,...,d$. These new set of operators obey the $SO(d+1,1)$ commutation relations:

$$[J_{ab},J_{cd}] = i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}) \tag{2.13}$$

where η_{ab} has Minkowski signature, $(-1,1...,1)$ and is constructed by adding two dimensions of signature $(-1,+1)$ to the d Euclidean dimensions. This exhibits the isomorphism between the conformal group in d dimensions and the group $SO(d+1,1)$. Note that the d dimensional Euclidean group along with dilations form a subgroup $SO(d)\times SO(1,1)$ of the full conformal group. If we work in d dimension Minkowski space-time, the conformal group will be $SO(d,2)$.

2.2 Conformal Invariance in Classical Field Theory

We now define the effect of conformal transformations on classical fields. Given an infinitesimal conformal transformation parameterized by w_g, a multicomponent field $\Phi(x)$ transforms as

$$\Phi'(x') = (1 - iw_g T_g)\Phi(x) \tag{2.14}$$

The generator T_g are added to the space-time part given in 2.13 to obtain the full generator of the symmetry. The recipe to find the allowed form of these generators is to start by studying the subgroup of the Poincare group that leaves the origin, $x = 0$ invariant, which is the Lorentz group. The action of infinitesimal Lorentz transformation on the $\Phi(0)$ is given as:

$$L_{\mu\nu}\Phi(0) = S_{\mu\nu}\Phi(0) \tag{2.15}$$

$S_{\mu\nu}$ is the spin associated with the field Φ. Next, by commutation relations of Poincare group, we translate $L_{\mu\nu}$ to a nonzero value of x:

$$e^{ix^\rho P_\rho} L_{\mu\nu} e^{-ix^\rho P_\rho} = S_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu \tag{2.16}$$

where we used the Hausdorff formula to evaluate the lhs. We proceed in similar fashion to evaluate the whole group and obtain the following form for all the generators:

$$\begin{aligned} P_\mu\Phi(x) &= -i\partial_\mu\Phi(x) \\ D\Phi(x) &= -i(x^\mu\partial_\mu + \Delta)\Phi(x) \\ L_{\mu\nu}\Phi(x) &= i(x_\mu\partial_\nu - x_\mu\partial_\nu)\Phi(x) + S_{\mu\nu}\Phi(x) \\ K_\mu\Phi(x) &= -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu + 2x_\mu\Delta - ix^\nu S_{\mu\nu})\Phi(x) \end{aligned} \tag{2.17}$$

Δ is the scaling dimension of the field $\Phi(x)$. Under a conformal transformation $x \to x' = \lambda x$, a spinless field ϕ ($S_{\mu\nu} = 0$) of scaling dimension Δ, transforms as

$$\phi(x) \to \phi'(x') = \lambda^{-\Delta}\phi(x) \tag{2.18}$$

We can write this expression in terms of the scale factor Λ from 2.1 as

$$\phi'(x') = \Lambda^{\Delta/2}\phi(x) \tag{2.19}$$

which comes from the invariance of line element ds^2.

$$
\begin{aligned}
ds'^2 &= ds^2 \\
\Rightarrow \quad g'_{\mu\nu}(x')dx'^{\mu}dx'^{\nu} &= g_{\mu\nu}(x)dx^{\mu}dx^{\nu} \\
\Rightarrow \Lambda g_{\mu\nu}(x)\lambda^2 dx^{\mu}dx^{\nu} &= g_{\mu\nu}(x)dx^{\mu}dx^{\nu} \\
\Rightarrow \Lambda &= \lambda^{-2}
\end{aligned}
\tag{2.20}
$$

Now since the infinitesimal volume element should be invariant under conformal transformation, we have

$$
d^dx'(g'(x'))^{1/2} = d^dx(g(x))^{1/2} \tag{2.21}
$$

where $g(x)$ is the determinant of the metric, the above equation is equivalent to

$$
\left|\frac{\partial x'}{\partial x}\right| d^dx\ \Lambda(x)^{d/2}\ g(x)^{1/2} = d^dx\ g(x)^{1/2} \tag{2.22}
$$

where $|\partial x'/\partial x|$ is the Jacobian of conformal transformation. This gives the following relation between the Jacobian and the scale factor:

$$
\left|\frac{\partial x'}{\partial x}\right| = \Lambda(x)^{-d/2} \tag{2.23}
$$

We can use the above identity and rewrite 2.19 as

$$
\phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d} \phi(x) \tag{2.24}
$$

A field transforming in above form is called a ”quasi-primary” field.

2.3 The Energy-Momentum Tensor

Under an arbitrary transformation $x \to x' = x + \epsilon(x)$, the action changes as:

$$
\delta S = \int d^dx\ \ T^{\mu\nu}\ \partial_\mu \epsilon_\nu = \frac{1}{2}\int d^dx\ \ T^{\mu\nu}\ (\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu) \tag{2.25}
$$

where $T^{\mu\nu}$ is the energy momentum tensor. We assume here that $T^{\mu\nu}$ is symmetric as we know it can always be made symmetric by transforming it to Belinfante form (worked out in detail in [2]). Now using equation 2.3 and 2.4 for an infinitesimal conformal transformation, we reduce the above equation to the following form:

$$\delta S = \frac{1}{d}\int d^d x \;\; T^{\mu}_{\;\;\mu}\, \partial_\nu \epsilon^\nu \tag{2.26}$$

The tracelessness of the energy momentum tensor implies the invariance of the action under conformal transformation. The converse is however not true because $\partial_\mu \epsilon^\mu$ is not an arbitrary function as $\epsilon^\mu(x)$ can only be of form shown in 2.9, i.e. at most quadratic in x.

Current for a general infinitesimal transformation is given as

$$j^\mu_a = \left(\frac{\partial L}{\partial(\partial_\mu \Phi)}\partial_\nu \Phi - \delta^\mu_\nu L\right)\frac{\delta x^\nu}{\delta w_a} - \frac{\partial L}{\partial(\partial_\mu \Phi)}\frac{\delta F}{\delta w_a} \tag{2.27}$$

where

$$\begin{aligned} x'^\mu &= x^\mu + w_a \frac{\delta x^\mu}{\delta w_a} \\ \Phi'(x') &= \Phi(x) + w_a \frac{\delta F}{\delta w_a}(x) \end{aligned} \tag{2.28}$$

$\{w_a\}$ is a set of infinitesimal parameters. For an infinitesimal dilation, we can write $\frac{\delta x^\mu}{\delta w_a} \to x^\mu$ and $\frac{\delta F}{\delta w_a} \to \Delta\Phi$. Inserting this in 2.27, the associated dilation current can be written

$$\begin{aligned} j^\mu_D &= -Lx^\mu + \frac{\partial L}{\partial(\partial_\mu \Phi)} x^\nu \partial_\nu \Phi + \frac{\partial L}{\partial(\partial_\mu \Phi)}\Delta\Phi \\ &= T^{\mu}_{c\;\nu} x^\nu + \frac{\partial L}{\partial(\partial_\mu \Phi)}\Delta\Phi \end{aligned} \tag{2.29}$$

$T^{\mu}_{c\;\nu}$ being the canonical energy-momentum tensor. Now one can show that by using the symmetries of a theory, we can write down a symmetric and traceless energy momentum tensor which is related to the dilation current as

$$j^\mu_D = T^{\mu}_{\;\;\nu} x^\nu \tag{2.30}$$

We notice that $\partial_\mu j^\mu = 0$ since the energy-momentum tensor is traceless. It can be shown that in 2 dimensions no modification is required to the canonical energy-momentum tensor and it is already traceless and similarly related to the current as in above equation.

2.4 Conformal Invariance in Quantum Field Theory

2.4.1 Transformation of the Correlations Functions

At quantum level, correlation functions are the main object of study, and a continuous symmetry leads to constraints relating different correlation functions. Consider a theory involving a collection of fields Φ with action $S[\Phi]$ invariant under a transformation of type $x \to x'$ and $\phi(x) \to \phi'(x')$. A general correlation function is written as

$$\langle \Phi(x_1)\Phi(x_2)...\Phi(x_n)\rangle = \frac{1}{Z}\int [d\Phi]\Phi(x_1)\Phi(x_2)...\Phi(x_n)exp - S[\Phi] \qquad (2.31)$$

where Z is the vacuum functional. The consequence of the symmetry of action and of the invariance of the functional integration measure under the above transformation is the following identity.

$$\langle \Phi(x_1')\Phi(x_2')...\Phi(x_n')\rangle = \langle F(\Phi(x_1))F(\Phi(x_2))...F(\Phi(x_n))\rangle \qquad (2.32)$$

where the mapping F describes the functional change of the field under the transformation.

2.4.2 2, 3 and 4-Point Functions

Conformal invariance puts restriction on the form of the two and three-point correlation functions of quasi-primary fields. We know how a quasi primary field should transform under conformal transformation and this will put a set of constraints on the structure of the correlation functions. Let us consider a two-point function

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{1}{Z}\int [d\Phi] \ \ \phi_1(x_1)\phi_2(x_2) \ e^{-S[\Phi]} \qquad (2.33)$$

where ϕ_1 and ϕ_2 are quasi-primary fields and $S[\Phi]$ is the action which is conformally invariant. We also assume that the functional integration measure is conformally invariant. The correlation function will transform in the following manner

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/d} \left|\frac{\partial x'}{\partial x}\right|_{x=x_2}^{\Delta_2/d} \langle \phi_1(x'_1)\phi_2(x'_2)\rangle \tag{2.34}$$

Here Δ_i is the scaling dimension of field ϕ_i. Now translation and rotation invariance require that

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = f(|x_1 - x_2|) \tag{2.35}$$

Under a scale transformation $x \to \lambda x$ we get

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2}\langle \phi_1(\lambda x_1)\phi_2(\lambda x_2)\rangle \tag{2.36}$$

In other words we can write

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \tag{2.37}$$

where C_{12} is a constant coefficient. Recall that the function should also be invariant under special conformal transformation. Under such a transformation

$$\left|\frac{\partial x'}{\partial x}\right| = \frac{1}{(1 - 2b.x + b^2x^2)^d}$$

$$|x'_i - x'_j| = \frac{|x_i - x_j|}{(1 - 2b.x_i + b^2x_i^2)^{1/2}(1 - 2b.x_j + b^2x_j^2)^{1/2}} = \frac{|x_i - x_j|}{\gamma_i\gamma_j}$$

Covariance of 2.37 under special conformal transformation implies

$$\frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} = \frac{C_{12}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}} \frac{(\gamma_1\gamma_2)^{(\Delta_1+\Delta_2)/2}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \tag{2.38}$$

This is true only when $\Delta_1 = \Delta_2$. This means that two quasi-primary fields are correlated only if they have same scaling dimensions. Similarly we can argue that a three-point correlation function should have the following form:

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}^{(abc)}}{x_{12}^a x_{23}^b x_{31}^c} \tag{2.39}$$

where $x_{ij} = |x_i - x_j|$ and a,b,c are such that

$$a + b + c = \Delta_1 + \Delta_2 + \Delta_3 \tag{2.40}$$

Demanding invariance under special conformal transformation, we have

$$\frac{C_{123}^{(abc)}}{x_{12}^a x_{23}^b x_{31}^c} = \frac{C_{123}^{(abc)}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}\gamma_3^{\Delta_3}} \frac{(\gamma_1\gamma_2)^{a/2}(\gamma_2\gamma_3)^{b/2}(\gamma_1\gamma_3)^{c/2}}{x_{12}^a x_{23}^b x_{31}^c} \tag{2.41}$$

This is true when we have the following set of constraints

$$\begin{aligned} a &= \Delta_1 + \Delta_2 - \Delta_3 \\ b &= \Delta_2 + \Delta_3 - \Delta_1 \\ c &= \Delta_3 + \Delta_1 - \Delta_2 \end{aligned} \tag{2.42}$$

Therefore the correlator of three quasi-primary fields can be written as

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{x_{12}{}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{31}^{\Delta_3+\Delta_1-\Delta_2}} \tag{2.43}$$

The conformal invariance however does not provide enough constraints to fix the structure of four-point correlation function or beyond. This is because the global conformal transformations only allow us to fix three coordinates, so the best we can do is to take say $x_1, x_2, x_3, x_4 = \infty, 1, x, 0$. Indeed, with four points, it is possible to construct conformal invariants, the anharmonic ratios

$$\frac{|x_1 - x_2||x_3 - x_4|}{|x_1 - x_3||x_2 - x_4|} \qquad \frac{|x_1 - x_2||x_3 - x_4|}{|x_2 - x_3||x_1 - x_4|} \tag{2.44}$$

and the residual x will be a function of these ratios. Then we write the four-point correlation function in terms of these ratios

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = f(\frac{x_{12}x_{34}}{x_{13}x_{24}}) \prod_{i<j}^{4} x_{ij}^{\Delta/3-\Delta_i-\Delta_j} \tag{2.45}$$

where $\Delta = \sum_{i=1}^{4} \Delta_i$ and the function f is undetermined.

2.5 Ward Identities

The Ward identities are used to exhibit the consequences of the symmetry of action and measure on the correlation functions. As shown in the appendix B, the ward identity for a current j_a^μ is given as

$$\frac{\partial}{\partial x^\mu} \langle j_a^\mu \Phi(x_1)\Phi(x_2)....\Phi(x_n)\rangle = -i \sum_i \delta(x - x_i)\langle \Phi(x_1)...G_a\Phi(x_i)...\Phi(x_n)\rangle \tag{2.46}$$

G_a is the generator of symmetry transformation. We use this equation to write down the Ward identity for translation invariance. The energy momentum tensor is the current associated with translation invariance and the corresponding generator is $G_\mu = \frac{\partial}{\partial x^\mu}$. Hence if we define X as a product of n local fields, we have

$$\partial_\mu \langle T^\mu{}_\nu X\rangle = -\sum_i \delta(x - x_i)\frac{\partial}{\partial x_i^\mu}\langle X\rangle \tag{2.47}$$

Now we find the Ward identity associated with Lorentz invariance. The current is written as

$$j^{\mu\nu\rho} = T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu \tag{2.48}$$

where $T^{\mu\nu}$ is the symmetrized tensor. The generator of Lorentz transformation is given as

$$L^{\mu\nu} = (x^\mu\partial^\nu - x^\nu\partial^\mu) + S^{\mu\nu} \tag{2.49}$$

$S^{\mu\nu}$ is the spin generator. Consequently the Ward identity is

$$\partial_\mu\langle(T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu)X\rangle = \sum_i \delta(x - x_i)[(x_i^\mu\partial_i^\nu - x_i^\nu\partial_i^\mu)\langle X\rangle - iS_i^{\mu\nu}\langle X\rangle] \tag{2.50}$$

Index i is for the i_{th} field in X. On the lhs, we use 2.47 to reduce the above identity to

$$\langle(T^{\rho\nu} - T^{\nu\rho})X\rangle = -i\sum_i \delta(x - x_i) \quad S_i^{\nu\rho}\langle X\rangle \tag{2.51}$$

which states that the energy-momentum tensor is symmetric within correlation functions except at the positions of the fields.

Finally we derive the Ward identity for the scale invariance. As shown in 2.30 and 2.17, the dilation current is $j^\mu_D = T^\mu_\nu x^\nu$ and the dilation generator $D = -ix^\mu\partial_\mu - i\Delta$, respectively. We insert these in 2.46

$$\partial_\mu\langle T^\mu{}_\nu x^\nu X\rangle = -\sum_i \delta(x-x_i)(x_i^\nu \frac{\partial}{\partial x_i^\nu}\langle X\rangle + \Delta_i\langle X\rangle) \tag{2.52}$$

Again using the 2.47, this reduces to

$$\langle T^\mu{}_\mu X\rangle = -\sum_i \delta(x-x_i)\Delta_i\langle X\rangle \tag{2.53}$$

The trace of the energy-momentum tensor vanishes except at the location of the fields. The 3 equations 2.47, 2.51, 2.53 are the Ward identities associated with the conformal symmetry.

Chapter 3

Conformal transformation in 2 dimensions

Unlike other dimensions, where the conformal group is finite dimensional and global, in 2D, it is local and therefore infinite dimensional. The condition for a transformation to be conformal is the same as the Cauchy-Riemann condition for an analytic function. Since there are an infinite number of analytic functions on a plane, this implies that the conformal group is infinite dimensional. We develop the language of holomorphic and antiholomorphic coordinates on a plane which is a prominent tool in 2-d CFT. We exhibit the distinction between local and global transformations and introduce generators for local conformal transformations.

3.1 Conformal Group in 2D

In 2 dimensions, it is convenient to work with complex coordinates. We introduce complex coordinates

$$z = x^1 + ix^2, \quad \bar{z} = x^1 - ix^2 \tag{3.1}$$

In this coordinate system, the metric tensor and its inverse look like

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \quad g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \tag{3.2}$$

where μ takes the values z and $\bar{z}$. Hence, the line element is written as $ds^2 = dzd\bar{z}$.

Also the antisymmetric tensor $\varepsilon_{\mu\nu}$ and $\varepsilon^{\mu\nu}$ are

$$\varepsilon_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2}i \\ -\frac{1}{2}i & 0 \end{pmatrix} \quad \varepsilon^{\mu\nu} = \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix} \tag{3.3}$$

We consider the coordinates (z^0, z^1) on a plane. Now under a coordinate transformation $z^\mu \to w^\mu(z)$, the metric tensor transforms as

$$g^{\mu\nu}(z) \to \left(\frac{\partial w^\mu}{\partial z^\alpha}\right)\left(\frac{\partial w^\nu}{\partial z^\beta}\right) g^{\alpha\beta}(w) \tag{3.4}$$

using the form of metric tensor as in 3.2, we find

$$\left(\frac{\partial w^0}{\partial z^0}\right)^2 + \left(\frac{\partial w^0}{\partial z^1}\right)^2 = \left(\frac{\partial w^1}{\partial z^0}\right)^2 + \left(\frac{\partial w^1}{\partial z^1}\right)^2 \tag{3.5}$$

$$\frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^0} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^1} = 0 \tag{3.6}$$

Solving these equations, we find they are equivalent either to

$$\frac{\partial w^1}{\partial z^0} = \frac{\partial w^0}{\partial z^1} \quad \& \quad \frac{\partial w^0}{\partial z^0} = -\frac{\partial w^1}{\partial z^1} \tag{3.7}$$

or

$$\frac{\partial w^1}{\partial z^0} = -\frac{\partial w^0}{\partial z^1} \quad \& \quad \frac{\partial w^0}{\partial z^0} = \frac{\partial w^1}{\partial z^1} \tag{3.8}$$

We notice these are the Cauchy-Riemann equations for holomorphic and antiholomorphic functions.

$$\begin{aligned} \partial_z w(z, \bar{z}) &= 0 \\ \partial_{\bar{z}} w(z, \bar{z}) &= 0 \end{aligned} \tag{3.9}$$

The solutions to these two equations are any anti-holomorphic or holomorphic mapping respectively.

$$\begin{aligned} \bar{z} &\to \bar{w}(\bar{z}) \\ z &\to w(z) \end{aligned} \tag{3.10}$$

Two dimensional conformal transformations thus coincide with the analytic coordinate transformations, the local algebra of which is infinite dimensional.

3.2 Conformal Generators

To find the commutation relations of the generators of the conformal algebra, i.e. the mapping of the form 3.10, we take the infinitesimal transformation

$$z \to z' = z - \sum_{-\infty}^{\infty} c_n \epsilon_n(z) \qquad \bar{z} \to \bar{z}' = \bar{z} - \sum_{-\infty}^{\infty} c'_n \bar{\epsilon}_n(\bar{z}) \tag{3.11}$$

where

$$\epsilon_n(z) = -z^{n+1} \qquad \bar{\epsilon}_n(\bar{z}) = -\bar{z}^{n+1} \tag{3.12}$$

The corresponding infinitesimal generators are

$$l_n = -z^{n+1}\partial_z \quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \tag{3.13}$$

These generators obey the following commutation relations:

$$\begin{aligned} [l_n, l_m] &= (n-m)l_{n+m} \\ [\bar{l}_n, \bar{l}_m] &= (n-m)\bar{l}_{n+m} \\ [\bar{l}_n, l_m] &= 0 \end{aligned}$$

This is called the Witt's algebra. In quantum case the algebra will be corrected by adding an extra term known as the central extension. Since l_n's commute with $\bar{l}_m$'s, the conformal algebra is direct sum of two isomorphic algebras.

We notice that the above algebra is a local algebra as the generators are not all well-defined globally on the Riemann sphere. Holomorphic conformal transformations are generated by

$$v(z) = -\sum_{-\infty}^{\infty} a_n l_n = \sum_{-\infty}^{\infty} a_n z^{n+1}\partial_z \tag{3.14}$$

Non-singularity of $v(z)$ as $z \to 0$ allows non-zero a_n only for $n \geq -1$. To check the behavior of $v(z)$ as $z \to \infty$, we carry out a transformation $z = -1/w$,

$$v(z) = \sum_{-\infty}^{\infty} a_n \left(-\frac{1}{w}\right)^{n+1} \left(\frac{dz}{dw}\right)^{-1} \partial_w = \sum_{-\infty}^{\infty} a_n \left(-\frac{1}{w}\right)^{n-1} \partial_w \qquad (3.15)$$

Non-singularity as $w \to 0$ demands $a_n \neq 0$ only for $n \leq 1$. Hence we find that only conformal transformations generated by $a_n l_n$ for $n = 0, \pm 1$ are defined globally. These transformations form a subgroup of the whole conformal group and are well-defined and invertible over the Riemann sphere (i.e. the whole complex plane plus the point at infinity). The globally defined generators are $\{l_{-1}, l_0, l_1\} \cup \{\bar{l}_{-1}, \bar{l}_0, \bar{l}_1\}$ and they form the special conformal group. From 3.13, we can identify the l_{-1} and $\bar{l}_{-1}$ as generators of translations, $l_0 + \bar{l}_0$ and $i(l_0 - \bar{l}_0)$ as generators of scale transformation and rotation and $l_1, \bar{l}_1$ as generators of special conformal transformations. A complete set of such global conformal transformations can be written as

$$f(z) = \frac{az + b}{cz + d} \quad \text{with} \quad \mathrm{ad} - \mathrm{bc} = 1 \qquad (3.16)$$

where a,b,c,d are complex numbers. These mappings are called the projective transformations and they form the $SL(2, C)$ group .To each global conformal transformation we can associate the matrix,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad (3.17)$$

In $SL(2, C)$ language, we can write the transformations as

$$Translations : \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \qquad Rotations : \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$$

$$Dilations : \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \qquad Special\ Conformal : \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$$

where $A = a^1 + ia^2$ and $B = b^1 + ib^2$.

3.3 Primary fields

The global conformal algebra generated by $\{l_{-1}, l_0, l_1\} \cup \{\bar{l}_{-1}, \bar{l}_0, \bar{l}_1\}$ is useful for characterizing properties of physical states. If we work in a basis of eigenstates of the two operators l_0 and $\bar{l}_0$, and denote their eigenvalues by h and $\bar{h}$ respectively (here h and $\bar{h}$ are meant to indicate independent (real) quantities, not complex conjugates of one another), they are known as the conformal weights of the state. Since $l_0 + \bar{l}_0$ and $i(l_0 - \bar{l}_0)$ generate dilations and rotations respectively, the scaling dimension Δ and the planar spin s of the state are given by $\Delta = h+\bar{h}$ and $s = h-\bar{h}$. Or we can say for given a field with scaling dimensions Δ and spin s, we write the holomorphic conformal dimension h and its anti-holomorphic counterpart $\bar{h}$ as

$$h = \frac{1}{2}(\Delta + s) \qquad \bar{h} = \frac{1}{2}(\Delta - s) \tag{3.18}$$

Under a conformal map $z \to w(z)$, $\bar{z} \to \bar{w}(\bar{z})$, a quasi-primary field transforms as

$$\phi'(w, \bar{w}) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}) \tag{3.19}$$

If the map $z \to w$ is close to the identity, i.e. $w = z + \epsilon(z)$ and $\bar{w} = \bar{z} + \bar{\epsilon}(\bar{z})$ with ϵ and $\bar{\epsilon}$ very small, the variation of the quasi-primary field is

$$\delta_{\epsilon,\bar{\epsilon}}\phi = \phi'(z, \bar{z}) - \phi(z, \bar{z}) = (h\phi\partial_z\epsilon + \epsilon\partial_z\phi) - (\bar{h}\phi\partial_{\bar{z}}\bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}}\phi) \tag{3.20}$$

A field whose variation under *any* local transformation is given by above equation is called a primary field. All primary fields are quasi-primary but reverse is not true.

3.4 Correlation Functions

Expressed in terms of holomorphic and anti-holomorphic coordinates, the conformal transformation of correlation function of n primary fields ϕ_i with conformal dimensions h_i and $\bar{h}_i$ is written as:

$$\langle\phi_1(w_i, \bar{w}_1)....\phi_n(w_n, \bar{w}_n)\rangle = \prod_{i=1}^{n} \left(\frac{dw}{dz}\right)^{-h_i}_{w=w_i} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}_i}_{\bar{w}=\bar{w}_i} \langle\phi_1(z_i, \bar{z}_1)....\phi_n(z_n, \bar{z}_n)\rangle \tag{3.21}$$

This relation fixes the form of two- and three-point functions. We can use the conformal constraints similar to how we did in section 2.4.2 and find the two- and three-point functions. However here we follow another method just to get a better insight. Now a two-point function $G^{(2)}(z_i, \bar{z}_i) = \langle \Phi_1(z_1, \bar{z}_1)\Phi_2(z_2, \bar{z}_2)\rangle$ will satisfy the following equation under infinitesimal transformation

$$\delta_{\epsilon,\bar{\epsilon}} G^{(2)}(z_i, \bar{z}_i) = \langle \delta_{\epsilon,\bar{\epsilon}}\Phi_1(z_1, \bar{z}_1), \Phi_2\rangle + \langle \Phi_1, \delta_{\epsilon,\bar{\epsilon}}\Phi_2(z_2, \bar{z}_2)\rangle = 0 \tag{3.22}$$

Now using the equation 3.20, this gives

$$\begin{aligned}&\Big((\epsilon(z_1)\partial_{z_1} + h_1\epsilon\partial(z_1)) + (\epsilon(z_2)\partial_{z_2} + h_2\partial\epsilon(z_2)) + \\ &(\bar{\epsilon}(\bar{z}_1)\partial_{\bar{z}_1} + \bar{h}_1\bar{\epsilon}\partial(\bar{z}_1)) + (\bar{\epsilon}(\bar{z}_2)\partial_{\bar{z}_2} + \bar{h}_2\partial\bar{\epsilon}(\bar{z}_2)\Big)G^{(2)}(z_i, \bar{z}_i) = 0\end{aligned} \tag{3.23}$$

We know ϵ and $\bar{\epsilon}$ are at most quadratic in the coordinates. Let us first take $\epsilon(z) = 1$ and $\bar{\epsilon}(\bar{z}) = 1$. This shows that $G^{(2)}$ depends only on $z_{12} = z_1 - z_2$ and $\bar{z}_{12} = \bar{z}_1 - \bar{z}_2$. Then we use $\epsilon(z) = z$ and $\bar{\epsilon}(\bar{z}) = \bar{z}$ to find $G^{(2)} = C_{12}/(z_{12}^{h_1+h_2}\bar{z}_{12}^{\bar{h}_1+\bar{h}_2})$. Finally we use $\epsilon(z) = z^2$ and $\bar{\epsilon}(\bar{z}) = \bar{z}^2$ which puts the constraints $h_1 = h_2 = h$ and $\bar{h}_1 = \bar{h}_2 = \bar{h}$. This results in the following form of the two-point function

$$G^{(2)}(z_i, \bar{z}_i) = \frac{C_{12}}{z_{12}^{2h}\bar{z}_{12}^{2\bar{h}}} \tag{3.24}$$

For a spinless field, from 3.18, we get $h = \bar{h} = \Delta/2$. Then the above equation is equivalent to

$$G^{(2)}(z_i, \bar{z}_i) = \frac{C_{12}}{|z_{12}|^{2\Delta}} \tag{3.25}$$

We similarly find the three-point function $G^{(3)} = \langle \Phi_1\Phi_2\Phi_3\rangle$ takes the form [3]

$$\begin{aligned}G^{(3)}(z_i, \bar{z}_i) = C_{123}&\left\{\frac{1}{z_{12}^{h_1+h_2-h_3}\, z_{23}^{h_2+h_3-h_1}\, z_{31}^{h_3+h_1-h_2}}\right\} \times \\ &\left\{\frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3}\, \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1}\, \bar{z}_{31}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}}\right\}\end{aligned} \tag{3.26}$$

Note that any three points z_1, z_2, z_3 can always be mapped to three reference points $\infty, 1, 0$, using the $SL(2,C)$ or the global conformal transformations, which will reduce the above equation to $\lim_{z_1\to\infty} z_1^{2h_1} \bar{z}_1^{2\bar{h}_1} G^{(3)} = C_{123}$.

As discussed earlier for higher dimensions, even in 2 dimensions the conformal invariance doesn't fix the precise form of four-point function. The four-point function can be written down as a real function of ratios of the four variables $z_{12}, z_{23}, z_{34}, z_{41}$ also known as anharmonic ratios η. The general expression for four-point function can be written as

$$\langle \phi_1(x_1)...\phi_4(x_4)\rangle = f(\eta, \bar{\eta}) \prod_{i<j}^{4} z_{ij}^{h/3-h_i-h_j} \bar{z}_{ij}^{\bar{h}/3-\bar{h}_i-\bar{h}_j} \tag{3.27}$$

where $\eta = \frac{z_{12}z_{34}}{z_{13}z_{24}}$. Now we can use a global conformal map to send z_1 to 1, z_2 to ∞ and z_3 to 0 and this will make $\eta = -z_4$ and a generic four-point function will depend on this last point.

3.5 Ward Identities

In previous chapter we derived the Ward identities corresponding to translation, rotation and scale invariance in $2.47, 2.51$ and 2.53 respectively. Let us assemble the three equations here:

$$\begin{aligned}
\partial_\mu \langle T^\mu{}_\nu X\rangle &= -\sum \delta(x-x_i) \frac{\partial}{\partial x_i^\nu}\langle X\rangle \\
\epsilon_{\rho\nu}\langle (T^{\rho\nu} - T^{\nu\rho})X\rangle &= -i\sum_i \delta(x-x_i)\ s_i\langle X\rangle \\
\langle T^\mu{}_\mu X\rangle &= -\sum_i \delta(x-x_i)\Delta_i\langle X\rangle
\end{aligned}$$

Here s_i stands for the spin of the field ϕ_i. Now we want to rewrite these equations in terms of complex coordinates and complex components. We use the expressions 3.2 and 3.3 for the metric and antisymmetric tensor. For the delta function $\delta(x)$, we use the identity (derived in Appendix [C])

$$\delta(x) = \frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z} = \frac{1}{\pi}\partial_z\frac{1}{\bar{z}} \tag{3.28}$$

One may in principle use either one of the above representations but we will use the first one if the integrand is holomorphic and vice-versa. The Ward identities can now be written as

$$
\begin{aligned}
2\pi\partial_z\langle T_{\bar{z}z}X\rangle + 2\pi\partial_{\bar{z}}\langle T_{zz}X\rangle &= -\sum_{i=1}^{n}\partial_{\bar{z}}\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle \\
2\pi\partial_z\langle T_{\bar{z}\bar{z}}X\rangle + 2\pi\partial_{\bar{z}}\langle T_{z\bar{z}}X\rangle &= -\sum_{i=1}^{n}\partial_{z}\frac{1}{\bar{z}-\bar{w}_i}\partial_{\bar{w}_i}\langle X\rangle \\
-2\langle T_{z\bar{z}}X\rangle + 2\langle T_{\bar{z}z}X\rangle &= -\sum_{i=1}^{n}\delta(x-x_i)\Delta_i\langle X\rangle \\
2\langle T_{z\bar{z}}X\rangle + 2\langle T_{\bar{z}z}X\rangle &= -\sum_{i=1}^{n}\delta(x-x_i)s_i\langle X\rangle
\end{aligned}
$$

We add and subtract the last two equations to get

$$
\begin{aligned}
2\pi\langle T_{\bar{z}z}X\rangle &= -\sum_{i=1}^{n}\partial_{\bar{z}}\frac{1}{z-w_i}h_i\langle X\rangle \\
2\pi\langle T_{z\bar{z}}X\rangle &= -\sum_{i=1}^{n}\partial_{z}\frac{1}{\bar{z}-\bar{w}_i}\bar{h}_i\langle X\rangle
\end{aligned}
\tag{3.29}
$$

where now the holomorphic and anti-holomorphic seperate and we use the appropriate definition of delta function from 3.28 and conformal dimensions from 3.18. Inserting these relations into the first two equations from 3.29, we get

$$
\begin{aligned}
\partial_{\bar{z}}\left(\langle T(z,\bar{z})X\rangle - \sum_{i=1}^{n}\left(\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle + \frac{h_i}{(z-w_i)^2}\langle X\rangle\right)\right) &= 0 \\
\partial_{z}\left(\langle \bar{T}(z,\bar{z})X\rangle - \sum_{i=1}^{n}\left(\frac{1}{\bar{z}-\bar{w}_i}\partial_{\bar{w}_i}\langle X\rangle + \frac{\bar{h}_i}{(\bar{z}-\bar{w}_i)^2}\langle X\rangle\right)\right) &= 0
\end{aligned}
\tag{3.30}
$$

where we have introduced the normalized energy-momentum tensor

$$
T(z,\bar{z}) = -2\pi T_{zz} \qquad \bar{T}(z,\bar{z}) = -2\pi T_{\bar{z}\bar{z}} \tag{3.31}
$$

Now by definition, the expressions inside braces in the above equations are holomorphic and anti-holomorphic respectively. Hence we can write

$$\langle T(z)X\rangle = \sum_{i=1}^{n}\left(\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle + \frac{h_i}{(z-w_i)^2}\langle X\rangle\right) + reg. \tag{3.32}$$

where $reg.$ refers to holomorphic functions of z, regular at $z = w_i$. And we will have a similar expression for anti-holomorphic term too.

3.6 Conformal Ward Identity

We can bring all the derived Ward identities 3.28 into a single equation as follows. Given a conformal transformation $x \to x + \epsilon(x)$, we can write

$$\begin{aligned}\partial_\mu(\epsilon_\nu T^{\mu\nu}) &= \epsilon_\nu\partial_\mu T^{\mu\nu} + \frac{1}{2}(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu)T^{\mu\nu} + \frac{1}{2}(\partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu)T^{\mu\nu} \\ &= \epsilon_\nu\partial_\mu T^{\mu\nu} + \frac{1}{2}(\partial_\rho\epsilon^\rho)\eta_{\mu\nu}T^{\mu\nu} + \frac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha\epsilon_\beta\varepsilon_{\mu\nu}T^{\mu\nu}\end{aligned} \tag{3.33}$$

where we have used

$$\begin{aligned}\frac{1}{2}(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu) &= \frac{1}{2}(\partial_\rho\epsilon^\rho)\eta_{\mu\nu} \\ \frac{1}{2}(\partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu) &= \frac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha\epsilon_\beta\varepsilon_{\mu\nu}\end{aligned} \tag{3.34}$$

We note here, the first term is the translation transformation, $\frac{1}{2}\partial_\rho\epsilon^\rho$ is the local scaling while $\frac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha\epsilon_\beta$ is the local rotation angle. Integrating both sides of the equation above, the three Ward identities can be encapsulated into

$$\delta_\epsilon\langle X\rangle = \int_M d^2x\partial_\mu\langle T^{\mu\nu}(x)\epsilon_\nu(x)X\rangle \tag{3.35}$$

where $\delta_\epsilon\langle X\rangle$ is the variation of X under local conformal transformation. Here integral is taken over a domain M which contains the position of all the fields in the string X. Since the integrand is a divergence, we use the Gauss's theorem to reduce this to the form

$$\delta_{\epsilon,\bar{\epsilon}}\langle X\rangle = \frac{i}{2}\int_C \{-dz\langle T^{\bar{z}\bar{z}}\epsilon_{\bar{z}}X\rangle + d\bar{z}\langle T^{zz}\epsilon_z X\rangle\} \tag{3.36}$$

where we have used 3.3 and 3.28. Further using the 3.31, we write down the conformal Ward identity:

$$\delta_{\epsilon,\bar{\epsilon}}\langle X\rangle = \frac{1}{2\pi i}\oint_C dz\ \epsilon(z)\langle T(z)X\rangle + \frac{1}{2\pi i}\oint_C d\bar{z}\ \bar{\epsilon}(\bar{z})\langle \bar{T}(\bar{z})X\rangle \tag{3.37}$$

The contour C needs to include all the positions $(w_i, \bar{w}_i)$ of the fields contained in X.

3.7 Operator Product Expansion

The equation 3.32 delivers an important information about the product of energy momentum tensor with primary fields. It yields a singular behaviour of the correlator of the field $T(z)$ with a primary field $\phi_i(w_i, \bar{w}_i)$, as $z \to w_i$. For a single primary field of conformal dimensions h and $\bar{h}$, this is written as

$$\begin{aligned} T(z)\phi(w,\bar{w}) &\sim \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w}) \\ \bar{T}(\bar{z})\phi(w,\bar{w}) &\sim \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\phi(w,\bar{w}) \end{aligned} \tag{3.38}$$

Here "$\sim$" means equality up to terms which are regular as $z \to w$. The operator product expansion (OPE) implies that two local operators inserted at nearby points can be closely approximated by a strings of operators at one of these points [5]. In general, if we denote all the operators of CFT by O_i , then the OPE is

$$O_i(z,\bar{z})O_j(w,\bar{w}) = \sum_k C_{ij}^k(z-w,\bar{z}-\bar{w})O_k(w,\bar{w}) \tag{3.39}$$

here C_{ij}^k are a set of functions which on ground of translation invariance depend only on the separation between two operators. It is important to note that the above equations are always to be understood as operator insertions between correlation functions. The singular behaviour of OPE as $z \to w$ will really be the only thing we care about! Just like in the OPE of energy momentum tensor with a primary field in 3.38. It will turn out to contain the same information as commutation relations, as well as telling us how operators transform under symmetries. We now proceed with specific example to familiarize ourselves with simple but important system and basic techniques.

3.8 The Free Boson

A simplest example of CFT is a free massless bosonic theory. The action for such a bosonic field is

$$S = \frac{1}{2} \int d^2x \; \partial_\mu \phi \; \partial^\mu \phi \tag{3.40}$$

The two-point function or the propagator for this action is given by

$$\langle \phi(x)\phi(y) \rangle = -\frac{1}{4\pi} ln(x-y)^2 + const. \tag{3.41}$$

This can be rewritten in terms of complex coordinates as

$$\langle \phi(z,\bar{z})\phi(w,\bar{w}) \rangle = -\frac{1}{4\pi} \left(ln(z-w) + ln(\bar{z}-\bar{w}) \right) + const. \tag{3.42}$$

The *log* means $\phi(z,\bar{z})$ doesn't transform as a primary field under conformal transformation. We can separate the holomorphic and the anti-holomorphic parts of above equation by taking derivatives with respect to holomorphic and anti-holomorphic coordinates, ∂_z and $\partial_{\bar{z}}$ and in the process we will discover important aspects of the theory

$$\begin{aligned} \langle \partial_z \phi(z,\bar{z}) \partial_w \phi(w,\bar{w}) \rangle &= -\frac{1}{4\pi} \frac{1}{(z-w)^2} \\ \langle \partial_{\bar{z}} \phi(z,\bar{z}) \partial_{\bar{w}} \phi(w,\bar{w}) \rangle &= -\frac{1}{4\pi} \frac{1}{(\bar{z}-\bar{w})^2} \end{aligned} \tag{3.43}$$

This OPE reflects the bosonic character of the fields as interchanging the two fields doesn't change the correlator. The energy momentum tensor for this theory is given by

$$T_{\mu\nu} = \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} \partial_\rho \phi \partial^\rho \phi \right) \tag{3.44}$$

In terms of complex coordinates, we write the energy-momentum defined in 3.31 as

$$T(z) = -2\pi : \partial\phi\partial\phi : \tag{3.45}$$

where we have normal ordered the fields. More explicitly, the above expression means

$$T(z) = -2\pi \lim_{w\to z} \left(\partial\phi(z)\partial\phi(w) - \langle\partial\phi(z)\partial\phi(w)\rangle\right) \tag{3.46}$$

We now find the OPE of $T(z)$ with $\partial\phi$. This is written as

$$T(z)\partial\phi(w) = -2\pi : \partial\phi(z)\partial\phi(z) : \partial\phi(w) \tag{3.47}$$

Using Wick's theorem and 3.43, this is

$$\sim \frac{\partial\phi(z)}{(z-w)^2} \tag{3.48}$$

then expanding $\partial\phi(z)$ around w, we get

$$T(z)\partial\phi(w) \sim \frac{\partial\phi(w)}{(z-w)^2} + \frac{\partial_w^2\phi(w)}{(z-w)} \tag{3.49}$$

This tells us $\partial\phi$ is a primary field with conformal dimension $h = 1$ which is true as ϕ has spin $s = 0$ and scaling dimension $\Delta = 0$ and the derivative operator ∂ has spin $s = 1$ and $\Delta = 1$.
We can also find the OPE of the energy-momentum tensor with itself

$$T(z)T(w) = 4\pi^2 : \partial\phi(z)\partial\phi(z) :: \partial\phi(w)\partial\phi(w) :$$

$$\sim \frac{1/2}{(z-w)^4} - \frac{4\pi : \partial\phi(z)\partial\phi(w) :}{(z-w)^2} \tag{3.50}$$

We get the first term by two double contractions and second term by four single contractions. Again by expanding the second term around w, we find the above equation is equivalent to

$$\sim \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \tag{3.51}$$

We note that the energy-momentum tensor is not a primary field as its OPE has an anomalous term which doesn't appear in 3.38

3.9 The Central Charge

In CFT, the most prominent example of an operator which is not primary is, as we saw earlier, the energy-momentum tensor. We worked it out for the free scalar model, but the fact remains true for all CFTs. It is in fact a quasi-primary operator of weight $(h, \bar{h}) = (2, 0)$. Generally in any CFT, the TT OPE takes the following form

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + ... \quad (3.52)$$

and similarly

$$\bar{T}(\bar{z})\bar{T}(\bar{w}) = \frac{\tilde{c}/2}{(\bar{z}-\bar{w})^4} + \frac{2\bar{T}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\bar{T}(\bar{w})}{(\bar{z}-\bar{w})} + ... \quad (3.53)$$

The constants c and $\tilde{c}$ are called the central charges. They are perhaps the most important numbers characterizing a CFT. We can already get some intuition for the information contained in these two numbers. Looking back at the free scalar field, we see that it has $c = \tilde{c} = 1$. If we instead considered D non-interacting free scalar fields, we would get $c = \tilde{c} = D$. This gives us a hint: c and $\tilde{c}$ are somehow measuring the number of degrees of freedom in the CFT. Note that c is not necessarily an integer i.e. $c = 1/2$ for a free fermion theory.

From the definition it seems natural that the energy-momentum tensor should transform as a rank-2 covariant tensor,

$$T(z) \to T'(w) = \left(\frac{dw}{dz}\right)^{-2} T(z) \quad (3.54)$$

under a transformation $z \to w(z)$. However it turns out, because of the presence of the anomalous term in the TT OPE, the above equation is corrected to

$$T'(w) = \left(\frac{dw}{dz}\right)^{-2} [T(z) - \frac{c}{12}\{w; z\}] \quad (3.55)$$

where c is the central charge and

$$\{w; z\} = \frac{d^3w/dz^3}{dw/dz} - \frac{3}{2}\left(\frac{d^2w/dz^2}{dw/dz}\right)^2 \quad (3.56)$$

is called the Schwarzian derivative. This additional term appearing during the transformation of T is called the Schwinger term. Notice that the

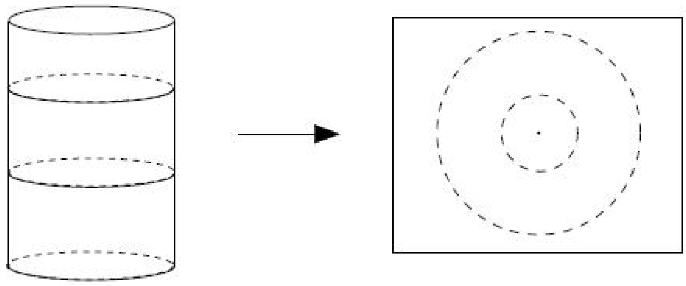

Figure 3.1: Conformal mapping from cylinder to complex plane

Schwarzian derivative vanishes for a global conformal map which is true for a quasi-primary field. Now the Schwinger term is independent of T and only effects the constant term or the zero mode in the energy. In other words, it is related to the Casimir energy as we will shortly explain.

We look at an example to understand the physical significance of the central charge. Consider a Euclidean cylinder, parameterized by

$$w = \sigma + i\tau, \qquad \sigma \in [0, 2\pi] \tag{3.57}$$

We make a conformal transformation from cylinder to a complex plane by

$$z = e^{-iw} \tag{3.58}$$

The fact that the cylinder and the plane are related by a conformal map means that if we understand a given CFT on the cylinder, then we immediately understand it on the plane. Notice that constant time slices on the cylinder are mapped to circles of constant radii. The origin, $z = 0$, is the distant past, $\tau \to -\infty$.

Now the Schwarzian for this mapping can be calculated and it gives, $S(z, w) = 1/2$. This gives us the way T transforms

$$T_{cylinder}(w) = -z^2 T_{plane}(z) + \frac{c}{24} \tag{3.59}$$

Now suppose the ground state energy vanishes when the theory is defined on the plane, $\langle T_{plane} \rangle = 0$. We then calculate the energy on the cylinder

$$H = \int d\sigma T_{\tau\tau} = -\int d\sigma (T_{ww} + \bar{T}_{\bar{w}\bar{w}}) \tag{3.60}$$

The conformal transformation then tells us that the ground state energy on the cylinder is

$$E = -\frac{2\pi(c + \tilde{c})}{24} \tag{3.61}$$

This is indeed the (negative) Casimir energy on a cylinder. For a free scalar field, we have $c = \tilde{c} = 1$ and the energy density $E/2\pi = -1/12$.

Chapter 4

The Operator Formalism

Throughout the previous sections, all our manipulations were assumed to hold inside correlation functions. The consequences of conformal symmetry on two-dimensional field theories were embodied in constraints imposed on these correlation functions via the Ward identities. These Ward identities were most easily expressed in the form of an OPE of the energy-momentum tensor with local fields. Up to now, we only used the path-integral representation of the theory in which all correlation functions could in principle be obtained. We would now like to give an operator interpretation in terms of states in a Hilbert space.

4.1 Radial Quantization

The operator formalism distinguishes a time direction from a space direction. This in natural in Minkowski space-time, but somewhat arbitrary in Euclidean space. This allows to choose the radial direction from the origin as time direction, and the space direction being orthogonal to it. This choice leads to the so-called radial quantization of two-dimensional conformal field theories.

We may start from a two-dimensional Minkowski space with coordinates t and σ. One usually takes the space direction σ to be periodic, $\sigma \in 2[0, L]$, defining this way the theory on a cylinder. We continue to Euclidean space, $t = -i\tau$, and then perform the conformal transformation

$$z = e^{(\tau + i\sigma)\frac{2\pi}{L}} \quad , \quad \bar{z} = e^{(\tau - i\sigma)\frac{2\pi}{L}} \tag{4.1}$$

which maps the cylinder onto the complex plane $C \cup \{\infty\}$, topologically the Riemann sphere. Surfaces of equal Euclidean time τ on the cylinder will

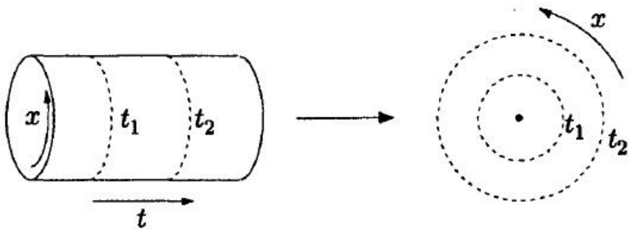

Figure 4.1: Conformal mapping sends constant time to constant radii

become circles of equal radii on the complex plane. This means that the infinite past $\tau = -\infty$ gets mapped onto the origin of the plane (z = 0) and the infinite future becomes $z = \infty$. Time reversal becomes $z \to z^*(= e^{-\tau+i\sigma})$ on the complex plane.

We assume the existence of a vacuum state $|0\rangle$ upon which a Hilbert space is constructed by application of creation operators. In free field theories, the vacuum may be defined as the state annihilated by the positive frequency part of the field. For an interacting field ϕ, we assume the same Hilbert space except that the actual energy eigenstates are different. We suppose then that the interaction is attenuated as $t \to \pm\infty$ and the asymptotic field

$$\phi = \lim_{t \to -\infty} \phi(x, t) \tag{4.2}$$

is free. Within radial quantization, this asymptotic field reduces to a single operator, which upon acting on $|0\rangle$, creates a single asymptotic "in" state:

$$|\phi_{in}\rangle = \lim_{z, \bar{z} \to 0} \phi(z, \bar{z})|0\rangle \tag{4.3}$$

We define a bilinear product in this Hilbert space. This can be done by defining an asymptotic "out" state together with action of Hermitian conjugation on the conformal fields. In Minkowski space, Hermitian conjugation does not effect the space-time coordinates. However in Euclidean space since the time $\tau = it$, it must be reversed upon Hermitian conjugation if t is to be left unchanged. In radial quantization this can be done by mapping $z \to 1/z^*$. This will justify the following definition of Hermitian conjugation on the real surface $\bar{z} = z^*$

$$[\phi(z, \bar{z})]^\dagger = \bar{z}^{-2h} z^{-2\bar{h}} \phi(1/\bar{z}, 1/z) \tag{4.4}$$

where ϕ is a quasi-primary field of dimensions h and $\bar{h}$. The above equation can be justified by demanding that the asymptotic "out" state

$$\langle \phi_{out} | = | \phi_{in} \rangle^\dagger \tag{4.5}$$

has a well defined inner product with the $|\phi_{in}\rangle$. Using the above defined formula for Hermitian conjugation, we get the inner product to be

$$\begin{aligned}
\langle \phi_{out} | \phi_{in} \rangle &= \lim_{z,\bar{z},w,\bar{w}\to 0} \langle 0 | \phi(z,\bar{z})^\dagger \phi(w,\bar{w}) | 0 \rangle \\
&= \lim_{z,\bar{z},w,\bar{w}\to 0} \bar{z}^{-2h} z^{-2\bar{h}} \langle \phi(1/\bar{z}, 1/z) \phi(w,\bar{w}) | 0 \rangle \\
&= \lim_{\rho,\bar{\rho}\to\infty} \bar{\rho}^{2h} \rho^{2\bar{h}} \langle 0 | \phi(\bar{\rho},\rho) \phi(0,0) | 0 \rangle
\end{aligned}$$

Now according to form of two-point function as shown in 3.24, the above expression is independent of ρ. This justifies the presence of the prefactors in defining the Hermitian conjugation for if they were absent, the inner product $\langle \phi_{out} | \phi_{in} \rangle$ would not have been well defined as $\rho \to \infty$.

4.2 Mode Expansion

A conformal field $\phi(z,\bar{z})$ of dimensions $(h,\bar{h})$ can be expanded in modes in following manner:

$$\begin{aligned}
\phi(z,\bar{z}) &= \sum_{m\in Z} \sum_{n\in Z} z^{-m-h} \bar{z}^{-n-\bar{h}} \phi_{m,n} \\
\phi_{m,n} &= \frac{1}{2\pi i} \oint dz z^{m+h-1} \frac{1}{2\pi i} \oint d\bar{z} \bar{z}^{m+\bar{h}-1} \phi(z,\bar{z})
\end{aligned} \tag{4.6}$$

Taking the Hermitian conjugate, we get

$$\phi(z,\bar{z})^\dagger = \sum_{m\in Z} \sum_{n\in Z} \bar{z}^{-m-h} z^{-n-\bar{h}} \phi^\dagger_{m,n} \tag{4.7}$$

However the definition 4.4 gives

$$\begin{aligned}\phi(z,\bar{z})^{\dagger} &= \bar{z}^{-2h}z^{-2\bar{h}}\phi(1/\bar{z},1/z) \\ &= \bar{z}^{-2h}z^{-2\bar{h}}\sum_{m\in Z}\sum_{n\in Z}\phi_{m,n}\bar{z}^{m+h}z^{n+\bar{h}} \\ &= \sum_{m\in Z}\sum_{n\in Z}\phi_{-m,-n}\bar{z}^{-m-h}z^{-n-\bar{h}} \qquad (4.8)\end{aligned}$$

For the above two expression for the Hermitian conjugate of modes to match, we must have

$$\phi^{\dagger}_{m,n} = \phi_{-m,-n} \qquad (4.9)$$

Also if the "in" and "out" are to be well-defined, the vacuum state must satisfy

$$\phi_{m,n}|0\rangle = 0 \qquad (m+h>0, n+\bar{h}>0) \qquad (4.10)$$

We use the fact that holomorphic and anti-holomorphic degrees of freedom decouple and we will drop the dependence of fields upon the anti-holomorphic coordinates for sake of ease. We rewrite the mode expansion in simplified form

$$\phi(z) = \sum_{m\in Z} z^{-m-h}\phi_m \qquad (4.11)$$

and

$$\phi_m = \frac{1}{2\pi i}\oint dz z^{m+h-1}\phi(z) \qquad (4.12)$$

4.3 Radial Ordering

In a quantum field theory, we are interested in time-ordered correlation functions. Time ordering on the cylinder becomes radial ordering on the plane. Operators in correlation functions are ordered so that those inserted at larger

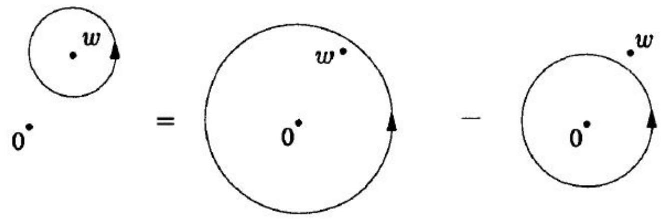

Figure 4.2: Subtraction of contours

radial distance are moved to the left. The definition of radially ordered correlation functions becomes

$$R\phi_1(z)\phi_2(w) = \begin{cases} \phi_1(z)\phi_2(w) \text{ if } |\mathrm{z}| > |\mathrm{w}| \\ \phi_2(w)\phi_1(z) \text{ if } |\mathrm{z}| < |\mathrm{w}| \end{cases} \tag{4.13}$$

If the two fields are fermions, a minus sign is added in front of the second expression. Since all fields within correlation functions must be radially ordered, so must be the l.h.s. of any OPE if it is to have an operator meaning. We now relate OPEs to commutation relations. Let $a(z)$ and $b(z)$ be two holomorphic fields, and consider the integral

$$\oint_w dz\ a(z)b(w) \tag{4.14}$$

wherein the integration contour circles counterclockwise around w. This expression has an operator meaning within correlation function as long as it is radially ordered. Accordingly we split the contour into two fixed time circles going in opposite directions. Then the above integral resembles a commutator:

$$\oint_w dz\ a(z)b(w) = \oint_{C_1} dz\ a(z)b(w) - \oint_{C_2} dz\ b(w)a(z)$$

$$= [A, b(w)]$$

where C_1 and C_2 are the fixed time contour of radii respectively equal to

$|w| + \epsilon$ and $|w| - \epsilon$, ϵ being infinitesimal and the operator A is the integral over fixed time of the field $a(z)$:

$$A = \oint dz\ a(z) \tag{4.15}$$

We take $\epsilon \to 0$ as $b(w)$ should be the only field present between the two contours as in general there can be arbitrary number of fields between $a(z)$ and $b(w)$. This makes the commutator an equal-time commutator. In practice, the integral in 4.14 can be evaluated by substituting the OPE of $a(z)$ and $b(w)$, of which on the term $1/(z-w)$ contributes, by theorem of residues.

The commutator [A,B] of two operators, each the integral of a holomorphic field is obtained by integrating 4.15 over w:

$$[A, B] = \oint_0 dw \oint_w dza(z)b(w) \tag{4.16}$$

where the integral over z is taken around w and the integral over w is around the origin. Also

$$A = \oint dz\ a(z) \qquad B = \oint dw\ b(w) \tag{4.17}$$

We have managed to find a relation between OPEs and commutators and this allows us to translate into operator language the symmetry statements contained in the OPEs.

4.4 Virasoro Algebra and Conformal Generators

Let $\epsilon(z)$ be the holomorphic component of an infinitesimal conformal change in the coordinates. We define the conformal charge as

$$Q_\epsilon = \frac{1}{2\pi i} \oint dz\ \epsilon(z)T(z) \tag{4.18}$$

Now from the conformal Ward identity 3.37,

$$\delta_\epsilon \Phi(w) = \frac{1}{2\pi i} \oint_w dz\epsilon(z)T(z)\Phi(w) \tag{4.19}$$

Using 4.15, this identity translates into

$$\delta_\epsilon \Phi(w) = -[Q_\epsilon, \Phi(w)] \tag{4.20}$$

which means that the operator Q_ϵ is the generator of conformal transformation.

We now expand the energy-momentum tensor according to 4.6:

$$\begin{aligned} T(z) &= \sum_{n\in Z} z^{-n-2} L_n \qquad L_n = \frac{1}{2\pi i}\oint dz\ z^{n+1} T(z) \\ \bar{T}(\bar{z}) &= \sum_{n\in Z} \bar{z}^{-n-2} \bar{L}_n \qquad \bar{L}_n = \frac{1}{2\pi i}\oint d\bar{z}\ \bar{z}^{n+1} \bar{T}(\bar{z}) \end{aligned} \tag{4.21}$$

We also expand the infinitesimal conformal change $\epsilon(z)$ as

$$\epsilon(z) = \sum_{n\in Z} z^{n+1}\epsilon_n \tag{4.22}$$

We can then write the conformal charge 4.18 in terms of these modes as

$$Q_\epsilon = \sum_{n\in Z} \epsilon_n L_n \tag{4.23}$$

This displays that the mode operators L_n and $\bar{L}_n$ of the energy momentum operator are the generators of the local conformal transformations on the Hilbert space, exactly like the l_n and $\bar{l}_n$ of 3.14 are generators of conformal mappings on the space of functions. Similarly L_{-1}, L_0 and L_1 are the generators of SL(2,C) . In particular the operator $L_0 + \bar{L}_0$ generates dilation $(z, \bar{z}) \to \lambda(z, \bar{z})$ which is in fact time translation in radial quantization. Thus $L_0 + \bar{L}_0$ is proportional to the Hamiltonian of the system. These quantum generators obey the following algebra

$$
\begin{aligned}
&[L_m, L_n] = (n-m)L_{m+n} + \frac{c}{12}n(n^2-1)\delta_{m+n,0} \\
&[L_n, \bar{L}_m] = 0 \\
&[\bar{L}_m, \bar{L}_n] = (n-m)\bar{L}_{m+n} + \frac{c}{12}n(n^2-1)\delta_{m+n,0}
\end{aligned}
\tag{4.24}
$$

which is similar to the classical Witt algebra except for the appearance of central charge of the theory. This is called the Virasoro algebra. Notice that only the global subgroup SL(2,C) is not effected by the central charge.

4.5 The Hilbert Space and Verma Module

The vacuum state $|0\rangle$ must be invariant under global conformal transformation. This means that it must be annihilated L_{-1}, L_0 and L_1 which in turn can be recovered by requiring the regularity of $T(z)|0\rangle$ at $z = 0$. Evidently only the terms with $n < -1$ are allowed. So we learn that

$$
\begin{aligned}
&L_n|0\rangle = 0, \qquad n \geq -1 \\
&\bar{L}_n|0\rangle = 0, \qquad n \geq -1
\end{aligned}
\tag{4.25}
$$

This includes the invariance of vacuum $|0\rangle$ with respect to the global conformal group. It also implies the vanishing of the vacuum expectation value of the energy momentum tensor:

$$
\langle 0|T(z)|0\rangle = \langle 0|\bar{T}(\bar{z})|0\rangle = 0 \tag{4.26}
$$

Action of primary fields on vacuum state creates asymptotic states which are eigenstates of the Hamiltonian. To show this, let us first find the commutator of L_n with a primary field $\phi(z, \bar{z})$ of conformal dimensions $(h, \bar{h})$.

$$
\begin{aligned}
[L_n, \phi(w, \bar{w})] &= \frac{1}{2\pi i}\oint_w dz\ z^{n+1} T(z)\phi(w, \bar{w}) \\
&= \frac{1}{2\pi i}\oint_w dz\ z^{n+1}[\frac{h\phi(w, \bar{w})}{(z-w)^2} + \frac{\partial\phi(w, \bar{w})}{z-w} + reg.] \\
&= h(n+1)w^n\phi(w, \bar{w}) + w^{n+1}\partial\phi(w, \bar{w}) \qquad (4.27)
\end{aligned}
$$

The anti-holomorphic counterpart of this equation gives,

$$
[\bar{L}_n, \phi(w, \bar{w})] = \bar{h}(n+1)\bar{w}^n\phi(w, \bar{w}) + \bar{w}^{n+1}\bar{\partial}\phi(w, \bar{w}) \qquad (4.28)
$$

Now using the fact

$$
|h, \bar{h}\rangle = \phi(0, 0)|0\rangle \qquad (4.29)
$$

we apply both sides of 4.27 on the vacuum $|0\rangle$ for $n = 0$:

$$
[L_0, \phi(0, 0)]|0\rangle = h\phi(0, 0)|0\rangle
$$

$$
\Rightarrow L_0\phi(0, 0)|0\rangle - \phi(0, 0)L_0|0\rangle = h|h, \bar{h}\rangle
$$

$$
\Rightarrow L_0|h, \bar{h}\rangle = h|h, \bar{h}\rangle \qquad (4.30)
$$

And similarly

$$
\bar{L}_0|h, \bar{h}\rangle = \bar{h}|h, \bar{h}\rangle \qquad (4.31)
$$

Thus $|h, \bar{h}\rangle$ is an eigenstate of the Hamiltonian. Similarly, we have

$$
\begin{aligned}
L_n|h, \bar{h}\rangle &= 0 \\
\bar{L}_n|h, \bar{h}\rangle &= 0 \qquad (4.32)
\end{aligned}
$$

For $n > 0$. Now to find the ladder operators which on application give the excited states above the asymptotic state $|h, \bar{h}\rangle$, we compute the commutators of the modes of a holomorphic field $\phi(w)$ with L_ns. We find $[L_n, \phi_m]$ by multiplying 4.27 with w^{m+h-1} and then integrating over w. This gives

$$[L_n, \phi_m] = [n(h-1) - m]\phi_{m+n} \tag{4.33}$$

which for $n = 0$ gives

$$[L_0, \phi_m] = -m\phi_m \tag{4.34}$$

This means that

$$L_0\phi_m|h, \bar{h}\rangle = (h - m)\phi_m|h, \bar{h}\rangle \tag{4.35}$$

ϕ_m reduces the conformal dimension of the state by m. Similarly ϕ_{-m} acts as a raising operator.
The generators $L_{-m}, (m > 0)$ also increase the conformal dimension as we have from 4.24

$$[L_0, L_{-m}] = mL_{-m} \tag{4.36}$$

This means that excited states can by obtained by successive applications of these operators on the asymptotic state $|h\rangle$. This results in an infinite tower of states and all the states obtained in this way are called the 'descendants' of state $|h\rangle$. From this initial primary state, the tower fans out:

$$\begin{aligned} |h\rangle \\ L_{-1}|h\rangle \\ L_{-1}^2, L_{-2}|h\rangle \\ L_{-1}^3, L_{-1}L_{-2}, L_{-3}|h\rangle \end{aligned}$$

The whole set of states is called the 'Verma module'. They are the irreducible representations of the Virasoro algebra. This means that if we know the spectrum of primary states, then we know the spectrum of the whole theory. We are not guaranteed however that all the above states are linearly independent. That depends on the structure of the Virasoro algebra for given values of h and c. A linear combination of states that vanishes is known as a

null state, and the representation of the Virasoro algebra with highest weight is constructed from the above Verma module by removing all null states (and their descendants).
At N_{th} level in this tower, there will be P(N) fields with conformal dimension $h+N$, where P(N) is the number of partitions of N into positive integer parts. P(N) is given in terms of generating function

$$\frac{1}{\prod_{n=1}^{\infty}(1-q^n)} = \sum_{N=0}^{\infty} P(N) q^N \tag{4.37}$$

where P(0) = 1.

Chapter 5

Discussions

In this thesis, we investigated the structure of Conformal Field Theory in $2d$ which is a recipe for further studies of critical behavior of various systems in statistical mechanics and quantum field theory. We briefly reviewed CFT in d dimensions which plays a prominent role for example in the well-known duality AdS/CFT in string theory where the CFT lives on the AdS boundary.

We studied the generators of conformal transformations and derived their commutation relations. The associated group in d dimensions was identified with the non-compact group $SO(d+1,1)$. The notion of a "quasi primary field" was developed which are covariant under global conformal transformations. Using the constraints of conformal symmetry, we derived the forms of $2-, 3-$ and $4-$point correlation functions. We further derived the Ward identities associated with the translation, rotation and scale invariance of the theory.

Main goal of our study was to develop the structure of a conformal field theory in 2 dimensions. 2 dimension case is special as the conformal symmetry becomes a local symmetry of theory and we have an infinite set of mapping from the 2d complex plane onto itself. The earlier derived group of global conformal symmetry turns out to be a subset of the 2d CFT algebra. We developed the essential language of holomorphic and anti-holomorphic coordinates which is a very helpful tool in the study of 2d CFT. We defined the notion of "primary fields" and evaluated their correlation functions. We also introduced the concept of short-distance product of operators and applied this language to specific example of free bosons. Transformation properties of the energy-momentum tensor were explored for this example and the idea of central charge c was explained in brief.

In the last chapter, we mapped our theory from the cylinder to a complex plane which helped us gain an insight into the process of radial quantization and radial ordering. Then carrying out the mode expansion of fields led to development of the "Virasoro algebra". Once we had the algebra and the generators in Hilbert space, we could develop the whole tower of descendant fields given a primary field. This infinite tower of states generated from a given asymptotic state form a representation of the Virasoro algebra which is the well known "Verma module".

The last chapter ends with an introduction to the Verma module which provides the basic tool to study particularly simple conformal field theories called 'minimal models'. These theories are characterized by a Hilbert space made of a finite number of representations of Virasoro algebra and are used to describe discrete statistical models (e.g.Ising model) at their critical points.

Appendix A: Conformal Generators

Let us write down the change in the distance x under the earlier discussed infinitesimal conformal transformations:

$$
\begin{aligned}
Translation\ \ & \delta x^\mu = a^\mu \\
Dilation\ \ & \delta x^\mu = \lambda x^\mu \\
Rotation\ \ & \delta x^\mu = w^\mu{}_\nu x^\nu \\
SCT\ \ & \delta x^\mu = 2(x.b)x^\mu - b^\mu x^2
\end{aligned}
$$

Now we will demonstrate how the conformal generators when exponentiated and operated upon the coordinates reproduce exactly the above given expressions. Given a generator $G(x)$, we know the coordinates transform as:

$$x'^\mu = e^{iaG(x)} x^\mu \tag{1}$$

which, for an infinitesimal transformation gives us

$$\delta x^\mu = iaG(x)x^\mu \tag{2}$$

We first check for the translation generator P_μ by inserting its form in the above equation. Now

$$P_\mu = -i\partial_\mu \tag{3}$$

which, when inserted in 2 gives

$$\delta x^\mu = ia^\nu(-i\partial_\nu)x^\mu = a^\mu \tag{4}$$

which is what we expect. Now let's work out the same steps with other generators. Dilation generator $D = -ix^\mu\partial_\mu$, which gives

$$\delta x^\mu = ia(-ix^\nu\partial_\nu)x^\mu = ax^\mu \tag{5}$$

again same as we expected. Repeating the steps using rotation generator, $L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$ -

$$\delta x^\mu = i w^{\alpha\beta}(i x_\alpha \partial_\beta - i x_\beta \partial_\alpha) x^\mu = W^{\mu\nu} x_\nu \tag{6}$$

where W is introduced to absorb the factor of 2 and the minus sign. The above equation is similar to the form of δx^μ for rotation showed above. Finally we verify the generator formula for the special conformal transformation. SCT generator is given as $K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)$. Inserting this in 2,

$$\delta x^\mu = i b^\alpha(-i(2x_\alpha x^\nu \partial_\nu - x^2 \partial_\alpha)) x^\mu$$

$$= b^\alpha(2x_\alpha x^\mu - x^2 \delta^\mu_\alpha)$$

$$= 2(b.x)x^\mu - b^\mu x^2 \tag{7}$$

which shows the correctness of the generator formula.

Appendix B: Ward Identity for current

An infinitesimal transformation may be written in terms of the generators as

$$\Phi'(x) = \Phi(x) - iw_a G_a \Phi(x) \tag{8}$$

where w_a is a set of infinitesimal constant parameters. For a given correlation function, if we make a change of functional integration variables in the above defined manner, the function changes as

$$\langle X' \rangle = \frac{1}{Z} \int [\Phi'](X + \delta X) \; exp - \left(S[\Phi] + \int dx \; \partial_\mu j_a^\mu w_a(x) \right) \tag{9}$$

We assume the functional integration measure is invariant under local transformation (i.e.$[d\Phi'] = [d\Phi]$). Expanding the equation above to first order in $w_a(x)$, we get:

$$\langle \delta X \rangle = \int dx \; \partial_\mu \langle j_a^\mu(x) X \rangle w_a(x) \tag{10}$$

Now the variation of X can also be explicitly written as

$$\begin{aligned} \langle \delta X \rangle &= -i \sum_{i=1}^{n} \langle \Phi(x_1)...G_a\Phi(x_i)...\Phi(x_n) \rangle w_a(x_i) \\ &= -i \int dx \; w_a(x) \sum_{i=1}^{n} \{ \Phi(x_1)...G_a\Phi(x_i)...\Phi(x_n) \rangle \} \delta(x - x_i) \end{aligned} \tag{11}$$

Now 10 satisfies for any infinitesimal function $w_a(x)$, so we can write, using 10 and 11, the Ward identity for j_a^μ

$$\partial_\mu \langle j_a^\mu(x) \Phi(x_1)...\Phi(x_n) \rangle = -i \sum_{i=1}^{n} \delta(x - x_i) \langle \Phi(x_1)...G_a\Phi(x_i)...\Phi(x_n) \rangle \tag{12}$$

Appendix C: Delta Function in complex coordinates

In terms of holomorphic and anti-holomorphic coordinates, delta function $\delta(x)$ can be written in following way

$$\delta(x) = \frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z} = \frac{1}{\pi}\partial_{z}\frac{1}{\bar{z}} \tag{13}$$

To justify this identity, let us consider a vector F^{μ} whose divergence is integrated within a region M of the complex plane bounded by a contour ∂M. Applying the Gauss's theorem, this gives us

$$\int_M d^2x \partial_\mu F^\mu = \int_{\partial M} d\xi_\mu F^\mu \tag{14}$$

where ξ is an outward-directed differential of circumference, orthogonal to the boundary ∂M. Let us use a counterclockwise differential ds^ρ, parallel to $\partial M \to d\xi_\mu = \varepsilon_{\mu\rho} ds^\rho$. In terms of complex coordinates we write the above equation as

$$\begin{aligned}\int_M d^2x \partial_\mu F^\mu &= \int_{\partial M} \{dz \varepsilon_{\bar{z}z} F^{\bar{z}} + d\bar{z} \varepsilon_{z\bar{z}} F^z\} \\ &= \frac{1}{2} i \int_{\partial M} \{-dz F^{\bar{z}} + d\bar{z} F^z\}\end{aligned} \tag{15}$$

where the contour ∂M circles counterclockwise. We now consider a holomorphic function $f(z)$ and check if the above defined delta function works fine by integrating it against $f(z)$ within a neighbourhood M of the origin:

$$
\begin{aligned}
\int_M d^2x\delta(x)f(z) &= \frac{1}{\pi}\int_M d^2x f(z)\partial_{\bar{z}}\frac{1}{z} \\
&= \frac{1}{\pi}\int_M d^2x\partial_{\bar{z}}\left(\frac{f(z)}{z}\right) \\
&= \frac{1}{2\pi i}\int_{\partial M} dz\frac{f(z)}{z} \\
&= f(0)
\end{aligned} \tag{16}
$$

which displays the correctness of our delta function. We can similarly prove the second representation by using an antiholomorphic function $\bar{f}(\bar{z})$ instead.

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