Studies in $3d$ QFT and $5d$ gravity

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Declaration

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

The work was done under the guidance of Prof. Shiraz Minwalla, at the Tata Institute of Fundamental Research, Mumbai.

V Umesh

[Candidate's name and signature]

In my capacity as supervisor of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

Prof. Shiraz Minwalla

[Guide's name and signature]

Date:

To the great thinkers of this ancient land,

who delved into the recesses of their mind:

कस्मिन्नु भगवो विज्ञाते सर्वमिदं विज्ञातं भवति।

What is That, through the knowledge of which, everything becomes known?
Mundakopanishad, 1.1.3

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Chapter 0

Synopsis

0.1 Introduction

Quantum field theories (QFTs) form the bedrock on which our understanding of nature is based. They are ubiquitous in physics, underlying most of the phenomena ranging from high energy particle physics to condensed matter many body physics. The diverse range of application of QFTs can hardly be overstated - much like calculus - which leads one to believe that it is a very deep idea. Whenever it has been supposed that QFTs have outlived their usefulness, the field has made remarkable comebacks and surprised the physics community with its depth, power and richness. To quote one historical example, when it was thought that QFTs are pretty much useless for describing strongly interacting phenomena - as was the case when numerous strongly interacting particles were discovered - non-abelian gauge theories, with their remarkable property of asymptotic freedom finally provided the correct UV description of the QCD sector of the standard model of particle physics. Not to mention that the standard model itself is a special example of a non-abelian gauge theory. The remarkable ways in which QFTs have changed our understanding of nature - which continues to date - warrants their investigation in as much detail as possible.

The main tool which is used extensively in the study of QFTs is perturbation theory - where one starts with a free field vacuum and constructs the interacting theory Hilbert space order by order in an expansion parameter which is small. Although many powerful and major results have been obtained this way, many interesting phenomena are in regimes where this approximation scheme fails to be of any use. Also, there are examples of QFTs which do not posses any parameter that can be made small so that one may identify a free limit. In the vast literature on QFT, there are very few instances known where one has been able to extract exact results and therefore have an access to strongly coupled phenomena. Thus, exact results in QFTs should be taken seriously when available and studied in detail so that we can understand phenomena beyond perturbation theory and learn qualitative lessons.

This thesis is devoted to three different studies of QFT. In the next section we report the results of the study of the exact S-matrices of supersymmetric Chern-Simons theories with matter in the fundamental representation. This is based on a recently completed work with various collaborators [1]. We have computed the S-matrix for 2×2 scattering to all orders in the 't Hooft coupling (in the large-N limit) and our results are in perfect agreement with the conjectured duality of [2, 3], which confirms that this is a bose-fermi duality. Even more striking, perhaps, is the requirement of modified crossing symmetry rules to achieve unitarity, first observed in [4].

One motivation to understand supersymmetric Chern-Simons-matter theories is to extend the results of [5, 6] to the superconformal context. In the subsequent section, we elaborate the results of a work with collaborators [7] where we discuss superconformal field theories with fundamental matter and construct the spectrum of single trace operators of free theories. We present a conjecture for the three point function of such operators and provide evidence for the same. This study sets the stage for further investigation of superconformal Chern-Simons-matter theories so that one can understand better theories with bi-fundamental matter such as ABJ [8] and ABJM [9].

In the final section we take a different route - we investigate the Einstein-Maxwell system in AdS_5. Albeit this might at first glance seem removed from the main thrust of this thesis, this is not the case as explained below. Motivated by the remarkable AdS/CFT correspondence [10, 11, 12] - which posits that a quantum field theory (at strong coupling) is dual to gravity - this study is the study of QFTs too, from a different (and very surprising) vantage point. This deep correspondence implies the equivalence of QFTs and gravity. It is of interest therefore, to understand gravity in AdS space, where the field theory duals provide for interesting interpretations of the strongly coupled regime. We elaborate on the construction of the 'hairy black hole' solution in perturbation theory and analyse the phase diagram of this system. The hairy black hole is interpreted in terms of the boundary field theory in terms of a finite temperature system with a bose condensate. This section is based on work with collaborators appearing in [13]. We conclude with some general remarks and outlook.

0.2 Exact S-matrix of supersymmetric Chern-Simons-matter theories

We study Chern-Simons-matter theories with matter in the fundamental representation which possess $\mathcal{N} = 1, 2$ supersymmetry. This work was completed with Karthik Inbasekar, Sachin Jain, Subhajit Mazumdar, Shiraz Minwalla and Shuichi Yokoyama, and appears in [1]. We compute the 2×2 S-matrix for these theories and present explicit expressions for the exact (to all orders in the 't Hooft coupling $\lambda = N/\kappa$) S-matrix (in the large-N limit) in these theories by summing all the planar graphs for the process in question. We verify the strong-weak bose-fermi duality proposed in [3] for the (supersymmetric) S-matrix. We analyse the unitarity of the singlet channel amplitude and verify that various conjectures made in [4] in the non-supersymmetric case continue to hold in these theories as well. Preliminary analysis of the results obtained suggests that extension to higher supersymmetry is straightforward.

Supersymmetric Chern-Simons-matter theories are interesting for several reasons. Theories like ABJM are dual to string theory, and play a very important role in the AdS/CFT correspondence. A proper understanding of scattering in supersymmetric Chern-Simons-matter theories is therefore vital and may help in shedding light on various puzzling features of scattering amplitudes (viz. unitarity) in theories like ABJM. The properties of amplitudes observed in [4] seem to be universally shared by all theories with Chern-Simons gauge fields interacting with vector matter. Also, these theories possess interesting dualities between bosons and fermions and one can hope to achieve an analytic understanding because of the availability of exact results of computable quantities (to all orders in the coupling). Apart from this, supersymmetric Chern-Simons-matter theories are dual to (supersymmetric) Vasiliev theories of higher spin in AdS_4, which may be thought of as a highly symmetric phase of string theory [14]. One can therefore hope to understand features of the bulk higher-spin dual theories through studies of Chern-Simons-matter theories.

Furthermore, Chern-Simons-matter theories have many applications in the field of condensed matter physics. Particles in $2+1$ dimensions have have anyonic statistics - these have been shown to be important in condensed matter systems [15, 16, 17, 18, 19, 20] - and therefore, study of these field theories makes contact with anyonic physics. The presence of anyonic particles can also be used to explain, at least heuristically, the unexpected transformation of the S-matrix under crossing symmetry. Motivated by these considerations, we proceed to investigate the S-matrix of these supersymmetric Chern-Simons-matter theories.

The $\mathcal{N}=1$ supersymmetric theory is described by the classical action (written in supersymmetric light cone gauge $\Gamma_-=0$)

$$S_{tree} = -\int d^3x d^2\theta \Big[-\frac{\kappa}{8\pi}\mathrm{Tr}(\Gamma^- i\partial_{--}\Gamma^-) - \frac{1}{2}D^\alpha\bar{\Phi}D_\alpha\Phi - \frac{i}{2}\Gamma^-(\bar{\Phi}D_-\Phi - D_-\bar{\Phi}\Phi) + m_0\bar{\Phi}\Phi + \frac{4\pi w}{4\kappa}(\bar{\Phi}\Phi)^2 \Big] . \tag{0.1}$$

The above action is written in $\mathcal{N}=1$ superspace which consists of an ordinary bosonic spacetime co-ordinate x^μ and a real Grassmannian spinor co-ordinate θ_α. This theory contains a matter superfield Φ in the fundamental of the gauge group and interacts with a gauge field Γ_α and has a quartic self interaction as well. These superfields are arbitrary functions of the spacetime co-ordinates x^μ as well as the superspace co-ordinates θ_α. This theory has a well defined large-N limit with 't Hooft coupling $\lambda = N/\kappa$. This theory has a conjectured duality [3] to a theory with parameters λ', ω' and m_0' under the map

$$\lambda' = \lambda - \mathrm{Sgn}(\lambda) \ , \ w' = \frac{3-w}{1+w} \ , m_0' = \frac{-2m_0}{1+w} \ , m' = -m, \tag{0.2}$$

provided

$$\mathrm{Sgn}(\lambda)\mathrm{Sgn}(m) = 1 \tag{0.3}$$

is satisfied. Here m is the invariant pole mass

$$m = \frac{2m_0}{2 + (-1+w)\lambda \, \mathrm{Sgn}(m)}. \tag{0.4}$$

It should be mentioned here that at $w = 1$, the theory has $\mathcal{N} = 2$ supersymmetry. As we will show by quoting a sample result from the work, under this map, the S-matrix of the theory is invariant - i.e., 'self-dual'.

We wish to compute the S-matrix for 2×2 scattering of fundamental (and anti-fundamental) quanta in the above theory. Therefore, we study scattering between two in and two out states where each in (out) state can either be a boson or a fermion. As a scattering amplitude represents the transition between free incoming and free outgoing on-shell particles,it is sufficient to study the superfields Φ_i subject to the free equation of motion

$$\left(\frac{1}{2}D^\alpha D_\alpha + m\right)\Phi = 0. \tag{0.5}$$

The solutions to the above equation are free fields defined by the oscillators

$$\begin{aligned} A_i(\mathbf{p}) &= a_i(\mathbf{p}) + \alpha_i(\mathbf{p})\theta_i \\ A_i^\dagger(\mathbf{p}) &= a_i^\dagger(\mathbf{p}) + \theta_i \alpha_i^\dagger(\mathbf{p}). \end{aligned} \tag{0.6}$$

Here θ_i is a new formal superspace parameter (in particular θ_i has nothing to do with the θ_α that appear in the superfield action (0.1)). Here $a/a^\dagger$ are annihilation/creation operators for the bosonic particles and $\alpha/\alpha^\dagger$ are annihilation/creation operators for the fermionic particles respectively[1].

We are interested in the S-matrix

$$\begin{aligned} S(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4)\sqrt{(2p_1^0)(2p_2^0)(2p_3^0)(2p_4^0)} = \\ \langle 0|A_4(\mathbf{p}_4, \theta_4)A_3(\mathbf{p}_3, \theta_3)U(\infty, -\infty)A_2^\dagger(\mathbf{p}_2, \theta_2)A_1^\dagger(\mathbf{p}_1, \theta_1)|0\rangle, \end{aligned} \tag{0.7}$$

where the RHS denotes the transition amplitude from the in state with particles 1 and 2 to the out state with particles 3 and 4, and U is the time evolution operator. As mentioned above, the in and out states can be either bosons or fermions. Therefore, there naively seem to be eight independent processes (not sixteen, because only processes with an even number of fermions are allowed) which make up the full 2×2 S-matrix. However, the condition of invariance of the on-shell S-matrix under supersymmetry determines six of the eight amplitudes in terms of two scattering functions f_1 and F_2, which are the the amplitudes for

[1] Similarly $a^c/a^{c\dagger}$ and $\alpha^c/\alpha^{c\dagger}$ are the annihilation/creation operators for the bosonic and fermionic anti-particles respectively.

2 boson to 2 boson scattering and 2 fermion to 2 fermion scattering respectively. In formulae,

$$\begin{aligned} S(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4) &= \mathcal{S}_B + \mathcal{S}_F\ \theta_1\theta_2\theta_3\theta_4 + \left(\frac{1}{2}C_{12}\mathcal{S}_B - \frac{1}{2}C^*_{34}\mathcal{S}_F\right)\ \theta_1\theta_2 \\ + \left(\frac{1}{2}C_{13}\mathcal{S}_B - \frac{1}{2}C^*_{24}\mathcal{S}_F\right)\ \theta_1\theta_3 &+ \left(\frac{1}{2}C_{14}\mathcal{S}_B + \frac{1}{2}C^*_{23}\mathcal{S}_F\right)\ \theta_1\theta_4 + \left(\frac{1}{2}C_{23}\mathcal{S}_B + \frac{1}{2}C^*_{14}\mathcal{S}_F\right)\ \theta_2\theta_3 \\ &+ \left(\frac{1}{2}C_{24}\mathcal{S}_B - \frac{1}{2}C^*_{13}\mathcal{S}_F\right)\ \theta_2\theta_4 + \left(\frac{1}{2}C_{34}\mathcal{S}_B - \frac{1}{2}C^*_{12}\mathcal{S}_F\right)\ \theta_3\theta_4, \quad (0.8) \end{aligned}$$

where the C_{ij}s are functions of the in-going and out-going momenta whose precise form has been worked out; these shall be elaborated on in chapter 2 of the thesis. The rule to read off a scattering amplitude from the above expression is as follows. The presence of a θ_i refers to a fermion in (out) state when the index i is 1,2 (3,4) and one should consider the other two states (the absence of θ_i) as indicating a boson in (out) state. Therefore, the coefficient of the $\theta_1\theta_2$ term is the scattering amplitude for a two fermion in to two boson out process and so on.

The 2×2 scattering of elementary quanta with only fundamental matter (of the gauge group $U(N)$) can proceed in the following 'channels': for fundamental-fundamental scattering[2], we either have a 'symmetric' representation (called the direct-U channel, two boxes in the first row and no boxes in any other row of the Young tableaux) or we have an 'anti-symmetric' representation (called the exchanged-U channel, two boxes in the first column and no boxes in any other column of the Young's tableaux); for fundamental-antifundamental scattering, we have an adjoint channel (we call this the T-channel) and the singlet channel (called the S-channel).

In order to obtain the scattering amplitude, one computes the off-shell 4-point correlator (Green's function) of the superfields and then takes the external legs on-shell. To obtain the exact correlator, one needs to sum all the (planar) graphs of the process in question - fortunately, it turns out that this can be done. This sum is shown in diagrammatic form in Fig. 0.1 below. The resulting Schwinger-Dyson equations are obtained as integral equations that are then solved to obtain the exact Green's function.

We have been able to solve the integral equation only in the special limit $q_\pm = 0$. Thus the momentum q has only a spacelike component q_3. This is not a problem as long as it doesn't play the role of the centre-of-mass momentum which can never be spacelike. In channels where q isn't the centre-of-mass momentum (like the T (adjoint) channel), q plays the role of momentum transfer - this is always spacelike - therefore by a rotation can always be made to lie in the 3-direction. After expressing the amplitude in covariant form, we can always obtain Lorentz invariant S-matrices. However, when q is the centre-of-mass momentum (in our case, this happens to be in the S (singlet) channel), which can never be spacelike in any on-shell scattering process, this special limit imposes a severe limitation - one cannot

[2]The scattering of two anti-fundamentals is simply related to the scattering of two fundamentals, and will not be considered separately.

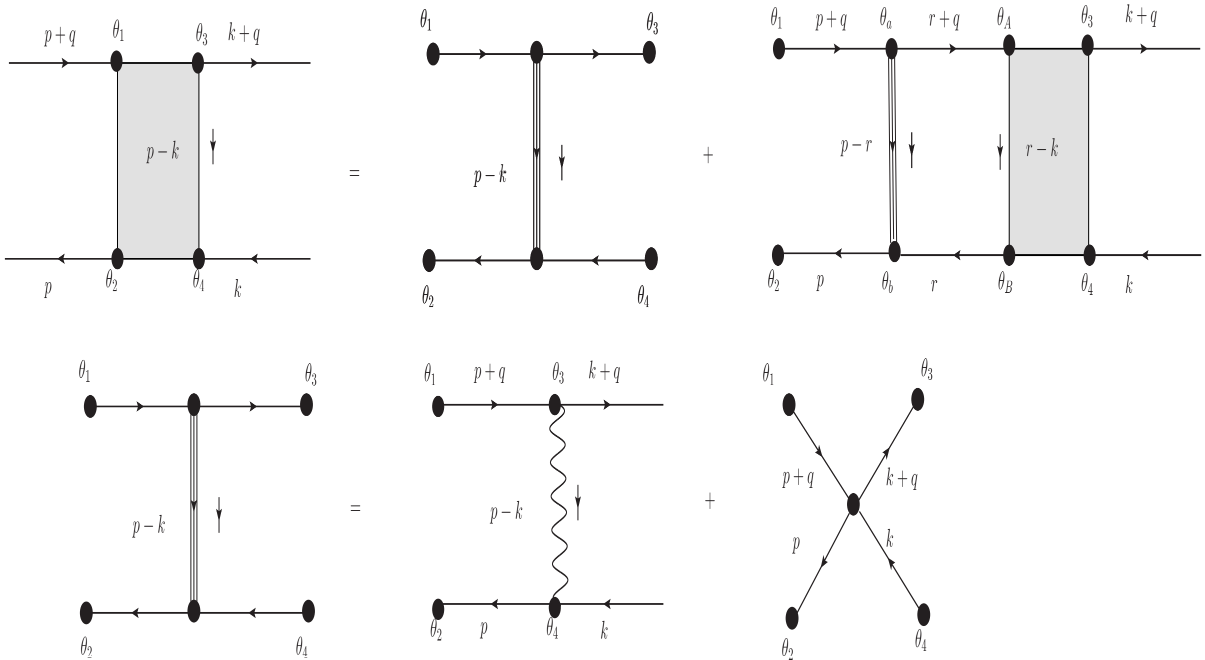

Figure 0.1: Feynman graph for 2×2 scattering in the large-N limit.

compute the S (singlet) channel amplitude directly - one can only obtain it indirectly - by analytically continuing the amplitude obtained in the T-channel. Below we present, in covariant form, the expressions for the T-matrix (see the line below (0.13) for definition of the T-matrix) for particle - particle scattering in the symmetric and antisymmetric channels (U_d and U_e) and particle - antiparticle scattering in the T (adjoint) channel

$$\mathcal{T}_B = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2} + J_B(|q|,\lambda)\ , \tag{0.9}$$

$$\mathcal{T}_F = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2} + J_F(|q|,\lambda)\ , \tag{0.10}$$

where the J functions are [3]

$$\begin{aligned} J_B(|q|,\lambda) &= \frac{4\pi|q|}{\kappa}\frac{(\tilde{n}_1+\tilde{n}_2+\tilde{n}_3)}{(\tilde{d}_1+\tilde{d}_2+\tilde{d}_3)}\ , \\ J_F(|q|,\lambda) &= \frac{4\pi|q|}{\kappa}\frac{(-\tilde{n}_1+\tilde{n}_2+\tilde{n}_3)}{(\tilde{d}_1+\tilde{d}_2+\tilde{d}_3)}\ , \end{aligned} \tag{0.11}$$

[3]See chapter 2 for details about these functions.

where,

$$
\begin{aligned}
\tilde{n}_1 =& 16m|q|(w+1)e^{i\lambda\left(2\tan^{-1}\frac{2|m|}{|q|}+\pi\right)} \ , \\
\tilde{n}_2 =& (w-1)(|q|+2im)(2m(w-1)+i|q|(w+3))\left(-e^{2i\pi\lambda}\right) \ , \\
\tilde{n}_3 =& (w-1)(2m+i|q|)(|q|(w+3)+2im(w-1))e^{4i\lambda\tan^{-1}\frac{2|m|}{|q|}}) \ , \\
\tilde{d}_1 =& (w-1)\left(4m^2(w-1)-8im|q|+|q|^2(w+3)\right)e^{4i\lambda\tan^{-1}\frac{2|m|}{|q|}} \ , \\
\tilde{d}_2 =& (w-1)\left(4m^2(w-1)+8im|q|+|q|^2(w+3)\right)e^{2i\pi\lambda} \ , \\
\tilde{d}_3 =& -2\left(4m^2(w-1)^2+|q|^2(w(w+2)+5)\right)e^{i\lambda\left(2\tan^{-1}\frac{2|m|}{|q|}+\pi\right)}.
\end{aligned}
\tag{0.12}
$$

We mention here that the amplitudes (0.9) and (0.10) can be written in terms of the Mandelstam invariants s, t, u as well. Here $|q| = \sqrt{q^2}$ is, for instance $\sqrt{-t}$ in the adjoint channel. It becomes $\sqrt{-s}$ after analytic continuation to the singlet channel.

We have checked that under the duality map (0.2), these two functions transform into each other, providing striking evidence for the duality. This confirms the observations in [2, 3, 4] that this is indeed a bose-fermi duality.

We next proceed to analyse the unitarity of the S-matrix obtained above. The unitarity condition is

$$SS^\dagger = \mathbb{1}. \tag{0.13}$$

In actual computations, we recast the above equation for the T matrix, which is $S = \mathbb{1} + iT$. Thus, we have

$$i(T^\dagger - T) = TT^\dagger. \tag{0.14}$$

Note that, however, in the T (adjoint) channel, the R.H.S. of this equation is $\mathcal{O}(1/N^2)$, since the T-channel T matrix ((0.9),(0.10)) are themselves $\mathcal{O}(1/N)$. This just asserts that the adjoint channel S-matrix is real. We have checked that the amplitudes (0.9),(0.10) satisfy these conditions.

Unitarity in the S (singlet) channel is, however, a more subtle matter, as both sides of the unitarity equation are of the same order and it must be shown that (0.14) is satisfied. As was observed in [4], a naive analytic continuation of the adjoint channel amplitude to obtain the singlet channel amplitude results in a failure of unitarity. To cure this, a proposal was made which involved a modification of the naive crossing symmetry (analytic continuation) rules and adding a non-analytic δ function piece localised on forward scattering. Here we present the supersymmetric version of the appropriately modified S (singlet) channel amplitude (after performing the analytic continuation) first in schematic form:

$$
\begin{aligned}
\mathcal{T}_B^S(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= -i(\cos(\pi\lambda)-1)\mathbb{1}_B + \frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}_B^{S;\text{naive}}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \ , \\
\mathcal{T}_F^S(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= -i(\cos(\pi\lambda)-1)\mathbb{1}_F + \frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}_F^{S;\text{naive}}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \ ,
\end{aligned}
\tag{0.15}
$$

where $\mathcal{T}_B^{S;\text{naive}}$, $\mathcal{T}_F^{S;\text{naive}}$ defines the T matrix obtained from naive crossing symmetry rules (see chapter 2, (2.157) for details) and $\mathbb{1}_B$ and $\mathbb{1}_F$ are the identity 2×2 boson and fermion S-matrices respectively. In the centre-of-mass frame, these amplitudes can be expressed in terms of the Mandelstam variable s and the scattering angle θ

$$\begin{aligned}\mathcal{S}_B^S(s,\theta) &= 8\pi\sqrt{s}\delta(\theta) + i\mathcal{T}_B^S(s,\theta) \ , \\ \mathcal{S}_F^S(s,\theta) &= 8\pi\sqrt{s}\delta(\theta) + i\mathcal{T}_F^S(s.\theta) \ ,\end{aligned} \tag{0.16}$$

where

$$\mathcal{T}_B^S(s,\theta) = -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + \frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}_B^{S;\text{naive}}(s,\theta) \ , \tag{0.17}$$

$$\mathcal{T}_F^S(s,\theta) = -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + \frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}_F^{S;\text{naive}}(s,\theta) \ . \tag{0.18}$$

The naive analytically continued T matrices are

$$\begin{aligned}\mathcal{T}_B^{S;\text{naive}}(s,\theta) &= 4\pi i\lambda\sqrt{s}\cot(\theta/2) + J_B(\sqrt{s},\lambda) \ , \\ \mathcal{T}_F^{S;\text{naive}}(s,\theta) &= 4\pi i\lambda\sqrt{s}\cot(\theta/2) + J_F(\sqrt{s},\lambda) \ ,\end{aligned} \tag{0.19}$$

where the J functions are as defined in (0.11). We have verified that these amplitudes satisfy highly non-trivial unitarity equations. We have also checked that the above S (singlet) channel amplitude respects the conjectured duality.

As with the analysis of scattering, we obtain a manifestly supersymmetric equation for unitarity in terms of the amplitudes $\mathcal{S}_B$ and $\mathcal{S}_F$ of the supersymmetric S-matrix (0.8). The unitarity condition (0.13) takes the following form on on-shell superspace

$$\begin{aligned}&\int d\Gamma S(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{k}_3,\phi_1,\mathbf{k}_4,\phi_2)\exp(\phi_1\phi_3+\phi_2\phi_4) \\ &\quad 2k_1^0(2\pi)^2\delta^{(2)}(\mathbf{k}_3-\mathbf{k}_1)2k_2^0(2\pi)^2\delta^{(2)}(\mathbf{k}_4-\mathbf{k}_2)S^\dagger(\mathbf{k}_1,\phi_3,\mathbf{k}_2,\phi_4,\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4) = \\ &\qquad \exp(\theta_1\theta_3+\theta_2\theta_4)2p_3^0(2\pi)^2\delta^{(2)}(\mathbf{p}_1-\mathbf{p}_3)2p_4^0(2\pi)^2\delta^{(2)}(\mathbf{p}_2-\mathbf{p}_4),\end{aligned} \tag{0.20}$$

where by $S^\dagger(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4)$ we mean

$$S^\dagger(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4) = S^*(\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4,\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2). \tag{0.21}$$

The invariant measure $d\Gamma$ is

$$d\Gamma = \frac{d^2k_3}{2k_3^0(2\pi)^2}\frac{d^2k_4}{2k_4^0(2\pi)^2}\frac{d^2k_1}{2k_1^0(2\pi)^2}\frac{d^2k_2}{2k_2^0(2\pi)^2}d\phi_1 d\phi_3 d\phi_2 d\phi_4. \tag{0.22}$$

Here $\mathbf{p}_i$ spatial parts of the on-shell momenta p_i and $p_1+p_2=p_3+p_4$. On plugging in (0.8) (and using the definition (0.21)) into (0.20), we have obtained the unitarity conditions in a general form in terms of the functions $\mathcal{S}_B$ and $\mathcal{S}_F$. Below we present the form most suited to check unitarity of the S (singlet) channel amplitudes[4] ((0.17), (0.18))

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big[\mathcal{T}_B^S(s,\theta)\mathcal{T}_B^{S*}(s,-(\alpha-\theta)) - Y(s)\Big(\mathcal{T}_B^S(s,\theta)-\mathcal{T}_F^S(s,\theta)\Big) \Big(\mathcal{T}_B^{S*}(s,-(\alpha-\theta))-\mathcal{T}_F^{S*}(s,-(\alpha-\theta))\Big)\Big] = i(\mathcal{T}_B^{S*}(s,-\alpha)-\mathcal{T}_B^S(s,\alpha)), \tag{0.23}$$

and

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big[-\mathcal{T}_F^S(s,\theta)\mathcal{T}_F^{S*}(s,-(\alpha-\theta)) + Y(s)\Big(\mathcal{T}_B^S(s,\theta)-\mathcal{T}_F^S(s,\theta)\Big) \Big(\mathcal{T}_B^{S*}(s,-(\alpha-\theta))-\mathcal{T}_F^{S*}(s,-(\alpha-\theta))\Big)\Big] = i(\mathcal{T}_F^{S}(s,\alpha)-\mathcal{T}_F^{S*}(s,-\alpha)). \tag{0.24}$$

These two equations assert that when these are met, the S-matrix is unitary. Since the procedure to obtain these is manifestly supersymmetric, we have also checked that once these are satisfied, the unitarity for the other six amplitudes is automatic.

The kinematic factor $Y(s) = \frac{-s+4m^2}{16m^2}$. Although algebraically highly non-trivial, we have checked that the appropriately modified S (singlet) channel amplitudes ((0.17), (0.18)) do indeed satisfy the above equations.

We have already mentioned that at the point $w = 1$ the supersymmetry of the theory is enhanced to $\mathcal{N} = 2$. The scattering amplitudes at this point take extremely simple forms.

$$\mathcal{T}_B^{\mathcal{N}=2} = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2} - \frac{8\pi m}{\kappa}\ , \tag{0.25}$$

$$\mathcal{T}_F^{\mathcal{N}=2} = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2} + \frac{8\pi m}{\kappa}\ , \tag{0.26}$$

whereas for the S (singlet) channel we have (in the centre-of-mass variables)

$$\mathcal{T}_B^{S;\mathcal{N}=2}(s,\theta) = 4i\sqrt{s}\,\sin(\pi\lambda)\cot(\theta/2) - 8m\,\sin(\pi\lambda) - 8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta), \tag{0.27}$$

$$\mathcal{T}_F^{S;\mathcal{N}=2}(s,\theta) = 4i\sqrt{s}\,\sin(\pi\lambda)\cot(\theta/2) + 8m\,\sin(\pi\lambda) - 8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta). \tag{0.28}$$

As we can infer from the results ((0.25), (0.26)), the $\mathcal{N} = 2$ scattering amplitudes are the same as that of the tree-level answers (up to the shift term proportional to the mass); they don't get renormalised, i.e., receive corrections in any of the channels except the S (singlet) channel ((0.27), (0.28)). There are indications that this might well continue to be the case for higher extended supersymmetry. We have also checked that the $\mathcal{N} = 2$ amplitudes respect the duality and they satisfy the unitarity equations ((0.23), (0.24)) (in the case of S (singlet) channel amplitude ((0.27), (0.28))) respectively.

[4] These conditions are analogues of (0.14). Similar equations which are analogues of (0.13) have also been obtained and shall be presented in chapter 2 of the thesis.

One last point is in order regarding the poles of the S-matrix. These poles indicate the presence of bound states in the theory. We have analysed this aspect and find the following. For a fixed λ, we vary the parameter w (we vary it along negative values). At $w = -1 - \delta w$ for some small (and positive) δw, we find a pole near threshold, i.e., $\sqrt{s} = 2m$. As we start going toward more and more negative values, we hit a critical value of w at which this pole becomes massless. This value of w_c as a function of λ is

$$w_c(\lambda) = 1 - \frac{2}{|\lambda|} \tag{0.29}$$

As we continue to higher negative values, this pole finally slides back to the threshold as w tends to $-\infty$. We have checked that this critical point is duality invariant. The emergence of massless bound states coming out of a theory which was massive elementary excitations to begin with is a striking phenomenon and illustrates the richness of phenomena at strong coupling. Also, that the pole doesn't go tachyonic at any point is in agreement with the well known fact that supersymmetric theories have stable vacuua.

Before closing this section, it is instructive to try and sketch a logical explanation for the modified crossing symmetry rule which had been obtained by 'guessing' the correct (i.e., modified) form required to maintain consistency - to restore the unitarity of the S-matrix and also to respect the duality. This explanation originally appeared in [4]. We present the arguments given in [4] below.

Since the central features of the phenomena (viz., failure of naive crossing symmetry to produce a unitary S-matrix) discussed at length in this section essentially arise from the Chern-Simons gauge fields, the precise form of the matter content of the theory (also other details such as the presence/ absence of supersymmetry) won't play a crucial role. Therefore, we consider, for definiteness the purely bosonic theory with 't Hooft coupling λ_B. When λ_B is set to zero, the bosonic theory effectively reduces to a theory of scalars with global $U(N)$ symmetry. In this theory the off-shell correlator

$$C = \langle \phi_i(x_1)\bar{\phi}^j(x_2)\bar{\phi}^k(x_3)\phi_m(x_4)\rangle \tag{0.30}$$

is a well-defined meromorphic function of its arguments. By $U(N)$ invariance this correlator is given by

$$C^{jk}_{im}(x_1, x_2, x_3, x_4) = A(x_1, x_2, x_3, x_4)\delta^j_i\delta^k_m + B(x_1, x_2, x_3, x_4)\delta^k_i\delta^j_m \tag{0.31}$$

where the coefficient functions A and B are functions of the insertion points $x^1 \ldots x^4$. crossing symmetry follows from the observation that distinct scattering amplitudes are simply distinct on-shell limits of the same correlators.

This statement is usually made precise in momentum space, but we will find it more convenient to work in position space. Consider an S^2 of size R, inscribed around the origin in Euclidean R^3 (we will eventually be interested in the limit $R \to \infty$). The S-matrices S_{U_d} and S_S may both be obtained from the correlator A as follows. Consider free incoming particles of momentum p_i and p_m starting out at very early times and focused so that their worldlines will both intersect the origin of R^3. These two world lines intersect the S^2 described above at easily determined locations x_1 and x_4 respectively. Similarly the coordinates x_2 and x_3 are chosen to be the intercepts of the world lines of particles with index j and k, starting out from the origin of R^3 and proceeding to the future along world lines of momentum p_2 and p_3 respectively. Having now chosen the insertion points of all operators as definite functions of momenta, the correlator $A(x_1, x_2, x_3, x_4)$ is now a function only of the relevant particle-particle scattering data; the particle-particle S-matrix may in fact be read off from this correlator in the limit $R \to \infty$ after we strip off factors pertaining to free propagation of our particles from the surface of the S^2 to the origin of R^3. Particle- antiparticle scattering may be obtained in an identical manner, by choosing x_1 and x_2 to lie along the trajectory of incoming particles or antiparticles of momentum p_1 and p_2 respectively, while x_3 and x_4 lie along particle trajectories of outgoing particles and antiparticles of momentum p_3 and p_4 respectively. Intuitively we expect that crossing symmetry follows from the analyticity of the correlator A as a function of x_1, x_2, x_3 and x_4 on the large S^2.

In the large-N limit A may be obtained from the correlator C^{jk}_{im} in (0.31) from the identity

$$A = \frac{1}{N^2} C^{jk}_{im} \delta^i_j \delta^m_k \tag{0.32}$$

At nonzero λ_B the correlator C^{jk}_{im} no longer makes sense as it is not gauge invariant. In order to construct an appropriate gauge invariant quantity let W_{12} denote an open Wilson line, in the fundamental representation, starting at x_1, ending at x_2 and running entirely outside the S^2 one which the operators are inserted. In a similar manner let W_{43} denote an open Wilson starting at x_4 and ending at x_3, once again traversing a path that lies entirely outside the S^2 on which operators are inserted. Then the quantity

$$A' = C^{jk}_{im} (W_{12})^i_j (W_{43})^m_k \tag{0.33}$$

is a rough analogue of A in the gauged theory. The precise relationship is that A' reduces to A in the limit $\lambda_B \to 0$ in which gauge dynamics decouples from matter dynamics. A' is clearly gauge invariant at all λ_B; moreover there seems no reason to doubt that A' is an analytic function of $x_1 \ldots x_4$.

We can now evaluate A' in the same two on-shell limits discussed in the paragraph above; as in the paragraph above this yields two functions of on-shell momenta that are analytic continuations of each other. In the limit $\lambda_B \to 0$ these two functions are simply the direct channel and singlet channel S-matrices. We will now address the following question: what is the interpretation of these two functions, obtained out of A', at finite λ_B?

The path integral that evaluates the quantity A' may conceptually be split up into three parts. The path integral inside the S^2 may be thought of as defining a ket $|\psi_1\rangle$ of the field theory that lives on S^2. The path integral outside the S^2 defines a bra of the field theory on S^2, lets call it $\langle\psi_2|$. And, finally, the path integral on S^2 evaluates $\langle\psi_2|\psi_1\rangle$.

The key observation here is that the inner product occurs in the direct product of the matter Hilbert space, and the pure gauge Hilbert Space. The pure gauge Hilbert space is

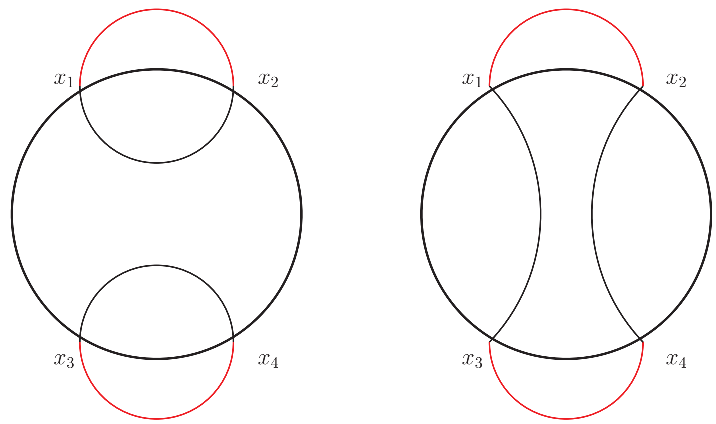

Figure 0.2: The full effective Wilson lines for S and U_d channels

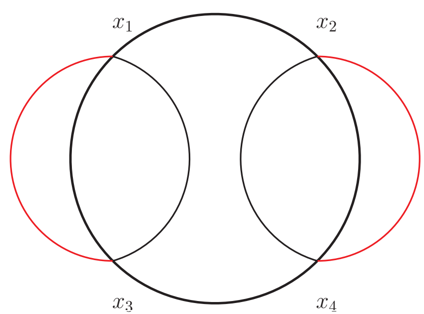

Figure 0.3: The full effective Wilson lines for T-channel

the two dimensional Hilbert Space of conformal blocks of pure Chern-Simons theory on S^2 with two fundamental and two antifundamental Wilson line insertions.

The inner product in the gauge sector depends only on the topology of the paths of matter particles inside the S^2. The distinct topological sectors are distinguished by a relative winding number of the two scattering particles around each other. In the large-N limit where the probability for reconnections in the Skein relations (see Eq. 4.22 of [21]) vanishes, the gauge theory inner product in a sector of winding number w differs from the inner product in a sector of winding number zero merely by the relevant Aharonov-Bohm phase. This relative weighting is, of course, a very important part of the scattering amplitude of the theory, producing all the nontrivial behaviour. However the gauge theory inner product is nontrivial even at $w = 0$. The details of this extra factor depend on the apparently unphysical external Wilson lines. This extra factor is not present in the 'S-matrix' computed in this paper (as we had no external Wilson lines connecting the various particles). In order to compare with the S-matrices presented in this paper, we must remove this overall inner product factor.

The gauge inner product $\langle \psi_2^G | \psi_1^G \rangle$ corresponding to identity matter scattering (i.e. the geodesic paths of the matter particles from production to annihilation) depends on the scattering channel. Let us first study scattering in the identity channel. The initial particle created at x_1 connects up to the final particle at x_3, while the particle created at x_2 connects up with the final particle at x_4. Combining with the external lines, the full effective Wilson line is topologically a circle, see the second of Fig. 0.2. On the other hand, in the case of particle-particle scattering, the dominant dynamical trajectories are from the initial insertion at x_1 to the final insertion at x_2 and from the initial insertion at x_4 to the final insertion at x_3. Including the external lines, the net effective Wilson line has the topology of two circles, see the first of Fig. 0.2.

As the topology of the effective Wilson loops in the first and second of Fig. 0.2 differs, it follows that the gauge theory inner product (even at zero winding) is different in the two sectors. It was demonstrated by Witten in [21] that the ratio of the path integral with two circular Wilson lines to the path integral with a single circular Wilson line is in fact given by

$$\frac{k\sin(\pi\lambda_B)}{\pi} = N\frac{\sin(\pi\lambda_B)}{\pi\lambda_B}$$

in the large-N limit. It follows that we should expect that

$$T_S = \frac{k\sin(\pi\lambda_B)}{\pi}T_{U_d} \tag{0.34}$$

in perfect agreement with (0.15) after replacing $\lambda_B \to \lambda$ (the δ function piece in (0.15) is presumably related to a contact term in the correlators described in this subsection).

A similar argument relates T_{U_e} to T_T without any relative factor, as in this case the closed Wilson lines described above has the topology of two circles in both cases.

0.3 Superconformal invariance, BPS spectrum and current correlators

Continuing with our study of Chern-Simons-matter theories, in this section we wish to understand aspects of these theories when all the mass terms of these theories are switched off. When this is done, a theory with supersymmetry invariance becomes superconformally invariant. Such superconformal field theories play an important role in the AdS/CFT correspondence. It turns out that when matter is in the fundamental representation, the entire single-trace gauge invariant local operator spectrum of these theories are conserved currents. Since one can sprinkle arbitrary number of derivatives on the fields, we can construct currents of arbitrarily high spin (by ‘high spin’ we mean all spins $s > 2$). It is a well known fact that free theories have such conserved currents of increasing spin, which, however, are not conserved anymore as soon as we consider interactions. A remarkable conjecture was stated and proved, in the non-supersymmetric (but conformal) context in [5], that whenever such higher spin conserved currents exist in a theory (with the assumption that the theory has a unique conserved current of spin 2 which is the stress tensor), the theory is necessarily free. It is therefore of interest to understand this in the context of superconformal quantum field theories. The interesting thing to note is, that even after the inclusion of interactions, under certain circumstances, such higher spin currents continue to be 'almost' conserved. Here, by 'almost' we mean that the conservation is broken weakly in the $1/N$ expansion. Using this notion of 'weakly broken' higher spin currents with great efficiency, Maldacena and Zhiboedov in [6] were able to pin down the form of the three point functions of these current operators in conformal field theories to leading order in N and furthermore showed that they are constrained to lie on a one parameter family. This is of tremendous interest and thus one is naturally led to the analogous question for superconformal field theories. One expects the structure to be even more constraint on account of increased symmetry. Also, it is well known that such theories have as bulk duals weakly coupled four dimensional higher spin gravity theory in AdS_4. Before proceeding to the case of weakly broken conservation which we make precise in what follows, we have to understand the free theories first. Therefore, we begin by constructing these current operators explicitly in various superconfomal field theories with fundamental matter in three dimensions. We use an on-shell superspace formalism, as off-shell superspace techniques become cumbersome as we go to higher extended superconformal symmetry. For carrying out a Maldacena-Zhiboedov type analysis we don't need an off-shell formalism. This work was carried out in collaboration with Amin A. Nizami, Tarun K. Sharma, and is based on [7].

We now proceed to understand in general terms the structure, in superspace, of the current superfield in terms of on-shell free superfields in each case. We use the condition of conservation to obtain the general structure. Let us start by first describing the structure of the $\mathcal{N}=1$ supercurrents. A general spin s supercurrent multiplet can be written as a superfield carrying $2s$ spacetime spinor indices and can be expanded in components as follows

$$\Phi^{\alpha_1\alpha_2\ldots\alpha_{2s}} = \phi^{\alpha_1\alpha_2\ldots\alpha_{2s}} + \theta_\alpha\psi^{\alpha\alpha_1\alpha_2\ldots\alpha_{2s}} + \frac{i}{4}\theta^\alpha\theta_\alpha\partial^{\{\alpha_1}_{\ \ \beta}\phi^{|\beta|\alpha_2\ldots\alpha_{2s}\}}, \tag{0.35}$$

where all the indices α_1,α_2,$\ldots\alpha_{2s}$ are symmetrised, and $\phi^{\alpha_1\alpha_2\ldots\alpha_{2s}}$, $\psi^{\alpha\alpha_1\alpha_2\ldots\alpha_{2s}}$ are spin s and spin $s+\frac{1}{2}$ conserved currents respectively. It can be checked that the above current superfield satisfies the conservation condition (shortening) condition

$$D_{\alpha_1}\Phi^{\alpha_1\alpha_2\ldots\alpha_{2s}} = 0, \tag{0.36}$$

where D_α is the supercovariant derivative. Also, the individual component equations resulting from the above are nothing but the conservation conditions for the component currents, i.e.,

$$\partial_{\alpha_1\alpha_2}\phi^{\alpha_1\alpha_2\ldots\alpha_{2s}} = 0 \text{ and } \partial_{\alpha\alpha_1}\psi^{\alpha\alpha_1\ldots\alpha_{2s}} = 0. \tag{0.37}$$

The general structure of the current superfield described above goes through for higher supersymmetries as well. For higher supersymmetries the conservation equation reads

$$D^a_{\ \alpha_1}\Phi^{\alpha_1\alpha_2\ldots\alpha_{2s}} = 0, \tag{0.38}$$

where $a = 1, 2 \ldots \mathcal{N}$ is the R-symmetry index[5]. In the case of an $\mathcal{N} = m$ spin-s current multiplet, the currents $\phi^{\alpha_1\alpha_2\ldots\alpha_{2s}}$ and $\psi^{\alpha\alpha_1\alpha_2\ldots\alpha_{2s}}$ are themselves $\mathcal{N} = m-1$ spin s and spin $s+\frac{1}{2}$ conserved current superfields (depending on the Grassmann coordinates θ^a_α: $a = 1,\ldots m-1$) while the θ_α in (0.35) is the left over Grassmann coordinate θ^m_α. Thus we see the general structure of the supercurrent multiplets: An $\mathcal{N} = m$ spin s supercurrent multiplet breaks up into two $\mathcal{N} = m-1$ supercurrents with spins s and $s+\frac{1}{2}$ respectively.

This structure can be used to express higher supercurrents superfields in terms of components. For instance, the $\mathcal{N} = 2$ spin s currents superfield can be expanded in components as follows

$$\begin{aligned}\Phi^{\alpha_1\alpha_2\ldots\alpha_{2s}} = \varphi^{\alpha_1\alpha_2\ldots\alpha_{2s}} + \theta^a_\alpha(\psi^a)^{\alpha\alpha_1\alpha_2\ldots\alpha_{2s}} + \frac{1}{2}\epsilon_{ab}\theta^a_\alpha\theta^b_\beta\mathcal{A}^{\alpha\beta\alpha_1\alpha_2\ldots\alpha_{2s}} \\ + \text{term involving derivatives of } \varphi, \psi^a \text{ and } \mathcal{A},\end{aligned} \tag{0.39}$$

where a, b are R-symmetry indices and take values in $\{1, 2\}$, and $\varphi^{\alpha_1\alpha_2\ldots\alpha_{2s}}$ and $(\psi^a)^{\alpha\alpha_1\alpha_2\ldots\alpha_{2s}}$ are the single spin s and two spin $s+\frac{1}{2}$ component currents which appear in the $\mathcal{N} = 2$ multiplet. We have checked that the conformal state content so obtained, namely $(\varphi, \psi^1, \psi^2, \mathcal{A})$ above, match exactly with the decomposition of spin s supercurrent multiplet into conformal multiplets as required by representation theory.

As mentioned above, such higher spin conserved currents exist in free theories. We have constructed these currents explicitly in terms of the free on-shell superfields. The explicit results are given in table[6]0.1. We have also checked that the spectrum of conserved currents indeed agrees with the full BPS operator spectrum of the theories in question. We have listed the full operator spectrum of $\mathcal{N} = 1, 2, 3, 4$ & 6 theories below in table[7][8]0.2.

The above current superfields have been constructed in terms of free superfields, which can themselves be expressed in terms of component fields which obey their (free) equations of motion. As explicit examples of on-shell superfields (obtained by solving the chirality constraints and the equations of motion), we present below the expression for the on-shell $\mathcal{N} = 1$ and $\mathcal{N} = 3$ superfields, where in the first case there is no R-symmetry and the latter example transforms in the doublet of the $SU(2)_R$ representation. The on-shell $\mathcal{N} = 1$

[5]Note that for $\mathcal{N} > 1$, (0.38) is true only for R-symmetry singlet currents. For currents carrying non-trivial R-symmetry representation the shortening condition is different. In this chapter we will only need the shortening condition (0.38).

[6]Here $J^{(s)} = \lambda^{\alpha_1}\lambda^{\alpha_2}\cdots\lambda^{\alpha_{2s}}J_{\alpha_1\alpha_2\cdots\alpha_{2s}}$, $\partial = i\lambda^\alpha\gamma^\mu_{\alpha\beta}\lambda^\beta\partial_\mu$, $D = \lambda^\alpha D_\alpha$ and λ_αs are polarisation spinors. All the currents are conserved, i.e., $\frac{\partial}{\partial\lambda^\alpha}D^\alpha J^{(s)} = 0$.

[7]Superconformal primaries are labelled by $(\Delta, j, \{h\})$where Δ is the scaling dimension, j the spin and $\{h\}$ is the R-charge. For R-symmetry quantum numbers taking values in $SU(2)_R$, we give the dimension of the representation while writing down the quantum numbers (Δ, j, h). For example, $(1, 0, 1)$ corresponds to $\Delta = 1$, spin-0 and a singlet under R. In other words, instead of writing the highest weight j for the R-symmetry representation, we write $2j + 1$ as the third quantum number. For the case of $\mathcal{N} = 4$, (j_1, j_2) is a representation of the $SO(4) \sim SU(2) \times SU(2)$ R-symmetry, where j_1 and j_2 correspond to the spin quantum number (the highest weight of the representation) of each of the two $SU(2)$s respectively.

[8]In specifying the $SO(6)$ R-symmetry representation we use the following notation $1 \to$ Singlet, $6 \to$ Vector, $15 \to$ Second rank symmetric traceless tensor, $10 \to$ (anti) Self-dual 3 form.

(On-shell) superfield	Current superfield
Complex Φ, no R-symmetry, stress tensor in $s=\frac{3}{2}$ multiplet.	$J^{(s)}_{\mathcal{N}=1}=\sum_{r=0}^{2s}(-1)^{\frac{r(r+1)}{2}}\binom{2s}{r}D^r\bar{\Phi}D^{2s-r}\Phi$, $s=0,\frac{1}{2},1\ldots$
Complex Φ, transforms by $U(1)_R$ phase, stress tensor in $s=1$ multiplet.	$J^{(s)}_{\mathcal{N}=2}=\sum_{r=0}^{s}\left\{(-1)^{r(2r+1)}\binom{2s}{2r}\partial^r\bar{\Phi}\partial^{s-r}\Phi +(-1)^{(r+1)(2r+1)}\binom{2s}{2r+1}\partial^r\bar{D}\bar{\Phi}\partial^{s-r-1}D\Phi\right\}$, $s=0,1,2\ldots$
Complex Φ^i, $SO(3)_R\sim SU(2)_R$ spinor, stress tensor in $s=\frac{1}{2}$ multiplet.	$J^{(s)}_{\mathcal{N}=3}=\sum_{r=0}^{s}(-1)^r\binom{2s}{2r}\partial^r\bar{\Phi}^i\partial^{s-r}\Phi_i +\frac{2}{9}\sum_{r=0}^{s-1}(-1)^{r+1}\binom{2s}{2r+1}\partial^r D_i{}^j\bar{\Phi}^i\partial^{s-r-1}D_j{}^k\Phi_k$ $J^{(s+\frac{1}{2})}_{\mathcal{N}=3}=\sum_{r=0}^{s}\left\{(-1)^r\binom{2s+1}{2r}\partial^r\bar{\Phi}^i\partial^{s-r}D_i{}^j\Phi_j +(-1)^{r+1}\binom{2s+1}{2r+1}\partial^r D_i{}^j\bar{\Phi}^i\partial^{s-r}\Phi_j\right\}$, $s=0,1,2\ldots$
Complex Φ^i, Weyl spinor of $SO(4)_R$, stress tensor in $s=0$ multiplet.	$J^{(s)}_{\mathcal{N}=4}=\sum_{r=0}^{s}(-1)^r\binom{2s}{2r}\ \partial^r\bar{\Phi}^i\ \partial^{s-r}\Phi_i +\frac{1}{8}\sum_{r=0}^{s-1}(-1)^r\binom{2s}{2r+1}\ \partial^r D^{\tilde{i}i}\bar{\Phi}_i\ \partial^{s-r-1}D_{\tilde{i}j}\Phi^j$ $s=0,1,2\ldots$
Complex Φ^i, 4 of $SO(6)_R\sim SU(4)_R$, stress tensor in $s=0$ multiplet.	$J^{(s)}_{\mathcal{N}=6}=\sum_{r=0}^{s}(-1)^r\binom{2s}{2r}\ \partial^r\bar{\Phi}_p\ \partial^{s-r}\Phi^p -\frac{1}{24}\sum_{r=0}^{s-1}(-1)^{r+1}\binom{2s}{2r+1}\ \epsilon_{ijkl}\ \partial^r D^{ij}\bar{\Phi}_p\ \partial^{s-r-1}D^{kl}\Phi^p$ $s=0,1,2\ldots$

Table 0.1: Current superfields of various fundamental matter theories in $3d$.

superfield is

$$\Phi=\phi+\theta\psi+\frac{\theta^2}{2}F, \tag{0.40}$$

where

$$F=0,\quad \partial^2\phi=0,\quad p_\mu\gamma^\mu\psi=0.$$

The on-shell $\mathcal{N}=3$ superfield is

$$\Phi^k=\phi^k-\frac{1}{\sqrt{2}}\theta^{kl\alpha}\psi_{l\alpha}-\frac{1}{4}\epsilon^{abc}\theta^{a\alpha}\theta^{b\beta}(\sigma^c)^{kl}\partial_{\alpha\beta}\phi_l+\frac{1}{12\sqrt{2}}\epsilon^{abc}\theta^{a\alpha}\theta^{b\beta}\theta^{c\gamma}\partial_{\alpha\beta}\psi^k_\gamma, \tag{0.41}$$

where ϕ^k and $\psi_{l\alpha}$ are the free scalar doublet and fermion (under $SU(2)_R$) fields respectively. In the last term the α,β,γ indices are completely symmetrised and $k=1,2$. Here a,b are vector $SO(3)$ indices and i,j,k are spinor indices and σ^c are Pauli matrices of $SU(2)_R$. Note that (0.41) holds only when the component fields obey the free equations of motion.

We now focus on $\mathcal{N}=1$ theories. Here we provide evidence for a conjectured form we propose for the three point function of the higher spin conserved currents. We propose that the structure of the three point function of the higher spin currents takes the following form

$$\langle J_{s_1}J_{s_2}J_{s_3}\rangle=\frac{1}{\tilde{X}_{12}\tilde{X}_{23}\tilde{X}_{31}}\left(a\langle J_{s_1}J_{s_2}J_{s_3}\rangle\text{even}+b\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\text{odd}}\right), \tag{0.42}$$

Theory	Minimal (on-shell) field content	Superconformal spectrum
$\mathcal{N}=1$	1 complex scalar, 1 complex fermion	$\bigoplus_{j=0,\frac{1}{2},1,\dots}^{\infty}(j+1,j)_{\mathcal{N}=1}$
$\mathcal{N}=2$	1 complex scalar, 1 complex fermion	$\bigoplus_{j=0,1,\dots}^{\infty}(j+1,j,0)_{\mathcal{N}=2}$
$\mathcal{N}=3$	2 complex scalars, 2 complex fermions	$\left[2\bigoplus_{j=0,\frac{1}{2},1,\dots}^{\infty}[(j+1,j,1)\oplus(j+1,j,3)]\right]$ $\oplus(1,0,1)\oplus(1,0,3)\oplus(2,0,1)\oplus(2,0,3)$
$\mathcal{N}=4$	2 complex scalars, 2 complex fermions	$\left[\bigoplus_{j=0,1,\dots}^{\infty}(j+1,j,\{0,0\})_{\mathcal{N}=4}\right]$ $\oplus(1,0,\{1,0\})_{\mathcal{N}=4}$
$\mathcal{N}=6$	4 complex scalars, 4 complex fermions	$\left[\bigoplus_{j=1,2,\dots}^{\infty}(j+1,j;1)_{\mathcal{N}=6}\right]\oplus(1,0;15)_{\mathcal{N}=6}$

Table 0.2: Field content and operator spectrum of various fundamental matter theories in $3d$.

where a and b are independent constants, and the 'even' structure arises from free field theory.

As a first step, we construct superconformally invariant structures as functions on superspace (we have worked these out for the case of general extended supersymmetry, we provide below the definitions that apply to the special case of $\mathcal{N}=1$). We adopt the polarisation spinor formalism of [22].

As a warm up, we present the result for the two point function of the conserved currents (of arbitrary spin s) in terms of the invariants computed in table 0.3. It is

$$\langle J_s(1)J_s(2)\rangle = c(s)\frac{P_3^{2s}}{\tilde{X}_{12}^2}, \tag{0.43}$$

where[9] 1,2 refer to superspace operator insertion points and $c(s)=\left(\frac{i}{2}\right)^{2s}\frac{\sqrt{\pi}}{s!\Gamma(s+\frac{1}{2})}$ for all $s\geq 0$.

Moving on to the case of three point functions, after writing down the structures that can appear for a given three point function, we use on-shell conservation laws of the currents to constrain the coefficients appearing in front of the structures. We find that there exist new structures for both the parity even and odd part of $\langle J_{s_1}J_{s_2}J_{s_3}\rangle$ which were not present in the nonsupersymmetric case. The parity-odd superconformal invariants are of special interest as they arise in interacting $3d$ SCFTs.

Here,

$$\begin{aligned}(X_{ij+})_\alpha^{\ \beta} &= (X_{i+})_\alpha^{\ \beta} - (X_{j-})_\alpha^{\ \beta} + i\theta_{i\alpha}\theta_j^{\beta},\\ (X_{ij-})_\alpha^{\ \beta} &= (X_{i-})_\alpha^{\ \beta} - (X_{j+})_\alpha^{\ \beta} - i\theta_{j\alpha}\theta_i^{\beta},\end{aligned} \tag{0.44}$$

[9]This constant depends on the conventions used; this follows from the conventions of [7].

	Parity even	Parity odd
Bosonic	$P_1 = \lambda_2 X_{23-}^{-1} \lambda_3$ $Q_1 = \lambda_1 X_{12-}^{-1} X_{23+} X_{31-}^{-1} \lambda_1$ and cyclic	$S_1 = \frac{\lambda_3 X_{31+} X_{12+} \lambda_2}{\tilde{X}_{12} \tilde{X}_{23} \tilde{X}_{31}}$ and cyclic
Fermionic	$R_1 = \lambda_1 \Theta_1$ and cyclic	$T = \tilde{X}_{31} \frac{\Theta_1 X_{12+} X_{23+} \Theta_3}{\tilde{X}_{12} \tilde{X}_{23}}$

Table 0.3: Invariant structures in $\mathcal{N} = 1$ superspace.

and

$$\begin{aligned} (X_{ij\pm})^{-1} &= \frac{X_{ij\mp}}{\tilde{x}_{ij}^2 + \frac{1}{16}(\theta_{ij}\theta_{ij})^2}, \\ \Theta_{1\alpha} &= \left((X_{21+}^{-1}\theta_{21})_\alpha - (X_{31+}^{-1}\theta_{31})_\alpha\right), \end{aligned} \tag{0.45}$$

and Θ_2, Θ_3 are defined similarly. The modulus squared is $\tilde{X}_{ij} \equiv \sqrt{(\tilde{X}_{ij})_\alpha^{\ \beta} (\tilde{X}_{ij})_\beta^{\ \alpha}}$ [10].

With the explicit expressions for the currents at hand (for the free field theories), and by using methods similar to [22], we can check proposed form (0.42). Below we tabulate (in table 0.4) the results of a few low-spin examples.

Three-pt function	Even	Odd
$\langle J_{\frac{1}{2}} J_{\frac{1}{2}} J_0 \rangle$	$P_3 - \frac{i}{2} R_1 R_2$	$S_3 - \frac{i}{2} P_3 T$
$\langle J_1 J_{\frac{1}{2}} J_0 \rangle$	$P_3 R_1 + \frac{1}{2} Q_1 R_2$	0
$\langle J_1 J_1 J_0 \rangle$	$\frac{1}{2} Q_1 Q_2 + P_3^2 - i R_1 R_2 P_3$	$S_3 P_3$ $+\frac{i}{2}(S_3 R_1 R_2 - Q_1 Q_2 T)$
$\langle J_{\frac{3}{2}} J_{\frac{1}{2}} J_0 \rangle$	$P_3 Q_1 - \frac{i}{2} Q_1 R_1 R_2$	$Q_1 S_3 - i Q_1 P_3 T$
$\langle J_{\frac{3}{2}} J_{\frac{1}{2}} J_{\frac{1}{2}} \rangle$	$Q_1 R_1 P_1 + Q_1 (R_2 P_2 + R_3 P_3)$ $+2 R_1 P_2 P_3$	0
$\langle J_2 J_{\frac{1}{2}} J_{\frac{1}{2}} \rangle$	$Q_1^2 P_1 - 4 Q_1 P_2 P_3 - \frac{5i}{2} R_2 R_3 Q_1^2$	$Q_1 (P_2 S_3 + P_3 S_2)$ $+\frac{i}{2}(Q_1^2 P_1 - 3 Q_1 P_2 P_3) T$

Table 0.4: Explicit examples of conserved three point functions.

It must be emphasised that the even structures obtained in table 0.4 match with the expressions obtained from free field theory (up to overall constants). We thus have some evidence for the claim that the three point function of conserved currents has a parity even part (generated by a free field theory) and a parity odd piece.

[10]Note that throughout $\tilde{X}_{12}$ denotes this scalar object. The matrix will always be denoted with the indices: $(\tilde{X}_{12})_\alpha^{\ \beta}$. All spinor indices are summed in the NW-SE convention.

One of the main motivations for the study embarked upon in this work is to perform a Maldacena-Zhiboedov type study of superconformal Chern-Simons vector matter theories. As we shall explain below, the structure of terms violating higher spin current conservation is much more constrained in superconformal case as compared to the conformal case suggesting that higher spin correlators in superconformal case must be more severely constrained. For this purpose it will be useful to extend the analysis of three point functions presented here for $\mathcal{N} = 1$ case to extended supersymmetry.

We have so far dealt with exactly conserved higher spin currents. However, as we have already mentioned, it is of great interest, and one of the motivations of this work to be able to constrain the form of three point functions of current operators of superconformal field theories using the notion of 'weak breaking' alluded to earlier. The free superconformal theories discussed above have an exact higher spin symmetry algebra generated by the charges corresponding to the infinite number of conserved currents that these theories possess. These free theories can be deformed into interacting theories by turning on $U(N)$ ($SU(N)$) Chern-Simons (CS) gauge interactions, in a supersymmetric fashion and preserving the conformal invariance of free CFTs, under which the matter fields transform in the fundamental representation. The CS gauge interactions do not introduce any new local degrees of freedom so the spectrum of local operators in the theory remains unchanged. Turning on the interactions breaks the higher spin symmetry of the free theory but in a controlled way which we discuss below. These interacting CS vector models are interesting in there own right as non trivial interacting quantum field theories.

As already mentioned, in [5, 6] theories with exact conformal symmetry but weakly broken higher spin symmetry were studied. It was first observed in [23], and later used in [6], that the anomalous 'conservation' equations are of the schematic form

$$\partial \cdot J_{(s)} = \frac{a}{N} J_{(s_1)} J_{(s_2)} + \frac{b}{N^2} J_{(s'_1)} J_{(s'_2)} J_{(s'_3)} \tag{0.46}$$

plus derivatives sprinkled appropriately. The structure of this equation is constrained on symmetry grounds - the twist $(\Delta_i - s_i)$ of the L.H.S. is 3. If each J_s has conformal dimension $\Delta = s+1+O(1/N)$, and thus twist $\tau = 1+O(1/N)$, the two terms on the R.H.S. are the only ones possible by twist matching. Thus we can have only double or triple trace deformations in the case of weakly broken conservation and terms with four or higher number of currents are not possible.

In the superconformal case that we are dealing with, since D has dimension $\frac{1}{2}$, $D \cdot J_{(s)}$ is a twist 2 operator. Thus in this case the triple trace deformation is forbidden and the only possible structure is more constrained:

$$D \cdot J_{(s)} = \frac{a}{N} J_{(s_1)} J_{(s_2)}. \tag{0.47}$$

In view of this, it is feasible that in large-N superconformal Chern-Simons theories the structure of correlation functions is much more constrained (compared to the non-supersymmetric (but conformal) case).

0.4 Hairy black holes in global AdS_5

Motivated by the AdS/CFT correspondence, in this work we study gravity in AdS_5 minimally coupled to a charged massless scalar field. The collaborators on this project were Pallab Basu, Jyotirmoy Bhattacharya, Sayantani Bhattacharyya, R. Loganayagam and Shiraz Minwalla, and appears in [13]. This Einstein-Maxwell system minimally coupled to a scalar field is an interesting system for several reasons. It is described by the action

$$\begin{aligned} S &= \frac{1}{8\pi G_5}\int d^5x\sqrt{g}\left[\frac{1}{2}\left(\mathcal{R}[g]+12\right)-\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}-|D_\mu\phi|^2\right], \\ D_\mu\phi &= \nabla_\mu\phi - ieA_\mu\phi, \end{aligned} \tag{0.48}$$

where G_5 is the Newton's constant and the radius of AdS_5 is set to unity. This system, (sometimes called the massless Abelian Higgs model) admits a well known set of charged black brane solutions which are asymptotically Poincaré AdS. It was argued in [24] that, at large scalar-gauge field coupling, and when they are near enough to extremality, these black branes are unstable. The end point of the tachyon condensation sparked by this instability is a so called hairy black brane - a solution with a planar horizon immersed in a charged scalar condensate.

Under the AdS/CFT correspondence, these planar AdS_5 solutions are dual to the states of a conformal field theory living on the flat spacetime $R^{3,1}$. Another natural arena to study $3+1$ dimensional conformal field theories is to work on $S^3 \times R_{\text{time}}$ instead. States of such a boundary field theory living on S^3 are dual to gravitational solutions that asymptote to global AdS_5 instead of planar AdS_5. The corresponding charged black holes in global AdS_5 spacetime are characterised by their radius in units of the AdS_5 radius and their charge. At large horizon radius, these black holes are locally well approximated by black branes and we expect their physics to be qualitatively similar to the Poincaré AdS charged branes. It is natural to enquire about the opposite limit: do small hairy black holes exist, and what are their qualitative properties? In this work we answer this question by explicitly constructing a set of spherically symmetric hairy charged black holes whose radii are small compared to the AdS_5 radius. It permits an analytic construction of the microcanonical phase diagram of our system at small mass and charge.

These small Reissner-Nordström black holes in asymptotically AdS spacetimes (RNAdS black holes for short) are well known to suffer from instabilities called superradiant instabilities. It a well known fact that when a mode with a frequency ω is incident on a black hole in flat space, the reflection coefficient exceeds unity when $\omega < e\mu$. Since there is no 'reflection from infinity' in the case of flat space, the wave is not re-incident on the black hole. This phenomenon, called superradiance [25], has immediate and well known implications for the stability of small RNAdS black holes. In AdS, however, the wave is reflected off from infinity and is re-incident on the RNAdS black hole in the centre; each time this process happens, more comes out of the black hole than comes in - therefore, the black hole emits into the mode, gradually losing charge and mass - we wish to study this instability, called 'superradiance' for small RNAdS black holes.

As the spectrum of frequencies of a minimally coupled charged scalar field (in a gauge where $A_t^{(r=\infty)} = 0$) in AdS_5 is bounded from below $\omega \geq \Delta_0 \equiv 4$, we expect small charged black holes in AdS_5 space to exhibit superradiant instabilities whenever the condition $e\mu \geq \omega \geq \Delta_0$ is satisfied. Now the chemical potential μ of a small black hole is bounded from above by the chemical potential of the extremal black hole; i.e., $\mu^2 \leq \mu_c^2 = \frac{3}{2}$. It follows that small charged AdS black holes are always stable when $e^2 \leq \frac{\Delta_0^2}{\mu_c^2} \equiv e_c^2 = \frac{32}{3}$. When $e^2 \geq e_c^2$, however, small black holes that are near enough to extremality suffer from a superradiant instability.

A physical picture for how this instability proceeds is as follows. The black hole emits into a scalar condensate, thereby losing mass and charge itself. As the charge to mass ratio of the condensate (i.e superradiant mode), $\frac{e}{\Delta_0}$, exceeds $\frac{1}{\mu}$, the chemical potential of the black hole also decreases as this emission proceeds. Now the decay rate of the black hole is proportional to $(\Delta_0 - \mu e)$ and so slows down as μ approaches $\frac{\Delta_0}{e}$. It seems intuitively plausible that the system asymptotes to a configuration consisting of a $\mu \approx \frac{\Delta_0}{e}$ stationary charged black hole core surrounded by a diffuse AdS scale charge condensate, i.e., a hairy black hole.

In the discussion of the previous paragraph we have ignored both the backreaction of the scalar field on the geometry as well as the effect of the charged black hole core on the scalar condensate. However these effects turn out to be small whenever the starting black hole is small enough. In other words the end point of the superradiant instability of a small charged black hole is given approximately by a non-interacting mix of the black hole core and the condensate cloud at leading order. We will now pause to explain why this is the case.

First note that the charge and energy density of the superradiant mode is contained in an AdS radius scale cloud. As the charge and mass of the initial unstable black hole is small, the same is true for the charge and mass of the eventual the scalar condensate. Consequently, the scalar condensate is of low density and so backreacts only weakly on the geometry everywhere.[11] For this reason the metric of the final solution is a small deformation of the RNAdS black hole with $\mu = \frac{\Delta_0}{e}$, and the scalar condensate does not significantly affect the properties of the RNAdS black hole. On the other hand the condensate cloud is very large compared to the RNAdS black hole at its core. This difference in scales ensures that the charged black hole also does not significantly affect the properties of the scalar condensate.

Motivated by these considerations, in this work we construct the hairy black hole that marks the end point of the superradiant tachyon condensation process in a perturbative expansion around a small RNAdS black hole with $\mu = \frac{\Delta_0}{e}$ and small but arbitrary radius. Unfortunately, it turns out that it is not possible to solve the equations of motion in full generality. It is easy, however, to solve the equations of motion separately in two regimes: at large r (in an expansion in $\frac{R}{r}$ which we call as the far-field expansion and mark by a superscript 'out') and at small r (in an expansion in r which we call the near-field expansion and mark by a superscript 'in'). Here r is the radial coordinate (that is zero at the black hole

[11]Note that, in contrast, the small charged black hole at the core has its mass and charge concentrated within a small Schwarzschild radius. Consequently even a black hole of very small mass and charge is a large perturbation about the AdS vacuum at length scales comparable to its Schwarzschild radius.

singularity and infinity at the boundary of AdS) and R is the Schwarzschild radius of the unperturbed RNAdS black hole solution. The first expansion is valid when $r \gg R$, while the second expansion works when $r \ll 1$. As we are interested in $R \ll 1$, the validity domains of these two approximations overlap. Consequently, we are able to solve the resultant linear equations everywhere, in a power series expansion in R^2.

Our calculations allow us to determine the microcanonical phase diagram of our system, as a function of mass and charge at small values of these parameters; our results are plotted for $e = 5$ in Fig. 0.4 below (the results are qualitatively similar for every e provided $e^2 \geq \frac{32}{3}$, and may also be simply generalised to the study of (0.48) with a mass term).

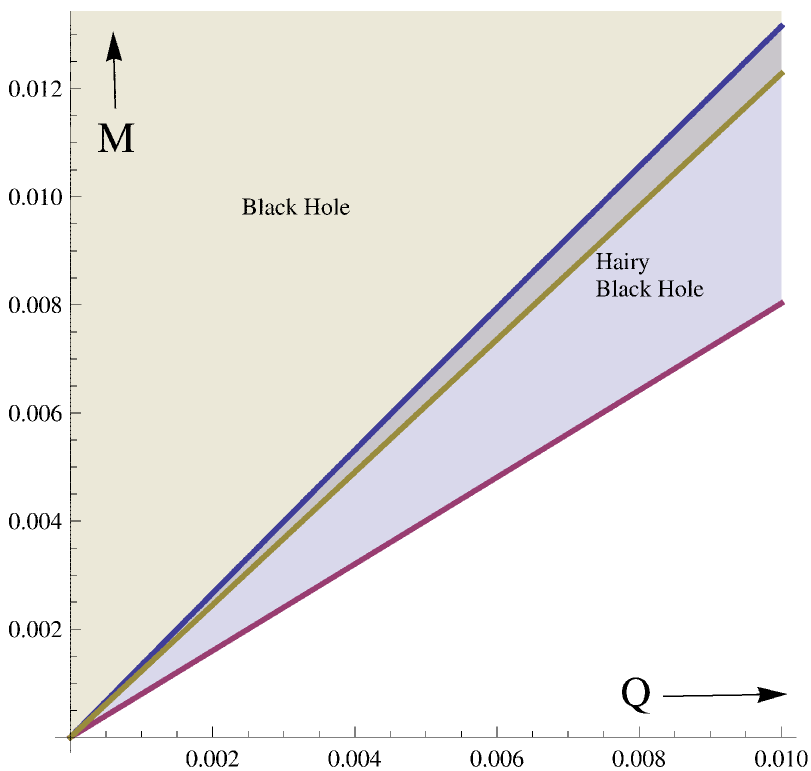

Figure 0.4: Microcanonical phase diagram at small mass and charge. The overlapping region is dominated by the hairy black hole.

As summarised in Fig. 0.4, hairy black holes exist only in the mass range

$$\frac{4}{e}Q + \frac{(9e^2-64)}{7\pi e^2}Q^2 + \mathcal{O}(Q^3) \leq M \leq \frac{3e}{16}\left(1+\frac{32}{3e^2}\right)Q - \frac{3\,(5e^4+64e^2-1024)}{64\pi e^2}Q^2 + \mathcal{O}(Q^3), \tag{0.49}$$

where M and Q are the mass and the charge of the hairy black hole.

Above the upper bound in (0.49) (i.e., in the shaded grey region above the blue line in Fig. 0.4), RNAdS black holes are stable and are the only known stationary solutions. The upper end of (0.49) coincides with the onset of superradiant instabilities for RNAdS black holes. The lower bound in (0.49) is marked by the lowest (i.e., red) line in Fig. 0.4. The extremality line for RNAdS black holes (the yellow line in Fig. 0.4) lies in the middle of this range in (0.49). At masses below lower bound of (0.49) (red line in Fig. 0.4), the system presumably has no states.

In summary, the hairy black hole interpolates between pure black hole and pure condensate as we scan from the upper to the lower bound of (0.49) (or down from the blue line to the red line in Fig. 0.4).Throughout the range of its existence, the hairy black hole is the only known stable solution[12]. It is also the thermodynamically dominant solution, as the entropy of the hairy black hole exceeds that of the RNAdS black hole of the same mass and charge, whenever both solutions exist.

One important advantage of the analytic procedure described above to construct hairy black hole solutions is the following. Within its regime of validity our perturbative procedure is very powerful. It allows us, once and for all, to compute the phase diagram and thermodynamics of all relevant solutions - including each of the infinite number of excited state hairy black holes - as analytic function of the parameters of the problem (e.g. the mass and charge of the scalar field). Perhaps more importantly our procedure gives us qualitative intuitive insight into the nature of hairy black holes. For instance, as we have explained many times, the hairy black hole is an approximately non interacting mix of a RNAdS black hole and the scalar condensate. This picture together with a few lines of algebra, immediately yields a formula for the entropy of the hairy black hole, to leading order in its mass and charge,

$$S_{BH} = \frac{\pi^2}{2} R^3.$$

In other words the perturbative approach employed in this work gives more than numerical answers; it helps us to understand why small hairy black holes behave the way they do.

To conclude, in this work we have demonstrated that very small charged hairy black holes of the Lagrangian (0.48) are extremely simple objects. To leading order in an expansion of the mass and charge, these objects may be thought of as an *non interacting* superposition of a small RNAdS black hole and a charged soliton. The different components of this mixture interact only weakly for two related reasons. The black hole does not affect the soliton because it is parametrically smaller than the soliton. The soliton does not backreact on the black hole because its energy density is parametrically small. Under the AdS/CFT correspondence, the interpretation of the hairy black hole solution is that of a strongly coupled field theory at finite temperature consisting of an approximately non interacting mix of a normal charged phase and a bose condensate.

[12]The reader may wonder whether it is possible to construct an excited hairy black hole solution that is a weakly interacting mix of a RNAdS black hole and an excited state of the scalar field. It turns out that these excited hairy black holes are all unstable to the superradiant decay of the scalar mode with energy $\Delta_0 = 4$. They presumably decay to the ground state hairy black hole, in comparison to which they are all turn out to be entropically sub dominant.

As an example of extension of the methods of this work to more general and realistic settings, in [26] a similar system which was a truncation of $\mathcal{N} = 8$ gauged supergravity was studied. Analysis of this system suggests that it might have a rich family of rotating hairy black holes, including new hairy supersymmetric black holes.

0.5 Conclusions and outlook

In this synopsis we have surveyed a small corner of the vast arena of quantum field theories. In the first two sections, we focused on Chern-Simons-matter theories with supersymmetry and with superconformal symmetry respectively. The third section was devoted to a more indirect approach to understanding QFTs - via the gauge/ gravity duality. The results of section 0.2 help illustrate the rich physics hidden in the strongly coupled regime of QFTs. The exact S-matrix of these theories posses many striking and remarkable features such as self-duality (a bose-fermi duality) and seem to require modified crossing symmetry rules. This study led us on to investigate theories which posses superconformal symmetry in section 0.3. Motivated by results in conformal field theories, we set the stage for the analysis of superconfomral theories by constructing tools needed to fully understand the large-N limit of the three point functions of operators of these theories. We have already provided evidence that these theories would be more constrained than their non-supersymmetric cousins. The explicit results outlined in section 0.2 can also be used as powerful checks of such conjectures, therefore paving the way for a better understanding of superconformal Chern-Simons-matter theories. Proceeding along these lines, we can hope to understand the physics of theories such as ABJ, which aren't fundamental matter theories, but can be approached from the purely fundamental matter theories in some sort of expansion in the ratio of the rank of the gauge groups.

In section 0.4, we took a completely different route - the understanding of gravity is also the understanding of QFT because of the magnificent AdS/CFT correspondence. Study of this system set the stage for a more sophisticated analysis of gravity systems which allow for interpretations of the phenomena studied in terms of a boundary field theory as has been explained.

The conceptual depth and richness of the structure of QFTs - which began as attempts at bringing together special relativity and quantum mechanics - cannot be overstated. QFT beyond perturbation theory is a topic whose surface has barely been scratched. For this reason QFTs are going to remain a very active and productive area of research well into the foreseeable future. The very surprising properties of QFT - like containing gravity (AdS/CFT correspondence) which was till recently thought impossible - and its deep connections to many areas across physics and mathematics make it an extremely exciting and productive field in theoretical physics.

Chapter 1

Introduction

This thesis is devoted to the study of quatum field theories in three spacetime dimensions. Quantum field theories (QFTs) are indispensable in any branch of modern theoretical physics - they appear everywhere from routine humdrum materials to events we witness in high energy accelerators. It was thought for a long time that only gravitation lies outside the scope of QFTs. With the advent of string theory, one could hope to achieve a UV complete theory which ‘contained’ QFT as its low energy limit. The AdS/CFT revolution, however, firmly put gravity within the reach of QFTs at least for a certain sector of quantum gravity.

Given such a vast arena of application, it cannot be overstated that understanding and solving QFTs in various situations (viz., dimensions etc) is a primary goal of theoretical physics. In practise however, this is a very difficult task. One therefore looks for situations where one can gain enough control so that one can compute quantities of interest reliably. Since analytic exact results are really hard to come by, the tool physicists use widely to understand QFTs is perturbation theory - an approximation scheme where one first identifies a free field limit (in which case the dynamics is trivial) and then takes into account the quantum corrections order by order in a power series in a dimensionless quantity that is small in the regime of interest. This method has been very successful; the standard model of particle physics, in many sectors is weakly coupled, and perturbation theory has been utilised to great effect here, which has shed a lot of light on elementary particles and their interactions.

On the other hand, however, the most interesting phases and features QFT lie in domains where such a scheme is of no use. This is the regime where the quantum effects are strong - most known exotic effects in nature viz, confinement and chiral symmetry breaking in QCD and so on - are the result of strong dynamics. It is of great interest therefore, to have available at least toy examples where one can determine with complete accuracy, certain quantities of interest and study them. Non-ableian gauge theories like QCD are particularly interesting in this respect, not only because they govern most exotic aspects of nature, but also admit a very interesting limit - the large-N limit, where N is the rank of the gauge group). Unlike in the weak coupling scenario, where there is a general method to generate a perturbation expansion, in the strongly coupled case there is no such known general method to handle and compute the physical effects reliably. What one could hope to do, however, is that at least in a given simplifying regime (like large-N) hope to sum all the graphs that arise in perturbation theory and thus arrive at an exact, closed from solution for quantities where such a sum is feasible.

As mentioned in the outset, this thesis largely studies QFTs in three dimensions and is structured as follows. In the next chapter we identify a particular QFT - a supersymmetric non-abelian Chern-Simons-matter theory with fundamental matter - and sum all the planar graphs for 2×2 scattering. This gives us an exact (to all orders in the 't Hooft coupling λ) answer for the 2×2 S-matrix. In three dimensions, the nature of particle statistics - anyons - produce exotic non-perturbative physics. Apart from exhibiting a strong-weak coupling duality, what is even more striking is the failure of unitarity (in the anyonic channel - where the anyonic phase is $\mathcal{O}(1)$) if one makes use of 'naive' crossing symmetry rules from traditional QFT. A prescription to cure this, first observed in [4] is shown to hold even for this class of theories, establishing the universal nature of such a modification. Already one can see the highly non-trivial effects of strong coupling in QFTs. The pole structure of the S-matrix is interesting as well. Here we find the existence of massless bound states as we vary a single parameter in the theory. This striking because though we start with a theory of massive elementary interactions, we end up producing a massless bound state. This result helps establish that one cannot ignore the non-trivial physics of non-ableian anyons, and has potential consequences for the scattering amplitudes computed in more complicated theories like ABJ [8] and ABJM [9]. The treatment of the failure of unitarity in these theories should lie on the lines sketched in the computations of this chapter.

In chapter three we investigate fundamental matter theories with supersymmetry with all mass parameters turned off. Such theories are superconformal. Superconformal vector matter theories are of interest again because they are tractable and can yield general results. One such very interesting general result was the Maldacena-Zhiboedov 'theorem' [5, 6]. With the knowledge of the gauge invariant operator spectrum of such theories - which are higher spin conserved currents - Maldacena and Zhiboedov were able to prove general results about the three point functions of such operators. Thus, it is of interest to understand these results in the supersymmetric context. In this chapter we list out the full operator spectrum of such theories with varying amounts of supersymmetry. We also present evidence for a conjectured form of the three point function these higher spin conserved currents. The work described in this chapter forms the groundwork for a complete analysis of superconformal fundamental matter theories. If preliminary analysis using twist arguments is an indication, the structure of the three point functions of these current operators is more constrained.

In the final chapter we change our vantage point and analyse QFTs from the point of view of the AdS/CFT correspondence. We study the minimally coupled Einstein-Maxwell system in global AdS_5. We construct the 'hairy black hole' in a perturbative expansion in the radius of the black hole. We elucidate and study in detail the phase diagram of this system at small mass and charge. The hairy black hole is reinterpreted in the QFT context as a strongly coupled field theory at finite temperature consisting of an approximately non interacting mix of a normal charged phase and a bose condensate. This study sets the stage for a more detailed analysis of such hairy black holes in realistic settings like $AdS_5 \times S^5$.

The subject of QFT is fascinating in so many respects. It is fair to say that we have barely begun to scratch the surface of strongly coupled phenomena in QFTs. That simple toy models contain extremely rich physics is an indication of the power and depth of QFT. As emphasised already, QFTs are a very deep idea about nature the true depths of which we have only begun to explore, and exact results in tractable QFTs are one such window through which we can learn much about the nature of quantum field theory.

Chapter 2

Exact S-matrix of supersymmetric Chern-Simons-matter theories

2.1 Introduction

Non-Abelian $U(N)$ gauge theories in three spacetime dimensions are dynamically rich. At low energies parity preserving gauge self interactions are generically governed by the Yang-Mills action

$$\frac{1}{g_{YM}^2}\int d^3x \ \mathrm{Tr}\ F_{\mu\nu}^2 \ . \tag{2.1}$$

As g_{YM}^2 has the dimensions of mass, gluons are strongly coupled in the IR. In the absence of parity invariance the gauge field Lagrangian generically includes an additional Chern-Simons term and schematically takes the form

$$\frac{i\kappa}{4\pi}\int \mathrm{Tr}\left(AdA+\frac{2}{3}A^3\right)-\frac{1}{4g_{YM}^2}\int d^3x \ \mathrm{Tr}\ F_{\mu\nu}^2 \ . \tag{2.2}$$

The Lagrangian (2.2) describes a system of *massive* gluons; with mass $m \propto \kappa g_{YM}^2$. At energies much lower than g_{YM}^2 (2.2) has no local degrees of freedom. The effective low energy dynamics is topological, and is governed by the action (2.2) with the Yang-Mills term set to zero. This so called pure-Chern-Simons theory was solved over twenty five years ago by Witten [21]; his beautiful and nontrivial exact solution has had several applications in the study of two dimensional conformal field theories and the mathematical study of knots on three manifolds.

Let us now add matter fields with standard, minimally coupled kinetic terms, (in any representation of the gauge group) to (2.2). The resulting low energy dynamics is particularly simple in the limit in which all matter masses are parametrically smaller than g_{YM}^2. In order to focus on this regime we take the limit $g_{YM}^2 \to \infty$ with masses of matter fields held fixed. In this limit the Yang Mills term in (2.2) can be ignored and we obtain a Chern-Simons self coupled gauge theory minimally coupled to matter fields. While the gauge fields are non propagating, they mediate non-local interactions between matter fields.

In order to gain intuition for these interactions it is useful to first consider the special case $N = 1$, i.e. the case of an Abelian gauge theory interacting with a unit charge scalar field. The gauge equation of motion

$$\kappa \varepsilon^{\mu\nu\rho} F_{\nu\rho} = 2\pi J^{\mu} \tag{2.3}$$

ensures that each matter particle traps $\frac{1}{\kappa}$ units of flux (where $i \int F = 2\pi$ is defined as a single unit of flux). It follows as a consequence of the Aharonov-Bohm effect that exchange of two unit charge particles results in a phase $\frac{\pi}{\kappa}$; in other words the Chern-Simons interactions turns the scalars into anyons with anyonic phase $\pi\nu = \frac{\pi}{\kappa}$.

The interactions induced between matter particles by the exchange of non-abelian Chern-Simons gauge bosons are similar with one additional twist. In close analogy with the discussion of the previous paragraph, the exchange of two scalar matter quanta in representations R_1 and R_2 of $U(N)$ results in the phase $\frac{\pi T_{R_1} . T_{R_2}}{\kappa}$ where T_R is the generator of $U(N)$ in the representation R. The new element in the non-abelian theory is that the phase obtained upon interchanging two particles is an operator (in $U(N)$ representation space) rather than a number. The eigenvalues of this operator are given by

$$\nu'_R = \frac{c_2(R_1) + c_2(R_2) - c_2(R')}{2\kappa} \tag{2.4}$$

where $c_2(R)$ is the quadratic Casimir of the representation R and R' runs over the finite set of representations that appear in the Clebsh-Gordon decomposition of the tensor product of R_1 and R_2. In other words the interactions mediated by non-abelian Chern-Simons coupled gauge fields turns matter particles into non-abelian anyons.

In some ways anyons are qualitatively different from either bosons or fermions. For example anyons (with fixed anyonic phases) are never free: there is no limit in which the multi particle anyonic Hilbert space can be regarded as a 'Fock space' of a single particle state space. Thus while matter Chern-Simons theories are regular relativistic quantum field theories from a formal viewpoint, it seems possible that they will display dynamical features never before encountered in the study of quantum field theories. This possibility provides one motivation for the intensive study of these theories.

Over the last few years matter Chern-Simons theories have been intensively studied in two different contexts. The $\mathcal{N} = 6$ supersymmetric ABJ and ABJM theories [8, 9] have been exhaustively studied from the viewpoint of the AdS/CFT correspondence [10, 27]. Several other supersymmetric Chern-Simons theories with $\mathcal{N} \geq 2$ supersymmetry have also been intensively studied, sometimes motivated by brane constructions in string theory. The technique of supersymmetric localisation has been used to perform exact computations of several supersymmetric quantities [28, 29, 30, 31, 32, 33] (indices, supersymmetric Wilson loops, three sphere partition functions). These studies have led, in particular, to the conjecture and detailed check for 'Seiberg like' Giveon-Kutasov dualities between Chern-Simons matter theories with $\mathcal{N} \geq 2$ supersymmetry [34, 35]. Most of these impressive studies have, however, focused on observables [1] that are not directly sensitive to the anyonic nature of of the underlying excitations and have exhibited no qualitative surprises.

Qualitative surprises arising from the effectively anyonic nature of the matter particles seem most likely to arise in observables built out of the matter fields themselves rather than gauge invariant composites of these fields. There exists a well defined gauge invariant observable of this sort; the S-matrix of the matter fields. While this quantity has been somewhat studied for highly supersymmetric Chern-Simons theories, the results currently available (see e.g. [36, 37, 38, 39, 40, 41, 42]) have all be obtained in perturbation theory. Methods based on supersymmetry have not yet proved powerful enough to obtain results for S-matrices at all orders in the coupling constant, even for the maximally supersymmetric ABJ theory. For a very special class of matter Chern-Simons theories, however, it has recently been demonstrated that large-N techniques are powerful enough to compute S-matrices at all orders in a 't Hooft coupling constant, as we now pause to review.

Consider large-N Chern-Simons coupled to a finite number of matter fields in the fundamental representation of $U(N)$. [2] It was realised in [23] that the usual large-N techniques are roughly as effective in these theories as in vector models even in the absence of supersymmetry (see [43, 44, 45, 2, 46, 47, 48, 3, 14, 49, 50, 51, 52, 53, 54, 55] for related works). In particular large-N techniques have recently been used in [4] to compute the $2 \to 2$ S-matrices of the matter particles in purely bosonic/fermionic fundamental matter theories coupled to a Chern-Simons gauge field.

Before reviewing the results of [4] let us pause to work out the effective anyonic phases for two particle systems of quanta in the fundamental/ antifundamental representations at large-N. [3] Following [4] we refer to any matter quantum that transforms in the (anti)fundamental of $U(N)$ a(n) (anti)particle. A two particle system can couple into two representations R' (see (2.4)); the symmetric representation (two boxes in the first row of the Young Tableaux) and the antisymmetric representation (two boxes in the first column of the Young Tableaux). It is easily verified that the anyonic phase $\nu_{R'}$ (see (2.4)) is of order $\frac{1}{N}$ (and so negligible in the large-N limit) for both choices of R'. On the other hand a particle - antiparticle system can couple into R' which is either the adjoint of the singlet. $\nu_{R'}$ once again vanishes in the large-N limit when R' is the adjoint. However when R' is the singlet representation it turns out that $\nu_{sing} = \frac{N}{\kappa} = \lambda$ and so is of order unity in the large-N limit. In summary

[1] These observables include partition functions, indices, Wilson lines and correlation functions of local gauge invariant operators. Note that gauge invariant operators do not pick up anyonic phases when they go around each other precisely because they are gauge singlets.

[2] These theories were initially studied because of their conjectured dual description in terms of Vasiliev equations of higher spin gravity.

[3] The application of large-N techniques to these theories has led to conjectures for strong weak coupling dualities between classes of these theories. The simplest such duality relates a Chern-Simons theory coupled to a single fundamental bosonic multiplet to another Chern-Simons theory coupled to a single fermionic multiplet. The discovery of a three dimensional Bose-Fermi duality was the first major qualitative surprise in the study of Chern-Simons matter theories, and is intimately connected with the effectively anyonic nature of the matter excitations, as explained, for instance, in [4].

two particle systems are always non anyonic in the large-N limit of these special theories. Particle - antiparticle systems are also non anyonic in the adjoint channel. However they are effectively anyonic - with an interesting finite anyonic phase- in the singlet channel. See [4] for more details. This preparation makes clear that qualitative surprises related to anyonic physics in the two quantum scattering in these theories might occur only in particle - antiparticle scattering in the singlet sector.

The authors of [4] used large-N techniques to explicitly evaluate the S-matrices in all three non-anyonic channels in the theories they studied (see below for more details of this process). They also used a mix of consistency checks and physical arguments involving crossing symmetry to conjecture a formula for the particle - antiparticle S-matrix in the singlet channel. The conjecture of [4] for the S-matrix in the singlet channel has two unexpected novelties related to the anyonic nature of the two particle state

- 1. The singlet S-matrix in both the bosonic and fermion theories has a contact term localised on forward scattering. In particular the S-matrix is not an analytic function of momenta.

- 2. The analytic part of the singlet S-matrix is given by the analytic continuation of the S-matrix in any of the other three channels $\times \frac{\sin \pi\lambda}{\pi\lambda}$. In other words the usual rules of crossing symmetry to the anyonic channel are modified by a factor determined by the anyonic phase.

The modification of the usual rules of analyticity and crossing symmetry in the anyonic channel of 2×2 scattering was a major surprise of the analysis of [4]. The authors of [4] offered physical explanations - involving the anyonic nature of scattering in the singlet channel for both these unusual features of the S-matrix. The simple (though non rigorous) explanations proposed in [4] are universal in nature; they should apply equally well to all large-N Chern-Simons theories coupled to fundamental matter, and not just the particular theories studied in [4]. This fact suggests a simple strategy for testing the conjectures of [4] which we employ here. We simply redo the S-matrix computations of [4] in a different class of Chern-Simons theory coupled to fundamental matter and check that the conjectures of [4] - unmodified in all details - indeed continue to yield sensible results (i.e. results that pass all necessary consistency checks) in the new system. We now describe the system we study and the nature of our results in much more detail.

The theories we study are the most general power counting renormalisable $\mathcal{N} = 1$ $U(N)$ gauge theories coupled to a single fundamental multiplet (see (2.5) below). In order to study scattering in these theories we imitate the strategy of [4]. The authors of [4] worked in light cone gauge; in this chapter we work in a supersymmetric generalisation of light cone gauge (2.3.1). In this gauge (which preserves manifest off-shell supersymmetry) the gauge self interaction term vanishes. This fact - together with planarity at large-N - allows us to find a manifestly supersymmetric Schwinger-Dyson equation for the exact propagator of the matter supermultiplet. This equation turns out to be easy to solve; the solution gives simple exact expression for the all orders propagator for the matter supermultiplet (see subsection §2.3.3).

With the exact propagator in hand, we then proceed to write down an exact Schwinger-Dyson equation for the off-shell four point function of the matter supermultiplet. The resultant integral equation is quite complicated; as in [4] we have been able to solve this equation only in a restricted kinematic range ($q_{\pm} = 0$ in the notation of Fig. 2.4). In this kinematic regime, however, we have been able to find a completely explicit (if somewhat complicated) solution of the resulting equation (see subsection §2.3.5-§2.3.6).

In order to evaluate the S-matrices we then proceed to take the on-shell limit of our explicit off-shell results. As explained in detail in [4], the 3 vector q^μ has the interpretation of momentum transfer for both channels of particle- particle scattering and also for particle antiparticle scattering in the adjoint channel. In these channels the fact that we know the off-shell four point amplitudes only when $q_{\pm} = 0$ forces us to study scattering in a particular Lorentz frame; any frame in which momentum transfer happens along the spatial q^3 direction. In any such frame we obtain explicit results for all 2×2 scattering matrices in these three channels. The results are then covariantised to formulae that apply to any frame. Following this method we have obtained explicit results for the S-matrices in these three channels. Our results are presented in detail in subsections §2.3.7 - §2.3.11. As we explain in detail below, our explicit results have exactly the same interplay with the proposed strong weak coupling self duality of the set of $\mathcal{N} = 1$ Chern-Simons fundamental matter theories (see subsection 2.2.2) as that described in [4]; duality maps particle - particle S-matrices in the symmetric and antisymmetric channels to each other, while it maps the particle - antiparticle S-matrix in the adjoint channel to itself.

As in [4] our explicit off-shell results do not permit a direct computation of the S-matrix for particle - particle scattering in the singlet channel. This is because the three vector q^μ is the centre of mass momentum for this scattering process and so must be timelike, which is impossible if $q^{\pm} = 0$. Our explicit results for the S-matrices in the other channels, together with the conjectured modified crossing symmetry rules of [4], however, yield a conjectured formula for the S-matrix in this channel.

In section 2.4 we subject our conjecture for the particle - antiparticle S-matrix to a very stringent consistency check; we verify that it obeys the nonlinear unitarity equation (2.64)[4]. From the purely algebraic point of view the fact that our complicated S matrices are unitary appears to be a minor miracle- one that certainly fails very badly for the S-matrix obtained using the usual rules of crossing symmetry. We view this result as very strong evidence for the correctness of our formula, and indirectly for the modified crossing symmetry rules of [4].

Our proposed formula for particle - antiparticle scattering in the singlet channel has an interesting analytic structure. As a function of s (at fixed t) our S-matrix has the expected two particle cut starting at $s = 4m^2$. In a certain range of interaction parameters it also has poles at smaller (though always positive) values of s. These poles represent bound states;

[4]At large-N this equation may be shown to close on 2×2 scattering.

when they exist these bound states must be absolutely stable even at large but finite N, simply because they are the lightest singlet sector states (baring the vacuum) in the theory; recall that our theory has no gluons. Quite remarkably it turns out that the mass of this bound state supermultiplet vanishes at $w = w_c(\lambda)$ where w is the superpotential interaction parameter of our theory (see (2.5)) and $w_c(\lambda)$ is the simple function listed in (2.216). In other words a one parameter tuning of the superpotential is sufficient to produce massless bound states in a theory of massive 'quarks'; we find this result quite remarkable. Scaling w to w_c permits a parametric separation between the mass of this bound state and all other states in the theory. In this limit there must exist a decoupled QFT description of the dynamics of these light states even at large but finite N; it seems likely to us that this dynamics is governed by a $\mathcal{N} = 1$ Wilson-Fisher fixed point.

The S-matrices computed and conjectured in this chapter turn out to simplify dramatically at $w = 1$, at which point the system (2.5) turns out to enjoy an enhanced $\mathcal{N} = 2$ supersymmetry. In the three non-anyonic channels our S-matrix reduces simply to its tree level counterpart at $w = 1$. It follows, in other words, that the S-matrix is not renormalised as a function of λ in these channels. This result illustrates the conflict between naive crossing symmetry and unitarity in a simple setting. Naive crossing symmetry would yield a singlet channel S-matrix that is also tree level exact. However tree level S-matrices by themselves can never obey the unitarity equations (they do not have the singularities needed to satisfy the Cutkosky's rules obtained by gluing them together). The resolution to this paradox appears simply to be that the naive crossing symmetry rules are wrong in the current context. Applying the conjectured crossing symmetry rules of [4] we find a singlet channel S-matrix that continues to be very simple, but is not tree level exact, and in fact satisfies the unitarity equation.

In this work we have limited our attention to the study of $\mathcal{N} = 1$ theories with a single fundamental matter multiplet. Were we to extend our analysis to theories with two multiplets we would encounter, in particular, the $\mathcal{N} = 3$ theory. Extending to the study of a theory with four multiplets (and allowing for the the gauging of a $U(1)$ global symmetry) would allow us to study the $\mathcal{N} = 6$ $U(N) \times U(1)$ ABJ theory. We believe it would not be difficult to adapt the methods of this chapter to find explicit all orders results for the S-matrices of all these theories at leading order in large-N. We expect to find scattering matrices that are unitary precisely because they transform under global symmetry in the unusual manner conjectured in [4]. It would be particularly interesting to find explicit results for the $\mathcal{N} = 6$ theory in order to facilitate a detailed comparison with the perturbative computations of S-matrices in ABJM theory [36, 37, 38, 39, 40, 41, 42], which appear to report results that are crossing symmetric but (at least naively) conflict with unitarity.

2.2 Review of background material

2.2.1 Renormalizable $\mathcal{N} = 1$ theories with a single fundamental multiplet

In this chapter we study 2×2 scattering in the most general renormalizable $\mathcal{N} = 1$ supersymmetric $U(N)$ Chern-Simons theory coupled to a single fundamental matter multiplet. Our theory is defined in superspace by the Euclidean action [56, 57]

$$\mathcal{S}_{\mathcal{N}=1} = -\int d^3x d^2\theta \Big[\frac{\kappa}{2\pi}\text{Tr}\Big(-\frac{1}{4}D_\alpha\Gamma^\beta D_\beta\Gamma^\alpha - \frac{1}{6}D^\alpha\Gamma^\beta\{\Gamma_\alpha,\Gamma_\beta\} - \frac{1}{24}\{\Gamma^\alpha,\Gamma^\beta\}\{\Gamma_\alpha,\Gamma_\beta\}\Big) \\ -\frac{1}{2}(D^\alpha\bar{\Phi} + i\bar{\Phi}\Gamma^\alpha)(D_\alpha\Phi - i\Gamma_\alpha\Phi) + m_0\bar{\Phi}\Phi + \frac{\pi w}{\kappa}(\overline{\Phi}\Phi)^2\Big] . \tag{2.5}$$

The integration in (2.5) is over the three Euclidean spatial coordinates and the two anticommuting spinorial coordinates θ^α (the $SO(3)$ spinorial indices α range over two allowed values $\pm$). The fields Φ and Γ^α in (2.5) are, respectively, complex and real superfields [5]. They may be expanded in components as

$$\begin{aligned} \Phi &= \phi + \theta\psi - \theta^2 F \ , \\ \bar{\Phi} &= \bar{\phi} + \theta\bar{\psi} - \theta^2\bar{F} \ , \\ \Gamma^\alpha &= \chi^\alpha - \theta^\alpha B + i\theta^\beta A_\beta{}^\alpha - \theta^2(2\langle^\alpha - i\partial^{\alpha\beta}\chi_\beta) \ , \end{aligned} \tag{2.6}$$

where Γ_α is an $N \times N$ matrix in colour space, while Φ is an N dimensional column.

The superderivative D_α in (2.5) is defined by

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\theta^\beta\partial_{\alpha\beta} \ , D^\alpha = C^{\alpha\beta}D_\beta \ , \tag{2.7}$$

where $C^{\alpha\beta}$ is the charge conjugation matrix. See appendix §A.1.1 for notations and conventions.

The theories (2.5) are characterised by one dimensionless coupling constant w, a dimensionful mass scale m_0, and two integers N (the rank of the gauge group $U(N)$) and κ, the level of the Chern-Simons theory. [6] In the large-N limit of interest to us in this chapter, the 't Hooft coupling $\lambda = \frac{N}{\kappa}$ is a second effectively continuous dimensionless parameter.

The action (2.5) enjoys invariance under the super gauge transformations

$$\begin{aligned} \delta\Phi &= iK\Phi \ , \\ \delta\bar{\Phi} &= -i\bar{\Phi}K \ , \\ \delta\Gamma_\alpha &= D_\alpha K + [\Gamma_\alpha, K] \ , \end{aligned} \tag{2.8}$$

---

[5] See appendix §A.1.2 for our conventions for superspace

[6] The precise definition of κ is defined as follows. Let k denote the level of the WZW theory related to Chern-Simons theory after all fermions have been integrated out. κ is the related to k by $\kappa = k + \text{sgn}(k)N$

where K is a real superfield (it is an $N \times N$ matrix in color space).

(2.5) is manifestly invariant under the two supersymmetry transformations generated by the supercharges Q_α

$$Q_\alpha = i(\frac{\partial}{\partial\theta^\alpha} - i\theta^\beta \partial_{\beta\alpha}) \tag{2.9}$$

that act on Φ and Γ_α as

$$\begin{aligned} \delta_\alpha \Phi &= Q_\alpha \Phi \ , \\ \delta_\alpha \Gamma_\beta &= Q_\alpha \Gamma_\beta \ . \end{aligned} \tag{2.10}$$

The differential operators Q_α and D_α obey the algebra

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= 2i\partial_{\alpha\beta} \ , \\ \{D_\alpha, D_\beta\} &= 2i\partial_{\alpha\beta} \ , \\ \{Q_\alpha, D_\beta\} &= 0 \ . \end{aligned} \tag{2.11}$$

At the special value $w = 1$, the action (2.5) actually has enhanced supersymmetry; it is $\mathcal{N} = 2$ (four supercharges) supersymmetric. [7]

The physical content of the theory (2.5) is most transparent when the Lagrangian is expanded out in component fields in the so called Wess-Zumino gauge - defined by the requirement

$$B = 0 \ , \chi = 0 \ . \tag{2.12}$$

Imposing this gauge, integrating over θ and eliminating auxiliary fields we obtain the component field action [8]

$$\begin{aligned} \mathcal{S}_{\mathcal{N}=1} = \int d^3x \Big(&- \frac{\kappa}{2\pi}\epsilon^{\mu\nu\rho}\mathrm{Tr}\big(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\big) + \mathcal{D}^\mu\bar{\phi}\mathcal{D}_\mu\phi + m_0^2\bar{\phi}\phi - \bar{\psi}(i\not{\mathcal{D}} + m_0)\psi \\ &+ \frac{4\pi w m_0}{\kappa}(\bar{\phi}\phi)^2 + \frac{4\pi^2 w^2}{\kappa^2}(\bar{\phi}\phi)^3 - \frac{2\pi}{\kappa}(1+w)(\bar{\phi}\phi)(\bar{\psi}\psi) - \frac{2\pi w}{\kappa}(\bar{\psi}\phi)(\bar{\phi}\psi) \\ &+ \frac{\pi}{\kappa}(1-w)\big((\bar{\phi}\psi)(\bar{\phi}\psi) + (\bar{\psi}\phi)(\bar{\psi}\phi)\big)\Big) \end{aligned} \tag{2.15}$$

[7] This may be confirmed, for instance, by checking that (2.15) at $w = 1$ is identical to the $\mathcal{N} = 2$ superspace Chern-Simons action coupled to a single chiral multiplet in the fundamental representation with no superpotential (see Eq 2.3 of [58]) expanded in components in Wess-Zumino gauge.

[8] Our trace conventions are

$$Tr(T^aT^b) = \frac{1}{2}\delta^{ab} \ , \ \sum_a (T^a)_i^{\ j}(T^a)_k^{\ l} = \frac{1}{2}\delta_i^{\ l}\delta_k^{\ j} \ . \tag{2.13}$$

The gauge covariant derivatives in (2.15) are

$$\begin{aligned} &\mathcal{D}^\mu\bar{\phi} = \partial^\mu\bar{\phi} + i\bar{\phi}A^\mu \ , \ \mathcal{D}_\mu\phi = \partial_\mu\phi - iA_\mu\phi \ , \\ &\not{\mathcal{D}}\bar{\psi} = \gamma^\mu(\partial_\mu\bar{\psi} + i\bar{\psi}A_\mu) \ , \ \not{\mathcal{D}}\psi = \gamma^\mu(\partial_\mu\psi - iA_\mu\psi). \end{aligned} \tag{2.14}$$

displaying that (2.5) is the action for one fundamental boson and one fundamental fermion coupled to a Chern-Simons gauge field. Supersymmetry sets the masses of the bosonic and fermionic fields equal, and imposes several relations between a priori independent coupling constants.

2.2.2 Conjectured duality

It has been conjectured [3] that the theory (2.5) enjoys a strong weak coupling self duality. The theory (2.5) with 't Hooft coupling λ and self coupling parameter w is conjectured to be dual to the theory with 't Hooft coupling λ' and self coupling w' where

$$\lambda' = \lambda - \text{Sgn}(\lambda) \ , \ w' = \frac{3-w}{1+w} \quad m_0' = \frac{-2m_0}{1+w} \ . \tag{2.16}$$

As we will explain below, the pole mass for the matter multiplet in this theory is given by

$$m = \frac{2m_0}{2+(-1+w)\lambda \ \text{Sgn(m)}} \ . \tag{2.17}$$

It is easily verified that under duality

$$m' = -m \ . \tag{2.18}$$

The concrete prior evidence for this duality is the perfect matching of S^2 partition functions of the two theories. This match works provided [3]

$$\lambda m(m_0, w) \geq 0 \ , \tag{2.19}$$

The quantity m has the physical interpretation as the exact pole mass of the matter multiplet Φ (in particular the fermionic field ψ in the supermultiplet Φ obeys the Dirac equation with m_0 replaced by m). Through this chapter we will assume that (2.19) is obeyed. Note that the condition (2.19) is preserved by duality (i.e. a theory and its conjectured dual either both obey or both violate (2.19)).

Note that $w = 1$ is a fixed point for the duality map (2.16); this was necessary on physical grounds (recall that our theory has enhanced $\mathcal{N} = 2$ supersymmetry only at $w = 1$). In the special case $w = 1$ and $m_0 = 0$, the duality conjectured in this subsection reduces to the previously studied duality [34] (a variation on Giveon- Kutasov duality [35]). Over the last few years this supersymmetric duality has been subjected to (and has successfully passed) several checks performed with the aid of supersymmetric localisation, including the matching of three sphere partition function, superconformal indices and Wilson loops on both sides of the duality [28, 29, 30, 31, 32, 33, 46].

2.2.3 Properties of free solutions of the Dirac equation

In subsequent subsections we will investigate the constraints imposed supersymmetry on the S-matrices of the the theory (2.5). Our analysis will make heavy use of the properties of the free solutions to Dirac's equations, which we review in this subsection.

Let u_α and v_α are positive and negative energy solutions to Dirac's equations with mass m. Let $p^\mu = (\sqrt{m^2+\mathbf{p}^2}, \mathbf{p})$. Then u_α and v_α obey

$$\begin{aligned}(\not{p} - m)u(p) &= 0 \ , \\ (\not{p} + m)v(p) &= 0 \ .\end{aligned} \tag{2.20}$$

We choose to normalise these spinors so that

$$\begin{aligned}\bar{u}(\mathbf{p})\cdot u(\mathbf{p}) &= -2m \qquad & \bar{v}(\mathbf{p})\cdot v(\mathbf{p}) &= 2m \\ u(\mathbf{p})u^*(\mathbf{p}) &= -(\not{p}+m)C \qquad & v(\mathbf{p})v^*(\mathbf{p}) &= -(\not{p}-m)C \ .\end{aligned} \tag{2.21}$$

C in (2.21) is the charge conjugation matrix defined to obey the equation

$$C\gamma^\mu C^{-1} = -(\gamma^\mu)^T \ . \tag{2.22}$$

Throughout this chapter we use γ matrices that obey the algebra[9]

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu} \ . \tag{2.23}$$

We also choose all three γ^μ matrices to be purely imaginary [10] and to obey

$$(\gamma^\mu)^\dagger = -\eta^{\mu\mu}\gamma^\mu \quad \text{no sum} \tag{2.24}$$

with these conventions it is easily verified that $C = \gamma^0$ obeys (2.22) and so we choose

$$C = \gamma^0 \ .$$

Using the conventions spelt out above, it is easily verified that $u(\mathbf{p})$ and $v^*(\mathbf{p})$ obey the same equation (i.e. complex conjugation flips the two equations in (2.20)), and have the same normalisation. It follows that it is possible to pick the (as yet arbitrary) phases of $u(\mathbf{p})$ and $v(\mathbf{p})$ to ensure that

$$u_\alpha(\mathbf{p}) = -v^*_\alpha(\mathbf{p}), \quad v_\alpha(\mathbf{p}) = -u^*_\alpha(\mathbf{p}) \tag{2.25}$$

[11]. We will adopt the choice (2.25) throughout.

---

[9] We use the mostly plus convention for $\eta_{\mu\nu}$, the corresponding Euclidean algebra obeys $\{\gamma^\mu,\gamma^\nu\} = -2\delta^{\mu\nu}$. See appendix §A.1.1 for explicit representations of the γ matrices and charge conjugation matrix C.

[10] This is possible in 3 dimensions; recall the unconventional choice of sign in (2.23).

[11] Note that $\bar{u}^\alpha = u^{*\alpha} = C^{\alpha\beta}u^*_\beta$ and not $(u^\alpha)^*$. Thus, $(u^{*\alpha})^* = -u^\alpha$, where we have used the fact that $C = \gamma^0$ is imaginary. Similarly $(u^\alpha)^* = -u^{*\alpha}$. Likewise for v. Care should be taken while complex conjugating dot products of spinors, for instance $(v^*(\mathbf{p}_i)v^*(\mathbf{p}_j))^* = -(v(\mathbf{p}_i)v(\mathbf{p}_j))$, $(u(\mathbf{p}_i)u(\mathbf{p}_j))^* = -(u^*(\mathbf{p}_i)u^*(\mathbf{p}_j))$, and so on.

Notice that the replacement $m \to -m$ interchanges the equations for u and v. It follows that $u(m) \propto v(-m)$. At least with the choice of phase that we adopt in here (see below) we find

$$u(m,p) = -v(-m,p), \quad v(m,p) = -u(-m,p) \ . \tag{2.26}$$

To proceed further it is useful to make a particular choice of γ matrices and to adopt a particular choice of phase for u. We choose the γ^μ matrices listed in §A.1.1 and take $u(\mathbf{p})$ and $v(\mathbf{p})$ to be given by

$$\begin{aligned} u(\mathbf{p}) &= \begin{pmatrix} -\sqrt{p^0 - p^1} \\ \frac{p^3 + im}{\sqrt{p^0 - p^1}} \end{pmatrix} , \ \bar{u}(\mathbf{p}) = \begin{pmatrix} \frac{ip^3 + m}{\sqrt{p^0 - p^1}} & i\sqrt{p^0 - p^1} \end{pmatrix} , \\ v(\mathbf{p}) &= \begin{pmatrix} \sqrt{p^0 - p^1} \\ \frac{-p^3 + im}{\sqrt{p^0 - p^1}} \end{pmatrix} , \ \bar{v}(\mathbf{p}) = \begin{pmatrix} \frac{-ip^3 + m}{\sqrt{p^0 - p^1}} & -i\sqrt{p^0 - p^1} \end{pmatrix} , \end{aligned} \tag{2.27}$$

where

$$p^0 = +\sqrt{m^2 + \mathbf{p}^2} \ .$$

Notice that the arguments of the square roots in (2.27) are always positive; the square roots in (2.27) are defined to be positive (i.e. $\sqrt{x^2} = |x|$). It is easily verified that the solutions (2.27) respect (2.26) as promised.

In the rest of this section we discuss an analytic rotation of the spinors to complex (and in particular negative) values of the p^μ (and in particular p^0). This formal construction will prove useful in the study of the transformation properties of the S-matrix under crossing symmetry.

Let us define

$$\sqrt{ae^{i\alpha}} = |\sqrt{a}|e^{i\frac{\alpha}{2}}.$$

Clearly our function is single valued only on a double cover of the complex plane. In other words our square root function is well defined if α is specified modulo 4π, but is not well defined if α is specified modulo 2π. We define

$$\begin{aligned} u(\mathbf{p},\alpha) = u(e^{i\alpha}p^\mu) &= \begin{pmatrix} -e^{i\frac{\alpha}{2}}\sqrt{p^0 - p^1} \\ \frac{p^3 e^{i\frac{\alpha}{2}} + ime^{-i\frac{\alpha}{2}}}{\sqrt{p^0 - p^1}} \end{pmatrix} , \\ v(\mathbf{p},\alpha) = v(e^{i\alpha}p^\mu) &= -\begin{pmatrix} -e^{i\frac{\alpha}{2}}\sqrt{p^0 - p^1} \\ \frac{p^3 e^{i\frac{\alpha}{2}} - ime^{-i\frac{\alpha}{2}}}{\sqrt{p^0 - p^1}} \end{pmatrix} , \\ u^*(\mathbf{p},\alpha) &= \begin{pmatrix} -e^{-i\frac{\alpha}{2}}\sqrt{p^0 - p^1} \\ \frac{p^3 e^{-i\frac{\alpha}{2}} - ime^{i\frac{\alpha}{2}}}{\sqrt{p^0 - p^1}} \end{pmatrix} , \\ v^*(\mathbf{p},\alpha) &= -\begin{pmatrix} -e^{-i\frac{\alpha}{2}}\sqrt{p^0 - p^1} \\ \frac{p^3 e^{-i\frac{\alpha}{2}} + ime^{+i\frac{\alpha}{2}}}{\sqrt{p^0 - p^1}} \end{pmatrix} , \end{aligned} \tag{2.28}$$

with $\alpha \in [0, 4\pi)$. It follows immediately from these definitions that

$$\begin{aligned} u(\mathbf{p},\alpha+\pi) &= -iv(\mathbf{p},\alpha)\ ,\ v(\mathbf{p},\alpha+\pi) &= -iu(\mathbf{p},\alpha)\ , \\ u(\mathbf{p},\alpha-\pi) &= iv(\mathbf{p},\alpha)\ ,\ v(\mathbf{p},\alpha-\pi) &= iu(\mathbf{p},\alpha)\ , \\ u^*(\mathbf{p},\alpha) &= -v(\mathbf{p},-\alpha)\ ,\ v^*(\mathbf{p},\alpha) &= -u(\mathbf{p},-\alpha)\ . \end{aligned} \tag{2.29}$$

Notice, in particular, that the choice $\alpha = \pi$ and $\alpha = -\pi$ both amount to the replacement of p^μ with - p^μ. Note also that the complex conjugation of $u(p, \alpha)$ is equal to the function $u^*(p)$ with p rotated by $-\alpha$.

2.2.4 Constraints of supersymmetry on scattering

In this chapter we will study 2×2 scattering of particles in an $\mathcal{N} = 1$ supersymmetric field theory. In this subsection we set up our conventions and notations and explore the constraints of supersymmetry on scattering amplitudes.

Let us consider the scattering process

$$1 + 2 \to 3 + 4 \tag{2.30}$$

where $1, 2$ represent initial state particles and $3, 4$ are final state particles. Let the i^{th} particle be associated with the superfield Φ_i. As a scattering amplitude represents the transition between free incoming and free outgoing on-shell particles, the initial and final states of Φ_i are effectively subject to the free equation of motion

$$\left(D^2 + m_i\right)\Phi_i = 0 \tag{2.31}$$

where $D^2 = \frac{1}{2}D^\alpha D_\alpha$. The general solution to this free equation of motion is

$$\begin{aligned} \Phi(x,\theta) = \int \frac{d^2p}{\sqrt{2p^0}(2\pi)^2}\Bigg[&\Big(a(\mathbf{p})(1+m\theta^2) + \theta^\alpha u_\alpha(\mathbf{p})\alpha(\mathbf{p})\Big)e^{ip.x} \\ &+ \Big(a^{c\dagger}(\mathbf{p})(1+m\theta^2) + \theta^\alpha v_\alpha(\mathbf{p})\alpha^{c\dagger}(\mathbf{p})\Big)e^{-ip.x}\Bigg] \end{aligned} \tag{2.32}$$

where $a/a^\dagger$ are annihilation/creation operator for the bosonic particles and $\alpha/\alpha^\dagger$ are annihilation/creation operators for the fermionic particles respectively [12]. The bosonic and fermionic oscillators obey the commutation relations

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^2\delta^2(\mathbf{p}-\mathbf{p}'), \quad [a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^2\delta^2\left(\mathbf{p}-\mathbf{p}'\right)\ . \tag{2.33}$$

(a^c and α^c obey analogous commutation relations).

[12] Similarly $a^c/a^{c\dagger}$ and $\alpha^c/\alpha^{c\dagger}$ are the annihilation/creation operators for the bosonic and fermionic anti-particles respectively.

The action of the supersymmetry operator on a free on-shell superfield is simple

$$\begin{aligned}
&[Q_\alpha, \Phi_i] = \\
&\quad Q_\alpha \Phi_i = i \int \frac{d^2p}{(2\pi)^2 \sqrt{2p^0}} \Bigg[\Big(u_\alpha(\mathbf{p})(1 + m\theta^2)\alpha(\mathbf{p}) + \theta^\beta(-u_\beta(\mathbf{p}) u^*_\alpha(\mathbf{p})) a(\mathbf{p}) \Big) e^{ip.x} \\
&\qquad + \Big(v_\alpha(\mathbf{p})(1 + m\theta^2)\alpha^{c\dagger}(\mathbf{p}) + \theta^\beta(v_\beta(\mathbf{p}) v^*_\alpha(\mathbf{p})) a^{c\dagger}(\mathbf{p}) \Big) e^{-ip.x} \Bigg] .
\end{aligned}$$

In other words, the action of the supersymmetry generator on on-shell superfields is given by

$$\begin{aligned}
-iQ_\alpha &= u_\alpha(\mathbf{p}_i) \left(a\partial_\alpha + a^c \partial_{\alpha^c} \right) + u^*_\alpha(\mathbf{p}_i) \left(-\alpha\partial_a + \alpha^c \partial_{a^c} \right) \\
&\quad + v_\alpha(\mathbf{p}_i) \left(a^\dagger \partial^\dagger_\alpha + (a^c)^\dagger \partial_{(\alpha^c)^\dagger} \right) + v^*_\alpha(\mathbf{p}_i) \left(\alpha^\dagger \partial^\dagger_a + (\alpha^c)^\dagger \partial_{(a^c)^\dagger} \right) .
\end{aligned} \tag{2.34}$$

The explicit action of Q_α on on-shell superfields may be repackaged as follows. Let us define a superfield of annihilation operators, and another superfield for creation operators:

$$\begin{aligned}
A_i(\mathbf{p}) &= a_i(\mathbf{p}) + \alpha_i(\mathbf{p})\theta_i \ , \\
A^\dagger_i(\mathbf{p}) &= a^\dagger_i(\mathbf{p}) + \theta_i \alpha^\dagger_i(\mathbf{p}) \ .
\end{aligned} \tag{2.35}$$

Here θ_i is a new formal superspace parameter (θ_i has nothing to do with the θ_α that appear in the superfield action (2.5)). It follows from (2.34) and (2.35) that

$$\begin{aligned}
[Q_\alpha, A_i(\mathbf{p}_i, \theta_i)] &= Q^1_\alpha A_i(\mathbf{p}_i, \theta_i) \\
[Q_\alpha, A^\dagger_i(\mathbf{p}_i, \theta_i)] &= Q^2_\alpha A^\dagger_i(\mathbf{p}_i, \theta_i)
\end{aligned} \tag{2.36}$$

where

$$\begin{aligned}
Q^1_\beta &= i \left(-u_\beta(\mathbf{p}) \frac{\overrightarrow{\partial}}{\partial\theta} + u^*_\beta(\mathbf{p})\theta \right) \\
Q^2_\beta &= i \left(v_\beta(\mathbf{p}) \frac{\overrightarrow{\partial}}{\partial\theta} + v^*_\beta(\mathbf{p})\theta \right) .
\end{aligned} \tag{2.37}$$

We are interested in the S-matrix

$$\begin{aligned}
&S(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) \sqrt{(2p^0_1)(2p^0_2)(2p^0_3)(2p^0_4)} = \\
&\qquad \langle 0 | A_4(\mathbf{p}_4, \theta_4) A_3(\mathbf{p}_3, \theta_3) U A^\dagger_2(\mathbf{p}_2, \theta_2) A^\dagger_1(\mathbf{p}_1, \theta_1) | 0 \rangle
\end{aligned} \tag{2.38}$$

where U is an evolution operator (the RHS denotes the transition amplitude from the in state with particles 1 and 2 to the out state with particles 3 and 4).

The condition that the S-matrix defined in (2.38) is invariant under supersymmetry follows from the action of supersymmetries on oscillators given in (2.34). The resultant equation for the S-matrix may be written in terms of the operators defined in (2.37) as

$$\Big(\overrightarrow{Q}^1_\alpha(\mathbf{p}_1,\theta_1)+\overrightarrow{Q}^1_\alpha(\mathbf{p}_2,\theta_2) + \overrightarrow{Q}^2_\alpha(\mathbf{p}_3,\theta_3)+\overrightarrow{Q}^2_\alpha(\mathbf{p}_4,\theta_4)\Big)S(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4)=0\ . \tag{2.39}$$

We have explicitly solved (2.39); the solution[13] is given by

$$\begin{aligned} S(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4) &= \mathcal{S}_B+\mathcal{S}_F\ \theta_1\theta_2\theta_3\theta_4+\left(\frac{1}{2}C_{12}\mathcal{S}_B-\frac{1}{2}C^*_{34}\mathcal{S}_F\right)\ \theta_1\theta_2 \\ &+\left(\frac{1}{2}C_{13}\mathcal{S}_B-\frac{1}{2}C^*_{24}\mathcal{S}_F\right)\ \theta_1\theta_3+\left(\frac{1}{2}C_{14}\mathcal{S}_B+\frac{1}{2}C^*_{23}\mathcal{S}_F\right)\ \theta_1\theta_4+\left(\frac{1}{2}C_{23}\mathcal{S}_B+\frac{1}{2}C^*_{14}\mathcal{S}_F\right)\ \theta_2\theta_3 \\ &+\left(\frac{1}{2}C_{24}\mathcal{S}_B-\frac{1}{2}C^*_{13}\mathcal{S}_F\right)\ \theta_2\theta_4+\left(\frac{1}{2}C_{34}\mathcal{S}_B-\frac{1}{2}C^*_{12}\mathcal{S}_F\right)\ \theta_3\theta_4 \end{aligned} \tag{2.40}$$

where

$$\begin{aligned} \frac{1}{2}C_{12} &= -\frac{1}{4m}v^*(\mathbf{p}_1)v^*(\mathbf{p}_2) & \frac{1}{2}C_{23} &= -\frac{1}{4m}v^*(\mathbf{p}_2)u^*(\mathbf{p}_3) \\ \frac{1}{2}C_{13} &= -\frac{1}{4m}v^*(\mathbf{p}_1)u^*(\mathbf{p}_3) & \frac{1}{2}C_{24} &= -\frac{1}{4m}v^*(\mathbf{p}_2)u^*(\mathbf{p}_4) \\ \frac{1}{2}C_{14} &= -\frac{1}{4m}v^*(\mathbf{p}_1)u^*(\mathbf{p}_4) & \frac{1}{2}C_{34} &= -\frac{1}{4m}u^*(\mathbf{p}_3)u^*(\mathbf{p}_4) \end{aligned} \tag{2.41}$$

and

$$\begin{aligned} \frac{1}{2}C^*_{12} &= \frac{1}{4m}v(\mathbf{p}_1)v(\mathbf{p}_2) & \frac{1}{2}C^*_{23} &= \frac{1}{4m}v(\mathbf{p}_2)u(\mathbf{p}_3) \\ \frac{1}{2}C^*_{13} &= \frac{1}{4m}v(\mathbf{p}_1)u(\mathbf{p}_3) & \frac{1}{2}C^*_{24} &= \frac{1}{4m}v(\mathbf{p}_2)u(\mathbf{p}_4) \\ \frac{1}{2}C^*_{14} &= \frac{1}{4m}v(\mathbf{p}_1)u(\mathbf{p}_4) & \frac{1}{2}C^*_{34} &= \frac{1}{4m}u(\mathbf{p}_3)u(\mathbf{p}_4) \end{aligned} \tag{2.42}$$

Note that the general solution to (2.39) is given in terms of two arbitrary functions $\mathcal{S}_B$ and $\mathcal{S}_F$ of the four momenta; (2.39) determines the remaining six functions in the general

[13]The superspace S-matrix (2.40) encodes different processes allowed by supersymmetry in the theory. In particular, the presence of Grassmann parameters indicates fermionic in (θ_1,θ_2) and fermionic out (θ_3,θ_4) states. The absence of Grassmann parameter indicates a bosonic in/out state. Thus, the no θ term $\mathcal{S}_B$ encodes the $2\to 2$ S-matrix for a purely bosonic process, while the four θ term $\mathcal{S}_F$ encodes the $2\to 2$ S-matrix of a purely fermionic process. Note in particular that S-matrices corresponding to all other $2\to 2$ processes that involve both bosons and fermions are completely determined in terms of the S-matrices $\mathcal{S}_B$ and $\mathcal{S}_F$ together with (2.41) and (2.42).

expansion of the S-matrix in terms of these two functions. See appendix A.2 for a check of these relations from another viewpoint (involving off-shell supersymmetry of the effective action, see section §2.3.4)

Although we are principally interested in $\mathcal{N}=1$ supersymmetric theories in this chapter, we will sometimes study the special limit $w=1$ in which (2.5) enjoys an enhanced $\mathcal{N}=2$ supersymmetry. In this case the additional supersymmetry further constrains the S-matrix. In appendix A.3 we demonstrate that the additional supersymmetry determines $\mathcal{S}_B$ in terms of $\mathcal{S}_F$. In the $\mathcal{N}=2$ case, in other words, all components of the S-matrix are determined by supersymmetry in terms of the four boson scattering matrix.

2.2.5 Supersymmetry and dual supersymmetry

The strong weak coupling duality we study is conjectured to be a Bose-Fermi duality. In other words

$$a^D = \alpha, \quad \alpha^D = a \tag{2.43}$$

together with a similar exchange of bosons and fermions for creation operators (the superscript D stands for 'dual'). Suppose we define

$$\begin{aligned} A_i^D(\mathbf{p}) &= a_i^D(\mathbf{p}) + \alpha_i^D(\mathbf{p})\theta_i \ , \\ (A^D)_i^\dagger(\mathbf{p}) &= (a^D)_i^\dagger(\mathbf{p}) + \theta_i(\alpha_i^D)^\dagger(\mathbf{p}) \ . \end{aligned} \tag{2.44}$$

The dual supersymmetries must act in the same way on A^D and $(A^D)^\dagger$ as ordinary supersymmetries act on A and A^D. In other words the action of dual supersymmetries on A^D and $(A^D)^\dagger$ is given by

$$\begin{aligned} [Q_\alpha^D, A_i^D(\mathbf{p}_i,\theta_i)] &= (Q^D)^1_\alpha A_i^D(\mathbf{p}_i,\theta_i) \ , \\ [Q_\alpha^D, (A^D)_i^\dagger(\mathbf{p}_i,\theta_i)] &= (Q^D)^2_\alpha (A^D)_i^\dagger(\mathbf{p}_i,\theta_i) \ , \end{aligned} \tag{2.45}$$

where

$$\begin{aligned} (Q^D)^1_\beta &= i\left(-u_\beta(\mathbf{p},-m)\frac{\overrightarrow{\partial}}{\partial\theta} - v_\beta(\mathbf{p},-m)\theta\right) \ , \\ (Q^D)^2_\beta &= i\left(v_\beta(\mathbf{p},-m)\frac{\overrightarrow{\partial}}{\partial\theta} - u_\beta(\mathbf{p},-m)\theta\right) \ . \end{aligned} \tag{2.46}$$

The spinors in (2.46) are all evaluated at $-m$ as duality flips the sign of the pole mass.

The action of the dual supersymmetries on A and $A^\dagger$ is obtained from (2.46) upon performing the interchange $\theta \leftrightarrow \partial_\theta$ (this accounts for the interchange of bosons and fermions). Using also (2.26) we find that

$$\begin{aligned} [Q_\alpha^D, A_i(\mathbf{p}_i,\theta_i)] &= -Q^1_\alpha A_i^D(\mathbf{p}_i,\theta_i) \ , \\ [Q_\alpha^D, A_i^\dagger(\mathbf{p}_i,\theta_i)] &= Q^2_\alpha (A^D)_i^\dagger(\mathbf{p}_i,\theta_i) \ . \end{aligned} \tag{2.47}$$

It follows, in particular, that an S-matrix invariant under the usual supersymmetries is automatically invariant under dual supersymmetries. In other words on-shell supersymmetry 'commutes' with duality.

2.2.6 Naive crossing symmetry and supersymmetry

Let us define the analytically rotated supersymmetry operators [14]

$$\begin{aligned} Q^1_\beta(\mathbf{p},\alpha,\theta) &= i\left(-u_\beta(\mathbf{p},\alpha)\frac{\overrightarrow{\partial}}{\partial\theta} + u^*_\beta(\mathbf{p},-\alpha)\theta\right) \ , \\ Q^2_\beta(\mathbf{p},\alpha,\theta) &= i\left(v_\beta(\mathbf{p},\alpha)\frac{\overrightarrow{\partial}}{\partial\theta} + v^*_\beta(\mathbf{p},-\alpha)\theta\right) \ . \end{aligned} \tag{2.48}$$

It is easily verified from these definitions that

$$Q^2_\alpha(\mathbf{p},0,-i\theta) = Q^1_\alpha(\mathbf{p},\pi,\theta) \ . \tag{2.49}$$

Using (2.49) the equation (2.39) may equivalently be written as

$$\begin{aligned} \Big(\overrightarrow{Q}{}^1_\alpha(\mathbf{p}_1,0,\theta_1) + \overrightarrow{Q}{}^1_\alpha(\mathbf{p}_2,0,\theta_2) \\ + \overrightarrow{Q}{}^1_\alpha(\mathbf{p}_3,\pi,\theta_3) + \overrightarrow{Q}{}^1_\alpha(\mathbf{p}_4,\pi,\theta_4)\Big) S(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,-i\theta_3,\mathbf{p}_4,-i\theta_4) = 0 \end{aligned} \tag{2.50}$$

with $p_1 + p_2 = p_3 + p_4$.

The constraints of supersymmetry on the S-matrix are consistent with (naive) crossing symmetry. In order to make this manifest, we define a 'master' function S_M

$$S_M(\mathbf{p}_1,\phi_1,\theta_1,\mathbf{p}_2,\phi_2,\theta_2,\mathbf{p}_3,\phi_3,\theta_3,\mathbf{p}_4,\phi_4,\theta_4) \ .$$

The master function S_M is defined so that

$$S(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,-i\theta_3,\mathbf{p}_4,-i\theta_4) = S_M(\mathbf{p}_1,0,\theta_1,\mathbf{p}_2,0,\theta_2,\mathbf{p}_3,\pi,\theta_3,\mathbf{p}_4,\pi,\theta_4) \tag{2.51}$$

In other words S_M is S with the replacement $-i\theta_3 \to \theta_3$, $-i\theta_4 \to \theta_4$, analytically rotated to general values of the phase ϕ_1, ϕ_2, ϕ_3 and ϕ_4. It follows from (2.50) that the master equation S_M obeys the completely symmetrical supersymmetry equation

[14] Note that the notation $u^*_\beta(\mathbf{p},-\alpha)$ means that the analytically rotated function of u^* in (2.28) is evaluated at the phase $-\alpha$.

$$\Big(\overrightarrow{Q}^1_\alpha(\mathbf{p}_1,\phi_1,\theta_1)+\overrightarrow{Q}^1_\alpha(\mathbf{p}_2,\phi_2,\theta_2)+\overrightarrow{Q}^1_\alpha(\mathbf{p}_3,\phi_3,\theta_3)$$
$$+\overrightarrow{Q}^1_\alpha(\mathbf{p}_4,\phi_4,\theta_4)\Big)S_M(\mathbf{p}_1,\phi_1,\theta_1,\mathbf{p}_2,\phi_2,\theta_2,\mathbf{p}_3,\phi_3,\theta_3,\mathbf{p}_4,\phi_4,\theta_4)=0 \quad (2.52)$$

The function S_M encodes the scattering matrices in all channels. In order to extract the S-matrix for $p_i+p_j \to p_k+p_m$ with $p_i+p_j = p_k+p_m$ (with (i, j, k, m) being any permutation of (1, 2, 3, 4)) we simply evaluate the function S_M with ϕ_i and ϕ_j set to zero, ϕ_k and ϕ_m set to π, θ_i and θ_j left unchanged and θ_k and θ_m replaced by $i\theta_k$ and $i\theta_m$. The fact that the master equation obeys an equation that is symmetrical in the labels $1,2,3,4$ is the statement of (naive) crossing symmetry.

The solution to the differential equation (2.52) is

$$S_M(\mathbf{p}_1,\phi_1,\theta_1,\mathbf{p}_2,\phi_2,\theta_2,\mathbf{p}_3,\phi_3,\theta_3,\mathbf{p}_4,\phi_4,\theta_4)=\tilde{\mathcal{S}_B}+\tilde{\mathcal{S}_F}\theta_1\theta_2\theta_3\theta_4$$
$$+\frac{\tilde{\mathcal{S}_B}}{4}\sum_{i,j=1}^{4}D_{ij}(\mathbf{p}_i,\phi_i,\mathbf{p}_j,\phi_j)\theta_i\theta_j-\frac{\tilde{\mathcal{S}_F}}{8}\sum_{i,j,k,l=1}^{4}\epsilon^{ijkl}\tilde{D}_{ij}(\mathbf{p}_i,\phi_i,\mathbf{p}_j,\phi_j)\theta_k\theta_l \quad (2.53)$$

where

$$\frac{1}{2}D_{ij}(\mathbf{p}_i,\phi_i,\mathbf{p}_j,\phi_j)=-\frac{1}{4m}u^*(\mathbf{p}_i,-\phi_i)u^*(\mathbf{p}_j,-\phi_j)\ ,$$
$$\frac{1}{2}\tilde{D}_{ij}(\mathbf{p}_i,\phi_i,\mathbf{p}_j,\phi_j)=\frac{1}{4m}u(\mathbf{p}_i,\phi_i)u(\mathbf{p}_j,\phi_j)\ . \quad (2.54)$$

In the above equations '$*$' means complex conjugation and the spinor indices are contracted from NW-SE as usual. To summarise, S_M obeys the supersymmetric ward identity and is completely solved in terms of two analytic functions $\tilde{\mathcal{S}_B}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)$ and $\tilde{\mathcal{S}_F}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)$ of the momenta.

As we have explained under (2.52), the S-matrix corresponding to scattering processes in any given channel can be simply extracted out of S_M. For example, let S denote the the S-matrix in the channel with p_1,p_2 as in-states and p_3,p_4 as out-states. Then

$$S(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4)=S_M(\mathbf{p}_1,\pi,i\theta_1,\mathbf{p}_2,\pi,i\theta_2,\mathbf{p}_3,0,\theta_3,\mathbf{p}_4,0,\theta_4)\ . \quad (2.55)$$

It is easily verified that (2.54) together with (2.29) imply (2.41).

Notice that (2.55) maps $\tilde{\mathcal{S}_B}$ to $\mathcal{S}_B$ while $\tilde{\mathcal{S}_F}$ is mapped to $-\mathcal{S}_F$ [15]. The minus sign in the continuation of $\mathcal{S}_F$ has an interesting explanation. The four fermion amplitude $\mathcal{S}_F$ has a phase ambiguity. This ambiguity follows from the fact that $\mathcal{S}_F$ is the overlap of initial and final fermions states. These initial and final states are written in terms of the spinors u_α

[15] Of course $\tilde{\mathcal{S}_B}$ and $\tilde{\mathcal{S}_F}$ are evaluated at $\phi_1=\phi_2=\pi$ while $\mathcal{S}_B$ and $\mathcal{S}_F$ are evaluated at $\phi_1=\phi_2=0$; roughly speaking this amounts to the replacement $p_1^\mu \to -p_1^\mu$, $p_2^\mu \to -p_2^\mu$.

and v_α, which are defined as appropriately normalised solutions of the Dirac equation are inherently ambiguous upto a phase. It is easily verified that the quantity

$$(u^*(\mathbf{p}_1, -\phi_1)u(\mathbf{p}_3, \phi_3))\,(u^*(\mathbf{p}_2, -\phi_2)u(\mathbf{p}_4, \phi_4))$$

has the same phase ambiguity as $\mathcal{S}_F$. If we define an auxiliary quantity $\tilde{\mathcal{S}_f}$ by the equation

$$\tilde{\mathcal{S}_F} = -\frac{1}{4m^2}\,(u^*(\mathbf{p}_1, -\phi_1)u(\mathbf{p}_3, \phi_3))\,(u^*(\mathbf{p}_2, -\phi_2)u(\mathbf{p}_4, \phi_4))\,\tilde{\mathcal{S}_f} \tag{2.56}$$

and $\mathcal{S}_f$ by

$$\mathcal{S}_F = -\frac{1}{4m^2}\,(u^*(\mathbf{p}_1)u(\mathbf{p}_3))\,(u^*(\mathbf{p}_2)u(\mathbf{p}_4))\,\mathcal{S}_f \tag{2.57}$$

then the phases of $\mathcal{S}_f$ and $\tilde{\mathcal{S}_f}$ are unambiguous and so potentially physical. As the quantity

$$(u^*(\mathbf{p}_1, -\phi_1)u(\mathbf{p}_3, \phi_3))\,(u^*(\mathbf{p}_2, -\phi_2)u(\mathbf{p}_4, \phi_4))$$

picks up a minus sign under the phase rotation that takes us from S_M to S. It follows that $\tilde{\mathcal{S}_f}$ rotates to $\mathcal{S}_f$ with no minus sign.

2.2.7 Properties of the convolution operator

Like any matrices, S-matrices can be multiplied. The multiplication rule for two S-matrices, S_1 and S_2, expressed as functions in on-shell superspace is given by

$$\begin{aligned} S_1 \star S_2 \equiv \int d\Gamma S_1(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{k}_3, \phi_1, \mathbf{k}_4, \phi_2) \exp(\phi_1\phi_3 + \phi_2\phi_4) 2k_1^0(2\pi)^2\delta^{(2)}(\mathbf{k}_3 - \mathbf{k}_1) \\ 2k_2^0(2\pi)^2\delta^{(2)}(\mathbf{k}_4 - \mathbf{k}_2) S_2(\mathbf{k}_1, \phi_3, \mathbf{k}_2, \phi_4, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) \end{aligned} \tag{2.58}$$

where the measure $d\Gamma$ is

$$d\Gamma = \frac{d^2k_3}{2k_3^0(2\pi)^2}\frac{d^2k_4}{2k_4^0(2\pi)^2}\frac{d^2k_1}{2k_1^0(2\pi)^2}\frac{d^2k_2}{2k_2^0(2\pi)^2} d\phi_1 d\phi_3 d\phi_2 d\phi_4 \ . \tag{2.59}$$

It is easily verified that the on-shell superfield I

$$\begin{aligned} I(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) &= \exp(\theta_1\theta_3 + \theta_2\theta_4) I(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) \\ I(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) &= 2p_3^0(2\pi)^2\delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_3) 2p_4^0(2\pi)^2\delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_4) \end{aligned} \tag{2.60}$$

is the identity operator under this multiplication rule, i.e.

$$S \star I = I \star S = S \tag{2.61}$$

for any S. It may be verified that I defined in (2.60) obeys (2.39) and so is supersymmetric.

In appendix §A.4 we demonstrate that if S_1 and S_2 are on-shell superfields that obey (2.39), then $S_1 \star S_2$ also obeys (2.39). In other words the product of two supersymmetric S-matrices is also supersymmetric.

The on-shell superfield corresponding to $S^\dagger$ is given in terms of the on-shell superfield corresponding to S by the equation

$$S^\dagger(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) = S^*(\mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4, \mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2) \ . \tag{2.62}$$

The equation satisfied by $S^\dagger$ can be obtained by complex conjugating and interchanging the momenta in the supersymmetry invariance condition for S (see (A.68)). It follows from the anti-hermiticity of Q that

$$\left(Q^*_{u(\mathbf{p}_1)} + Q^*_{u(\mathbf{p}_2)} + Q_{u(\mathbf{p}_3)} + Q_{u(\mathbf{p}_4)}\right) S^*(\mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4, \mathbf{p}_1, \theta_3, \mathbf{p}_2, \theta_4) = 0 \tag{2.63}$$

which implies $[Q, S^\dagger] = 0$. Thus $S^\dagger$ is supersymmetric if and only if S is supersymmetric.

2.2.8 Unitarity of scattering

The unitarity condition

$$SS^\dagger = \mathbb{I} \tag{2.64}$$

may be rewritten in the language of on-shell superfields as

$$(S \star S^\dagger - I) = 0 \ . \tag{2.65}$$

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It follows from the general results of the previous subsection that the LHS of (2.65) is supersymmetric, i.e it obeys (2.39). Recall that any on-shell superfield that obeys (2.39) must take the form (2.40) where $\mathcal{S}_B$ and $\mathcal{S}_F$ are the zero theta and 4 theta terms in the expansion of the corresponding object. In particular, in order to verify that the LHS of (2.65) vanishes, it is sufficient to verify that its zero and 4 theta components vanish.

Inserting the explicit solutions for S and $S^\dagger$, one finds that the no-theta term of (2.65) is proportional to (we have used that $k_3 \cdot k_4 = p_3 \cdot p_4$ on-shell)

$$\begin{aligned}
&\int \frac{d^2k_3}{2k_3^0(2\pi)^2}\frac{d^2k_4}{2k_4^0(2\pi)^2} \left[\mathcal{S}_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4)\right.\\
&-\frac{1}{16m^2}\left(2(p_3 \cdot p_4 + m^2)\mathcal{S}_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4)\right.\\
&+u^*(\mathbf{k}_3)u^*(\mathbf{k}_4)\ v^*(\mathbf{p}_3)v^*(\mathbf{p}_4)\mathcal{S}_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4)\\
&+v(\mathbf{p}_1)v(\mathbf{p}_2)\ u(\mathbf{k}_3)u(\mathbf{k}_4)\mathcal{S}_F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4)\\
&\left.\left.+v(\mathbf{p}_1)v(\mathbf{p}_2)\ v^*(\mathbf{p}_3)v^*(\mathbf{p}_4)\mathcal{S}_F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4)\right)\right]\\
&= 2p_3^0(2\pi)^2\delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_3)2p_4^0(2\pi)^2\delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_4) \ .
\end{aligned} \tag{2.66}$$

[16] As explained in [4], the unitarity equation for 2×2 does not receive contributions from $2 \times n$ scattering in the large-N limits studied here as well.

The four theta term in (2.65) is proportional to

$$\begin{aligned}
&\int \frac{d^2k_3}{2k_3^0(2\pi)^2}\frac{d^2k_4}{2k_4^0(2\pi)^2}\,[-\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\\
&+\frac{1}{16m^2}\left(2(p_3\cdot p_4+m^2)\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\right.\\
&+u(\mathbf{k}_3)u(\mathbf{k}_4)\ v(\mathbf{p}_3)v(\mathbf{p}_4)\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\\
&+v^*(\mathbf{p}_1)v^*(\mathbf{p}_2)\ u^*(\mathbf{k}_3)u^*(\mathbf{k}_4)\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\\
&\left.+v^*(\mathbf{p}_1)v^*(\mathbf{p}_2)\ v(\mathbf{p}_3)v(\mathbf{p}_4)\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\right)]\\
&=-2p_3^0(2\pi)^2\delta^{(2)}(\mathbf{p}_1-\mathbf{p}_3)2p_4^0(2\pi)^2\delta^{(2)}(\mathbf{p}_2-\mathbf{p}_4)\ . \qquad (2.67)
\end{aligned}$$

The equations (2.66) and (2.67) are necessary and sufficient to ensure unitarity.

(2.66) and (2.67) may be thought of as constraints imposed by unitarity on the four boson scattering matrix $\mathcal{S}_B$ and the four fermion scattering matrix $\mathcal{S}_F$. These conditions are written in terms of the on-shell spinors u and v (rather than the momenta of the scattering particles for a reason we now pause to review. Recall that the Dirac equation and normalisation conditions define u_α and v_α only upto an undetermined phase (which could be a function of momentum). An expression built out of u's and v's can be written unambiguously in terms of on-shell momenta if and only if all undetermined phases cancel out. The phases of terms involving $\mathcal{S}_F$ in (2.66) and (2.67) do not cancel. This might at first appear to be a paradox; surely the unitarity (or lack) of an S-matrix cannot depend on the unphysical choice of an arbitrary phase. The resolution to this 'paradox' is simple; the function $\mathcal{S}_F$ is itself not phase invariant, but transforms under phase transformations like $(u(\mathbf{p}_1)u(\mathbf{p}_2))\,(v(\mathbf{p}_3)v(\mathbf{p}_4))$. It is thus useful to define

$$\mathcal{S}_F=\frac{1}{4m^2}\,(u(\mathbf{p}_1)u(\mathbf{p}_2))\,(v(\mathbf{p}_3)v(\mathbf{p}_4))\,\mathcal{S}_f\ . \qquad (2.68)$$

The utility of this definition is that $\mathcal{S}_f$ does not suffer from a phase ambiguity. Rewritten in terms of $\mathcal{S}_B$ and $\mathcal{S}_f$, the unitarity equations may be written entirely in terms of participating momenta (with no spinors) [17]. In terms of the quantity

$$Y(\mathbf{p}_3,\mathbf{p}_4)=\frac{2(p_3\cdot p_4+m^2)}{16m^2} \qquad (2.69)$$

and

$$d\Gamma'=\frac{d^2k_3}{2k_3^0(2\pi)^2}\frac{d^2k_4}{2k_4^0(2\pi)^2}$$

[17]See §A.5 for a derivation of this result.

$$\int d\Gamma' \Big[\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)$$
$$-Y(\mathbf{p}_3,\mathbf{p}_4)\Big(\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)+4Y(\mathbf{p}_1,\mathbf{p}_2)\mathcal{S}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\Big)$$
$$\Big(\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)+4Y(\mathbf{p}_3,\mathbf{p}_4)\mathcal{S}_f^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\Big)\Big] = 2p_3^0(2\pi)^2\delta^{(2)}(\mathbf{p}_1-\mathbf{p}_3)2p_4^0(2\pi)^2\delta^{(2)}(\mathbf{p}_2-\mathbf{p}_4) \tag{2.70}$$

and

$$\int d\Gamma' \Big[-16Y^2(\mathbf{p}_3,\mathbf{p}_4)\mathcal{S}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_f^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)$$
$$+Y(\mathbf{p}_3,\mathbf{p}_4)\Big(\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)+4Y(\mathbf{p}_1,\mathbf{p}_2)\mathcal{S}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\Big)$$
$$\Big(\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)+4Y(\mathbf{p}_3,\mathbf{p}_4)\mathcal{S}_f^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\Big)\Big] = -2p_3^0(2\pi)^2\delta^{(2)}(\mathbf{p}_1-\mathbf{p}_3)2p_4^0(2\pi)^2\delta^{(2)}(\mathbf{p}_2-\mathbf{p}_4). \tag{2.71}$$

The equations (2.70) and (2.71) followed from (2.64). It is useful to rephrase the above equations in terms of the "T matrix" that represents the actual interacting part of the "S-matrix". Using the definition of the Identity operator (2.60) we can write a superfield expansion to define the "T matrix" as

$$\begin{aligned} S(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{k}_3,\theta_3,\mathbf{k}_4,\theta_4) =& I(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{k}_3,\theta_3,\mathbf{k}_4,\theta_4) \\ &+ i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)T(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{k}_3,\theta_3,\mathbf{k}_4,\theta_4)\ . \end{aligned} \tag{2.72}$$

The identity operator is defined in (2.60) is a supersymmetry invariant. It follows that the " T matrix" is also invariant under supersymmetry. In other words the "T matrix" obeys (2.39) and has a superfield expansion [18]

$$\begin{aligned} T(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4) =& \mathcal{T}_B + \mathcal{T}_F\ \theta_1\theta_2\theta_3\theta_4 + \left(\frac{1}{2}C_{12}\mathcal{T}_B - \frac{1}{2}C_{34}^*\mathcal{T}_F\right)\ \theta_1\theta_2 \\ +\left(\frac{1}{2}C_{13}\mathcal{T}_B - \frac{1}{2}C_{24}^*\mathcal{T}_F\right)\ \theta_1\theta_3 &+ \left(\frac{1}{2}C_{14}\mathcal{T}_B + \frac{1}{2}C_{23}^*\mathcal{T}_F\right)\ \theta_1\theta_4 + \left(\frac{1}{2}C_{23}\mathcal{T}_B + \frac{1}{2}C_{14}^*\mathcal{T}_F\right)\ \theta_2\theta_3 \\ &+ \left(\frac{1}{2}C_{24}\mathcal{T}_B - \frac{1}{2}C_{13}^*\mathcal{T}_F\right)\ \theta_2\theta_4 + \left(\frac{1}{2}C_{34}\mathcal{T}_B - \frac{1}{2}C_{12}^*\mathcal{T}_F\right)\ \theta_3\theta_4 \end{aligned} \tag{2.73}$$

[18] The matrices $\mathcal{T}_B$ and $\mathcal{T}_F$ correspond to the T matrices of the four boson and four fermion scattering respectively.

where

$$\mathcal{T}_F = \frac{1}{4m^2}\left(u(\mathbf{p}_1)u(\mathbf{p}_2)\right)\left(v(\mathbf{p}_3)v(\mathbf{p}_4)\right)\mathcal{T}_f \ . \tag{2.74}$$

and the coefficients C_{ij} are given as before in (2.41) and (2.42).

It follows from (2.72) that

$$\begin{aligned}\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \ ,\\ \mathcal{S}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \ .\end{aligned} \tag{2.75}$$

Substituting the definitions (2.75) into (2.70) and (2.71) the unitarity conditions can be rewritten as

$$\begin{aligned}&\int d\tilde{\Gamma}\Big[\mathcal{T}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{T}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\\ &-Y(\mathbf{p}_3,\mathbf{p}_4)\Big(\mathcal{T}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)+4Y(\mathbf{p}_1,\mathbf{p}_2)\mathcal{T}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\Big)\\ &\Big(\mathcal{T}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)+4Y(\mathbf{p}_3,\mathbf{p}_4)\mathcal{T}_f^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\Big)\Big] = i(\mathcal{T}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)-\mathcal{T}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{p}_1,\mathbf{p}_2))\end{aligned} \tag{2.76}$$

and

$$\begin{aligned}&\int d\tilde{\Gamma}\Big[-16Y^2(\mathbf{p}_3,\mathbf{p}_4)\mathcal{T}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{T}_f^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\\ &+Y(\mathbf{p}_3,\mathbf{p}_4)\Big(\mathcal{T}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)+4Y(\mathbf{p}_1,\mathbf{p}_2)\mathcal{T}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\Big)\\ &\Big(\mathcal{T}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)+4Y(\mathbf{p}_3,\mathbf{p}_4)\mathcal{T}_f^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\Big)\Big]\\ &= 4iY(\mathbf{p}_3,\mathbf{p}_4)\left(\mathcal{T}_f^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{p}_1,\mathbf{p}_2)-\mathcal{T}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\right)\end{aligned} \tag{2.77}$$

where

$$d\tilde{\Gamma} = (2\pi)^3\delta^3(p_1+p_2-k_3-k_4)\frac{d^2k_3}{2k_3^0(2\pi)^2}\frac{d^2k_4}{2k_4^0(2\pi)^2} \ .$$

The equations (2.76) and (2.77) can be put in a more user friendly form by going to the centre of mass frame with the definition

$$\begin{aligned}&p_1 = \left(\sqrt{p^2+m^2},p,0\right),\ p_2 = \left(\sqrt{p^2+m^2},-p,0\right)\\ &p_3 = \left(\sqrt{p^2+m^2},p\cos(\theta),p\sin(\theta)\right),\ p_4 = \left(\sqrt{p^2+m^2},-p\cos(\theta),-p\sin(\theta)\right)\end{aligned} \tag{2.78}$$

where θ is the scattering angle between p_1 and p_3. In terms of the Mandelstam variables

$$s = -(p_1+p_2)^2 \ , t = -(p_1-p_3)^2, \ u = (p_1-p_4)^2, \ s+t+u = 4m^2 \ ,$$
$$s = 4(p^2+m^2) \ , t = -2p^2(1-\cos(\theta)) \ , u = -2p^2(1+\cos(\theta)) \ . \tag{2.79}$$

Using the definitions we see that (2.69) becomes

$$Y = \frac{2(p_3 \cdot p_4 + m^2)}{16m^2} = \frac{-s+4m^2}{16m^2} = Y(s) \ . \tag{2.80}$$

Then (2.76) and (2.77) can be put in the form (See for instance eq 2.58-eq 2.59 of [4])

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(- Y(s)(\mathcal{T}_B(s,\theta) + 4Y(s)\mathcal{T}_f(s,\theta))(\mathcal{T}_B^*(s,-(\alpha-\theta)) + 4Y(s)\mathcal{T}_f^*(s,-(\alpha-\theta))) + \mathcal{T}_B(s,\theta)\mathcal{T}_B^*(s,-(\alpha-\theta))\Big) = i(\mathcal{T}_B^*(s,-\alpha) - \mathcal{T}_B(s,\alpha)) \tag{2.81}$$

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(Y(s)(\mathcal{T}_B(s,\theta) + 4Y(s)\mathcal{T}_f(s,\theta))(\mathcal{T}_B^*(s,-(\alpha-\theta)) + 4Y(s)\mathcal{T}_f^*(s,-(\alpha-\theta))) - 16Y(s)^2\mathcal{T}_f(s,\theta)\mathcal{T}_f^*(s,-(\alpha-\theta))\Big) = i4Y(s)(-\mathcal{T}_f(s,\alpha) + \mathcal{T}_f^*(s,-\alpha)) \tag{2.82}$$

In a later section §2.4 we will use the simplified equations (2.81) and (2.82) for the unitarity analysis.

2.3 Exact computation of the all orders S-matrix

In this section we will present results and conjectures for the the 2×2 S-matrix of the general $\mathcal{N}=1$ theory (2.84) at all orders in the 't Hooft coupling. In §2.3.2 we recall the action for our theory and determine the bare propagators for the scalar and vector superfields. At leading order in the $\frac{1}{N}$ the vector superfield propagator is exact (it is not renormalised). However the propagator of the scalar superfield does receive corrections. In §2.3.3, we determine the all orders propagator for the superfield Φ by solving the relevant Schwinger-Dyson equation. We will then turn to the determination of the exact off-shell four point function of the superfield Φ. As in [4], we demonstrate that this four point function is the solution to a linear integral equation which we explicitly write down in §2.3.5. In a particular kinematic regime we present an exact solution to this integral equation in §2.3.6. In order to obtain the S-matrix, in §2.3.7 we take the on-shell limit of this answer. The kinematic restriction on our off-shell result turns out to be inconsistent with the on-shell limit in one of the four channels of scattering (particle - antiparticle scattering in the singlet channel) and so we do not have an explicit computation of the S-matrix in this channel. In the other three channels, however, we are able to extract the full S-matrix (with no kinematic restriction) albeit in a particular Lorentz frame. In §2.3.7 we present the unique covariant expressions for the S-matrix consistent with our results. In §2.3.8 we report our result that the covariant S-matrix reported in §2.3.7 is duality invariant. We present explicit exact results for the S-matrices in the T and U channels of scattering in §2.3.9. In §2.3.10 we present the explicit conjecture for the S-matrix in the singlet (S) channel. In §2.3.11 we report the explicit S-matrices for the $\mathcal{N}=2$ theory.

2.3.1 Supersymmetric light cone gauge

We study the general $\mathcal{N}=1$ theory (2.5). Wess Zumino gauge, employed in subsection §2.2.1 to display the physical content of our theory, is inconvenient for actual computations as it breaks manifest supersymmetry. In other words if Γ_α is chosen to lie in Wess Zumino gauge, it is in general not the case that $Q_\beta \Gamma_\alpha$ also respects this gauge condition. In all calculations presented in this chapter we will work instead in 'supersymmetric light cone gauge'

$$\Gamma_- = 0 \tag{2.83}$$

As Γ_- transforms homogeneously under supersymmetry (see (2.10)) it is obvious that this gauge choice is supersymmetric. It is also easily verified that all gauge self interactions in (2.5) vanish in our light cone gauge and the action (2.5) simplifies to

$$\begin{aligned} S_{tree} = -\int d^3x d^2\theta \Big[-\frac{\kappa}{8\pi} Tr(\Gamma^- i\partial_{--}\Gamma^-) - \frac{1}{2} D^\alpha \bar{\Phi} D_\alpha \Phi - \frac{i}{2}\Gamma^-(\bar{\Phi} D_- \Phi - D_- \bar{\Phi}\Phi) \\ + m_0 \bar{\Phi}\Phi + \frac{\pi w}{\kappa}(\bar{\Phi}\Phi)^2 \Big] \ . \end{aligned} \tag{2.84}$$

Note in particular that (2.84) is quadratic in Γ_+.

The condition (2.83) implies, in particular, that the component gauge fields in Γ_α obey

$$A_- = A_1 + iA_2 = 0$$

(see appendix §A.6 for more details and further discussion about this gauge). In other words the gauge (2.83) is a supersymmetric generalisation of ordinary light cone gauge.

2.3.2 Action and bare propagators

The bare scalar propagator that follows from (2.84) is

$$\langle \bar{\Phi}(\theta_1, p)\Phi(\theta_2, -p')\rangle = \frac{D^2_{\theta_1,p} - m_0}{p^2 + m_0^2}\delta^2(\theta_1 - \theta_2)(2\pi)^3\delta^3(p - p') \ . \tag{2.85}$$

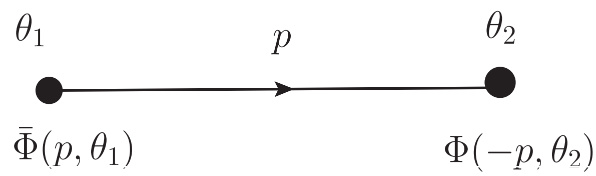

Figure 2.1: Scalar superfield propagator

where m_0 is the bare mass. We have chosen the convention for the momentum flow direction to be from $\bar{\Phi}$ to Φ (see Fig2.1). Our sign conventions are such that the momenta leaving a vertex have a positive sign. The notation $D^2_{\theta_1,p}$ means that the operator depends on θ_1 and the momentum p, the explicit form for D^2 and some useful formulae are listed in §A.1.2. The gauge superfield propagator in momentum space is

$$\langle \Gamma^-(\theta_1,p)\Gamma^-(\theta_2,-p')\rangle = -\frac{8\pi}{\kappa}\frac{\delta^2(\theta_1-\theta_2)}{p_{--}}(2\pi)^3\delta^3(p-p') \tag{2.86}$$

where $p_{--} = -(p_1+ip_2) = -p_-$. Inserting the expansion (2.6) into the LHS of (2.86) and

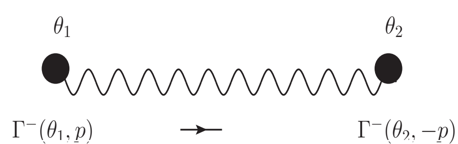

Figure 2.2: Gauge superfield propagator, the arrow indicates direction of momentum flow

matching powers of θ, we find in particular that

$$\langle A_+(p)A_3(-p')\rangle = \frac{4\pi i}{\kappa}\frac{1}{p_-}(2\pi)^3\delta^3(p-p')\ ,\ \langle A_3(p)A_+(-p')\rangle = -\frac{4\pi i}{\kappa}\frac{1}{p_-}(2\pi)^3\delta^3(p-p')\ , \tag{2.87}$$

is in perfect agreement with the propagator of the gauge field in regular (non supersymmetric) light cone gauge (see appendix A ,Eq A.7 of [23])

2.3.3 The all orders matter propagator

Constraints from supersymmetry

The exact propagator of the matter superfield Φ enjoys invariance under supersymmetry transformations which implies that

$$(Q_{\theta_1,p}+Q_{\theta_2,-p})\langle\bar{\Phi}(\theta_1,p)\Phi(\theta_2,-p)\rangle = 0 \tag{2.88}$$

where the supergenerators $Q_{\theta_1,p}$ were defined in (2.9). This constraint is easily solved. Let the exact scalar propagator take the form

$$\langle\bar{\Phi}(p,\theta_1)\Phi(-p',\theta_2)\rangle = (2\pi)^3\delta^3(p-p')P(\theta_1,\theta_2,p) \ . \tag{2.89}$$

The condition (2.88) implies that the function F obeys the equation

$$\left[\frac{\partial}{\partial\theta_1^\alpha}+\frac{\partial}{\partial\theta_2^\alpha}-p_{\alpha\beta}(\theta_1^\beta-\theta_2^\beta)\right]P(\theta_1,\theta_2,p)=0 \ . \tag{2.90}$$

The most general solution to (2.90) is

$$C_1(p^\mu)\exp(-\theta_1^\alpha p_{\alpha\beta}\theta_2^\beta)+C_2(p^\mu)\delta^2(\theta_1-\theta_2) \tag{2.91}$$

or equivalently

$$P(\theta_1,\theta_2,p)=\exp(-\theta_1^\alpha p_{\alpha\beta}\theta_2^\beta)\left(C_1(p^\mu)+C_2(p^\mu)\delta^2(\theta_1-\theta_2)\right) \tag{2.92}$$

[19] where $C_1(p^\mu)$ is an arbitrary function of p^μ of dimension m^{-2}, while $C_2(p^\mu)$ is another function of p^μ of dimension m^{-1}.

It is easily verified using the formulae (A.21) that the bare propagator (2.85) can be recast in the form (2.92) with

$$C_1=\frac{1}{p^2+m_0^2} \ , \ C_2=\frac{m_0}{p^2+m_0^2} \ . \tag{2.93}$$

In a similar manner supersymmetry constrains the terms quadratic in Φ and $\bar{\Phi}$ in the quantum effective action. In momentum space the most general supersymmetric quadratic effective action takes the form

$$S=-\int\frac{d^3p}{(2\pi)^3}d^2\theta\bar{\Phi}(p,\theta)\left(A(p)D^2+B(p)\right)\Phi(-p,\theta) \tag{2.94}$$

$$=-\int\frac{d^3p}{(2\pi)^3}d^2\theta_1d^2\theta_2\bar{\Phi}(p,\theta_1)\exp(-\theta_1^\alpha p_{\alpha\beta}\theta_2^\beta)(A(p)+B(p)\delta^2(\theta_1-\theta_2))\Phi(-p,\theta_2) \tag{2.95}$$

[20] The tree level quadratic action of our theory is clearly of the form (2.94) with $A(p)=1$ and $B(p)=m_0$.

[19] The equivalence of (2.92) and (2.91) follows from the observation that $\theta^a A_{ab}\theta^b$ vanishes if A_{ab} is symmetric in a and b.

[20] In going from the first line to the second line of (2.94) we have integrated by parts and used the identity (A.21). See appendix §A.1.2 for the expressions of superderivatives and operator D^2 in momentum space.

All orders two point function

Let the exact 1PI quadratic effective action take the form

$$S_2 = \int \frac{d^3p}{(2\pi)^3} d^2\theta_1 d^2\theta_2 \bar{\Phi}(-p,\theta_1) \left(\exp(-\theta_1^\alpha p_{\alpha\beta}\theta_2^\beta) + m_0 \delta^2(\theta_1-\theta_2) + \Sigma(p,\theta_1,\theta_2) \right) \Phi(p,\theta_2) \ . \tag{2.96}$$

It follows from (2.94) that the supersymmetric self energy Σ is of the form

$$\Sigma(p,\theta_1,\theta_2) = C(p)\exp(-\theta_1^\alpha p_{\alpha\beta}\theta_2^\beta) + D(p)\delta^2(\theta_1-\theta_2) \tag{2.97}$$

where $C(p)$ and $D(p)$ are as yet unknown functions of momenta.

Imitating the steps described in section 3 of [23], the self energy Σ defined in (2.96) may

$\Sigma(p,\theta_1,\theta_2) =$

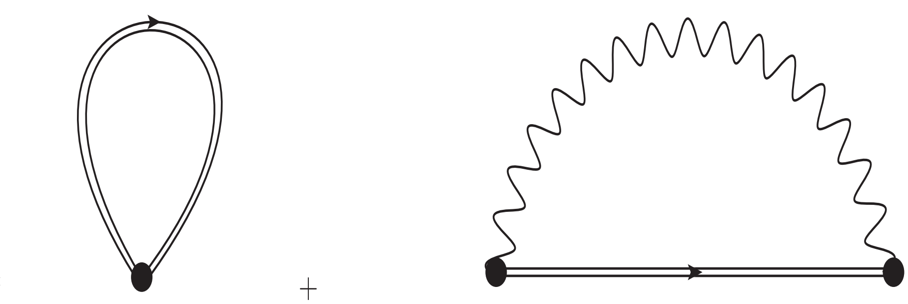

Figure 2.3: Integral equation for self energy

be shown to obey the integral equation [21]

$$\begin{aligned}\Sigma(p,\theta_1,\theta_2) = {} & 2\pi\lambda w \int \frac{d^3r}{(2\pi)^3} \delta^2(\theta_1-\theta_2) P(r,\theta_1,\theta_2) \\ & - 2\pi\lambda \int \frac{d^3r}{(2\pi)^3} D_-^{\theta_2,-p} D_-^{\theta_1,p} \left(\frac{\delta^2(\theta_1-\theta_2)}{(p-r)_{--}} P(r,\theta_1,\theta_2) \right) \\ & + 2\pi\lambda \int \frac{d^3r}{(2\pi)^3} \frac{\delta^2(\theta_1-\theta_2)}{(p-r)_{--}} D_-^{\theta_1,r} D_-^{\theta_2,-r} P(r,\theta_1,\theta_2)\end{aligned} \tag{2.98}$$

where $P(p,\theta_1,\theta_2)$ is the exact superfield propagator. [22] Note that the propagator P depends on Σ (in fact P is obtained by inverting quadratic term in effective action (2.96)). In other words Σ appears both on the LHS and RHS of (2.98); we need to solve this equation to determine Σ.

[21] We work at leading order in the large-N limit

[22] The first line in the RHS of (2.98) comes from the quartic interaction in Fig 2.3 while the second and third lines in (2.98) comes from the gaugesuperfield interaction in Fig. 2.3 . Note that each vertex in the diagram corresponding to the gaugesuperfield interaction in Fig. 2.3 contains one factor of D, resulting in the two powers of D in the second and third line of (2.98).

Using the equations (A.22), the second and third lines on the RHS if (2.98) may be considerably simplified (see appendix §A.7) and we find

$$\begin{aligned}\Sigma(p,\theta_1,\theta_2) &= 2\pi\lambda w \int \frac{d^3r}{(2\pi)^3}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2)\\ &- 2\pi\lambda \int \frac{d^3r}{(2\pi)^3}\frac{p_{--}}{(p-r)_{--}}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2)\\ &+ 2\pi\lambda \int \frac{d^3r}{(2\pi)^3}\frac{r_{--}}{(p-r)_{--}}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2)\end{aligned} \tag{2.99}$$

Combining the second and third lines on the RHS of (2.99) we see that the factors of p_{--} and r_{--} cancel perfectly between the numerator and denominator, and (2.99) simplifies to

$$\Sigma(p,\theta_1,\theta_2) = 2\pi\lambda(w-1)\int \frac{d^3r}{(2\pi)^3}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2)\ . \tag{2.100}$$

Notice that the RHS of (2.100) is independent of p, so it follows that

$$\Sigma(p,\theta_1,\theta_2) = (m-m_0)\delta^2(\theta_1-\theta_2)$$

for some as yet undetermined constant m. It follows that the exact propagator P takes the form of the tree level propagator with m_0 replaced by m i.e.

$$P(p,\theta_1,\theta_2) = \frac{D^2-m}{p^2+m^2}\delta^2(\theta_1-\theta_2)\ . \tag{2.101}$$

Plugging (2.101) into (2.100) and simplifying we find the equation

$$m-m_0 = 2\pi\lambda(w-1)\int \frac{d^3r}{(2\pi)^3}\frac{1}{r^2+m^2}\ . \tag{2.102}$$

The integral on the RHS diverges. Regulating this divergence using dimensional regularisation, we find that (2.102) reduces to

$$m-m_0 = \frac{\lambda|m|}{2}(1-w) \tag{2.103}$$

and so

$$m = \frac{2m_0}{2+(-1+w)\lambda\ \mathrm{Sgn(m)}}\ . \tag{2.104}$$

Let us summarise. The *exact* 1PI quadratic effective action for the Φ superfield has the same form as the tree level effective action but with the bare mass m_0 replaced by the exact mass m given in (2.104). [23] As explained in §2.2.2 the exact mass (2.104) is duality invariant.

Note also that the $\mathcal{N} = 2$ point, $w = 1$ there is no renormalisation of the mass, and the bare propagator is exact and the bare mass (which equals the pole mass) is itself duality invariant.

2.3.4 Constraints from supersymmetry on the off-shell four point function

Much as with the two point function, the off-shell four point function of matter superfields is constrained by the supersymmetric Ward identities. Let us define

$$\begin{aligned}\langle\bar{\Phi}((p+q+\frac{l}{4}),\theta_1)\Phi(-p+\frac{l}{4},\theta_2)\Phi(-(k+q)+\frac{l}{4},\theta_3)\bar{\Phi}(k+\frac{l}{4},\theta_4)\rangle \\ = (2\pi)^3\delta(l)V(\theta_1,\theta_2,\theta_3,\theta_4,p,k,q).\end{aligned} \tag{2.105}$$

It follows from the invariance under supersymmetry that

$$(Q_{\theta_1,p+q} + Q_{\theta_2,-p} + Q_{\theta_3,-k-q} + Q_{\theta_4,k})V(\theta_1,\theta_2,\theta_3,\theta_4,p,k,q) = 0 \ . \tag{2.106}$$

The general solution to (2.106) is easily obtained (see appendix §A.8.1). Defining

$$\begin{aligned} X &= \sum_{i=1}^{4}\theta_i \ , \\ X_{12} &= \theta_1 - \theta_2 \ , \\ X_{13} &= \theta_1 - \theta_3 \ , \\ X_{43} &= \theta_4 - \theta_3 \ . \end{aligned} \tag{2.107}$$

we find

$$V = \exp\left(\frac{1}{4}X.(p.X_{12} + q.X_{13} + k.X_{43})\right)F(X_{12},X_{13},X_{43},p,q,k) \ . \tag{2.108}$$

where F is an unconstrained function of its arguments. In other words supersymmetry fixes the transformation of V under a uniform shift of all θ parameters $\theta_i \to \theta_i + \gamma$. (for $i = 1\ldots 4$

[23] Note that propagator for the fermion in the superfield Φ is the usual propagator for a relativistic fermion of mass m. Recall, of course, that the propagator of Φ is not gauge invariant, and so its form depends on the gauge used in the computation. If we had carried out all computations in Wess-Zumino gauge (which breaks off-shell supersymmetry) we would have found the much more complicated expression for the fermion propagator reported in section 2.1 of [3]. Note however that the gauge invariant physical pole mass m of (2.104) agrees perfectly with the pole mass (reported in eq 1.6 of [3]) of the complicated propagator of [3]. The agreement of gauge invariant quantities in these rather different computations constitutes a nontrivial consistency check of the computations presented in this subsection.

where γ is a constant Grassman parameter). The undetermined function F is a function of shift invariant combinations of the four θ_i.

Let us now turn to the structure of the exact 1PI effective action for scalar superfields in our theory. The most general effective action consistent with global $U(N)$ invariance and supersymmetry takes the form

$$\begin{aligned} S_4 = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3} d^2\theta_1 d^2\theta_2 d^2\theta_3 d^2\theta_4 \\ \big(V(\theta_1,\theta_2,\theta_3,\theta_4,p,k,q)\Phi_m(-(p+q),\theta_1)\bar{\Phi}^m(p,\theta_2)\bar{\Phi}^n(k+q,\theta_3)\Phi_n(-k,\theta_4)\big) \ . \end{aligned} \tag{2.109}$$

It follows from the definition (2.109) that the function V may be taken to be invariant under the Z_2 symmetry

$$\begin{aligned} & p \to k+q, k \to p+q, q \to -q \ , \\ & \theta_1 \to \theta_4, \theta_2 \to \theta_3, \theta_3 \to \theta_2, \theta_4 \to \theta_1 \ . \end{aligned} \tag{2.110}$$

As in the case of two point functions, it is easily demonstrated that the invariance of this action under supersymmetry constraints the coefficient function V that appears in (2.109) to obey the equation (2.106). As we have already explained above, the most general solution to this equation is given in equation (2.108) for a general shift invariant function F.

2.3.5 An integral equation for the off-shell four point function

The coefficient function V of the quartic term of the exact IPI effective action may be shown to obey the integral equation (see Fig. 2.4 for a diagrammatic representation of this equation)

$$\begin{aligned} V(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k) = {} & V_0(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k) \\ & + \int \frac{d^3r}{(2\pi)^3} d^2\theta_a d^2\theta_b d^2\theta_A d^2\theta_B \Big(N V_0(\theta_1,\theta_2,\theta_a,\theta_b,p,q,r) \\ & P(r+q,\theta_a,\theta_A)P(r,\theta_B,\theta_b)V(\theta_A,\theta_B,\theta_3,\theta_4,r,q,k)\Big) \end{aligned} \tag{2.111}$$

In (2.111) V_0 is the tree level contribution to V. V_0 receives contributions from the two diagrams depicted in Fig. 2.4. The explicit evaluation of V_0 is a straightforward exercise and we find (see appendix §A.8.2 for details)

$$\begin{aligned} V_0(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k) = {} & \exp\left(\frac{1}{4}X\cdot(p\cdot X_{12} + q\cdot X_{13} + k\cdot X_{43})\right) \\ & \Big(-\frac{i\pi w}{\kappa} X_{12}^- X_{12}^+ X_{13}^- X_{13}^+ X_{43}^- X_{43}^+ \\ & - \frac{4\pi i}{\kappa(p-k)_{--}} X_{12}^+ X_{13}^+ X_{43}^+ (X_{12}^- + X_{34}^-)\Big) \ . \end{aligned} \tag{2.112}$$

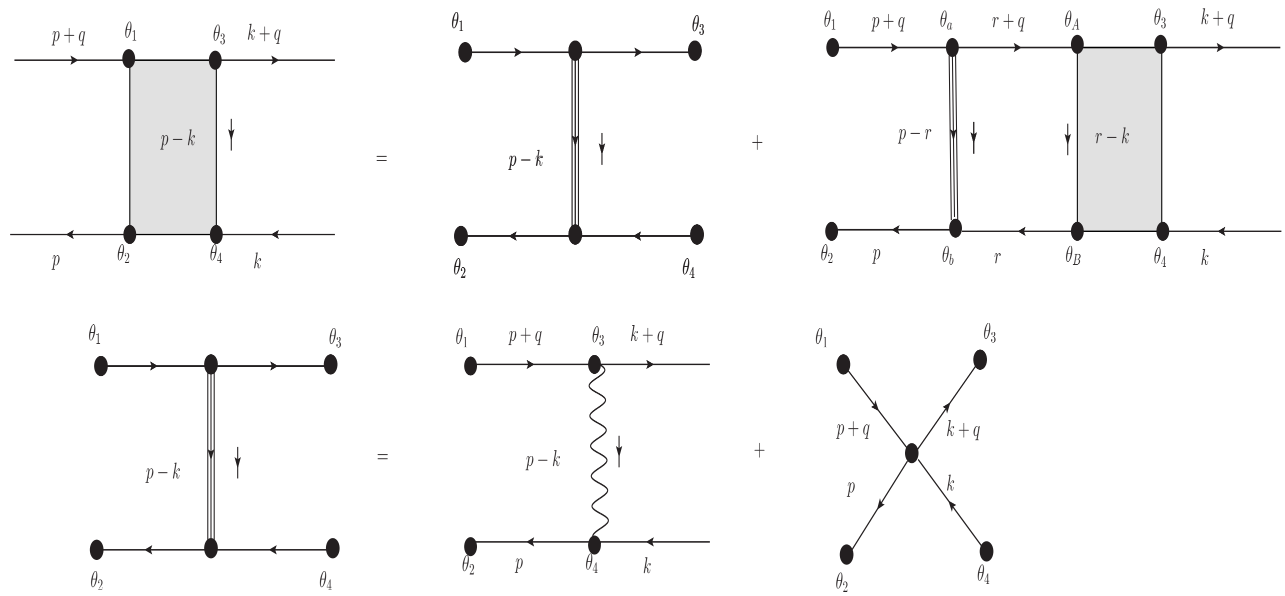

Figure 2.4: The diagrams in the first line pictorially represents the Schwinger-Dyson equation for off-shell four point function (see (2.111)). The second line represents the tree level contributions from the gauge superfield interaction and the quartic interactions.

In the above, the first term in the bracket is the delta function from the quartic interaction, the second term is from the tree diagram due to the gauge superfield exchange computed in §A.8.2.

We now turn to the evaluation of the coefficient V in the exact 1PI effective action. There are 2^6 linearly independent functions of the six independent shift invariant Grassmann variables $X^{\pm}_{12}$, $X^{\pm}_{13}$ and $X^{\pm}_{43}$. Consequently the most general V consistent with supersymmetry is parameterised by 64 unknown functions of the three independent momenta. V (and so F) is necessarily an even function of these variables. It follows that the most general function F can be parameterised in terms of 32 bosonic functions of p, k and q. In principle one could insert the most general supersymmetric F into the integral equation (2.111) and equate equal powers of θ_i on the two sides of (2.111) to obtain 32 coupled integral equations for the 32 unknown complex valued functions. One could, then, attempt to solve this system of equations. This procedure would obviously be very complicated and difficult to implement in practice. Focusing on the special kinematics $q^{\pm} = 0$ we were able to shortcircuit this laborious process, in a manner we now describe.

After a little playing around we were able to demonstrate that V of the form [24]

$$V(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k) = \exp\left(\frac{1}{4}X\cdot(p\cdot X_{12}+q\cdot X_{13}+k\cdot X_{43})\right)F(X_{12},X_{13},X_{43},p,q,k)$$
$$F(X_{12},X_{13},X_{43},p,q,k) = \begin{array}{c} X_{12}^+X_{43}^+\Big(A(p,k,q)X_{12}^-X_{43}^-X_{13}^+X_{13}^- + B(p,k,q)X_{12}^-X_{43}^- \\ + C(p,k,q)X_{12}^-X_{13}^+ + D(p,k,q)X_{13}^+X_{43}^-\Big)\end{array}, \tag{2.113}$$

is closed under the multiplication rule induced by the RHS of (2.111) (see appendix §A.8.3). Plugging in the general form of V (2.113) in the integral equation (2.111) and performing the Grassmann integration, (2.111) turns into to the following integral equations for the coefficient functions A, B, C and D:

$$\begin{aligned} &A(p,k,q)+\frac{2\pi i w}{\kappa} \\ &+i\pi\lambda\int\frac{d^3r_E}{(2\pi)^3}\frac{2A(q_3p_-+2(q_3-im)r_-)+(q_3r_-+2imp_-)(2Bq_3+Ck_-)-Dr_-(q_3p_-+2imr_-)}{(r^2+m^2)((r+q)^2+m^2)(p-r)_-} \\ &-i\pi\lambda w\int\frac{d^3r_E}{(2\pi)^3}\frac{4iAm+2Bq_3^2+Cq_3k_-+2D(q_3+im)r_-}{(r^2+m^2)((r+q)^2+m^2)}=0 \end{aligned} \tag{2.114}$$

$$\begin{aligned} B(p,k,q)&+i\pi\lambda\int\frac{d^3r_E}{(2\pi)^3}\frac{2A(p+r)_-+4B(q_3r_-+im(p-r)_-)-Ck_-(p+r)_--Dr_-(p-3r)_-}{(r^2+m^2)((r+q)^2+m^2)(p-r)_-} \\ &-i\pi\lambda w\int\frac{d^3r_E}{(2\pi)^3}\frac{2A+4imB-Ck_--Dr_-}{(r^2+m^2)((r+q)^2+m^2)}=0 \end{aligned} \tag{2.115}$$

$$\begin{aligned} C(p,k,q)-\frac{4\pi i}{\kappa(p-k)_-}&+i\pi\lambda\int\frac{d^3r_E}{(2\pi)^3}\frac{2C\big(q_3(p+3r)_-+2im(p-r)_-\big)}{(r^2+m^2)((r+q)^2+m^2)(p-r)_-} \\ &-i\pi\lambda w\int\frac{d^3r_E}{(2\pi)^3}\frac{2C(q_3+2im)}{(r^2+m^2)((r+q)^2+m^2)}=0 \end{aligned} \tag{2.116}$$

$$\begin{aligned} &D(p,k,q)-\frac{4\pi i}{\kappa(p-k)_-} \\ &+i\pi\lambda\int\frac{d^3r_E}{(2\pi)^3}\frac{-A(4q_3-8im)+(q_3-2im)(4Bq_3+2Ck_-)+2D(3q_3+2im)r_-}{(r^2+m^2)((r+q)^2+m^2)(p-r)_-}=0\ . \end{aligned} \tag{2.117}$$

[24] The variables X, X_{ij} are defined in terms of θ_i in (2.107).

We will sometimes find it useful to view the four integral equations above as a single integral equation for a four dimensional column vector E whose components are the functions A, B, C, D, i.e.

$$E(p,k,q) = \begin{pmatrix} A(p,k,q) \\ B(p,k,q) \\ C(p,k,q) \\ D(p,k,q) \end{pmatrix} . \tag{2.118}$$

The integral equations take the schematic form

$$E = R + IE \tag{2.119}$$

where R is a 4 column of functions and I is a matrix of integral operators acting on E. The integral equation (2.119) may be converted into a differential equation by differentiating both sides of (2.119) w.r.t p_+. Using (A.129) and performing all d^3r integrals (using (A.127) for the integral over r_3) we obtain the differential equations

$$\partial_{p_+} E(p,k,q) = S(p,k,q) + H(p,k_-,q)E(p,k,q) \ , \tag{2.120}$$

where

$$S(p,k,q) = -\frac{8i\pi^2}{\kappa}\delta^2((p-k)_-, (p-k)_+) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \tag{2.121}$$

$$H(p,k_-,q_3) = \frac{1}{a(p_s,q_3)} \begin{pmatrix} (6q_3 - 4im)p_- & 2q_3(2im+q_3)p_- & (2im+q_3)k_-p_- & -(2im+q_3)p_-^2 \\ 4p_- & 4q_3p_- & -2k_-p_- & 2p_-^2 \\ 0 & 0 & 8q_3p_- & 0 \\ 8im - 4q_3 & 4q_3(q_3-2im) & 2(q_3-2im)k_- & (4im+6q_3)p_- \end{pmatrix} \tag{2.122}$$

and

$$a(p_s,q_3) = \frac{\sqrt{m^2+p_s^2}\,(4m^2+q_3^2+4p_s^2)}{2\pi} \ . \tag{2.123}$$

As we have explained above, the exact vertex V enjoys invariance under the $\mathbb{Z}_2$ transformation (2.110). In terms of the functions A, B, C, D, the $\mathbb{Z}_2$ action is given by

$$E(p,k,q) = TE(k,p,-q) \ , \tag{2.124}$$

where

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} . \tag{2.125}$$

The differential equations (2.120) do not manifestly respect the invariance (2.124). In fact in appendix §A.8.4 we have demonstrated that the differential equations (2.120) admit solutions that enjoy the invariance (2.124) if and only if the following consistency condition is obeyed:

$$[H(p,k_-,q), TH(k,p_-,-q)T] = 0 \ . \tag{2.126}$$

In the same appendix we have also explicitly verified that this integrability condition is in fact obeyed; this is a consistency check on (2.120) and indirectly on the underlying integral equations.

2.3.6 Explicit solution for the off-shell four point function

In this subsection, we solve the system of integral equations for the unknown functions A, B, C, D presented in the previous subsection. We propose the ansatz

$$\begin{aligned}
A(p,k,q) &= A_1(p_s,k_s,q_3) + \frac{A_2(p_s,k_s,q_3)k_-}{(p-k)_-} \ , \\
B(p,k,q) &= B_1(p_s,k_s,q_3) + \frac{B_2(p_s,k_s,q_3)k_-}{(p-k)_-} \ , \\
C(p,k,q) &= -\frac{C_2(p_s,k_s,q_3) - C_1(p_s,k_s,q_3)k_+p_-}{(p-k)_-} \ , \\
D(p,k,q) &= -\frac{D_2(p_s,k_s,q_3) - D_1(p_s,k_s,q_3)k_-p_+}{(p-k)_-} \ .
\end{aligned} \tag{2.127}$$

Our ansatz (2.127) [25] fixes the solution in terms of 8 unknown functions of p_s, k_s and q_3.

Plugging the ansatz (2.127) into the integral equations (2.114)-(2.117), one can do the angle and r_3 integrals (using the formulae (A.128) and (A.127) respectively) leaving only the r_s integral to be performed. Differentiating this expression w.r.t. to p_s turns out to kill the r_s integral yielding differential equations in p_s for the eight equations above. [26] The resulting differential equations turn out to be exactly solvable. Assuming that the solution respects the symmetry (2.124), it turns out to be given in terms of two unknown functions of k_s and q_3. These can be thought of as the integration constants that are not fixed by the symmetry requirement (2.124). Plugging the solutions back into the integral equations we were able to determine these two integration functions of k_s and q_3 completely. We now report our results.

[25] We were able to arrive at this ansatz by first explicitly computing the one loop answer and observing the functional forms. Moreover, in previous work a very similar ansatz was already used to solve the integral equations for the fermions (see appendix F of [4]).

[26] Another way to obtain these differential equations is to plug the ansatz (2.127) directly into the differential equations (2.120).

The solutions for A and B are

$$
\begin{aligned}
A_1(p_s,k_s,q_3) =& e^{-2i\lambda \tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}}\Bigg(G_1(k_s,q_3) \\
&+\frac{2\pi(w-1)(2m-iq_3)e^{2i\lambda\left(\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}+\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}\right)}}{\kappa(e^{\frac{i\pi\lambda q_3}{|q_3|}}(q_3(w+3)-2im(w-1))+i(w-1)(2m+iq_3)e^{2i\lambda\tan^{-1}\frac{2|m|}{q_3}})}\Bigg)\ , \\
A_2(p_s,k_s,q_3) =& e^{-2i\lambda\tan^{-1}\left(\frac{2\sqrt{m^2+p_s^2}}{q_3}\right)}G_2(k_s,q_3)\ ,
\end{aligned}
$$

$$
\begin{aligned}
B_1(p_s,k_s,q_3) =& \frac{2\pi A_1(p_s,k_s,q_3)}{q_3} \\
&+\frac{2\pi}{b_1 b_2}\Bigg(-i(w-1)^2(4m^2+q_3^2)e^{i\lambda\left(\frac{\pi q_3}{|q_3|}-2\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}+4\tan^{-1}\frac{2|m|}{q_3}\right)} \\
&+i(w-1)^2(-4m^2+8imq_3+3q_3^2)e^{i\lambda\left(\frac{\pi q_3}{|q_3|}+2\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}\right)} \\
&-8iq_3^2(w+1)e^{i\lambda\left(\frac{\pi q_3}{|q_3|}+2(\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}-\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}+\tan^{-1}\frac{2|m|}{q_3})\right)} \\
&+(w-1)(q_3+2im)(2m(w-1)+iq_3(w+3))+e^{2i\lambda(\frac{\pi q_3}{|q_3|}-\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}+\tan^{-1}\frac{2|m|}{q_3})} \\
&+(w-1)(2m-3iq_3)(q_3(w+3)+2im(w-1))+e^{2i\lambda\left(\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}+\tan^{-1}\frac{2|m|}{q_3}\right)}\Bigg)\ ,
\end{aligned}
$$

$$
\begin{aligned}
B_2(p_s,k_s,q_3) =& \frac{A_2(p_s,k_s,q_3)}{q_3}\ , \\
G_1(k_s,q_3) =& -\frac{2\pi}{\kappa}\frac{1}{g_1}\Bigg(-8iq_3^2(w+1)e^{i\lambda\left(\frac{\pi q_3}{|q_3|}+2(\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}+\tan^{-1}\frac{2|m|}{q_3})\right)} \\
&+i(w-1)^2(q_3-2im)^2e^{i\lambda\left(\frac{\pi q_3}{|q_3|}+4\tan^{-1}\frac{2|m|}{q_3}\right)} \\
&-(w-1)(q_3-2im)(2m(w-1)+iq_3(w+3))e^{2i\lambda\left(\frac{\pi q_3}{|q_3|}+\tan^{-1}\frac{2|m|}{q_3}\right)}\Bigg)\ , \\
G_2(k_s,q_3) =& 0\ ,
\end{aligned}
\tag{2.128}
$$

where we have defined some parameters as given below for ease of presentation.

$$
\begin{aligned}
g_1 =&(w-1)(q_3+2im)e^{\frac{2i\pi\lambda q_3}{|q_3|}}(q_3(w+3)-2im(w-1)) \ , \\
&+(w-1)(4m^2(w-1)-8imq_3+q_3^2(w+3))e^{4i\lambda \tan^{-1}\frac{2|m|}{q_3}} \ , \\
&-2(4m^2(w-1)^2+q_3^2(w^2+2w+5))e^{i\lambda(\frac{\pi q_3}{|q_3|}+2\tan^{-1}\frac{2|m|}{q_3})} \ , \\
b_1 =&\kappa q_3((w-1)(q_3+2im)e^{\frac{i\pi\lambda q_3}{|q_3|}}+(-q_3(w+3)-2im(w-1))e^{2i\lambda\tan^{-1}\frac{2|m|}{q_3}}) \ , \\
b_2 =&e^{\frac{i\pi\lambda q_3}{|q_3|}}(q_3(w+3)-2im(w-1))+i(w-1)(2m+iq_3)e^{2i\lambda\tan^{-1}\frac{2|m|}{q_3}} \ ,
\end{aligned}
\tag{2.129}
$$

The solutions for C and D are

$$
\begin{aligned}
C_1(p_s,k_s,q_3) =&\frac{4\pi(q_3+2im)(e^{2i\lambda\tan^{-1}\frac{2|m|}{q_3}}-e^{2i\lambda\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}})e^{i\lambda(\frac{\pi q_3}{|q_3|}-2\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3})}}{\kappa k_s^2(i(q_3+2im)e^{\frac{i\pi\lambda q_3}{|q_3|}}+(2m-iq_3\left(\frac{w+3}{w-1}\right))e^{2i\lambda\tan^{-1}\frac{2|m|}{q_3}})} \ , \\
C_2(p_s,k_s,q_3) =&\frac{4\pi e^{2i\lambda(\tan^{-1}\frac{2|m|}{q_3}-\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3})}((q_3+2im)e^{\frac{i\pi\lambda q_3}{|q_3|}}-(q_3\left(\frac{w+3}{w-1}\right)+2im)e^{2i\lambda\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}})}{\kappa(i(q_3+2im)e^{\frac{i\pi\lambda q_3}{|q_3|}}+(2m-iq_3\left(\frac{w+3}{w-1}\right))e^{2i\lambda\tan^{-1}\frac{2|m|}{q_3}})} \ ,
\end{aligned}
$$

$$
\begin{aligned}
D_1(p_s,k_s,q_3) =&C_1(k_s,p_s,-q_3) \ , \\
D_2(p_s,k_s,q_3) =&C_2(k_s,p_s,-q_3) \ .
\end{aligned}
\tag{2.130}
$$

It is straightforward to show that the above solutions satisfy the various symmetry requirements that follow from (2.124).

Although the solutions (2.128) and (2.130) are quite complicated, a drastic simplification occurs at the $\mathcal{N}=2$ point $w=1$

$$
\begin{aligned}
A &= -\frac{2i\pi e^{2i\lambda\left(\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}-\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}\right)}}{\kappa} \ , \\
B &= 0 \ , \\
C &= -\frac{4i\pi e^{2i\lambda\left(\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}-\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}\right)}}{\kappa(k-p)_-} \ , \\
D &= -\frac{4i\pi e^{2i\lambda\left(\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}-\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}\right)}}{\kappa(k-p)_-} \ .
\end{aligned}
\tag{2.131}
$$

It is satisfying that the complicated results of the general $\mathcal{N}=1$ theory collapse to an extremely simple form at the $\mathcal{N}=2$ point.

2.3.7 On-shell limit and the S-matrix

The explicit solution for the functions A, B, C and D, presented in the previous subsection, completely determine V in (2.109), and so the quadratic part of the exact (large-N) IPI effective action. The most general 2×2 S-matrix may now be obtained from (2.109) as follows. We simply substitute the on-shell expressions

$$\Phi(p,\theta) = (2\pi)\delta(p^2+m^2)\Big[\theta(p^0)\Big(a(\mathbf{p})(1+m\theta^2)+\theta^\alpha u_\alpha(\mathbf{p})\alpha(\mathbf{p})\Big) + \theta(-p^0)\Big(a^{c\dagger}(-\mathbf{p})(1+m\theta^2)+\theta^\alpha v_\alpha(-\mathbf{p})\alpha^{c\dagger}(-\mathbf{p})\Big)\Big] \tag{2.132}$$

into (2.109) (here a and α are the effectively free oscillators that create and destroy particles at very early or very late times; these oscillators obey the commutation relations (2.33)). Performing the integrals over θ^α reduces (2.109) to a quartic form (let us call it L) in bosonic and fermionic oscillators. The S-matrix is obtained by sandwiching the resultant expression between the appropriate in and out states, and evaluating the resulting matrix elements using the commutation relations (2.33).

It may be verified that the quartic form in oscillators takes the form [27]

$$L = \sum_{\phi_i=0,\pi}\int\prod_{i=1}^{4} d\theta_i \frac{d^3p_i}{((2\pi)^3)^4}\delta(p_i^2+m^2)S_M(p_1,\phi_1,\theta_1,p_2,\phi_2,\theta_2,p_3,\phi_3,\theta_3,p_4,\phi_4,\theta_4)$$
$$\Big(\delta_{\phi_i,0}\theta(p_i^0)A(p_i,\phi_i,\theta_i)+\delta_{\phi_i,\pi}\theta(-p_i^0)\tilde{A}(-p_i,\phi_i,\theta_i)\Big)(2\pi)^3\delta^3(p_1+p_2+p_3+p_4)$$

where

$$\begin{aligned} A(p_i,\phi_i,\theta_i) &= a(\mathbf{p}_i)+\alpha(\mathbf{p}_i)e^{-\frac{i\phi_i}{2}}\theta_i \ , \\ \tilde{A}(p_i,\phi_i,\theta_i) &= a^\dagger(\mathbf{p}_i)+e^{-\frac{i\phi_i}{2}}\theta_i\alpha^\dagger(\mathbf{p}_i) \ , \end{aligned} \tag{2.133}$$

where the one component fermionic variables θ_i are the fermionic variables that parameterise on-shell superspace (see §2.2.4) and the master formula is defined in (2.53). Note that the phase variables ϕ_i are summed over two values 0 and π; the symbol $\delta_{\phi,0}$ is unity when $\phi = 0$ but zero when $\phi = \pi$ and $\delta_{\phi,\pi}$ has an analogous definition. (2.133) compactly identifies the coefficient of every quartic form in oscillators. For instance it asserts that the coefficient of $a_1a_2a_3^\dagger a_4^\dagger$ is the S-matrix for scattering bosons with momentum p_1, p_2 to bosons with momenta p_3, p_4, while the the coefficient of $\alpha_2\alpha_4a_1^\dagger a_3^\dagger$ is minus the S-matrix for scattering fermions with momentum p_2, p_4 to bosons with momentum p_1, p_3, etc.

We can use the δ function in (2.133) to perform the integral over one of the four momenta; the integral over the remaining momenta may be recast as an integral over the momenta p k and q employed in the previous section; specifically (see Fig 2.4)

$$p_1 = p+q \ , \ p_2 = -k-q \ , \ p_3 = -p \ , \ p_4 = k \ . \tag{2.134}$$

[27] The definition of A and $\tilde{A}$ reduces to the definition (2.35) for $\phi = 0$. While for $\phi = \pi$, it reduces to (2.35) together with the identification $\theta \to i\theta$. With these definitions $\tilde{A} = A^\dagger$ both at $\phi = 0, \pi$.

From the explicit results we get by substituting (2.132) into (2.109) we can read off all S-matrices at $q_\pm = 0$.

To start with, let us restrict our attention to the bosonic sector. From direct computation [28] we find that in this sector (2.133) reduces to

$$\begin{aligned}L_B = \sum_{\phi_i=0,\pi} \int & \frac{d^3p}{(2\pi)^3}\frac{dq_3}{(2\pi)}\frac{d^3k}{(2\pi)^3}\delta((p+q)^2+m^2)\delta((k+q)^2+m^2) \\ & \delta(p^2+m^2)\delta(k^2+m^2)\mathcal{T}_B(p,k,q_3) \\ & \big(\delta_{\phi_i,0}\theta(p^0)a(\mathbf{p}+\mathbf{q})+\delta_{\phi_i,\pi}\theta(-p^0)a^\dagger(-\mathbf{p}-\mathbf{q})\big) \\ & \big(\delta_{\phi_i,0}\theta(-k^0)a(-\mathbf{k}-\mathbf{q})+\delta_{\phi_i,\pi}\theta(k^0)a^\dagger(\mathbf{k}+\mathbf{q})\big) \\ & \big(\delta_{\phi_i,0}\theta(-p^0)a(-\mathbf{p})+\delta_{\phi_i,\pi}\theta(p^0)a^\dagger(\mathbf{p})\big) \\ & \big(\delta_{\phi_i,0}\theta(k^0)a(\mathbf{k})+\delta_{\phi_i,\pi}\theta(-k^0)a^\dagger(-\mathbf{k})\big)\end{aligned} \tag{2.136}$$

while for the purely fermionic sector (2.133) reduces to

$$\begin{aligned}L_F = \sum_{\phi_i=0,\pi} \int & \frac{d^3p}{(2\pi)^3}\frac{dq_3}{(2\pi)}\frac{d^3k}{(2\pi)^3}\delta((p+q)^2+m^2)\delta((k+q)^2+m^2) \\ & \delta(p^2+m^2)\delta(k^2+m^2)\mathcal{T}_F(p,k,q_3) \\ & \big(\delta_{\phi_i,0}\theta(p^0)\alpha(\mathbf{p}+\mathbf{q})+\delta_{\phi_i,\pi}\theta(-p^0)\alpha^\dagger(-\mathbf{p}-\mathbf{q})\big) \\ & \big(\delta_{\phi_i,0}\theta(-k^0)\alpha(-\mathbf{k}-\mathbf{q})+\delta_{\phi_i,\pi}\theta(k^0)\alpha^\dagger(\mathbf{k}+\mathbf{q})\big) \\ & \big(\delta_{\phi_i,0}\theta(-p^0)\alpha(-\mathbf{p})+\delta_{\phi_i,\pi}\theta(p^0)\alpha^\dagger(\mathbf{p})\big) \\ & \big(\delta_{\phi_i,0}\theta(k^0)\alpha(\mathbf{k})+\delta_{\phi_i,\pi}\theta(-k^0)\alpha^\dagger(-\mathbf{k})\big)\end{aligned} \tag{2.137}$$

[28] Note that the on-shell delta functions in the equations (2.136) and (2.137) ensure that

$$p_3 = k_3 = -\frac{q_3}{2}\ ,\ p_s = k_s\ ,\ k_s = \frac{i}{2}\sqrt{q_3^2+4m^2}\ . \tag{2.135}$$

where [29]

$$\mathcal{T}_B = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2} + J_B(q,\lambda\ , \tag{2.138}$$

$$\mathcal{T}_F = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2} + J_F(q,\lambda\ , \tag{2.139}$$

where the J functions[30] are

$$\begin{aligned} J_B(q,\lambda) =& \frac{4\pi q}{\kappa}\frac{N_1 N_2 + M_1}{D_1 D_2}\ , \\ J_F(q,\lambda) =& \frac{4\pi q}{\kappa}\frac{N_1 N_2 + M_2}{D_1 D_2}\ , \end{aligned} \tag{2.140}$$

where

$$\begin{aligned} N_1 =& \left(\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}(w-1)(2m+iq) + (w-1)(2m-iq)\right)\ , \\ N_2 =& \left(\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}(q(w+3)+2im(w-1)) + (q(w+3)-2im(w-1))\right)\ , \\ M_1 =& -8mq((w+3)(w-1)-4w)\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}\ , \\ M_2 =& -8mq(1+w)^2\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}\ , \\ D_1 =& \left(i\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}(w-1)(2m+iq) - 2im(w-1) + q(w+3)\right)\ , \\ D_2 =& \left(\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}(-q(w+3)-2im(w-1)) + (w-1)(q+2im)\right)\ . \end{aligned} \tag{2.141}$$

[29]Our actual computations gave the functions J_B and J_F in the special case $q^\pm = 0$. We obtained the answers reported in (2.138) and (2.139) by determining the unique covariant expression that reduce to our answers for our special kinematics. While this procedure is completely correct (with standard conventions) for J_B, it is a bit inaccurate for J_F. The reason for this is that J_F is Lorentz invariant only up to a phase. As we have explained around (2.56), the phase of J_F depends on the (arbitrary) phase of the u and v spinors of the particles in the scattering process. The accurate answer is obtained by covariantising the unambiguous $\mathcal{S}_f$ defined in (2.57). $\mathcal{S}_F$ is obtained by multiplying this result by the quadrilinear term in spinor wavefunctions as defined in (2.68). This gives an explicit but cumbersome expression for $\mathcal{S}_F$, which agrees with the result presented above up to an overall convention dependent phase. This phase vanishes near identity scattering (where it could have interfered with identity), and we have dealt with this issue carefully in deriving the unitarity equation. In the equation above we have simply ignored the phase in order to aid readability of formulas.

[30]The J functions are quite complicated and can be written in many avatars. In this section we have written the most elegant form of the J function, the other forms are reported in appendix §A.9

The equations (2.138) and (2.139) capture purely bosonic and purely fermionic S-matrices in all channels (particle-particle scattering in the symmetric and antisymmetric channels as well as particle-antiparticle scattering in the adjoint channel) restricted to the kinematics $q_\pm = 0$. Recall that supersymmetry (see §2.2.4) determines all other scattering amplitudes in terms of the four boson and four fermion amplitudes, so the formulae (2.138) and (2.139) are sufficient to determine all $2 \to 2$ scattering processes restricted to our special kinematics. In other words S_M in (2.133) is completely determined by (2.138) and (2.139) together with (2.53).

2.3.8 Duality of the S-matrix

Under the duality transformation (see (2.16))

$$w' = \frac{3-w}{w+1}, \lambda' = \lambda - \text{sgn}(\lambda), m' = -m, \kappa' = -\kappa \tag{2.142}$$

we have verified that

$$\begin{aligned} J_B(q, \kappa', \lambda', w', m') &= -J_F(q, \kappa, \lambda, w, m) \ , \\ J_F(q, \kappa', \lambda', w', m') &= -J_B(q, \kappa, \lambda, w, m) \ . \end{aligned} \tag{2.143}$$

provided (2.19) is respected. In other words duality maps the purely bosonic and purely fermionic S-matrices into one another. It follows that (2.138) and (2.139) map to each other under duality upto a phase. As we have explained in subsection §2.2.5, this result is sufficient to guarantee that the full S-matrix (including, for instance, the S-matrix for Bose-Fermi scattering) is invariant under duality, once we interchange bosons with fermions.

2.3.9 S-matrices in various channels

In this subsection we explicitly list the purely bosonic and purely fermionic S-matrices in every channel, as functions of the Mandelstam variables of that channel. These results are, of course, easily extracted from (2.136) and (2.137). There is a slight subtlety here; even though (2.138) and (2.139) are manifestly Lorentz invariant, it is not possible to write them entirely in terms of Mandelstam variables. [31] This is because (as was noted in [4]) $2+1$ dimensional kinematics allows for an additional Z_2 valued invariant (in addition to the Mandelstam variables)

$$E(q, p-k, p+k) = \text{Sign}\left(\epsilon_{\mu\nu\rho} q^\mu (p-k)^\nu (p+k)^\rho\right) \ . \tag{2.145}$$

[32] The sign of the first term in (2.138) and (2.139) is given by this new invariant as we will see in more detail below.

[31] We define the Mandelstam variables as usual

$$s = -(p_1+p_2)^2 \ , \ t = -(p_1-p_3)^2 \ , \ u = -(p_1-p_4)^2 \ . \tag{2.144}$$

[32] Note, in particular that the expression (2.145) changes sign under the interchange of any two vectors.

U channel

For particle-particle scattering

$$P_i(p_1) + P_j(p_2) \to P_i(p_3) + P_j(p_4)$$

we have the direct scattering referred to as the U_d (symmetric) channel. [33] Our momenta assignments (see LHS of Fig2.4) are

$$p_1 = p + q \ , \ p_2 = k \ , \ p_3 = p \ , \ p_4 = k + q \ . \tag{2.146}$$

In terms of the Mandelstam variables

$$s = -(p + q + k)^2 \ , \ t = -q^2 \ , \ u = -(p - k)^2 \ , \tag{2.147}$$

the U_d channel T matrices for the boson-boson and fermion-fermion scattering are

$$\begin{aligned} \mathcal{T}_B^{U_d} &= E(q, p-k, p+k)\frac{4\pi i}{k}\sqrt{ts/u} + J_B(\sqrt{-t}, \lambda) \ , \\ \mathcal{T}_F^{U_d} &= E(q, p-k, p+k)\frac{4\pi i}{k}\sqrt{ts/u} + J_F(\sqrt{-t}, \lambda) \ . \end{aligned} \tag{2.148}$$

For the exchange scattering, referred to as the U_e (Antisymmetric) channel the momenta assignments are (see LHS of Fig2.4)

$$p_1 = k \ , \ p_2 = p + q \ , \ p_3 = p \ , \ p_4 = k + q \ . \tag{2.149}$$

In terms of the Mandelstam variables

$$s = -(p + q + k)^2 \ , \ t = -(p - k)^2 \ , \ u = -q^2 \ , \tag{2.150}$$

the U_e channel T matrices for the boson-boson and fermion-fermion scattering are

$$\begin{aligned} \mathcal{T}_B^{U_e} &= E(q, p-k, p+k)\frac{4\pi i}{k}\sqrt{us/t} + J_B(\sqrt{-u}, \lambda) \ , \\ \mathcal{T}_F^{U_e} &= E(q, p-k, p+k)\frac{4\pi i}{k}\sqrt{us/t} + J_F(\sqrt{-u}, \lambda) \ . \end{aligned} \tag{2.151}$$

T channel

For particle-antiparticle scattering

$$P_i(p_1) + A^j(p_2) \to P_i(p_3) + A^j(p_4)$$

[33] We adopt the terminology of [4] in specifying scattering channels; we refer the reader to that paper for a more complete definition of the U_d, U_e, T, and S channels that we will repeatedly refer to below.

S-matrix in the adjoint channel is referred to as the T channel. The momentum assignments are (see LHS of fig 2.4)

$$p_1 = p + q \ , \ p_2 = -k - q \ , \ p_3 = p \ , \ p_4 = -k \ . \tag{2.152}$$

In terms of the Mandelstam variables

$$s = -(p-k)^2 \ , \ t = -q^2 \ , \ u = -(p+q+k)^2 \ , \tag{2.153}$$

the T channel T matrices for the boson-boson and fermion-fermion scattering are

$$\begin{aligned} \mathcal{T}_B^T &= E(q, p-k, p+k) \frac{4\pi i}{k} \sqrt{tu/s} + J_B(\sqrt{-t}, \lambda) \ , \\ \mathcal{T}_F^T &= E(q, p-k, p+k) \frac{4\pi i}{k} \sqrt{tu/s} + J_F(\sqrt{-t}, \lambda) \ . \end{aligned} \tag{2.154}$$

In particle-anti particle scattering there is also the singlet channel that we describe below.

2.3.10 The singlet (S) channel

We now turn to the most interesting scattering process; the scattering of particles with antiparticles in the S (singlet) channel. In this channel the external lines on the LHS of Fig. 2.4 are assigned positive energy (and so represent initial states) while those on the right of the diagram are assigned negative energy (and so represent final states). It follows that we must make the identifications

$$p_1 = p + q \ , \ p_2 = -p \ , \ p_3 = k + q \ , p_4 = -k \ , \tag{2.155}$$

so that the Mandelstam variables for this scattering process are

$$s = -q^2 \ , \ t = -(p-k)^2 \ , \ u = -(p+k)^2 \ . \tag{2.156}$$

Note, in particular, that $s = -q^2$, and so is always negative when $q^\pm = 0$. As we have been able to evaluate the off-shell correlator V (see (2.113)) only for $q^\pm = 0$, it follows that we cannot specialise our off-shell computation to an on-shell scattering process in the S channel in which $s \geq 4m^2$. In other words we do not have a direct computation of S channel scattering in any frame.

It is nonetheless tempting to simply assume that (2.138) and (2.139) continue to apply at every value of q^μ and not just when $q^\pm = 0$; indeed this is what the usual assumptions of analyticity of S-matrices (and crossing symmetry in particular) would inevitably imply. Provisionally proceeding with this 'naive' assumption, it follows upon performing the appropriate analytic continuation ($q^2 \to -s$ for positive s; see sec 4.4 of [4]) that

$$\begin{aligned} \mathcal{T}_B^{S;\text{naive}} &= E(q, p-k, p+k) 4\pi i \lambda \sqrt{su/t} + J_B(\sqrt{s}, \lambda) \ , \\ \mathcal{T}_F^{S;\text{naive}} &= E(q, p-k, p+k) 4\pi i \lambda \sqrt{su/t} + J_F(\sqrt{s}, \lambda) \ , \end{aligned} \tag{2.157}$$

where

$$\begin{aligned} J_B(\sqrt{s},\lambda) =& -4\pi i\lambda\sqrt{s}\frac{N_1N_2+M_1}{D_1D_2} \ , \\ J_F(\sqrt{s},\lambda) =& -4\pi i\lambda\sqrt{s}\frac{N_1N_2+M_2}{D_1D_2} \ , \end{aligned} \tag{2.158}$$

where

$$\begin{aligned} N_1 =& \left((w-1)(2m+\sqrt{s})+(w-1)(2m-\sqrt{s})e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^{\lambda}\right) \ , \\ N_2 =& \left((-i\sqrt{s}(w+3)+2im(w-1))+(-i\sqrt{s}(w+3)-2im(w-1))e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^{\lambda}\right) \ , \\ M_1 =& 8mi\sqrt{s}((w+3)(w-1)-4w)e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^{\lambda} \ , \\ M_2 =& 8mi\sqrt{s}(1+w)^2e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^{\lambda} \ , \\ D_1 =& \left(i(w-1)(2m+\sqrt{s})-(2im(w-1)+i\sqrt{s}(w+3))e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^{\lambda}\right) \ , \\ D_2 =& \left((\sqrt{s}(w+3)-2im(w-1))+(w-1)(-i\sqrt{s}+2im)e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^{\lambda}\right) \ . \end{aligned} \tag{2.159}$$

Including the identity factors, the naive S channel S-matrix that follows from the usual rules of crossing symmetry are

$$\begin{aligned} \mathcal{S}_B^{S;\text{naive}}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \ &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)+i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_B^{S;\text{naive}}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \ , \\ \mathcal{S}_F^{S;\text{naive}}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \ &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)+i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_F^{S;\text{naive}}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \ , \end{aligned} \tag{2.160}$$

where the identity operator is defined in (2.60).

We pause here to note a subtlety. The quantity $\mathcal{S}_F^{S;\text{naive}}$ quoted above equals the S-matrix in the S channel only up to phase. In order to obtain the fully correct S-matrix we analytically continue the phase unambiguous quantity $\mathcal{S}_f^{S;\text{naive}}$ [34]. The result of that continuation is given by

$$\mathcal{S}_f^{S;\text{naive}} = \frac{\mathcal{S}_F^{S;\text{naive}}}{X(s)} \tag{2.161}$$

[34] Indeed it does not make sense to analytically continue $\mathcal{S}_F$ as the ambiguous phases of this quantity are not necessarily Lorentz invariant, and so are not functions only of the Mandelstam variables.

where[35]

$$X(s) = -\frac{-s+4m^2}{4m^2} = -4Y(s) \ . \tag{2.163}$$

The full four fermion amplitude in the S channel, including phase is then given by

$$A_F^{S;\text{naive}} = \mathcal{S}_f^{S;\text{naive}} X(p,k,q)$$

where[36]

$$X(p,k,q) = \frac{1}{4m^2}\left(u(p+q)u(-p)\right)\left(v(k+q)v(-k)\right) \ . \tag{2.164}$$

It is not difficult to check that

$$|X(p,k,q)| = X(s) \ .$$

It follows that the S channel 4 fermion amplitude agrees with $\mathcal{S}_F$ upto a convention dependent phase. This phase factor may be shown to vanish near the identity momentum configuration ($p_1 = p_3$, $p_2 = p_4$) and so does not affect the interference with identity, and in general has no physical effect; it follows we would make no error if we simply regarded $\mathcal{S}_F$ as the four fermion scattering amplitude. At any rate we have been careful to express the unitarity relation in terms of the phase unambiguous quantity $\mathcal{S}_f$ given unambiguously by (2.57).

The naive S channel S-matrix (2.160) is not duality (2.16) invariant. In later section, we also show that it also does not obey the constraints of unitarity, leading to an apparent paradox.

A very similar paradox was encountered in [4] where it was conjectured that the usual rules of crossing symmetry are modified in matter Chern-Simons theories. It was conjectured in [4] that the correct transformation rule under crossing symmetry for *any* matter Chern-Simons theory with fundamental matter in the large-N limit is given by

$$\begin{aligned}\mathcal{S}_B^S(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_B^S(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \ ,\\ \mathcal{S}_F^S(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_F^S(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \ ,\end{aligned} \tag{2.165}$$

where

$$\begin{aligned}\mathcal{T}_B^S(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= -i(\cos(\pi\lambda)-1)I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + \frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}_B^{S;\text{naive}}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \ ,\\ \mathcal{T}_F^S(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= -i(\cos(\pi\lambda)-1)I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + \frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}_F^{S;\text{naive}}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \ ,\end{aligned} \tag{2.166}$$

---

[35]The factor of $X(s)$ is the analytic continuation of (see (2.57))

$$\left(\bar{u}(p)u(p+q)\right)\left(\bar{v}(-k-q)v(-k)\right) = X(q) = -\frac{q^2+4m^2}{4m^2} \ . \tag{2.162}$$

The analytic continuation of the above formula is same as $-4Y(s)$ (see (2.80).)

[36]The spinor quadrilinear is as defined in (2.68) with momentum assignments corresponding to the S channel (2.155).

where (2.157) defines the T matrices obtained from naive crossing rules. In the centre of mass frame the conjectured S-matrix (2.165) has the form

$$\begin{aligned} \mathcal{S}_B^S(s,\theta) &= 8\pi\sqrt{s}\delta(\theta) + i\mathcal{T}_B^S(s,\theta) \ , \\ \mathcal{S}_F^S(s,\theta) &= 8\pi\sqrt{s}\delta(\theta) + i\mathcal{T}_F^S(s.\theta) \ , \end{aligned} \tag{2.167}$$

where

$$\begin{aligned} \mathcal{T}_B^S(s,\theta) &= -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + \frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}_B^{S;\text{naive}}(s,\theta) \ , \\ \mathcal{T}_F^S(s,\theta) &= -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + \frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}_F^{S;\text{naive}}(s,\theta) \ . \end{aligned} \tag{2.168}$$

The naive analytically continued T matrices are

$$\begin{aligned} \mathcal{T}_B^{S;\text{naive}}(s,\theta) &= 4\pi i\lambda\sqrt{s}\cot(\theta/2) + J_B(\sqrt{s},\lambda) \ , \\ \mathcal{T}_F^{S;\text{naive}}(s,\theta) &= 4\pi i\lambda\sqrt{s}\cot(\theta/2) + J_F(\sqrt{s},\lambda) \ , \end{aligned} \tag{2.169}$$

where the J functions are as defined in (2.158). In other words the conjectured S-matrix takes the following form

$$\begin{aligned} \mathcal{S}_B^S(s,\theta) &= 8\pi\sqrt{s}\cos(\pi\lambda)\delta(\theta) + i\frac{\sin(\pi\lambda)}{\pi\lambda}\left(4\pi i\lambda\sqrt{s}\cot(\theta/2) + J_B(\sqrt{s},\lambda)\right) \ , \\ \mathcal{S}_F^S(s,\theta) &= 8\pi\sqrt{s}\cos(\pi\lambda)\delta(\theta) + i\frac{\sin(\pi\lambda)}{\pi\lambda}\left(4\pi i\lambda\sqrt{s}\cot(\theta/2) + J_F(\sqrt{s},\lambda)\right) \ . \end{aligned} \tag{2.170}$$

It was demonstrated in [4] that the conjecture (2.166) yields an S channel S-matrix that is both duality invariant and consistent with unitarity in the the systems under study in that paper. Here we will follow [4] to conjecture that (2.166) continues to define the correct S channel S-matrix for the theories under study. In the next section we will demonstrate that (2.166) obeys the nonlinear unitarity equations (2.76) and (2.77). We regard this fact as highly nontrivial evidence in support of the conjecture (2.166). As (2.166) appears to work in at least two rather different classes of large-N fundamental matter Chern-Simons theories (namely the purely bosonic and fermionic theories studied in [4] and the supersymmetric theories studied here in this chapter) it seems likely that (2.166) applies universally to all Chern-Simons fundamental matter theories, as suggested in [4].

Straightforward non-relativistic limit

The conjectured S channel S-matrix has a simple non-relativistic limit leading to the known Aharonov-Bohm result (see section 2.6 of [4] for details). In this limit we take (in the centre of mass frame) $\sqrt{s} \to 2m$ in the T matrix (2.168) with all other parameters held fixed. In this limit we find

$$\begin{aligned} \mathcal{T}_B^S(s,\theta) &= -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + 4\sqrt{s}\sin(\pi\lambda)\left(i\cot(\theta/2)-1\right) \ , \\ \mathcal{T}_F^S(s,\theta) &= -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + 4\sqrt{s}\sin(\pi\lambda)\left(i\cot(\theta/2)+1\right) \ . \end{aligned} \tag{2.171}$$

The non-relativistic limit also coincides with the $\mathcal{N}=2$ limit of the S-matrix (2.165) as we show in the following subsection. In §2.5.5 we describe a slightly modified non-relativistic limit of the S-matrix.

2.3.11 S-matrices in the $\mathcal{N}=2$ theory

As discussed in §2.2.1 the $\mathcal{N}=1$ theory (2.5) has an enhanced $\mathcal{N}=2$ supersymmetric regime when the Φ^4 coupling constant takes a special value $w=1$. We have already seen that the momentum dependent functions in the off-shell four point function simplify dramatically (2.131), and so it is natural to expect that the S-matrices at $w=1$ are much simpler than at generic w. This is indeed the case as we now describe.

By taking the limit $w \to 1$ in the S-matrix formulae presented in (2.138) and (2.139), we find that the four boson and four fermion $\mathcal{N}=2$ S-matrices take the very simple form [37]

$$\mathcal{T}_B^{\mathcal{N}=2} = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2} - \frac{8\pi m}{\kappa} \ , \tag{2.172}$$

$$\mathcal{T}_F^{\mathcal{N}=2} = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2} + \frac{8\pi m}{\kappa} \ . \tag{2.173}$$

The S-matrices above are simply those for tree level scattering. It follows that the tree level S-matrices in the three non-anyonic channels are not renormalised, at any order in the coupling constant, in the $\mathcal{N}=2$ theory.

There is an immediate (but rather trivial) check of this result. Recall that according to §A.3 the four boson and four fermion scattering amplitudes are not independent in the $\mathcal{N}=2$ theory; supersymmetry determines the former in terms of the latter. The precise relation is derived in A.3 and is given by (A.57) for particle-antiparticle scattering and (A.62) for particle-particle scattering. It is easy to verify that (2.172) and (2.173) trivially satisfy (A.57) (or (A.62)) using (2.41),(2.42) and appropriate momentum assignments for the channels of scattering discussed in section §2.3.9. [38]

For completeness we now present explicit formulae for the S-matrices of the $\mathcal{N}=2$ theory in the three non-anyonic channels.

---

[37] This is because the J functions reported in (2.138) and (2.139) have an extremely simple form at $w=1$ (see (A.142)).

[38] As an example, in the T channel (see (2.152)) we substitute the coefficients (2.41), (2.42) into (A.57) and evaluate it to get

$$\mathcal{S}_B = \mathcal{S}_F \frac{-2m(k-p)_- + iq_3(k+p)_-}{2m(k-p)_- + iq_3(k+p)_-} \ . \tag{2.174}$$

It is clear that the covariant form of the S-matrices given in (2.172) and (2.173) trivially satisfy (2.174). Similarly it can be easily checked that the result (2.174) follows from (A.62) for particle-particle scattering.

For the U_d channel

$$\begin{aligned}\mathcal{T}_B^{U_d;\mathcal{N}=2} &= E(q,p-k,p+k)\frac{4\pi i}{k}\sqrt{ts/u}-\frac{8\pi m}{\kappa}\ ,\\ \mathcal{T}_F^{U_d;\mathcal{N}=2} &= E(q,p-k,p+k)\frac{4\pi i}{k}\sqrt{ts/u}+\frac{8\pi m}{\kappa}\ .\end{aligned}\tag{2.175}$$

For the U_e channel

$$\begin{aligned}\mathcal{T}_B^{U_e;\mathcal{N}=2} &= E(q,p-k,p+k)\frac{4\pi i}{k}\sqrt{us/t}-\frac{8\pi m}{\kappa}\ ,\\ \mathcal{T}_F^{U_e;\mathcal{N}=2} &= E(q,p-k,p+k)\frac{4\pi i}{k}\sqrt{us/t}+\frac{8\pi m}{\kappa}\ .\end{aligned}\tag{2.176}$$

For the T channel

$$\begin{aligned}\mathcal{T}_B^{T;\mathcal{N}=2} &= E(q,p-k,p+k)\frac{4\pi i}{k}\sqrt{tu/s}-\frac{8\pi m}{\kappa}\ ,\\ \mathcal{T}_F^{T;\mathcal{N}=2} &= E(q,p-k,p+k)\frac{4\pi i}{k}\sqrt{tu/s}+\frac{8\pi m}{\kappa}\ .\end{aligned}\tag{2.177}$$

Let us now turn to the singlet channel. As described in §2.3.10, we cannot compute the S channel S-matrix directly because of our choice of the kinematic regime $q_\pm = 0$. The naive analytic continuation of (2.172) and (2.173) to the S channel gives

$$\begin{aligned}\mathcal{T}_B^{S;\text{naive};\mathcal{N}=2} &= E(q,p-k,p+k)4\pi i\lambda\sqrt{su/t}-8\pi m\lambda\ ,\\ \mathcal{T}_F^{S;\text{naive};\mathcal{N}=2} &= E(q,p-k,p+k)4\pi i\lambda\sqrt{su/t}+8\pi m\lambda\ .\end{aligned}\tag{2.178}$$

Thus the naive S channel S-matrix for the $\mathcal{N}=2$ theory is

$$\begin{aligned}\mathcal{S}_B^{S;\text{naive};\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) =& I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\\ &+i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_B^{S;\text{naive};\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\ ,\\ \mathcal{S}_F^{S;\text{naive};\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) =& I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\\ &+i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_F^{S;\text{naive};\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\ .\end{aligned}\tag{2.179}$$

As explained in the introduction §2.1, this result is obviously non-unitary. Applying the modified crossing symmetry transformation rules (2.165) we obtain our conjecture for the $\mathcal{N}=2$ S-matrix in the singlet channel

$$\begin{aligned}\mathcal{S}_B^{S;\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)+i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_B^{S;\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\ ,\\ \mathcal{S}_F^{S;\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)+i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_F^{S;\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\ ,\end{aligned}\tag{2.180}$$

where

$$
\begin{aligned}
\mathcal{T}_B^{S;\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= -i(\cos(\pi\lambda)-1)I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + \frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}_B^{S;\text{naive};\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \ , \\
\mathcal{T}_F^{S;\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= -i(\cos(\pi\lambda)-1)I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + \frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}_F^{S;\text{naive};\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \ .
\end{aligned}
\tag{2.181}
$$

In the centre of mass frame the conjectured S channel S-matrix in the $\mathcal{N}=2$ theory takes the form

$$
\begin{aligned}
\mathcal{S}_B^{S;\mathcal{N}=2}(s,\theta) &= 8\pi\sqrt{s}\delta(\theta) + i\mathcal{T}_B^S(s,\theta) \ , \\
\mathcal{S}_F^{S;\mathcal{N}=2}(s,\theta) &= 8\pi\sqrt{s}\delta(\theta) + i\mathcal{T}_F^S(s.\theta) \ ,
\end{aligned}
\tag{2.182}
$$

where

$$
\begin{aligned}
\mathcal{T}_B^{S;\mathcal{N}=2}(s,\theta) &= -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + \sin(\pi\lambda)(4i\sqrt{s}\cot(\theta/2) - 8m) \ , \\
\mathcal{T}_F^{S;\mathcal{N}=2}(s,\theta) &= -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + \sin(\pi\lambda)(4i\sqrt{s}\cot(\theta/2) + 8m) \ .
\end{aligned}
\tag{2.183}
$$

Note that as $\sqrt{s} \to 2m$ (2.183) reproduces the straightforward non-relativistic limit of the $\mathcal{N}=1$ theory (2.171).

In other words the conjectured S channel S-matrix for the $\mathcal{N}=2$ theory takes the following form in the centre of mass frame

$$
\begin{aligned}
\mathcal{S}_B^{S;\mathcal{N}=2}(s,\theta) &= 8\pi\sqrt{s}\cos(\pi\lambda)\delta(\theta) + i\sin(\pi\lambda)\left(4i\sqrt{s}\cot(\theta/2) - 8m\right) \ , \\
\mathcal{S}_F^{S;\mathcal{N}=2}(s,\theta) &= 8\pi\sqrt{s}\cos(\pi\lambda)\delta(\theta) + i\sin(\pi\lambda)\left(4i\sqrt{s}\cot(\theta/2) + 8m\right) \ .
\end{aligned}
\tag{2.184}
$$

We explicitly show that the conjectured S channel S-matrix is unitary in the following section.

2.4 Unitarity

In this section, we first show that the S-matrices in the T and U channel obey the unitarity conditions (2.76) and (2.77) at leading order in the large-N limit. As the relevant unitarity equations are linear, the unitarity equation is a relatively weak consistency check of the S-matrices.

We then proceed to demonstrate that the S-matrix (2.165) also obeys the constraints of unitarity. As the unitarity equation is nonlinear in the S channel, this constraint is highly nontrivial, we believe it provides an impressive consistency check of the conjecture (2.165).

2.4.1 Unitarity in the T and U channels

We begin by discussing the unitarity condition for the T (adjoint) and U (particle - particle) channels. Firstly we note that the S-matrices in these channels are $O(1/N)$. Therefore the LHS of (2.76) and (2.77) are $O(1/N^2)$. It follows that the unitarity equations (2.76) and (2.77) are obeyed at leading order in the large-N limit provided

$$\begin{aligned}\mathcal{T}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= \mathcal{T}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{p}_1,\mathbf{p}_2)\ ,\\ \mathcal{T}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= \mathcal{T}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{p}_1,\mathbf{p}_2)\ .\end{aligned} \tag{2.185}$$

The four boson and four fermion S-matrices in the T channel are given in terms of the universal functions in (2.138) and (2.139) after applying the momentum assignments (2.152). It follows that (2.185) holds in the T channel provided

$$\begin{aligned}\mathcal{T}_B^T(p+q,-k-q,p,-k) &= \mathcal{T}_B^{T*}(p,-k,p+q,-k-q)\ ,\\ \mathcal{T}_F^T(p+q,-k-q,p,-k) &= \mathcal{T}_F^{T*}(p,-k,p+q,-k-q)\ .\end{aligned} \tag{2.186}$$

This equation may be verified to be true (see below for some details).

Similarly the U_d channel S-matrix is obtained via the momentum assignments (2.146); It follows that (2.185) is obeyed provided

$$\begin{aligned}\mathcal{T}_B^{U_d}(p+q,k,p,k+q) &= \mathcal{T}_B^{U_d*}(p,k+q,p+q,k)\ ,\\ \mathcal{T}_F^{U_d}(p+q,k,p,k+q) &= \mathcal{T}_F^{U_d*}(p,k+q,p+q,k)\ ,\end{aligned} \tag{2.187}$$

which can also be checked to be true.

Finally in the U_e channel it follows from the momentum assignments (2.149) that (2.185) holds provided

$$\begin{aligned}\mathcal{T}_B^{U_e}(k,p+q,p,k+q) &= \mathcal{T}_B^{U_e*}(p,k+q,k,p+q)\ ,\\ \mathcal{T}_F^{U_e}(k,p+q,p,k+q) &= \mathcal{T}_F^{U_e*}(p,k+q,k,p+q)\ ,\end{aligned} \tag{2.188}$$

which we have also verified.

The T matrices for all the above channels of scattering are reported in §2.3.9. Note that the starring of the T matrices in (2.185) also involves a momentum exchange $p_1 \Leftrightarrow p_3$ and $p_2 \Leftrightarrow p_4$. It follows that under this exchange $q \to -q$. [39]

In verifying (2.186), (2.187) and (2.188) we have used the fact that the functions J_B and J_F are both invariant under the combined operation of complex conjugation accompanied by the flip $q \to -q$ (see (A.138)). We also use the fact that in each case (T, U_d and U_e) the

[39] For instance in the T channel, we get the equations

$$p'+q'=p\ ,\ p'=p+q\ ,\ -k'-q'=-k\ ,\ -k'=-k-q\ . \tag{2.189}$$

It follows that $q'=-q$.

factor $E(q, p-k, p+k)$ flips sign under the momentum exchange $p_1 \Leftrightarrow p_3$ and $p_2 \Leftrightarrow p_4$; the sign obtained from this process compensates the minus sign from complex conjugating the explicit factor of i. [40]

2.4.2 Unitarity in the S channel

The S-matrix in the S channel is of $O(1)$ and one has to use the full non-linear unitarity conditions (2.81) and (2.82) . We reproduce them here for convenience.

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(-Y(s)(\mathcal{T}_B^S(s,\theta)+4Y(s)\mathcal{T}_f^S(s,\theta))(\mathcal{T}_B^{S*}(s,-(\alpha-\theta))+4Y(s)\mathcal{T}_f^{S*}(s,-(\alpha-\theta))) + \mathcal{T}_B^S(s,\theta)\mathcal{T}_B^{S*}(s,-(\alpha-\theta))\Big) = i(\mathcal{T}_B^{S*}(s,-\alpha)-\mathcal{T}_B^S(s,\alpha)) \ , \tag{2.190}$$

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(Y(s)(\mathcal{T}_B^S(s,\theta)+4Y(s)\mathcal{T}_f^S(s,\theta))(\mathcal{T}_B^{S*}(s,-(\alpha-\theta))+4Y(s)\mathcal{T}_f^{S*}(s,-(\alpha-\theta))) - 16Y(s)^2\mathcal{T}_f^S(s,\theta)\mathcal{T}_f^{S*}(s,-(\alpha-\theta))\Big) = i4Y(s)(-\mathcal{T}_f^S(s,\alpha)+\mathcal{T}_f^{S*}(s,-\alpha)) \ , \tag{2.191}$$

where

$$Y(s) = \frac{-s+4m^2}{16m^2} \tag{2.192}$$

is as defined in (2.69), and $\mathcal{T}_B^S$ corresponds to the bosonic T matrix while $\mathcal{T}_f^S$ corresponds to the phase unambiguous part of the fermionic T matrix in the Singlet (S) channel given in (2.166) (also see (2.161)). In centre of mass coordinates it takes the form

$$\mathcal{T}_f^S(s,\theta) = -\frac{\mathcal{T}_F^S(s,\theta)}{4Y(s)} \ . \tag{2.193}$$

Substituting the above into (2.190) and (2.191), the conditions for unitarity may be rewritten as

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(-Y(s)(\mathcal{T}_B^S(s,\theta)-\mathcal{T}_F^S(s,\theta))(\mathcal{T}_B^{S*}(s,-(\alpha-\theta))-\mathcal{T}_F^{S*}(s,-(\alpha-\theta))) + \mathcal{T}_B^S(s,\theta)\mathcal{T}_B^{S*}(s,-(\alpha-\theta))\Big) = i(\mathcal{T}_B^{S*}(s,-\alpha)-\mathcal{T}_B^S(s,\alpha)) \ , \tag{2.194}$$

[40] The unitarity conditions in these channels are simply the statement that the S-matrices are real. The reality of S-matrices is tightly connected to the absence of two particle branch cuts in the S-matrices in these channels at leading order in large-N.

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Bigg(Y(s)(\mathcal{T}_B^S(s,\theta)-\mathcal{T}_F^S(s,\theta))(\mathcal{T}_B^{S*}(s,-(\alpha-\theta))-\mathcal{T}_F^{S*}(s,-(\alpha-\theta)))$$
$$-\mathcal{T}_F^S(s,\theta)\mathcal{T}_F^{S*}(s,-(\alpha-\theta))\Bigg)=i(\mathcal{T}_F^S(s,\alpha)-\mathcal{T}_F^{S*}(s,-\alpha))\ . \quad (2.195)$$

Let us pause to note that under duality $\mathcal{T}_B \to \mathcal{T}_F$ and vice versa; it follows then (2.194) and (2.195) map to each other under duality. In other words the unitarity conditions are compatible with duality.

We will now verify that our S channel S-matrix is indeed compatible with unitarity. Let us recall that the angular dependence of the S-matrix, in the centre of mass frame is given by

$$\begin{aligned}\mathcal{T}_B^S &= H_B T(\theta) + W_B - iW_2\delta(\theta)\ ,\\ \mathcal{T}_F^S &= H_F T(\theta) + W_F - iW_2\delta(\theta)\ ,\end{aligned} \quad (2.196)$$

where

$$T(\theta) = i\cot(\theta/2).$$

We will list the particular values of the coefficient functions $H_B(s)$ etc below; we will be able to proceed for a while leaving these functions unspecified.

Substituting (2.196) in (2.194) and doing the angle integrations (The relevant formulae have been worked out in eq 2.63 of [4].) we find that (2.194) is obeyed if and only if

$$\begin{aligned}H_B - H_B^* &= \frac{1}{8\pi\sqrt{s}}(W_2H_B^* - H_BW_2^*)\ ,\\ W_2 + W_2^* &= -\frac{1}{8\pi\sqrt{s}}(W_2W_2^* + 4\pi^2H_BH_B^*)\ ,\\ W_B - W_B^* &= \frac{1}{8\pi\sqrt{s}}(W_2W_B^* - W_2^*W_B) - \frac{i}{4\sqrt{s}}(H_BH_B^* - W_BW_B^*) - \frac{iY}{4\sqrt{s}}(W_B - W_F)(W_B^* - W_F^*)\ .\end{aligned} \quad (2.197)$$

Similarly (2.195) is obeyed if and only if

$$\begin{aligned}H_F - H_F^* &= \frac{1}{8\pi\sqrt{s}}(W_2H_F^* - H_FW_2^*)\ ,\\ W_2 + W_2^* &= -\frac{1}{8\pi\sqrt{s}}(W_2W_2^* + 4\pi^2H_FH_F^*)\ ,\\ W_F - W_F^* &= \frac{1}{8\pi\sqrt{s}}(W_2W_F^* - W_2^*W_F) - \frac{i}{4\sqrt{s}}(H_FH_F^* - W_FW_F^*) - \frac{iY}{4\sqrt{s}}(W_B - W_F)(W_B^* - W_F^*)\ .\end{aligned} \quad (2.198)$$

The first two equations of (2.197) and (2.198) are entirely identical to the first two equations of equation 2.66 in [4] for the non-supersymmetric case. The third equation has an additional contribution due to supersymmetry. Note that (2.197) and (2.198) are compatible with duality under $H_B \to H_F$ and $W_B \to W_F$ and vice versa.

Let us now proceed to verify that the equations (2.197) and (2.198) are indeed obeyed; for this purpose we need to use the specific values of the coefficient functions in (2.196). These functions are easily read off from the formulae (2.168) (that we reproduce here for convenience)

$$\mathcal{T}_B^S = -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + \frac{\sin(\pi\lambda)}{\pi\lambda}\left(4\pi i\lambda\sqrt{s}\cot(\theta/2) + J_B(\sqrt{s},\lambda)\right) ,$$
$$\mathcal{T}_F^S = -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + \frac{\sin(\pi\lambda)}{\pi\lambda}\left(4\pi i\lambda\sqrt{s}\cot(\theta/2) + J_F(\sqrt{s},\lambda)\right) , \tag{2.199}$$

from which we find

$$W_B = J_B(\sqrt{s},\lambda)\frac{\sin(\pi\lambda)}{\pi\lambda} ,$$
$$W_F = J_F(\sqrt{s},\lambda)\frac{\sin(\pi\lambda)}{\pi\lambda} , \tag{2.200}$$

where the explicit form of the J functions are given in (2.158). While we also identify

$$H_B = H_F = 4\sqrt{s}\sin(\pi\lambda),\ W_2 = 8\pi\sqrt{s}(\cos(\pi\lambda)-1),\ T(\theta) = i\cot(\theta/2) . \tag{2.201}$$

Using the above relations it is very easy to see that the first two equations in each of (2.197) and (2.198) are satisfied. The first equation in each of (2.197) and (2.198) holds because H_B, H_F and W_2 are all real. The second equation in each case boils down to a true trigonometric identity.

The functions W_B and W_F occur only in the third equation in (2.197) and (2.198). These equations assert two nonlinear identities relating the (rather complicated) J_B and J_F functions. We have verified by explicit computation that these identities are indeed obeyed. It follows that the conjectured S-matrix (2.165) is indeed unitary.

At the algebraic level, the satisfaction of the unitarity equation appears to be a minor miracle. A small mistake of any sort (a factor or two or an incorrect sign) causes this test to fail badly. In particular, unitarity is a very sensitive test of the conjectured form (2.165) of the S matrix. Let us recall again that this conjecture was first made in [4], where it was shown that it leads to a unitary $2 \to 2$ S-matrix. The supersymmetric S matrices considered here are more complicated than the S-matrices of the purely bosonic or purely fermionic theories of [4]. In particular the unitarity equation for four boson and four fermion S-matrices is different here from the corresponding equations in [4] (the difference stems from the fact that two bosons can scatter not just to two bosons but also to two fermions, and this second process also contributes to the quadratic part of the unitarity equations). Nonetheless the prescription (2.165) adopted from [4] turns out to give results that obey the modified unitarity equation presented here. In our opinion this constitutes a very nontrivial check of the crossing symmetry relation (2.165) proposed in [4].

The unitarity equation is satisfied for the arbitrary $\mathcal{N} = 1$ susy theory, and so is, in particular obeyed for the $\mathcal{N} = 2$ theory. Recall that the $\mathcal{N} = 2$ theory has a particularly simple S-matrix (2.183). In fact in the T and U channels the $\mathcal{N} = 2$ S-matrix is tree level exact at leading order in large-N. According to the rules of naive crossing symmetry the S channel S-matrix would also have been tree level exact. This result is in obvious conflict with the unitarity equation: in the equation $-i(T - T^\dagger) = TT^\dagger$ the LHS vanishes at tree level while the RHS is obviously nonzero. The modified crossing symmetry rules (2.165) resolve this paradox in a very beautiful way. According to the rules (2.165), the T matrix is not Hermitian even if T^{naive} is; as the term in (2.165) proportional to identity is imaginary. It follows from (2.165) that both LHS and the RHS of the unitarity equation are nonzero; they are infact equal, as we now pause to explicitly demonstrate. In the $\mathcal{N} = 2$ limit (see (2.183)) we have

$$\begin{aligned} H_B &= H_F = 4\sqrt{s}\sin(\pi\lambda) \ , \\ W_B &= -8m\sin(\pi\lambda) \ , \\ W_F &= 8m\sin(\pi\lambda) \ , \\ W_2 &= 8\pi\sqrt{s}(\cos(\pi\lambda) - 1) \ . \end{aligned} \tag{2.202}$$

The first equation in (2.197) is satisfied because everything is real. We have checked that the second equation is satisfied using a trigonometric identity. [41] The third equation works because we have

$$(H_B H_B^* - W_B W_B^*) = -16\sin^2(\pi\lambda)(-s + 4m^2) \tag{2.203}$$

and

$$Y(W_B - W_F)(W_B^* - W_F^*) = 16\sin^2(\pi\lambda)(-s + 4m^2) \tag{2.204}$$

the other terms don't matter because everything else is real. The same thing is true for (2.198) since

$$(H_F H_F^* - W_F W_F^*) = -16\sin^2(\pi\lambda)(-s + 4m^2) \tag{2.205}$$

and thus the unitarity conditions are satisfied by the conjectured S-matrix (2.166) in the $\mathcal{N} = 2$ theory as well.

2.5 Pole structure of the S-matrix in the S channel

The S channel S-matrix studied in the last two sections turns out to have an interesting analytic structure. In this section we will demonstrate that the S-matrix has a pole whenever $w < -1$. As we demonstrate below the pole is at threshold at $w = -1$, migrates to lower masses as w is further reduced until it actually occurs at zero mass at a critical value $w = w_c(\lambda) < -1$. As w is further reduced, the squared mass of the pole increases again, until the pole mass returns to threshold at $w = -\infty$.

[41] This is the only equation in which the LHS and RHS are both nonzero. The LHS is the imaginary part of the coefficient of identity.

In order to establish all these facts let us recall the structure of four boson and four fermion S-matrix in the S channel. The S-matrices take the form (see (2.158))

$$\mathcal{T}_B^S = \frac{n_b}{d_1 d_2}, \qquad \mathcal{T}_F^S = \frac{n_f}{d_1 d_2} , \tag{2.206}$$

where

$$\begin{aligned} d_1 =& -4|m|^2 \left(\mathrm{sgn}(\lambda)(w-1) \left(\left(\frac{1+y}{1-y}\right)^\lambda - 1 \right) + y \left(-w \left(\frac{1+y}{1-y}\right)^\lambda + w + \left(\frac{1+y}{1-y}\right)^\lambda + 3 \right) \right) , \\ d_2 =& \ \mathrm{sgn}(\lambda)(w-1) \left(\left(\frac{1+y}{1-y}\right)^\lambda - 1 \right) + y \left(w \left(\left(\frac{1+y}{1-y}\right)^\lambda - 1 \right) + 3 \left(\frac{1+y}{1-y}\right)^\lambda + 1 \right) , \end{aligned} \tag{2.207}$$

$$\begin{aligned} n_b =& -32|m|^3 y \sin(\pi\lambda) \Bigg(8 \ \mathrm{sgn}(\lambda)(w+1) y \left(\frac{1+y}{1-y}\right)^\lambda \\ & + (w-1)(\mathrm{sgn}(\lambda) - y) \left(\frac{1+y}{1-y}\right)^{2\lambda} (\mathrm{sgn}(\lambda)(w-1) + (w+3)y) \\ & - (w-1)(\mathrm{sgn}(\lambda) + y)(\mathrm{sgn}(\lambda)(w-1) - (w+3)y) \Bigg) , \\ n_f =& 32|m|^3 y \sin(\pi\lambda) \Bigg(8 \ \mathrm{sgn}(\lambda)(w+1) y \left(\frac{1+y}{1-y}\right)^\lambda \\ & - (w-1)(\mathrm{sgn}(\lambda) - y) \left(\frac{1+y}{1-y}\right)^{2\lambda} (\mathrm{sgn}(\lambda)(w-1) + (w+3)y) \\ & + (w-1)(\mathrm{sgn}(\lambda) + y)(\mathrm{sgn}(\lambda)(w-1) - (w+3)y) \Bigg) , \end{aligned} \tag{2.208}$$

where $y = \sqrt{s}/2|m|$. Through this discussion we assume that $\lambda m > 0$ (recall this condition was needed for duality invariance).

The denominators d_1, d_2 and the numerators are all polynomials of y and the quantity

$$X = \left(\frac{1+y}{1-y}\right)^\lambda .$$

Most of the interesting scaling behaviors we will encounter below are a consequence of the dependence of all quantities on X. Note that d_1 and d_2 are linear functions of X while n_b and n_f are quadratic functions of X. It is consequently possible to recast n_b and n_f in the form

$$\begin{aligned} n_b &= a_b d_1 d_2 + b_b d_1 + c_b d_2 , \\ n_f &= a_f d_1 d_2 + b_f d_1 + c_f d_2 . \end{aligned}$$

Here a_b, b_b, c_b, a_f, b_f and c_f are polynomials of y (but are independent of X) and are given by

$$\begin{aligned}
a_b &= y \ , \\
b_b &= (w-1)(\mathrm{sgn}(\lambda)+y)^2 \ , \\
c_b &= -4|m|^2(\mathrm{sgn}(\lambda)-y)(\mathrm{sgn}(\lambda)(w-1)-(w+3)y) \ , \\
a_f &= y \ , \\
b_f &= -(w-1)\left(1-y^2\right) \ , \\
c_f &= 4|m|^2(\mathrm{sgn}(\lambda)+y)(\mathrm{sgn}(\lambda)(w-1)-(w+3)y) \ .
\end{aligned} \tag{2.209}$$

In order to study the poles of the S-matrix we need to investigate the zeroes of the functions d_1 and d_2. Let us first consider the case $\lambda > 0$. In this case it turns out that $d_1(y)$ has a zero for $w \in (-\infty, w_c]$, while $d_2(y)$ has a zero in the range $w \in [w_c, -1]$ where

$$w_c(\lambda) = 1 - \frac{2}{|\lambda|} \ . \tag{2.210}$$

At $w = -\infty$ the zero of d_1 occurs at $y = 1$. As w is increased the y value of the zero decreases, until it reaches $y = 0$ at $w = w_c$. At larger values of w, d_1 no longer has a zero. However $d_2(y)$ develops a zero. The zero of $d_2(y)$ starts out at $y = 0$ when $w = w_c$, and then increases, reaching $y = 1$ at $w = -1$. At larger values of w neither d_1 nor d_2 have a zero.

When $\lambda < 0$ we have an identical situation except that the roles of d_1 and d_2 are reversed. $d_2(y)$ has a zero for $w \in (-\infty, w_c]$, while $d_1(y)$ has a zero in the range $w \in [w_c, -1]$. At $w = -\infty$ the zero of d_2 occurs at $y = 1$. As w is increased the y value of the zero decreases, until it reaches $y = 0$ at $w = w_c$. At larger values of w, d_2 no longer has a zero. However $d_1(y)$ develops a zero. The zero of $d_1(y)$ starts out at $y = 0$ when $w = w_c$, and then increases, reaching $y = 1$ at $w = -1$. At larger values of w neither d_1 nor d_2 have a zero.

In summary our S-matrix has a pole for $w \in (-\infty, -1]$. The pole lies at threshold at the end points of this range, and becomes massless at $w = w_c$. There are clearly three special values of w in this range: $w = -1$, $w = w_c$ and $w = -\infty$. In the rest of this section we examine the neighborhood of three special points in turn.

2.5.1 Behavior near $w = -1 - \delta w$

In this subsection we study the pole in the neighborhood of $w = -1$. When $w \to -1 - \delta w$ with $0 < \delta w << 1$, we also expand $y \to 1 - \delta y$ (where $0 < \delta y << 1$) and find that

$$\begin{aligned}
d_1 \sim & 4|m|^2 \left((\mathrm{sgn}(\lambda) - 1) \left(\delta w - 2 \left(\frac{2}{\delta y} \right)^{\lambda} \right) + 2(\mathrm{sgn}(\lambda) + 1) \right) , \\
d_2 \sim & (\mathrm{sgn}(\lambda) + 1) \left(2 - \left(\frac{2}{\delta y} \right)^{\lambda} \delta w \right) - 2 \left(\frac{2}{\delta y} \right)^{\lambda} (\mathrm{sgn}(\lambda) - 1) , \\
a_b \sim & 1 - \delta y , \\
b_b \sim & - (2 + \delta w)(\mathrm{sgn}(\lambda) + 1 - \delta y)^2 , \\
c_b \sim & 4|m|^2 (\mathrm{sgn}(\lambda) - 1 + \delta y)(\mathrm{sgn}(\lambda)(2 + \delta w) + (2 - \delta w)(1 - \delta y)) , \\
a_f \sim & 1 - \delta y , \\
b_f \sim & 2\delta y (2 + \delta w)(2 - \delta y) , \\
c_f \sim & - 4|m|^2 (\mathrm{sgn}(\lambda) + 1 - \delta y)(\mathrm{sgn}(\lambda)(\delta w + 2) + (2 - \delta w)(1 - \delta y)) .
\end{aligned} \tag{2.211}$$

Let us first consider the case $\lambda > 0$. In this case d_1 equals $16m^2$ at leading order and so does not have a zero for δw and δy small. On the other hand

$$d_2 \propto \left(2 - \left(\frac{2}{\delta y} \right)^{\lambda} \delta w \right)$$

and so vanishes when

$$\frac{\delta w}{2} = \left(\frac{\delta y}{2} \right)^{|\lambda|} , \quad \frac{\delta y}{2} = \left(\frac{\delta w}{2} \right)^{\frac{1}{|\lambda|}} . \tag{2.212}$$

When $\lambda < 0$, d_2 is a monotonic function that never vanishes. However d_1 vanishes provided the condition (2.212) is met. It follows that the S-matrix has a pole when (2.212) is satisfied for both signs of λ.

The pole in the S-matrix occurs due to the vanishing of the denominator $d_1 d_2$. As this denominator is the same for both the boson boson $\to$ boson boson and the fermion fermion $\to$ fermion fermion S-matrices, both these scattering processes have a pole at the value of y listed in (2.212). The residue of this pole is, however, significantly different in the four boson and four fermion scattering processes. Let us first consider the four boson scattering term. The residue of the pole is determined by b_b evaluated at (2.212) (in the case $\lambda > 0$) and c_b evaluated at the same pole (in the case $\lambda < 0$). In either case we find the structure of the pole for four boson scattering to be

$$\mathcal{T}_B \sim \frac{\left(\frac{\delta y}{2} \right)^{|\lambda|}}{\delta w - 2 \left(\frac{\delta y}{2} \right)^{|\lambda|}} . \tag{2.213}$$

In a similar manner the residue of the pole for four fermion scattering is determined by b_f evaluated at (2.212) (in the case $\lambda > 0$) and c_f evaluated at the same pole (in the case $\lambda < 0$). In either case we find that

$$\mathcal{T}_F \sim \frac{\left(\frac{\delta y}{2}\right)^{1+|\lambda|}}{\delta w - 2\left(\frac{\delta y}{2}\right)^{|\lambda|}} \ . \tag{2.214}$$

Notice that while the residue of the pole for four fermion scattering is suppressed compared to the residue of the same pole for four boson scattering by a factor of $(\delta w)^{\frac{1}{|\lambda|}}$.

2.5.2 Pole near $y = 0$

There exists a critical value, $w = w_c(\lambda)$, at which both d_1 and d_2 have zeroes at $y = 0$. In order to locate w_c we expand d_1 and d_2 about $y = 0$. To linear order we find

$$d_1 = d_2 \sim y(\lambda \mathrm{sgn}(\lambda)(w-1) + 2) \ . \tag{2.215}$$

Clearly d_1 and d_2 have a common zero at $y = 0$ provided

$$w = w_c(\lambda) = 1 - \frac{2}{|\lambda|} \ . \tag{2.216}$$

In order to study this pole in the neighbourhood of $w = w_c$ we set $w = w_c + \delta w$ (with $|\delta w| < 1$) near $y = \delta y$ (with $0 < \delta y << 1$); expanding in δw and δy we find

$$\begin{aligned}
d_1 &\sim \frac{8|m|^2 \delta y \left(\delta w \lambda + 2\delta y(1-|\lambda|)\right)}{\mathrm{sgn}(\lambda)} \ , \\
d_2 &\sim \frac{\delta y \left(\delta w \lambda - 2\delta y(1-|\lambda|)\right)}{\mathrm{sgn}(\lambda)} \ , \\
n_b &\sim -\frac{512|m|^3 \sin(\pi\lambda)\delta y^2(-1+|\lambda|)}{\lambda} \ , \\
n_f &\sim \frac{512|m|^3 \sin(\pi\lambda)\delta y^2(-1+|\lambda|)}{\lambda} \ .
\end{aligned} \tag{2.217}$$

The product $d_1 d_2$ vanishes when

$$\delta y = \frac{|\lambda \delta w|}{2(1-|\lambda|)}, \quad \text{i.e.} \quad \delta y^2 = \frac{\lambda^2 \delta w^2}{4(1-|\lambda|)^2} \ . \tag{2.218}$$

[42] The residue of the pole at $y = 0$ for any sign of λ is given by substituting (2.218) into the functions n_b and n_f in (2.217). We find the pole structure of the bosonic S-matrix near $y = 0$ to be

$$\mathcal{T}_B \sim -\frac{64|m| \sin(\pi\lambda)(-1+|\lambda|)}{|\lambda| \left(\delta w^2 \lambda^2 - 4\delta y^2(1-|\lambda|)^2\right)} \ . \tag{2.219}$$

[42] $d_1 d_2$ also vanishes quadratically at $\delta y = 0$. Note however that both n_b and n_f are proportional to δy^2. Consequently the factors of δy^2 cancel between the numerator and denominator.

In a similar manner we find the pole structure of the fermion S-matrix near $y = 0$ to be

$$\mathcal{T}_F \sim \frac{64|m|\sin(\pi\lambda)(-1+|\lambda|)}{|\lambda|\left(\delta w^2\lambda^2 - 4\delta y^2(1-|\lambda|)^2\right)} \ . \tag{2.220}$$

2.5.3 Behavior at $w \to -\infty$

We now turn to the analysis of the pole structure at $w \to -\infty$. This is easily achieved by setting $w = -\frac{1}{\delta w}$ with $0 < \delta w << 1$ and $y \to 1 - \delta y$ with $0 < \delta y << 1$. The various functions (2.207) in the S-matrix (2.206) have the behavior

$$\begin{aligned}
d_1 \sim & \frac{4|m|^2}{\delta w}\left((\delta y + \mathrm{sgn}(\lambda) - 1)\left(1 - \left(\frac{2}{\delta y}\right)^{\lambda}\right) + (\mathrm{sgn}(\lambda)+3)\delta w\right) \ , \\
d_2 \sim & \frac{1}{\delta w}\left((-\delta y + \mathrm{sgn}(\lambda) + 1) - \left(\frac{2}{\delta y}\right)^{\lambda}((\mathrm{sgn}(\lambda)-3)\delta w + \mathrm{sgn}(\lambda)+1)\right) \ , \\
a_b \sim & 1 - \delta y \ , \\
b_b \sim & -\frac{1}{\delta w}(\mathrm{sgn}(\lambda) + 1 - \delta y)^2 \ , \\
c_b \sim & -4|m|^2(\mathrm{sgn}(\lambda) - 1 + \delta y)(-\mathrm{sgn}(\lambda)(1 + \frac{1}{\delta w}) - (3 - \frac{1}{\delta w})(1 - \delta y)) \ , \\
a_f \sim & 1 - \delta y \ , \\
b_f \sim & \frac{\delta y}{\delta w}(2 - \delta y) \ , \\
c_f \sim & 4|m|^2(\mathrm{sgn}(\lambda) + 1 - \delta y)(-\mathrm{sgn}(\lambda)(1 + \frac{1}{\delta w}) - (3 - \frac{1}{\delta w})(1 - \delta y)) \ .
\end{aligned} \tag{2.221}$$

Let us first consider the case $\lambda > 0$. In this case d_2 is a monotonic function that never vanishes and so does not have a zero for δw and δy small. On the other hand

$$d_1 \propto \left(\delta w - \frac{1}{2}\left(\frac{\delta y}{2}\right)^{1-|\lambda|}\right)$$

and so vanishes when

$$\delta w = \frac{1}{2}\left(\frac{\delta y}{2}\right)^{1-|\lambda|}, \quad \delta y = \left(\frac{4\delta w}{2^{|\lambda|}}\right)^{\frac{1}{1-|\lambda|}} \ . \tag{2.222}$$

When $\lambda < 0$, d_1 is a constant $-8m^2$. However d_2 vanishes provided the condition (2.222) is met. It follows that the S-matrix has a pole when (2.222) is satisfied for both signs of λ.

The pole in the S-matrix occurs due to the vanishing of the denominator $d_1 d_2$. As this denominator is the same for both the boson boson $\to$ boson boson and the fermion fermion $\to$ fermion fermion S-matrices, both these scattering processes have a pole at the value of

y listed in (2.222). The residue of this pole is different in the four boson and four fermion scattering processes as before. Let us first consider the four boson scattering term. The residue of the pole is determined by c_b evaluated at (2.222) (in the case $\lambda > 0$) and b_b evaluated at the same pole (in the case $\lambda < 0$). In either case we find the structure of the pole for four boson scattering to be

$$\mathcal{T}_B \sim \frac{\left(\frac{\delta y}{2}\right)^{2-|\lambda|}}{\delta w - \frac{1}{2}\left(\frac{\delta y}{2}\right)^{1-|\lambda|}} \ . \tag{2.223}$$

In a similar manner the residue of the pole for four fermion scattering is determined by c_f evaluated at (2.212) (in the case $\lambda > 0$) and b_f evaluated at the same pole (in the case $\lambda < 0$). In either case we find that

$$\mathcal{T}_F \sim \frac{\left(\frac{\delta y}{2}\right)^{1-|\lambda|}}{\delta w - \frac{1}{2}\left(\frac{\delta y}{2}\right)^{1-|\lambda|}} \ . \tag{2.224}$$

Notice that the residue of the pole for four boson scattering is supressed by a factor of $(\delta w)^{\frac{1}{1-|\lambda|}}$ compared to the residue for four fermion scattering.

2.5.4 Duality invariance

It is most interesting to note that the statements and results obtained in the above sections ((2.212), (2.216) and (2.222)) are all duality invariant. This is most transparent from the observation that under the duality transformation (2.16)

$$\begin{aligned} d_1 &\leftrightarrow d_1 \ , \\ d_2 &\leftrightarrow d_2 \ . \end{aligned} \tag{2.225}$$

Hence the zeroes of d_1 and d_2 ((2.212) and (2.222)) should map to themselves, and w_c (2.216) should be duality invariant. Also recollect that under duality the bosonic and fermionic S-matrices map to one another. Thus it is natural to expect that the pole in the bosonic S-matrix at $w = -1$ (2.212) should map to the pole of the fermionic S matrix at $w = -\infty$ (2.222) and vice versa. Since both the bosonic and fermionic S matrices have a pole at $w = w_c$ (2.216) at $y = 0$, this pole should be self dual.

Upon using (2.16) on (2.216) it is straightforward to see that it is duality invariant. The slightly non-trivial part is the mapping of the two scaling regimes (2.212) and (2.222). It is straightforward to obtain the identification from $w = -\infty$ to $w = -1$ from (2.16)

$$-\frac{1}{\delta w_\infty} = \frac{3 - (-1 - \delta w_{-1})}{1 + (-1 - \delta w_{-1})} \sim -\frac{4}{\delta w_{-1}} \tag{2.226}$$

Using the above result in (2.222) and applying (2.16) for λ it is easy to check that (2.212) follows (and vice versa).

2.5.5 Scaling limit of the S-matrix

In this subsection we discuss a particularly interesting near-threshold limit of the S-matrix. It was shown in [59] that in this limit the S-matrices for the boson-boson and fermion-fermion reduce to the ones that are obtained by solving the Schrodinger equation with Amelino-Camelia-Bak boundary conditions [60, 61]. In this subsection we illustrate that the analysis of [59] applies for our results as well. We consider the near threshold region

$$y = 1 + \frac{k^2}{2m^2} \tag{2.227}$$

with $k << 1$ and

$$w = -1 - \delta w \tag{2.228}$$

where $0 < \delta w << 1$. In the limit

$$k \to 0, \quad \delta w \to 0, \quad , \frac{k^2}{4m^2}\left(\frac{\delta w}{2}\right)^{-\frac{1}{|\lambda|}} = \text{fixed} \tag{2.229}$$

the J function in the bosonic S-matrix ((2.206)) reduces to [43]

$$J_B = 8|m\sin(\pi\lambda)|\frac{1 + e^{i\pi|\lambda|}\frac{A_R}{k^{2|\lambda|}}}{1 - e^{i\pi|\lambda|}\frac{A_R}{k^{2|\lambda|}}}. \tag{2.230}$$

where

$$A_R = \frac{4^{|\lambda|}}{2}|m|^{2|\lambda|}\delta w \ . \tag{2.231}$$

Comparing our Lagrangian (2.15) with that of eq 1.1 of [59] we make the parameter identifications

$$\delta w = \frac{\delta b_4}{8|m|\pi\lambda} \ .$$

Substituting δw in (2.231) we see that (2.230) matches exactly with eq 1.12 of [59].

2.5.6 Effective theory near $w = w_c$?

As we have explained above, our theory develops a massless bound state at $w = w_c$; the mass of this bound state scales like $w - w_c$ in units of the mass of the scattering particles. [44] When $w - w_c \ll 1$ there is a separation of scales between the new bound state and all other excitations in our theory. In this regime the effective dynamics of the nearly massless particles should be governed by an autonomous quantum field theory that makes no reference

[43] Here we work in the regime $\sqrt{s} > 2m$ i.e $y > 1$ and hence the appearance of the factors of $e^{i\pi\lambda}$.

[44] We expect all of these results to continue to hold at finite N at least when N is large; in the rest of the discussion we assume that N is finite, and so the interactions between two bound state particles is not parametrically suppressed.

to UV degrees of freedom. It seems likely that the superfield that creates the bound states is a real $\mathcal{N} = 1$ superfield. The fixed point that governs the dynamics of this field presumably has a single relevant deformation; as it was possible to approach this theory with a single fine tuning (setting $w = w_c$). These considerations suggest that the dynamics of the light bound state is governed by an $\mathcal{N} = 1$ Wilson-Fisher theory built out of a single real superfield. If this suggestion is correct it would imply that the long distance dynamics of the light bound states is independent of λ. Given that the bound states are gauge neutral this possibility does not seem absurd to us. It would be interesting to study this further.

2.6 Discussion

In this chapter we have presented computations and conjectures for the all orders S-matrix in the most general renormalizable $\mathcal{N} = 1$ Chern-Simons matter theory with a single fundamental matter multiplet. Our results are consistent with unitarity if and only if we assume that the usual results of crossing symmetry are modified in precisely the manner proposed in [4]; we view this fact as a nontrivial consistency check of the crossing symmetry rules proposed in [4].

The 'particle - antiparticle' S-matrix in the singlet channel conjectured here has an interesting analytic structure. In a certain range of superpotential parameters the S-matrix has a bound state pole; a one parameter tuning of superpotential parameters can be used to set the pole mass to zero. We find the existence of a massless bound state in a theory whose elementary excitations are all massive fascinating. It would be interesting to further investigate the low energy dynamics of these massless bound states. It would also be interesting to investigate if these bound states are 'visible' in the explicit results for the partition functions of Chern-Simons matter theories.

As we have explained in the previous section, our singlet sector particle - antiparticle S-matrix has a simple non-relativistic limit. It would be useful to reproduce this scattering amplitude from the solution of a manifestly supersymmetric Schrodinger equation.

The results of this chapter suggest many natural extensions and questions. First it would be useful to generalise the computations of this chapter to the mass deformed $\mathcal{N} = 3$ and especially to the mass deformed $\mathcal{N} = 6$ susy gauge theories (the later is necessarily a $U(N) \times U(M)$ theory; the methods of this chapter are likely to be useful in the limit $N \to \infty$ with M held fixed). This generalisation should allow us to make contact with earlier studies of scattering in ABJ theory [36, 37, 38, 39, 40, 41, 42] that were performed arbitrary values of M and N but perturbatively (to given loop order) in λ.

At the $\mathcal{N} = 2$ point the S-matrices presented here are tree level exact in the three non anyonic channels, and depend on λ in a very simple way in the singlet channel. It is possible that this very simple result can be deduced in a more structural manner using only general principles and $\mathcal{N} = 2$ supersymmetry. It would be interesting if this were the case.

As an intermediate step in the computation of the S-matrix we evaluated the off-shell four point function of four superfields. This four point correlator was rather complicated in the general $\mathcal{N} = 1$ theory, but extremely simple at the $\mathcal{N} = 2$ point. The four point correlator (or sum of ladder diagrams) is a useful intermediate piece in the evaluation of two, three and four point functions of gauge invariant operators [44, 45, 62, 55]. The simplicity of the $\mathcal{N} = 2$ results suggest that it would be rather easy to explicitly evaluate such correlators, at least in special kinematic limits. Such computations could be used as independent checks of duality as well as well as inputs into $\mathcal{N} = 2$ generalisations of the Maldacena-Zhiboedov solutions of Chern-Simons fundamental matter theories [5, 6].

All of the computations presented here have been performed under the assumption $\lambda m \geq 0$. At least naively all of the checks of duality (including earlier checks involving the partion function) fail when $\lambda m < 0$. It would be interesting to understand why this is the case. It is possible that our theory undergoes a phase transition as λm changes sign (see [46, 3] for related discussion). It would be interesting to understand this better.

We believe that the results of this chapter put the crossing symmetry relations conjectured in [4] on a firm footing. It would be interesting to find a rigorous proof of these crossing relations, and even more interesting to hit upon a plausible generalisation of these relations to finite N and k. From a traditional perturbative point of view the modified crossing symmetry rules are presumably related to infrared divergences. It thus seems likely that one route to a proof and generalisation of these relations lies in a detailed study of the infrared divergences of the relevant Feynman graphs.

Chapter 3

Superconformal invariance, BPS spectrum and current correlators

3.1 Introduction

In this chapter we continue our investigation of three dimensional quantum field theories with all mass terms turned off. In such cases, the resulting theory is superconformal. This chapter is devoted to the study of such theories where matter is in the fundamental representation of the gauge group. We begin by understanding the gauge invariant operator content of these theories and list the spectrum of local gauge invariant operators - these are conserved currents of arbitrarily high spin (by which we mean spin $s > 2$). Next, we construct the operators in the case of free theories where such currents exist and are conserved, but are not conserved as soon as we introduce interactions. We use superconformal invariance and current conservation to constrain the form of the three point function of these current operators and present evidence for a conjecture about the structure of these three point functions.

It has recently been realised that non-abelian Chern-Simons theories coupled to fundamental matter fields in 3 dimensions are exactly solvable in the large-N limit. These theories have an interesting 'current algebra' structure involving almost conserved higher spin fields, appear to enjoy invariance under nontrivial level-rank type strong-weak coupling dualities, and also appear to admit a bulk dual description in terms of Vasiliev's equations for higher spin fields [43, 23, 5, 6, 63, 14, 44, 2, 64, 65, 49, 45, 46, 47, 48, 3].

The new results obtained for large-N vector Chern-Simons theories are exciting partly because they apply to non-supersymmetric theories. Most of the results obtained in [22, 44], however, have simple extensions to the supersymmetric counterparts of the theories studied there (see e.g. [2]). For instance, it should be possible to extend the results of Maldacena and Zhiboedov [5, 6] to obtain the exact form of the higher spin current algebra and the correlation functions of supersymmetric Chern-Simons coupled to fundamental matter fields with minimal matter content.

In order to extend recent results in the study of matter Chern-Simons theories to their supersymmetric counterparts, it would be convenient to have a formulation of these theories in superspace. off-shell superspace formulations of theories with extended supersymmetry are complicated and very messy. Moreover the abstract study of supersymmetric matter Chern Simons theories, along the lines of [5, 6], does not need an off-shell formalism. In this chapter we initiate the development of on-shell superspace techniques to study superconformal field theories. In particular we present a detailed study of free superconformal field theories in superspace using on-shell techniques. We present a superspace construction of higher spin supercurrents in free theories, and describe the structural form of the current algebra of the corresponding higher spin currents once we include the effect of interactions. We also study the correlation functions of higher spin currents in superspace; in particular we conjecture that superconformal invariance and current conservation constrains the form of the three point functions of higher spin currents to a linear combination of the unique 'free' structure and a parity odd structure; we present evidence in favour of this conjecture.

This chapter is structured as follows. In section 3.2 we consider $3d$ superspace, and the differential form of various operators which act in it. The construction of superconformally covariant structures in superspace is reviewed. Section 3.3 deals with specifics of $\mathcal{N} = 1, 2, 3, 4$ and 6 superconformal symmetry in superspace and the construction of superfield multiplets. In section 3.4 on-shell supercurrent multiplets for higher spin currents in the free theory are constructed out of the superfields. In section 3.5 we make a few remarks about the structure of anomalous conservation equations for $3d$ CFTs and SCFTs with weakly broken higher spin symmetry. In sections 3.6 and 3.7, which are essentially independent of sections 3.3, 3.4 and 3.5 and can be read independently, we turn to correlation functions of $\mathcal{N} = 1$ $3d$ SCFTs. In section 3.6 we give the form of the two point function of a spin s operator and give an elementary derivation, on the basis of symmetry and dimensional arguments, of the two point function of two spin half operators and explicitly compute a two point correlator in the free theory. In section 3.7 we turn to three point correlation functions - we first construct parity even and odd superconformal invariants in superspace, determine the myriad non-linear relations between them and then use these results (in subsection 3.7.3) to determine the independent invariant structures which can arise in various three point functions of higher spin operators. Sections 3.6 and 3.7 are essentially an extension, to the superconformal case, of many of the results of [22]. We build the invariants using the superconformal covariant structures constructed by J-H Park and H. Osborn [66, 67, 68, 69, 70] augmented by the polarisation spinor formalism used by [22]. In appendices B.1 and B.2 we list our conventions and some useful identities. In appendix B.3 we give single trace conformal primary decomposition of a free $U(N)$ theory of a single complex scalar and complex fermion. In appendix B.4 we present the full single trace superconformal primary spectrum of the theories discussed in section 3.3.

3.2 Superspace

We begin by reviewing superspace in three dimensions and the covariant structures that it admits, relying heavily on the paper of Park [69]. Our conventions are summarised in appendix B.1.

In order to study $\mathcal{N} = m$ superconformal field theories in 3 dimensions we employ a superspace whose coordinates are the 3 spacetime coordinates x^μ together with the $2m$ fermionic coordinates θ^a_α. Here $\alpha = 1, 2$ is a spacetime spinor index while $a = 1 \ldots m$ is the R-symmetry index, where the θs (and the supercharges Q^a_αs) are Majorana spinors that lie in the vector representation of the R-symmetry group $SO(\mathcal{N})$. The superconformal algebra, listed in (B.7) in appendix B.1.1, is implemented in superspace by the construction

$$
\begin{aligned}
P_\mu &= -i\partial_\mu, \\
M_{\mu\nu} &= -i\left(x_\mu\partial_\nu - x_\nu\partial_\mu - \frac{1}{2}\epsilon_{\mu\nu\rho}(\gamma^\rho)_\alpha{}^\beta\theta^a_\beta\frac{\partial}{\partial\theta^a_\alpha}\right) + \mathcal{M}_{\mu\nu}, \\
D &= -i\left(x^\nu\partial_\nu + \frac{1}{2}\theta^{\alpha a}\frac{\partial}{\partial\theta^\alpha_a}\right) + \Delta, \\
K_\mu &= -i\left(\left(x^2 + \frac{(\theta^a\theta^a)^2}{16}\right)\partial_\mu - 2x_\mu\left(x\cdot\partial + \theta^{\alpha a}\frac{\partial}{\partial\theta^\alpha_a}\right) + (\theta^a X_+\gamma_\mu)^\beta\frac{\partial}{\partial\theta^\beta_a}\right) \\
&= x^\nu M_{\nu\mu} - x_\mu D + \frac{i}{2}(\theta^a\gamma_\mu X)^\alpha\frac{\partial}{\partial\theta^\alpha_a} - \frac{i}{16}(\theta^a\theta^a)^2\partial_\mu + \frac{(\theta^a\theta^a)}{4}(\theta^b\gamma_\mu)^\alpha\frac{\partial}{\partial\theta^\alpha_b}, \\
Q^a_\alpha &= \frac{\partial}{\partial\theta^\alpha_a} - \frac{i}{2}\theta^{\beta a}(\gamma^\mu)_{\beta\alpha}\partial_\mu, \\
S^a_\alpha &= -(X_+)_\alpha{}^\beta Q^a_\beta - i\theta^a\theta^b\frac{\partial}{\partial\theta^\alpha_b} - i\theta^a_\alpha\theta^{b\beta}\frac{\partial}{\partial\theta^\beta_b} + \frac{i}{2}(\theta^b\theta^b)\frac{\partial}{\partial\theta^\alpha_a} \\
&= -(X_-)_\alpha{}^\beta\frac{\partial}{\partial\theta^\beta_a} + \frac{\theta^a_\alpha}{2}D + \frac{1}{4}\epsilon_{\mu\nu\rho}(\gamma^\rho\theta^a)_\alpha M^{\mu\nu} - \frac{(\theta^b\theta^b)}{8}\theta^{a\beta}\partial_{\beta\alpha} - \frac{3i}{4}\left(\theta^a_\alpha\theta\frac{\partial}{\partial\theta} + \theta^a\theta^b\frac{\partial}{\partial\theta^\alpha_b}\right), \\
I^{ab} &= -i\left(\theta^a\frac{\partial}{\partial\theta_b} - \theta^b\frac{\partial}{\partial\theta_a}\right) + \mathcal{I}^{ab}\ .
\end{aligned}
\tag{3.1}
$$

Here the derivative expressions act on superspace coordinates while the operators $\mathcal{M}$, Δ and $\mathcal{I}^{ab}$ act on the operators (states) which carry tensor structure, non-zero scaling dimensions and transform non-trivially under R-symmetry. All indices are contracted in matrix notation (the spinors are contracted from north-west to south-east, see appendix B.1.1) and the definitions of X_+, X_- are given in (3.9). Note that $x^2 + \frac{(\theta^a\theta^a)^2}{16} = \frac{1}{2}(X_+X_-)_\alpha{}^\alpha$ (this combination appears in the expression for K_μ above). Below we will often have occasion to use a 'supersymmetric' derivative operator D^a_α defined by

$$D^a_\alpha = \frac{\partial}{\partial\theta^\alpha_a} + \frac{i}{2}\theta^{a\beta}\partial_{\beta\alpha}, \tag{3.2}$$

The operator D^i_α has the property that it anticommutes will all supersymmetry generators

$$\{D^a_\alpha, Q^b_\beta\} = 0 \tag{3.3}$$

Note also that

$$\{D^a_\alpha, D^b_\beta\} = -P_{\alpha\beta}\delta^{ab} \tag{3.4}$$

In the sequel we will sometimes require to construct functions built out of coordinates in superspace that are invariant under superconformal transformations. Given two points in superspace, (x_1, θ_1) and (x_2, θ_2), it is obvious that $\theta_{12} = \theta_1 - \theta_2$ is annihilated by the supersymmetry generators. It is also easy to verify that the supersymmetrised coordinate difference

$$\tilde{x}^\mu_{12} = x^\mu_{12} + \frac{i}{2}\theta_1^{a\alpha}(\gamma^\mu)_\alpha{}^\beta\theta^a_{2\beta} \tag{3.5}$$

is also annihilated by all Q_α.

Any vector of $SO(2,1)$ may equally be regarded as a symmetrised bispinor. So x^μ may be represented in terms of bispinors by the 2×2 matrix $X = x \cdot \gamma$. In this notation (3.5) may be rewritten as

$$(\tilde{X}_{12})_\alpha{}^\beta = (X_{12})_\alpha{}^\beta + i\theta^a_{1\alpha}\theta_2^{a\beta} + \frac{i}{2}(\theta^a_1\theta^a_2)\delta_\alpha{}^\beta \tag{3.6}$$

While an arbitrary function of θ_{12} and $\tilde{X}_{12}$ is annihilated by the supersymmetry operator, it is not, in general, annihilated by the generator of superconformal transformations. In order to build superconformally invariant expressions it is useful to note that

$$S^a_\alpha = IQ^a_\alpha I \tag{3.7}$$

where I is the superinversion operator, whose action on the coordinates of superspace is given by

$$I(x^\mu) = \frac{x^\mu}{x^2 + \frac{(\theta^a\theta^a)^2}{16}} \tag{3.8}$$

To define the superinversion properties of spinors, it is useful to define the objects

$$X_\pm = X \pm \frac{i}{4}(\theta^a\theta^a)\mathbb{1}. \tag{3.9}$$

It follows from (3.8) that this object transforms homogeneously under inversions

$$\begin{aligned} I(X_\pm) &= X_\pm^{-1} \\ I(\theta^a_\alpha) &= (X_+^{-1}\theta^a)_\alpha \\ I(\theta^{a\beta}) &= -(\theta^a X_-^{-1})^\beta \end{aligned} \tag{3.10}$$

(Here X is the 2×2 matrix corresponding to a particular superspace point, not a coordinate difference).

Using these rules it follows that the following objects (see e.g. [66, 67, 68, 69, 70] transform homogeneously under inversions:

$$(X_{ij+})_\alpha{}^\beta = (X_{i+})_\alpha{}^\beta - (X_{j-})_\alpha{}^\beta + i\theta^a_{i\alpha}\theta_j^{\beta a} \tag{3.11}$$

$$(X_{ij-})_\alpha{}^\beta = (X_{i-})_\alpha{}^\beta - (X_{j+})_\alpha{}^\beta - i\theta^a_{j\alpha}\theta_i^{\beta a} \tag{3.12}$$

For example,

$$I\left(X_{ij+}\right)_{\alpha}^{\ \beta} = I\left((X_{i+})_{\alpha}^{\ \beta} - (X_{j-})_{\alpha}^{\ \beta} + i\theta^{a}_{i\alpha}\theta^{a\beta}_{j}\right) = -(X^{-1}_{i+})_{\alpha}^{\ \gamma}(X_{ij+})_{\gamma}^{\ \delta}(X^{-1}_{j-})_{\delta}^{\ \beta} \tag{3.13}$$

Moreover it may be demonstrated [66, 67, 68, 69, 70] that

$$X_{ij\pm} = \tilde{X}_{ij} \pm \frac{i}{4}\theta^{2}_{ij}\mathbb{1} \tag{3.14}$$

In other words $X_{ij\pm}$ transform homogeneously under inversions and are also annihilated by the generators of supersymmetry. In performing various manipulations it is useful to note that

$$X_{+}X_{-} = (x^{2} + \frac{1}{16}(\theta^{a}\theta^{a})^{2})\mathbb{1} \tag{3.15}$$

$$X_{ij+}X_{ij-} = (\tilde{x}^{2}_{ij} + \frac{1}{16}(\theta^{a}_{ij}\theta^{a}_{ij})^{2})\mathbb{1} \tag{3.16}$$

so that

$$\begin{aligned}(X_{\pm})^{-1} &= \frac{X_{\mp}}{x^{2} + \frac{1}{16}(\theta^{a}\theta^{a})^{2}} \\ (X_{ij\pm})^{-1} &= \frac{X_{ij\mp}}{\tilde{x}^{2}_{ij} + \frac{1}{16}(\theta^{a}_{ij}\theta^{a}_{ij})^{2}}\end{aligned} \tag{3.17}$$

(note that the the R-symmetry index a is summed over but that, throughout, i, j $(= 1, 2, 3)$ label points in superspace and are not summed over).

There also exist fermionic covariant structures (which are identically zero in the non-supersymmetric case) which are constructed out of the superspace co-ordinates as follows [66, 67, 68, 69, 70]:

$$\Theta^{a}_{1\alpha} = \left((X^{-1}_{21+}\theta^{a}_{21})_{\alpha} - (X^{-1}_{31+}\theta^{a}_{31})_{\alpha}\right) \tag{3.18}$$

Θ_2, Θ_3 are defined similarly. Its transformation properties under superinversion are

$$\Theta^{a}_{i\alpha} \to -(X_{i-})_{\alpha}^{\ \beta}\Theta^{b}_{i\beta}I_{b}^{\ a} \qquad \Theta^{\alpha\, a}_{i} \to I^{T a}_{\ \ b}\,\Theta^{\beta\, b}_{i}(X_{i+})_{\beta}^{\ \alpha} \tag{3.19}$$

The basic covariant structures $X_{ij\pm}$, $\Theta^{a}_{i\alpha}$ are annihilated by the generators of supersymmetry. For this reason they form the basic building blocks for the construction of superconformal invariants, as we will explain in a later section.

Polarisation spinors: Since we will be dealing extensively with higher spin operators and their correlators, it will be useful to adopt a formalism, developed in [22][1], in which the information about the tensor structure is encoded in *polarisation spinors*: λ_α. These auxiliary objects are book-keeping devices to keep track of the tensorial nature of correlators in an efficient manner. They are defined to be real, bosonic, two-component objects transforming

[1] See also [71] for a similar approach.

as spinors of the $3d$ Lorentz group (see [22]). Being spinors in 2+1 dimensions fixes their transformation law under superinversions:

$$\lambda_\alpha \to (X_+^{-1}\lambda)_\alpha \quad , \quad \lambda^\beta \to -(\lambda X_-^{-1})^\beta \tag{3.20}$$

(This is the same as the transformation law of the θ's).

A higher spin primary operator $J_{\mu_1\mu_2.....\mu_{s_i}}$ with spin s_i can be represented in spinor components by $J_{\alpha_1\alpha_2.....\alpha_{2s_i}} \equiv (\sigma^{\mu_1})_{\alpha_1\alpha_2}(\sigma^{\mu_2})_{\alpha_3\alpha_4}....(\sigma^{\mu_s})_{\alpha_{2s-1}\alpha_{2s}}J_{\mu_1\mu_2.....\mu_{s_i}}$. We note that this represents an operator supermultiplet in contradistinction to [22] where the non-supersymmetric conformal case was considered (also, J need not necessarily be a conserved current). We then define $J_{s_i} \equiv \lambda^{\alpha_1}\lambda^{\alpha_2}...\lambda^{\alpha_{2s_i}}J_{\alpha_1\alpha_2.....\alpha_{2s_i}}$.

The three point function $\langle J_{s_1}(x_1,\theta_1,\lambda_1)J_{s_2}(x_2,\theta_2,\lambda_2)J_{s_3}(x_3,\theta_3,\lambda_3)\rangle$ is then a superconformal invariant constructed out of three points in (augmented) superspace with co-ordinates labeled by (x_i,θ_i,λ_i). The tensor structure of the correlator, instead of being represented by indices, is encoded by the polynomial in λ's (the three point function being a multinomial with degree $\lambda_1^{2s_1}\lambda_2^{2s_2}\lambda_3^{2s_3}$ for each term).

3.3 Free superconformal theories in superspace[2]

In this section we study free superconformal theories, with $\mathcal{N} = 1, 2, 3, 4$ and 6 supersymmetry in superspace[3].

3.3.1 $\mathcal{N} = 1$

$\mathcal{N} = 1$ superspace consists of points $z^A = (x^\mu, \theta_\alpha)$, where θ_α is a Majorana spinor. There are two real supercharges Q_α; these operators are implemented in superspace by the expressions (3.1) with $\mathcal{N} = 1$.

The 'minimal' free $\mathcal{N} = 1$ theory consists of a single complex scalar field together with a single complex fermion. These fields are packaged together into a single complex $\mathcal{N} = 1$ superfield Φ subject to the supersymmetric equation of motion

$$D^\alpha D_\alpha \Phi = 0 \tag{3.21}$$

Note that Φ, like any scalar $\mathcal{N} = 1$ superfield, may be expanded in components as

$$\begin{aligned}\Phi &= \phi + \theta\psi + \frac{\theta^2}{2}F \\ \bar{\Phi} &= \bar{\phi} - \theta\psi^* - \frac{\theta^2}{2}\bar{F}.\end{aligned} \tag{3.22}$$

[2]In this chapter we deal exclusively with on-shell superspace. For off-shell $3d$ superspace and multiplets in theories with and without gravity, see [72, 73].

[3]Sections 3.3, 3.4 and 3.5 have been worked out in collaboration with Shiraz Minwalla.

By expanding (3.21) in components it is not difficult to verify that (3.21) implies that

$$F = 0, \quad \partial^2\phi = 0, \quad p_\mu\gamma^\mu\psi = 0.$$

It follows that the superfield Φ subject to the equation of motion (3.21) actually describes a free massless scalar and fermion.

In the case of $\mathcal{N} = 1$ supersymmetry it is, of course, not difficult to find a manifestly supersymmetric off-shell description of the theory. The equation of motion (3.21) follows by extremizing the action

$$S = \int d^2\theta d^3x D_\alpha\bar{\Phi}D^\alpha\Phi \tag{3.23}$$

w.r.t. Φ. One way of adding interactions to the system (3.23) is to add a 'superpotential' term ($\int d^2\theta W(\Phi)$) to the action; however we will not investigate off-shell superspace in this chapter.

3.3.2 $\mathcal{N} = 2$

In this case the fermionic coordinates of superspace consist of two copies of the minimal $\mathcal{N} = 1$ Majorana spinor which can be labeled as θ^i_α ($i = 1, 2$). It is sometimes useful to group these coordinates into the complex pairs

$$\theta_\alpha = \frac{1}{\sqrt{2}}(\theta^1_\alpha + i\theta^2_\alpha), \quad \bar{\theta}_\alpha = \frac{1}{\sqrt{2}}(\theta^1_\alpha - i\theta^2_\alpha).$$

In a similar manner there are two natural choices for a basis in the space of supersymmetries. One natural choice is to work with the supersymmetry operators defined in (3.1). The commutation relations of the supersymmetries (and associated supersymmetric derivatives) is given by

$$\begin{aligned} \{Q^i_\alpha, Q^j_\beta\} &= P_{\alpha\beta}\delta^{ij} \\ \{D^i_\alpha, D^j_\beta\} &= -P_{\alpha\beta}\delta^{ij} \end{aligned} \tag{3.24}$$

Another choice is to work with complex supersymmetries; if we define

$$Q_\alpha = \frac{1}{\sqrt{2}}(Q^1_\alpha - iQ^2_\alpha), \quad D_\alpha = \frac{1}{\sqrt{2}}(D^1_\alpha - iD^2_\alpha)$$

we have

$$\begin{aligned} \{Q_\alpha, \bar{Q}_\beta\} &= P_{\alpha\beta} \\ \{D_\alpha, \bar{D}_\beta\} &= -P_{\alpha\beta} \end{aligned} \tag{3.25}$$

(also $\{Q_\alpha, Q_\beta\} = \{D_\alpha, D_\beta\} = 0$). In this basis the supersymmetry operators and super-covariant derivatives are most naturally written in terms of the complex variables θ_α; in particular for supercovariant derivatives we have

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} + \frac{i}{2}\bar{\theta}^\beta\partial_{\beta\alpha} \\ \bar{D}_\alpha &= \frac{\partial}{\partial\bar{\theta}^\alpha} + \frac{i}{2}\theta^\beta\partial_{\beta\alpha} \end{aligned} \tag{3.26}$$

It is sometimes useful to utilize 'chiral' and anti chiral coordinates (y_R, θ_α), $(y_L, \bar{\theta}_\alpha)$ where

$$y_R^\mu = x^\mu - \frac{i}{2}\theta\gamma^\mu\bar{\theta}, \quad y_L^\mu = x^\mu + \frac{i}{2}\theta\gamma^\mu\bar{\theta}$$

These coordinates are useful because

$$\bar{D}_\alpha y_R = 0, \quad D_\alpha y_L = 0$$

It is easily verified that

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} + i\bar{\theta}^\beta\partial^{y_R}_{\beta\alpha} \\ \bar{D}_\alpha &= \frac{\partial}{\partial\bar{\theta}^\alpha} \end{aligned} \tag{3.27}$$

Analogous expressions may also be obtained if we choose $y_L, \theta, \bar{\theta}$ as our coordinates.

$\mathcal{N} = 2$ theories posses a $U(1)$ R-symmetry under which we can assign charges to operators. We normalize this symmetry by assigning the charges 1 and -1 to θ and $\bar{\theta}$ respectively. It follows that the operators D_α and $\bar{D}_\alpha$ respectively have charges -1 and $+1$ under R-symmetry. Below we will sometimes use the notation $D \leftrightarrow D^{--}$ and $\bar{D} \leftrightarrow D^{++}$, notation that emphasizes these charge assignments.

The minimal free $\mathcal{N} = 2$ theory has the same field content as the minimal $\mathcal{N} = 1$ theory, i.e., the propagating degrees of freedom are a single complex scalar and complex fermion. The manifestly supersymmetric form of these equations of motion is given as follows. The basic dynamical superfield Φ is required to be chiral

$$\bar{D}_\alpha\Phi = D_\alpha\bar{\Phi} = 0 \tag{3.28}$$

In addition it is required to obey the equations of motion (of a free theory):

$$D^\alpha D_\alpha\Phi = \bar{D}^\alpha\bar{D}_\alpha\bar{\Phi} = 0 \tag{3.29}$$

These equations are solved by

$$\Phi = \phi(y_R) + \theta\psi(y_R) = \phi + \theta\psi - \frac{i}{2}\theta\gamma^\mu\bar{\theta}\partial_\mu\phi \tag{3.30}$$

and its complex conjugate (an anti-chiral field) is

$$\bar{\Phi} = \bar{\phi}(y_L) - \theta\psi^*(y_L) = \bar{\phi} - \bar{\theta}\psi^* + \frac{i}{2}\theta\gamma^\mu\bar{\theta}\partial_\mu\bar{\phi} \tag{3.31}$$

where ϕ and ψ obey the free Klein-Gordon and Dirac equations respectively (here $\theta\gamma^\mu\bar{\theta} = \theta^\alpha(\gamma^\mu)_\alpha{}^\beta\bar{\theta}_\beta$).

As the field component of the minimal $\mathcal{N}=2$ theory is the same as that of the $\mathcal{N}=1$ theory, it is possible to write the $\mathcal{N}=2$ superfield in terms of the $\mathcal{N}=1$ superfield; explicitly

$$\begin{aligned}\Phi_{\mathcal{N}=2} &= \Phi_{\mathcal{N}=1} + i\theta^{(2)}D^{(1)}\Phi_{\mathcal{N}=1}\\ \bar{\Phi}_{\mathcal{N}=2} &= \bar{\Phi}_{\mathcal{N}=1} - i\theta^{(2)}D^{(1)}\bar{\Phi}_{\mathcal{N}=1}\end{aligned} \tag{3.32}$$

(here $\theta^{(2)}$ is the second Majorana Grassmann co-ordinate - the coordinate that belongs to $\mathcal{N}=2$ but not to $\mathcal{N}=1$ superspace - and the $\Phi_{\mathcal{N}=1}$ field has the usual expansion in the $\theta^{(1)}$ Grassmann co-ordinate.

3.3.3 $\mathcal{N}=3$

The fermionic coordinates of superspace consist of three Majorana spinors, θ^a_α in this case. The indices a transform in the vector representation of the $SO(3)$ R-symmetry. It is sometimes useful to regard vectors of the $SO(3)$ R-symmetry as bispinors, or 2×2 matrices. Vectors are easily converted to matrices by dotting their components with the Pauli matrices $(\sigma^a)_i{}^j$.

The field content of the minimal $\mathcal{N}=3$ free theory consists of two free complex scalars and two free complex fermions. These fields may be packaged together into a doublet of complex superfields that transform in the spin-$\frac{1}{2}$ of the R-symmetry group. The free theory is a trivial example of a superconformal field theory. Primary operators in any superconformal field theory are labeled by (Δ, j, h) where Δ is the scaling dimension, j is the spin and h is the 'R-symmetry spin' (i.e. the quantum number that describes the R-symmetry representation of the primary operator). In this notation the free superfield described above transforms in the representation $(\frac{1}{2}, 0, \frac{1}{2})$. The doublet of free superfields obey the 'equations of motion'

$$D_\alpha^{\{ij}\Phi^{k\}} = 0, \tag{3.33}$$

This equation of motion has a simple interpretation; it follows from the analysis of unitary representations of the superconformal algebra that a representation with quantum numbers $(\frac{1}{2}, 0, \frac{1}{2})$ has a null state with quantum numbers $(1, \frac{1}{2}, \frac{3}{2})$; the equation (3.33) is simply the assertion that this null state vanishes.

The equations of motion (3.33) may be analysed as follows. Let us denote the two components of the doublet superfield Φ by Φ^+ and (superscripts denote R-symmetry charge; a single + denotes charge $\frac{1}{2}$). The equations of motion assert that

$$\begin{aligned}2D_\alpha^{(3)}\Phi^+ - \sqrt{2}D_\alpha^{++}\Phi^- &= 0\\ 2D_\alpha^{(3)}\Phi^- + \sqrt{2}D_\alpha^{--}\Phi^+ &= 0\end{aligned} \tag{3.34}$$

It is possible to solve for Φ^+ and Φ^- in terms of a single $\mathcal{N}=2$ chiral superfield φ^+ and a single antichiral superfield φ^-; we find

$$\begin{aligned}\Phi^+ &= \varphi^+ + \frac{1}{\sqrt{2}}\theta^{(3)}D^{++}\varphi^- \\ \Phi^- &= \varphi^- - \frac{1}{\sqrt{2}}\theta^{(3)}D^{--}\varphi^+\end{aligned} \tag{3.35}$$

These $\mathcal{N}=2$ superfields in turn obey the free $\mathcal{N}=2$ equations of motion

$$D^\alpha D_\alpha \varphi^+ = \bar{D}^\alpha \bar{D}_\alpha \varphi^- = 0 \tag{3.36}$$

demonstrating that the propagating degrees of freedom are twice that of the $\mathcal{N}=2$ theory.

The final expression of the $\mathcal{N}=3$ superfield in terms of the component fields, after we have solved for the (3.36), is given by

$$\Phi^k = \phi^k - \frac{1}{\sqrt{2}}\theta^{kl\alpha}\psi_{l\alpha} - \frac{1}{4}\epsilon^{abc}\theta^{a\alpha}\theta^{b\beta}(\sigma^c)^{kl}\partial_{\alpha\beta}\phi_l + \frac{1}{12\sqrt{2}}\epsilon^{abc}\theta^{a\alpha}\theta^{b\beta}\theta^{c\gamma}\partial_{\alpha\beta}\psi^k_\gamma \tag{3.37}$$

In the last term the α, β, γ indices are completely symmetrised and $k = 1, 2$. Here a, b are vector $SO(3)$ indices and i, j, k are spinor indices. Note that (3.37) hold only when the component fields obey the free equations of motion.

3.3.4 $\mathcal{N}=4$

In this case we have four Majorana spinor coordinates θ^a_α lying in the 4 of the R-symmetry group $SO(4)$. The superfield Φ^i is a Weyl spinor of $SO(4)$[4]. The $\mathcal{N}=4$ chirality constraint is

$$D^{\tilde{i}\{j}_\alpha \Phi^{k\}} = D^{\tilde{i}j}_\alpha \Phi^k + D^{\tilde{i}k}_\alpha \Phi^j = 0. \tag{3.38}$$

To understand the field content of the minimal $\mathcal{N}=4$ theory, we split the $\mathcal{N}=4$ chirality constraint into a part that constrains the $\theta^{(4)}_\alpha$ dependence and a part that's purely $\mathcal{N}=3$. We begin by choosing an $\mathcal{N}=3$ subspace, which we take as the $1, 2, 3$ directions. The remaining 4 direction is the orthogonal direction. A chiral (top-half) part of a $SO(4)$ Weyl spinor is the Dirac spinor in three dimensions. The $SO(4)$ vector $D^{(a)}_\alpha$ decomposes into an $SO(3)$ vector $D^{(a)}$ for $a = 1, 2, 3$ and a scalar $D^{(4)}_\alpha$. This can be seen as the symmetric and antisymmetric part of the matrix $D^{i\tilde{i}}_\alpha$ respectively. The antisymmetric part contains only $D^{(4)}_\alpha$ and the symmetric part is the D^{ij}_α which is purely along the 1,2 and 3 directions. When the above chirality constraint is analysed, one finds

$$D^{(4)}_\alpha \Phi^k = -\frac{i}{3}D^{ki}_\alpha \Phi_i \tag{3.39}$$

[4]See appendix B.1.2 for $SO(4)$ conventions.

where on the L.H.S. we have the the supercovariant derivative along the 4 direction in the $SO(4)$ R-symmetry space; on the R.H.S. we have the symmetric part of the $D_\alpha^{i\tilde{i}}$ supercovariant derivative, which is purely along the (orthogonal) $SO(3)$ subspace. All the spinor indices are now thought of as $SO(3)$ (Dirac) $\mathcal{N}=3$ spinor indices. This equation is the analog of (3.34) in the present case.

Solving (3.39) shows that the chiral $\mathcal{N}=4$ superfield Φ^k is completely determined in terms of a single $\mathcal{N}=3$ chiral superfield φ_i as

$$\Phi^k = \varphi^k - \frac{i}{3}\theta^{(4)} D^{ki}\varphi_i. \tag{3.40}$$

Thus, we see that the minimal field content of the $\mathcal{N}=4$ theory is the same as that of $\mathcal{N}=3$. An explicit component field expression can now be obtained from (3.40) by using (3.37) for the φ^k.

3.3.5 $\mathcal{N}=6$

In this case we have six Majorana spinor coordinates θ_α^a lying in the vector representation of the R-symmetry group $SO(6)$ ($\equiv SU(4)$). The superfield Φ^I is a Weyl spinor of $SO(6)$[5], which is the 4 of $SU(4)$. The field Φ^I satisfies the 'chirality constraint' [6]

$$D_\alpha^{IJ}\Phi^K = D_\alpha^{JK}\Phi^I = D_\alpha^{KI}\Phi^J \tag{3.41}$$

To understand the field content of the minimal $\mathcal{N}=6$ theory, we proceed as above and split the $\mathcal{N}=6$ chirality constraint into an $\mathcal{N}=4$ part and another piece which describes the $\theta^{(5)}$ and $\theta^{(6)}$ dependences. We begin by choosing an $\mathcal{N}=4$ subspace, which we take as the $1,2,3$ and 4 directions. The remaining $5,6$ directions are the orthogonal directions. In the conventions we have chosen, it may be checked that a Weyl spinor Φ^I of $SO(6)$ decomposes as one chiral Φ^i spinor ($i=1,2$) and one anti-chiral (bottom half of the $SO(4)$ spinor) $\Phi^{\tilde{i}}$ ($\tilde{i}=3,4$) of the $SO(4)$ sub-group [7]. Similarly, the $SO(6)$ vector (the $(4\times4)_{\text{antisym}}$ of $SU(4)$) decomposes into two scalars and one $SO(4)$ vector. In matrix language, one can construct the antisymmetric matrix D_α^{IJ} and observe that the two scalars ($D_\alpha^{(5)}$ and $D_\alpha^{(6)}$) form the linear combinations $i\sqrt{2}D_\alpha = D_\alpha^{(5)} - iD_\alpha^{(6)}$ and $i\sqrt{2}\bar{D}_\alpha = D_\alpha^{(5)} + iD_\alpha^{(6)}$, when $I,J=\tilde{i},\tilde{j}$ and $I,J=i,j$ respectively. On the other hand, when $I,J=\tilde{i},j$ (or vice-verse) we get the (single) vector which involves only $D_\alpha^{(a)}$ where $a=1,\dots 4$. We can pick any two terms from the above equation (we choose the first and third) and analyze them as follows

$$\begin{array}{llll} D_\alpha^{ij}\Phi^k & = D_\alpha^{ki}\Phi^j & \qquad D_\alpha^{\tilde{i}\tilde{j}}\Phi^{\tilde{k}} & = D_\alpha^{\tilde{k}\tilde{i}}\Phi^{\tilde{j}} \\ D_\alpha^{\tilde{i}j}\Phi^k & = D_\alpha^{k\tilde{i}}\Phi^j & \qquad D_\alpha^{i\tilde{j}}\Phi^{\tilde{k}} & = D_\alpha^{\tilde{k}i}\Phi^{\tilde{j}} \\ D_\alpha^{ij}\Phi^{\tilde{k}} & = D_\alpha^{\tilde{k}i}\Phi^j & \qquad D_\alpha^{\tilde{i}\tilde{j}}\Phi^k & = D_\alpha^{k\tilde{i}}\Phi^{\tilde{j}} \end{array} \tag{3.42}$$

[5] See appendix B.1.2 for $SO(6)$ conventions.

[6] We briefly use upper case I,J which take values $1,\dots 4$ for the $SU(4)$ indices in (3.41) to avoid confusion with the lower case i,j which appear in the $\mathcal{N}=4$ equations.

[7] We adopt the convention wherein the un-tilded indices i take values 1,2 and the tilded indices $\tilde{i}$ take values 3,4.

The second equation in each of the two sets above is just the $\mathcal{N} = 4$ chirality condition (3.38) for each of the fields Φ^i and $\Phi^{\tilde{i}}$. It remains to analyze the first and third equations from each of the two sets. The first set reads

$$\bar{D}_\alpha \Phi^k = 0 \qquad D_\alpha \Phi^{\tilde{k}} = 0 \tag{3.43}$$

where $\bar{D}_\alpha = \frac{1}{\sqrt{2}}(D_\alpha^{(5)} + iD_\alpha^{(6)})$ and $D_\alpha = \frac{1}{\sqrt{2}}(D_\alpha^{(5)} - iD_\alpha^{(6)})$. Thus, Φ^k and $\Phi^{\tilde{k}}$ can be thought of as two independent 'chiral' and 'anti-chiral' superfields and we can accordingly expand them in the $\theta_\alpha = \frac{1}{\sqrt{2}}(\theta_\alpha^{(5)} + i\theta_\alpha^{(6)})$ and $\bar{\theta}_\alpha = \frac{1}{\sqrt{2}}(\theta_\alpha^{(5)} - i\theta_\alpha^{(6)})$ coordinates. Let's now analyze the third equation from the above set. They are

$$\bar{D}_\alpha \Phi^{\tilde{k}} = \frac{i}{2\sqrt{2}} D_\alpha^{\tilde{k}i} \Phi_i \qquad D_\alpha \Phi^k = \frac{i}{2\sqrt{2}} D_\alpha^{k\tilde{i}} \Phi_{\tilde{i}} \tag{3.44}$$

Solving the above equations leads us to the following result for Φ^k and $\Phi^{\tilde{k}}$

$$\begin{aligned} \Phi^k &= \varphi^k + \frac{i}{2\sqrt{2}} \theta D^{k\tilde{i}} \varphi_{\tilde{i}} - \frac{i}{2} \theta \gamma^\mu \bar{\theta} \partial_\mu \varphi^k \\ \Phi^{\tilde{k}} &= \varphi^{\tilde{k}} + \frac{i}{2\sqrt{2}} \bar{\theta} D^{\tilde{k}i} \varphi_i + \frac{i}{2} \theta \sigma^\mu \bar{\theta} \partial_\mu \varphi^{\tilde{k}} \end{aligned} \tag{3.45}$$

Where $k = 1, 2$ and $\tilde{k} = 3, 4$ make up the full $\mathcal{N} = 6$ multiplet, and $\theta_\alpha = \frac{1}{\sqrt{2}}(\theta_\alpha^{(5)} + i\theta_\alpha^{(6)})$ and $\bar{\theta}_\alpha = \frac{1}{\sqrt{2}}(\theta_\alpha^{(5)} - i\theta_\alpha^{(6)})$. Thus, we see that the field content of the minimal $\mathcal{N} = 6$ theory consists of two independent $\mathcal{N} = 4$ fields, φ^k and $\varphi^{\tilde{k}}$. (3.37) and (3.36) can now be used in (3.45) to obtain explicit component field expression for Φ^K.

3.4 Currents

In this section we describe the construction of conserved currents in the theories discussed above. These currents constitute the full local gauge invariant operator spectrum of the theories considered. In the non-supersymmetric case the bosonic conserved currents and the violation, due to interactions, of their conservation by $\frac{1}{N}$ effects play a central role in the solution of three point functions in these theories [5, 6]. The currents we consider in this section are the supersymmetric extension of the bosonic currents considered in [22, 5, 6]. We construct the supercurrents, using the on-shell superspace described in sections 3.2 and 3.3, in terms of on-shell superfields and supercovariant derivatives.

3.4.1 General structure of the current superfield

Let us start by first describing the structure of the $\mathcal{N} = 1$ supercurrents. A general spin s supercurrent multiplet can be written as a superfield carrying $2s$ spacetime spinor indices and can be expanded in components as follows

$$\Phi^{\alpha_1\alpha_2\ldots\alpha_{2s}} = \phi^{\alpha_1\alpha_2\ldots\alpha_{2s}} + \theta_\alpha \psi^{\alpha\alpha_1\alpha_2\ldots\alpha_{2s}} + \theta^{\{\alpha_1}\chi^{\alpha_2\ldots\alpha_{2s}\}} + \theta^\alpha\theta_\alpha B^{\alpha_1\alpha_2\ldots\alpha_{2s}} \tag{3.46}$$

where all the indices α_1,α_2, ... α_{2s} are symmetrised. The conservation (shortening) condition for the supercurrent is

$$D_{\alpha_1}\Phi^{\alpha_1\alpha_2\ldots\alpha_{2s}} = 0 \tag{3.47}$$

where D_α is the supercovariant derivative given by

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{i}{2}\theta^\beta\partial_{\beta\alpha} \tag{3.48}$$

Using eqs.(3.48) and (3.46) we obtain

$$\begin{aligned}\delta_{\alpha_1}^{\{\alpha_1}\chi^{\alpha_2\ldots\alpha_{2s}\}} &+ \theta_{\alpha_1}(2B^{\alpha_1\alpha_2\ldots\alpha_{2s}} - \frac{i}{2}\partial_\beta{}^{\alpha_1}\phi^{\beta\alpha_2\ldots\alpha_{2s}}) \\ &- \frac{i}{2}\theta^2\partial_{\alpha\alpha_1}\psi^{\ \alpha\alpha_1\alpha_2\ldots\alpha_{2s}} + \frac{i}{2}\theta^\beta\partial_{\beta\alpha_1}\theta^{\{\alpha_1}\chi^{\alpha_2\ldots\alpha_{2s}\}} = 0\end{aligned} \tag{3.49}$$

This implies

$$\chi^{\alpha_2\ldots\alpha_{2s}} = 0 \tag{3.50}$$

while the symmetric part of the θ component gives

$$B^{\alpha_1\alpha_2\ldots\alpha_{2s}} = \frac{i}{4}\partial_\beta{}^{\{\alpha_1}\phi^{|\beta|\alpha_2\ldots\alpha_{2s}\}} \tag{3.51}$$

whereas the antisymmetric part gives

$$\epsilon_{\alpha_1\alpha_2}\partial^{\alpha_1}_\beta\phi^{\beta\alpha_2\ldots\alpha_{2s}} = 0 \Rightarrow \partial_{\alpha_1\alpha_2}\phi^{\alpha_1\alpha_2\ldots\alpha_{2s}} = 0 \tag{3.52}$$

which is the current conservation equation for the current ϕ. Since $\chi = 0$, the $\theta\theta$ component gives the current conservation equation for ψ

$$\partial_{\alpha\alpha_1}\psi^{\alpha\alpha_1\ldots\alpha_{2s}} = 0 \tag{3.53}$$

Thus the form of the supercurrent multiplet for a spin s conserved current is

$$\Phi^{\alpha_1\alpha_2\ldots\alpha_{2s}} = \phi^{\alpha_1\alpha_2\ldots\alpha_{2s}} + \theta_\alpha\psi^{\alpha\alpha_1\alpha_2\ldots\alpha_{2s}} + \frac{i}{4}\theta^\alpha\theta_\alpha\partial^{\{\alpha_1}_\beta\phi^{|\beta|\alpha_2\ldots\alpha_{2s}\}} \tag{3.54}$$

The general structure of the current superfield described above goes through for higher supersymmetries as well. For higher supersymmetries the conservation equation reads

$$D^a_{\ \alpha_1}\Phi^{\alpha_1\alpha_2\ldots\alpha_{2s}} = 0 \tag{3.55}$$

where $a = 1, 2\ldots\mathcal{N}$ is the R-symmetry index[8]. In the case of an $\mathcal{N} = m$ spin-s current multiplet, the currents $\phi^{\alpha_1\alpha_2\ldots\alpha_{2s}}$ and $\psi^{\alpha\alpha_1\alpha_2\ldots\alpha_{2s}}$ are themselves $\mathcal{N} = m-1$ spin s and

[8]Note that for $\mathcal{N} > 1$, (3.55) is true only for R-symmetry singlet currents. For currents carrying non-trivial R-symmetry representation the shortening condition is different. In this chapter we will only need the shortening condition (3.55).

spin $s+\frac{1}{2}$ conserved current superfields (depending on the grassmann coordinates θ^a_α: $a = 1,\dots m-1$) while the θ_α in (3.54) is the left over grassmann coordinate θ^m_α. Thus we see the general structure of the supercurrent multiplets: An $\mathcal{N}=m$ spin s supercurrent multiplet breaks up into two $\mathcal{N}=m-1$ supercurrents with spins s and $s+\frac{1}{2}$ respectively.

This structure can be used to express higher supercurrents superfields in term of components. For instance, the $\mathcal{N}=2$ spin s currents superfield can be expanded in components as follows

$$\begin{aligned}\Phi^{\alpha_1\alpha_2\dots\alpha_{2s}} &= \varphi^{\alpha_1\alpha_2\dots\alpha_{2s}} + \theta^a_\alpha(\psi^a)^{\alpha\alpha_1\alpha_2\dots\alpha_{2s}} + \frac{1}{2}\epsilon_{ab}\theta^a_\alpha\theta^b_\beta\mathcal{A}^{\alpha\beta\alpha_1\alpha_2\dots\alpha_{2s}}\\ &+ \text{term involving derivatives of } \varphi, \psi^a \text{ and } \mathcal{A}\end{aligned} \tag{3.56}$$

where a,b are R-symmetry indices and take values in $\{1,2\}$. The conformal state content so obtained, namely $(\varphi,\psi^1,\psi^2,\mathcal{A})$ above, match exactly with the decomposition of spin s supercurrent multiplet into conformal multiplets presented in appendix B.4.2.

3.4.2 Free field construction of currents

In this section we give explicit construction of the conserved supercurrent discussed in previous subsection in terms of free superfields.

$\mathcal{N}=1$

The spin s supercurrent here can be expressed in term of the $\mathcal{N}=1$ superfield Φ as follows

$$J^{(s)} = \sum_{r=0}^{2s}(-1)^{\frac{r(r+1)}{2}}\binom{2s}{r}D^r\bar{\Phi}D^{2s-r}\Phi \tag{3.57}$$

where $J^{(s)} = \lambda^{\alpha_1}\lambda^{\alpha_2}\cdots\lambda^{\alpha_{2s}}J_{\alpha_1\alpha_2\cdots\alpha_{2s}}$ and $D=\lambda^\alpha D_\alpha$, and λ_αs are polarisation spinors and $s=0,\frac{1}{2},1\dots$. The currents are of both integral and half-integral spins. It can be verified that the above is the unique expression for the conserved spin-s current in $\mathcal{N}=1$ free field theory. In equations, the following holds

$$\frac{\partial}{\partial\lambda^\alpha}D^\alpha J^{(s)} = 0. \tag{3.58}$$

We note here that the stress tensor lies in the spin $\frac{3}{2}$ current supermultiplet (which also contains the supersymmetry current), and thus is conserved exactly even in interacting theory.

$\mathcal{N}=2$

We give the expression of the conserved current in terms of the free $\mathcal{N}=2$ superfield Φ and its complex conjugate $\bar{\Phi}$.

$$J^{(s)} = \sum_{r=0}^{s} \left\{ (-1)^{r(2r+1)} \binom{2s}{2r} \partial^r \bar{\Phi} \partial^{s-r} \Phi + (-1)^{(r+1)(2r+1)} \binom{2s}{2r+1} \partial^r \bar{D} \bar{\Phi} \partial^{s-r-1} D\Phi \right\} \quad (3.59)$$

where $\partial = i\lambda^\alpha \gamma^\mu_{\alpha\beta} \lambda^\beta \partial_\mu$, $D = \lambda^\alpha D_\alpha$ and $s = 0, 1, 2 \ldots$. The spin 1 supercurrent multiplet contains the stress tensor, supersymmetry current and R-current, and its conservation holds even in the interacting superconformal theory.

As described above in subsection 3.4.1, these $\mathcal{N}=2$ currents can be decomposed into $\mathcal{N}=1$ currents. It is straightforward to check that the currents 3.59 when expanded in θ^2_α as in (3.54) correctly reproduce the $\mathcal{N}=1$ currents (3.57). This give a consistency check of these $\mathcal{N}=2$ currents.

$\mathcal{N}=3$

The $\mathcal{N}=3$ chirality constraint on the matter superfield Φ^k is

$$\begin{aligned} D^{\{ij}\Phi^{k\}} &= D^{ij}\Phi^k + D^{ik}\Phi^j + D^{jk}\Phi^i = 0 \\ \text{or equivalently} \qquad D^{ij}\Phi^k &= -\frac{1}{3}\left(D^{il}\Phi_l \epsilon^{jk} + D^{jl}\Phi_l \epsilon^{ik}\right) \end{aligned} \quad (3.60)$$

where $D^{ij}_\alpha = (\sigma^a)^{ij} D^a_\alpha$.

From this chirality constraint the following identities, which would be useful in proving current conservation, can be derived[9]

$$D^{ij}_\alpha D^{mn}_\beta \Phi^k = \frac{1}{2}\left(i\partial_{\alpha\beta}\Phi^i \epsilon^{jm}\epsilon^{nk} + i\partial_{\alpha\beta}\Phi^i \epsilon^{jn}\epsilon^{mk} + i\partial_{\alpha\beta}\Phi^j \epsilon^{im}\epsilon^{nk} + i\partial_{\alpha\beta}\Phi^j \epsilon^{in}\epsilon^{mk}\right) \quad (3.61)$$

Contracting various indices, the following relations can be obtained from (3.61) as corollaries

$$\begin{aligned} D^{\alpha ij} D^{mn}_\alpha \Phi^k &= 0 \\ D^{ij}_\alpha D^{mk}_\beta \Phi_k &= -\frac{3}{2}\left(i\partial_{\alpha\beta}\Phi^i \epsilon^{jm} + i\partial_{\alpha\beta}\Phi^j \epsilon^{im}\right) \\ D^{ij}_\alpha D_{ij\beta}\Phi^k &= -3i\partial_{\alpha\beta}\Phi^k = \frac{2}{3} D^k_j D^{ji}\Phi_i \end{aligned} \quad (3.62)$$

We give here the expression for the conserved currents in terms of the $\mathcal{N}=3$ superfield Φ^i.

$$\begin{aligned} J^{(s)} &= \sum_{r=0}^{s} (-1)^r \binom{2s}{2r} \partial^r \bar{\Phi}^i \partial^{s-r} \Phi_i + \frac{2}{9} \sum_{r=0}^{s-1} (-1)^{r+1} \binom{2s}{2r+1} \partial^r D_i{}^j \bar{\Phi}^i \partial^{s-r-1} D_j{}^k \Phi_k \\ J^{(s+\frac{1}{2})} &= \sum_{r=0}^{s} \left\{ (-1)^r \binom{2s+1}{2r} \partial^r \bar{\Phi}^i \partial^{s-r} D_i{}^j \Phi_j + (-1)^{r+1} \binom{2s+1}{2r+1} \partial^r D_i{}^j \bar{\Phi}^i \partial^{s-r} \Phi_j \right\} \end{aligned} \quad (3.63)$$

[9]See appendix B.1.2 for $SO(3)$ conventions.

where $\partial = i\lambda^\alpha\gamma^\mu_{\alpha\beta}\lambda^\beta\partial_\mu$, $D = \lambda^\alpha D_\alpha$ and $s = 0, 1, 2\ldots$. The stress energy tensor in this case lies the spin $\frac{1}{2}$ supercurrent multiplet along with the R-current and supersymmetry currents. The conservation of this supercurrent holds exactly even in the interacting superconformal theory.

$\mathcal{N} = 4$

The R-symmetry in this case is $SO(4)$ (equivalently $SU(2)_l \times SU(2)_r$)[10]. The supercharges $Q^{\tilde{i}i}_\alpha$ transform in the 4 of $SO(4)$(equivalently $(2,2)$ of $SU(2)_l \times SU(2)_r$). The two matter superfields transform in the $(2,0)$ representation which implies that the scalar transforms in the $(2,0)$ while the fermions transform in $(0,2)$. The matter multiplet again satisfies a 'chirality' constraint

$$\begin{aligned} D^{\tilde{i}\{i}\Phi^{j\}} &= D^{\tilde{i}i}\Phi^j + D^{\tilde{i}j}\Phi^i = 0, \\ \text{or equivalently} \qquad D^{\tilde{i}i}\Phi^j &= -\frac{1}{2}\epsilon^{ij}D^{\tilde{i}k}_\alpha\Phi_k. \end{aligned} \tag{3.64}$$

where $D^{\tilde{i}j}_\alpha = (\bar{\sigma}^a)^{\tilde{i}j}D^a_\alpha$.

From this chirality constraint the following identities, useful in proving current conservation, can be derived[1]

$$D^{\tilde{i}i}_\alpha D^{\tilde{j}j}_\beta\Phi^k = 2i\partial_{\alpha\beta}\Phi^i\epsilon^{\tilde{i}\tilde{j}}\epsilon^{jk} \tag{3.65}$$

Contracting various indices, the following equations can be obtained from (3.65) as corollaries

$$\begin{aligned} D^{\alpha\tilde{i}i}D^{\tilde{j}j}_\alpha\Phi^k &= 0 \\ D^{\tilde{i}i}_\alpha D^{\tilde{j}j}_\beta\Phi_j &= -4i\partial_{\alpha\beta}\Phi^i\epsilon^{\tilde{i}\tilde{j}} \\ D^{\tilde{i}j}_\alpha D_{\beta\tilde{i}k}\Phi^k &= 2D^{\tilde{i}i}_\alpha D_{\beta\tilde{i}i}\Phi^j = 8i\partial_{\alpha\beta}\Phi^j. \end{aligned} \tag{3.66}$$

Using these equations it is straightforward to show that the following currents are conserved.

$$J^{(s)} = \sum_{r=0}^{s}(-1)^r\binom{2s}{2r}\,\partial^r\bar{\Phi}^i\,\partial^{s-r}\Phi_i + \frac{1}{8}\sum_{r=0}^{s-1}(-1)^r\binom{2s}{2r+1}\,\partial^r D^{\tilde{i}i}\bar{\Phi}_i\,\partial^{s-r-1}D_{\tilde{i}j}\Phi^j. \tag{3.67}$$

where $\partial = i\lambda^\alpha\gamma^\mu_{\alpha\beta}\lambda^\beta\partial_\mu$, $D = \lambda^\alpha D_\alpha$ and $s = 0, 1, 2\ldots$. In this theory the stress energy tensor lies in the R-symmetry singlet spin zero supercurrent multiplet $(1, 0, \{0, 0\})$.

$\mathcal{N} = 6$

The field content of this theory is double of the field content of the $\mathcal{N} = 4$ theory. In $\mathcal{N} = 2$ language the field content is 2 chiral and 2 antichiral multiplets in fundamental

[10]The indices $a, b..$ take values $1, 2, 3, 4$ and represent the vector indices of $SO(4)$ while the fundamental indices of the $SU(2)_l$ and $SU(2)_r$ are denoted by $i, j...$ and $\tilde{i}, \tilde{j}....$

of the gauge group. The R-symmetry in this theory is $SO(6)$ ($\equiv SU(4)$) under which the supercharges transform in vector representation (6 of $SO(6)$) while the 2+2 chiral and antichiral multiplets transform in chiral spinor representation (4 of $SU(4)$).

The $\mathcal{N}=6$ shortening (chirality) condition on the matter multiplet is[11]

$$\begin{aligned} D^{ij}_\alpha \Phi^k &= D^{jk}_\alpha \Phi^i = D^{ki}_\alpha \Phi^j \\ \text{or equivalently} \quad D^a_\alpha \Phi^k &= -\frac{1}{10} D^b_\alpha \Phi^l (\bar\gamma^{ab})_l{}^k \end{aligned} \tag{3.68}$$

From this chirality constraint the following identities, which are useful in proving current conservation, can be derived[2]

$$\begin{aligned} D^a_\alpha D^b_\beta \Phi^k &= \frac{i}{2}\partial_{\alpha\beta}\Phi^k \delta^{ab} + \frac{i}{4}\partial_{\alpha\beta}\Phi^l (\bar\gamma^{ab})_l{}^k, \\ \text{or equivalently} \quad D^{ij}_\alpha D^{mn}_\beta \Phi^k &= -i\partial_{\alpha\beta}\left(\epsilon^{ijmn}\Phi^k + \epsilon^{kjmn}\Phi^i + \epsilon^{ikmn}\Phi^j - \epsilon^{ijkn}\Phi^m - \epsilon^{ijmk}\Phi^n\right) \end{aligned} \tag{3.69}$$

Taking the complex conjugate of equations (3.68) and (3.69), and using the property that γ^{ab} and $\bar\gamma^{ab}$ are antihermitian, we get

$$\begin{aligned} D^{ij}_\alpha \bar\Phi_k &= \frac{1}{3}\left(D^{il}_\alpha \bar\Phi_l \delta^j_k - D^{jl}_\alpha \bar\Phi_l \delta^i_k\right) \\ \text{or equivalently} \quad D^a_\alpha \bar\Phi_k &= \frac{1}{10} D^b_\alpha (\bar\gamma^{ab})_k{}^l \bar\Phi_l \end{aligned} \tag{3.70}$$

and

$$\begin{aligned} D^a_\alpha D^b_\beta \bar\Phi_k &= \frac{i}{2}\partial_{\alpha\beta}\bar\Phi_k \delta^{ab} - \frac{i}{4}\partial_{\alpha\beta}(\bar\gamma^{ab})_k{}^l \bar\Phi_l, \\ \text{or equivalently} \quad D^{ij}_\alpha D^{mn}_\beta \bar\Phi_k &= -i\partial_{\alpha\beta}\left(\epsilon^{ijmn}\Phi^k - \epsilon^{ljmn}\bar\Phi_l\delta^i_k - \epsilon^{ilmn}\bar\Phi_l\delta^j_k + \epsilon^{ijln}\bar\Phi_l\delta^m_k + \epsilon^{ijml}\bar\Phi_l\delta^n_k\right) \end{aligned} \tag{3.71}$$

Using the above relation a straightforward computation shows that the following R-symmetry singlet integer spin currents are conserved

$$J^{(s)} = \sum_{r=0}^{s}(-1)^r \binom{2s}{2r}\, \partial^r \bar\Phi_p\, \partial^{s-r}\Phi^p - \frac{1}{24}\sum_{r=0}^{s-1}(-1)^{r+1}\binom{2s}{2r+1}\, \epsilon_{ijkl}\, \partial^r D^{ij}\bar\Phi_p\, \partial^{s-r-1} D^{kl}\Phi^p. \tag{3.72}$$

where $\partial = i\lambda^\alpha \gamma^\mu_{\alpha\beta}\lambda^\beta \partial_\mu$, $D = \lambda^\alpha D_\alpha$ and $s = 0, 1, 2 \ldots$. The stress-energy tensor of this theory lies, as in the $\mathcal{N}=4$ theory, in the R-symmetry singlet spin zero multiplet $(1, 0, \{0, 0, 0\})$.

[11] Here we revert back to lower case letters for the $SU(4)$ indices i, j (taking values $1, \ldots 4$) as there is no confusion with other R indices.

3.5 Weakly broken conservation

The free superconformal theories discussed above have an exact higher spin symmetry algebra generated by the charges corresponding to the infinite number of conserved currents that these theories possess. These free theories can be deformed into interacting theories by turning on $U(N)(SU(N))$ Chern-Simons(CS) gauge interactions, in a supersymmetric fashion and preserving the conformal invariance of free CFTs, under which the matter fields transform in fundamental representations. The CS gauge interactions do not introduce any new local degrees of freedom so the spectrum of local operators in the theory remains unchanged. Turning on the interactions breaks the higher spin symmetry of the free theory but in a controlled way which we discuss below. These interacting CS vector models are interesting in there own right as non trivial interacting quantum field theories. Exploring the phase structure of these theories at finite temperature and chemical potential, provides a platform for studying a lot of interesting physics, at least in the large-N limit, using the techniques developed in [23].

From a more string theoretic point of view, a very interesting example of this class of theories is the $U(N) \times U(M)$ ABJ theory in the vector model limit $\frac{M}{N} \to 0$. ABJ theory in this vector model limit has recently been argued to be holographically dual a non-abelian supersymmetric generalisation of the non-minimal Vasiliev theory in AdS_4 [14]. The ABJ theory thus connects, as its holographic duals, Vasiliev theory at one end to a string theory at another end. Increasing $\frac{M}{N}$ from 0 corresponds to increasing the coupling of $U(M)$ gauge interactions in the bulk Vasiliev theory. Thus, understanding the ABJ theory away from the vector model limit in an expansions in $\frac{M}{N}$ would be a first step towards understanding of how string theory emerges from 'quantum' Vasiliev theory.[12]

In [5, 6] theories with exact conformal symmetry but weakly broken higher spin symmetry were studied. It was first observed in [23], and later used with great efficiency in [6], that the anomalous "conservation" equations are of the schematic form

$$\partial \cdot J_{(s)} = \frac{a}{N} J_{(s_1)} J_{(s_2)} + \frac{b}{N^2} J_{(s'_1)} J_{(s'_2)} J_{(s'_3)} \tag{3.73}$$

plus derivatives sprinkled appropriately. The structure of this equation is constrained on symmetry grounds - the twist $(\Delta_i - s_i)$ of the L.H.S. is 3. If each J_s has conformal dimension $\Delta = s+1+O(1/N)$, and thus twist $\tau = 1+O(1/N)$, the two terms on the R.H.S. are the only ones possible by twist matching. Thus we can have only double or triple trace deformations in the case of weakly broken conservation and terms with four or higher number of currents are not possible.

In the superconformal case that we are dealing with, since D has dimension 1/2 , $D \cdot J_{(s)}$ is a twist 2 operator. Thus in this case the triple trace deformation is forbidden and the only possible structure is more constrained:

$$D \cdot J_{(s)} = \frac{a}{N} J_{(s_1)} J_{(s_2)} \tag{3.74}$$

[12] See [74] for a very recent attempt in this direction.

In view of this, it is feasible that in large-N supersymmetric Chern-Simons theories the structure of correlation functions is much more constrained (compared to the non-supersymmetric case).

3.6 Two point functions

The two point function of two spin-s operators in a $3d$ SCFT has a form completely determined (up to overall multiplicative constants) by superconformal invariance. Since, as we saw in section 2, $X_{12\pm}$ is the only superconformally covariant structure built out of two points in superspace, the only possible expression for the two point function which also has the right dimension and homogeneity in λ is:

$$\langle J_s(1)J_s(2)\rangle \propto \frac{P_3^{2s}}{\tilde{X}_{12}^2} \tag{3.75}$$

where P_3 is the superconformal invariant defined on two points, given in Table 3.1. The overall constant can be determined in free field theory, see below.

As an illustrative example, we consider the two point function of two spin half supercurrents. On the basis of symmetry and dimension matching we can have the following possible structure for the two point function:

$$\langle J_{1/2}(x_1,\theta_1,\lambda_1)J_{1/2}(x_2,\theta_2,\lambda_2)\rangle = b\frac{\lambda_1\lambda_2}{\tilde{X}_{12}^{\Delta_1+\Delta_2}}\frac{\theta_{12}^2}{\tilde{X}_{12}} + \frac{\lambda_1\tilde{X}_{12}\lambda_2}{\tilde{X}_{12}^{\Delta_1+\Delta_2+1}}(c+d\frac{\theta_{12}^2}{\tilde{X}_{12}}) \tag{3.76}$$

where $\tilde{X}_{12} \equiv \sqrt{(\tilde{X}_{12})_\alpha^{\ \beta}(\tilde{X}_{12})_\beta^{\ \alpha}}$ [13]. The shortening condition on the above two point function gives

$$d = 0 \qquad b = \frac{ic}{4}(\Delta_1+\Delta_2-2) \tag{3.77}$$

For $J_{1/2}$ a superconformal primary $\Delta_1 = \Delta_2 = 3/2$ so $b = ic/4$ and the two point function (up to some undetermined overall normalisation) is given by

$$\langle J_{1/2}(x_1,\theta_1,\lambda_1)J_{1/2}(x_2,\theta_2,\lambda_2)\rangle \propto \frac{\lambda_1\tilde{X}_{12}\lambda_2}{\tilde{X}_{12}^4} + \frac{i}{4}\frac{\lambda_1\lambda_2\theta_{12}^2}{\tilde{X}_{12}^4} \tag{3.78}$$

A natural generalisation, that reduces correctly to the above equation for $s = 1/2$, is

$$\langle J_s(1)J_s(2)\rangle \propto \frac{(\lambda_1\tilde{X}_{12}\lambda_2)^{2s-1}}{\tilde{X}_{12}^{4s+2}}(\lambda_1\tilde{X}_{12}\lambda_2 + \frac{is}{2}\lambda_1\lambda_2\theta_{12}^2) \tag{3.79}$$

[13] Note that throughout $\tilde{X}_{12}$ denotes this scalar object. The matrix will always be denoted with the indices: $(\tilde{X}_{12})_\alpha^{\ \beta}$.

with $\langle J_0 J_0\rangle = 1/\tilde{X}_{12}^2$ (since the superconformal shortening condition is different for spin zero). Note that the above can be written as

$$\langle J_s(1)J_s(2)\rangle \propto \frac{(\lambda_1\tilde{X}_{12}\lambda_2 + \frac{i}{4}\lambda_1\lambda_2\theta_{12}^2)^{2s}}{\tilde{X}_{12}^{4s+2}} \tag{3.80}$$

which is the same as (3.75). The shortening condition on this is satisfied, as may be explicitly checked.

As a check, we also work out, by elementary field theory methods, the two point function of the spin $\frac{1}{2}$ current constructed out of the free $\mathcal{N}=1$ superfield which is defined as[14]

$$\begin{aligned}\Phi &= \phi + i\theta\psi \\ \bar{\Phi} &= \bar{\phi} + i\theta\psi^*\end{aligned} \tag{3.81}$$

We find that the twp point function computed explicitly in the free theory is in agreement with our result (3.75) above. The spin half supercurrent is

$$J_\alpha = \bar{\Phi}D_\alpha\Phi - (D_\alpha\bar{\Phi})\Phi \tag{3.82}$$

Using the equation of motion for Φ this obeys the shortening condition $D^\alpha J_\alpha = 0$. The two point function of two such currents can be obtained after doing Wick contractions to write 4-point functions in terms of two point functions. We use the free field propagator $\langle\bar{\Phi}\Phi\rangle = \frac{1}{\tilde{X}_{12}}$, and also that,

$$D_{1\alpha}D_{2\beta}\frac{1}{\tilde{X}_{12}} = \frac{-i(\tilde{X}_{12})_{\alpha\beta}}{(\tilde{X}_{12})^3} \quad , \quad D_{1\alpha}\frac{1}{\tilde{X}_{12}}D_{2\beta}\frac{1}{\tilde{X}_{12}} = \frac{\epsilon_{\alpha\beta}\theta_{12}^2}{4(\tilde{X}_{12})^4} \tag{3.83}$$

This gives (up to multiplicative factors which we neglect)

$$\langle J_\alpha(1)J_\beta(2)\rangle \propto \frac{((-\tilde{X}_{12})_{\alpha\beta} + \frac{i}{4}\theta_{12}^2\epsilon_{\alpha\beta})}{\tilde{X}_{12}^4}. \tag{3.84}$$

Contracting with λ_1^α and λ_2^β we find, in free field theory,

$$\langle J_{\frac{1}{2}}(1)J_{\frac{1}{2}}(2)\rangle = -i\frac{P_3}{\tilde{X}_{12}^2} \tag{3.85}$$

which, indeed, is what was expected. One can determine the constants appearing in front of the two point function in free field theory and we divide by it (so that the final result is normalised to one) which gives the following result for general spin s

$$\langle J_s(1)J_s(2)\rangle = c(s)\frac{P_3^{2s}}{\tilde{X}_{12}^2} \tag{3.86}$$

where $c(s) = \left(\frac{i}{2}\right)^{2s}\frac{\sqrt{\pi}}{s!\Gamma(s+\frac{1}{2})}$ for all $s \geq 0$.

[14] We insert a factor of i in this definition for convenience, which differs from the definition given in (3.22).

3.7 Three point functions

In this section we undertake the task of determining all the possible structures that can occur in the three point functions of higher spin operators $\langle J_{s_1} J_{s_2} J_{s_3} \rangle$. For the nonsupersymmetric case this was done in [22]. We will use superconformal invariance to ascertain what structures can occur in three point functions.

The structure of correlation functions in SCFTs has been earlier studied by J-H Park [69, 66, 68, 70] and H. Osborn [67].[15]. The structure of covariant objects which are used as building blocks for the construction of invariants in the present work was entirely laid out in the above references. However, our goal in the present work is to make use of these structures to study theories which have conserved currents of higher spin. For this purpose, it is convenient to adopt the polarisation spinor formalism of [22]. After writing down the structures that can appear for a given three point function, we use on-shell conservation laws of the currents to constrain the coefficients appearing in front of the structures.

We find that there exist new structures for both the parity even and odd part of $\langle J_{s_1} J_{s_2} J_{s_3} \rangle$ which were not present in the nonsupersymmetric case. The parity-odd superconformal invariants are of special interest as they arise in interacting $3d$ SCFTs. We will here restrict ourselves to the case of $\mathcal{N}=1$ SCFTs (no R-symmetry). The results are summarised in the table given below:

	Parity even	Parity odd
Bosonic	$P_1 = \lambda_2 X_{23-}^{-1} \lambda_3$ $Q_1 = \lambda_1 X_{12-}^{-1} X_{23+} X_{31-}^{-1} \lambda_1$ and cyclic	$S_1 = \frac{\lambda_3 X_{31+} X_{12+} \lambda_2}{\tilde{X}_{12} \tilde{X}_{23} \tilde{X}_{31}}$ and cyclic
Fermionic	$R_1 = \lambda_1 \Theta_1$ and cyclic	$T = \tilde{X}_{31} \frac{\Theta_1 X_{12+} X_{23+} \Theta_3}{\tilde{X}_{12} \tilde{X}_{23}}$

Table 3.1: Invariant structures in $\mathcal{N}=1$ superspace.

3.7.1 Superconformal invariants for three point functions of $\mathcal{N}=1$ higher spin operators

We need to determine all the superconformal invariants that can be constructed out of the co-ordinates of (augmented) superspace : x_i, θ_i and the (bosonic) polarisation spinors λ_i $(i=1,2,3)$. Using the covariant objects of section 2, which transformed homogeneously under superinversions, we can begin to write down the superconformal invariants constructed out of $(x_i, \theta_i, \lambda_i)$.

[15]Kuzenko [75] has also studied three point functions of the supercurrent and flavour currents of $\mathcal{N}=2$ 4d SCFTs.

We have

$$\lambda_i X_{ij-}^{-1} \lambda_j \to -(\lambda_i X_{i-}^{-1})(-X_{i-} X_{ij-}^{-1} X_{j-})(X_{j+}^{-1} \lambda_j) = \lambda_i X_{ij-}^{-1} \lambda_j \tag{3.87}$$

Thus we have the three superconformal invariants

$$P_1 = \lambda_2 X_{23-}^{-1} \lambda_3 \,, \quad P_2 = \lambda_3 X_{31-}^{-1} \lambda_1 \,, \quad P_3 = \lambda_1 X_{12-}^{-1} \lambda_2 \tag{3.88}$$

Also, under superinversion,

$$\mathfrak{X}_{1+} = X_{12-}^{-1} X_{23+} X_{31-}^{-1} \to -X_{1-} \mathfrak{X}_{1+} X_{1+} \tag{3.89}$$

and similarly for $\mathfrak{X}_{2+}, \mathfrak{X}_{3+}$, so we also have the following as superconformal invariants:

$$Q_1 = \lambda_1 \mathfrak{X}_{1+} \lambda_1 \,, \quad Q_2 = \lambda_2 \mathfrak{X}_{2+} \lambda_2 \,, \quad Q_3 = \lambda_3 \mathfrak{X}_{3+} \lambda_3 \tag{3.90}$$

Furthermore,

$$\lambda_3 X_{31+} X_{12+} \lambda_2 \to -\frac{1}{x_1^2 x_2^2 x_3^2} \lambda_3 X_{31+} X_{12+} \lambda_2 \,, \quad \tilde{X}_{ij}^2 \to \frac{\tilde{X}_{ij}^2}{x_i^2 x_j^2} \tag{3.91}$$

so there are the additional (parity odd) superconformal invariants

$$S_1 = \frac{\lambda_3 X_{31+} X_{12+} \lambda_2}{\tilde{X}_{12} \tilde{X}_{23} \tilde{X}_{31}} \,, \quad S_2 = \frac{\lambda_1 X_{12+} X_{23+} \lambda_3}{\tilde{X}_{12} \tilde{X}_{23} \tilde{X}_{31}} \,, \quad S_3 = \frac{\lambda_2 X_{23+} X_{31+} \lambda_1}{\tilde{X}_{12} \tilde{X}_{23} \tilde{X}_{31}} \tag{3.92}$$

which transform to minus themselves under inversion. Together these constitute the supersymmetric generalisations of the conformally invariant P, Q, S structures discussed in [22] [16]

Using the covariant Θ structures of section 2 it follows that we have the additional (parity even) fermionic invariants

$$R_1 = \lambda_1 \Theta_1 \,, \quad R_2 = \lambda_2 \Theta_2 \,, \quad R_3 = \lambda_3 \Theta_3 \tag{3.93}$$

It may be checked that

$$R_1^2 = R_2^2 = R_3^2 = R_1 R_2 R_3 = 0 \tag{3.94}$$

Construction of the parity odd fermionic invariant T

We can construct more superconformally covariant structures from the building blocks (X_{jk+}, $\mathfrak{X}_{i+}$, Θ_i, λ_i) - these are the fermionic analogues of P, S, Q. We define them below and also give there transformation under superinversion.

a) Fermionic analogues of P_i: Define

$$\pi_{ij} = \lambda_i X_{ij+} \Theta_j \tag{3.95}$$

[16] Note that the S_k in [22] has an extra factor of iP_k compared to ours.

Then under superinversion

$$\pi_{ij} \to -\lambda_i X_{i-}^{-1} X_{i+}^{-1} X_{ij+} X_{j-}^{-1} X_{j-} \Theta_j = -\frac{1}{x_i^2}\pi_{ij} \tag{3.96}$$

Similarly,

$$\Pi_{ij} = \Theta_i X_{ij+} \Theta_j \,, \quad \Pi_{ij} \to \Pi_{ij} \tag{3.97}$$

It turns out, however, that

$$\Pi_{ij} = 0 \tag{3.98}$$

b) Fermionic analogues of S_i:

$$\sigma_{13} = \frac{\lambda_1 X_{12+} X_{23+} \Theta_3}{\tilde{X}_{12} \tilde{X}_{23} \tilde{X}_{31}}, \quad \sigma_{13} \to x_3^2 \sigma_{13} \tag{3.99}$$

$$\Sigma_{13} = \frac{\Theta_1 X_{12+} X_{23+} \Theta_3}{\tilde{X}_{12} \tilde{X}_{23} \tilde{X}_{31}}\,, \quad \Sigma_{13} \to -x_1^2 x_3^2 \Sigma_{13} \tag{3.100}$$

$\sigma_{32}, \sigma_{21}, \Sigma_{32}, \Sigma_{21}$ are similarly defined through cyclic permutation of the indices. It follows that

$$\tilde{X}_{ij}^2 \Sigma_{ij} \to -\tilde{X}_{ij}^2 \Sigma_{ij} \tag{3.101}$$

c) Fermionic analogues of Q_i:

$$\omega_i = \lambda_i \mathfrak{X}_{i+} \Theta_i \,, \quad \omega_i \to -x_i^2 \omega_i \tag{3.102}$$

$$\Omega_i = \Theta_i \mathfrak{X}_{i+} \Theta_i \,, \quad \Omega_i \to x_i^4 \Omega_i \tag{3.103}$$

However, Ω_i is identically zero

$$\Omega_i = 0 \tag{3.104}$$

The invariants constructed out of the product of two parity odd (or two parity even) covariant structures would be parity even, and since we have already listed all the parity even invariants, would be expressible in terms of P_i, Q_i, R_i. Thus, we find the following relations for the above covariant structures

$$\pi_{ij}^2 = \sigma_{ij}^2 = \omega_i^2 = 0 \tag{3.105}$$

$$\pi_{ij}\omega_i = 0 \tag{3.106}$$

$$\frac{1}{\tilde{X}_{12}^2}\pi_{12}\pi_{23} = -R_1 R_2 \,, \quad \frac{1}{\tilde{X}_{23}^2}\pi_{23}\pi_{31} = -R_2 R_3 \,, \quad \frac{1}{\tilde{X}_{31}^2}\pi_{31}\pi_{12} = -R_3 R_1 \tag{3.107}$$

$$\frac{1}{\tilde{X}_{ij}^2}\pi_{ij}\pi_{ji} = R_i R_j = \tilde{X}_{ij}^2 \sigma_{ij}\sigma_{ji} \tag{3.108}$$

$$\tilde{X}_{12}^2 \sigma_{21}\sigma_{32} = R_2 R_3 \,, \quad \tilde{X}_{23}^2 \sigma_{32}\sigma_{13} = R_3 R_1 \,, \quad \tilde{X}_{31}^2 \sigma_{13}\sigma_{21} = R_1 R_2 \tag{3.109}$$

$$\tilde{X}_{ij}^2\,\omega_i\omega_j = -R_iR_j \tag{3.110}$$

From the above covariant structures it is possible to build additional parity odd fermionic invariants by taking products of a parity even and a parity odd covariant structure.[17]Thus, we have

$$T_{ij} = \pi_{ij}\sigma_{ji} \tag{3.111}$$

and under superinversion

$$T_{ij} \to -T_{ij} \tag{3.112}$$

Note that $\pi_{ij} \neq \pi_{ji}$ so $\{\pi_{12},\, \pi_{23},\, \pi_{31}\}$ is a different set of parity odd covariant structures than $\{\pi_{21},\, \pi_{32},\, \pi_{13}\}$ (the same is true for the even structures σ_{ij}). However, because the following relation is true

$$T_{ij} = -T_{ji} \tag{3.113}$$

it follows that we have only three odd invariant structures:

$$T_1 \equiv T_{23} = \pi_{23}\sigma_{32}\,, \quad T_2 \equiv T_{31} = \pi_{31}\sigma_{13}\,, \quad T_3 \equiv T_{12} = \pi_{12}\sigma_{21} \tag{3.114}$$

We may also define

$$T'_{23} = \pi_{12}\sigma_{31}\,, \quad T'_{31} = \pi_{23}\sigma_{12}\,, \quad T'_{12} = \pi_{31}\sigma_{23} \quad T'_{ij} \to -T'_{ij} \tag{3.115}$$

with $T'_{32} = \pi_{13}\sigma_{21}\,, \quad T'_{13} = \pi_{21}\sigma_{32}\,, \quad T'_{21} = \pi_{32}\sigma_{13}$ again being related to the above by

$$P_3T'_{21} = -P_2T'_{31}\,, \quad P_1T'_{32} = -P_3T'_{12}\,, \quad P_2T'_{13} = -P_1T'_{23} \tag{3.116}$$

Also

$$\bar{T}_{ij} = \tilde{X}_{ij}^2\sigma_{ji}\omega_j\,, \quad \bar{T}_{ij} \to -\bar{T}_{ij} \tag{3.117}$$

Again, we have the relation

$$\bar{T}_{ij}Q_i = \bar{T}_{ji}Q_j \tag{3.118}$$

thus we have only three $\bar{T}_{ij}$'s.

Likewise, we have

$$\hat{T}_{12} = \tilde{X}_{12}^2\sigma_{31}\omega_2\,, \hat{T}_{23} = \tilde{X}_{23}^2\sigma_{12}\omega_3\,\hat{T}_{31} = \tilde{X}_{31}^2\sigma_{23}\omega_1 \quad \hat{T}_{ij} \to -\hat{T}_{ij} \tag{3.119}$$

with $\hat{T}_{21}\,, \hat{T}_{32}\,, \hat{T}_{13}$ being related to the above by

$$P_j\hat{T}_{ij} = P_i\hat{T}_{ji} \tag{3.120}$$

We also have the following relations involving Σ_{ij}

$$\Sigma_{ij} = \Sigma_{ji}\,, \quad \tilde{X}_{12}^2\Sigma_{12} = \tilde{X}_{23}^2\Sigma_{32} = \tilde{X}_{31}^2\Sigma_{31} \tag{3.121}$$

[17]Note that structures like $x_i\omega_i$, π_{ij}/x_i would be parity odd invariants under inversion. However, these are not Poincare invariant (since correlation functions should depend only on differences (x_{ij}) of the coordinates). We could also construct structures like $U = \tilde{X}_{12}\tilde{X}_{23}\tilde{X}_{31}\omega_1\omega_2\omega_3$ which would be an odd invariant ($U \to -U$). However, it is identically zero because the product of three different Θ's vanishes.

Therefore, here we get just one parity odd invariant

$$T \equiv \tilde{X}_{ij}^2 \Sigma_{ij} \tag{3.122}$$

It turns out that T'_{ij}, $\bar{T}_{ij}$, $\hat{T}_{ij}$, $\tilde{X}_{ij}^2\Sigma_{ij}$ can be expressed in terms of T_i by means of the following relations

$$\begin{aligned}
&P_1 T'_{31} = P_3 T_1\,, \quad P_2 T'_{12} = P_1 T_2\,, \quad P_3 T'_{23} = P_2 T_3 \\
&P_3 \bar{T}_{12} = -Q_2 T_3\,, \quad P_1 \bar{T}_{23} = -Q_3 T_1\,, \quad P_2 \bar{T}_{31} = -Q_1 T_2 \\
&\frac{1}{2} P_2 \tilde{X}_{13}^2 \Sigma_{13} = T_2\,, \quad \frac{1}{2} P_3 \tilde{X}_{21}^2 \Sigma_{21} = T_3\,, \quad \frac{1}{2} P_1 \tilde{X}_{32}^2 \Sigma_{32} = T_1 \\
&P_1 \hat{T}_{23} = -P_2 T_1\,, \quad P_2 \hat{T}_{31} = -P_3 T_2\,, \quad P_3 \hat{T}_{12} = -P_1 T_3
\end{aligned} \tag{3.123}$$

Making use of the above equation and eq.(3.122) we can express all parity odd fermionic structures in terms of T

$$T_2 = \frac{1}{2} P_2 T\,, \quad T_3 = \frac{1}{2} P_3 T\,, \quad T_1 = \frac{1}{2} P_1 T \tag{3.124}$$

$$\bar{T}_{12} = -\frac{1}{2} Q_2 T\,, \quad \bar{T}_{23} = -\frac{1}{2} Q_3 T\,, \quad \bar{T}_{31} = -\frac{1}{2} Q_1 T \tag{3.125}$$

$$T'_{31} = \frac{1}{2} P_3 T\,, \quad T'_{12} = \frac{1}{2} P_1 T\,, \quad T'_{23} = \frac{1}{2} P_2 T \tag{3.126}$$

$$\hat{T}_{12} = -\frac{1}{2} P_1 T\,, \quad \hat{T}_{23} = -\frac{1}{2} P_2 T\,, \quad \hat{T}_{31} = -\frac{1}{2} P_3 T \tag{3.127}$$

To summarize, from our fermionic covariant structures we could construct five parity odd invariants T_i, T'_{ij}, $\bar{T}_{ij}$, $\hat{T}_{ij}$, T. However, only T suffices as the other four are related to it through the above simple relations [18].

Summary of this section: We have thus obtained the superconformal invariants P_i, Q_i, R_i, S_i, T (listed in tabular form at the beginning of this section) out of which the invariant structures for particular three point functions can be constructed as monomials in these variables. Before we do this, however, we need to determine all the relations between these variables using which we can get a linearly independent basis of monomial structures for three point functions.

[18] The structure $\frac{\tilde{X}_{31}\tilde{X}_{12}}{\tilde{X}_{23}}(\Theta_1)^2$ (along with its two cyclic permutations) is also a parity odd superconformal invariant (We thank Denis Bashkirov and Ran Yacoby for pointing it out) but it happens to be identical to T.

3.7.2 Relations between the invariant structures

Since the $\mathcal{N}=1$ superconformal group in 3 dimensions has 14 generators (10 bosonic, 4 fermionic), out of $(x_i,\, \theta_i,\, \lambda_i)$ $(i=1,2,3)$ we can construct $7\times 3-14=7$ superconformal invariants. Thus among the nine parity even structures $(P_i\,, Q_i\,, R_i)$ we must have two relations. One of them is the supersymmetrised version of the non-linear relation (2.14) in [22]

$$P_1^2Q_1+P_2^2Q_2+P_3^2Q_3-2P_1P_2P_3-Q_1Q_2Q_3-\frac{i}{2}(R_1R_2P_3Q_3+R_2R_3P_1Q_1+R_3R_1P_2Q_2)=0 \tag{3.128}$$

This cuts down the number of independent invariants by one. We also have the following triplet of relations which vanishes identically when the Grassmann variables are set to zero (fermionic relations) and reduces the number of invariants to seven :

$$\begin{aligned}
&P_2R_1R_2+Q_1R_2R_3+P_3R_3R_1=0\\
&P_3R_2R_3+Q_2R_3R_1+P_1R_1R_2=0\\
&P_1R_3R_1+Q_3R_1R_2+P_2R_2R_3=0
\end{aligned} \tag{3.129}$$

There are further non-linear relations involving the S's. Since the squares or products of S's are parity even, we expect them to be determined in terms of the parity even structures. Indeed, we find

$$\begin{gathered}
S_1^2=P_1^2-Q_2Q_3-iP_1R_2R_3\,,\quad S_2^2=P_2^2-Q_3Q_1-iP_2R_3R_1\,,\quad S_3^2=P_3^2-Q_1Q_2-iP_3R_1R_2\\
S_1S_2=P_3Q_3-P_1P_2\,,\quad S_2S_3=P_1Q_1-P_2P_3\,,\quad S_3S_1=P_2Q_2-P_3P_1
\end{gathered} \tag{3.130}$$

They imply that the most general odd structures that can occur in any three point function are linear in S_i. It turns out there exist further *linear* relations between the parity odd structures. We find the following basic linear relationships between the various parity odd invariant structures:

At $O(\lambda_1\lambda_2\lambda_3)$:

$$R_1S_1+R_2S_2+R_3S_3=0 \tag{3.131}$$

At $O(\lambda_1^2\lambda_2\lambda_3,\, \lambda_1\lambda_2^2\lambda_3,\, \lambda_1\lambda_2\lambda_3^2)$:

$$\begin{aligned}
&Q_1S_1+P_2S_3+P_3S_2-\frac{i}{2}P_2P_3T=0\\
&Q_2S_2+P_3S_1+P_1S_3-\frac{i}{2}P_1P_3T=0\\
&Q_3S_3+P_1S_2+P_2S_1-\frac{i}{2}P_1P_2T=0
\end{aligned} \tag{3.132}$$

and

$$S_2R_1R_2+S_3R_3R_1+T(Q_1P_1-P_2P_3)=0$$

$$S_3R_2R_3 + S_1R_1R_2 + T(Q_2P_2 - P_3P_1) = 0 \tag{3.133}$$
$$S_1R_3R_1 + S_2R_2R_3 + T(Q_3P_3 - P_1P_2) = 0$$

From eq. (3.131) follows:

$$\begin{aligned} S_2R_1R_2 - S_3R_3R_1 &= 0 \\ S_3R_2R_3 - S_1R_1R_2 &= 0 \\ S_1R_3R_1 - S_2R_2R_3 &= 0 \end{aligned} \tag{3.134}$$

From these follow other linear relations at higher orders in $\lambda_1\,,\lambda_2\,,\lambda_3$:

$$\begin{aligned} Q_1P_1S_1 + Q_2P_2S_2 - Q_3P_3S_3 + 2P_1P_2S_3 - \frac{i}{2}TP_1P_2P_3 &= 0 \\ Q_2P_2S_2 + Q_3P_3S_3 - Q_1P_1S_1 + 2P_2P_3S_1 - \frac{i}{2}TP_1P_2P_3 &= 0 \\ Q_3P_3S_3 + Q_1P_1S_1 - Q_2P_2S_2 + 2P_3P_1S_2 - \frac{i}{2}TP_1P_2P_3 &= 0 \end{aligned} \tag{3.135}$$

Adding the above equations gives

$$Q_1P_1S_1 + Q_2P_2S_2 + Q_3P_3S_3 - \frac{3i}{2}TP_1P_2P_3 + 2(P_1P_2S_3 + P_2P_3S_1 + P_3P_1S_2) = 0 \tag{3.136}$$

Also, we get

$$R_1R_2(S_1P_2 + \frac{1}{2}Q_3S_3) + R_2R_3(S_2P_3 + \frac{1}{2}Q_1S_1) + R_3R_1(S_3P_1 + \frac{1}{2}Q_2S_2) = 0 \tag{3.137}$$

$$\begin{aligned} (P_1^2Q_1 - P_2^2Q_2)P_3S_3 + (P_3^2 - Q_1Q_2 - iP_3R_1R_2)(Q_1P_1S_1 - Q_2P_2S_2) &= 0 \\ (P_2^2Q_2 - P_3^2Q_3)P_1S_1 + (P_1^2 - Q_2Q_3 - iP_1R_2R_3)(Q_2P_2S_2 - Q_3P_3S_3) &= 0 \\ (P_3^2Q_3 - P_1^2Q_1)P_2S_2 + (P_2^2 - Q_3Q_1 - iP_2R_3R_1)(Q_3P_3S_3 - Q_1P_1S_1) &= 0 \end{aligned} \tag{3.138}$$

and so on. All these relations can be put to use in eliminating linearly dependent structures in three point functions. The above relations between the invariant structures extend the corresponding non-supersymmetric ones in [22].

We also have the following relations

$$T^2 = 0\,, \quad FT = 0\,, \quad S_iT = -\epsilon_{ijk}R_jR_k \quad \textit{sum over } j,k \tag{3.139}$$

where F stands for any of the fermionic covariant/invariant structures. This implies that for any three point function it suffices to consider parity odd structures linear in T, S_i. Thus S_i, T comprise all the parity odd invariants we need in writing down possible odd structures in the three point functions of higher spin operators and we need only terms linear in these invariants.

3.7.3 Simple examples of three point functions

Independent invariant structures for three point functions

Below we write down the possible superconformal invariant structures that can occur in specific three point functions $\langle J_{s_1}(1)J_{s_2}(2)J_{s_3}(3)\rangle$. We consider the case of abelian currents so that, when some spins are equal, the correlator is (anti-) symmetric under pairwise exchanges of identical currents. We use only superconformal invariance to constrain the correlators, so the results of this section apply even if the higher spin symmetry is broken (that is, if J_s is not conserved for $s > 2$). All that is required is that J_s are higher spin operators transforming suitably under superconformal transformations[19].

Under the pairwise exchange $2 \leftrightarrow 3$ we have

$$A_1 \to -A_1\,, \quad A_2 \to -A_3\,, \quad A_3 \to -A_2\,, \quad T \to T \tag{3.140}$$

where A stands for any of P, Q, R, S.

$\langle J_{\frac{1}{2}} J_{\frac{1}{2}} J_0\rangle$: It is clear that any term that can occur is of order $\lambda_1\lambda_2$. Thus the possible structures that can occur in this correlator are:

$$P_3\,,\; R_1 R_2\,,\; S_3\,,\; P_3 T \tag{3.141}$$

We also computed this correlator explicitly in the free field theory (like the $\langle J_{\frac{1}{2}} J_{\frac{1}{2}}\rangle$ correlator in the previous section) and the result is (with $\Delta_1 = \Delta_2 = \frac{3}{2}\,,\; \Delta_3 = \frac{1}{2}$):

$$\frac{1}{\tilde{X}_{12}^{3/2}\tilde{X}_{23}^{1/2}\tilde{X}_{31}^{1/2}}(P_3 - \frac{i}{2}R_1R_2) \tag{3.142}$$

The odd piece can not occur in the free field case.

$\langle J_{\frac{1}{2}} J_{\frac{1}{2}} J_{\frac{1}{2}}\rangle$: Note that this has to be antisymmetric under exchange of any two currents. However the only two possible structures $\sum R_i P_i\,, \sum R_i S_i$ are symmetric under this exchange. Thus $\langle J_{\frac{1}{2}} J_{\frac{1}{2}} J_{\frac{1}{2}}\rangle$ vanishes.

$\langle J_s J_0 J_0\rangle$: For s an even integer, the possible structures are

$$Q_1^s\,,\; Q_1^s T \tag{3.143}$$

For s odd the correlator is zero. It is non-zero for half-integral s of the form: $s = n + \frac{1}{2}$ with n being an odd integer, the unique possible structure in this case being: $Q_1^n R_1$.

$\langle J_s J_{\frac{1}{2}} J_{\frac{1}{2}}\rangle$: For s an even integer, the possible structures are

[19] We take $J_{\alpha_1\alpha_2.....\alpha_{s_i}}$ to be a primary with arbitrary conformal dimension Δ_i so that $J_{s_i} \equiv \lambda^{\alpha_1}\lambda^{\alpha_2}...\lambda^{\alpha_{s_i}} J_{\alpha_1\alpha_2.....\alpha_{s_i}}$ has dimension $\Delta_i - s_i$. In general J_{s_i} need not be conserved. However, if the unitarity bound is attained - $\Delta_i = s_i + 1$ for $s_i \geq \frac{1}{2}$; $\Delta_i = \frac{1}{2}$ for $s_i = 0$ - then J_{s_i}, being a short primary, is necessarily conserved: $D_{(i)\alpha}\frac{\partial}{\partial\lambda_{(i)\alpha}} J_{s_i} = 0$.

$$Q_1^s P_1 \,,\; Q_1^{s-1} P_2 P_3 \,,\; R_2 R_3 Q_1^s \,,$$
$$Q_1^{s-1}(P_2 S_3 + P_3 S_2) \,,\; Q_1^s P_1 T \,,\; Q_1^{s-1} P_2 P_3 T$$

The structure $R_1 Q_1^{s-1}(R_2P_2 - R_3P_3)$ is also possible but using eq.(3.129) equals $-R_2R_3Q_1^s$ and hence can be eliminated while writing down independent superconformal invariant structures. Similarly, the structure $Q_1^s S_1$ can be written in terms of others listed above by using eq. (3.132) and $R_1Q_1^{s-1}(R_2S_2 - R_3S_3)$ in terms of the last two structures above by using eq. (3.133)

For s odd, antisymmetry under the exchange $2 \leftrightarrow 3$ allows only the following possible structures

$$R_1 Q_1^{s-1}(R_2P_2 + R_3P_3) \,,\; Q_1^{s-1}(P_2S_3 - P_3S_2)$$

The structure $R_1Q_1^{s-1}(R_2S_2 + R_3S_3)$ vanishes on using eq. (3.131).

$\langle J_1 J_1 J_0 \rangle$: The possible structures are

$$Q_1Q_2 \,,\; P_3^2 \,,\; R_1R_2P_3 \,,\; R_1R_2S_3 \,,\; P_3S_3 \,,\; Q_1Q_2T \,,\; P_3^2T$$

$\langle J_1 J_1 J_1 \rangle$: Note that all the parity even structures that can occur in $\langle J_1 J_1 J_1 \rangle$ are those that are present in the non-linear relation eq.(3.128) but all these structures are antisymmetric under the exchange of any two currents whereas this correlator is symmetric under the same exchange. Hence the parity even part of $\langle J_1 J_1 J_1 \rangle$ vanishes. For the same reason no possible parity odd structures can occur either. Thus $\langle J_1 J_1 J_1 \rangle$ vanishes in general.

$\langle J_{\frac{3}{2}} J_{\frac{1}{2}} J_0 \rangle$: Here the possible structures are

$$Q_1P_3 \,,\; R_1R_2Q_1 \,,\; Q_1S_3 \,,\; Q_1P_3T$$

$\langle J_{\frac{3}{2}} J_{\frac{1}{2}} J_{\frac{1}{2}} \rangle$: The linearly independent structures are

$$R_1Q_1P_1 \,,\; R_1P_2P_3 \,,\; Q_1(R_2P_2 + R_3P_3) \,,\; R_1Q_1S_1$$

Two other possible fermionic parity odd structures can be eliminated using eqs. (3.131,3.132)

$\langle J_{\frac{3}{2}} J_{\frac{1}{2}} J_1 \rangle$: After eliminating some structures using the relations in sec. (7.2) we get the following linearly independent structures:

$$Q_1Q_2P_2 \,,\; Q_1P_1P_3 \,,\; P_3^2P_2 \,,\; R_1R_2Q_1P_1 \,,\; R_1R_2P_2P_3 \,,\; R_3R_1Q_1Q_2,$$
$$Q_1P_1S_3 \,,\; Q_1P_3S_1 \,,\; P_2P_3S_3 \,,\; R_1R_2P_2S_3 \,,\; Q_1Q_2P_2T \,,\; Q_1P_1P_3T \,,\; P_3^2P_2T$$

$\langle J_{\frac{3}{2}} J_{\frac{3}{2}} J_{\frac{3}{2}} \rangle$:

$$Q_1Q_2Q_3\sum_i R_iP_i \,,\quad \sum_{cyclic} R_1Q_2Q_3P_2P_3 \,,\quad \sum_i R_iQ_iP_i^3 \,,\quad P_1P_2P_3\sum_i R_iP_i \,,$$
$$\sum_i R_iQ_iP_i^2S_i$$

The structure $\sum_{cyc} R_1P_1(P_2^2Q_2 + P_3^2Q_3)$ can, by using the non-linear identity eq.(3.128), be expressed in terms of the above structures and hence need not be included. The structure $\sum_{cyclic} R_1Q_2Q_3(P_2S_3 + P_3S_2)$ vanishes on using eqs. (3.132,3.131)

$\langle J_2J_1J_1\rangle$: The possible linearly independent structures are

$$\begin{aligned} Q_1^2Q_2Q_3\,,\ Q_1^2P_1^2\,,\ Q_1P_1P_2P_3\,,\ P_2^2P_3^2\,,\\ R_2R_3P_1Q_1^2\,,\ R_2R_3P_2P_3Q_1\,,\\ Q_1Q_2P_2S_2+Q_1Q_3P_3S_3\,,\ P_2^2P_3S_3+P_3^2P_2S_2\,,\\ R_1R_2P_2^2S_3+R_3R_1P_3^2S_2\,,\\ Q_1^2Q_2Q_3T\,,\ Q_1^2P_1^2T\,,\ Q_1P_1P_2P_3T\,,\ P_2^2P_3^2T \end{aligned}$$

Other structures are possible, but can be written in terms of the other structures listed above by using the relations in section 3.7.2.

$\langle J_3J_1J_1\rangle$: As before, after eliminating some structures which are antisymmetric under the exchange $2 \leftrightarrow 3$ we are left with the following linearly independent basis for $\langle J_3J_1J_1\rangle$:

$$\begin{aligned} Q_1^2(P_2^2Q_2-P_3^2Q_3)\,,\\ Q_1^2(R_1R_2P_1P_2-R_3R_1P_3P_1)\,,\ Q_1(R_1R_2P_2^2P_3-R_3R_1P_3^2P_2)\,,\\ Q_1^2(P_2Q_2S_2-P_3Q_3S_3)\,,\ Q_1(P_3^2P_2S_2-P_2^2P_3S_3)\,,\\ Q_1(R_1R_2P_2^2S_3-R_3R_1P_3^2S_2)\,,\ Q_1^2(P_2^2Q_2-P_3^2Q_3)T \end{aligned}$$

Again, linearly dependent structures have been eliminated using the relations of section 3.7.2.

$\langle J_4J_1J_1\rangle$: The structures that occur here are the same as Q_1^2 times the structures in $\langle J_2J_1J_1\rangle$.

$\langle J_sJ_1J_1\rangle$: For s even this again equals $Q_1^{s-2}\langle J_2J_1J_1\rangle$ (this was noted, for the non-supersymmetric case, in ref. [22]- it continues to hold in our case). For s odd and greater than three this correlator equals $Q_1^{s-2}\langle J_3J_1J_1\rangle$. Thus the number of possible tensor structures in $\langle J_sJ_1J_1\rangle$ does not increase with s.

$\langle J_2J_2J_2\rangle$: The following are the possible independent invariant structures

$$\begin{aligned} Q_1^2Q_2^2Q_3^2\,,\quad P_1^2P_2^2P_3^2\,,\quad Q_1Q_2Q_3P_1P_2P_3\,,\quad \sum_i Q_i^2P_i^4\,,\\ Q_1Q_2Q_3\sum_{cyclic}Q_3P_3R_1R_2\,,\ \ P_1P_2P_3\sum_{cyclic}Q_3P_3R_1R_2\,,\\ P_1P_2P_3\sum_{cyclic}P_1P_2S_3\,,\quad \sum_i Q_i^2P_i^3S_i\,,\\ Q_1Q_2Q_3\sum_{cyclic}Q_3S_3R_1R_2\,,\quad P_1P_2P_3\sum_{cyclic}Q_3S_3R_1R_2\,,\\ Q_1^2Q_2^2Q_3^2T\,,\quad P_1^2P_2^2P_3^2T\,,\quad Q_1Q_2Q_3P_1P_2P_3T\,,\quad \sum_i Q_i^2P_i^4T \end{aligned}$$

Many other linearly dependent structures have been eliminated using the relations in sec. (7.2).

As is evident, the number of invariant structures needed to construct the three point correlator increases rapidly as the spins of the operators increase and we will not consider more examples.

It is clear from the above examples that the general structure of the three point function is the following:

$$\langle J_{s_1} J_{s_2} J_{s_3} \rangle = \frac{1}{\tilde{X}_{12}^{m_{123}} \tilde{X}_{23}^{m_{231}} \tilde{X}_{31}^{m_{312}}} \sum_n \mathcal{F}_n(P_i,\, Q_i,\, R_i,\, S_i,\, T) \tag{3.144}$$

where $m_{ijk} \equiv (\Delta_i - s_i) + (\Delta_j - s_j) - (\Delta_k - s_k)$ and the sum is over all the independent invariant structures $\mathcal{F}_n$, each of homogeneity $\lambda_1^{2s_1}\lambda_2^{2s_2}\lambda_3^{2s_3}$. Since the three point function is linear in the parity odd invariants and linear or bilinear in the R's (either R_i or $R_j R_k$, $j \neq k$), we have the following structure for $\mathcal{F}_n$:

$$\begin{aligned}\mathcal{F}_n = F_n^{(1)}(P_i, Q_i) + a_n^{(1)} F_n^{(1)}(P_i, Q_i) T + a_n^{(2)} F_n^{(2)}(P_i, Q_i) S_i + a_n^{(3)} F_n^{(3)}(P_i, Q_i) R_i \\ + a_n^{(4)} F_n^{(4)}(P_i, Q_i) R_i S_j + a_n^{(5)} F_n^{(5)}(P_i, Q_i) R_j R_k + a_n^{(6)} F_n^{(6)}(P_i, Q_i) R_j R_k S_l\end{aligned}$$

Here each $F_n^{(a)}(P_i, Q_i)$ is a monomial in P's and Q's such that each term on the r.h.s above has homogeneity $\lambda_1^{2s_1}\lambda_2^{2s_2}\lambda_3^{2s_3}$.[20]

Three point functions of conserved currents

We have so far considered the constraints on the structure of the three point functions of higher spin operators arising due to superconformal invariance alone. We will now see how the structure is further constrained by current conservation, i.e, when the operators are actually conserved higher spin currents. In this section we present evidence for the claim that the three point function of the conserved higher spin currents in $\mathcal{N} = 1$ superconformal field theory consists of two linearly independent parts, i.e.,

$$\langle J_{s_1} J_{s_2} J_{s_3} \rangle = \frac{1}{\tilde{X}_{12}\tilde{X}_{23}\tilde{X}_{31}} \left(a \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\text{even}} + b \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\text{odd}} \right) \tag{3.145}$$

where a and b are independent constants, and the 'even' structure arises from free field theory.

The procedure, quite similar to that used by [22], is as follows. For any particular three point function we first consider the linearly independent basis of monomial structures (listed in section 3.7.3) and take an arbitrary linear combination of these structures.

$$\langle J_{s_1} J_{s_2} J_{s_3} \rangle = \frac{1}{\tilde{X}_{12}\tilde{X}_{23}\tilde{X}_{31}} \sum_n a_n \mathcal{F}_n \tag{3.146}$$

[20]The six $F_n^{(a)}(P_i, Q_i)$ are not independent functions. $F_n^{(2)}$, $F_n^{(4)}$, $F_n^{(6)}$ can be obtained from $F_n^{(1)}$, $F_n^{(3)}$, $F_n^{(5)}$, respectively, by replacing a P_i^p in the latter by $P_i^{p-1} S_i$ (suitably (anti-)symmetrised if some spins are equal in the three point function).

Current conservation $D_{\alpha_1} J^{\alpha_1\alpha_2.....\alpha_{2s}} = 0$ is tantamount to the following equation on the contracted current $J_s(x,\lambda)$:

$$D_\alpha \frac{\partial}{\partial \lambda_\alpha} J_s = 0 \tag{3.147}$$

Thus the equation

$$D_i \frac{\partial}{\partial \lambda_i} \langle J_{s_1} J_{s_2} J_{s_3} \rangle = 0 \tag{3.148}$$

for each $i = 1,2,3$ gives additional constraints in the form of linear equations in the a_n's- some of these constants can thus be determined. The algebraic manipulations get quite unwieldy- we used superconformal invariance to set some co-ordinates to particular values and took recourse to Mathematica. The results obtained are given below (the known $\tilde{X}_{ij}$ dependent factors in the denominator are not listed below):

Three-pt function	Even	Odd
$\langle J_{\frac{1}{2}} J_{\frac{1}{2}} J_0 \rangle$	$P_3 - \frac{i}{2} R_1 R_2$	$S_3 - \frac{i}{2} P_3 T$
$\langle J_1 J_{\frac{1}{2}} J_0 \rangle$	$P_3 R_1 + \frac{1}{2} Q_1 R_2$	0
$\langle J_1 J_1 J_0 \rangle$	$\frac{1}{2} Q_1 Q_2 + P_3^2 - i R_1 R_2 P_3$	$S_3 P_3 + \frac{i}{2}(S_3 R_1 R_2 - Q_1 Q_2 T)$
$\langle J_{\frac{3}{2}} J_{\frac{1}{2}} J_0 \rangle$	$P_3 Q_1 - \frac{i}{2} Q_1 R_1 R_2$	$Q_1 S_3 - i Q_1 P_3 T$
$\langle J_{\frac{3}{2}} J_{\frac{1}{2}} J_{\frac{1}{2}} \rangle$	$Q_1 R_1 P_1 + Q_1(R_2 P_2 + R_3 P_3) + 2 R_1 P_2 P_3$	0
$\langle J_2 J_{\frac{1}{2}} J_{\frac{1}{2}} \rangle$	$Q_1^2 P_1 - 4 Q_1 P_2 P_3 - \frac{5i}{2} R_2 R_3 Q_1^2$	$Q_1(P_2 S_3 + P_3 S_2)$ $+\frac{i}{2}(Q_1^2 P_1 - 3 Q_1 P_2 P_3) T$

Table 3.2: Explicit examples of conserved three point functions.

Using expression (3.57) for the currents in the $\mathcal{N} = 1$ free theory, some three point functions were explicitly evaluated (again using Mathematica, s the computations get quite cumbersome beyond a few lower spin examples). It must be emphasised that the (tabulated) even structures obtained above match with the expressions obtained from free field theory (up to overall constants). We thus have some evidence for the claim that the three point function of conserved currents has a parity even part (generated by a free field theory) and a parity odd piece.

3.8 Summary and outlook

In this chapter we have presented the study of superconformal Chern-Simons matter theories in an on-shell superspace formalism. To conclude we summarize the main results of this chapter below.

- An explicit construction of higher spin conserved supercurrents in terms of higher spin component currents in section 3.4.1.

- An explicit construction of higher spin conserved supermultiplets in terms of on shell elementary superfields in free superconformal field theories in section 3.4.2.
- A decomposition of the state content of single trace operators in large-N vector Chern-Simons superconformal theories into multiplets of the superconformal algebra in the theories with $\mathcal{N} = 1, 2, 3, 4, 6$ superconformal symmetry in appendix B.4.
- Determination of the form of two point functions of conserved higher spin supercurrents, and the explicit computation of these two point functions in free theories in section 3.6.
- Classification of superconformal invariants formed out of 3 polarisation spinors and 3 superspace insertion points (following [69]) and use thereof to constrain three point functions of higher spin operators in $3d$ superconformal field theories in section 3.7.1.
- A conjecture - and evidence - that there are exactly two structures allowed in the three point functions of the conserved higher spin currents for $\mathcal{N} = 1$ in section 3.7.3.
- The superspace structure of higher spin symmetry breaking on adding interactions to large-N gauge theories in section 3.5.

One of the main motivations for the study described in this chapter is to perform a Maldacena-Zhiboedov type study of superconformal Chern-Simons vector matter theories. As argued in section 3.5, the structure of terms violating higher spin current conservation is much more constrained in superconformal case as compared to the conformal case suggesting that higher spin correlators in superconformal case must be more severely constrained. For this purpose it will be useful to extend the analysis of three point functions presented here for $\mathcal{N} = 1$ case to extended supersymmetry. Besides describing a variety of renormalisation group fixed points in 3 dimensions, theories of this type are also expected to be holographic duals to supersymmetric higher spin Vasiliev theories in 4 dimensions. It may also be worth extending this formalism for 4 and higher point functions by using polarisation spinor techniques, perhaps together with the embedding formalism, in view of implementing the (super)conformal bootstrap for higher spin operators.

Chapter 4

Hairy black holes in global AdS_5

4.1 Introduction

This thesis has been mainly focused on studies of quantum field theories in three dimensions. We now turn to the study of a *classical* field theory in five dimensions - an Einstein-Maxwell system minimally coupled to a scalar field. Albeit far removed from the main thrust of this thesis it may seem at first glance, this is not really the case for reasons explained in the introduction. The AdS/CFT correspondence implies a deep connection between quantum field theories and classical gravity systems, and thus provides us with a vista from which we can study strong coupling regimes of non-gravitational QFTs.

In this chapter we investigate the physics of charged black brane solutions of the Lagrangian[1]

$$\begin{aligned} S &= \frac{1}{8\pi G_5}\int d^5x\sqrt{g}\left[\frac{1}{2}\left(\mathcal{R}[g]+12\right)-\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}-|D_\mu\phi|^2\right] \\ D_\mu\phi &= \nabla_\mu\phi - ieA_\mu\phi \end{aligned} \tag{4.1}$$

where G_5 is the Newton's constant and the radius of AdS_5 is set to unity[2]. This system (sometimes called the massless Abelian Higgs model) admits a well known set of charged black brane solutions which are asymptotically Poincaré AdS. Recent interest in this system is due to Gubser's observation [24] that, at large e and when they are near enough to extremality, these black branes are unstable. The end point of the tachyon condensation sparked by this instability is a so called hairy black brane - a solution with a planar horizon immersed in a charged scalar condensate. Black branes interacting with such matter condensates are novel and interesting, and have been studied intensively over the last few years

[1]We have chosen to work in 5 spacetime dimensions, and chosen the scalar field below to be massless merely for simplicity. The analysis of this chapter carries over, without qualitative modification, in arbitrary spacetime and for arbitrary scalar potential.

[2]See appendix C.8 for a summary of notation employed in this chapter.

(see [76, 77, 78, 79, 80] and references therein). Unfortunately almost all constructions of these solutions have been numerical[3].

Under the AdS/CFT correspondence, these planar AdS_5 solutions are dual to the states of a conformal field theory living on the flat spacetime $R^{3,1}$. Another natural arena to study $3+1$ dimensional conformal field theories is to work on $S^3 \times R_{\rm time}$ instead. States of such a boundary field theory living on S^3 are dual to gravitational solutions that asymptote to global AdS_5 instead of planar AdS_5. The corresponding charged black holes in global AdS_5 spacetime are characterised by their radius in units of the AdS_5 radius and their charge. At large horizon radius, these black holes are locally well approximated by black branes and we expect their physics to be qualitatively similar to the Poincaré AdS charged branes. It is natural to enquire about the opposite limit: do small hairy black holes exist, and what are their qualitative properties? In this chapter we answer this question by explicitly constructing a set of spherically symmetric hairy charged black holes whose radii are small compared to the AdS_5 radius[4]. Our construction is perturbative in the radii of our solutions, but is otherwise analytic. It permits an analytic construction of the microcanonical phase diagram of our system at small mass and charge. In the rest of this introduction, we will describe in detail our construction of small hairy black holes, their properties, and the phase diagram of our system.

To begin with, we start our discussion with a consideration that may, at first, seem unrelated to the study of AdS black holes. Consider a spherically symmetric shell of a scalar field of frequency ω incident on a charged black hole in flat spacetime. One might naively expect a part of this wave to be absorbed by the black hole while the rest is reflected back to infinity. It is, however, a well known fact that the reflection coefficient for this process actually exceeds unity when $\omega < e\mu$ (μ is the chemical potential of the black hole). Under these conditions more of the incident wave comes out than was sent in. This phenomenon, called superradiance [25], has immediate and well known implications for the stability of small RNAdS black holes, as we now explain.

Consider a superradiant wave incident on a small charged black hole sitting at the centre of global AdS spacetime. Such a wave reflects off the black hole, propagates out to large r, but unlike the flat spacetime case, bounces back from the boundary of AdS_5 and then finds itself re-incident on the black hole. This process continues indefinitely. As every reflection increases the amplitude of this wave by a fixed factor, this process constitutes an instability of the charged black hole. A closely related instability, the so called black hole bomb, was discussed (in the context of a flat spacetime black hole surrounded by mirrors) as early as the 1970s [85].

As the spectrum of frequencies of a minimally coupled charged scalar field (in a gauge where $A_t^{(r=\infty)} = 0$) in AdS_5 is bounded from below $\omega \geq \Delta_0 \equiv 4$, we expect small charged black holes in AdS_5 space to exhibit superradiant instabilities whenever the condition $e\mu \geq$

[3]See however [81, 82, 83] for analytical studies in a related context.

[4]See [84] for earlier work on scalar condensation in black hole backgrounds in global AdS_5.

$\omega \geq \Delta_0$ is satisfied[5]. Now the chemical potential μ of a small black hole is bounded from above by the chemical potential of the extremal black hole; i.e. $\mu^2 \leq \mu_c^2 = \frac{3}{2}$. It follows that small charged AdS black holes are always stable when $e^2 \leq \frac{\Delta_0^2}{\mu_c^2} \equiv e_c^2 = \frac{32}{3}$. When $e^2 \geq e_c^2$, however, small black holes that are near enough to extremality suffer from a superradiant instability.

The superradiant instability described above admits a very simple thermodynamical interpretation. Notice that the Boltzmann factor for a mode of energy Δ_0 and charge e is given by $e^{-T^{-1}(\Delta_0 - e\mu)}$ where T is the temperature of the black hole. Now this factor leads to an exponential enhancement (rather than the more usual suppression) whenever $\mu e \geq \Delta_0$. In other words, a small charged black hole with $\mu e \geq \Delta_0$ is unstable against bose condensation of the lightest scalar mode. Indeed the leading unstable mode of a small charged black hole with $\mu \geq \frac{\Delta_0}{e}$ is a small deformation of the lightest scalar mode in global AdS_5 space.

The considerations outlined above suggests that superradiant tachyon condensation proceeds in the following manner. The black hole emits into a scalar condensate, thereby losing mass and charge itself. As the charge to mass ratio of the condensate (i.e superradiant mode), $\frac{e}{\Delta_0}$, exceeds $\frac{1}{\mu}$, the chemical potential of the black hole also decreases as this emission proceeds. Now the decay rate of the black hole is proportional to $(\Delta_0 - \mu e)$ and so slows down as μ approaches $\frac{\Delta_0}{e}$. It seems intuitively plausible that the system asymptotes to a configuration consisting of a $\mu \approx \frac{\Delta_0}{e}$ stationary charged black hole core surrounded by a diffuse AdS scale charge condensate, i.e. a hairy black hole. We will provide substantial quantitative evidence for the correctness of this picture in this chapter.

In the discussion of the previous paragraph we have ignored both the backreaction of the scalar field on the geometry as well as the effect of the charged black hole core on the scalar condensate. However these effects turn out to be small whenever the starting black hole is small enough. In other words the end point of the superradiant instability of a small charged black hole is given approximately by a non-interacting mix of the black hole core and the condensate cloud at leading order. We will now pause to explain why this is the case.

First note that the charge and energy density of the superradiant mode is contained in an AdS radius scale cloud. As the charge and mass the initial unstable black hole is small, the same is true charge and mass of the eventual the scalar condensate. Consequently, the scalar condensate is of low density and so backreacts only weakly on the geometry everywhere.[6] For this reason the metric of the final solution is a small deformation of the RNAdS black hole with $\mu = \frac{\Delta_0}{e}$, and the scalar condensate does not significantly affect the properties of the RNAdS black hole. On the other hand the condensate cloud is very large compared to

[5]In appendix C.1, we verify this expectation by direct computation of the lowest quasinormal mode of this system. We find that for small R, the imaginary part of the frequency of this mode is given by $3R^3(e\mu - 4) = 3R^3(e\mu - \Delta_0)$, where R is the Schwarzschild radius of the black hole. This imaginary part changes sign precisely where we expect the instability.

[6]Note that, in contrast, for the small charged black hole at the core has its mass and charge concentrated within a small Schwarzschild radius. Consequently even a black hole of very small mass and charge is a large perturbation about the AdS vacuum at length scales comparable to its Schwarzschild radius.

the RNAdS black hole at its core. This difference in scales ensures that the charged black hole also does not significantly affect the properties of the scalar condensate.

Motivated by these considerations, we construct the hairy black hole that marks the end point of the superradiant tachyon condensation process in a perturbative expansion around a small RNAdS black hole with $\mu = \frac{\Delta_0}{e}$ and small but arbitrary radius. The perturbative procedure we employ in our construction is completely standard except for one twist, which we now explain. As is usual in perturbation theory, we expand out the metric, gauge field and scalar field in a power series in ϵ which is the small parameter of our expansion[7]

$$\begin{aligned} g_{\mu\nu} &= g_{0\mu\nu} + \epsilon^2 g_{2\mu\nu} + \dots \\ A_t &= A_{0t} + \epsilon^2 A_{2t} + \dots \\ \phi &= \epsilon\phi_1 + \epsilon^3\phi_3 + \dots \end{aligned} \tag{4.2}$$

Here $g_{0\mu\nu}$ and $A_{0\mu}$ are the metric and gauge field of our starting RNAdS black hole solution. We then plug this expansion into the equations of motion, expand the latter in a power series in ϵ, and attempt to solve the resultant equations recursively. Unfortunately, the linear ordinary differential equations that appear in this process do not appear to be analytically solvable in full generality. However it turns out to be easy to solve these equations separately in two regimes: at large r (in an expansion in $\frac{R}{r}$ which we call as the far-field expansion and mark by a superscript 'out') and at small r (in an expansion in r which we call the near-field expansion and mark by a superscript 'in'). Here r is the radial coordinate (that is zero at the black hole singularity and infinity at the boundary of AdS) and R is the Schwarzschild radius of the unperturbed RNAdS black hole solution. The first expansion is valid when $r \gg R$, while the second expansion works when $r \ll 1$. As we are interested in $R \ll 1$, the validity domains of these two approximations overlap. Consequently, we are able to solve the resultant linear equations everywhere, in a power series expansion in R^2.

When the dust has settled we are thus able to solve for the hairy black holes only in a double expansion in ϵ^2 and R^2. This expansion is sufficient to understand small hairy black holes [8]. In section 4.2 below we have explicitly implemented this expansion to $\mathcal{O}(\epsilon^m R^2)$ for $m \leq 5$. Our calculations allow us to determine the microcanonical phase diagram of our system, as a function of mass and charge at small values of these parameters[9]; our results are

[7] As we explain in Section 4.2 below, we find it convenient to choose ϵ to be the coefficient of the $\frac{1}{r^4}$ decay of the scalar field at infinity, i.e. the vacuum expectation value of the operator dual to the scalar field.

[8] The technical obstruction to solving the equations at arbitrary R has a physical interpretation. Even at arbitrarily small ϵ, the black hole reacts significantly on the condensate at finite R. Our system can be regarded as a non interacting mix of the black hole and the condensate only at very small R. Effectively, we perturb around this non interacting limit.

[9] The microcanonical ensemble is well suited to our purposes. We discuss the phase diagram in other ensembles, in particular the canonical and grand canonical ensemble, in appendix C.6 and C.7 below. We are able to make less definite statements in these ensembles because it turns out that the system at a given fixed chemical potential and temperature often receives contributions both from small as well as big black holes. As the approximation techniques of this chapter do not apply to big black holes, we are unable to quantitatively assess the relative importance of these saddle points.

plotted for $e = 5$ in Fig. 4.1 below (the results are qualitatively similar for every e provided $e^2 \geq \frac{32}{3}$, and may also be simply generalised to the study of (4.1) with a mass term added for the scalar field- see section 4.6.4).

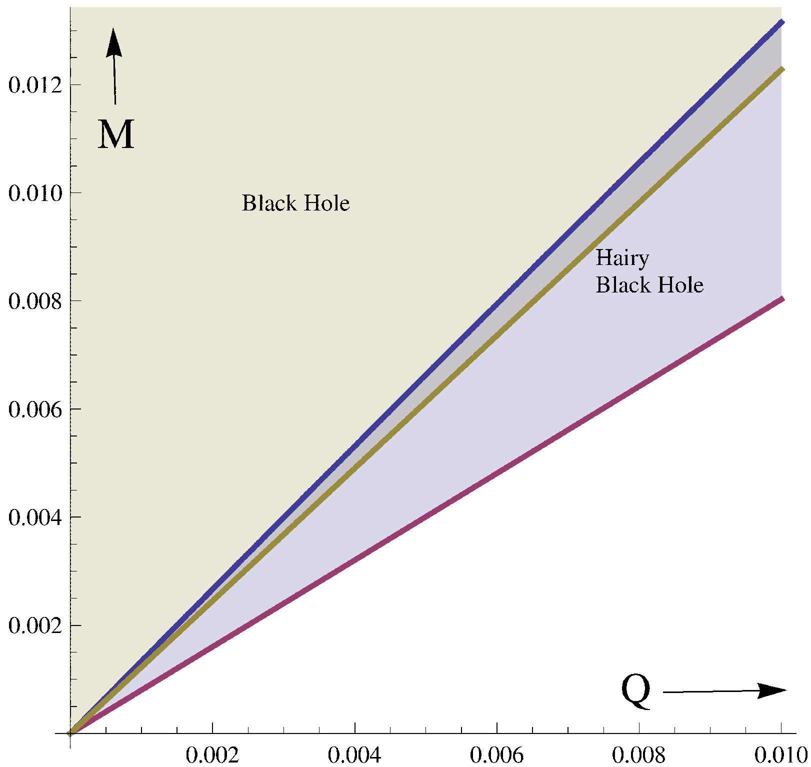

Figure 4.1: Microcanonical phase diagram at small mass and charge. The overlapping region is dominated by the hairy black hole.

As summarised in Fig. 4.1, hairy black holes exist only in the mass range

$$\frac{4}{e}Q + \frac{(9e^2-64)}{7\pi e^2}Q^2 + \mathcal{O}(Q^3) \leq M \leq \frac{3e}{16}\left(1+\frac{32}{3e^2}\right)Q - \frac{3\,(5e^4+64e^2-1024)}{64\pi e^2}Q^2 + \mathcal{O}(Q^3), \tag{4.3}$$

where M and Q are the mass and the charge of the hairy black hole.

Above the upper bound in (4.3) (i.e. in the shaded grey region above the blue line in Fig. 4.1.), RNAdS black holes are stable and are the only known stationary solutions. The upper end of (4.3) coincides with the onset of superradiant instabilities for RNAdS black holes. The lower bound in (4.3) is marked by the lowest (i.e. red) line in Fig. 4.1. The extremality line for RNAdS black holes (the yellow line in Fig. 4.1.) lies in the middle of this range in (4.3). At masses below lower bound of (4.3) (red line in Fig. 4.1.), the system presumably has no states.

As we have emphasised, a hairy black hole may approximately be thought of as a non interacting superposition of a RNAdS black hole and a scalar bose condensate. At the upper bound of (4.3) the condensate vanishes, and hairy black holes reduce to a RNAdS black holes. As we decrease the mass of a hairy black hole at fixed charge (or increase the charge at fixed mass), the fraction of the condensate increases. Eventually, at the lower end of (4.3) - the red line in Fig. 4.1. - the black hole shrinks to zero. In this limit [10] the solution reduces to a regular horizon free soliton (see [86] for the appearance of a similar solitonic solution in a qualitatively similar context). The solitonic solution, on the red line of Fig. 4.1 is simpler than the hairy black hole solution. As $R = 0$ on this solution, it may be generated as an expansion in a single parameter ϵ; in appendix C.3 we have carried out this expansion to 17th order. Hairy black holes may be thought of as solitons with small RNAdS solutions in their centre (see the related general analysis[11] of [87]).

In summary, the hairy black hole interpolates between pure black hole and pure condensate as we scan from the upper to the lower bound of (4.3) (or down from the blue line to the red line in Fig. 4.1.).Throughout the range of its existence, the hairy black hole is the only known stable solution[12]. It is also the thermodynamically dominant solution, as the entropy of the hairy black hole exceeds that of the RNAdS black hole of the same mass and charge, whenever both solutions exist.

Note that the solitonic solutions described above have some similarities to so called boson stars[13], which have been extensively discussed in the General Relativity literature, mainly in asymptotically flat space (see e.g. [88]) but also in asymptotically AdS spaces (see e.g. [89]). However boson stars usually have scalar fields with a harmonic time dependence, which obstructs placing black holes in their centre (the scalar field would oscillate an infinite number of times as it approaches the horizon). Our solitonic solutions are genuinely stationary[14] in a particular gauge. In this gauge the temporal component of the gauge field takes a particular non zero value at the origin of the soliton. This allows us to construct stationary hairy black hole solutions by placing charged black holes at the centre of the soliton (roughly via the procedure of [87].) if and only if the black holes are chosen so that their chemical potential matches the gauge field at the centre of the soliton. The last condition has a simple

[10] We take the $R \to 0$ limit purely within classical relativity. Of course stringy and quantum gravity effects (including the one loop energy density of the gas outside the black hole) become important when the black hole becomes parametrically small. Such effects are important only in an infinitesimal wedge above the red line in Fig. 4.1 and depend on the detailed microphysics of the system (they would be different, for instance, in string theory and M theory). We ignore all such effects in here. We thank K. Papadodimas for a discussion on this issue.

[11] We thank G. Horowitz and H. Reall for drawing our attention to this reference.

[12] Except for excited solitons, see below for details.

[13] We thank G. Horowitz, M. Rangamani and K. Papadodimas for drawing our attention to the literature on boson stars and explaining their properties to us. K. Papadodimas has further drawn our attention to the fact that our soliton reduces precisely to a traditional boson star in the limit $e \to 0$. Of course hairy black holes exist only when $e > e_c$ and so do not exist in the small e limit.

[14] The presence of the gauge field allows us to evade Derrick's theorem. We thank K. Papadodimas pointing this out.

and intuitive thermodynamical interpretation; solitons and black holes can be put into an approximately non interacting mix only when their chemical potentials match!

The hairy black hole we have described so far in this introduction is a weakly interacting mix of the RNAdS black hole and a condensate of the ground state of the scalar field. The reader may wonder whether it is possible to construct an excited hairy black hole solution that is a weakly interacting mix of a RNAdS black hole and an excited state of the scalar field. This is indeed the case. The set of spherically symmetric linearised excitations of a massless scalar field appear in a one parameter family[15] labelled by an integer n. The energy of the n^{th} state is $\Delta_n \equiv 4 + 2n$ where $n = 0 \ldots \infty$. It turns out to be possible to mimic the construction described above to construct excited hairy black holes that reduce, at small masses, to the superposition of a RNAdS black hole with $\mu = \frac{\Delta_n}{e} = \frac{4+2n}{e}$ with a condensate of the n^{th} scalar excited state. It turns out that these excited hairy black holes are all unstable to the superradiant decay of the scalar mode with energy $\Delta_0 = 4$. They presumably decay to the ground state hairy black hole, in comparison to which they are all turn out to be entropically sub dominant.

Each excited hairy black hole exists in a limited mass range which turns out to be a subset of the mass range (4.3). At the lower end of this range, each excited hairy black hole reduces to a horizon free scalar condensate, i.e. an excited soliton. Here we find a surprise. Recall that the instability of excited state hairy black holes is superradiant in nature. Superradiant instability rates scale like R^3, and so go to zero as $R \to 0$. It thus appears that excited state scalars are actually stable to small fluctuations, and so cannot decay classically. The discussion of this paragraph actually leaves open the possibility that excited solitons may have an independent non super radiant instability mode. It is, however, possible to demonstrate that the spectrum of small oscillations about arbitrary unstable solitons has no exponentially growing eigenmode, at least within the assumption of spherical symmetry (see 4.6.3). While this result does not rigorously prove the stability of excited solitons [90, 91], it at least suggests that they are stable. We find this surprising and do not have a clear sense for its implications.

It may be worth pausing to consider the relative merits and disadvantages of the perturbative procedure employed here versus the numerical approach more usually used to study hairy black branes. On the negative side our perturbative procedure gives us no information about the regime of large masses and charges (where the perturbative expansion breaks down). Within its regime of validity, however, our perturbative procedure is very powerful. It allows us, once and for all, to compute the phase diagram and thermodynamics of all relevant solutions - including each of the infinite number of excited state hairy black holes - as analytic function of the parameters of the problem (e.g. the mass and charge of the scalar field). Perhaps more importantly our procedure gives us qualitative intuitive insight into the nature of hairy black holes. For instance, as we have explained many times, the hairy black hole is an approximately non interacting mix of a RNAdS black hole and the

[15]These modes are dual, under the state operator map, to the operator $\partial^{2n}O$, where O is the dimension 4 operator that corresponds to the bulk field ϕ according to the rules of the AdS/CFT dictionary.

scalar condensate. This picture together with a few lines of algebra, immediately yields a formula for the entropy of the hairy black hole, to leading order in its mass and charge (see 4.6.4). In other words the perturbative approach employed here gives more than numerical answers; it helps us to understand why small hairy black holes behave the way they do.

In this work we have focused on the black holes in global AdS at small mass and charge. Almost all previous studies of the Lagrangian (4.1) have studied the system in the Poincaré patch[16]. The local dynamics of black holes in global AdS reduces to the dynamics of black branes of Poincaré AdS at large mass and charge. It follows that the phase diagram displayed in Fig. 4.1 should make contact with the results of previous analyses at large mass and charge. In section 4.7 we present a conjectural phase diagram that interpolates between the small mass and charge behaviour derived here and the large mass and charge behaviour determined in previous work. The analysis of that section makes it clear that something special happens to the phase diagram of hairy black holes at charges, in units of the inverse Newton constant, of order unity. It would be interesting to analyse these aspects further.

The reader who is interested in asymptotically AdS gravitational dynamics principally because of the AdS/CFT correspondence might legitimately complain that our choice of the Lagrangian (4.1) was arbitrary; the dynamics of charged scalar fields in any given example of the AdS/CFT correspondence is unlikely to be given by (4.1). Our attitude to this is the following: we regard (4.1) as a toy model which we have chosen to study (in common with much earlier work on the subject), largely because it is a simple system that possesses several of the ingredients that are qualitatively important for hairy black hole dynamics. The study presented in this chapter of the toy model sets the stage for a similar analysis of small hairy black holes in ‘realistic’ theories such as IIB theory on $AdS_5 \times S^5$. Small black holes in this special bulk theory share many of the qualitative features discussed in this chapter, but also have some properties that result from dynamical features special to it.[17] These and related matters have been studied and appear in [26].

Note added in the paper: After the first version of the paper which forms the basis of this chapter appeared, we were made aware of similar work by Maeda.et.al. [93].

4.2 The basic setup for the hairy black holes

4.2.1 Basic equations of motion

As mentioned in the introduction, we study the Lagrangian (4.1). This action describes the interaction of a massless minimally coupled scalar field, of charge e, interacting with a negative cosmological constant Einstein Maxwell system. Through most of this work we

[16]See however [84] for a study in global AdS.

[17]For instance, it is important to keep in mind that small black holes in IIB theory on $AdS_5 \times S^5$ sometimes suffer from Gregory Laflamme instabilities [92] (in addition to potential superradiant instabilities), an additional feature that complicates (but enriches) the dynamics of small black holes in a ‘realistic’ theory like IIB SUGRA on $AdS_5 \times S^5$.

will be interested in stationary, spherically symmetric solutions of this system . However, in appendix C.1 we will generalise to the study of time-dependent configurations to investigate the stability against small fluctuations. We adopt a Schwarzschild like gauge and set

$$\begin{aligned} ds^2 &= -f(r)dt^2 + g(r)dr^2 + r^2 d\Omega_3^2 \\ A_t &= A(r) \\ A_r &= A_i = 0 \\ \phi &= \phi(r) \end{aligned} \tag{4.4}$$

The four unknown functions $f(r)$, $g(r)$, $A(r)$ and $\phi(r)$ are constrained by Einstein's equations, the Maxwell equations and the minimally coupled scalar equations. It is possible to demonstrate that f, g, A, ϕ are solutions to the equations of motion if and only if

$$\begin{aligned} & r\left(3f'(r) - 2e^2 r g(r) A(r)^2 \phi(r)^2 + rA'(r)^2\right) - 2f(r)\left(\left(6r^2+3\right)g(r) + r^2\phi'(r)^2 - 3\right) = 0 \\ & f(r)\left(3rg'(r) - 2g(r)\left(r^2\phi'(r)^2 + 3\right)\right. \\ & \qquad \left. +6\left(2r^2+1\right)g(r)^2\right) - r^2 g(r)\left(2e^2 g(r) A(r)^2 \phi(r)^2 + A'(r)^2\right) = 0 \\ & rg(r)f'(r)A'(r) + f(r)\left(rg'(r)A'(r) + 4e^2 r g(r)^2 A(r)\phi(r)^2\right. \\ & \qquad \left. -2g(r)\left(rA''(r) + 3A'(r)\right)\right) = 0 \\ & g(r)\left(\left(rf'(r) + 6f(r)\right)\phi'(r) + 2rf(r)\phi''(r)\right) - rf(r)g'(r)\phi'(r) + 2e^2 r g(r)^2 A(r)^2 \phi(r) = 0. \end{aligned} \tag{4.5}$$

The four equations listed in (4.5) are the rr and tt components of Einstein's equations, the Maxwell equation and the minimally coupled scalar equation, in that order.

The equations (4.5) contain only first derivatives of f and g, but depend on derivatives up to the second order for ϕ and A. It follows that (4.5) admit a 6 parameter set of solutions. One of these solutions is empty AdS_5 space, given by $f(r) = r^2 + 1$, $g(r) = \frac{1}{1+r^2}$, $A(r) = \phi(r) = 0$. We are interested in those solutions to (4.5) that asymptote to AdS spacetime, i.e. solutions whose large r behaviour is given by

$$\begin{aligned} f(r) &= r^2 + 1 + \mathcal{O}(1/r^2) \\ g(r) &= \frac{1}{1+r^2} + \mathcal{O}(1/r^6) \\ A(r) &= \mathcal{O}(1) + \mathcal{O}(1/r^2) \\ \phi(r) &= \mathcal{O}(1/r^4) \end{aligned} \tag{4.6}$$

It turns out that these conditions effectively impose two conditions on the solutions of (4.5), so that the system of equations admits a four parameter set of asymptotically AdS solutions[18]. We will also be interested in solutions that are regular (in a suitable sense) in the interior. As we will see below, this requirement will cut down solution space to distinct classes of two parameter space of solutions (the parameters may be thought of as the mass

[18] For example, the equations above are easily solved in linearisation about AdS_5 ;the six dimensional

and charge of the solutions). In particular, we are seeking the hairy black hole solutions of the above equations that constitute the endpoint of the superradiant instability of small RNAdS black holes. To set the stage and notations for our computation we first briefly review the charged RNAdS black hole solutions in global AdS spacetime.

4.2.2 RNAdS black holes and their superradiance

The AdS-Reissner-Nordstrom black holes constitute a very well known two parameter set of solutions to the equations (4.5). These solutions are given by

$$\begin{aligned} ds^2 &= -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_3^2 \\ V(r) &\equiv 1 + r^2 - \frac{R^2}{r^2}\left[1 + R^2 + \frac{2}{3}\mu^2\right] + \frac{2}{3}\mu^2\frac{R^4}{r^4} \\ &= \left[1 - \frac{R^2}{r^2}\right]\left[1 + r^2 + R^2 - \frac{2}{3}\frac{\mu^2 R^2}{r^2}\right] \\ A(r) &= \mu\left[1 - \frac{R^2}{r^2}\right] \\ \phi(r) &= 0 \end{aligned} \tag{4.8}$$

where μ is the chemical potential of the RNAdS black hole. The function $V(r)$ in (4.8) vanishes at $r = R$ and consequently this solution has a horizon at $r = R$. In fact, it can be shown that R is the outer event horizon provided

$$\mu^2 \leq \frac{3}{2}(1 + 2R^2). \tag{4.9}$$

We will review later the thermodynamics of these solutions in more detail with a particular focus on small charged black holes whose $R \ll 1$. Consider the small RNAdS black hole solutions of the system described by the Lagrangian in (4.1). As we have explained in the introduction, in the limit $R \ll 1$ we expect the solution in (4.8) to be unstable to superradiant decay provided $e\mu \geq \Delta_0 = 4$. In appendix C.1, we verify this intuitive expectation by

solution space is given by

$$\begin{aligned} \delta f(r) &= a_1(1 + r^2) - \frac{a_2}{r^2} \\ \delta g(r) &= \frac{a_2}{r^2(1 + r^2)^2} \\ \delta A(r) &= a_3 + \frac{a_4}{r^2} \\ \delta\phi(r) &= a_5 + a_6\int\frac{dr}{r^3(1 + r^2)} \end{aligned} \tag{4.7}$$

The asymptotically AdS condition set $a_1 = a_5 = 0$.

determining the lowest quasinormal mode of this system in a power series in R. In a gauge where $A_t^{(r=R)} = 0$, we find that the time dependence of this lowest mode is given by $e^{-i\omega t}$ where

$$\begin{aligned} \omega &= (\Delta_0 - e\mu) + R^2\ (-6 + 3e\mu - 4\mu^2) - i\ 3R^3(\Delta_0 - e\mu) + \mathcal{O}(R^4) \\ &= (4 - e\mu) + R^2\ (-6 + 3e\mu - 4\mu^2) - i\ 3R^3(4 - e\mu) + \mathcal{O}(R^4). \end{aligned} \tag{4.10}$$

Note in particular that

$$\mathrm{Im}\,(\omega) = -3R^3(\Delta_0 - e\mu) + \mathcal{O}(R^4)$$

it follows that the time dependence $e^{-i\omega t}$ of this mode represents an exponential damping when $\mu e < \Delta_0$ but an exponential growth when $\mu e > \Delta_0$. Consequently, small charged black holes are unstable when $\mu e > \Delta_0$, in agreement with the intuitive expectations outlined in the introduction. Further, note that the decay (or growth) constant of the lowest quasinormal mode is given by $3R^3|\Delta - e\mu|$, and goes to zero either when R goes to zero or as μ goes near $\frac{\Delta_0}{e}$. As we have argued in the introduction, this motivates us to seek a hairy black hole solution which is constructed in a perturbation theory about these RNAdS black holes.

4.2.3 Setting up the perturbation theory

The starting point of our construction is a small RNAdS black hole

$$\begin{aligned} ds^2 &= -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_3^2 \\ V(r) &= \left[1 - \frac{R^2}{r^2}\right]\left[1 + r^2 + R^2 - \frac{2}{3}\frac{\mu^2 R^2}{r^2}\right] \\ A(r) &= \mu\left[1 - \frac{R^2}{r^2}\right] \end{aligned} \tag{4.11}$$

at arbitrary but small R, and

$$\begin{aligned} \mu = \mu(\epsilon, R) &= \sum_{n=0} \epsilon^{2n}\mu_{2n}(R) \\ \mu_{2n}(R) &= \sum_{k=0}^{\infty} \mu_{(2n,2k)} R^{2k} \\ \mu_{(0,0)} &= \frac{4}{e} \end{aligned} \tag{4.12}$$

Here $\mu = \mu(R, \epsilon)$ is the as yet unknown chemical potential of our final solution. Note that, at the leading order in the perturbative expansion, $\mu = \frac{4}{e}$.

To proceed we simply expand every unknown function

$$
\begin{aligned}
f(r,R,\epsilon) &= \sum_{n=0}^{\infty} \epsilon^{2n} f_{2n}(r,R) \\
g(r,R,\epsilon) &= \sum_{n=0}^{\infty} \epsilon^{2n} g_{2n}(r,R) \\
A(r,R,\epsilon) &= \sum_{n=0}^{\infty} \epsilon^{2n} A_{2n}(r,R) \\
\phi(r,R,\epsilon) &= \sum_{n=0}^{\infty} \epsilon^{2n+1} \phi_{2n+1}(r,R)
\end{aligned}
\tag{4.13}
$$

Here f_0, g_0 and A_0 are the values of the functions f, g and A for a RNAdS black hole with radius R and chemical potential $\mu = \mu_0(R)$. given in (4.12). We expand our equations in a power series in ϵ. At each order in ϵ we have a set of linear differential equations (see below for the explicit form of the equations), which we solve subject to the requirements of the normalisability of $\phi(r)$ and $f(r)$ at infinity together with the regularity of $\phi(r)$ and the metric at the horizon. These four physical requirements turn out to automatically imply that $A(r = R) = 0$ i.e. the gauge field vanishes at the horizon, as we would expect of a stationary solution. These four physical requirements determine 4 of the six integration constants in the differential equation, yielding a two parameter set of solutions. We fix the remaining two integration constants by adopting the following conventions to label our solutions: we require that $\phi(r)$ fall off at infinity like $\frac{\epsilon}{r^4}$ (definition of ϵ), that the horizon area of our solution is $2\pi^2 R^3$ (definition of R). This procedure completely determines our solution as a function of R and ϵ. We can then read of the value of μ in (4.12) on our solution from the value of the gauge field at infinity.

As we have explained in the introduction, the linear differential equations that arise in perturbation theory are difficult to solve exactly, but are easily solved in a power series expansion in R, by matching near field and far field solutions. At every order in ϵ we thus have a solution as an expansion in R. Our final solutions are, then presented in a double power series expansion in ϵ and R.

In the next few sections, we present a detailed description of the implementation of this perturbation expansion at order ϵ and ϵ^2. In appendix C.2 we present explicit results for this perturbation expansion at higher orders.

4.3 Perturbation theory at $\mathcal{O}(\epsilon)$

We will now present a detailed description of the implementation of our perturbative expansion at $\mathcal{O}(\epsilon)$. The procedure described in this subsection applies, with minor modifications, to the perturbative construction at $\mathcal{O}(\epsilon^{2m+1})$ for all m.

In this section we wish to construct the first order correction around the black hole

$$\begin{aligned} f_0(r,R) &= V(r), \quad g_0(r,R) = \frac{1}{V(r)} \\ A_0(r,R) &= \mu_0(1-\frac{R^2}{r^2}) \\ V(r) &= 1 + r^2\left(1 - \frac{\frac{2R^4\mu_0^2}{3} + R^4 + R^6}{r^4R^2} + \frac{2R^4\mu_0^2}{3r^6}\right) \end{aligned} \tag{4.14}$$

Plugging in (4.13), we expand the equations of motion in a power series in ϵ to $\mathcal{O}(\epsilon)$. Of course all equations are automatically obeyed at $\mathcal{O}(\epsilon^0)$. The only nontrivial equation at $\mathcal{O}(\epsilon)$ is $D^2\phi = 0$ where $D_\mu = \nabla_\mu - ieA_\mu$ is the linearised gauge covariantised Laplace equation about the background (4.14). We will now solve this equation subject to the constraints of normalisability at infinity, regularity at the horizon, and the requirement that $\phi(r) \sim \frac{\epsilon}{r^4}$ at large r.

4.3.1 Far field region

Let us first focus on the region $r \gg R$. In this region the background (4.14) is a small perturbation about global AdS space. For this reason we expand

$$\phi_1^{out}(r) = \sum_{k=0}^{\infty} R^{2k}\phi_{(1,2k)}^{out}(r), \tag{4.15}$$

where the superscript *out* emphasises that this expansion is good at large r. In the limit $R \to 0$, (4.14) reduces to global AdS spacetime with $A_t = \frac{4}{e}$. A stationary linearised fluctuation about this background is gauge equivalent to a linearised fluctuation with time dependence e^{-4it} about global AdS space with $A_t = 0$ (A_t is the temporal component of the gauge field). The required solution is simply the ground state excitation of a massless minimally coupled scalar field about global AdS

$$\phi_{(1,0)}^{out}(r) = \frac{1}{(1+r^2)^2}. \tag{4.16}$$

The overall normalisation of the mode is set by the requirement

$$\phi_{(1,0)}^{out}(r) = \frac{1}{r^4} + \mathcal{O}(1/r^6).$$

We now plug (4.15) into the equations of motion $D^2\phi = 0$ and expand to $\mathcal{O}(R^2)$ to solve for $\phi_{1,2}^{out}$. Here D^2 is the gauge covariant Laplacian about the background (4.11). Now

$$(D^2)^{out} = (D_0^2)^{out} + R^2(D_2^2)^{out} + \dots$$

where $(D_0^2)^{out}$ is the gauge covariant Laplacian about global AdS spacetime with background gauge field $A_t = \frac{4}{e}$. It follows that, at $\mathcal{O}(R^2)$,

$$(D_0^2)^{out}\phi^{out}_{(1,2)} = -(D_2^2)^{out}\phi^{out}_{(1,0)} = -(D_2^2)^{out}\left[\frac{1}{(1+r^2)^2}\right]$$

This equation is easily integrated and we find

$$\begin{aligned}\phi^{out}_{(1,2)}(r) =& \frac{2\left(-3e^2+6\left(e^2-32\right)\left(r^2+1\right)\log(r)-3\left(e^2-32\right)\left(r^2+1\right)\log\left(r^2+1\right)-32\right)}{3e^2\left(r^2+1\right)^3}\\&+\left(\mu_{0,2}-\frac{6e^2-64}{e^3}\right)\left(\frac{e\left(6\log(r)r^2-3\log\left(r^2+1\right)r^2-1\right)}{6\left(r^3+r\right)^2}\right)\end{aligned} \tag{4.17}$$

We could iterate this process to generate $\phi^{out}_{(1,2k)}$ till any desired k. As in (4.17), it turns out that the expressions $\phi^{out}_{(1,2k)}$ are increasingly singular as $r \to 0$. In fact it may be shown that the most singular piece of $\phi^{out}_{(1,2k)}$ scales like $\frac{1}{r^{2k}}$, up to logarithmic corrections. In other words the expansion of ϕ^{out} in powers of R^2 is really an expansion in $\frac{R^2}{r^2}$ (up to log corrections) and breaks down at $r \sim R$.

In summary we have found that, to $\mathcal{O}(R^2)$

$$\begin{aligned}\phi_1^{out}(r) =& \frac{1}{(r^2+1)^2}\\&+R^2\left[\frac{2\left(-3e^2+3\left(e^2-32\right)\left(r^2+1\right)\log(r^2/(r^2+1))-32\right)}{3e^2\left(r^2+1\right)^3}\right.\\&\left.+\left(\mu_{(0,2)}-\frac{6e^2-64}{e^3}\right)\left(\frac{e\left(6\log(r)r^2-3\log\left(r^2+1\right)r^2-1\right)}{6\left(r^3+r\right)^2}\right)\right]\\&+\mathcal{O}(R^4/r^4)\end{aligned} \tag{4.18}$$

The small r expansion of this result is given by

$$\begin{aligned}\phi_1^{out}(r) =& \left[1-2r^2+\mathcal{O}(r^4)\right]+R^2\left[\frac{4}{e^2}(e^2-32)\log(r)-2\left(1+\frac{32}{3e^2}\right)+\mathcal{O}(r^2)\right]\\&-R^2\left(\mu_{(0,2)}-\frac{6e^2-64}{e^3}\right)\left[-\frac{e}{6r^2}+e\log(r)+\frac{e}{3}+\mathcal{O}(r^2)\right]+\mathcal{O}(R^4)\end{aligned} \tag{4.19}$$

Note that this result depends on the as yet unknown parameter $\mu_{(0,2)}$. This quantity will be determined below by matching with the near field solution.

4.3.2 Near field region

Let us now turn to inner region $r \ll 1$. Over these length scales the small black hole is far from a small perturbation about AdS_5 space. Instead the simplification in this region arises from the fact that background gauge field, which is of order unity, is negligible compared to the mass scale set by the horizon radius $\frac{1}{R}$. In other words the gauge field is a small perturbation about the black hole background in this region. To display this fact it is convenient to work in a rescaled radial coordinate $y = \frac{r}{R}$ and a rescaled time coordinate $\tau = \frac{t}{R}$. Note that the near field region consists of spacetime points with y of order unity. In these coordinates the background black hole solution takes the form

$$\begin{aligned} ds^2 &= R^2 \left(-V(y) d\tau^2 + \frac{dy^2}{V(y)} + y^2 d\Omega_3^2 \right) \\ V(r) &= \left[1 - \frac{1}{y^2} \right] \left[1 - \frac{2}{3} \frac{\mu^2}{y^2} + R^2 \left(1 + y^2 \right) \right] \\ A_\tau &= R\mu_0 (1 - \frac{1}{y^2}) \end{aligned} \tag{4.20}$$

The explicit factor of R in A_τ in (4.20) demonstrates the effective weakness of the gauge field. This justifies an expansion of the near field solution in a power series in R

$$\phi_1^{in}(y) = \sum_{k=0}^{\infty} R^{2k} \phi_{(1,2k)}^{in}(y) \tag{4.21}$$

To determine the unknown functions in this expansion, we must solve the equation $D^2 \phi^{in} = 0$, where D^2 is the gauge covariant Laplacian about the background (4.20). Our solutions are subject to the constraint of regularity at the horizon. Further, they must match with the far field expansion in equations (4.18) and (4.19) above.

Note

$$(D^2)^{in} = \frac{1}{R^2} (D_0^2)^{in} + (D_2^2)^{in} + \dots$$

where $(D_0^2)^{in}$ in is the leading part of $(D^2)^{in}$ in an R expansion. At leading order we find $D_0^2 \phi_{(1,0)}^{in}(y) = 0$ The two linearly independent solutions of this equation are easily obtained by integration. The only solution regular at the horizon is the constant

$$\phi_{(1,0)}^{in}(y) = 1$$

where we have determined the value of the constant by matching with equations (4.18) and (4.19) (more below about the matching).

At next order in R^2 we obtain an equation of the form

$$D_0^2 \phi_{(1,2)}^{in} = -D_2^2 \phi_{(1,0)}^{in}(y).$$

Even though $\phi_{(1,0)}^{in}(y)$ is a constant, the RHS of this equation is nonzero because of the gauge field in (4.14). The equation is easily solved by integration; imposing regularity of the solution at the horizon we find[19]

$$\begin{aligned}\phi^{in}_{(1,2)}(y) =& \alpha + \frac{1}{3e^2}\Bigg[- 6e^2y^2 - 128\log\left(3e^2-32\right)\log\left(\frac{y^2-1}{3e^2y^2-32}\right) - 192\log\left(3e^2y^2-32\right)\\ &+ 6\log\left(3e^2y^2-32\right)e^2 + 128\log\left(-\frac{3e^2\left(y^2-1\right)}{3e^2-32}\right)\log\left(3e^2y^2-32\right)\\ &+ 64\log^2\left(3e^2y^2-32\right) + 128\text{Li}_2\left(\frac{32-3e^2y^2}{32-3e^2}\right)\Bigg]\end{aligned} \tag{4.22}$$

where $\text{Li}_2(x)$ is the polylog function as defined in Mathematica 6

$$\text{Li}_2(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^2}$$

The single unknown parameter α in this solution will be determined by matching below.

The perturbative procedure described above may be iterated to arbitrary order. It turns out that the fields $\phi^{in}_{(1,2m)}$ at high m are increasingly singular at large y. In fact it may be shown that the dominant growth of $\phi^{in}_{(1,2m)}$ is generically y^{2m}. It follows that the near field perturbative expansion is an expansion in $(yR)^2 = r^2$.

In summary

$$\begin{aligned}\phi^{in}_1(y) =& 1 + R^2\alpha\\ &+ \frac{R^2}{3e^2}\Bigg[- 6e^2y^2 - 128\log\left(3e^2-32\right)\log\left(\frac{y^2-1}{3e^2y^2-32}\right) - 192\log\left(3e^2y^2-32\right)\\ &+ 6\log\left(3e^2y^2-32\right)e^2 + 128\log\left(-\frac{3e^2\left(y^2-1\right)}{3e^2-32}\right)\log\left(3e^2y^2-32\right)\\ &+ 64\log^2\left(3e^2y^2-32\right) + 128\text{Li}_2\left(\frac{32-3e^2y^2}{32-3e^2}\right)\Bigg] + \mathcal{O}(yR)^4\end{aligned} \tag{4.23}$$

The large y expansion of this result is given by

$$\begin{aligned}\phi^{in}_1(y) =& 1 + R^2\Bigg[\left(-2y^2 + \frac{4}{e^2}(e^2-32)\log(y)\right) + \alpha - \frac{64\pi^2}{3e^2} + 6\left(1-\frac{32}{e^2}\right)\log(3)\\ &+ \frac{1}{3e^2}\Bigg(- 192\log^2\left(\frac{1}{32-3e^2}\right) + 384\log(3)\log\left(3e^2-32\right)\\ &+ 12\log(e)\left(3e^2 + 64\log\left(3e^2-32\right) - 96\right)\Bigg) + \mathcal{O}\left(\frac{1}{y^2}\right)\Bigg] + \mathcal{O}(Ry)^4\end{aligned} \tag{4.24}$$

In (4.24) we have determined $\phi^{in}_1(y)$ in terms of the as yet unknown parameter α which will be determined by matching below.

[19]The apparent logarithmic singularities at $y = 1$, in two of the terms of (4.22), actually cancel.

4.3.3 Matching

In order to match the near and far field results, we substitute $y = \frac{r}{R}$ in (4.24) (the large y expansion of ϕ_1^{in}) and view the resultant expression as an expansion about small r and small R. As we have explained above, the resultant expression is reliable to all order in R but only to order $\mathcal{O}(r^2)$; all terms of order $\mathcal{O}(r^4)$ or higher receive contributions from as yet undetermined fourth order terms in the perturbation expansion of ϕ_1^{in} (4.24).

We then compare this expression with the small r expansion of ϕ^{out}, (4.19). We can generate this expansion to any order in r that we desire (merely by Taylor expanding (4.19)); however the resultant expression is clearly valid only to $\mathcal{O}(R^2)$ in R (terms of $\mathcal{O}(R^4)$ obviously receive contributions from as yet undetermined fourth order terms in the expansion of ϕ^{out}). Terms of the form r^0R^0, r^2R^0 and r^0R^2 (together with logarithmic corrections) are reliably computed by both expansions and so must agree. The unknown parameters α and $\mu_{(0,2)}$ are determined to ensure this (as we have more conditions than variables we also obtain valuable consistency checks). We find

$$\begin{aligned}
\mu_{(0,2)} =& \frac{6e^2-64}{e^3} \\
\alpha =& \frac{2\left(-9e^2-192\log\left(3e^2-32\right)+288\right)}{9e^2}\log(3) - \frac{2\left(3e^2-32\log^2\left(32-3e^2\right)+32\right)}{3e^2} \\
&+\frac{64\pi^2}{9e^2} - 18\left(e^2-32\right)\log(R) + 6\log(e)\left(3e^2+64\log\left(3e^2-32\right)-96\right)
\end{aligned}$$

This completes our determination of our solution to $\mathcal{O}(R^2)$. The procedure described in this subsection can be iterated to obtain the solution at higher orders.

4.4 Perturbation theory at $\mathcal{O}(\epsilon^2)$

We now briefly outline the procedure used to evaluate the solution at $\mathcal{O}(\epsilon^2)$. We proceed in close imitation to the previous subsection. The main difference is that at this (and all even orders) in the ϵ expansion, perturbation theory serves to determine the corrections to the functions f, g and A rather than the function ϕ. The procedure described here applies, with minor modifications, to the perturbative construction at $\mathcal{O}(\epsilon^{2m})$ for all m.

4.4.1 Far field region

In the far field region $r \gg R$ we expand

$$\begin{aligned} f_2^{out}(r) &= \sum_{m=0}^{\infty} R^{2m} f_{(2,2m)}^{out}(r) \\ g_2^{out}(r) &= \sum_{m=0}^{\infty} R^{2m} g_{(2,2m)}^{out}(r) \\ A_2^{out}(r) &= \sum_{m=0}^{\infty} R^{2m} A_{(2,2m)}^{out}(r) \end{aligned} \tag{4.25}$$

Plugging this expansion into the equations of motion and expanding to $\mathcal{O}(R^{2m})$ we find equations of the form

$$\begin{aligned} \frac{d}{dr}\left(r^2(1+r^2)^2 g_{(2,2m)}^{out}(r)\right) &= P_{(2,2m)}^g(r) \\ \frac{d}{dr}\left(\frac{f_{(2,2m)}^{out}(r)}{1+r^2}\right) &= \frac{2(1+2r^2)}{r} g_{(2,2m)}^{out}(r) + P_{(2,2m)}^f(r) \\ \frac{d}{dr}\left(r^3 \frac{dA_{(2,2m)}^{out}(r,R)}{dr}\right) &= P_{(2,2m)}^A(r) \end{aligned} \tag{4.26}$$

Where $P_{(2,2m)}^g(r),\ P_{(2,2m)}^f(r),\ P_{(2,2m)}^A(r)$ are the source terms at order $\epsilon^2 R^{2m}$ which are obtained from terms quadratic in ϕ_1^{out}. The most general normalisable solution to these equations takes the form

$$\begin{aligned} g_{(2,2m)}^{out}(r) =& \frac{b_{(2,2m)}}{r^2(1+r^2)^2} - \frac{1}{r^2(1+r^2)^2}\left(\int_r^\infty dx\ P_{(2,2m)}^g(x)\right) \\ f_{(2,2m)}^{out}(r) =& -(1+r^2)\left(\int_x^\infty dx\ \left[\frac{2(1+2x^2)}{r} g_{(2,2m)}^{out}(x) + P_{(2,2m)}^f(x)\right]\right) \\ A_{(2,2m)}^{out}(r) =& \frac{h_{(2,2m)}}{r^2} + k_{(2,2m)} + \int_r^\infty \frac{dx}{x^3}\left[\int_x^\infty dw\ P_{(2,2m)}^A(w)\right] \end{aligned} \tag{4.27}$$

Note that this solution has three undetermined integration constants $b_{(2,2m)}, h_{(2,2m)}$ and $k_{(2,2m)}$.

Let us first focus on $\mathcal{O}(R^0)$. The constants $b_{(2,0)}$ and $h_{(2,0)}$ are determined by the requirement that the expansion of $g_{(2,0)}^{out}$ and $A_{(2,0)}^{out}$ at small r starts out regular (i.e. has no term that goes like $\frac{1}{r^2}$). This requirement follows from matching with the near field solution. For example, a term in $g_{(2,0)}^{out} \propto \frac{1}{r^2}$ results would match onto a term in g_2^{in} that scales like $\frac{1}{y^2R^2}$.

However $g_2^{in}(y,R)$ has a regular power series expansion in R and so does not have such a term [20]. The constant $k_{(2,2m)} = \mu_{(2,2m)}$, is as yet undetermined.

4.4.2 Near field region

We now turn to the solution in the inner region $r \ll 1$. As in the previous section we find it convenient to solve the equations here in the rescaled y and τ coordinates. We expand

$$\begin{aligned} f_2^{in}(y) &= \sum_{m=0}^{\infty} R^{2m} f_{(2,2m)}^{in}(y) \\ g_2^{in}(y) &= \sum_{m=0}^{\infty} R^{2m} g_{(2,2m)}^{in}(y) \\ A_2^{in}(y) &= \sum_{m=0}^{\infty} R^{2m} A_{(2,2m)}^{in}(y) \end{aligned} \tag{4.28}$$

The equations are slightly simpler when rewritten in terms of a new function

$$K_{(2,2m)}(y) = V_0(y) g_{(2,2m)}^{in}(y) + \frac{f_{(2,2m)}^{in}(y)}{V_0(y)}$$

where

$$V_0(y) = \frac{(y^2-1)(y^2-\frac{2}{3}\mu_{(0,0)}^2)}{y^4}$$

In terms of this function the final set of equations take the following form.

$$\begin{aligned} \frac{dK_{(2,2m)}(y)}{dy} &= S_{(2,2m)}^{(K)}(y) \\ \frac{d}{dy}\left(y^3 \frac{dA_{(2,2m)}^{in}(y)}{dy}\right) &= S_{(2,2m)}^{(A)}(y) + \mu_{(0,0)}\left(\frac{dK_{(2,2m)}(y)}{dy}\right) \\ \frac{d}{dy}\left(y^2 V_0(y) f_{(2,2m)}^{in}(y)\right) &= S_{(2,2m)}^{(f)}(y) + 2yK_{(2,2m)}(y) - \frac{4\mu_{(0,0)}}{3}\left(\frac{dA_{(2,2m)}^{in}(y)}{dy}\right) \end{aligned} \tag{4.29}$$

Where $S_{(2,2m)}^{(K)}(y)$, $S_{(2,2m)}^{(A)}(y)$ and $S_{(2,2m)}^{(f)}(y)$ are the source terms which depend on the solutions

[20]At higher orders in the expansion in R, similar reasoning will not set $b_{(2,2m)}$ and $h_{(2,2m)}^{(2)}$ to zero but will instead determine them by matching with $g_{(2,2m-2)}^{in}$.

at all previous orders. The most general solution to these equations is given by

$$\begin{aligned} K_{(2,2m)}(y) =& f_{(2,2m)} + \int_1^y dx\ S^{(K)}_{(2,2m)}(x) \\ A^{in}_{(2,2m)}(y) =& \tilde{h}_{(2,2m)}(1-\frac{1}{y^2}) + \int_1^y \frac{dx}{x^3}\left[\int_1^x dw\ \left(S^{(A)}_{(2,2m)}(w) + \mu_{(0,0)}S^{(K)}_{(2,2m)}(w)\right)\right] \\ f^{in}_{(2,2m)}(y) =& \frac{4\mu_{(0,0)}}{3}A^{in}_{(2,2m)}(y) + \int_1^y dx\left(S^{(f)}_{(2,2m)}(x) + 2xK_{(2,2m)}(x)\right) \end{aligned} \tag{4.30}$$

In the above solution two of the four integration constants are chosen such that the solution obeys the requirement that $A(1) = 0$ vanishes at the horizon (regularity of the gauge field at the horizon in Euclidean space) and that $f(1) = 0$ (this is the requirement that the horizon is located at $y = 1$, which follows from our definition of R). It may also be shown that the remaining two constants in the inner solution at $\mathcal{O}(R^{2k})$ may be determined by matching to the outer solution at the same order ($\mathcal{O}(R^{2k})$).

In particular, the inner solution at order R^0 is completely determined by matching with the $\mathcal{O}(R^0)$ outer solution that we have already determined above, in terms of a single unknown $\mu_{(2,0)}$. This yields the complete solution at order $\mathcal{O}(R^0)$ in terms of this one unknown number.

4.4.3 Iteration

This process may now be iterated. Our determination of the inner solution at $\mathcal{O}(R^0)$ permits an unambiguous determination of the integration constants in the outer solution at $\mathcal{O}(R^2)$. This then allows the complete determination of the inner solution at $\mathcal{O}(R^2)$, which, in turn, permits the determination of the outer solution at $\mathcal{O}(R^4)$ and so on. This procedure may be iterated indefinitely.

In summary the procedure described in this subsection permits the complete determination of the $\mathcal{O}(\epsilon^2)$ correction to our solution (order by order in R^2), as a function of the shift in the chemical potential μ at $\mathcal{O}(\epsilon^2)$, i.e. in terms of the as yet unknown numbers $\mu_{(2,2m)}$. These numbers are left undetermined by $\mathcal{O}(\epsilon^2)$ analysis, but turn out to be fixed by the requirement that there exist regular solutions of the scalar equation at $\mathcal{O}(\epsilon^3)$. This is completely analogous to the fact that the $\mathcal{O}(\epsilon^0)$ shift in the chemical potential was determined from the analysis of the scalar equation at $\mathcal{O}(\epsilon)$.

It is relatively straightforward (though increasingly tedious) to carry out our perturbative expansion to higher orders in perturbation theory. The equations at odd orders in the ϵ expansion serve to determine scalar field corrections, while equations at even orders serve to determine corrections to the metric and gauge field. In appendix C.2 we list explicit results for the correction to the metric, gauge field and scalar field at low orders in perturbation theory. We will analyse the thermodynamics of these hairy black holes in more detail in later sections.

4.5 The soliton

The hairy black hole solitons of the previous section appear in a two parameter family labelled by R and ϵ. In general our solutions may be thought of as a small RNAdS black hole surrounded by a cloud of scalar condensate. As we have described in the introduction, the limit $\epsilon \to 0$ switches off the condensate cloud. In this limit (the blue line of Fig. 4.1) hairy black holes reduce to RNAdS black holes. On the other hand, in the limit $R \to 0$ the black hole at the centre of the condensate cloud shrinks to zero size, apparently leaving behind a horizon free scalar cloud. This is indeed the case. Indeed the solitonic solutions so obtained are considerably simpler than hairy black holes, as they may be generated as a single expansion in ϵ. The linear differential equations that we encounter at every order in this process are exactly solvable without recourse to the elaborate matching procedure described in the previous section.

In this section we will directly construct the hairy black hole at $R = 0$ in a perturbation expansion in ϵ. We refer to the solution of this section as the 'soliton'. In order to construct the soliton, we search for all stationary charged solutions that are everywhere completely singularity (and horizon) free. We use global AdS as a starting point for these solutions, which we construct in a perturbative expansion in the scalar amplitude. As in the previous section we will only study rotationally invariant solutions, i.e. solutions that preserve the full $SO(4)$ symmetry group of AdS_5 .

At linear order the complete set of regular, asymptotically AdS_5, $SO(4)$ symmetric fluctuations about global AdS is given by[21]

$$\begin{aligned} &\delta\phi = \sum_n \frac{a_n e^{-i\omega_n t}}{(1+r^2)^{n+2}}\ {}_2F_1\left[-n,\ -(n+2),\ 2,\ -r^2\right], \qquad \text{with } \omega_n \equiv 4+2n-e\mathfrak{a} \\ &A_t = \mathfrak{a}, \qquad \text{and} \qquad \delta g_{\mu\nu} = \delta A_i = 0. \end{aligned} \tag{4.31}$$

The equation (4.31) is simply the most general rotationally invariant normalisable and regular solution to the equation $\partial^2\phi$. The constant $\mathfrak{a}$ in (4.31) can be set to any desired value

[21] We remind the reader that the Gauss's hypergeometric function ${}_2F_1\left[a,\ b,\ c,\ z\right]$ is a solution to the equation

$$\left[\left(z\frac{d}{dz}+a\right)\left(z\frac{d}{dz}+b\right)-\left(z\frac{d}{dz}+c\right)\frac{d}{dz}\right]{}_2F_1\left[a,\ b,\ c,\ z\right]=0$$

defined by the series

$${}_2F_1\left[a,\ b,\ c,\ z\right] \equiv \sum_{k=0}^{k=\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}$$

where the 'Pochhammer symbol' $(a)_k$ is defined by the raising factorial

$$(a)_k \equiv a(a+1)(a+2)\dots(a+k-1)$$

Note also that for n an integer, the function ${}_2F_1\left[-n,-n-2,2,\ z\right]$ is actually an n^{th} degree polynomial in the variable z.

by a choice of gauge. While (4.31) is generically time dependent, it reduces to a stationary solution when only one of the modes is turned on and $\mathfrak{a}$ is chosen accordingly.i.e.,

$$\mathfrak{a} = \frac{2k+4}{e} \qquad \text{and} \qquad a_k \propto \delta_{nk}$$

for some non-negative integer k. This yields a discrete set of stationary, one parameter, solutions to the equations of motion, labelled by their amplitude. In the rest of this section we describe the construction of the nonlinear counterparts of these solutions in a power series expansion in the amplitude. We refer to these stationary solutions as nonlinear 'solitons'.

Unlike the two parameter set of black hole solutions described in the previous section, the solitonic solutions of this subsection appear in a one parameter family, labelled by their charge; the soliton mass is a determined function of its charge. The fact that their are fewer solitonic than black hole solutions is related to the fact that the solitons we construct in this subsection have no horizons, and therefore carry no macroscopic entropy.

The ground state soliton (i.e. the soliton at $n = 0$) has a specially simple interpretation. It may be thought of as the nonlinear version of the bose condensate, that forms when a macroscopic number of scalar photons each occupies the scalar ground state 'wave function'. It also represents the $R \to 0$ limit of the hairy black hole solution of the previous section. We will construct this solution in this section, postponing discussion of excited solitons to the next section.

4.5.1 Perturbation theory for the soliton

To initiate the perturbative construction of the ground state soliton we set

$$\begin{aligned} f(r) &= 1 + r^2 + \sum_n \epsilon^{2n} f_{2n}(r) \\ g(r) &= \frac{1}{1+r^2} + \sum_{n=1}^{\infty} \epsilon^2 n g_{2n}(r) \\ A(r) &= \frac{4}{e} + \sum_{n=1}^{\infty} \epsilon^{2n} A_{2n}(r) \\ \phi(r) &= \frac{\epsilon}{(1+r^2)^2} + \sum_{n=1}^{\infty} \phi_{2n+1}(r) \epsilon^{2n+1} \end{aligned} \tag{4.32}$$

and plug these expansions into (4.5). We then expand out and solve these equations order by order in ϵ. All equations are automatically solved up to $\mathcal{O}(\epsilon)$. At order ϵ^{2n} the last equation in (4.5) is trivial while the first three take the form

$$\begin{aligned} \frac{d}{dr}\left(r^2(1+r^2)^2 g_{2n}(r)\right) &= P_{2n}^{(g)}(r) \\ \frac{d}{dr}\left(\frac{f_{2n}(r)}{1+r^2}\right) &= \frac{2(1+2r^2)}{r} g_{2n}(r) + P_{2n}^{(f)}(r) \\ \frac{d}{dr}\left(r^3 \frac{dA_{2n}(r)}{dr}\right) &= P_{2n}^{(A)}(r). \end{aligned} \tag{4.33}$$

On the other hand, at order ϵ^{2n+1} the first three equations in (4.5) is trivial while the last equation reduces to

$$\frac{d}{dr}\left(\frac{r^3}{(1+r^2)^3}\frac{d}{dr}\left[(1+r^2)^2\phi_{2n+1}(r)\right]\right) = P_{2n+1}^{(\phi)}(r) \tag{4.34}$$

Here the source terms $P_{2n}^{(g)}(r),\ P_{2n}^{(f)}(r),\ P_{2n}^{(A)}(r)$ and $P_{2n+1}^{(\phi)}(r)$ are the source terms which are completely determined by the solution to lower orders in perturbation theory, and so should be thought of as known functions, in terms of which we wish to determine the unknowns f_{2n}, g_{2n}, A_{2n} and ϕ_{2n+1}.

Note that (4.33) are identical to the equations that appear in the far field expansion of the hairy black hole solution of the previous section. This is intuitive; in the limit $R \to 0$ all of the hairy black hole spacetime lies within the far field region. The soliton is simpler to construct than the hairy black hole precisely because it has no separate near field region. The differential equations that arise, at any given order of perturbation theory, may simply be solved once and for all, with no need for an elaborate matching procedure.

The equations (4.33) are all easily integrated. It also turns out that all the integration constants in these equations are uniquely determined by the requirements of regularity, normalisability and our definition of ϵ, as we now explain.

The integration constant in the first equation of (4.33) is determined by the requirement that $g(r)$ is regular at the origin. The integration constant in the second equation is fixed by requirement of normalisability for $f_{2n}(r)$. The constant for the first of the two integrals needed to solve the third equation is fixed by the regularity of $A_{2n}(r)$ at the origin. The constant in the second integral (an additive shift in A_{2n}) is left unfixed at $\mathcal{O}(\epsilon^{2n})$ but is fixed at $\mathcal{O}(\epsilon^{2n+1})$ (see below).

The equation (4.34) is also easily solved by integration. The constant in the first integration needed to solve this equation is determined by the requirement of regularity of $\phi_{2n+1}(r)$ at the origin. Once we have fixed this constant, it turns out that the solution for ϕ_{2n+1} is generically non normalisable for every value of the second integration constant. In fact normalisability is achieved only when the previously undetermined constant shift of A_{2n} takes a specific value, a condition that determines this quantity. The constant in the last integral that determines ϕ from (4.34) is determined by our definition of ϵ which implies that

$$\phi_{2n+1} \sim \mathcal{O}(1/r^6)$$

for $n \geq 1$.

4.5.2 The soliton up to $\mathcal{O}(\epsilon^4)$

In summary, the perturbative procedure outlined in this subsection is very easily implemented to arbitrary order in perturbation theory. In fact, by automating the procedure described above, we have implemented this perturbative series to 17th order in a Mathematica program.

We present some of the results, to this order, in appendix C.3. In the rest of this subsection we content ourselves with a presentation of our results to $\mathcal{O}(\epsilon^4)$.

$$\begin{aligned} \phi(r) =& \frac{\epsilon}{\left(r^2+1\right)^2}+\frac{\epsilon^3}{63\left(r^2+1\right)^6}\Big(-e^2\left(9r^6+30r^4+34r^2+13\right)+64r^6+260r^4 \\ &+360r^2+150\Big)+\mathcal{O}(\epsilon^5) \end{aligned} \tag{4.35}$$

$$\begin{aligned} f(r) =& \left(r^2+1\right)-\frac{8\left(r^4+3r^2+3\right)\epsilon^2}{9\left(r^2+1\right)^3}+\frac{\epsilon^4}{39690\left(r^2+1\right)^7}\Big(e^2\left(6767r^{12}+48104r^{10}\right. \\ &\left.+147252r^8+256816r^6+271348r^4+163008r^2+42426\right)-32\left(2448r^{12}+17136r^{10}\right. \\ &\left.+51408r^8+86688r^6+87794r^4+50014r^2+11213\right)\Big)+\mathcal{O}(\epsilon^5) \end{aligned} \tag{4.36}$$

$$\begin{aligned} g(r) =& \frac{1}{r^2+1}+\frac{8r^2\left(r^2+3\right)\epsilon^2}{9\left(r^2+1\right)^5}-\frac{\epsilon^4}{39690\left(r^2+1\right)^9}\Big(r^2\left(e^2\left(6767r^{10}+48104r^8+147252r^6\right.\right. \\ &\left.+229600r^4+180460r^2+58800\right)-64\left(1224r^{10}+8568r^8+26194r^6+43260r^4\right. \\ &\left.\left.+37065r^2+11025\right)\right)\Big)+\mathcal{O}(\epsilon^5) \end{aligned} \tag{4.37}$$

$$\begin{aligned} A(r) =& \frac{4}{e}+\epsilon^2\left(-\frac{e}{6r^2}+\frac{e}{6r^2\left(r^2+1\right)^3}+\frac{3e}{14}-\frac{32}{21e}\right)+\epsilon^4\Bigg(\frac{1}{105840r^2\left(r^2+1\right)^7}\Big(e^3\left(945r^8\right. \\ &\left.+315r^6-5691r^4-8917r^2-3856\right)+16e\left(241e^2-2658\right)\left(r^2+1\right)^7 \\ &-32e\left(210r^8+21r^6-1967r^4-3527r^2-1329\right)\Big) \\ &-\frac{6383817e^4-122400480e^2+574944256}{97796160e}\Bigg)+\mathcal{O}(\epsilon^5) \end{aligned} \tag{4.38}$$

4.5.3 Excited state solitons and hairy black holes

As explained around (4.31), the stationary solitonic solution constructed in the previous section is simply one (albeit a special one, as it has the smallest mass to charge ratio) of an infinite class of stationary solitonic solutions, each of which may be constructed in a perturbative expansion in ϵ, exactly as in the previous section. We label solitonic solutions by an integer n; the n^{th} stationary soliton has chemical potential $\mu = 4 + 2n$ at small amplitude.

We have explicitly constructed the excited solitons with $n = 1$ and $n = 2$ up to a high order in perturbation theory. In appendix C.4 below we present some of the details of our results at low orders. Further, we can further construct a large class of excited hairy black holes which reduce to these excited state solitons as their horizon size goes to zero. These black holes may be thought of as a mixture of the excited solitons and a small RNAdS black holes with $\mu \approx \frac{4+2n}{e}$. In appendix C.5, we construct this excited state hairy black hole at $n = 1$. We have a simple program in Mathematica that may be used to generate the excited hairy black hole solution at any fixed value of n. It should prove possible to generalise this construction once and for all at arbitrary n, but we have not attempted this generalisation. We will present a detailed analysis of the thermodynamics (and stability) of the excited solitons and black holes in the next section.

4.6 Thermodynamics in the microcanonical ensemble

In this section we compare the entropies of the various solutions constructed as a function of their mass and charge. We find it convenient to present all formulae in terms of the rescaled mass m and the rescaled charge q. The physical mass and charge of the system, M and Q, differ from m and q by the rescaling

$$\begin{aligned} Q &= \frac{\pi q}{2} \\ M &= \frac{3\pi}{8} m \end{aligned} \tag{4.39}$$

The grand canonical partition function for the system is defined by the formula

$$Z_{GC} = \mathrm{Tr} \exp\left[-T^{-1}\left\{M - \mu Q\right\}\right].$$

where T is the temperature and μ is the chemical potential.

4.6.1 RNAdS black hole

The basic thermodynamics for an RNAdS black hole is summarised by the following formulae[22]

$$\begin{aligned}
M &\equiv \frac{3\pi}{8}m = \frac{3\pi}{8}R^2\left[1+R^2+\frac{2}{3}\frac{q^2}{R^4}\right] \\
&= \frac{3\pi}{8}R^2\left[1+R^2+\frac{2}{3}\left(\frac{2Q}{\pi R^2}\right)^2\right] = \frac{3\pi}{8}R^2\left[1+R^2+\frac{2}{3}\mu^2\right] \\
Q &\equiv \frac{\pi}{2}q = \frac{\pi}{2}\mu R^2 \\
S &= \frac{A_H}{4} = \frac{1}{4}(2\pi^2 R^3) = \frac{\pi^2}{2}R^3 \\
T &= \frac{V'(R)}{4\pi} = \frac{1}{2\pi R}\left[1+2R^2-\frac{2}{3}\frac{q^2}{R^4}\right] \\
&= \frac{1}{2\pi R}\left[1+2R^2-\frac{2}{3}\left(\frac{2Q}{\pi R^2}\right)^2\right] = \frac{1}{2\pi R}\left[1+2R^2-\frac{2}{3}\mu^2\right] \\
\mu &= A_t^{(r=\infty)} - A_t^{(r=R)} = \frac{q}{R^2} = \frac{2Q}{\pi R^2}.
\end{aligned} \tag{4.40}$$

were Q is the charge, M is the mass of the black hole, S is its entropy, T its temperature and μ its chemical potential.We use the symbol A_H to denote the area of the outer horizon. The condition for R to be the outer horizon radius is

$$\frac{q^2}{R^4} = \mu^2 \leq \frac{3}{2}(1+2R^2). \tag{4.41}$$

We are mainly interested in small RNAdS black holes with $R \ll 1$. and the thermodynamic expressions can be simplified in this limit. The mass of RNAdS black holes at fixed charge is bounded from below; at small m and q, we have

$$m \geq 2\sqrt{2/3}q + \left(\sqrt{2/3}q\right)^2 - \left(\sqrt{2/3}q\right)^3 + \mathcal{O}(q^4)$$

For every pair (m,q) that obeys this inequality, there exists a unique black hole solution.

At small mass and charge (with mass and charge taken to be of the same order) the entropy and the radius of the black hole is given by

$$\begin{aligned}
S &= \frac{\pi^2 R^3}{2} \\
R^2 &= \frac{m+\sqrt{m^2-\frac{8}{3}q^2}}{2} + \mathcal{O}(m^2, q^2, mq)
\end{aligned} \tag{4.42}$$

[22] All throughout, we find it convenient to consistently omit a factor of G_5^{-1} from all our extensive quantities.

At leading order in mass and charge, the chemical potentials of these black holes are given as solutions to the equation

$$\frac{m}{q} = \frac{1}{\mu}\left(1 + \frac{2\mu^2}{3}\right) \tag{4.43}$$

while the temperature is given by

$$T = \frac{1}{2\pi R}\left[\frac{m}{R^2} - \frac{4q^2}{3R^4}\right] = \frac{1}{2\pi R}\left[1 - \frac{2q^2}{3R^4}\right] \tag{4.44}$$

where R^2 is given in (4.42).

4.6.2 The soliton - ground and excited states

Using the soliton solution in the previous section, the mass of the ground state soliton can be easily determined as a function of its charge. We find

$$m = \frac{16q}{3e} + \frac{2}{21}\left(9 - \frac{64}{e^2}\right)q^2 + O\left(q^3\right) \tag{4.45}$$

In appendix C.3.1, we give the relation of the mass and the charge implicitly up to higher orders.

Upon continuing to Euclidean space, our soliton yields a regular solution for arbitrary periodicity of the Euclidean time coordinate (it is similar to global AdS spacetime in this respect). It follows that, within the classical gravity approximation, this soliton can be in thermodynamical equilibrium at arbitrary temperature. As the soliton has no horizon, its entropy vanishes in the classical gravity approximation. This implies that the free energy of the soliton is equal to its mass.

The chemical potential of the soliton is given by the value of the gauge potential at infinity

$$\mu = \frac{4}{e} + \left(\frac{9}{7} - \frac{64}{7e^2}\right)q + O\left(q^2\right) \tag{4.46}$$

Note that the coefficient of q in the formula above is positive when $e^2 > \frac{32}{3} \equiv e_c^2$ so that the chemical potential of the soliton increases with charge whenever hairy black holes exist. It is plausible that the only classical gravity state in the system with $\mu < \frac{4}{e}$ is the vacuum (or more precisely a thermal gas about the vacuum; this gas is absent in classical gravity).

The grand free energy of the ground state soliton is given by

$$G(\mu) \equiv M - TS - \mu Q = -\frac{343\pi e^2(e\mu - 4)^4}{4\left(9e^2 - 64\right)^3} + \mathcal{O}\left((\mu - 4/e)^3\right). \tag{4.47}$$

The above analysis is easily generalised to excited state solitons. For the general excited state solitons, we present thermodynamical formulae only at leading order. The mass and chemical potential of the soliton are given by

$$\begin{aligned} m &= \frac{4(4+2n)q}{3e} + \mathcal{O}(q^2) \\ \mu &= \frac{4+2n}{e} + \mathcal{O}(q) \end{aligned} \tag{4.48}$$

We have explicitly constructed the excited solitons with $n = 1$ and $n = 2$ up to a high order in perturbation theory. In appendix C.4 we present some of the details of our results at low orders. This allows us to give the thermodynamic formulae for these cases up to a higher order At $n = 1$ we find

$$\begin{aligned} M(q) &= \frac{3\pi q}{e} + \frac{1}{308}\pi\left(109 - \frac{2544}{e^2}\right)q^2 + \mathcal{O}\left(q^3\right), \\ \mu &= \frac{6}{e} + \frac{1}{77}\left(109 - \frac{2544}{e^2}\right)q + O\left(q^2\right) \\ G(\mu) &= -\frac{77\pi(e\mu-6)^2}{436e^2 - 10176} + \mathcal{O}\left(\mu - \frac{6}{e}\right)^3. \end{aligned} \tag{4.49}$$

while at $n = 2$

$$\begin{aligned} M(q) &= \frac{4\pi q}{e} + \frac{\pi\left(4741 - \frac{228352}{e^2}\right)q^2}{12012} + O\left(q^3\right) \\ \mu(q) &= \frac{8}{e} + \frac{\left(4741 - \frac{228352}{e^2}\right)q}{3003} + O\left(q^2\right) \\ G(\mu) &= -\frac{3003\pi(e\mu-8)^2}{18964e^2 - 913408} + \mathcal{O}\left(\mu - \frac{8}{e}\right)^3 \end{aligned} \tag{4.50}$$

Note that the coefficient of q in the expansion of μ in the expansion of (4.49) is positive whenever $e^2 > 24$ so that the first excited hairy black hole exists. On the other hand, the coefficient of q in the expansion of μ is negative at $e^2 = \frac{128}{3}$, the threshold for the existence of the second excited hairy black hole (see below).

4.6.3 Dynamical stability of the solitons

In this subsection, we comment on the dynamical stability of the excited solitons. In particular we will prove below that the spectrum of small fluctuations about excited solitons have no $SO(4)$ symmetric exponentially growing modes, within the ϵ perturbative expansion. This result suggests (but does not strictly prove [90, 91]) that small excited state solitons are all dynamically stable against small fluctuations.

As we have mentioned above, the normal modes of the scalar field constitute the only $SO(4)$ symmetric fluctuations of (4.1) about global AdS spacetime. At small ϵ the solitonic solution is everywhere a small perturbation around global AdS spacetime. It follows that the $SO(4)$ symmetric perturbations about the soliton, at small ϵ, are small perturbations of spherically symmetric scalar normal modes about global AdS spacetime. These modes obey the equation

$$D^2\phi = 0 \tag{4.51}$$

where D is the gauge covariant derivative about the soliton background. We study perturbations of the form

$$\phi(r,t) = \psi(r)e^{-i\omega t}$$

and wish to investigate whether the frequencies ω (which are all real about global AdS) can develop a small imaginary piece about the solitonic background. We will now demonstrate that this is impossible in the ϵ expansion. To establish this we multiply the equation (4.51) by ϕ^* and integrate the resultant scalar over AdS spacetime. We find

$$\int \sqrt{g}|g^{00}|\left(\omega - eA_t(r)\right)^2 |\psi|^2 = \int \sqrt{g}g^{rr}|\partial_r\psi|^2$$

Here $g_{\mu\nu}$ is the soliton metric and $A_t(r)$ is the gauge field of the solitonic solution.

Now recall that $A_t = \frac{4}{e} + \mathcal{O}(\epsilon^2)$. It follows that

$$(\omega - 4)^2 = \frac{\int \sqrt{g}g^{rr}|\partial_r\psi|^2}{\int \sqrt{g}|g^{00}||\psi|^2} + \mathcal{O}(\epsilon^2).$$

As the leading term on the RHS is $\mathcal{O}(\epsilon^0)$ and positive, it follows that ω is real within the ϵ expansion. Consequently the spectrum of spherically symmetric small fluctuations about the soliton background does not have exponentially growing modes in the ϵ expansion. This suggests that all excited solitons are classically stable. We find this result surprising, and think that it warrants further study.

4.6.4 Massive scalar: Hairy black hole thermodynamics

We concluded at the end of the previous subsection that the excited solitons seem to be classically stable. In contrast, our calculations in the appendix C.1 indicate that the RNAdS black hole can become superradiantly unstable. It is an interesting question to ask what is the thermodynamics of the resultant hairy black hole. Since, we have an explicit construction of the hairy black hole solutions, we can directly go ahead to compute the thermodynamic quantities for these hairy solutions. Before doing that however we wish to present an argument in this subsection which gives us some intuition about the kind of thermodynamics we should expect at the leading order.

We will present this argument in a slightly more general framework than we have been working till now - we wish to consider the effect of adding a scalar mass term to the Lagrangian(4.1).,i.e., we work with a more general system

$$\begin{aligned} S &= \frac{1}{8\pi G_5}\int d^5x\sqrt{g}\left[\frac{1}{2}\left(\mathcal{R}[g] + 12\right) - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - |D_\mu\phi|^2 - m_\phi^2|\phi|^2\right] \\ \mathcal{F}_{\mu\nu} &\equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu \qquad \text{and} \qquad D_\mu\phi \equiv \nabla_\mu\phi - ieA_\mu\phi \end{aligned} \tag{4.52}$$

This system has a minimally coupled charge scalar with mass m_ϕ and charge e in AdS_5. By the standard rules of AdS/CFT, the dual boundary operator $\mathcal{O}_\phi$ has a scaling dimension

$$\Delta_0 = \left[d/2 + \sqrt{(d/2)^2 + m_\phi^2}\,\right]_{d=4} = 2 + \sqrt{4 + m_\phi^2}$$

In a gauge where $A_t^{r=\infty} = 0$, this is also the energy of the lowest ϕ mode in vacuum AdS_5. For the case $m_\phi = 0$, this reduces to $\Delta_0 = 4$.

The other spherically symmetric modes of ϕ (dual to the descendants $\partial^{2n}\mathcal{O}_\phi$)have an energy

$$\Delta_n \equiv \Delta_0 + 2n = 2 + \sqrt{4 + m_\phi^2} + 2n$$

For the case $m_\phi = 0$, $\Delta_n = 4 + 2n$. Hence, in a gauge where $A_t^{r=\infty} = 0$, the energy of the n-th excited state is also given by Δ_n. We can form a non-linear bose condensate by dumping a charge Q_{sol} into this n-th excited state - this is equivalent to populating this excited state with Q_{sol}/e number of charged bosons. To the leading order, where we neglect self-interaction between these bosons, the mass of such a soliton is given by

$$M_{sol} = \frac{Q_{sol}}{e}\Delta_n + \mathcal{O}(Q_{sol}^2)$$

This sets the chemical potential of the soliton to be

$$\mu_{sol} \equiv \frac{\partial M_{sol}}{\partial Q_{sol}} = \frac{\Delta_n}{e} + \mathcal{O}(Q_{sol})$$

The entropy of such a solution is zero $S_{sol} = 0$. This in particular means that within this approximation, this solution exists at arbitrary temperatures T_{sol}.

Now, let us form a hairy black hole by placing at the core of this non-linear bose condensate a small ordinary RNAdS black hole with a small outer horizon radius R and chemical potential μ_{BH}. Such a black hole has a mass

$$M_{BH} = \frac{3\pi}{8}R^2\left[1 + \frac{2}{3}\mu_{BH}^2\right] + \mathcal{O}(R)^4$$

a charge

$$Q_{BH} = \frac{\pi}{2}\mu_{BH}R^2$$

an entropy

$$S_{BH} = \frac{1}{4}(2\pi^2 R^3) = \frac{\pi^2}{2}R^3$$

and a temperature

$$T_{BH} = \frac{1}{2\pi R}\left[1 - \frac{2}{3}\mu_{BH}^2\right] + \mathcal{O}(R)$$

If the number of bosons Q_{sol}/e is small, then the condensate outside is a small perturbation on the RNAdS black hole. And if the radius R of the black hole is small, then it is a small perturbation on the soliton on length scales large compared to R. Hence, if both these conditions are met, it is legitimate at the leading order to assume that there is no interaction between the core and the condensate parts of the hairy black hole. In this regime, since the core and the condensate can still exchange charge and energy, all that is needed for a stationary solution is that the core and the condensate be at a thermal and chemical equilibrium,i.e.,

$$\bar{T} = T_{sol} = T_{BH} = \frac{1}{2\pi R}\left[1 - \frac{2}{3}\mu_{BH}^2\right] + \mathcal{O}(R)$$

and

$$\bar{\mu} = \mu_{BH} = \mu_{sol} = \frac{\Delta_n}{e} + \mathcal{O}(Q_{sol})$$

Using these equilibrium conditions we want to figure out the 'mole fractions' of these two phases at equilibrium as a function of total mass and charge

$$M = M_{sol} + M_{BH} \qquad \text{and} \qquad Q = Q_{sol} + Q_{BH}$$

This is easily done and we get the mass fractions of the core and the condensate are given by

$$\begin{aligned}
M_{BH} &= \frac{(1+\frac{2}{3}\bar{\mu}^2)}{(1-\frac{2}{3}\bar{\mu}^2)}(M - \bar{\mu}\ Q) + \mathcal{O}\left(M^2, Q^2, MQ\right)\\
m_{BH} &\equiv \frac{3}{8\pi}M_{BH} = \frac{(1+\frac{2}{3}\bar{\mu}^2)}{(1-\frac{2}{3}\bar{\mu}^2)}(m - \frac{4}{3}\bar{\mu}\ q) + \mathcal{O}\left(m^2, q^2, mq\right)\\
&= \frac{(1+\frac{2\Delta_n^2}{3e^2})}{(1-\frac{2\Delta_n^2}{3e^2})}(m - \frac{4}{3}\bar{\mu}\ q) + \mathcal{O}\left(m^2, q^2, mq\right)\\
M_{sol} &= \frac{(1+\frac{2}{3}\bar{\mu}^2)\bar{\mu}\ Q - \frac{4}{3}\bar{\mu}^2\ M}{(1-\frac{2}{3}\bar{\mu}^2)} + \mathcal{O}\left(M^2, Q^2, MQ\right)\\
m_{sol} &\equiv \frac{3}{8\pi}M_{sol} = \frac{4\bar{\mu}}{3}\frac{(1+\frac{2}{3}\bar{\mu}^2)\ q - \bar{\mu}\ m}{(1-\frac{2}{3}\bar{\mu}^2)} + \mathcal{O}\left(m^2, q^2, mq\right)\\
&= \frac{4\Delta_n}{3e}\frac{(1+\frac{2\Delta_n^2}{3e^2})\ q - \frac{\Delta_n}{e}\ m}{(1-\frac{2\Delta_n^2}{3e^2})} + \mathcal{O}\left(m^2, q^2, mq\right)
\end{aligned} \tag{4.53}$$

The charge fractions of the core and the condensate are given by

$$\begin{aligned}
Q_{BH} &= \frac{4\bar{\mu}}{3}\frac{(M-\bar{\mu}\ Q)}{(1-\frac{2}{3}\bar{\mu}^2)} + \mathcal{O}\left(M^2,Q^2,MQ\right) \\
q_{BH} \equiv \frac{2}{\pi}Q_{BH} &= \bar{\mu}\frac{(m-\frac{4\bar{\mu}}{3}\ q)}{(1-\frac{2}{3}\bar{\mu}^2)} + \mathcal{O}\left(m^2,q^2,mq\right) \\
&= \frac{\Delta_n}{e}\frac{(m-\frac{4\Delta_n}{3e}\ q)}{(1-\frac{2\Delta_n^2}{3e^2})} + \mathcal{O}\left(m^2,q^2,mq\right) \\
Q_{sol} &= \frac{(1+\frac{2}{3}\bar{\mu}^2)\ Q - \frac{4}{3}\bar{\mu}\ M}{(1-\frac{2}{3}\bar{\mu}^2)} + \mathcal{O}\left(M^2,Q^2,MQ\right) \\
q_{sol} \equiv \frac{2}{\pi}Q_{sol} &= \frac{(1+\frac{2}{3}\bar{\mu}^2)\ q - \bar{\mu}\ m}{(1-\frac{2}{3}\bar{\mu}^2)} + \mathcal{O}\left(m^2,q^2,mq\right) \\
&= \frac{(1+\frac{2\Delta_n^2}{3e^2})\ q - \frac{\Delta_n}{e}\ m}{(1-\frac{2\Delta_n^2}{3e^2})} + \mathcal{O}\left(m^2,q^2,mq\right)
\end{aligned} \tag{4.54}$$

The radius of the black hole at the core is

$$\begin{aligned}
R &= \left[\frac{8}{3\pi}\frac{(M-\bar{\mu}\ Q)}{(1-\frac{2}{3}\bar{\mu}^2)} + \mathcal{O}\left(M^2,Q^2,MQ\right)\right]^{1/2} = \left[\frac{m-\frac{4}{3}\bar{\mu}\ q}{1-\frac{2}{3}\bar{\mu}^2} + \mathcal{O}\left(m^2,q^2,mq\right)\right]^{1/2} \\
&= \left[\frac{m-\frac{4\Delta_n}{3e}\ q}{1-\frac{2\Delta_n^2}{3e^2}} + \mathcal{O}\left(m^2,q^2,mq\right)\right]^{1/2}
\end{aligned} \tag{4.55}$$

and the entropy of the hairy black hole is given by

$$\begin{aligned}
S = \frac{\pi^2}{2}R^3 &= \frac{\pi^2}{2}\left[\frac{8}{3\pi}\frac{(M-\bar{\mu}\ Q)}{(1-\frac{2}{3}\bar{\mu}^2)} + \mathcal{O}\left(M^2,Q^2,MQ\right)\right]^{3/2} \\
&= \frac{\pi^2}{2}\left[\frac{m-\frac{4}{3}\bar{\mu}\ q}{1-\frac{2}{3}\bar{\mu}^2} + \mathcal{O}\left(m^2,q^2,mq\right)\right]^{3/2} \\
&= \frac{\pi^2}{2}\left[\frac{m-\frac{4\Delta_n}{3e}\ q}{1-\frac{2\Delta_n^2}{3e^2}} + \mathcal{O}\left(m^2,q^2,mq\right)\right]^{3/2}
\end{aligned} \tag{4.56}$$

The existence region of the hairy black holes is in between where the hairy black hole coincides with the RNAdS black hole on one side and where it coincides with the pure soliton on the other side. This gives the existence region as

$$\begin{aligned}
\frac{3e}{4\Delta_n}\left(1+\frac{2}{3}\frac{\Delta_n^2}{e^2}\right)Q + \mathcal{O}(Q^2) \geq M \geq \frac{\Delta_n}{e}Q + \mathcal{O}(Q^2) \\
\frac{e}{\Delta_n}\left(1+\frac{2}{3}\frac{\Delta_n^2}{e^2}\right)q + \mathcal{O}(q^2) \geq m \geq \frac{4}{3}\frac{\Delta_n}{e}q + \mathcal{O}(q^2)
\end{aligned} \tag{4.57}$$

and this happens only if

$$e \geq \sqrt{2/3}\Delta_n = \frac{\Delta_n}{\mu_c} \equiv e_c$$

where e_c is the critical charge above which the pure black hole becomes superradiantly unstable to radiation in the n-th excited state.

In this regime, the upper bound on M is a decreasing function of Δ_n whereas the lower bound is an increasing function of Δ_n. This implies that the existence region of n-th excited state hairy black hole is entirely inside the existence region of $(n-1)^{th}$ excited state black hole (see Fig. 4.2). Further, in this regime, one can show that the radius R is a decreasing function of Δ_n. Hence, the higher excited state hairy black holes have smaller cores and consequently are entropically subdominant to the lower excited state hairy black holes.

Armed with the above intuition, in the next few sections, we will derive the thermodynamics of hairy black holes with $m_\phi = 0$ directly from our solutions and show that their leading order behaviour is captured by the kind of non-interaction arguments that we have presented in this subsection.

4.6.5 Ground state hairy black hole

Once we have our solutions for hairy black holes from appendix C.2, the evaluation of their thermodynamic charges and potentials is a straight forward exercise. At low orders in the perturbative expansion we find[23]

[23] All throughout, we find it convenient to consistently omit a factor of G_5^{-1} from all our extensive quantities.

$$
\begin{aligned}
M =& \frac{3\pi}{8}\Bigg(\left[\left(1+\frac{32}{3e^2}\right)R^2 - \left(-1-\frac{32}{e^2}+\frac{1024}{3e^4}\right)R^4 + \mathcal{O}(R)^6\right] \\
&+ \epsilon^2\left[\frac{8}{9} - \left(\frac{1016}{189} - \frac{21760}{189e^2}\right)R^2 + \mathcal{O}(R)^4\right]\Bigg) + \mathcal{O}(\epsilon)^4 \\
Q =& \frac{\pi}{2}\Bigg(\left[\frac{4R^2}{e} - \left(\frac{64}{e^3} - \frac{6}{e}\right)R^4 + \mathcal{O}(R)^6\right] \\
&+ \epsilon^2\left[\frac{e}{6} - \left(\frac{317e}{252} - \frac{1528}{63e}\right)R^2 + \mathcal{O}(R)^4\right]\Bigg) + \mathcal{O}(\epsilon)^4 \\
\mu =& \Bigg[\frac{4}{e} + R^2\left(\frac{6}{e} - \frac{64}{e^3}\right) + R^4\Bigg(-\frac{21}{2e} - \frac{736}{3e^3} + \frac{40448}{9e^5} \\
&- \frac{256\log\left(1-\frac{32}{3e^2}\right)}{e^3} + \frac{8192\log\left(1-\frac{32}{3e^2}\right)}{3e^5} - \frac{512\log(R)}{e^3} + \frac{16384\log(R)}{3e^5}\Bigg) + \mathcal{O}(R)^6\Bigg] \\
&+ \epsilon^2\left[\left(\frac{9e^2-64}{42e}\right) + R^2\left(-\frac{75969e^4 - 2256672e^2 + 13746176}{26460e^3}\right) + \mathcal{O}(R)^4\right] + \mathcal{O}(\epsilon)^4 \\
T =& \frac{1}{4\pi R}\Bigg(\left[\left(2-\frac{64}{3e^2}\right) + \left(\frac{64\left(32-3e^2\right)}{3e^4} + 4\right)R^2 + \mathcal{O}(R)^4\right] + \epsilon^2\Bigg[\frac{8\left(e^2-32\right)}{21e^2} \\
&- R^2\Bigg(\frac{256\left(13357e^2 - 157376\right)}{6615e^4} + \frac{2048\left(3e^2-32\right)\left(\log\left(e^2-\frac{32}{3}\right) + 2\log\left(\frac{R}{e}\right)\right)}{27e^4}\Bigg) \\
&+ \mathcal{O}(R)^4\Bigg]\Bigg) + \mathcal{O}(\epsilon)^4
\end{aligned}
\tag{4.58}
$$

It may be verified that these quantities obey the first law of thermodynamics

$$dM = TdS + \mu dQ.$$

Equation (4.58) above lists formulae for the mass and charge of small hairy black holes as a function of R and ϵ. Inverting these relations we find

$$
\begin{aligned}
R^2 &= \left(\frac{e}{3e^2-32}\right)(3em - 16q) + \mathcal{O}(m^2, q^2, mq) \\
\epsilon^2 &= \left(\frac{6}{e\left(3e^2-32\right)}\right)\left[\left(3e^2+32\right)q - 12em\right] + \mathcal{O}(m^2, q^2, mq)
\end{aligned}
\tag{4.59}
$$

Hairy black holes exist for all positive values of R and ϵ. Of course R^2 and ϵ^2 are positive; this implies that the mass and charge of hairy black holes vary over the range (4.3). As we have mentioned in the introduction, it is possible to satisfy this inequality only when $e \geq \sqrt{\frac{32}{3}} = e_c$. Assuming this is the case, we have hairy black hole solutions only within the band (4.3). At the upper end of the band the solution reduces to a RNAdS black hole ($\epsilon = 0$). At the lower end of the band the solution reduces to the soliton with $R = 0$.

It is now a simple matter to plug (4.59) into (4.58) to determine the entropy, temperature and chemical potential of the black hole as a function of its mass and charge. We find

$$\begin{aligned} S =& \frac{\pi^2}{2} R^3 \\ =& \frac{\pi^2}{2} \left(\frac{e\left(3em - 16q\right)}{3e^2 - 32} \right)^{\frac{3}{2}} \Bigg[\\ & 1 + \frac{9}{14e\left(32 - 3e^2\right)^2 \left(3em - 16q\right)} \Big(3e^2 \left(21e^4 - 384e^2 + 5120\right) m^2 \\ & + 2\left(27e^6 - 64e^4 - 1024e^2 + 131072\right) q^2 pK - 8e\left(75e^4 - 1152e^2 + 17408\right) mq \Big) \\ & + \mathcal{O}(m^2, q^2, mq) \Bigg] \\ T =& \frac{2\left(3e^2 - 32\right)^{3/2}}{3\sqrt{e^5(3em - 16q)}} \left[1 + \mathcal{O}(m, q)\right] \\ \mu =& \frac{4}{e} + \frac{\left(576e - 18e^3\right) m + \left(-27e^4 + 576e^2 - 5120\right) q}{224e^2 - 21e^4} + \mathcal{O}(m^2, q^2, mq) \end{aligned} \tag{4.60}$$

As we have explained in the previous subsection, at leading order, these formulae have a very simple and intuitive explanation in terms of a noninteracting mixture of a RNAdS black hole and the ground state soliton. It is easily checked that the formulae in this section agree with the expressions that we had derived before if we put $\Delta_n = 4$.

4.6.6 Excited hairy black holes

We begin by reporting some of the basic thermodynamical formulae that follow from the formulae presented in appendix C.5. For the mass, charge and chemical potential we find

$$\begin{aligned} m = & \left(\left(1 + \frac{24}{e^2}\right) R^2 + \left(1 - \frac{304}{e^2} - \frac{1536}{e^4}\right) R^4 + O\left(R^5\right) \right) \\ & + \left(\frac{2}{9} + \left(\frac{26948}{231e^2} - \frac{8111}{1386} \right) R^2 + O\left(R^5\right) \right) \epsilon^2 + O\left(\epsilon^3\right) \end{aligned} \tag{4.61}$$

$$\begin{aligned} q = & \left(\frac{6R^2}{e} + \left(\frac{12}{e} - \frac{432}{e^3} \right) R^4 + O\left(R^5\right) \right) \\ & + \left(\frac{e}{36} + \left(\frac{7661}{462e} - \frac{1061e}{3696} \right) R^2 + O\left(R^5\right) \right) \epsilon^2 + O\left(\epsilon^3\right) \end{aligned} \tag{4.62}$$

$$\mu = \left(\frac{6}{e} + \left(\frac{12}{e} - \frac{432}{e^3} \right) R^2 + O(R)^4 \right) + \epsilon^2 \left(\left(\frac{109e}{2772} - \frac{212}{231e} \right) + O(R)^2 \right) + O\left(\epsilon^3\right) \tag{4.63}$$

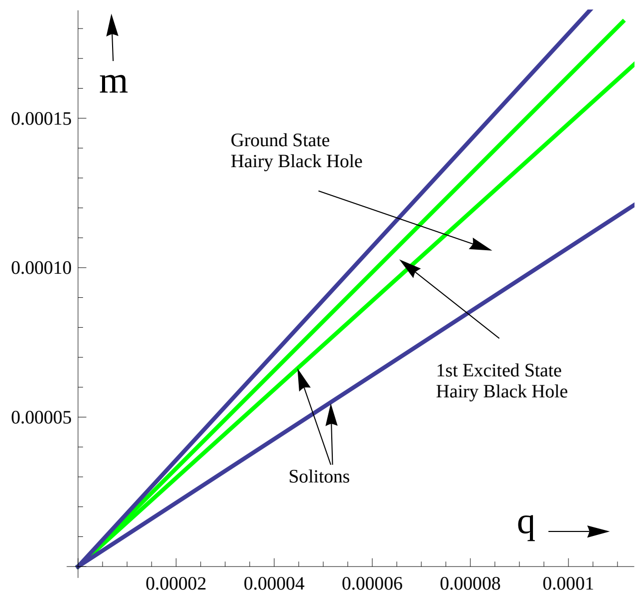

Figure 4.2: Microcanonical ensemble for $e = 5.4$: Existence region of ground state and excited state hairy black holes.

The n^{th} excited state is a small deformation of a small RNAdS black hole at $\mu = \frac{4+2n}{e}$. RNAdS black holes at this value of the chemical potential have $n-1$ superradiant instabilities. It follows that the n^{th} excited hairy black hole also has $n-1$ unstable linear fluctuation modes, which tend to flow the black hole to lower excited (generically ground state) hairy black holes.

The n^{th} excited hairy black hole exists only when $e^2 \geq \frac{2(4+n)^2}{e}$. When this condition is fulfilled, the n^{th} excited state black hole exists only when

$$\frac{8}{3}(n+2)q + \mathcal{O}(q^2) \leq m \leq \frac{(3e^2 + 8(n+2)^2)\,q}{6e(n+2)} + \mathcal{O}(q^2)$$

This is completely in accordance with our non-interaction argument as expected.

4.7 Discussion

In the work discussed in this chapter we have demonstrated that very small charged hairy black holes of the Lagrangian (4.1) are extremely simple objects. To leading order in an expansion of the mass and charge, these objects may be thought of as an *non interacting* superposition of a small RNAdS black hole and a charged soliton. The different components of this mixture interact only weakly for two related reasons. The black hole does not affect the soliton because it is parametrically smaller than the soliton. The soliton does not backreact on the black hole because its energy density is parametrically small.

We have constructed an infinite class of hairy black hole solutions labelled by a single parameter n. Excepting the ground state hairy black hole, each of these solutions is classically unstable. The time scale for this instability is proportional to the area of the RNAdS black hole that sits inside the hairy solution, and goes to zero in the limit that this area goes to zero. In particular all excited state solitons could well be stable configurations (see 4.6.3). We find the likely existence of an infinite number of classically stable classical solutions surprising, and do not have a good feeling for the implications of this observation. Of course these excited solitons will all eventually decay to the ground state hairy black hole via quantum tunnelling, but the rate for this decay will be exponentially suppressed. It would be interesting to construct the instanton that mediates this decay process.

We have constructed hairy black holes in a perturbative expansion in their mass and charge. As we increase the mass and charge, the soliton and the black hole begin to interact with each other. At large mass and charge (where this system has been intensively previously investigated) there is probably no sense in which the hairy black hole can usefully be regarded as a mix of two independent entities. In fact we suspect that the soliton does not even exist as an independent object at large enough charge. It is very natural to wonder how the phase diagram of Fig. 4.1 continues to large mass and charge. We sketch one possibility for this continuation in Fig. 4.3

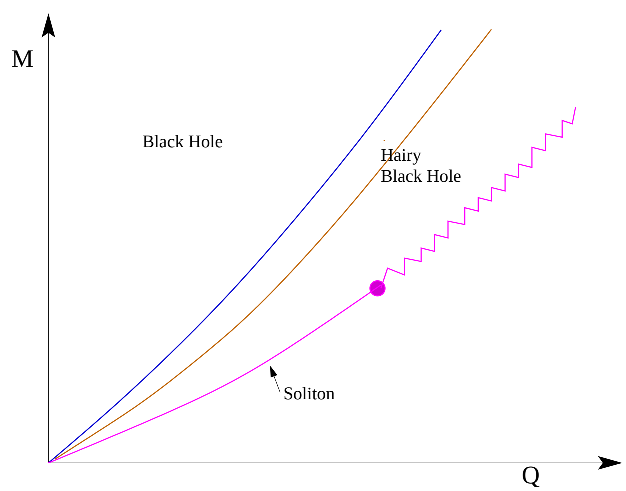

Figure 4.3: Proposed microcanonical phase diagram for large M and Q.

The main point of interest of the conjectured phase diagram of Fig. 4.3 is the lower edge of the diagram. As we have explained below, the hairy black hole phase is bounded from below by the solitonic solution at small mass and charge. The temperature of the hairy black hole also tends to infinity as we approach this line. On the other hand, it seems plausible that the solitonic solution goes singular past its 'Chandrasekhar' critical charge $q = q_c$ (the dot in Fig. 4.3). If this is indeed the case, it is of great interest to know the nature of the lower bound of Fig. 4.3 at higher values of the charge (the jagged line in Fig. 4.3). We get some information from studies of the system in the Poincaré patch limit (see for instance [94, 95, 96]), so asymptotically large charges in Fig. 4.3. In this limit prior studies appear to suggest that the jagged line is the zero temperature limit of hairy black brane solution[24]. If the phase diagram of Fig. 4.3 is indeed correct, it is natural to suppose that the jagged line everywhere represents the zero temperature limit of a hairy black hole. This suggests that the neighbourhood of the limiting soliton solution (the dot in Fig. 4.3) is extremely interesting. Hairy black holes on the solid lower line in the neighbourhood of this dot are at infinite temperature. On the other hand, points on the jagged line in the neighbourhood of this dot are at zero temperature. As we circle the dot from the solid to the jagged line we presumably pass through all temperatures in between. While all this is very speculative, it would be very interesting to investigate it further.

Although all the analysis presented here focused on the concrete and especially simple context of charged black holes in the system (4.1), the basic physical picture of small hairy black holes as a linear combination of approximately non interacting pieces is based on very general considerations, should apply equally to the study of any bulk gravitational asymptotically AdS system that hosts RNAdS black holes which suffer from a superradiant instability. It should be straightforward to generalise the calculations of this paper to systems in which (4.1) is modified by the addition of a potential for the scalar field, and/ or is studied in different dimensions. It may also be possible to generalise the constructions presented here to superradiant instabilities in charged rotating black holes [97, 98].

As we have explained in section 4.2, at leading order in perturbation theory, small hairy black holes exist for $e^2 > \frac{32}{3}$, but do not exist when $e^2 < \frac{32}{3}$. The case $e^2 = \frac{32}{3}$ lies on the edge, and is particularly interesting. The question of whether hairy black holes exist at this critical value is determined by a second rather than first order calculation. In the system under study here, it turns out that hairy black holes do exist for $e^2 < \frac{32}{3}$. One way of understanding this statement goes as follows.

At any fixed value of the charge q, hairy black holes exist if and only if $e^2 \geq e_c^2(q)$ where

$$e_c^2(q) = \frac{32}{3} - \frac{16}{21}q + \mathcal{O}(q^2)$$

(one may derive this result by comparing the mass of the soliton with the extremal RNAdS black hole at equal charge). It follows that small charge hairy black holes do exist at $e^2 = \frac{32}{3}$, but we have to go beyond leading order in perturbation theory to see this[25]. Although we haven't elaborated in detail here, it turns out that the properties of small charged black

[24] It is interesting that the zero temperature limit of hairy black branes appears to depend qualitatively on the mass of the charged scalar field, and $m = 0$ case is rather special. In the small black hole limit, on the other hand, we do not expect a qualitative dependence of our phase diagram on the mass of the scalar field. Indeed the leading order thermodynamics of hairy black holes in the Lagrangian (4.1) supplemented by a mass term for the scalar field can be easily obtained based on general non-interaction arguments (see section 4.6.4) and it gives results that are qualitatively similar to the $m = 0$ case. We thank A. Yarom for a discussion about this point.

[25] On the other hand, the study of the near horizon BF bound in highly charged extremal black hole

holes at $e^2 = \frac{32}{3}$ differ qualitatively from the properties of the same objects at larger e^2. This observation is particularly relevant as it turns out that charged scalar fields are forced to sit at this critical value of charge in some natural supersymmetric bulk theories.

As an example of an extension of the work discussed here, we point the reader to [26] where small hairy black holes in IIB theory on $AdS_5 \times S^5$ were constructed and studied. The analysis was carried out in a consistent truncation of $\mathcal{N} = 8$ gauged supergravity and the results suggest a rich structure for the space of (yet to be constructed) hairy charged rotating black holes in $AdS_5 \times S^5$, including new hairy supersymmetric black holes.

As another possible extension, recall that IIB supergravity on $AdS_5 \times S^5$ hosts a 4 parameter set of one sixteenth BPS supersymmetric black hole solutions [99, 100, 101]. It is possible that there exist new BPS hairy black holes consisting of a non interacting mix of these SUSY black holes with a SUSY graviton condensate. It would be very interesting to investigate this further.

---

backgrounds (which are locally well-approximated by extremal black branes) indicates that as $q \to \infty$,

$$e_c^2(q) \approx 3 + \frac{3^{1/3}}{2q^{1/3}} + \mathcal{O}(q^{-4/3})$$

Note that the leading deviation from the black brane result of $e_c^2 = 3$ is positive and in the large q limit e_c^2 continues to be monotonically decreasing. In other words, all available data is consistent with the conjecture that $e_c^2(q)$ is a monotonically decreasing function that interpolates between $\frac{32}{3}$ and 3 as q ranges from 0 to ∞.

Appendix A

Appendix to chapter 2

A.1 Notations and conventions

A.1.1 Gamma matrices

In this appendix, we list the various notations and conventions used in chapter 2. We follow those of [102]. We list them here for convenience.

The metric signature is $\eta_{\mu\nu} = \{-,+,+\}$. In three dimensions the Lorentz group is $SL(2,\mathbb{R})$ and it acts on two component real spinors ψ^α, where α are the spinor indices. A vector is represented by either a real and symmetric spinor $V_{\alpha\beta}$ or a symmetric traceless spinor V_α^β, where $V_{\alpha\beta} = V_\mu \gamma^\mu_{\alpha\beta}$. We will choose our gamma matrices in the real and symmetric form [103]

$$\gamma^\mu_{\alpha\beta} = \{\mathbb{I}, \sigma^3, \sigma^1\} \ . \tag{A.1}$$

The charge conjugation matrix $C_{\alpha\beta}$ is used to raise and lower the spinor indices

$$C_{\alpha\beta} = -C_{\beta\alpha} = \begin{bmatrix} 0 & -i \\ \Psi i & 0 \end{bmatrix} = -C^{\alpha\beta} \ . \tag{A.2}$$

In the above, note that $C_{\beta\alpha} = C^T$ and $C^{\alpha\beta} = (C^T)^{-1}$. It follows that

$$C_{\alpha\gamma} C^{\gamma\beta} = -\delta_\alpha^{\ \beta} \ , \tag{A.3}$$

where $\delta_\alpha^{\ \beta}$ is the usual identity matrix. The spinor indices are raised and lowered using the NW-SE convention

$$\psi^\alpha = C^{\alpha\beta}\psi_\beta \ , \psi_\alpha = \psi^\beta C_{\beta\alpha} \ . \tag{A.4}$$

We also use the notation $\psi^2 = \frac{1}{2}\psi^\alpha\psi_\alpha = i\psi^+\psi^-$. Note that ψ^2 is Hermitian. Since ψ^α is real, it is clear that ψ_α is imaginary since the charge conjugate matrix is imaginary.

The Clifford algebra is defined using the matrices $(\gamma^\mu)_\alpha^{\ \beta}$ and these can be obtained by raising the indices using $C^{\alpha\beta}$ as illustrated above

$$(\gamma^\mu)_\alpha^{\ \beta} = \{\sigma^2, -i\sigma^1, i\sigma^3\} \ . \tag{A.5}$$

Note that these matrices are purely imaginary. Choosing the $\gamma^{\mu}_{\alpha\beta}$ as real and symmetric always yields this and vice versa. Our $\mu = 0, 1, 3$, since at some point we will do an euclidean rotation from the $\mu = 0$ direction to $\mu = 2$. It is clear that $(\gamma^0)^2 = 1, (\gamma^1)^2 = -1, (\gamma^3)^2 = -1$, therefore with our metric conventions the Clifford algebra is satisfied by

$$(\gamma^{\mu})_{\alpha}{}^{\tau}(\gamma^{\nu})_{\tau}{}^{\beta} + (\gamma^{\nu})_{\alpha}{}^{\tau}(\gamma^{\mu})_{\tau}{}^{\beta} = -2\eta^{\mu\nu}\delta_{\alpha}{}^{\beta} \ . \tag{A.6}$$

Another very useful relation is

$$[\gamma^{\mu}, \gamma^{\nu}] = -2i\epsilon^{\mu\nu\rho}\gamma_{\rho} \ , \epsilon^{013} = -1 \ . \tag{A.7}$$

For completion we also note that

$$(\gamma^{\mu})^{\alpha\beta} = \{\mathbb{I}, \sigma^3, \sigma^1\} \ . \tag{A.8}$$

As a consequence of the Clifford algebra (A.6), we get a minus sign in the trace

$$k_{\alpha}{}^{\beta}k_{\beta}{}^{\alpha} = -2k^2 \ . \tag{A.9}$$

The Euclidean counterpart of (A.6) is obtained by the standard Euclidean rotation $\gamma^0 \to i\gamma^2$

$$(\gamma^{\mu})_{\alpha}{}^{\beta} = \{i\sigma^2, -i\sigma^1, i\sigma^3\} \ , \ \mu = 2, 1, 3, \tag{A.10}$$

and they satisfy the Euclidean Clifford algebra

$$(\gamma^{\mu})_{\alpha}{}^{\tau}(\gamma^{\nu})_{\tau}{}^{\beta} + (\gamma^{\nu})_{\alpha}{}^{\tau}(\gamma^{\mu})_{\tau}{}^{\beta} = -2\delta^{\mu\nu}\delta_{\alpha}{}^{\beta} \ . \tag{A.11}$$

where $\delta_{\mu\nu} = (+, +, +)$.

A.1.2 Superspace

The two component Grassmann parameters θ that appear in various places in superspace have the properties

$$\begin{gathered}\int d\theta = 0 \ , \int d\theta\theta = 1 \ , \int d^2\theta\theta^2 = -1 \ , \int d^2\theta\theta^{\alpha}\theta^{\beta} = \epsilon^{\alpha\beta} \ , \\ \frac{\partial\theta^{\alpha}}{\partial\theta^{\beta}} = \delta_{\beta}{}^{\alpha} \ , C^{\alpha\beta}\frac{\partial}{\partial\theta^{\beta}}\frac{\partial}{\partial\theta^{\alpha}}\theta^2 = -2 \ , \theta^{\alpha}\theta^{\beta} = -C^{\alpha\beta}\theta^2 \ , \theta_{\alpha}\theta_{\beta} = -C_{\alpha\beta}\theta^2 \ .\end{gathered} \tag{A.12}$$

The definition of the delta function in superspace follows from the relation

$$\int d^2\theta\theta^2 = -1 \implies \delta^2(\theta) = -\theta^2 \ . \tag{A.13}$$

Formally we write

$$\delta^2(\theta_1 - \theta_2) = -(\theta_1 - \theta_2)^2 = -(\theta_1^2 + \theta_2^2 - \theta_1\theta_2). \tag{A.14}$$

The superspace derivatives are defined as

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\theta^\beta \partial_{\alpha\beta} \ , D^\alpha = C^{\alpha\beta} D_\beta \ . \tag{A.15}$$

We will mostly use the momentum space version of the above in which we replace $i\partial_{\alpha\beta} \to k_{\alpha\beta}$

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \theta^\beta k_{\alpha\beta} \ . \tag{A.16}$$

Note that the choice of the real and symmetric basis in A.1 makes the momentum operator Hermitian. The superspace derivatives satisfy the algebra

$$\{D_\alpha, D_\beta\} = 2k_{\alpha\beta} \ . \tag{A.17}$$

The tracelessness of $(\gamma^\mu)_\alpha^{\ \beta}$ implies that

$$\{D^\alpha, D_\alpha\} = 0 \ . \tag{A.18}$$

Care has to be taken when integrating by parts with superderivatives due to their anticommuting nature. From the expression for D_α we can construct

$$D^2 = \frac{1}{2} D^\alpha D_\alpha = \frac{1}{2}\left(C^{\beta\alpha} \frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta} + 2\theta^\alpha k_\alpha^{\ \beta} \frac{\partial}{\partial\theta^\beta} + 2\theta^2 k^2 \right) \ . \tag{A.19}$$

From the above it is easy to verify

$$\begin{aligned} &(D^2)^2 = -k^2, \\ &D^2 D_\alpha = -D_\alpha D^2 = k_{\alpha\beta} D^\beta \\ &D^\alpha D_\beta D_\alpha = 0 \ . \end{aligned} \tag{A.20}$$

using the properties given in (A.12). Yet another extremely useful relation is the action of the superderivative square (A.19) on the delta function (A.14)

$$D^2_{\theta_1,k} \delta^2(\theta_1 - \theta_2) = 1 - \theta_1^\alpha \theta_2^\beta k_{\alpha\beta} - \theta_1^2\theta_2^2 k^2 = \exp(-\theta_1^\alpha\theta_2^\beta k_{\alpha\beta}) \ . \tag{A.21}$$

We will often suppress the spinor indices in the exponential with the understanding that the spinor indices are contracted as indicated above. Some useful formulae are

$$\begin{aligned} &\delta^2(\theta_1 - \theta_2)\delta^2(\theta_2 - \theta_1) = 0 \ , \\ &\delta^2(\theta_1 - \theta_2) D^\alpha_{\theta_2,k} \delta^2(\theta_2 - \theta_1) = 0 \ , \\ &\delta^2(\theta_1 - \theta_2) D^2_{\theta_2,k} \delta^2(\theta_2 - \theta_1) = \delta^2(\theta_1 - \theta_2) \ , \end{aligned} \tag{A.22}$$

and the transfer rule

$$D_\alpha^{\theta_1,p} \delta^2(\theta_1 - \theta_2) = -D_\alpha^{\theta_2,-p} \delta^2(\theta_2 - \theta_1) \ . \tag{A.23}$$

The supersymmetry generators

$$Q_\alpha^{\theta,k} = i\left(\frac{\partial}{\partial\theta^\alpha} - \theta^\beta k_{\alpha\beta}\right) , \tag{A.24}$$

satisfy the anticommutation relations

$$\begin{aligned}\{Q_\alpha, Q_\beta\} &= 2k_{\alpha\beta} ,\\ \{Q_\alpha, D_\beta\} &= 0 .\end{aligned} \tag{A.25}$$

It is also clear that the transfer rule (A.23) is the statement that the delta function of θ is a supersymmetric invariant.

A.1.3 Superfields

The scalar superfield $\Phi(x,\theta)$ contains a complex scalar ϕ, a complex fermion ψ^α, and a complex auxiliary field F. The vector superfield $\Gamma^\alpha(x,\theta)$ consists of the gauge field $V_{\alpha\beta}$, the gaugino λ^α, an auxiliary scalar B and an auxiliary fermion χ^α. The following superfield expansions are used repeatedly in several places. We list them here for easy reference.

$$\begin{aligned}&\Phi = \phi + \theta\psi - \theta^2 F ,\\ &\bar\Phi = \bar\phi + \theta\bar\psi - \theta^2 \bar F ,\\ &\bar\Phi\Phi = \bar\phi\phi + \theta^\alpha(\bar\phi\psi_\alpha + \bar\psi_\alpha\phi) - \theta^2(\bar F\phi + \bar\phi F + \bar\psi\psi) ,\\ &D_\alpha\Phi = \psi_\alpha - \theta_\alpha F + i\theta^2\partial_\alpha{}^\beta\psi_\beta + i\theta^\beta\partial_{\alpha\beta}\phi ,\\ &D^\alpha\bar\Phi D_\alpha\Phi\big|_{\theta^2} = \theta^2(2\bar F F + 2i\bar\psi^\alpha\partial_\alpha{}^\beta\psi_\beta - 2\partial\bar\phi\partial\phi) ,\\ &D^2_{q,\theta}(\bar\Phi\Phi) = (\bar\phi F + \bar F\phi + \bar\psi\psi) + \theta^\alpha q_\alpha{}^\beta(\bar\phi\psi + \bar\psi\phi) + \theta^2 q^2(\bar\phi\phi)^2 ,\\ &\Gamma^\alpha = \chi^\alpha - \theta^\alpha B + i\theta^\beta A_\beta{}^\alpha - \theta^2(2\lambda^\alpha - i\partial^{\alpha\beta}\chi_\beta) .\end{aligned} \tag{A.26}$$

A.2 A check on the constraints of supersymmetry on S-matrices

In §2.2.4 we demonstrated that the manifestly supersymmetric scattering of any $\mathcal{N}=1$ theory in three dimensions is described by two independent functions. In this section, we directly verify this result in theories whose off-shell effective action takes the form (2.109) with the function V that takes the particular supersymmetric form (2.113) (and so is determined by four unspecified functions A, B C and D).

We wish to use (2.109) to study scattering. In order to do this we evaluate (2.109) with the fields Φ and $\bar\Phi$ in that action chosen to be the most general linearised on-shell solutions to the equations of motion. In this appendix we focus on a particular scattering process - scattering in the adjoint channel. At leading order in the large N limit we can focus on this channel by choosing the solution for Φ_m and $\bar{\Phi}^m$ in (2.109) to be positive energy solutions (representing initial states), while $\bar{\Phi}^m$ and Φ_n are expanded in negative energy solutions (representing final states). The negative and positive energy solutions are both allowed to be an arbitrary linear combination of bosonic and fermionic solutions. Plugging these solutions into (2.109) yields a functional of the coefficients of the bosonic and fermionic solutions in the four superfields in (2.109). The coefficients of various terms in this functional are simply the S-matrices. For instance the coefficient of the term proportional to the product of four bosonic modes is the four boson scattering amplitude, etc.

Let us schematically represent the scattering process we study by

$$\begin{pmatrix} \Phi(\theta_1, p_1) \\ \bar{\Phi}(\theta_2, p_2) \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\Phi}(\theta_3, p_3) \\ \Phi(\theta_4, p_4) \end{pmatrix}$$

where the LHS represents the in-states and the RHS represents the out-states. The momentum assignments in (2.109) are

$$p_1 = p + q \ , \ p_2 = -k - q \ , \ p_3 = p \ , \ p_4 = -k \ . \tag{A.27}$$

In component form (A.27) encodes the following S-matrices

$$\mathcal{S}_B : \begin{pmatrix} \phi(p_1) \\ \bar{\phi}(p_2) \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\phi}(p_3) \\ \phi(p_4) \end{pmatrix} \ , \ \mathcal{S}_F : \begin{pmatrix} \psi(p_1) \\ \bar{\psi}(p_2) \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\psi}(p_3) \\ \psi(p_4) \end{pmatrix}$$

$$H_1 : \begin{pmatrix} \phi(p_1) \\ \bar{\phi}(p_2) \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\psi}(p_3) \\ \psi(p_4) \end{pmatrix} \ , \ H_2 : \begin{pmatrix} \psi(p_1) \\ \bar{\psi}(p_2) \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\phi}(p_3) \\ \phi(p_4) \end{pmatrix}$$

$$H_3 : \begin{pmatrix} \phi(p_1) \\ \bar{\psi}(p_2) \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\phi}(p_3) \\ \psi(p_4) \end{pmatrix} \ , \ H_4 : \begin{pmatrix} \psi(p_1) \\ \bar{\phi}(p_2) \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\psi}(p_3) \\ \phi(p_4) \end{pmatrix}$$

$$H_5 : \begin{pmatrix} \phi(p_1) \\ \bar{\psi}(p_2) \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\psi}(p_3) \\ \phi(p_4) \end{pmatrix} \ , \ H_6 : \begin{pmatrix} \psi(p_1) \\ \bar{\phi}(p_2) \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\phi}(p_3) \\ \psi(p_4) \end{pmatrix} \tag{A.28}$$

These S-matrix elements are all obtained by the process spelt out above in terms of the four unknown functions A, B, C, D (which we will take to be arbitrary and unrelated). The functions A,B,C,D are to be evaluated at the on-shell conditions that follow from taking the momenta on-shell, but that will play no role in what follows.

It is not difficult to demonstrate that the boson-boson $\rightarrow$ boson boson and the fermion-fermion $\rightarrow$ fermion fermion S-matrices are given in terms of the functions A, B, C and D

by [1]

$$
\begin{aligned}
\mathcal{S}_B &= (-4iAm + 4Bm^2 - Bq_3^2 - q_3(Ck_- + Dp_-)) \ , \\
\mathcal{S}_F &= (BC^{\beta\alpha}C^{\delta\gamma} - iC\ C^{\beta\alpha}C^{+\gamma}C^{+\delta} + iDC^{\delta\gamma}C^{+\alpha}C^{+\beta})\bar{u}_\alpha(p_3)u_\beta(p_1)v_\gamma(p_2)\bar{v}_\delta(p_4) \\
&= -B(4m^2 + q_3^2) + Ck_-(2im - q_3) - Dp_-(q_3 + 2im) \ .
\end{aligned} \tag{A.30}
$$

The S-matrices for the remaining processes in (A.28) are also easily obtained: we find

$$H_i = a_i\mathcal{S}_B + b_i\mathcal{S}_F \tag{A.31}$$

where the coefficients are given by

$$
\begin{aligned}
a_1 &= \frac{(4m^2+q_3^2)\,(q_3(p-k)_- + 2im(k+p)_-)}{32mk_-p_-\sqrt{k_+p_+}} &, \ b_1 &= \frac{(4m^2+q_3^2)\,(q_3(k-p)_- + 2im(k+p)_-)}{32mk_-p_-\sqrt{k_+p_+}} \\
a_2 &= \frac{(4m^2+q_3^2)\,(q_3(p-k)_- + 2im(k+p)_-)}{32mk_-p_-\sqrt{k_+p_+}} &, \ b_2 &= \frac{(4m^2+q_3^2)\,(q_3(k-p)_- + 2im(k+p)_-)}{32mk_-p_-\sqrt{k_+p_+}} \\
a_3 &= -\frac{2m+iq_3}{4m} &, \ b_3 &= \frac{2m+iq_3}{4m} \\
a_4 &= \frac{2m-iq_3}{4m} &, \ b_4 &= -\frac{2m-iq_3}{4m} \\
a_5 &= \frac{(4m^2+q_3^2)\,(q_3(k+p)_- - 2im(k-p)_-)}{32mk_-p_-\sqrt{k_+p_+}} &, \ b_5 &= -\frac{i\,(4m^2+q_3^2)\,(2m(k-p)_- - iq_3(k+p)_-)}{32mk_-p_-\sqrt{k_+p_+}} \\
a_6 &= \frac{i\,(4m^2+q_3^2)\,(2m(k-p)_- + iq_3(k+p)_-)}{32mk_-p_-\sqrt{k_+p_+}} &, \ b_6 &= \frac{(4m^2+q_3^2)\,(q_3(k+p)_- + 2im(k-p)_-)}{32mk_-p_-\sqrt{k_+p_+}}
\end{aligned} \tag{A.32}
$$

The above set of coefficients match with the coefficients directly evaluated from (2.41) and (2.42). This is a consistency check of the results of §2.2.4.

For the $\mathcal{N}=2$ theory the S-matrix (2.40) should also obey an additional constraint (see §A.3) that relates $\mathcal{S}_B$ and $\mathcal{S}_F$ through (A.57). For the T channel this relation was evaluated in (2.174), substituting this in (A.31) it is easy to verify that the $\theta_2\theta_3$ and $\theta_1\theta_4$ terms in (2.40)

$$H_5 : \begin{pmatrix} \phi(p_1) \\ \bar{\psi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\psi}(p_3) \\ \phi(p_4) \end{pmatrix} \ , \ H_6 : \begin{pmatrix} \psi(p_1) \\ \bar{\phi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\phi}(p_3) \\ \psi(p_4) \end{pmatrix} \tag{A.33}$$

vanish for the $\mathcal{N}=2$ theory. This is consistent with the fact that the corresponding terms in the tree level component Lagrangian (2.15) vanish at the $\mathcal{N}=2$ point $w=1$.

[1] For the T channel we have used

$$
\begin{aligned}
v_\alpha(-k) &= \begin{pmatrix} -\sqrt{k_+} \\ \frac{(q_3-2im)}{2\sqrt{k_+}} \end{pmatrix} \ , \ \bar{v}^\alpha(-k-q) = \begin{pmatrix} -\frac{2m+iq_3}{2\sqrt{k_+}} & i\sqrt{k_+} \end{pmatrix} \\
u_\alpha(p+q) &= \begin{pmatrix} -i\sqrt{p_+} \\ \frac{2m-iq_3}{2\sqrt{p_+}} \end{pmatrix} \ , \ \bar{u}^\alpha(p) = \begin{pmatrix} -\frac{(2im+q_3)}{2\sqrt{p_+}} & -\sqrt{p_+} \end{pmatrix}
\end{aligned} \tag{A.29}
$$

A.3 Manifest $\mathcal{N}=2$ supersymmetry invariance

In this appendix we discuss the general constraints on the S-matrix obtained by imposing $\mathcal{N}=2$ supersymmetry. In subsection 2.2.4 we have already solved the constraints coming from $\mathcal{N}=1$ supersymmetry. As an $\mathcal{N}=2$ theory is in particular also $\mathcal{N}=1$ supersymmetric, the results of this appendix will be a specialisation of those of subsection 2.2.4.

In the case of $\mathcal{N}=2$, we have to recall the notion of chirality. A 'chiral' (antichiral) $\mathcal{N}=2$ superfield Φ is defined as

$$\bar{D}_\alpha \Phi = 0, \qquad D_\alpha \bar{\Phi} = 0. \tag{A.34}$$

We define the following:

$$\theta_\alpha = \frac{1}{\sqrt{2}}(\theta_\alpha^{(1)} - i\theta_\alpha^{(2)}), \qquad \bar{\theta}_\alpha = \frac{1}{\sqrt{2}}(\theta_\alpha^{(1)} + i\theta_\alpha^{(2)}). \tag{A.35}$$

Where the superscripts (1) and (2) indicate the two (real) copies of the $\mathcal{N}=1$ superspace. With these definitions, we can define the supercharges

$$Q_\alpha = \frac{1}{\sqrt{2}}(Q_\alpha^{(1)} + iQ_\alpha^{(2)}) = i\left(\frac{\partial}{\partial\theta^\alpha} - i\bar{\theta}^\beta\partial_{\beta\alpha}\right), \tag{A.36}$$

$$\bar{Q}_\alpha = \frac{1}{\sqrt{2}}(Q_\alpha^{(1)} - iQ_\alpha^{(2)}) = i\left(\frac{\partial}{\partial\bar{\theta}^\alpha} - i\theta^\beta\partial_{\beta\alpha}\right). \tag{A.37}$$

Likewise, we can define the supercovariant derivative operators

$$D_\alpha = \frac{1}{\sqrt{2}}(D_\alpha^{(1)} + iD_\alpha^{(2)}) = \left(\frac{\partial}{\partial\theta^\alpha} + i\bar{\theta}^\beta\partial_{\beta\alpha}\right), \tag{A.38}$$

$$\bar{D}_\alpha = \frac{1}{\sqrt{2}}(D_\alpha^{(1)} - iD_\alpha^{(2)}) = \left(\frac{\partial}{\partial\bar{\theta}^\alpha} + i\theta^\beta\partial_{\beta\alpha}\right). \tag{A.39}$$

The solutions to the constraints (A.34) for (off-shell) chiral and anti-chiral fields are

$$\Phi = \phi + \sqrt{2}\theta\psi - \theta^2 F + i\theta\bar{\theta}\partial\phi - i\sqrt{2}\theta^2(\bar{\theta}\partial\!\!\!/\psi) + \theta^2\bar{\theta}^2\partial^2\phi, \tag{A.40}$$

$$\bar{\Phi} = \bar{\phi} + \sqrt{2}\bar{\theta}\bar{\psi} - \bar{\theta}^2\bar{F} - i\theta\bar{\theta}\partial\bar{\phi} - i\sqrt{2}\bar{\theta}^2(\theta\partial\!\!\!/\bar{\psi}) + \theta^2\bar{\theta}^2\partial^2\bar{\phi}. \tag{A.41}$$

Here $\theta\bar{\theta}\partial\phi = \theta^\alpha\bar{\theta}^\beta\partial_{\alpha\beta}$ and $\bar{\theta}\partial\!\!\!/\psi = \bar{\theta}^\alpha\partial_\alpha{}^\beta\psi_\beta$ and so on.

In the current context the chiral matter superfield transforms in the fundamental representation of the gauge group while the antichiral matter superfield transforms in the antifundamental representation of the gauge group. It follows that it is impossible to add a gauge invariant quadratic superpotential to our action (recall that an $\mathcal{N}=2$ superpotential can only depend on chiral multiplets) in order to endow our fields with mass. However it is possible to make the matter fields massive while preserving $\mathcal{N}=2$ supersymmetry; the fields can be made massive using a D term.

As our theory has no superpotential, it follows that $F = \bar{F} = 0$ on shell. We are interested in the action of supersymmetry on the on-shell component fields ϕ ($\bar{\phi}$) which are defined as

$$\phi(x) = \int \frac{d^2p}{(2\pi)^2\sqrt{2p^0}} \left[a(\mathbf{p})e^{ip\cdot x} + a^{c\dagger}(\mathbf{p})e^{-ip\cdot x}\right], \tag{A.42}$$

$$\bar{\phi}(x) = \int \frac{d^2p}{(2\pi)^2\sqrt{2p^0}} \left[a^c(\mathbf{p})e^{ip\cdot x} + a^{\dagger}(\mathbf{p})e^{-ip\cdot x}\right]. \tag{A.43}$$

Likewise, for ψ ($\psi^\dagger$) we have

$$\psi(x) = \int \frac{d^2p}{(2\pi)^2\sqrt{2p^0}} \left[u_\alpha(\mathbf{p})\alpha(\mathbf{p})e^{ip\cdot x} + v_\alpha(\mathbf{p})\alpha^{c\dagger}(\mathbf{p})e^{-ip\cdot x}\right], \tag{A.44}$$

$$\psi^\dagger(x) = \int \frac{d^2p}{(2\pi)^2\sqrt{2p^0}} \left[u_\alpha(\mathbf{p})\alpha^c(\mathbf{p})e^{ip\cdot x} + v_\alpha(\mathbf{p})\alpha^{\dagger}(\mathbf{p})e^{-ip\cdot x}\right]. \tag{A.45}$$

In order to obtain this action we used the transformation properties listed in equations F.16-F.20 of [2] and then specialised to the on-shell limit. [2] The results may be summarised as follows. As before, we define the (super) creation and annihilation operators

$$A(\mathbf{p}) = a(\mathbf{p}) + \alpha(\mathbf{p})\theta, \qquad A^c(\mathbf{p}) = a^c(\mathbf{p}) + \alpha^c(\mathbf{p})\theta, \tag{A.46}$$

$$A^\dagger(\mathbf{p}) = a^\dagger(\mathbf{p}) + \theta\alpha^\dagger(\mathbf{p}), \qquad A^{c\dagger}(\mathbf{p}) = a^{c\dagger}(\mathbf{p}) + \theta\alpha^{c\dagger}(\mathbf{p}). \tag{A.47}$$

The action of Q_α (and $\bar{Q}_\alpha$) on A and $A^\dagger$ is

$$[Q_\alpha, A(\mathbf{p})] = -i\sqrt{2}u_\alpha(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta}, \quad [\bar{Q}_\alpha, A(\mathbf{p})] = i\sqrt{2}u^*_\alpha(\mathbf{p})\theta,$$
$$[Q_\alpha, A^\dagger(\mathbf{p})] = i\sqrt{2}v^*_\alpha(\mathbf{p})\theta, \quad [\bar{Q}_\alpha, A^\dagger(\mathbf{p})] = i\sqrt{2}v_\alpha(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta}. \tag{A.48}$$

Similarly, the action of Q_α (and $\bar{Q}_\alpha$) on A^c and $A^{c\dagger}$ is

$$[Q_\alpha, A^c(\mathbf{p})] = i\sqrt{2}u^*_\alpha(\mathbf{p})\theta, \quad [\bar{Q}_\alpha, A^c(\mathbf{p})] = -i\sqrt{2}u_\alpha(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta},$$
$$[Q_\alpha, A^{c\dagger}(\mathbf{p})] = i\sqrt{2}v_\alpha(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta}, \quad [\bar{Q}_\alpha, A^{c\dagger}(\mathbf{p})] = i\sqrt{2}v^*_\alpha(\mathbf{p})\theta. \tag{A.49}$$

It is clear from (A.48) that $(Q_\alpha + \bar{Q}_\alpha)/\sqrt{2}$ produces the action of the first supercharge $Q^{(1)}_\alpha$, which we have seen earlier. That this action produces the correct differential operator given earlier is obvious as well. Therefore, in order to obtain the second supercharge $Q^{(2)}_\alpha$, we simply operate with the other linear combination $(Q_\alpha - \bar{Q}_\alpha)/i\sqrt{2}$.

[2] Note that the action of Q_α on the chiral field Φ is different from the action on the anti-chiral field $\bar{\Phi}$. Similar remarks apply for $\bar{Q}_\alpha$.

Note that for the $\mathcal{N}=1$ case, it doesn't matter if we used $A^\dagger$ or $A^{c\dagger}$ for the initial states (A or A^c for the final states), as is clear from (A.49). This agrees with the fact that the linear combination $(Q_\alpha+\bar{Q}_\alpha)/\sqrt{2}$ produces the same equation on all S-matrix elements. However other linear combinations of the two $\mathcal{N}=2$ supersymmetries act differently on A and A^c, and so the constraints of $\mathcal{N}=2$ supersymmetry are different depending on which scattering processes we consider.

A.3.1 Particle - antiparticle scattering

Let us first study the invariance of the following S-matrix element

$$S(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4)=\langle 0|A_4(\mathbf{p}_4,\theta_4)A_3^c(\mathbf{p}_3,\theta_3)A_2^\dagger(\mathbf{p}_2,\theta_2)A_1^{c\dagger}(\mathbf{p}_1,\theta_1)|0\rangle. \tag{A.50}$$

In the current context, this is the S-matrix for particle - antiparticle scattering. The full $\mathcal{N}=2$ invariance of the S-matrix is expressed as

$$\left(\sum_{i=1}^{4}Q_\alpha^i(\mathbf{p}_i,\theta_i)\right)S(\mathbf{p}_i,\theta_i)=0,\ \text{and}\ \left(\sum_{i=1}^{4}\bar{Q}_\alpha^i(\mathbf{p}_i,\theta_i)\right)S(\mathbf{p}_i,\theta_i)=0. \tag{A.51}$$

The above conditions (A.51) produce the following constraints for the S-matrix element (A.50)

$$\begin{aligned}
&\left(\sum_{i=1}^{4}Q_\alpha^i(\mathbf{p}_i,\theta_i)\right)S(\mathbf{p}_i,\theta_i)=0\Rightarrow\\
&\qquad\left(iv_\alpha(\mathbf{p}_1)\frac{\overrightarrow{\partial}}{\partial\theta_1}+iv^*(\mathbf{p}_2)\theta_2+iu_\alpha^*(\mathbf{p}_3)\theta_3-iu_\alpha(\mathbf{p}_4)\frac{\overrightarrow{\partial}}{\partial\theta_4}\right)S(\mathbf{p}_i,\theta_i)=0,\\
&\left(\sum_{i=1}^{4}\bar{Q}_\alpha^i(\mathbf{p}_i,\theta_i)\right)S(\mathbf{p}_i,\theta_i)=0\Rightarrow\\
&\qquad\left(iv_\alpha^*(\mathbf{p}_1)\theta_1+iv_\alpha(\mathbf{p}_2)\frac{\overrightarrow{\partial}}{\partial\theta_2}-iu_\alpha(\mathbf{p}_3)\frac{\overrightarrow{\partial}}{\partial\theta_3}+iu_\alpha^*(\mathbf{p}_4)\theta_4\right)S(\mathbf{p}_i,\theta_i)=0.
\end{aligned} \tag{A.52}$$

We check in what follows that the combination

$$\left(\frac{1}{\sqrt{2}}\sum_{i=1}^{4}Q_\alpha^i(\mathbf{p}_i,\theta_i)+\bar{Q}_\alpha^i(\mathbf{p}_i,\theta_i)\right)S(\mathbf{p}_i,\theta_i)=0 \tag{A.53}$$

produces the same equation (and therefore solution) of $\mathcal{N}=1$ which we have already found. We easily find that this gives

$$\left(iv_\alpha(\mathbf{p}_1)\frac{\overrightarrow{\partial}}{\partial\theta_1} + iv_\alpha(\mathbf{p}_2)\frac{\overrightarrow{\partial}}{\partial\theta_2} - iu_\alpha(\mathbf{p}_3)\frac{\overrightarrow{\partial}}{\partial\theta_3} - iu_\alpha(\mathbf{p}_4)\frac{\overrightarrow{\partial}}{\partial\theta_4} \right.$$
$$\left. + iv^*_\alpha(\mathbf{p}_1)\theta_1 + iv^*_\alpha(\mathbf{p}_2)\theta_2 + iu^*_\alpha(\mathbf{p}_3)\theta_3 + iu^*_\alpha(\mathbf{p}_4)\theta_4\right)S(\mathbf{p}_i,\theta_i)=0. \tag{A.54}$$

Now, we turn to the other linear combination, which is

$$\left(\frac{1}{i\sqrt{2}}\sum_{i=1}^{4} Q^i_\alpha(\mathbf{p}_i,\theta_i) - \bar{Q}^i_\alpha(\mathbf{p}_i,\theta_i)\right)S(\mathbf{p}_i,\theta_i)=0. \tag{A.55}$$

This readily gives the differential equation

$$\left(iv_\alpha(\mathbf{p}_1)\frac{\overrightarrow{\partial}}{\partial\theta_1} - iv_\alpha(\mathbf{p}_2)\frac{\overrightarrow{\partial}}{\partial\theta_2} + iu_\alpha(\mathbf{p}_3)\frac{\overrightarrow{\partial}}{\partial\theta_3} - iu_\alpha(\mathbf{p}_4)\frac{\overrightarrow{\partial}}{\partial\theta_4} \right.$$
$$\left. - iv^*_\alpha(\mathbf{p}_1)\theta_1 + iv^*_\alpha(\mathbf{p}_2)\theta_2 + iu^*_\alpha(\mathbf{p}_3)\theta_3 - iu^*_\alpha(\mathbf{p}_4)\theta_4\right)S(\mathbf{p}_i,\theta_i)=0. \tag{A.56}$$

The equation (A.54) is the same as it was for the $\mathcal{N}=1$ theory, whereas the second equation (A.56) must be obeyed by the same S-matrix in the $\mathcal{N}=2$ point. Thus (A.61) is an additional constraint obeyed by the $\mathcal{N}=2$ S-matrix (2.40). It follows that (A.56) gives a relation between $\mathcal{S}_B$ and $\mathcal{S}_F$

$$\mathcal{S}_B\left(C_{12}v_\alpha(\mathbf{p}_1) - C_{23}u_\alpha(\mathbf{p}_3) + C_{24}u_\alpha(\mathbf{p}_4) + v^*_\alpha(\mathbf{p}_2)\right) = \mathcal{S}_F(C^*_{13}u_\alpha(\mathbf{p}_4) + C^*_{14}u_\alpha(\mathbf{p}_3) + C^*_{34}v_\alpha(\mathbf{p}_1)) \tag{A.57}$$

Thus, the $\mathcal{N}=2$ S-matrix for particle-antiparticle scattering consists of only one independent function, with the other related by (A.57).

A.3.2 Particle - particle scattering

Now, consider the other S-matrix element (which was considered in the previous $\mathcal{N}=1$ computation)

$$S(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4) = \langle 0|A_4(\mathbf{p}_4,\theta_4)A_3(\mathbf{p}_3,\theta_3)A^\dagger_2(\mathbf{p}_2,\theta_2)A^\dagger_1(\mathbf{p}_1,\theta_1)|0\rangle. \tag{A.58}$$

The conditions (A.51) produce the following for the S-matrix element (A.58)

$$\begin{aligned}&\left(\sum_{i=1}^{4} Q_\alpha^i(\mathbf{p}_i,\theta_i)\right)S(\mathbf{p}_i,\theta_i)=0\Rightarrow\\ &\qquad\left(iv_\alpha^*(\mathbf{p}_1)\theta_1+iv^*(\mathbf{p}_2)\theta_2-iu_\alpha(\mathbf{p}_3)\frac{\overrightarrow{\partial}}{\partial\theta_3}-iu_\alpha(\mathbf{p}_4)\frac{\overrightarrow{\partial}}{\partial\theta_4}\right)S(\mathbf{p}_i,\theta_i)=0,\\ &\left(\sum_{i=1}^{4}\bar{Q}_\alpha^i(\mathbf{p}_i,\theta_i)\right)S(\mathbf{p}_i,\theta_i)=0\Rightarrow\\ &\qquad\left(iv_\alpha(\mathbf{p}_1)\frac{\overrightarrow{\partial}}{\partial\theta_1}+iv_\alpha(\mathbf{p}_2)\frac{\overrightarrow{\partial}}{\partial\theta_2}+iu_\alpha^*(\mathbf{p}_3)\theta_3+iu_\alpha^*(\mathbf{p}_4)\theta_4\right)S(\mathbf{p}_i,\theta_i)=0.\end{aligned} \tag{A.59}$$

For the combination (A.53) we get

$$\begin{aligned}&\left(iv_\alpha(\mathbf{p}_1)\frac{\overrightarrow{\partial}}{\partial\theta_1}+iv_\alpha(\mathbf{p}_2)\frac{\overrightarrow{\partial}}{\partial\theta_2}-iu_\alpha(\mathbf{p}_3)\frac{\overrightarrow{\partial}}{\partial\theta_3}-iu_\alpha(\mathbf{p}_4)\frac{\overrightarrow{\partial}}{\partial\theta_4}\right.\\ &\qquad\left.+iv_\alpha^*(\mathbf{p}_1)\theta_1+iv_\alpha^*(\mathbf{p}_2)\theta_2+iu_\alpha^*(\mathbf{p}_3)\theta_3+iu_\alpha^*(\mathbf{p}_4)\theta_4\right)S(\mathbf{p}_i,\theta_i)=0,\end{aligned} \tag{A.60}$$

and for the combination (A.55) we have

$$\begin{aligned}&\left(-iv_\alpha(\mathbf{p}_1)\frac{\overrightarrow{\partial}}{\partial\theta_1}-iv_\alpha(\mathbf{p}_2)\frac{\overrightarrow{\partial}}{\partial\theta_2}-iu_\alpha(\mathbf{p}_3)\frac{\overrightarrow{\partial}}{\partial\theta_3}-iu_\alpha(\mathbf{p}_4)\frac{\overrightarrow{\partial}}{\partial\theta_4}\right.\\ &\qquad\left.+iv_\alpha^*(\mathbf{p}_1)\theta_1+iv_\alpha^*(\mathbf{p}_2)\theta_2-iu_\alpha^*(\mathbf{p}_3)\theta_3-iu_\alpha^*(\mathbf{p}_4)\theta_4\right)S(\mathbf{p}_i,\theta_i)=0.\end{aligned} \tag{A.61}$$

Similar to the particle-anti particle case discussed in the previous section. The equation (A.60) is the same as it was for the $\mathcal{N}=1$ theory, whereas the second equation (A.61) must be obeyed by the same S-matrix in the $\mathcal{N}=2$ point. It follows that (A.61) gives a relation between $\mathcal{S}_B$ and $\mathcal{S}_F$

$$\mathcal{S}_B\left(C_{13}u_\alpha(\mathbf{p}_3)+C_{14}u_\alpha(\mathbf{p}_4)+C_{12}v_\alpha(\mathbf{p}_2)+v_\alpha^*(\mathbf{p}_1)\right)=\mathcal{S}_F(C_{24}^*u_\alpha(\mathbf{p}_3)-C_{23}^*u_\alpha(\mathbf{p}_4)+C_{34}^*v_\alpha(\mathbf{p}_2)) \tag{A.62}$$

The $\mathcal{N}=2$ S-matrix for particle-particle scattering consists of only one independent function, with the other related by (A.62).

Thus in the $\mathcal{N}=2$ theory the S-matrix is only made of one independent function. Note that the results of this section are true for *any* three dimensional $\mathcal{N}=2$ theory. It simply follows from the supersymmetric ward identity (A.51) and is independent of the details of the theory.

A.4 Identities for S-matrices in on-shell superspace

In this subsection we demonstrate that the product of two supersymmetric S-matrices is supersymmetric. In other words we demonstrate that

$$\left(\sum_{i=1}^{4} Q^i_\alpha(\mathbf{p}_i, \theta_i)\right) S_1 \star S_2 = 0. \tag{A.63}$$

provided S_1 and S_2 independently obey the same equation.

This can be analyzed as follows. We have the invariance (differential) equation for S_1 and S_2

$$\left(\overrightarrow{Q}_{\tilde{v}(\mathbf{p}_1)} + \overrightarrow{Q}_{\tilde{v}(\mathbf{p}_2)} + \overrightarrow{Q}_{u(\mathbf{p}_3)} + \overrightarrow{Q}_{u(\mathbf{p}_4)}\right) S_i(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) = 0 \quad \text{with } p_1 + p_2 = p_3 + p_4. \tag{A.64}$$

where the left-acting supercharges $\overrightarrow{Q}_{\tilde{v}(\mathbf{p})}$ are defined as

$$\overrightarrow{Q}_{\tilde{v}(\mathbf{p})} = i\left(v_\alpha(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta} + v^*_\alpha(\mathbf{p})\theta\right) \tag{A.65}$$

in contrast to (2.36), because we're acting from the left. It may be easily checked that this indeed produces the correct action of Q on $A^\dagger$. The reader is reminded that the (left- acting) supercharges $\overrightarrow{Q}_{u(\mathbf{p})}$ are defined as

$$\overrightarrow{Q}_{u(\mathbf{p})} = i\left(-u_\alpha(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta} + u^*_\alpha(\mathbf{p})\theta\right). \tag{A.66}$$

Note that

$$\begin{aligned}(\overrightarrow{Q}_{\tilde{v}(\mathbf{p})})^* &= \overrightarrow{Q}_{u(\mathbf{p})}\ , \\ (\overrightarrow{Q}_{u(\mathbf{p})})^* &= \overrightarrow{Q}_{\tilde{v}(\mathbf{p})}\ .\end{aligned} \tag{A.67}$$

We have used the fact that while complex conjugating, the Grassmannian derivatives acting from the left act from the right (and vice-versa) and to bring any such right acting derivative to the left involves introducing an extra minus sign. Armed with the definitions above, we can rewrite (A.64) as (all differential operators henceforth, unless noted otherwise, are taken to act from the left)

$$\left(Q^*_{u(\mathbf{p}_1)} + Q^*_{u(\mathbf{p}_2)} + Q_{u(\mathbf{p}_3)} + Q_{u(\mathbf{p}_4)}\right) S_i(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) = 0\ . \tag{A.68}$$

The next step is to observe that

$$\left(Q^*_{u(\mathbf{p}_1)} + Q^*_{u(\mathbf{p}_2)} + Q_{u(\mathbf{p}_3)} + Q_{u(\mathbf{p}_4)}\right) \exp(\theta_1\theta_3 + \theta_2\theta_4) 2p_3^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_3)$$
$$2p_4^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_4) = 0 \quad \text{(A.69)}$$

after we set $\mathbf{p}_1 = \mathbf{p}_3$ and $\mathbf{p}_2 = \mathbf{p}_4$. We now act on (2.58) with

$$\left(Q^*_{u(\mathbf{p}_1)} + Q^*_{u(\mathbf{p}_2)} + Q_{u(\mathbf{p}_3)} + Q_{u(\mathbf{p}_4)}\right) \int d\Gamma \Big[S_1(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{k}_3, \phi_1, \mathbf{k}_4, \phi_2)$$
$$\exp(\phi_1\phi_3 + \phi_2\phi_4) 2k_1^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_3 - \mathbf{k}_1) 2k_2^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_4 - \mathbf{k}_2)$$
$$S_2(\mathbf{k}_1, \phi_3, \mathbf{k}_2, \phi_4, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) \Big] . \quad \text{(A.70)}$$

Proceeding with (A.70), one finds

$$- \int d\Gamma \Big[\left(Q_{u(\mathbf{k}_3)} + Q_{u(\mathbf{k}_4)}\right) S_1(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{k}_3, \phi_1, \mathbf{k}_4, \phi_2) \exp(\phi_1\phi_3 + \phi_2\phi_4)$$
$$2k_1^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_3 - \mathbf{k}_1) 2k_2^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_4 - \mathbf{k}_2) S_2(\mathbf{k}_1, \phi_3, \mathbf{k}_2, \phi_4, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4)$$
$$+ S_1(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{k}_3, \phi_1, \mathbf{k}_4, \phi_2) \exp(\phi_1\phi_3 + \phi_2\phi_4) 2k_1^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_3 - \mathbf{k}_1)$$
$$2k_2^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_4 - \mathbf{k}_2) \left(Q^*_{u(\mathbf{k}_1)} + Q^*_{u(\mathbf{k}_2)}\right) S_2(\mathbf{k}_1, \phi_3, \mathbf{k}_2, \phi_4, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) \Big] . \quad \text{(A.71)}$$

We next integrate by parts keeping in mind that only the derivative parts of the Q change sign (as a consequence of the integration by parts). This gives

$$\int d\Gamma \Big[S_1(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{k}_3, \phi_1, \mathbf{k}_4, \phi_2)$$
$$\left(\tilde{Q}_{u(\mathbf{k}_3)} + \tilde{Q}_{u(\mathbf{k}_4)} + \tilde{Q}^*_{u(\mathbf{k}_1)} + \tilde{Q}^*_{u(\mathbf{k}_2)}\right) \exp(\phi_1\phi_3 + \phi_2\phi_4) 2k_1^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_3 - \mathbf{k}_1)$$
$$2k_2^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_4 - \mathbf{k}_2) S_2(\mathbf{k}_1, \phi_3, \mathbf{k}_2, \phi_4, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) \Big] . \quad \text{(A.72)}$$

Here, by $\tilde{Q}_{u(p)}$ and $\tilde{Q}^*_{u(p)}$ we mean

$$\tilde{Q}_{u(p)} = i \left(u_\alpha(\mathbf{p}) \frac{\overrightarrow{\partial}}{\partial\theta} + u^*_\alpha(\mathbf{p})\theta \right) , \quad \text{(A.73)}$$

$$\tilde{Q}^*_{u(p)} = i \left(u^*_\alpha(\mathbf{p}) \frac{\overrightarrow{\partial}}{\partial\theta} - u_\alpha(\mathbf{p})\theta \right) . \quad \text{(A.74)}$$

It can be easily checked (just like (A.69)) that (on setting $\mathbf{k}_3 = \mathbf{k}_1$ and $\mathbf{k}_4 = \mathbf{k}_2$)

$$\left(\tilde{Q}_{u(\mathbf{k}_3)} + \tilde{Q}_{u(\mathbf{k}_4)} + \tilde{Q}^*_{u(\mathbf{k}_1)} + \tilde{Q}^*_{u(\mathbf{k}_2)}\right) \exp(\phi_1\phi_3 + \phi_2\phi_4) 2k_1^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_3 - \mathbf{k}_1)$$
$$2k_2^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_4 - \mathbf{k}_2) = 0, \quad \text{(A.75)}$$

completing the proof.

A.5 Details of the unitarity equation

In this section, we simplify the unitarity equations (2.66) and (2.67). We define

$$Z(\mathbf{p}_i) = \frac{1}{4m^2} v^*(\mathbf{p}_1) v^*(\mathbf{p}_2)\ v(\mathbf{p}_3) v(\mathbf{p}_4)$$

and rewrite (2.66) and (2.67) as

$$\begin{aligned}
&\int d\Gamma' \, [\mathcal{S}_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{S}^*_B(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) \\
&- Y(\mathbf{p}_3, \mathbf{p}_4) \, (\mathcal{S}_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{S}^*_B(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) \\
&+ 4Y(\mathbf{p}_3, \mathbf{p}_4) \, (\mathcal{S}_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{S}^*_F(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) + \mathcal{S}_F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{S}^*_B(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4)) \\
&+ 16Y^2(\mathbf{p}_3, \mathbf{p}_4) \mathcal{S}_F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{S}^*_F(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4)) \big] = 2p_3^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_3) 2p_4^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_4)
\end{aligned} \quad \text{(A.76)}$$

and

$$\begin{aligned}
&Z(\mathbf{p}_i) \int d\Gamma' \, [-4Y(\mathbf{p}_3, \mathbf{p}_4) \mathcal{S}_F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{S}^*_F(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) \\
&+ \big(4Y^2(\mathbf{p}_3, \mathbf{p}_4) \mathcal{S}_F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{S}^*_F(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) \\
&+ Y(\mathbf{p}_3, \mathbf{p}_4) \, (\mathcal{S}_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{S}^*_F(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) + \mathcal{S}_F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{S}^*_B(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4)) \\
&+ \frac{1}{4} \mathcal{S}_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{S}^*_B(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) \Big) \Big] = -2p_3^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_3) 2p_4^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_4).
\end{aligned} \quad \text{(A.77)}$$

Since the factor $Z(\mathbf{p}_i)$ depends only on the external momenta $\mathbf{p}_i$, we may evaluate it on the delta functions and this simply yields $Z(\mathbf{p}_i) = 4Y(\mathbf{p}_3, \mathbf{p}_4)$. We finally arrive at

$$\begin{aligned}
&\int d\Gamma' \Bigg[\mathcal{S}_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{S}^*_B(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) \\
&- Y(\mathbf{p}_3, \mathbf{p}_4) \Bigg(\mathcal{S}_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{S}^*_B(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) \\
&+ 4Y(\mathbf{p}_3, \mathbf{p}_4) \big(\mathcal{S}_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{S}^*_F(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) + \mathcal{S}_F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{S}^*_B(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) \big) \\
&+ 16Y^2(\mathbf{p}_3, \mathbf{p}_4) \mathcal{S}_F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \mathcal{S}^*_F(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) \Bigg) \Bigg] = 2p_3^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_3) 2p_4^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_4)
\end{aligned} \quad \text{(A.78)}$$

and

$$\int d\Gamma' \Big[-16Y^2(\mathbf{p}_3,\mathbf{p}_4)\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)$$
$$+Y(\mathbf{p}_3,\mathbf{p}_4)\Big(\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)$$
$$+4Y(\mathbf{p}_3,\mathbf{p}_4)\big(\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)+\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\big)$$
$$+16Y^2(\mathbf{p}_3,\mathbf{p}_4)\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\Big)\Big] = -2p_3^0(2\pi)^2\delta^{(2)}(\mathbf{p}_1-\mathbf{p}_3)2p_4^0(2\pi)^2\delta^{(2)}(\mathbf{p}_2-\mathbf{p}_4). \tag{A.79}$$

The above equations can be more compactly written as (2.70) and (2.71) respectively (since $p_3 \cdot p_4 = p_1 \cdot p_2$).

A.6 Going to supersymmetric light cone gauge

In this brief appendix we will demonstrate that (upto the usual problem with zero modes) it is always possible to find a super gauge transformation that takes us to the supersymmetric lightcone gauge $\Gamma_- = 0$

Let us start with a gauge configuration that obeys our gauge condition $\Gamma_- = 0$. Starting with this gauge configuration, we will now demonstrate that we can perform a gauge transformation that will take Γ_- to any desired value, say $\tilde{\Gamma}_-$.

Performing the gauge transformation (2.8) we find that the new value of Γ_- is simply D_-K. Let

$$K = M + \theta\zeta - \theta^2 P, \tag{A.80}$$

where M, ζ^α, P are gauge parameters. It follows that

$$D_-K = \zeta_- - \theta_-(\partial_{-+}M + P) + \theta_+\partial_{--}M - i\theta_+\theta_-(\partial_{-+}\zeta_- - \partial_{--}\zeta_+) \tag{A.81}$$

Now let us suppose that

$$-\tilde{\Gamma}_- = \chi_- - \theta_-(B + A_{+-}) + \theta_+A_{--} + i\theta_+\theta_-(2\lambda_- + \partial_{--}\chi_+ - \partial_{-+}\chi_-)$$

We need to find K so that

$$D_-K = \tilde{\Gamma}_-$$

Equating coefficients on the two sides of this equation we find

$$\begin{aligned} \chi_- + \zeta_- &= 0 \ , \\ B + A_{+-} + P + \partial_{-+}M &= 0 \ , \\ A_{--} + \partial_{--}M &= 0 \ , \\ 2\lambda_- + \partial_{--}(\chi_+ + \zeta_+) - \partial_{-+}(\chi_- + \zeta_-) &= 0 \ , \end{aligned} \tag{A.82}$$

which are then solved to get,

$$\begin{aligned} \zeta_- &= -\chi_- \ , \\ \zeta_+ &= -2\partial_{--}^{-1}\lambda_- - \chi_+ \ , \\ M &= -\partial_{--}^{-1}A_{--} \ , \\ P &= -B - A_{+-} + \partial_{-+}(\partial_{--}^{-1}A_{--}) \ . \end{aligned} \tag{A.83}$$

Substituting the above expressions in the expansion for K, we can write

$$K = -\partial_{--}^{-1}A_{--} - i\theta_-(2\partial_{--}^{-1}\lambda_- + \chi_+) + i\theta_+\chi_- + i\theta_+\theta_-(\partial_{-+}\partial_{--}^{-1}A_{--} - B - A_{+-}) \ . \tag{A.84}$$

It can be checked that the form of K obtained above follows from

$$K = i\partial_{--}^{-1}D_-\Gamma_- \ , \tag{A.85}$$

which is a supersymmetric version of the gauge transformation used to generate an arbitrary A_- starting from usual lightcone gauge.

A.7 Details of the self energy computation

In this subsection, we will demonstrate that the self energy $\Sigma(p, \theta_1, \theta_2)$ is a constant independent of the momenta p. As discussed in §2.3.3 $\Sigma(p, \theta_1, \theta_2)$ obeys the integral equation

$$\begin{aligned} \Sigma(p, \theta_1, \theta_2) = {}& 2\pi\lambda w \int \frac{d^3r}{(2\pi)^3} \delta^2(\theta_1 - \theta_2) P(r, \theta_1, \theta_2) \\ & - 2\pi\lambda \int \frac{d^3r}{(2\pi)^3} D_-^{\theta_2, -p} D_-^{\theta_1, p} \left(\frac{\delta^2(\theta_1 - \theta_2)}{(p-r)_{--}} P(r, \theta_1, \theta_2) \right) \\ & + 2\pi\lambda \int \frac{d^3r}{(2\pi)^3} \frac{\delta^2(\theta_1 - \theta_2)}{(p-r)_{--}} D_-^{\theta_1, r} D_-^{\theta_2, -r} P(r, \theta_1, \theta_2,) \ . \end{aligned} \tag{A.86}$$

We will now simplify the second and third terms in (A.86). In §2.3.3 we already observed that the general form of the propagator is of the form given by (2.92). Using the formulae (A.21) and (A.22) we can write (2.92) as

$$P(p, \theta_1, \theta_2) = \left(C_1(p) D^2_{\theta_1, p} + C_2(p) \right) \delta^2(\theta_1 - \theta_2) \tag{A.87}$$

In the second term of (A.86) we have to evaluate

$$C_1(p) D_-^{\theta_2, -p} D_-^{\theta_1, p} \left(\delta^2(\theta_1 - \theta_2) D^2_{\theta_1, p} \delta^2(\theta_1 - \theta_2) \right) \ , \tag{A.88}$$

since the product of $\delta^2(\theta_1 - \theta_2)$ vanishes. We further use the formulae (A.22) and then the transfer rule (A.23) to get

$$\begin{aligned} -C_1(p) D_-^{\theta_2, -p} D_-^{\theta_2, -p} \delta^2(\theta_1 - \theta_2) &= p_{--} C_1(p) \delta^2(\theta_1 - \theta_2) \\ &= p_{--} \delta^2(\theta_1 - \theta_2) P(r, \theta_1, \theta_2) \ , \end{aligned} \tag{A.89}$$

where we have used the algebra (A.17) in the first line and (A.22) in the second.

Let us now proceed to simplify the third term in (A.86). We need to evaluate

$$\begin{aligned}\delta^2(\theta_1-\theta_2)D_-^{\theta_1,r}D_-^{\theta_2,-r}\left(C_1(p)D^2_{\theta_1,r}\delta^2(\theta_1-\theta_2)+C_2(p)\delta^2(\theta_1-\theta_2)\right)\\ =C_1(p)\delta^2(\theta_1-\theta_2)D_-^{\theta_1,r}D_-^{\theta_2,-r}D^2_{\theta_1,r}\delta^2(\theta_1-\theta_2)\ ,\end{aligned} \tag{A.90}$$

where we have used the transfer rule (A.23) and the fact that the product of $\delta^2(\theta_1-\theta_2)$ vanishes. We further simplify

$$\begin{aligned}C_1(p)\delta^2(\theta_1-\theta_2)D_-^{\theta_1,r}D_-^{\theta_2,-r}D^2_{\theta_1,r}\delta^2(\theta_1-\theta_2) &= -C_1(p)\delta^2(\theta_1-\theta_2)r_-^{\beta}D_-^{\theta_1,r}D_\beta^{\theta_2,-r}\delta^2(\theta_1-\theta_2)\\ &= C_1(p)\delta^2(\theta_1-\theta_2)r_-^{+}D_-^{\theta_1,r}D_+^{\theta_1,r}\delta^2(\theta_1-\theta_2)\\ &= C_1(p)\delta^2(\theta_1-\theta_2)(-ir_-^{+})D^2_{\theta_1,r}\delta^2(\theta_1-\theta_2)\\ &= r_{--}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2)\ ,\end{aligned} \tag{A.91}$$

where in the first line we have used (A.20), in the second line the expression is nonzero for $\beta=-$ and we have used the transfer rule (A.23), while the third line follows from the identity $-iD^2=D_-D_+$ and the last line follows from the arguments used before.

Thus, using the results (A.91) and (A.89) in (A.86) we get the final form as given in (2.99)

$$\begin{aligned}\Sigma(p,\theta_1,\theta_2) &= 2\pi\lambda w\int\frac{d^3r}{(2\pi)^3}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2)\\ &-2\pi\lambda\int\frac{d^3r}{(2\pi)^3}\frac{p_{--}}{(p-r)_{--}}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2)\\ &+2\pi\lambda\int\frac{d^3r}{(2\pi)^3}\frac{r_{--}}{(p-r)_{--}}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2)\ .\end{aligned} \tag{A.92}$$

From the above it is clear that the momentum dependence cancels between the second and third terms and we get

$$\Sigma(p,\theta_1,\theta_2)=2\pi\lambda w(w-1)\int\frac{d^3r}{(2\pi)^3}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2)\ . \tag{A.93}$$

A.8 Details relating to the evaluation of the off-shell four point function

A.8.1 Supersymmetry constraints on the off-shell four point function

In this section we will constrain the most general form of the four point function using supersymmetry (see Fig. A.1). Supersymmetric invariance of the four point function in

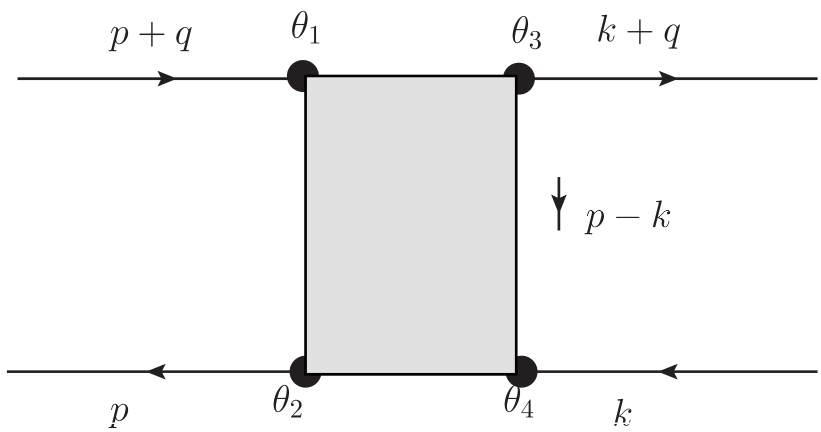

Figure A.1: Four point function in superspace

superspace (2.105) implies that

$$(Q_{\theta_1,p+q} + Q_{\theta_2,-p} + Q_{\theta_3,-k-q} + Q_{\theta_4,k})V(\theta_1,\theta_2,\theta_3,\theta_4,p,k,q) = 0 \ . \tag{A.94}$$

This can be simplified using (A.24) and written as

$$\sum_{i=1}^{4}\left(\frac{\partial}{\partial\theta_i^\alpha} - p_{\alpha\beta}(\theta_1-\theta_2)^\beta - q_{\alpha\beta}(\theta_1-\theta_3)^\beta - k_{\alpha\beta}(\theta_4-\theta_3)^\beta\right)V(\theta_1,\theta_2,\theta_3,p,k,q) = 0 \ . \tag{A.95}$$

We can make the following variable changes to simplify the equation (we suppress spinor indices for simplicity in notation)

$$\begin{aligned} X &= \sum_{i=1}^{4}\theta_i \ , \\ X_{12} &= \theta_1 - \theta_2 \ , \\ X_{13} &= \theta_1 - \theta_3 \ , \\ X_{43} &= \theta_4 - \theta_3 \ . \end{aligned} \tag{A.96}$$

The inverse coordinates are

$$\begin{aligned} \theta_1 &= \frac{1}{4}(X + X_{12} + 2X_{13} - X_{43}) \ , \\ \theta_2 &= \frac{1}{4}(X - 3X_{12} + 2X_{13} - X_{43}) \ , \\ \theta_3 &= \frac{1}{4}(X + X_{12} - 2X_{13} - X_{43}) \ , \\ \theta_4 &= \frac{1}{4}(X + X_{12} - 2X_{13} + 3X_{43}) \ . \end{aligned} \tag{A.97}$$

In terms of the new coordinates, the derivatives are then expressed as

$$\begin{aligned}
\frac{\partial}{\partial\theta_1} &= \frac{\partial}{\partial X} + \frac{\partial}{\partial X_{12}} + \frac{\partial}{\partial X_{13}} \ , \\
\frac{\partial}{\partial\theta_2} &= \frac{\partial}{\partial X} - \frac{\partial}{\partial X_{12}}, \\
\frac{\partial}{\partial\theta_3} &= \frac{\partial}{\partial X} - \frac{\partial}{\partial X_{13}} - \frac{\partial}{\partial X_{43}} \ , \\
\sum_{i=1}^{4} \frac{\partial}{\partial\theta_i} &= 4\frac{\partial}{\partial X} \ .
\end{aligned} \tag{A.98}$$

Using the above, one can rewrite (A.95) as

$$(4\frac{\partial}{\partial X} - p\cdot X_{12} - q\cdot X_{13} - k\cdot X_{43})V(X, X_{12}, X_{13}, X_{43}, p, q, k) = 0, \tag{A.99}$$

where $p\cdot X_{12} = p_{\alpha\beta}X_{12}^{\beta}$. The above equation can be thought of as a differential equation in the variables X_{ij} and is solved by

$$V(\theta_1,\theta_2,\theta_3,\theta_4,p,k,q) = \exp\left(\frac{1}{4}X\cdot(p\cdot X_{12} + q\cdot X_{13} + k\cdot X_{43})\right)F(X_{12}, X_{13}, X_{43}, p, q, k) \ . \tag{A.100}$$

This is the most general form of a four point function in superspace that is invariant under supersymmetry.

A.8.2 Explicitly evaluating V_0

In this subsection, we will compute the tree level diagram for the four point function due to the gauge superfield interaction. (see Fig. A.2). In Fig. A.2 the two diagrams are equivalent ways to represent the same process.

$$V_0(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k)^{gauge} = \frac{-2\pi}{\kappa(p-k)_{--}}(D_-^{\theta_2,p} - D_-^{\theta_4,k})(D_-^{\theta_1,p+q} - D_-^{\theta_3,-(k+q)})(\delta_{13}^2\delta_{24}^2\delta_{12}^2) \ , \tag{A.101}$$

where $\delta_{ij}^2 = \delta^2(\theta_i - \theta_j)$. [3]

It can be explicitly checked that (see (2.107) for definition of X_{ij})

$$(D_-^{\theta_2,p} - D_-^{\theta_4,k})(D_-^{\theta_1,p+q} - D_-^{\theta_3,-(k+q)})(\delta_{13}^2\delta_{24}^2\delta_{12}^2) = \exp\left(\frac{1}{4}X.(p.X_{12} + q.X_{13} + k.X_{43})\right) F_{tree}(X_{12}, X_{13}, X_{43}) \ , \tag{A.102}$$

[3]Note that each vertex factor in Fig. A.2 has a factor of D, resulting in two powers of D in (A.101).

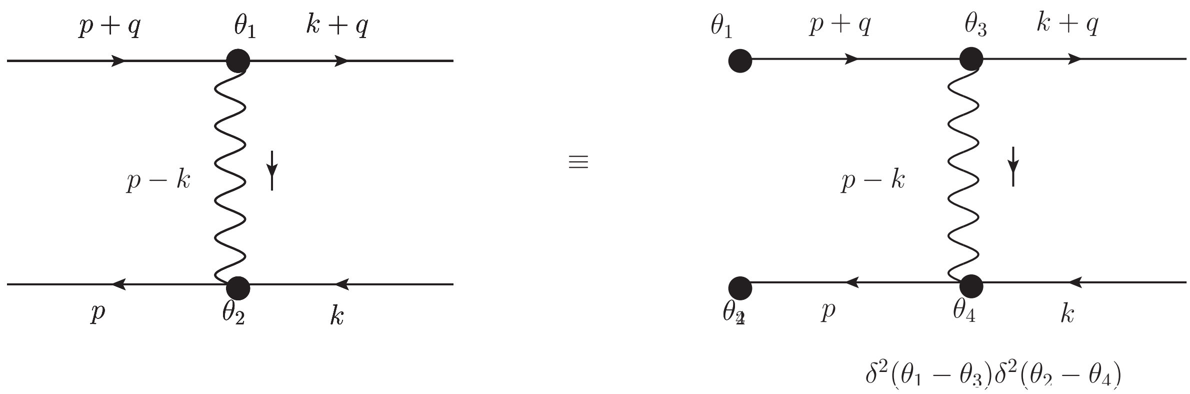

Figure A.2: Four point function for gauge interaction: Tree diagram

where

$$F_{tree} = 2iX^+_{12}X^+_{13}X^+_{43}(X^-_{12}+X^-_{34}) \ . \tag{A.103}$$

Thus the final result for the tree level diagram is given by

$$V_0(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k)^{gauge} = -\frac{4\pi i}{\kappa(p-k)_{--}}\exp\Big(\frac{1}{4}X_{1234}.(p.X_{12}+q.X_{13}+k.X_{43})\Big) \\ X^+_{12}X^+_{13}X^+_{43}(X^-_{12}+X^-_{34}) \ . \tag{A.104}$$

It is clear that the shift invariant function (A.103) has the general structure of (2.113), with the appropriate identification

$$A(p,q,k) = -\frac{4\pi i}{\kappa}\frac{1}{(p-k)_{--}} \ , \ B(p,q,k) = -\frac{4\pi i}{\kappa}\frac{1}{(p-k)_{--}} \tag{A.105}$$

Note that the Fig. A.2 has the $\mathbb{Z}_2$ symmetry (2.110). It is straightforward to check that (A.104) is invariant under (2.110).

A.8.3 Closure of the ansatz (2.113)

In this section, we establish the consistency of the ansatz (2.113) as a solution of the integral equation (2.111). Consistency is established by plugging the ansatz (2.113) into the RHS of this integral equation, and verifying that the resultant θ structure is once again of the form given in (2.113). In other words we will show that the dependence of

$$\int \frac{d^3r}{(2\pi)^3}d^2\theta_a d^2\theta_b d^2\theta_A d^2\theta_B\bigg(NV_0(\theta_1,\theta_2,\theta_a,\theta_b,p,q,r)P(r+q,\theta_a,\theta_A) \\ P(r,\theta_B,\theta_b)V(\theta_A,\theta_B,\theta_3,\theta_4,r,q,k)\bigg) \tag{A.106}$$

on θ_1, θ_2, θ_3 and θ_4 is given by the form (2.113) with appropriately identified functions A, B, C D.

The algebraic closure described above actually follows from a more general closure property that we now explain. Note that the tree level four point function V_0 (2.112) is itself of the form (2.113). The more general closure property (which we will explain below) is that the expression

$$V_{12} = V_1 \star V_2 \equiv \int \frac{d^3r}{(2\pi)^3} d^2\theta_a d^2\theta_b d^2\theta_A d^2\theta_B \Big(V_1(\theta_1, \theta_2, \theta_a, \theta_b, p, q, r) P(r + q, \theta_a, \theta_A) \\ P(r, \theta_B, \theta_b) V_2(\theta_A, \theta_B, \theta_3, \theta_4, r, q, k) \Big) \quad \text{(A.107)}$$

takes the form (2.113) whenever V_1 and V_2 are both also of the form (2.113). In other words (A.107) defines a closed multiplication rule on expressions of the form (2.113).

The explicit verification of the closure described the last paragraph follows from straightforward algebra. Let [4]

$$V_1(\theta_1, \theta_2, \theta_a, \theta_b, p, q, r) = \exp\big(\frac{1}{4} X_{12ab} \cdot (p \cdot X_{12} + q \cdot X_{1a} + r \cdot X_{ba})\big) F_1(X_{12}, X_{1a}, X_{ba}, p, q, r) \quad \text{(A.108)}$$

where

$$F_1(X_{12}, X_{1a}, X_{ba}, p, q, r) = X^+_{AB} X^+_{43} \Big(A_1(p, r, q) X^-_{12} X^-_{ba} X^+_{1a} X^-_{1a} + B_1(p, r, q) X^-_{12} X^-_{ba} \\ + C_1(p, r, q) X^-_{12} X^+_{1a} + D_1(p, r, q) X^+_{1a} X^-_{ba} \Big) \ . \quad \text{(A.109)}$$

and

$$V_2(\theta_A, \theta_B, \theta_3, \theta_4, r, q, k) = \exp\big(\frac{1}{4} X_{AB34} \cdot (r \cdot X_{AB} + q \cdot X_{A3} + k \cdot X_{43})\big) F_2(X_{AB}, X_{A3}, X_{43}, r, q, k) \ , \quad \text{(A.110)}$$

where

$$F_2(X_{AB}, X_{A3}, X_{43}, r, q, k) = X^+_{AB} X^+_{43} \Big(A_2(r, k, q) X^-_{AB} X^-_{43} X^+_{A3} X^-_{A3} + B_2(r, k, q) X^-_{AB} X^-_{43} \\ + C_2(r, k, q) X^-_{AB} X^+_{A3} + D_2(r, k, q) X^+_{A3} X^-_{43} \Big) \ . \quad \text{(A.111)}$$

[4] We have used the notations $X_{12ab} = \theta_1 + \theta_2 + \theta_a + \theta_b$ and $X_{AB34} = \theta_A + \theta_B + \theta_3 + \theta_4$.

Evaluating the integrals over $\theta_a, \theta_b, \theta_A, \theta_B$, we find that V_{12} in (A.107) is of the form (2.113) with

$$A_{12} = -\frac{1}{4}q_3 \int d^3\mathcal{R}\Big((C_1C_2k_- - D_1D_2p_- + 2B_2C_1q_3 - 2B_1D_2q_3)r_- \\ + 2A_2(D_1p_- + 2B_1q_3 + 2C_1r_-) + 2A_1(C_2k_- + 2B_2q_3 + 2D_2r_-)\Big),$$

$$B_{12} = -\frac{1}{4}\int d^3\mathcal{R}\Big((2A_2 - C_2k_-)(2A_1 + D_1p_-) + 4B_1B_2q_3^2 + 3C_1D_2r_-^2 \\ + (2A_2C_1 - 2A_1D_2 - C_1C_2k_- - D_1D_2p_- + 4B_2C_1q_3 + 4B_1D_2q_3)r_-\Big),$$

$$C_{12} = -\frac{1}{2}\int d^3\mathcal{R}C_2q_3(2A_1 + D_1p_- + 2B_1q_3 + 3C_1r_-) ,$$
$$D_{12} = -\frac{1}{2}\int d^3\mathcal{R}D_1q_3(-2A_2 + C_2k_- + 2B_2q_3 + 3D_2r_-) . \tag{A.112}$$

where

$$d^3\mathcal{R} = \frac{d^3r}{(2\pi)^3}\frac{1}{(r^2+m^2)((r+q)^2+m^2)}$$

It follows from (A.107) that

$$(V_1 \star V_2) \star V_3 = V_1 \star (V_2 \star V_3) \tag{A.113}$$

as both expressions in (A.113) are given by the same integral (the expressions differ only in the order in which the θ and internal momentum integrals are performed). In other words the product defined above is associative. We have directly checked that the explicit multiplication formula (A.112) defines an associative product rule.

A.8.4 Consistency check of the integral equation

In this section, we demonstrate that the integral equations (2.114)-(2.117) are consistent with the $\mathbb{Z}_2$ symmetry (2.110). First we note that the $\mathbb{Z}_2$ invariance (2.110) of (2.113) imposes the following conditions on the unknown functions of momenta

$$A(p,k,q) = A(k,p,-q) \ , B(p,k,q) = B(k,p,-q) \ ,$$
$$C(p,k,q) = -D(k,p,-q) \ , D(p,k,q) = -C(k,p,-q) \ . \tag{A.114}$$

These conditions can be written in the form of a matrix given by

$$E(p,k,q) = TE(k,p,-q) \ , \tag{A.115}$$

where

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \, , E(p,k,q) = \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} \tag{A.116}$$

The integral equations (2.114)-(2.117) can be written in differential form by taking derivatives of p_+ and using the formulae in appendix §A.8.5 [5]

$$\partial_{p_+} E(p,k,q) = S(p,k,q) + H(p,k_-,q)E(p,k,q) \tag{A.117}$$

where $S(p,k,q)$ is a source term. The equation for k_+ can be obtained from the above equation as follows

$$\begin{aligned} \partial_{k_+} E(p,k,q) &= T\partial_{k_+} E(k,p,-q) \ , \\ &= TS(k,p,-q) + TH(k,p_-,-q)E(k,p,-q) \ , \\ &= TS(k,p,-q) + TH(k,p_-,-q)TE(p,k,q) \ , \end{aligned} \tag{A.118}$$

where we have used (A.115). Applying k_+, p_+ derivative on (A.117) and (A.118) respectively and taking the difference we get

$$\begin{aligned} &\partial_{k_+} S(p,k,q) + H(p,k_-,q)\Big(TS(k,p,-q) + TH(k,p_-,-q)TE(p,k,q)\Big) \\ &= T\partial_{p_+} S(k,p,-q) + TH(k,p_-,-q)T\Big(S(p,k,q) + H(p,k_-,-q)E(p,k,q)\Big) \ . \end{aligned} \tag{A.119}$$

Comparing coefficients of $E(p,k,q)$ in the above equation we get the condition

$$[H(p,k_-,q), TH(k,p_-,-q)T] = 0 \ . \tag{A.120}$$

For the integral equations (2.114)-(2.117), the $H(p,k_-,q)$ are given by

$$H(p,k_-,q_3) = \frac{1}{a(p_s,q_3)} \begin{pmatrix} (6q_3 - 4im)p_- & 2q_3(2im+q_3)p_- & (2im+q_3)k_-p_- & -(2im+q_3)p_-^2 \\ 4p_- & 4q_3p_- & -2k_-p_- & 2p_-^2 \\ 0 & 0 & 8q_3p_- & 0 \\ 8im - 4q_3 & 4q_3(q_3-2im) & 2(q_3-2im)k_- & (4im+6q_3)p_- \end{pmatrix} \tag{A.121}$$

where

$$a(p_s,q_3) = \frac{\sqrt{m^2+p_s^2}\,(4m^2+q_3^2+4p_s^2)}{2\pi} \ . \tag{A.122}$$

[5] Taking derivatives with respect to p_+ eliminates the $r_\pm$ integrals because of the delta functions. The remaining r_3 integrals can be easily performed (see appendix §A.8.5).

The matrix $TH(k,p,-q_3)T$ is

$$TH(k,p,-q_3)T = \frac{1}{a(k_s,q_3)}\begin{pmatrix} -(4im+6q_3)k_- & 2q_3(q_3-2im)k_- & -(q_3-2im)k_-^2 & (q_3-2im)k_-p_- \\ 4k_- & -4q_3k_- & -2k_-^2 & 2k_-p_- \\ -8im-4q_3 & 4(-2im-q_3)q_3 & (4im-6q_3)k_- & -(4im+2q_3)p_- \\ 0 & 0 & 0 & -8q_3k_- \end{pmatrix}, \tag{A.123}$$

It is straightforward to check that (A.121) and (A.123) commute. Thus the system of differential equations (A.117) obey the integrability conditions (A.120). Thus the differential equations (A.117) will have solutions that respect the $\mathbb{Z}_2$ symmetry.

A.8.5 Useful formulae for r integrals

The Euclidean measure for the basic integrals are

$$\int \frac{(d^3r)_E}{(2\pi)^3} = \frac{1}{(2\pi)^3}\int r_s dr_s dr_3 d\theta \ , \tag{A.124}$$

where $r_s^2 = r_+r_- = r_1^2 + r_2^2$ and $r^2 = r_s^2 + r_3^2$. Here the integration limits are $-\infty \le r_3 \le \infty$, $0 \le r_s \le \infty$. Most often we encounter integrals of the type,

$$H(q) = \int \frac{d^3r}{(2\pi)^3}\frac{1}{(r^2+m^2)((r+q)^2+m^2)} = \frac{1}{4\pi|q_3|}\tan^{-1}\left(\left|\frac{q_3}{2m}\right|\right) \tag{A.125}$$

where we have set $q_\pm = 0$. Another frequently appearing integral is

$$\int \frac{d^3r}{(2\pi)^3}\frac{1}{r^2+m^2} = -\frac{|m|}{4\pi} \tag{A.126}$$

where we have regulated the divergence using dimensional regularisation.

In the integral equations (2.114)-(2.117), there are no explicit functions of r_3 appearing in the integral equations and the r_3 integral can be exactly done

$$\int_{-\infty}^{\infty} \frac{dr_3}{(r_s^2+r_3^2+m^2)(r_s^2+(r_3+q_3)^2+m^2)} = \frac{2\pi}{\sqrt{r_s^2+m^2}(4m^2+q_3^2+4r_s^2)} \ . \tag{A.127}$$

The results for the angle integrals are

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{(r-p)_-\,(k-r)_-} &= \frac{2\pi}{(k-p)_-}\left(\frac{k_+}{k_s^2}\theta[k_s-r_s] - \frac{p_+}{p_s^2}\theta[p_s-r_s]\right) , \\ \int_0^{2\pi} \frac{d\theta\ r_-}{(r-p)_-\,(k-r)_-} &= \frac{2\pi}{(k-p)_-}\left(\theta[k_s-r_s] - \theta[p_s-r_s]\right) , \\ \int_0^{2\pi} \frac{d\theta\ r_-^2}{(r-p)_-\,(k-r)_-} &= -\frac{2\pi}{(k-p)_-}\left(k_-(1-\theta[k_s-r_s]) - p_-(1-\theta[p_s-r_s])\right) . \end{aligned} \tag{A.128}$$

while the r_s integrals are done with the limits from 0 to ∞. We will also make use of the formula

$$\partial_{\bar{z}} \left(\frac{1}{z} \right) = 2\pi\delta^2(z, \bar{z}) \tag{A.129}$$

to derive the differential form of the integral equations.

A.9 Properties of the J functions

The J functions are given by

$$\begin{aligned} J_B(q_3, \lambda) &= \frac{4\pi q_3}{\kappa} \frac{n_1 + n_2 + n_3}{d_1 + d_2 + d_3} \ , \\ J_F(q_3, \lambda) &= \frac{4\pi q_3}{\kappa} \frac{-n_1 + n_2 + n_3}{d_1 + d_2 + d_3} \ , \end{aligned} \tag{A.130}$$

where the parameters are

$$\begin{aligned} n_1 =& 16 m q_3 (w+1) e^{i\lambda\left(2\tan^{-1}\frac{2|m|}{q_3} + \pi \mathrm{sgn}(q_3)\right)} \ , \\ n_2 =& (w-1)(q_3 + 2im)(2m(w-1) + iq_3(w+3)) \left(-e^{2i\pi\lambda \mathrm{sgn}(q_3)}\right) \ , \\ n_3 =& (w-1)(2m + iq_3)(q_3(w+3) + 2im(w-1)) e^{4i\lambda \tan^{-1}\frac{2|m|}{q_3}}) \ , \\ d_1 =& (w-1)\left(4m^2(w-1) - 8imq_3 + q_3^2(w+3)\right) e^{4i\lambda\tan^{-1}\frac{2|m|}{q_3}} \ , \\ d_2 =& (w-1)\left(4m^2(w-1) + 8imq_3 + q_3^2(w+3)\right) e^{2i\pi\lambda \mathrm{sgn}(q_3)} \ , \\ d_3 =& -2\left(4m^2(w-1)^2 + q_3^2(w(w+2)+5)\right) e^{i\lambda\left(2\tan^{-1}\frac{2|m|}{q_3} + \pi\mathrm{sgn}(q_3)\right)} \ . \end{aligned} \tag{A.131}$$

Both the J functions (A.130) are even functions of q_3

$$J_B(q_3, \lambda) = J_B(-q_3, \lambda) \ , \ J_F(q_3, \lambda) = J_F(-q_3, \lambda) \ . \tag{A.132}$$

Therefore in (A.130) we can replace q_3 with $|q_3|$ and rewrite them as

$$\begin{aligned} J_B(|q_3|, \lambda) &= \frac{4\pi|q_3|}{\kappa} \frac{(\tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3)}{(\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3)} \ , \\ J_F(|q_3|, \lambda) &= \frac{4\pi|q_3|}{\kappa} \frac{(-\tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3)}{(\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3)} \ , \end{aligned} \tag{A.133}$$

where

$$\begin{aligned}
\tilde{n}_1 =& 16m|q_3|(w+1)e^{i\lambda\left(2\tan^{-1}\frac{2|m|}{|q_3|}+\pi\right)} \ , \\
\tilde{n}_2 =& (w-1)(|q_3|+2im)(2m(w-1)+i|q_3|(w+3))\left(-e^{2i\pi\lambda}\right) \ , \\
\tilde{n}_3 =& (w-1)(2m+i|q_3|)(|q_3|(w+3)+2im(w-1))e^{4i\lambda\tan^{-1}\frac{2|m|}{|q_3|}}) \ , \\
\tilde{d}_1 =& (w-1)\left(4m^2(w-1)-8im|q_3|+|q_3|^2(w+3)\right)e^{4i\lambda\tan^{-1}\frac{2|m|}{|q_3|}} \ , \\
\tilde{d}_2 =& (w-1)\left(4m^2(w-1)+8im|q_3|+|q_3|^2(w+3)\right)e^{2i\pi\lambda} \ , \\
\tilde{d}_3 =& -2\left(4m^2(w-1)^2+|q_3|^2(w(w+2)+5)\right)e^{i\lambda\left(2\tan^{-1}\frac{2|m|}{|q_3|}+\pi\right)} \ . \qquad \text{(A.134)}
\end{aligned}$$

Another useful way to write the J function is to use the following identities

$$\begin{aligned}
\tan^{-1}\frac{2m}{q} &= \frac{\pi}{2}-\tan^{-1}\frac{q}{2m} \\
\tan^{-1}\frac{q}{2m} &= \frac{1}{2i}\log\left(\frac{1+\frac{iq}{2m}}{1-\frac{iq}{2m}}\right) \qquad \text{(A.135)}
\end{aligned}$$

Using this relations, it is easy to write the J functions in a factorised form as given in (2.140)

$$\begin{aligned}
J_B(q,\lambda) =& \frac{4\pi q}{\kappa}\frac{N_1N_2+M_1}{D_1D_2} \ , \\
J_F(q,\lambda) =& \frac{4\pi q}{\kappa}\frac{N_1N_2+M_2}{D_1D_2} \ , \qquad \text{(A.136)}
\end{aligned}$$

where

$$\begin{aligned}
N_1 =& \left(\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}(w-1)(2m+iq)+(w-1)(2m-iq)\right) \ , \\
N_2 =& \left(\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}(q(w+3)+2im(w-1))+(q(w+3)-2im(w-1))\right) \ , \\
M_1 =& -8mq((w+3)(w-1)-4w)\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda} \ , \\
M_2 =& -8mq(1+w)^2\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda} \ , \\
D_1 =& \left(i\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}(w-1)(2m+iq)-2im(w-1)+q(w+3)\right) \ , \\
D_2 =& \left(\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}(-q(w+3)-2im(w-1))+(w-1)(q+2im)\right) \ . \qquad \text{(A.137)}
\end{aligned}$$

Another useful property of the J function is manifest in the above form is its reality under complex conjugation

$$J_B(q,\lambda) = J_B^*(-q,\lambda) \ , \ J_F(q,\lambda) = J_F^*(-q,\lambda) \ . \tag{A.138}$$

Yet another useful way to write the J function is to note that the basic integral which appears in the four point function of scalars in an ungauged theory has the form

$$H(q) = \int \frac{d^3r}{(2\pi)^3}\frac{1}{(r^2+m^2)((r+q)^2+m^2)} = \frac{1}{4\pi|q_3|}\tan^{-1}\left(\left|\frac{q_3}{2m}\right|\right) \tag{A.139}$$

for $q_\pm = 0$. Thus we can also write

$$\begin{aligned} J_B(|q|,\lambda) =& \frac{4\pi|q|}{\kappa}\frac{N_1N_2+M_1}{D_1D_2} \ , \\ J_F(|q|,\lambda) =& \frac{4\pi|q|}{\kappa}\frac{N_1N_2+M_2}{D_1D_2} \ , \end{aligned} \tag{A.140}$$

where

$$\begin{aligned} N_1 =& \left(e^{-8\pi i\lambda|q|H(q)}(w-1)(2m+i|q|)+(w-1)(2m-i|q|)\right) \ , \\ N_2 =& \left(e^{-8\pi i\lambda|q|H(q)}(|q|(w+3)+2im(w-1))+(|q|(w+3)-2im(w-1))\right) \ , \\ M_1 =& -8m|q|((w+3)(w-1)-4w)e^{-8\pi i\lambda|q|H(q)} \ , \\ M_2 =& -8m|q|(1+w)^2e^{-8\pi i\lambda|q|H(q)} \ , \\ D_1 =& \left(ie^{-8\pi i\lambda|q|H(q)}(w-1)(2m+i|q|)-2im(w-1)+|q|(w+3)\right) \ , \\ D_2 =& \left(e^{-8\pi i\lambda|q|H(q)}(-|q|(w+3)-2im(w-1))+(w-1)(|q|+2im)\right) \ . \end{aligned} \tag{A.141}$$

A.9.1 Limits of the J function

$\mathcal{N}=2$ point

The $\mathcal{N}=1$ theory studied here enjoys a enhanced $\mathcal{N}=2$ supersymmetry when the ϕ^4 coupling $w=1$. Naturally in this limit we expect the J functions to have a simplification. In particular we get

$$\begin{aligned} J_B^{w=1} =& -\frac{8\pi m}{\kappa} \ , \\ J_F^{w=1} =& \frac{8\pi m}{\kappa} \ . \end{aligned} \tag{A.142}$$

Massless limit

There exists a consistent massless limit for the J functions

$$J_B^{m=0} = J_F^{m=0} = \frac{4\pi|q_3|}{\kappa}\frac{(w-1)(w+3)\sin(\pi\lambda)}{(w-1)(w+3)\cos(\pi\lambda)-w(w+2)-5} \ . \tag{A.143}$$

This expression is self dual under the duality map (2.16). Note that when $w = 1$ this vanishes and is consistent with the $m \to 0$ limit of (A.142).

Non relativistic limit in the singlet channel

The J functions for the S channel are given in (2.158). The non-relativistic limit of the J functions is obtained by taking $\sqrt{s} \to 2m$ with all the other parameters held fixed. In this limit, remarkably we recover the $\mathcal{N} = 2$ result.

$$\begin{aligned} J_B^{\sqrt{s}\to 2m} &= -\frac{8\pi m}{\kappa} \ , \\ J_F^{\sqrt{s}\to 2m} &= \frac{8\pi m}{\kappa} \ . \end{aligned} \tag{A.144}$$

Appendix B

Appendix to chapter 3

B.1 Conventions

B.1.1 Spacetime spinors

The Lorentz group in $D = 3$ is $SL(2,\mathbb{R})$ (see, for instance, the appendix of [103]) and we can impose the Majorana condition on spinors, i.e., the fundamental representation is a real two component spinor $\psi_\alpha = \psi^*_\alpha$ ($\alpha = 1, 2$). The metric signature is mostly plus. $D = 3$ superconformal theories with $\mathcal{N}$ extended supersymmetry posses an $SO(\mathcal{N})$ R-symmetry which is part of the superconformal algebra, whose generators are real antisymmetric matrices I^{ab}, where a, b are the vector indices of $SO(\mathcal{N})$. The supercharges carry a vector R-symmetry index, Q^a_α, as do the superconformal generators S^a_α.

In $D = 3$ we can choose a real basis for the γ matrices

$$(\gamma_\mu)_\alpha^{\ \beta} \equiv (i\sigma^2, \sigma^1, \sigma^3) = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \tag{B.1}$$

Gamma matrices with both indices up (or down) are symmetric

$$(\gamma_\mu)_{\alpha\beta} \equiv (\mathbb{1}, \sigma^3, -\sigma^1) \qquad (\gamma_\mu)^{\alpha\beta} \equiv (\mathbb{1}, -\sigma^3, \sigma^1) \tag{B.2}$$

The antisymmetric ϵ symbol is $\epsilon^{12} = -1 = \epsilon_{21}$. It satisfies

$$\begin{aligned} \epsilon\gamma^\mu\epsilon^{-1} &= -(\gamma^\mu)^T \\ \epsilon\Sigma^{\mu\nu}\epsilon^{-1} &= -(\Sigma^{\mu\nu})^T \end{aligned} \tag{B.3}$$

where $\Sigma^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]$ are the Lorentz generators. The charge conjugation matrix C can be chosen to be the identity, which we take to be

$$-\epsilon\gamma^0 = C^{-1} \qquad \gamma^0\epsilon^{-1} = C \tag{B.4}$$

$C^{\alpha\beta}$ denotes the inverse of $C_{\alpha\beta}$. Spinors transform as follows

$$\psi'_\alpha \to -(\Sigma_{\mu\nu})_\alpha{}^\beta \psi_\beta.$$

Spinors are naturally taken to have index structure down, i.e., ψ_α.

The raising and lowering conventions are

$$\begin{aligned} \psi^\beta &= \epsilon^{\beta\alpha}\psi_\alpha \\ \psi_\alpha &= \epsilon_{\alpha\beta}\psi^\beta \end{aligned} \tag{B.5}$$

There is now only one way to suppress contracted spinor indices,

$$\psi\chi = \psi^\alpha \chi_\alpha,$$

and this leads to a sign when performing Hermitian conjugation

$$(\psi\chi)^* = -\chi^*\psi^*.$$

The γ matrices satisfy

$$(\gamma_\mu\gamma_\nu)_\alpha{}^\beta = \eta_{\mu\nu}\delta_\alpha{}^\beta + \epsilon_{\mu\nu\rho}(\gamma^\rho)_\alpha{}^\beta \tag{B.6}$$

where $\epsilon_{\mu\nu\rho}$ is the Levi-Civita symbol, and we set $\epsilon_{012} = 1$ ($\epsilon^{012} = -1$). The superconformal algebra is given below:

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\lambda}] &= i\left(\eta_{\mu\rho}M_{\nu\lambda} - \eta_{\nu\rho}M_{\mu\lambda} - \eta_{\mu\lambda}M_{\nu\rho} + \eta_{\nu\lambda}M_{\mu\rho}\right), \\
[M_{\mu\nu}, P_\lambda] &= i(\eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu), \\
[M_{\mu\nu}, K_\lambda] &= i(\eta_{\mu\lambda}K_\nu - \eta_{\nu\lambda}K_\mu), \\
[D, P_\mu] &= iP_\mu \ , \quad [D, K_\mu] = -iK_\mu, \\
[P_\mu, K_\nu] &= 2i(\eta_{\mu\nu}D - M_{\mu\nu}), \\
[I_{ab}, I_{cd}] &= i\left(\delta_{ac}I_{bd} - \delta_{bc}I_{ad} - \delta_{ad}I_{bc} + \delta_{bd}I_{ac}\right), \\
\{Q^a_\alpha, Q^b_\beta\} &= (\gamma^\mu)_{\alpha\beta}P_\mu\delta^{ab}, \\
[I_{ab}, Q^\alpha_c] &= i(\delta_{ac}Q^\alpha_b - \delta_{bc}Q^\alpha_a), \\
\{S^a_\alpha, S^b_\beta\} &= (\gamma^\mu)_{\alpha\beta}K_\mu\delta^{ab}, \\
[I_{ab}, S^\alpha_c] &= i(\delta_{ac}S^\alpha_b - \delta_{bc}S^\alpha_a), \\
[K_\mu, Q^a_\alpha] &= i(\gamma_\mu)_\alpha{}^\beta S^a_\beta, \\
[P_\mu, S^a_\alpha] &= i(\gamma_\mu)_\alpha{}^\beta Q^a_\beta, \\
[D, Q^a_\alpha] &= \frac{i}{2}Q^a_\alpha \ , \quad [D, S^a_\alpha] = -\frac{i}{2}S^a_\alpha, \\
[M_{\mu\nu}, Q^a_\alpha] &= -(\Sigma_{\mu\nu})_\alpha{}^\beta Q^a_\beta, \\
[M_{\mu\nu}, S^a_\alpha] &= -(\Sigma_{\mu\nu})_\alpha{}^\beta S^a_\beta, \\
\{Q^a_\alpha, S^b_\beta\} &= \left(\epsilon_{\beta\alpha}D - \frac{1}{2}\epsilon_{\mu\nu\rho}(\gamma^\rho)_{\alpha\beta}M^{\mu\nu}\right)\delta^{ab} + \epsilon_{\beta\alpha}I^{ab}.
\end{aligned} \tag{B.7}$$

All other (anti)-commutators vanish.

B.1.2 R-symmetry

$SO(3)$

Gamma matrices are chosen to be the sigma matrices

$$(\sigma^a)_i{}^j = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right). \tag{B.8}$$

Indices are raised and lowered by $\epsilon^{12} = -1 = -\epsilon_{12}$. Note that σ matrices with both lower or both upper indices are symmetric.

The following identities are useful

$$\begin{aligned} \epsilon^{ij}\phi^k + \epsilon^{jk}\phi^i + \epsilon^{ki}\phi^j &= 0, \\ \epsilon^{ij}\epsilon_{kl} &= \delta^i_l \delta^j_k - \delta^i_k \delta^j_l, \\ \epsilon_{ij}\epsilon_{kl} &= \epsilon_{ik}\epsilon_{jl} - \epsilon_{il}\epsilon_{jk}, \quad \text{(same for upper indices)} \\ (\sigma^a)_i{}^j(\sigma^a)_k{}^l &= 2\delta^l_i \delta^j_k - \delta^j_i \delta^l_k \\ (\sigma^a)_{ij}(\sigma^a)_{kl} &= -(2\epsilon_{il}\epsilon_{jk} + \epsilon_{ij}\epsilon_{kl}) = -(\epsilon_{ik}\epsilon_{jl} + \epsilon_{il}\epsilon_{jk}) \end{aligned} \tag{B.9}$$

$SO(4)$

Gamma matrices are chosen to be

$$\begin{aligned} \Gamma^a &= \begin{pmatrix} 0 & \sigma^a \\ \bar{\sigma}^a & 0 \end{pmatrix} \text{ for } a = 1,2\ldots 4 \\ \text{where } (\sigma^a)_i{}^{\tilde{i}} &= (\sigma^1, \sigma^2, \sigma^3, i\mathbb{1}_2), \quad (\bar{\sigma}^a)_{\tilde{i}}{}^i = (\sigma^1, \sigma^2, \sigma^3, -i\mathbb{1}_2). \end{aligned} \tag{B.10}$$

Indices are raised and lowered by $\epsilon^{12} = -\epsilon_{12} = -1 = \tilde{\epsilon}^{12} = -\tilde{\epsilon}_{12}$. With these definitions, the following identities would be useful.

$$\begin{aligned} (\bar{\sigma}^a)^{\tilde{i}i} &= (\bar{\sigma}^{aT})^{\tilde{i}i} \quad \left((\bar{\sigma}^a)^T = -\epsilon\sigma^a\tilde{\epsilon}^{-1}\right), \\ (\sigma^a)_i{}^{\tilde{i}}(\bar{\sigma}^a)_{\tilde{j}}{}^j &= 2\delta^{\tilde{i}}_{\tilde{j}}\delta^j_i, \\ (\sigma^a)^{i\tilde{i}}(\bar{\sigma}^a)^{\tilde{j}j} &= -2\epsilon^{ij}\epsilon^{\tilde{i}\tilde{j}}, \quad (\sigma^a)_{i\tilde{i}}(\bar{\sigma}^a)_{\tilde{j}j} = -2\epsilon_{ij}\epsilon_{\tilde{i}\tilde{j}}. \end{aligned} \tag{B.11}$$

$SO(6)$

We choose the gamma matrices to be

$$\begin{aligned} &\Gamma^a = \begin{pmatrix} 0 & \gamma^a \\ \bar{\gamma}^a & 0 \end{pmatrix} \text{ for } a = 1, 2 \ldots 6 \\ &\text{where } \gamma^a = (\gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^5, i\mathbb{1}_4),\ \bar{\gamma}^a = (\gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^5, -i\mathbb{1}_4),\ \gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4. \\ &\text{and } \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \bar{\sigma}^i & 0 \end{pmatrix} \text{with } \sigma^i = (\sigma^1, \sigma^2, \sigma^3, i\mathbb{1}_2), \bar{\sigma}^i = (\sigma^1, \sigma^2, \sigma^3, -i\mathbb{1}_2) \text{ for } i = 1 \ldots 4 \end{aligned} \tag{B.12}$$

In these basis we the 'chirality' projection matrix is diagonal and is given by

$$\Gamma^7 = -i\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5 = \begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix} \tag{B.13}$$

The charge conjugation matrix is

$$C = \Gamma^0\Gamma^2\Gamma^4 = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix} \text{with } c = i\gamma^2\gamma^4 \tag{B.14}$$

which satisfies

$$\begin{aligned} C^* = C^{-1} = -C, &\quad (\Gamma^a)^* = C^{-1}\Gamma^a C \\ \Rightarrow \quad c = -c^* = c^{-1}, &\quad (\bar{\gamma}^a)^* = -c^{-1}\gamma^a c \\ \text{In index notation:} \quad (\bar{\gamma}^a{}_i{}^j)^* = (\bar{\gamma}^{a*})^i{}_j &= -c^{ik}(\gamma^a)_k{}^l c_{lj} = c^{ik}c_{jl}(\gamma^a)_k{}^l \end{aligned} \tag{B.15}$$

Indices are raised and lowered with using the charge conjugation matrix C for Γ^a and c for γ^a. With both indices up or down the γ matrices are antisymmetric[1]. The last equation in (B.15) implies the following useful properties for the generators Let us define

$$\gamma^{ab} = \gamma^a\bar{\gamma}^b - \gamma^b\bar{\gamma}^a, \quad \bar{\gamma}^{ab} = \bar{\gamma}^a\gamma^b - \bar{\gamma}^b\gamma^a,$$

then we have following useful relations

$$\begin{aligned} \gamma^{ab\dagger} = -\gamma^{ab}, &\quad \bar{\gamma}^{ab\dagger} = -\bar{\gamma}^{ab}, \\ (\bar{\gamma}^{ab*})^i{}_j = (c^{-1}\gamma^{ab}c)^i{}_j, &\quad (\bar{\gamma}^{ab})_i{}^j = -(c^{-1}\gamma^{ab}c)^j{}_i. \end{aligned} \tag{B.16}$$

The first line says that the generators of $SO(6)$ transformation are Hermitian[2] while the two equation in the second line follows from (B.15).

The following identities are useful[3]:

$$\begin{aligned} \bar{\gamma}^a_{ij} &= \gamma^a_{ij} + 2\delta^{a0}c_{ij}, \quad (\bar{\gamma}^a)_i{}^j = (\gamma^a)_i{}^j - 2\delta^{a0}\delta^j_i, \\ \gamma^a_{ij}\gamma^a_{kl} &= -2\epsilon_{ijkl} = 2(c_{ik}c_{jl} - c_{il}c_{jk} - c_{ij}c_{kl}), \\ \gamma^a_{ij}\bar{\gamma}^a{}_{kl} &= -2\epsilon_{ijkl} + 2c_{ij}c_{kl} = 2(c_{ik}c_{jl} - c_{il}c_{jk}), \\ (\gamma^a)^{ij}(\bar{\gamma}^a)_{kl} &= 2\delta^i_k\delta^j_l - 2\delta^i_l\delta^j_k, \\ (\gamma^{ab})^j_i(\gamma^{ab})^l_k &= -32\delta^l_i\delta^j_k + 8\delta^j_i\delta^l_k, \end{aligned} \tag{B.17}$$

[1]This should be the case as the vector of $SO(6)$ is $(4\times 4)_{\text{antisym}}$ of $SU(4)$.

[2]The generator of $SO(6)$ acting on chiral and antichiral transformation are respectively $-\frac{i}{4}\gamma^{ab}$ and $-\frac{i}{4}\bar{\gamma}^{ab}$

[3]Note that representation theory ($SU(4)$) wise C shouldn't be used to raise or lower indices as it is not an invariant tensor of $SU(4)$. Only ϵ^{ijkl} and ϵ_{ijkl}(which are specific combinations of product of c's) can be used to raise or lower $SU(4)$ indices. we will explicitly see that all the $SU(4)$ tensor equations can be written using just ϵ tensors.

B.2 Useful relations

Some useful relations and identities are given below

$$\epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^{\alpha\beta}} = -\frac{\partial}{\partial\theta^{a}_{\alpha}} \tag{B.18}$$

$$(\gamma^{\mu})_{\alpha}^{\ \beta}(\gamma_{\mu})_{\sigma}^{\ \rho} = 2\delta_{\alpha}^{\ \rho}\delta_{\sigma}^{\ \beta} - \delta_{\alpha}^{\ \beta}\delta_{\sigma}^{\ \rho} \tag{B.19}$$

$$\theta_{\alpha}\theta_{\beta} = \frac{1}{2}\epsilon_{\alpha\beta}\theta\theta, \quad \theta^{\alpha}\theta^{\beta} = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta \tag{B.20}$$

$$\theta_{1\alpha}\theta_{2}^{\beta} + \theta_{2\alpha}\theta_{1}^{\beta} + (\theta_1\theta_2)\delta_{\alpha}^{\beta} = 0 \tag{B.21}$$

$$X^2 \equiv X_{\alpha}^{\ \beta}X_{\beta}^{\ \alpha} = 2x_{\mu}x^{\mu} \equiv 2x^2\,, \quad X_{\alpha}^{\ \beta}X_{\beta}^{\ \gamma} = x^2\delta_{\alpha}^{\gamma} = \frac{X^2}{2}\delta_{\alpha}^{\gamma} \tag{B.22}$$

$$D_{1\alpha}\tilde{X}_{12}^{-\Delta} = i\Delta(\tilde{X}_{12})_{\alpha}^{\ \beta}(\theta_{12})_{\beta} \tag{B.23}$$

$$D_{1\alpha}(\tilde{X}_{12})_{\beta}^{\ \gamma} = -i\delta_{\alpha}^{\gamma}(\theta_{12})_{\beta} + \frac{i}{2}\delta_{\beta}^{\gamma}(\theta_{12})_{\alpha} \tag{B.24}$$

$$D_{1\alpha}(X_{12-})_{\beta}^{\ \gamma} = -i\delta_{\alpha}^{\gamma}\theta_{12\beta}\,, \quad D_{1\alpha}(X_{12+})_{\beta}^{\ \gamma} = i\epsilon_{\alpha\beta}\theta_{12}^{\gamma} \tag{B.25}$$

B.3 Conformal spectrum of free scalars and fermions

In this appendix we list the character decomposition of product of two short conformal representations into irreducible conformal representations for the particular cases of a complex scalar and complex fermions. Let us denote by $\chi(\phi)$ and $\chi(\psi)$ the conformal character of a free scalar field and a free fermion (in $D = 3$) respectively. Then we have

$$\begin{aligned}
\chi(\phi)\chi(\bar{\phi}) &= \frac{1}{(1-x)(1-xy)(1-xy^{-1})}\left(x + \sum_{k=1}^{\infty}\chi_{(sh)}(k+1,k)\right)\\
\chi(\bar{\phi})\chi(\psi) = \chi(\phi)\chi(\bar{\psi}) &= \frac{1}{(1-x)(1-xy)(1-xy^{-1})}\left(x^{\frac{3}{2}}\chi_{\frac{1}{2}}(y) + \sum_{k=1}^{\infty}\chi_{(sh)}(k+\frac{3}{2},k+\frac{1}{2})\right)\\
\chi(\psi)\chi(\bar{\psi}) &= \frac{1}{(1-x)(1-xy)(1-xy^{-1})}\left(x^2 + \sum_{k=1}^{\infty}\chi_{(sh)}(k+1,k)\right)
\end{aligned} \tag{B.26}$$

where the $\chi_{(sh)}(j+1,j)$ denotes the character of a *short* conformal representation with spin j.

Let us consider a free $\mathcal{N} = 1$ superconformal theory of a complex boson and a complex fermion transforming in N of $SU(N)$ gauge group. The spectrum of gauge invariant single trace operators in theory is then just the sum of the operators represented in (B.26). Using the decomposition in (B.28), the operators in (B.26) are easily combined into the representation of $\mathcal{N}=1$ supermultiplets. These are given as follows [4]

$$\left(\frac{1}{2},0\right)_1 \oplus \sum_{k=1}^{\infty}\left(\frac{k}{2}+1,\frac{k}{2}\right)_1 \tag{B.27}$$

i.e. along with the special short representation with spin zero there are superconformal short representation for every positive half integer spin starting from spin $\frac{1}{2}$. For convenience we list the decomposition of all short and long $\mathcal{N}=1$ superconformal representations below

$$\begin{aligned}
(\Delta,j)_{1,long} &= (\Delta,j)\oplus\left(\Delta+\frac{1}{2},j-\frac{1}{2}\right)\oplus\left(\Delta+\frac{1}{2},j+\frac{1}{2}\right)\oplus(\Delta+1,j),\\
(j+1,j)_1 &= (j+1,j)\oplus\left(j+\frac{3}{2},j+\frac{1}{2}\right),\\
\left(\frac{1}{2},0\right)_1 &= \left(\frac{1}{2},0\right)\oplus\left(1,\frac{1}{2}\right)\oplus\left(\frac{3}{2},0\right).
\end{aligned} \tag{B.28}$$

B.4 Superconformal spectrum of $\mathcal{N}=1,2,3,4,6$ theories

In this appendix we discuss the full single trace gauge invariant local operator spectrum of the free $U(N)$ superconformal Chern-Simons vector theories discussed in section 3.3. In subsequent subsections here we present the full conformal primary spectrum, using the conformal grouping discussed in appendix B.3, and then group these conformal primaries into representations of superconformal algebra of the respective theory[5].

B.4.1 $\mathcal{N}=1$

The minimal field content of this theory consists of a complex scalar and a complex fermion. The conformal content is easy to write down; there are both integer and half-integer spin currents in the theory. All of these group into short superconformal multiplets of both integer and half integer spin. Thus, the superconformal primary content of this theory is

$$\bigoplus_{j=0,\frac{1}{2},1,\ldots}^{\infty}(j+1,j)_{\mathcal{N}=1}, \tag{B.29}$$

[4]Here $(\Delta,j)_1$ denote $\mathcal{N}=1$ representation while (Δ,j) denotes a conformal representation.

[5]Representation of superconformal algebra are labeled by the scaling dimension, spin and R-symmetry representation of the superconformal primary. see e.g. section 3 of [104] a summary of unitary representations of superconformal algebra in 2+1d.

where $(j+1, j)$ denotes a dimension $j+1$, spin j short superconformal primary multiplet which contains the conserved spin j and spin $j+\frac{1}{2}$ conformal primaries. There is no R-symmetry quantum number in this case. The conformal content is

$$\begin{aligned} (j+1, j)_{\mathcal{N}=1} &\to (j+1, j) \oplus (j+\frac{3}{2}, j+\frac{1}{2}) \qquad j \neq 0 \\ (1,0)_{\mathcal{N}=1} &\to (1,0) \oplus \left(\frac{3}{2}, \frac{1}{2}\right) \oplus (2,0) \qquad j = 0. \end{aligned} \tag{B.30}$$

B.4.2 $\mathcal{N} = 2$

The field content of the $\mathcal{N} = 2$ theories is the same as that of $\mathcal{N} = 1$; the difference being that the spectrum of short superconformal multiplets consists only of integer spins. Thus, we can write the spectrum of short superconformal primaries in these theories as

$$\bigoplus_{j=0,1,\ldots}^{\infty} (j+1, j, 0)_{\mathcal{N}=2}. \tag{B.31}$$

The conformal content for a spin j $\mathcal{N} = 2$ short superconformal primary in terms of $\mathcal{N} = 1$ is

$$(j+1, j, 0)_{\mathcal{N}=2} \to (j+1, j)_{\mathcal{N}=1} \oplus (j+\frac{3}{2}, j+\frac{1}{2})_{\mathcal{N}=1} \tag{B.32}$$

from which he conformal content can be read off as

$$\begin{aligned} (j+1, j, 0)_{\mathcal{N}=2} &\to (j+1, j, 0) \oplus (j+\frac{3}{2}, j+\frac{1}{2}, 1) \oplus (j+\frac{3}{2}, j+\frac{1}{2}, -1) \oplus (j+2, j+1, 0) \quad j \neq 0 \\ (1,0,0)_{\mathcal{N}=2} &\to (1,0,0) \oplus (\frac{3}{2}, \frac{1}{2}, 1) \oplus (2,0,0) \oplus (\frac{3}{2}, \frac{1}{2}, -1) \oplus (2,1,0) \qquad j = 0. \end{aligned} \tag{B.33}$$

where the third quantum number is the $U(1)_R$ charge.

B.4.3 $\mathcal{N} = 3$

The conformal content of the $\mathcal{N} = 3$ theory is[6]

$$\left[2 \bigoplus_{j=0,\frac{1}{2},1,\ldots}^{\infty} [(j+1, j, 1) \oplus (j+1, j, 3)] \right] \oplus (1,0,1) \oplus (1,0,3) \oplus (2,0,1) \oplus (2,0,3) \tag{B.34}$$

[6] For R-symmetry quantum numbers taking values in $SU(2)_R$, we give the dimension of the representation while writing down the quantum numbers (Δ, j, h). For example, $(1,0,1)$ corresponds to $\Delta = 1$, spin-0 and a singlet under R. In other words, instead of writing the highest weight j for the R-symmetry representation, we write $2j+1$ as the third quantum number

The above conformal content can be grouped into $\mathcal{N}=3$ superconformal primary content as follows

$$\left[\bigoplus_{j=0,\frac{1}{2},1,\ldots}^{\infty} (j+1,j,1)_{\mathcal{N}=3}\right] \oplus (1,0,3)_{\mathcal{N}=3} \tag{B.35}$$

The decomposition of the $\mathcal{N}=3$ superconformal primaries into $\mathcal{N}=2$ superconformal primaries, given in [104] is[7]

$$(j+1,j,1)_{\mathcal{N}=3} \longrightarrow \quad (j+1,j,0)_{\mathcal{N}=2} \quad \oplus \qquad (j+\tfrac{3}{2},j+\tfrac{1}{2},0)_{\mathcal{N}=2} \tag{B.36}$$

We have the following result for the conformal content of a $\mathcal{N}=3$ superconformal primary of spin $j\in(0,\frac{1}{2},1,\ldots)$:

$$(j+1,j,1)_{\mathcal{N}=3} \to (j+1,j,1)\oplus(j+\frac{3}{2},j+\frac{1}{2},3)\oplus(j+2,j+1,3)\oplus(j+\frac{5}{2},j+\frac{3}{2},1) \tag{B.37}$$

The breakup of the $(1,0,3)_{\mathcal{N}=3}$ superconformal primary into $\mathcal{N}=2$ primaries is as follows[18]

$$(1,0,3)_{\mathcal{N}=3} \to \quad (1,0,1)_{\mathcal{N}=2}\oplus(1,0,0)_{\mathcal{N}=2}\oplus(1,0,-1)_{\mathcal{N}=2} \tag{B.38}$$

The conformal content of the $(1,0,3)_{\mathcal{N}=3}$ superconformal primary is[8]:

$$(1,0,3)_{\mathcal{N}=3} \to (1,0,3)\oplus(2,0,3)\oplus(\frac{3}{2},\frac{1}{2},1)\oplus(\frac{3}{2},\frac{1}{2},3)\oplus(2,1,1) \tag{B.39}$$

B.4.4 $\mathcal{N}=4$

The theory contains currents of integer spins only. It remains to comment about the $(1,0,3)$ superconformal primary which was obtained in equation B.35 above. This particular primary transforms in the antisymmetric $(1,0)$ representation of the $SO(4)\sim SU(2)\times SU(2)$ R-symmetry, (whose representations we label by (j_1,j_2)) where j_1 and j_2 correspond to the spin quantum number (the highest weight of the representation) of each of the two $SU(2)$s respectively. Therefore we have

$$\left[\bigoplus_{j=0,1,\ldots}^{\infty} (j+1,j,\{0,0\})_{\mathcal{N}=4}\right] \oplus (1,0,\{1,0\})_{\mathcal{N}=4} \tag{B.40}$$

where by $\{0,0\}$ we mean the singlet of the $SU(2)\times SU(2)$ R-symmetry.

[7] In the equation that follows note that the L.H.S. is written in terms of the $SU(2)_R$ quantum number whereas the R.H.S. has $U(1)_R$ quantum numbers.

[8] Written out in $SU(2)_R$ notation.

B.4.5 $\mathcal{N} = 6$

From the field content of the $\mathcal{N} = 6$ theory (see section 3.3.5) the conformal primary spectrum can be easily read off as

$$\left[2 \bigoplus_{j=0,1,\ldots}^{\infty} \Big((j+2, j+1; 1) \oplus (j+2, j+1; 15) \oplus (j+\frac{3}{2}, j+\frac{1}{2}; 6) \oplus (j+\frac{3}{2}, j+\frac{1}{2}; 10)\Big)\right]$$
$$\oplus (1, 0; 1) \oplus (1, 0; 15) \oplus (2, 0; 1) \oplus (2, 0; 15) \oplus \tag{B.41}$$

where the conformal primaries are labelled as ($\Delta\,, j\,; SO(6)$ representation). In specifying the $SO(6)$ R-symmetry representation we use the following notation

$$\begin{aligned} 1 &\rightarrow \text{Singlet}, \\ 6 &\rightarrow \text{Vector}, \\ 15 &\rightarrow \text{Second rank symmetric traceless tensor}, \\ 10 &\rightarrow \text{(anti) Self-dual 3 form}. \end{aligned} \tag{B.42}$$

The conformal primary spectrum can be grouped together into the following $\mathcal{N} = 6$ superconformal primary spectrum

$$\left[\bigoplus_{j=1,2,\ldots}^{\infty} (j+1, j; 1)_{\mathcal{N}=6}\right] \oplus (1, 0; 15)_{\mathcal{N}=6} \tag{B.43}$$

where again use the same labelling for the superconformal primary as above for conformal primaries with and extra subscript to distinguish from conformal primaries.

Appendix C

Appendix to chapter 4

C.1 Superradiant instability of small black holes

In this appendix we analyse the dynamical stability of RNAdS black holes in AdS_5 to superradiant emission in the presence of a massless charged minimally coupled scalar field. As we have explained in the introduction, we intuitively expect a very small black hole to be unstable whenever $\mu \geq \frac{4}{e}$. In this appendix we verify this expectation by computing the frequency of the lowest quasinormal mode of the black hole in a perturbative expansion R, the radius of the black hole. We find that the imaginary part of this frequency flips sign (from stable to unstable) as μ increases past $\frac{4}{e}$, exactly as we expected on intuitive grounds.

We wish to compute the lowest quasinormal mode of (4.11) at small R. By definition, quasinormal modes are regular at the future horizon, so it is useful to work in coordinates that are good at the future horizon. We choose to work in ingoing Eddington-Finkelstein coordinate; in other words we replace the Schwarzschild time t with the new time variable

$$v = t + \int \frac{1}{V(r)} dr,$$

where $V(r)$ given by

$$V(r) = \left(1 - \frac{R^2}{r^2}\right)\left(1 + r^2 + R^2 - \frac{2}{3}\mu^2\right) \tag{C.1}$$

In these new coordinates the background (4.11) takes the form

$$ds^2 = 2dvdr - V(r)dv^2 + r^2(d\theta^2 + \sin^2(\theta)d\psi^2 + \sin^2(\theta)\sin^2(\psi)d\lambda^2). \tag{C.2}$$

with

$$\begin{aligned} A_v &= \mu\left(1 - \frac{R^2}{r^2}\right). \\ A_r &= -\frac{\mu}{V(r)}\left(1 - \frac{R^2}{r^2}\right). \end{aligned} \tag{C.3}$$

A linearised scalar fluctuation about this background takes the form

$$D^\mu D_\mu \phi(v, r) = 0, \tag{C.4}$$

where

$$D_\mu \equiv \nabla_\mu - ieA_\mu,$$

with A_μ being the background gauge field (C.3)

In the rest of this appendix we will solve (C.4) separately in a far field region, $r \gg R$ and a near field region $r \ll 1$. In the limit $R \ll 1$, of interest here, the solution may then be determined everywhere by matching the two solutions in their overlapping domain of validity. The matching procedure may be carried out systematically in a power series in R, and turns out to determine the frequency of the quasinormal mode in a power series in R. We carry out this procedure to order R^3, the first order at which the quasinormal frequency develops an imaginary component. [1]

C.1.1 Solution in the near field region

When $r \ll 1$ it is useful to work with the rescaled coordinate y given by

$$r = Ry.$$

Let the scalar fluctuation take the form

$$\Phi^{in}(v, y) = \exp(-i\omega v)\, \Phi^{in}(y). \tag{C.5}$$

where

$$\Phi^{in}(y) = \Phi_0^{in}(y) + \Phi_1^{in}(y)R + \Phi_2^{in}(y)R^2 + \mathcal{O}(R^3). \tag{C.6}$$

and

$$\omega = 4 - \mu e + \omega^{(1)}R + \omega^{(2)}R^2 + \omega^{(3)}R^3 + \mathcal{O}(R^4) \tag{C.7}$$

Note we have chosen to study the quasinormal mode with the frequency $4 - \mu e + \mathcal{O}(R)$; here we have used the physical expectation that the lowest quasinormal mode should reduce to the lowest normal mode in the limit $R \to 0$. The energy of the lowest normal mode is 4.

[1] The fact that the quasinormal frequency first develops an imaginary piece at $\mathcal{O}(R^3)$ is simply related to the fact that the area - and so low frequency absorption cross section of a black hole in 5 dimensions - scales like R^3.

It is a simple matter to solve (C.4) perturbatively in R. Imposing the physical requirement of regularity at the horizon we find

$$
\begin{aligned}
\Phi_0^{in}(y) &= d_0, \\
\Phi_1^{in}(y) &= \frac{1}{6\left(2\mu^2-3\right)}\Big(6\left(2\mu^2-3\right)\left(d_1 - i d_0 y(e\mu-4)\right) \\
&\quad + i d_0(e\mu-4)\left(9\left(\log\left(3y^2-2\mu^2\right)-2\log(y+1)\right)+4\sqrt{6}\mu^3\tanh^{-1}\left(\frac{\sqrt{\frac{3}{2}}y}{\mu}\right)\right)\Big), \\
\Phi_2^{in}(y) &= -\frac{1}{2}d_0 y^2\left(e^2\mu^2-8e\mu+20\right) \\
&\quad + \frac{y(4-e\mu)\left(d_0(e\mu-4)\left(2\sqrt{6}\pi i\mu^3-9\log(3)\right)+6 i d_1\left(2\mu^2-3\right)\right)}{6\left(2\mu^2-3\right)} \\
&\quad - 4d_0\left(e\mu-2\mu^2-3\right)\log\left(\frac{1}{y}\right)+d_2+\mathcal{O}\left(\frac{1}{y}\right).
\end{aligned}
\tag{C.8}
$$

where d_0, d_1 and d_2 are as yet undetermined integration constants (they will be determined below by matching). For brevity we have also presented the result assuming $\omega^{(1)} = 0$, a result that turns out to be forced on us by matching with the far field expansion below.

C.1.2 Solution in the far field region

In the outer region the fluctuation takes the form

$$\Phi^{out}(v,r) = \exp\left(-i\omega v\right)\Phi^{out}(r). \tag{C.9}$$

where

$$\Phi^{out}(r) = \Phi_0^{out}(r) + \Phi_1^{out}(r)R + \Phi_2^{out}(r)R^2 + \Phi_3^{out}(r)R^3 + \mathcal{O}(R^4). \tag{C.10}$$

Solving the equation of motion subject to the requirement of normalisability at large r we find

$$
\begin{aligned}
\Phi_0^{out}(r) &= \frac{e^{-i(e\mu-4)\tan^{-1}(r)}}{\left(r^2+1\right)^2}, \\
\Phi_1^{out}(r) &= 0, \\
\Phi_2^{out}(r) &= \frac{e^{-i(e\mu-4)\tan^{-1}(r)}}{6r\left(r^2+1\right)^3}\Big(3ir^2\left(2\mu^2+3\right)(e\mu-4)+\frac{3}{2}\left(r^2+1\right)r\Big(-8e\mu\log\left(r^2+1\right) \\
&\quad +16\left(e\mu-2\mu^2-3\right)\log(r)+2i\left(e\mu\left(2\mu^2+9\right)-8\left(2\mu^2+3\right)\right)\tan^{-1}(r)-2i\pi e\mu^3 \\
&\quad -9i\pi e\mu+16\mu^2\log\left(r^2+1\right)+24\log\left(r^2+1\right)+16i\pi\mu^2+24i\pi\Big) \\
&\quad +2i\left(2\mu^2+3\right)(e\mu-4)-4r\left(2\mu^2+3\right)\Big),
\end{aligned}
\tag{C.11}
$$

and,

$$\Phi_3^{out}(r) = -\frac{e^{-i(e\mu-4)\tan^{-1}(r)}}{6\left(r^3+r\right)^2}\omega_3\Big(3i\pi r^2 + 3r^2\left(\log\left(r^2+1\right) - 2\log(r) - 2i\tan^{-1}(r)\right) + 1\Big) \tag{C.12}$$

where once again we have presented the results only for $\omega^{(1)} = 0$. We have also plugged $\omega^{(2)} = -6 + 3e\mu - 4\mu^2$ in the expression for $\Phi_2(r)$ and $\Phi_3(r)$ (this is forced on us by the matching condition below). Here besides imposing normalisability at infinity, we have also demanded that the coefficient of the leading normalisable piece is one.

C.1.3 Conditions for patch up

In order to complete our determination of the solution, we must now match the near and far field solutions. The logic for this matching procedure is exactly as described in subsection 4.3.3. Implementing this procedure we find

$$\begin{aligned}
d_0 &= 1,\\
d_1 &= \frac{i(e\mu-4)\left(2\sqrt{6}i\pi\mu^3 - 9\log(3)\right)}{6\left(2\mu^2-3\right)},\\
d_2 &= \frac{1}{12}\Big(4\left(2\mu^2+3\right)\left(e^2\mu^2 - 8e\mu + 14\right) + 48\left(e\mu - 2\mu^2 - 3\right)\log(R)\\
&\quad -3i\pi\left(2e\mu^3 + 9e\mu - 16\mu^2 - 24\right)\Big).
\end{aligned} \tag{C.13}$$

Also the quasinormal frequency is determined to be

$$\omega = 4 - e\mu - R^2(6 - 3e\mu + 4\mu^2) - R^3\left(3i(4-e\mu)\right) + \mathcal{O}\left(R^4\right). \tag{C.14}$$

Once these matching conditions are imposed, the large y expansion of the near field solution (with y substituted by $\frac{r}{R}$) and the small r expansion of the far field solution both share the common expansions

$$
\begin{aligned}
\Phi^{out}(r) =& \left(1 - ir(e\mu - 4) + r^2\left(-\frac{1}{2}e^2\mu^2 + 4e\mu - 10\right) + O\left(r^3\right)\right) \\
&+ \left(\frac{i\left(2\mu^2+3\right)(e\mu-4)}{3r} + (4(\mu(e-2\mu)-3)\log(r) \right. \\
&+\frac{1}{3}\left(2\mu^2+3\right)(e\mu(e\mu-8)+14) \\
&\left.\left. -\frac{1}{4}i\pi\left(2e\mu^3+9e\mu-16\mu^2-24\right)\right) + O\left(r\right)\right) R^2 \\
&- \left(\frac{i(e\mu-4)}{2r^2} + \mathcal{O}\left(\frac{1}{r}\right)\right)R^3 + \mathcal{O}\left(R^4\right). \qquad \text{(C.15)} \\
\Phi^{in}(r) =& \left(r^2\left(-\frac{1}{2}e^2\mu^2 + 4e\mu - 10\right) - ir(e\mu-4) + 1 + O\left(\frac{1}{r}\right)\right) \\
&+ \left(\left(4(\mu(e-2\mu)-3)\log(r) + \frac{1}{3}\left(2\mu^2+3\right)(e\mu(e\mu-8)+14)\right.\right. \\
&\left.\left.-\frac{1}{4}i\pi\left(2e\mu^3+9e\mu-16\mu^2-24\right)\right) + \frac{i\left(2\mu^2+3\right)(e\mu-4)}{3r} + O\left(\frac{1}{r^2}\right)\right) R^2 \\
&- \left(\frac{i(e\mu-4)}{2r^2} + \mathcal{O}\left(\frac{1}{r^3}\right)\right)R^3 + \mathcal{O}\left(R^4\right).
\end{aligned}
$$

Equation (C.14) is the main result of this appendix. Note that the imaginary part of ω turns positive as μ exceeds $\frac{4}{e}$, demonstrating that RNAdS black holes with $\mu \geq \frac{4}{e}$ suffer from a super radiant instability.

C.2 Results of the low order perturbative expansion of the hairy black hole

In this appendix we present explicit results for the perturbative expansion of the hairy black hole solution at low orders in perturbation theory. See section 4.2 for explanation of the notation etc.

C.2.1 Near field expansion

$$\begin{aligned}
f^{in}_{(0,0)}(y) =& \frac{(y^2-1)(3e^2y^2-32)}{3e^2y^4}\\
f^{in}_{(0,2)}(y) =& \frac{(y^2-1)(3(y^4+y^2)e^4-96e^2+1024)}{3e^4y^4}\\
f^{in}_{(0,4)}(y) =& \frac{32(y^2-1)\left[27e^4+1536e^2+384(3e^2-32)\log\left[\left(1-\frac{32}{3e^2}\right)R^2\right]-22528\right]}{27e^6y^4}\\
f^{in}_{(2,0)}(y) =& -\frac{8(y^2-1)(3e^2(7y^2-4)-64)}{63e^2y^4}
\end{aligned} \tag{C.16}$$

$$\begin{aligned}
g^{in}_{(0,0)}(y) =& \frac{3e^2y^4}{(y^2-1)(3e^2y^2-32)}\\
g^{in}_{(0,2)}(y) =& -\frac{3y^4(3(y^4+y^2)e^4-96e^2+1024)}{(y^2-1)(32-3e^2y^2)^2}\\
g^{in}_{(0,4)}(y) =& -\frac{2048y^4(9(5y^2-6)e^4+96(1-11y^2)e^2+6656)}{3e^2(y^2-1)(3e^2y^2-32)^3}\\
&+\frac{9e^2y^6\left(3(y^3+y)^2e^4-96(2y^2+3)e^2+2048y^2\right)}{(y^2-1)(3e^2y^2-32)^3}\\
&+\frac{y^4(y^8-12288(3e^2-32)(3e^2y^2-32))}{3e^2(y^2-1)(3e^2y^2-32)^3}\log\left[\left(1-\frac{32}{3e^2}\right)R^2\right]\\
g^{in}_{(2,0)}(y) =& \frac{32e^2(40-3e^2)y^4}{7(y^2-1)(32-3e^2y^2)^2}
\end{aligned} \tag{C.17}$$

$$\begin{aligned}
A^{in}_{(0,0)}(y) =& \frac{4}{e}\left(1-\frac{1}{y^2}\right)\\
A^{in}_{(0,2)}(y) =& \left(\frac{(6e^2-64)}{e^3}\right)\left(1-\frac{1}{y^2}\right)\\
A^{in}_{(0,4)}(y) =& \left(\frac{189e^4+4416e^2+1536(3e^2-32)\log\left(\left(1-\frac{32}{3e^2}\right)R^2\right)-80896}{18e^5}\right)\left(1-\frac{1}{y^2}\right)\\
A^{in}_{(2,0)}(y) =& -\frac{2(3e^2+16)}{21e}\left(1-\frac{1}{y^2}\right)
\end{aligned} \tag{C.18}$$

$$
\begin{aligned}
\phi^{in}_{(1,0)}(y) =& 1 \\
\phi^{in}_{(1,2)}(y) =& \alpha + \frac{1}{3e^2}\Bigg[-6e^2y^2 - 128\log\left(3e^2-32\right)\log\left(\frac{y^2-1}{3e^2y^2-32}\right) - 192\log\left(3e^2y^2-32\right) \\
& + 6\log\left(3e^2y^2-32\right)e^2 + 128\log\left(-\frac{3e^2\left(y^2-1\right)}{3e^2-32}\right)\log\left(3e^2y^2-32\right) \\
& + 64\log^2\left(3e^2y^2-32\right) + 128\mathrm{Li}_2\left(\frac{32-3e^2y^2}{32-3e^2}\right)\Bigg] \\
\phi^{in}_{(3,0)}(y) =& \frac{1}{63}\left(150-13e^2\right)
\end{aligned}
\tag{C.19}
$$

where

$$
\begin{aligned}
\alpha =& \frac{2\left(-9e^2-192\log\left(3e^2-32\right)+288\right)}{9e^2}\log(3) - \frac{2\left(3e^2-32\log^2\left(32-3e^2\right)+32\right)}{3e^2} \\
& + \frac{64\pi^2}{9e^2} - 18\left(e^2-32\right)\log(R) + 6\log(e)\left(3e^2+64\log\left(3e^2-32\right)-96\right)
\end{aligned}
\tag{C.20}
$$

C.2.2 Far field expansion

$$
\begin{aligned}
f^{out}_{(0,0)}(r) =& 1 + r^2 \\
f^{out}_{(0,2)}(r) =& \left(1+\frac{32}{3e^2}\right)\frac{1}{r^2} \\
f^{out}_{(0,4)}(r) =& \frac{32e^2+\left(1024-3e^2\left(e^2+32\right)\right)r^2}{3e^4r^4} \\
f^{out}_{(2,0)}(r) =& -\frac{8\left(r^4+3r^2+3\right)}{9\left(r^2+1\right)^3} \\
f^{out}_{(2,2)}(r) =& \frac{1}{189e^2r^2(1+r^2)^4}\left[-256\left(84r^{10}+463r^8+914r^6+755r^4+193r^2-6\right)\right. \\
& + 8e^2\left(252r^{10}+1261r^8+2419r^6+2169r^4+921r^2+99\right) \\
& \left.+84\left[\left(3e^2-32\right)r^2\left(r^2+1\right)^4+\left(32-e^2\right)r^2\left(r^4+3r^2+3\right)\right]\log\left(\frac{r^2}{r^2+1}\right)\right]
\end{aligned}
\tag{C.21}
$$

$$
\begin{aligned}
g^{out}_{(0,0)}(r) =& \frac{1}{1+r^2} \\
g^{out}_{(0,2)}(r) =& \left(1+\frac{32}{3e^2}\right)\frac{1}{r^2(1+r^2)^2} \\
g^{out}_{(0,4)}(r) =& \frac{9\left(r^4+r^2+1\right)e^4+96\left(3r^4+2r^2+1\right)e^2-1024\left(3\left(r^4+r^2\right)-1\right)}{9e^4r^4\left(r^2+1\right)^3} \\
g^{out}_{(2,0)}(r) =& \frac{8r^2\left(r^2+3\right)}{9\left(r^2+1\right)^5} \\
g^{out}_{(2,2)}(r) =& 8\left(-127e^2-\frac{448\left(6r^6+20r^4+14r^2+5\right)}{\left(r^2+1\right)^4}+2720\right) \\
&+\frac{56e^2\left(9r^6+36r^4+70r^2+13\right)}{\left(r^2+1\right)^4} \\
&+\frac{2\left(e^2\left(12r^8+57r^6+72r^4+70r^2+13\right)-384r^4\left(r^4+4r^2+3\right)\right)}{\left(r^2+1\right)^4}\log\left(\frac{r^2}{r^2+1}\right)
\end{aligned}
\tag{C.22}
$$

$$
\begin{aligned}
A^{out}_{(0,0)}(r) =& \frac{4}{e} \\
A^{out}_{(0,2)}(r) =& \frac{2\left(e^2\left(3-\frac{2}{r^2}\right)-32\right)}{e^3} \\
A^{out}_{(0,4)}(r) =& -\frac{189e^4+4416e^2+1536\left(3e^2-32\right)\log\left(\left(1-\frac{32}{3e^2}\right)R^2\right)-80896}{18e^5}+\frac{64-6e^2}{e^3r^2} \\
A^{out}_{(2,0)}(r) =& -\frac{e\left(r^4+3r^2+3\right)}{6\left(r^2+1\right)^3}+\frac{9e^2-64}{42e} \\
A^{out}_{(2,2)}(r) =& \frac{\left(\frac{33285}{r^2}-75969\right)e^4+96\left(23507-\frac{6685}{r^2}\right)e^2-13746176}{26460e^3} \\
&-\frac{8\left(24r^8+60r^6+20r^4-51r^2-29\right)}{9er^2\left(r^2+1\right)^4}+\frac{e\left(72r^8+228r^6+228r^4+55r^2-35\right)}{36r^2\left(r^2+1\right)^4} \\
&+\frac{2\left(-32\left(r^2+2\right)r^4+e^2\left(3r^4+8r^2+6\right)r^2+64\right)}{3e\left(r^2+1\right)^3}\log\left(\frac{r^2}{r^2+1}\right)
\end{aligned}
\tag{C.23}
$$

$$
\begin{aligned}
\phi^{out}_{(1,0)}(r) =& \frac{1}{(r^2+1)^2} \\
\phi^{out}_{(1,2)}(r) =& \frac{2\left(-3e^2+6\left(e^2-32\right)\left(r^2+1\right)\log(r)-3\left(e^2-32\right)\left(r^2+1\right)\log\left(r^2+1\right)-32\right)}{3e^2\left(r^2+1\right)^3} \\
\phi^{out}_{(3,0)}(r) =& \frac{64r^6+260r^4+360r^2-e^2\left(9r^6+30r^4+34r^2+13\right)+150}{63\left(r^2+1\right)^6}
\end{aligned}
\tag{C.24}
$$

C.3 The soliton at high orders in perturbation theory

C.3.1 Explicit results to $\mathcal{O}(\epsilon^{17})$

As we have mentioned in section 4.5, the perturbation theory that generates the soliton solution as a function of ϵ is straightforward and hence may be automated on Mathematica. We have implemented this automation and used it to generate the ground state soliton solution to $\mathcal{O}(\epsilon^{17})$. For what its worth, we present the resultant explicit formulae for all thermodynamical quantities: the mass, the charge and the chemical potential to $\mathcal{O}(\epsilon^{17})$. Later in this appendix we will also speculate that our solution develops a singularity at the origin at a finite value of ϵ. To aid this discussion we also present formulas for $f(r=0)$ and $\phi(r=0)$ to the same order in ϵ.

$$
\begin{aligned}
&m = \\
&0.888889\epsilon^2+\left(1.9737-0.170496e^2\right)\epsilon^4+\left(10.7168-1.77184e^2+0.0725209e^4\right)\epsilon^6 \\
&+\left(76.4861-18.5347e^2+1.48588e^4-0.0394005e^6\right)\epsilon^8 \\
&+\left(624.015-198.755e^2+23.5941e^4-1.23705e^6+0.0241682e^8\right)\epsilon^{10} \\
&+\left(5511.63-2173.08e^2+340.947e^4-26.6063e^6+1.03262e^8-0.0159449e^{10}\right)\epsilon^{12} \\
&+\left(51307.1-24103.6e^2+4697.17e^4-485.985e^6+28.1541e^8\right. \\
&\left.-0.865871e^{10}+0.0110442e^{12}\right)\epsilon^{14}+\left(495774.-270273.e^2+62898.2e^4-8099.8e^6\right. \\
&\left.+623.327e^8-28.6648e^{10}+0.729356e^{12}-0.0079209e^{14}\right)\epsilon^{16}+\mathcal{O}(\epsilon^{18})
\end{aligned}
$$

$$
\begin{aligned}
&q = \\
&0.166667e\ \epsilon^2 + \left(0.401814e - 0.0364324e^3\right)\epsilon^4 + \left(2.1931e - 0.373206e^3 + 0.0158055e^5\right)\epsilon^6 \\
&+ \left(15.6491e - 3.87045e^3 + 0.317549e^5 - 0.00864522e^7\right)\epsilon^8 \\
&+ \left(127.56e - 41.2748e^3 + 4.98583e^5 - 0.266507e^7 + 0.00531978e^9\right)\epsilon^{10} \\
&+ \left(1125.66e - 449.55e^3 + 71.5228e^5 - 5.66677e^7 + 0.223607e^9 - 0.00351592e^{11}\right)\epsilon^{12} \\
&+ \left(10470.4e - 4972.41e^3 + 980.312e^5 - 102.7e^7 + 6.03018e^9\right. \\
&\left.-0.188165e^{11} + 0.00243798e^{13}\right)\epsilon^{14} \\
&+ \left(101107.e - 55636.5e^3 + 13077.2e^5 - 1701.96e^7 + 132.464e^9 - 6.16548e^{11}\right. \\
&\left.+ 0.158912e^{13} - 0.00174981e^{15}\right)\epsilon^{16} + \mathcal{O}(\epsilon^{18})
\end{aligned}
$$

$$
\begin{aligned}
&\mu = \\
&\frac{1}{e}\left[4. + \left(-1.52381 + 0.214286e^2\right)\epsilon^2 + \left(-5.87901 + 1.25159e^2 - 0.0652768e^4\right)\epsilon^4\right. \\
&+ \left(-37.0661 + 10.6372e^2 - 1.00385e^4 + 0.0312297e^6\right)\epsilon^6 \\
&+ \left(-283.701 + 102.563e^2 - 13.7659e^4 + 0.813892e^6 - 0.0179038e^8\right)\epsilon^8 \\
&+ \left(-2410.37 + 1051.44e^2 - 182.027e^4 + 15.6419e^6 - 0.667552e^8 + 0.0113255e^{10}\right)\epsilon^{10} \\
&+ \left(-21860.7 + 11170.5e^2 - 2363.08e^4 + 264.989e^6 - 16.6181e^8\right. \\
&\left.+0.552789e^{10} - 0.00762253e^{12}\right)\epsilon^{12} \\
&+ \left(-207326. + 121451.e^2 - 30325.8e^4 + 4184.8e^6 - 344.747e^8 + 16.9581e^{10}\right. \\
&\left.-0.461289e^{12} + 0.00535403e^{14}\right)\epsilon^{14} \\
&+ \left(-2.03127 \times 10^6 + 1.34191 \times 10^6 e^2 - 386028.e^4 + 63167.2e^6 - 6431.58e^8 + 417.306e^{10}\right. \\
&\left.\left.-16.8524e^{12} + 0.387334e^{14} - 0.00387981e^{16}\right)\epsilon^{16} + \mathcal{O}(\epsilon^{18})\right]
\end{aligned}
$$

$$
\begin{aligned}
&f(r=0) = \\
&1. - 2.66667\epsilon^2 + \left(-9.04046 + 1.06893e^2\right)\epsilon^4 + \left(-55.7996 + 11.1848e^2 - 0.566798e^4\right)\epsilon^6 \\
&+ \left(-424.503 + 118.131e^2 - 10.9991e^4 + 0.34307e^6\right)\epsilon^8 \\
&+ \left(-3599.49 + 1276.91e^2 - 170.023e^4 + 10.0766e^6 - 0.224439e^8\right)\epsilon^{10} \\
&+ \left(-32626.9 + 14049.7e^2 - 2419.13e^4 + 208.257e^6 - 8.96698e^8 + 0.154554e^{10}\right)\epsilon^{12} \\
&+ \left(-309433. + 156626.e^2 - 33000.6e^4 + 3705.4e^6 - 233.893e^8\right. \\
&\left.+7.87144e^{10} - 0.110374e^{12}\right)\epsilon^{14} \\
&+ \left(-3.03237 \times 10^6 + 1.76343 \times 10^6 e^2 - 438933.e^4 + 60625.9e^6\right. \\
&\left.-5019.01e^8 + 249.084e^{10} - 6.86274e^{12} + 0.080996e^{14}\right)\epsilon^{16} + \mathcal{O}(\epsilon^{18})
\end{aligned}
$$

$$
\begin{aligned}
&\phi(r=0) = \\
&\epsilon + \left(2.38095 - 0.206349e^2\right)\epsilon^3 + \left(13.5366 - 2.20766e^2 + 0.0892759e^4\right)\epsilon^5 \\
&+ \left(99.3332 - 23.6891e^2 + 1.86986e^4 - 0.0488621e^6\right)\epsilon^7 \\
&+ \left(825.529 - 258.875e^2 + 30.261e^4 - 1.56287e^6 + 0.0300946e^8\right)\epsilon^9 \\
&+ \left(7388.22 - 2870.91e^2 + 443.951e^4 - 34.1507e^6 + 1.30689e^8 - 0.0199063e^{10}\right)\epsilon^{11} \\
&+ \left(69458.8 - 32195.6e^2 + 6190.33e^4 - 631.96e^6 + 36.1289e^8 \right.\\
&\left. -1.09675e^{10} + 0.0138126e^{12}\right)\epsilon^{13} \\
&+ \left(676349. - 364168.e^2 + 83703.e^4 - 10646.1e^6 + 809.226e^8 - 36.7616e^{10}\right.\\
&\left. +0.924182e^{12} - 0.00991936e^{14}\right)\epsilon^{15} + \left(6.76167\times 10^6 - 4.14712\times 10^6 e^2\right.\\
&+1.10887\times 10^6 e^4 - 168826.e^6 + 16007.9e^8 - 967.993e^{10} + 36.4553e^{12} \\
&\left. -0.781807e^{14} + 0.00731015e^{16}\right)\epsilon^{17} + \mathcal{O}(\epsilon^{19})
\end{aligned}
$$

C.3.2 Particular case $e = 4$

Of course, the formulae of the previous subsection are not immediately illuminating. In order to extract some (tentative) physical conclusions from those formulae, we specialise, in this section, to a particular value of e, namely $e = 4$. At this specific value of e we were able to coax Mathematica into producing results upto $\mathcal{O}(\epsilon^{30})$. We will not explicitly list our results, but use them to generate some plots that may carry qualitative lessons.

We are principally interested in the following question: does our solitonic solution develop a singularity (and so cease to exist) past a particular critical value of ϵ (or charge)? It would seem intuitively that this should be the case; no solitonic solution should exist at a mass greater than the 'Chandrasekhar limit' for this configuration.

Of course it is far from clear that perturbation theory can capture any phenomenon - particularly one as interesting as singularity formation - at finite values of ϵ. Nonetheless, in this subsection we will investigate the clues that we can glean from our perturbative analysis.

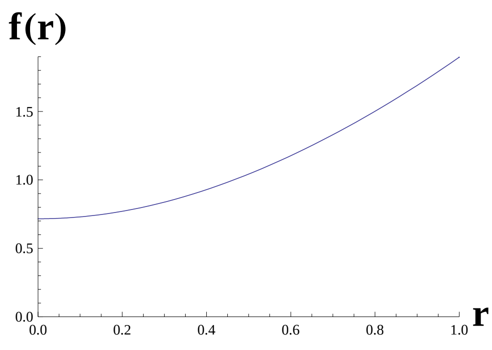

Figure C.1: $f(r)$ for $\epsilon = 0.4$ and $e = 4$

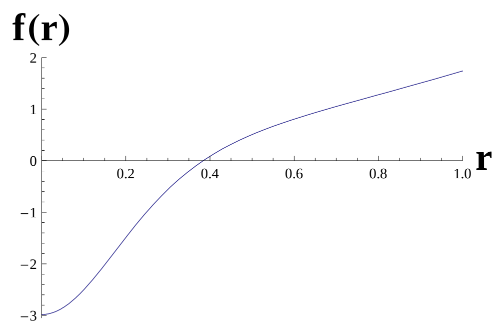

Figure C.2: $f(r)$ for $\epsilon = 0.7$ and $e = 4$

In Figs. C.1 and C.2 we present a plot of $f(r)$ at $\epsilon = 0.4$ and at $\epsilon = 0.7$. We also present a plot of the scalar field $\phi(r)$ at the origin $r = 0$ as a function of ϵ. Note that $f(r)$ is everywhere positive at $\epsilon = 0.4$ while it goes negative near the origin at $\epsilon = 0.7$. Note also that the scalar field behaves quite smoothly at the origin at $\epsilon = 0.4$ but shows a pronounced peak near the origin at $\epsilon = 0.7$.

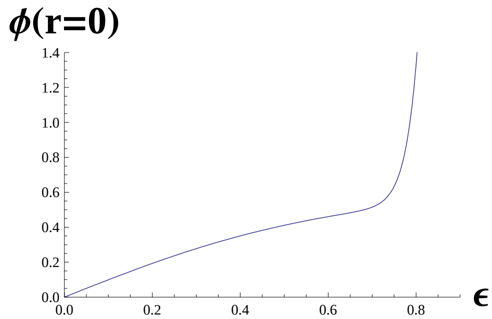

Figure C.3: $\phi(r = 0)$ as a function of ϵ for $e = 4$

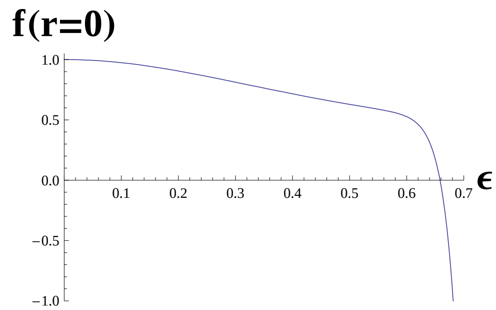

Figure C.4: $f(r = 0)$ as a function of ϵ for $e = 4$

We take these results to indicate that the actual solution develops a singularity at some value of ϵ between 0.4 and 0.7. Let us use the vanishing of $f(r)$ at the horizon as an estimator of the onset of this singularity. In Fig. C.4 we plot $f(r = 0)$ as a function of ϵ. Note that this graph goes through the origin at $\epsilon \approx 0.65$.

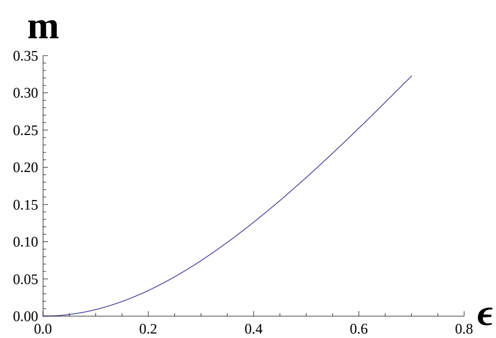

Figure C.5: Mass of soliton as a function of ϵ for $e = 4$

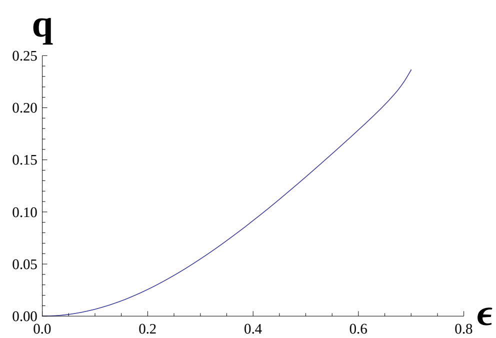

Figure C.6: Charge of soliton as function of ϵ for $e = 4$

In Figs. C.5 and C.6 we plot the mass and charge of the solution as a function of ϵ. Note that these graphs do not show a pronounced peak near $\epsilon = 0.65$. We take this result to indicate that the mass and charge of the soliton are finite as we approach the singularity. All of the ‘conclusions’ of this subsection are at best suggestive. A more serious analysis of the perturbative expansions described in this appendix could plausibly yield more solid indications as to the existence (or otherwise) of a singularity in the solution (rather than simply in the perturbative expansion) at $\epsilon = 0.65$. Numerical solutions to the differential equations would also likely yield valuable insights here.

C.4 Excited state solitons

In this section, we will present explicit solution for the solitons obtained by populating the first and the second excited state.

C.4.1 The first excited state soliton

The solution obtained by populating the first excited state takes the form

$$
\begin{aligned}
f(r) &= \left(r^2+1\right) - \frac{\left(r^8+5r^6+10r^4+6\right)\epsilon^2}{2\left(r^2+1\right)^5} \\
&\quad + \frac{\epsilon^4}{34151040\left(r^2+1\right)^{11}}\left(e^2\left(939978r^{20}+10428693r^{18}+52677075r^{16}+165426459r^{14}\right.\right. \\
&\quad +334808793r^{12}+425064222r^{10}+328198398r^8+171552590r^6+73128264r^4 \\
&\quad \left.+33080568r^2+13962524\right)-12\left(2568757r^{20}+28256327r^{18}+141281635r^{16}\right. \\
&\quad +434422857r^{14}+871830234r^{12}+1127373654r^{10}+916291332r^8+520240710r^6 \\
&\quad \left.\left.+186067332r^4+111081024r^2+26865882\right)\right)+O\left(\epsilon^5\right) \\
g(r) &= \frac{1}{r^2+1} + \frac{r^2\left(r^6+5r^4-2r^2+12\right)\epsilon^2}{2\left(r^2+1\right)^7} \\
&\quad - \frac{\epsilon^4}{11383680\left(r^2+1\right)^{13}}\left(r^2\left(e^2\left(313326r^{18}+3476231r^{16}+17559025r^{14}\right.\right.\right. \\
&\quad +44264825r^{12}+61767915r^{10}+58915626r^8+58567586r^6+66442530r^4 \\
&\quad \left.+35744940r^2+14941080\right)-4\left(2568757r^{18}+28256327r^{16}+141993115r^{14}\right. \\
&\quad +378069945r^{12}+580690770r^{10}+599020422r^8+676146702r^6+535404870r^4 \\
&\quad \left.\left.\left.+429618420r^2+121663080\right)\right)\right)+O\left(\epsilon^5\right) \\
A(r) &= \frac{6}{e} + \epsilon^2\left(-\frac{e\left(r^8+5r^6+10r^4+r^2+5\right)}{16\left(r^2+1\right)^5}+\frac{109e}{1232}-\frac{159}{77e}\right) \\
&\quad + \epsilon^4\left(\mathrm{C1}+\frac{1}{273208320r^2\left(r^2+1\right)^{11}}\left(e^3\left(2266110r^{16}-3679830r^{14}-57389178r^{12}\right.\right.\right. \\
&\quad -149388558r^{10}-172806150r^8-103232855r^6-29637025r^4-2330873r^2- \\
&\quad \left.1065873\right)+3e\left(355291e^2-11254468\right)\left(r^2+1\right)^{11}+12e\left(-4407480r^{16}+6142752r^{14}\right. \\
&\quad +116586624r^{12}+322636512r^{10}+364337655r^8+252086890r^6+50963352r^4 \\
&\quad \left.\left.\left.+7639614r^2+2813617\right)\right)\right)+O\left(\epsilon^5\right) \\
\phi(r) &= \frac{\left(2-3r^2\right)\epsilon}{2\left(r^2+1\right)^3} + \frac{\epsilon^3}{9856\left(r^2+1\right)^9}\left(e^2\left(1308r^{12}+6684r^{10}+13380r^8+12637r^6\right.\right. \\
&\quad \left.+5460r^4+117r^2-710\right)-4\left(7632r^{12}+41946r^{10}+90252r^8+87853r^6\right. \\
&\quad \left.\left.+41412r^4-1053r^2-5930\right)\right)+O\left(\epsilon^4\right)
\end{aligned}
\tag{C.25}
$$

Here C1 is a constant that will be determined by the regularity and normalisability conditions on ϕ at one higher order.

C.4.2 The second excited state soliton

The solution obtained by populating the second excited state is

$$\begin{aligned}
f(r) &= \left(r^2+1\right) - \frac{16\left(r^{12}+7r^{10}+21r^8-11r^6+82r^4-18r^2+9\right)\epsilon^2}{45\left(r^2+1\right)^7} \\
&\quad + \frac{\epsilon^4}{182614682250\left(r^2+1\right)^{15}}\left(11e^2\left(145257733r^{28}+2191163280r^{26}\right.\right. \\
&\quad +15436521240r^{24}+71111047280r^{22}+204078916860r^{20}+363116451984r^{18} \\
&\quad +447815797000r^{16}+559745609280r^{14}+629383835652r^{12}+760236964032r^{10} \\
&\quad +508886827176r^8+129434743200r^6+41946040800r^4-4941526368r^2 \\
&\quad \left.+3675474942\right)-128\left(809908894r^{28}+12148633410r^{26}+85040433870r^{24}\right. \\
&\quad +383937720530r^{22}+1113624190770r^{20}+2086530679662r^{18}+2767907054770r^{16} \\
&\quad +3367219159590r^{14}+3524605653426r^{12}+4551299991966r^{10}+1947299375658r^8 \\
&\quad \left.\left.+1248211079850r^6+39350124900r^4+27778828956r^2+14011941561\right)\right)+O\left(\epsilon^5\right) \\
g(r) &= \frac{1}{r^2+1} + \frac{16r^2\left(r^{10}+7r^8-9r^6+85r^4-50r^2+30\right)\epsilon^2}{45\left(r^2+1\right)^9} \\
&\quad + \frac{r^2\epsilon^4}{182614682250\left(r^2+1\right)^{17}}\left(256\left(404954447r^{26}+6074316705r^{24}+42610397025r^{22}\right.\right. \\
&\quad +146943860245r^{20}+332411328795r^{18}+736429945251r^{16}+1692683167175r^{14} \\
&\quad +2288373934275r^{12}+2999779439085r^{10}+693256048485r^8+761470058349r^6 \\
&\quad \left.+46619547975r^4+124153839810r^2+57083996970\right)-11e^2\left(145257733r^{26}\right. \\
&\quad +2191163280r^{24}+15436521240r^{22}+48745664240r^{20}+98672175420r^{18} \\
&\quad +226204362384r^{16}+539605666600r^{14}+922637944800r^{12}+647631553140r^{10} \\
&\quad +673336503840r^8-167785201584r^6+151437686400r^4-3202158960r^2 \\
&\quad \left.\left.+21249708480\right)\right)+O\left(\epsilon^5\right) \\
A(r) &= \frac{8}{e}+\epsilon^2\left(\frac{4741e^2-228352}{90090e}-\frac{e}{30r^2}+\frac{e\left(8\left(5r^4-4r^2+4\right)r^4+1\right)}{30r^2\left(r^2+1\right)^7}\right) \\
&\quad +\epsilon^4\left(\mathrm{C1}+\frac{1}{973944972000r^2\left(r^2+1\right)^{15}}\left(11e^3\left(776575800r^{24}-4643959320r^{22}\right.\right.\right. \\
&\quad -49995986040r^{20}-153300661800r^{18}-226524451725r^{16}-196951063595r^{14} \\
&\quad -81174009917r^{12}-35347997139r^{10}-47175005605r^8-22830663835r^6 \\
&\quad \left.-9008399805r^4-109267667r^2-82336064\right) \\
&\quad +64e\left(14151511e^2-890269174\right)\left(r^2+1\right)^{15}-128e\left(3214411200r^{24}\right. \\
&\quad -18271242990r^{22}-210141141210r^{20}-667533287850r^{18}-983744254350r^{16} \\
&\quad -972471806735r^{14} \\
&\quad -123178436381r^{12}-380041323207r^{10}-58304634115r^8-160254872155r^6 \\
&\quad \left.\left.\left.-31564145265r^4-979195571r^2-445134587\right)\right)\right)+O\left(\epsilon^5\right)
\end{aligned} \tag{C.26}$$

Here again C1 is determined at one higher order. Finally the scalar field in this case is given by

$$\begin{aligned}\phi(r) = {} & \frac{(2r^4-4r^2+1)\,\epsilon}{(r^2+1)^4} - \frac{2\epsilon^3}{675675\,(r^2+1)^{12}}\Big(11e^2\big(4310r^{18}+27530r^{16}+69735r^{14} \\ & +80920r^{12}+29848r^{10}-54222r^8-68110r^6-26098r^4-6504r^2+1149\big) \\ & -2\big(1141760r^{18}+7624220r^{16}+20257200r^{14}+24337600r^{12}+10397296r^{10} \\ & -17335794r^8-19274920r^6-9051766r^4-1800468r^2+416883\big)\Big)+O\left(\epsilon^4\right)\end{aligned} \tag{C.27}$$

C.5 The first excited hairy black hole

In this appendix we present the results of our construction of the first excited hairy black hole metric. We have written a Mathematica program that allows us to generate the corresponding solution for the n^{th} excited hairy black hole at any given value of n. We use the same conventions as in appendix C.2.

C.5.1 Near field expansion

$$\begin{aligned}f^{in}_{(0,0)}(y) = {} & \frac{(y^2-1)\,(e^2y^2-24)}{e^2y^4} \\ f^{in}_{(0,2)}(y) = {} & \frac{(y^2-1)\,(e^4\,(y^4+y^2)+304e^2+1536)}{e^4y^4} \\ f^{in}_{(0,4)}(y) = {} & \frac{8\,(y^2-1)\,(9\mathcal{C}_1e^5+361e^6y^2+57e^4\,(84y^2+19)+576e^2\,(26y^2+19)+27648)}{9e^6y^4} \\ f^{in}_{(2,0)}(y) = {} & -\frac{8\,(y^2-1)\,(e^2\,(154y^2-23)-212)}{231e^2y^4}\end{aligned} \tag{C.28}$$

$$g^{in}_{(0,0)}(y) = \frac{e^2 y^4}{(y^2-1)(e^2y^2-24)} \tag{C.29}$$

$$g^{in}_{(0,2)}(y) = -\frac{y^4\left(e^4\left(y^4+y^2\right)-96e^2+3456\right)}{(y^2-1)(e^2y^2-24)^2} \tag{C.30}$$

$$g^{in}_{(0,4)}(y) = \frac{y^4}{e^2(y^2-1)(e^2y^2-24)^3}\Big(8\mathcal{C}_1 e^7 y^2 - 192\mathcal{C}_1 e^5 + e^8 y^4\left(y^2+1\right)^2 - 96e^6\left(2y^4+y^2\right) \tag{C.31}$$

$$+6912e^4\left(y^4+1\right) + 124416e^2\left(y^2-4\right) + 8957952\Big) \tag{C.32}$$

$$B^{in}_{(2,0)}(y) = \frac{8e^2\left(712-23e^2\right)y^4}{231(y^2-1)(e^2y^2-24)^2} \tag{C.33}$$

$$A^{in}_{(0,0)}(y) = \frac{6\left(y^2-1\right)}{ey^2} \tag{C.34}$$

$$A^{in}_{(0,2)}(y) = \frac{12\left(e^2-36\right)\left(y^2-1\right)}{e^3y^2} \tag{C.35}$$

$$A^{in}_{(0,4)}(y) = \mathcal{C}_1\left(1-\frac{1}{y^2}\right) \tag{C.36}$$

$$A^{in}_{(2,0)}(y) = -\frac{\left(23e^2+212\right)\left(y^2-1\right)}{231ey^2} \tag{C.37}$$

$$\phi^{in}_{(1,0)}(y) = -\frac{2}{3} \tag{C.38}$$

$$\phi^{in}_{(1,2)}(y) = \frac{1}{2e^2}\Bigg(-288\,\mathrm{Li}_2\left(\frac{e^2y^2-24}{e^2-24}\right) - 12\left(e^2-72\right)\log(R) + e^2\left(6y^2-5\right) \tag{C.39}$$

$$-6\left(48\log\left(-\frac{e^2\left(y^2-1\right)}{e^2-24}\right) - 24\log\left(e^2y^2-24\right) + e^2 - 72\right)\log\left(e^2y^2-24\right) \tag{C.40}$$

$$-144\log^2\left(\frac{1}{24-e^2}\right) + 12\left(e^2-72\right)\log(e) - 48\pi^2 + 456\Bigg) \tag{C.41}$$

$$\phi^{in}_{(3,0)}(y) = \frac{5\left(71e^2-2372\right)}{16632}. \tag{C.42}$$

C.5.2 Far field expansion

$$
\begin{aligned}
f^{out}_{(0,0)}(r) =& r^2+1\\
f^{out}_{(0,2)}(r) =& -\frac{e^2+24}{e^2r^2}\\
f^{out}_{(0,4)}(r) =& \frac{1}{6e^2r^4}\left(144-6e^2\left(\frac{96\left(e^2-36\right)}{e^4}+1\right)r^2\right)\\
f^{out}_{(2,0)}(r) =& -\frac{2\left(r^8+5r^6+10r^4+6\right)}{9\left(r^2+1\right)^5}\\
f^{out}_{(2,2)}(r) =& \frac{1}{1386e^2r^2\left(r^2+1\right)^6}\Big(-2772r^2\left(r^2+1\right)\left(2\left(3e^2-88\right)r^{12}+12\left(3e^2-88\right)r^{10}\right.\\
&+\left(89e^2-2568\right)r^8+5\left(23e^2-632\right)r^6+80\left(e^2-24\right)r^4+12\left(3e^2-88\right)r^2\\
&\left.+256\right)\left(\log(1-ir)+\log(1+ir)-2\log(r)\right)+e^2\left(16632r^{14}+110675r^{12}\right.\\
&\left.+309542r^{10}+467241r^8+416782r^6+179030r^4+35422r^2+2952\right)\\
&-24\left(20328r^{14}+138869r^{12}+394622r^{10}+595703r^8+518662r^6+240194r^4\right.\\
&\left.+22250r^2-424\right)\Big)
\end{aligned}
\tag{C.43}
$$

$$
\begin{aligned}
g^{out}_{(0,0)}(r) =& \frac{1}{r^2+1}\\
g^{out}_{(0,2)}(r) =& \frac{e^2+24}{e^2r^2\left(r^2+1\right)^2}\\
g^{out}_{(0,4)}(r) =& \frac{e^4\left(r^4+r^2+1\right)+24e^2\left(4r^4+3r^2+1\right)-576\left(6r^4+6r^2-1\right)}{e^4r^4\left(r^2+1\right)^3}\\
g^{out}_{(2,0)}(r) =& \frac{2r^2\left(r^6+5r^4-2r^2+12\right)}{9\left(r^2+1\right)^7}\\
g^{out}_{(2,2)}(r) =& \frac{1}{1386e^2\left(r^3+r\right)^2}\Big(-\frac{77}{\left(r^2+1\right)^6}\left(36\left(e^2-72\right)\left(r^8+6r^6+3r^4+10r^2\right.\right.\\
&\left.+12\right)r^4(\log(1-ir)+\log(1+ir)-2\log(r))-e^2\left(32r^{10}+182r^8+856r^6\right.\\
&\left.\left.+393r^4+210r^2+19\right)+24\left(108r^{10}+586r^8+1292r^6+1471r^4+222r^2+69\right)\right)\\
&-2567e^2+161688\Big)
\end{aligned}
\tag{C.44}
$$

$$
\begin{aligned}
A^{out}_{(0,0)}(r) =& \frac{6}{e} \\
A^{out}_{(0,2)}(r) =& \frac{e^2\left(12-\frac{6}{r^2}\right)-432}{e^3} \\
A^{out}_{(0,4)}(r) =& \mathcal{C}_1 - \frac{12\left(e^2-36\right)}{e^3 r^2} \\
A^{out}_{(2,0)}(r) =& \frac{1}{2772}\left(-\frac{77e\left(r^8+5r^6+10r^4+r^2+5\right)}{\left(r^2+1\right)^5}+109e-\frac{2544}{e}\right) \\
A^{out}_{(2,2)}(r) =& \mathcal{C}_2 + \frac{1}{11088er^2\left(r^2+1\right)^6}\Big(e^2\left(\left(3\left(5681r^8+30852r^6+67813r^4+77507r^2\right.\right.\right. \\
& \left.\left.+39246\right)r^2+29647\right)r^2+1104\Big)+2772r^2\left(r^2+1\right)\Big(\left(\left(5e^2-104\right)r^{10}\right. \\
& +8\left(3e^2-56\right)r^8+5\left(9e^2-136\right)r^6+40\left(e^2-8\right)r^4 \\
& \left.+8\left(3e^2-56\right)r^2+256\right)\left(2\log(r)-\log\left(r^2+1\right)\right) \\
& -24\left(19673r^{12}+103716r^{10}+217325r^8+230143r^6\right. \\
& \left.+123462r^4+10777r^2-424\right)\Big)
\end{aligned}
\tag{C.45}
$$

$$
\begin{aligned}
\phi^{out}_{(1,0)}(r) =& \frac{3r^2-2}{3\left(r^2+1\right)^3} \\
\phi^{out}_{(1,2)}(r) =& \frac{1}{2e^2\left(r^2+1\right)^4}\Big(3\left(72-5e^2\right)r^2+6\left(e^2-72\right)\left(3r^4+r^2-2\right)\log(r) \\
& -3\left(e^2-72\right)\left(3r^4+r^2-2\right)\log\left(r^2+1\right)-5e^2+456\Big) \\
\phi^{out}_{(3,0)}(r) =& -\frac{1}{33264\left(r^2+1\right)^9}\Big(e^2\left(r^2+1\right)\left(1308r^{10}\right. \\
& \left.+5376r^8+8004r^6+4633r^4+827r^2-710\right) \\
& +4212r^2-4\left(7632r^8+41946r^6+90252r^4\right. \\
& \left.+87853r^2+41412\right)r^4+23720\Big)
\end{aligned}
\tag{C.46}
$$

In the above formulae $\mathcal{C}_1$ and $\mathcal{C}_2$ are constants which will only be determined by regularity and normalisability of the scalar field at one higher order.

C.6 Thermodynamics in the canonical ensemble

In the following subsections, we will describe the canonical phase diagrams that result from a competition between small RNAdS black holes, small hairy black holes and small solitons. Everywhere in this section we completely ignore large black holes and large hairy black holes.

In the microcanonical ensemble this was logically justified; large black holes never have small mass and charge. However large black holes can (and do) have temperatures (and or chemical potentials) comparable to their small counterparts. Consequently the phase diagrams we will draw in this appendix do not, in general, represent the true thermodynamical equilibrium of our system at finite temperature and chemical potential. The phase diagrams of this appendix should be regarded as formal; their purpose is to help us better understand the formal interrelationship between the phases constructed and studied in this work, ignoring all other phases that might exist in the system.

In this section we will study the interrelationship between small black hole, soliton and excited black hole phases at fixed charge and temperature. We find it convenient to work with the rescaled inverse temperature variable

$$\beta = \frac{1}{4\pi T}.$$

In this section we will assume that β and q are small. We also assume that q and β^2 are of the same order, and present all formulae only to leading order in q and β^2.

C.6.1 RNAdS black hole

At any fixed charge, the temperature of a small RNAdS black hole is given, as a function of its chemical potential $\mu = \frac{q}{R^2}$ by

$$\frac{\sqrt{q}}{\beta} = 2\sqrt{\mu}(1 - \frac{2}{3}\mu^2) \tag{C.47}$$

In Fig. C.7, we present a plot of $\frac{\sqrt{q}}{\beta}$ versus μ. Note that

$$\mu^2 \leq \frac{3}{2}$$

(the constraint follows directly from the requirement of positivity of the temperature in (C.47)). Note also that

$$\frac{\sqrt{q}}{\beta} \leq \frac{4}{5}\left[\frac{24}{5}\right]^{1/4} \tag{C.48}$$

This inequality is saturated at at $\mu = \mu_m \equiv \sqrt{3/10}$.

Note that there exist two small black holes (with different values of μ) for any given β that obeys (C.48) The free energy is given as a function on q and μ by

$$F_{sbh} = q\frac{\pi}{24}\left(10\mu + \frac{3}{\mu}\right) \tag{C.49}$$

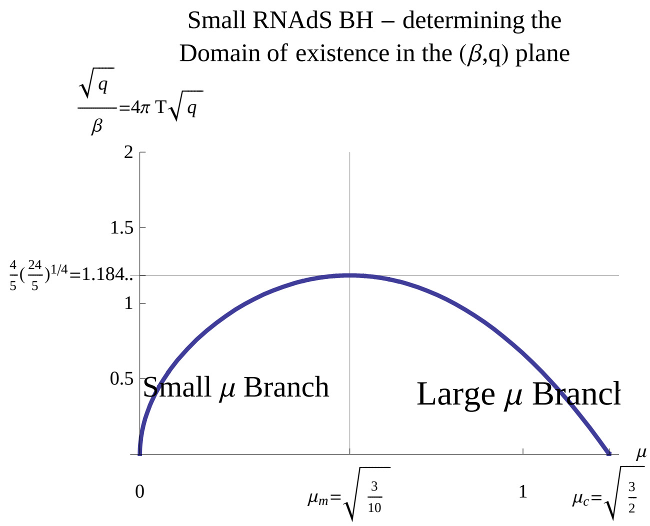

Figure C.7: Canonical ensemble: Small RNAdS BH - For any given charge and temperature there are two possible small black holes with two different chemical potentials (Large μ branch has a lower free-energy). And for a given temperature, there is a maximum possible charge which is attained by the black hole with $\mu = \mu_m = \sqrt{3/10}$.

In Fig. C.8 we present a plot of $\frac{F_{sbh}}{q}$ versus μ. Note that

$$\frac{F_{sbh}}{q} \geq \pi\sqrt{5/24}$$

and the minimum value occurs at $\mu = \mu_m \equiv \sqrt{3/10}$ (this is the same value of μ at which the temperature curve has a maximum). As is visually apparent from these graphs, the RNAdS black hole with the larger value of μ has lower free energy at any fixed β and q (see Fig. C.9). In this appendix we will refer to this solution as the small RNAdS black hole. We will completely ignore the free energetically subdominant RNAdS black hole in the rest of this appendix.

Let us briefly summarise. RNAdS black holes exist whenever the inequality (C.48) is obeyed. Their free energy is given as a function of β and q by (C.49) and (C.47), where we are instructed always to choose the larger of the two roots when inverting (C.47). The chemical potential of these black holes obey

$$\mu_m \equiv \sqrt{3/10} \geq \mu \geq \sqrt{3/2} \equiv \mu_c.$$

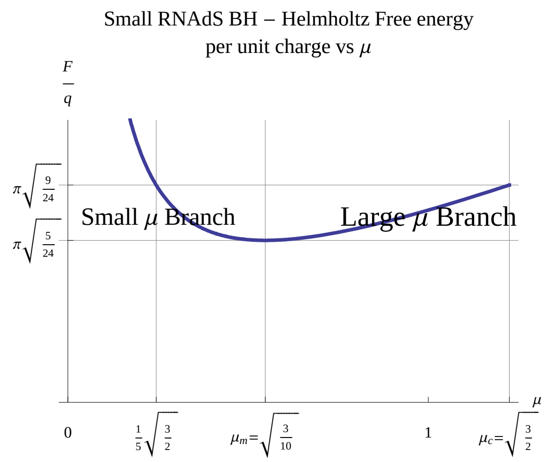

Figure C.8: Canonical ensemble: Small RNAdS BH - Free energy of the small BHs are positive and for the large μ branch, the free-energy varies over a bounded domain.

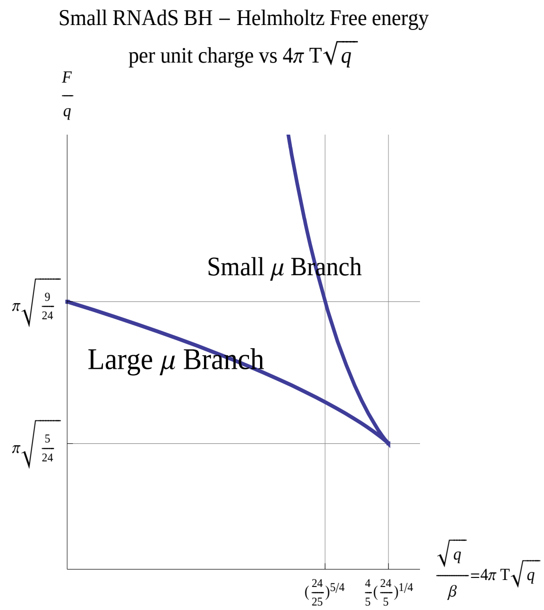

Figure C.9: Canonical ensemble: Small RNAdS BH - The large μ branch always has a lower free-energy.

C.6.2 Soliton

Solitonic solutions exist at all values of β and q. At leading order the Free energy and chemical potential of the soliton are given by

$$\begin{aligned} \frac{F_{sol}}{q} &= \frac{2\pi}{e} \\ \mu &= \frac{4}{e} \end{aligned} \tag{C.50}$$

Note that $\frac{2\pi}{e} \leq \sqrt{5/24}$ whenever

$$e^2 \geq \frac{32}{3} \times \frac{9}{5} = e_c^2 \times \frac{9}{5} \equiv e_1^2.$$

It follows that the soliton free energetically dominates that RNAdS black hole whenever this inequality is obeyed. At smaller values of e, on the other hand, the RNAdS black hole free energetically dominates the soliton at large enough temperatures (but temperatures that are small enough to be allowed by (C.48), i.e. whenever

$$\begin{aligned} &\frac{3}{2\sqrt{\mu_m}(3-2\mu_m^2)} \leq \frac{\beta}{\sqrt{q}} \leq \frac{3}{2\sqrt{\mu_*}(3-2\mu_*^2)} \\ &\mu_* \equiv \frac{4}{e} \times \frac{3}{5}\left(1+\sqrt{1-(5/9)\times(3e^2/32)}\right) = \frac{4}{e} \times \frac{3}{5}\left(1+\sqrt{1-(5/9)\times(e^2/e_c^2)}\right) \\ &\mu_m = \sqrt{3/10} \end{aligned} \tag{C.51}$$

We will return to a more detailed comparison of phases below.

C.6.3 Hairy black hole

Hairy black holes exist if and only if

$$\frac{\beta}{\sqrt{q}} \leq \sqrt{e^5/(16(e^2-(32/3))^2)} \tag{C.52}$$

In this regime their chemical potential, mass and free energy are given by

$$\begin{aligned}
m =& \left(\frac{4\left(3e^2-32\right)^3}{27e^6}\right)\beta^2 + \frac{16}{3e}q + \mathcal{O}(m^2, mq, q^2) \\
\mu =& \frac{4}{e} + \left(\frac{8\left(32-3e^2\right)^2\left(e^2-32\right)}{21e^7}\right)\beta^2 + \left(\frac{9}{7} - \frac{64}{7e^2}\right)q + \mathcal{O}(m^2, mq, q^2) \\
F(\beta, q) =& \frac{3\pi}{8}\left[\left(\frac{16q}{3e} + \frac{4\left(3e^2-32\right)^3\beta^2}{81e^6}\right) + \frac{16\left(32-3e^2\right)^4\left(21e^4-384e^2+5120\right)}{1701e^{12}}\beta^4\right. \\
&\left. + \frac{32\left(32-3e^2\right)^2\left(e^2-32\right)}{63e^7}\beta^2 q + \frac{2\left(9e^2-64\right)}{21e^2}q^2 + \mathcal{O}\left(\beta^6, \beta^4 q, \beta^2 q^2, q^3\right)\right]
\end{aligned} \tag{C.53}$$

(where we have listed perturbative corrections, but will only use leading order results in what follows). Note that the difference between the free energy of a hairy black hole and the soliton is a positive number times β^2, so that hairy black holes are free energetically always subdominant compared to the soliton.

It is also interesting to perform a comparison between RNAdS and hairy black holes. First notice that the existence ranges (C.52) and (C.48) overlap only when

$$e^2 \leq \frac{32}{3} \times 5 \equiv e_2^2$$

Moreover hairy black holes only exist for $e^2 \geq \frac{32}{3} \equiv e_c^2$. Within this range of e hairy and RNAdS black holes in an overlapping region in the β, q plane. It turns out that RNAdS black holes always dominate over hairy black holes in these overlapping regions.

C.6.4 Plots of phase existence and dominance

As we have explained above, the phase diagram of our system depends qualitatively on the value of e^2. In particular, there are three special values of e^2

$$e_c^2 = 32/3, \quad e_1^2 = e_c^2(9/5), \quad e_2^2 = 5e_c^2$$

Recall that RNAdS black holes are always stable - and no hairy black holes exist - for $e^2 \leq e_c^2$. In this regime the only phases of the system are the RNAdS black hole and the soliton. It turns out that the RNAdS black hole is free energetically dominant whenever it exists. The phase diagram is depicted in Fig. C.10 below.

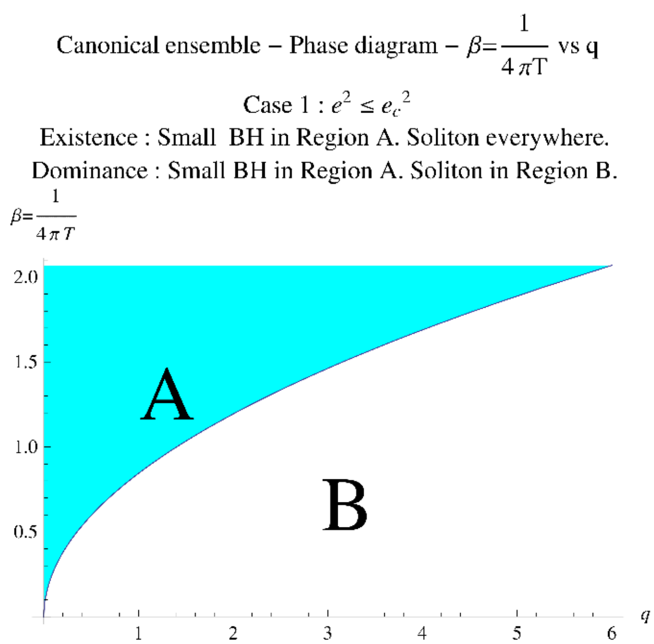

Figure C.10: Canonical ensemble: Case 1: $e^2 \leq e_c^2$.

RNAdS black holes exist only above the line drawn in Fig. C.10, and give the dominant phase when they exist. The soliton dominates the phase diagram elsewhere.

In the regime $e_c^2 \leq e^2 \leq e_1^2$ hairy black holes exist as a phase (below the topmost line in Fig. C.11) but are never dominant. In this regime the RNAdS black holes exist above the bottom most line but free energetically dominate the soliton only in the region between the bottom most line and the intermediate line. The soliton is free energetically dominant elsewhere. The RNAdS black hole is free energetically dominant over the hairy black hole over their common region of existence.

In the regime $e_1^2 \leq e^2 \leq e_2^2$ the phase diagram (shown in Fig. C.12) is very similar to that in Fig. C.11, except that there is no intermediate region; the soliton is the thermodynamically dominant solution everywhere. Note that in this regime the RNAdS and hairy black hole solutions continue to exist as phases; the RNAdS black hole is free energetically dominant over the hairy black hole over this region of overlap (i.e., region B in Fig. C.12).

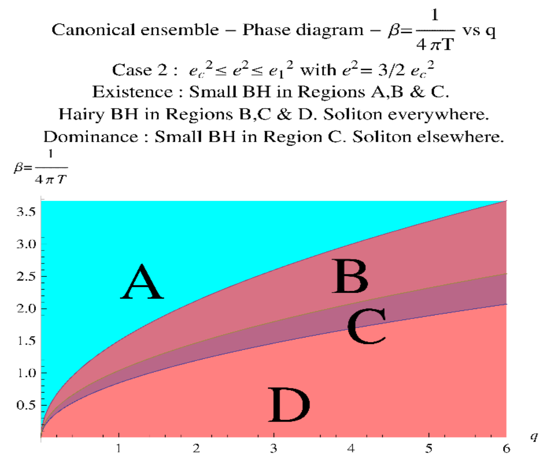

Figure C.11: Canonical ensemble: Case 2: $e_c^2 \le e^2 \le e_1^2$.

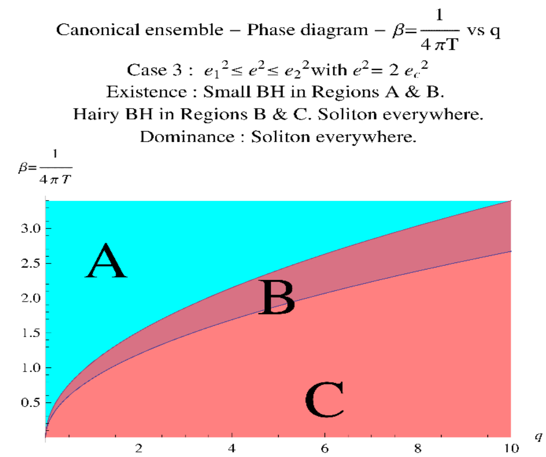

Figure C.12: Canonical ensemble: Case 3: $e_1^2 \le e^2 \le e_2^2$.

We turn, finally to the range $e^2 \ge e_2^2$. Note that in this range $\frac{4}{e} \le \sqrt{3/10} = \mu_m$. Recall that the chemical potential of a hairy black hole is equal to $\frac{4}{e}$ at leading order. It follows that the RNAdS black hole component of a hairy black hole, in this regime, has $\mu \le \mu_m$. In other words, in this regime, the RNAdS black hole that lies in the centre of a hairy black hole is of negative specific heat. In this regime, also, there is no overlap in the existence regimes of RNAdS black holes (C.48) and hairy black holes (C.52). At any given value of β and q we have at most two phases (soliton and black hole or soliton and hairy black hole) and the soliton is always the free energetically dominant phase. The phase diagram is displayed in Fig. C.13 on the next page.

C.7 Thermodynamics in the grand canonical ensemble

In the following subsections, we will describe the grand canonical phase diagrams that result from a competition between small RNAdS black holes, small hairy black holes and small solitons. Everywhere in this section we completely ignore large black holes and large hairy black holes. Like our discussion on canonical ensemble given in the previous appendix, this is not entirely justified if we are interested in the actual phase diagram of the system. However, the formal diagrams which we present in this appendix are still helpful in contrasting the various phases constructed in this work.

In this section we will study the thermodynamics of our system as a function of

$$\beta = \frac{1}{4\pi T}, \quad \delta\mu = \mu - \frac{4}{e}$$

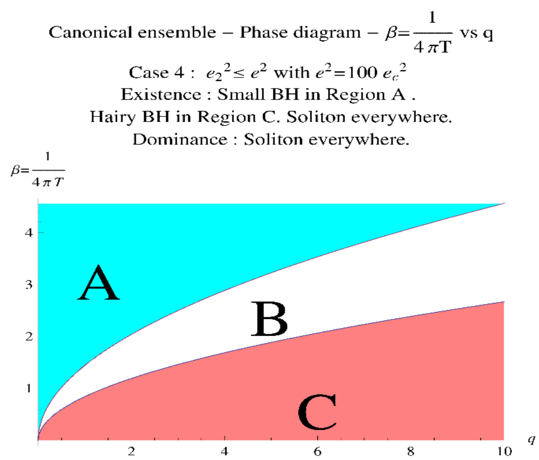

Figure C.13: Canonical ensemble: Case 4: $e_2^2 \leq e^2$.

C.7.1 RNAdS black holes

As we have seen above, small RNAdS black holes exist only for

$$\mu \leq \sqrt{3/2}.$$

Moreover these black holes also satisfy the inequality

$$\beta^2 \leq \frac{1}{32(1-\frac{2}{3}\mu^2)}$$

This last inequality is automatically obeyed for parametrically small values of β, of prime interest to us, and so will play no important role in the analysis below. Whenever these inequalities are satisfied, we have a unique small black hole.

The various important thermodynamical quantities of small RNAdS black holes, in the grand canonical ensemble, are given by

$$\begin{aligned} m =& \frac{4\left(32-3e^2\right)^2\left(3e^2+32\right)\beta^2}{27e^6} + \left[\frac{16\left(3e^2-32\right)^3\left(5e^2+32\right)}{27e^8}\right]\beta^4 \\ & - \left[\frac{64\left(e^2+32\right)\left(3e^2-32\right)}{9e^5}\right]\beta^2\delta\mu + \mathcal{O}\left(\beta^5, \beta^3\delta\mu, \delta\mu^2\beta\right) \\ q =& \frac{16\left(32-3e^2\right)^2\beta^2}{9e^5} + \left[\frac{256\left(3e^2-32\right)^3}{27e^7}\right]\beta^4 \\ & + \left[\frac{4\left(9\left(e^2-64\right)e^2+5120\right)}{9e^4}\right]\beta^2\delta\mu + \mathcal{O}\left(\beta^5, \beta^3\delta\mu, \delta\mu^2\beta\right) \end{aligned} \tag{C.54}$$

$$G(\beta,\delta\mu) = \frac{3\pi}{8}\left[\frac{4\left(3e^2-32\right)^3\beta^2}{81e^6} + \left(\frac{16\left(32-3e^2\right)^4}{81e^8}\right)\beta^4 - \left(\frac{64\left(32-3e^2\right)^2}{27e^5}\right)\beta^2\delta\mu + \mathcal{O}\left(\beta^5,\beta^3\delta\mu,\delta\mu^2\beta\right)\right] \tag{C.55}$$

C.7.2 Soliton

The soliton exists for all temperatures but for $\mu \geq \frac{4}{e}$. Its thermodynamical quantities are given by

$$\begin{aligned} m &= \frac{112e}{3\left(9e^2-64\right)}\,\delta\mu + \frac{e^2\left(2364219e^4-47285088e^2+244052992\right)}{1485\left(9e^2-64\right)^3}\,\delta\mu^2 + \mathcal{O}\left(\delta\mu^3\right) \\ q &= \frac{7e^2}{9e^2-64}\,\delta\mu + \frac{e^3\left(1802889e^4-39301728e^2+215667712\right)}{7920\left(9e^2-64\right)^3}\,\delta\mu^2 + \mathcal{O}\left(\delta\mu^3\right) \\ G(\mu,T) &= \frac{3\pi}{8}\left[\left(\frac{14e^2}{192-27e^2}\right)\delta\mu^2 + \left(\frac{\left(-1802889e^7+39301728e^5-215667712e^3\right)}{17820\left(9e^2-64\right)^3}\right)\delta\mu^3 + \mathcal{O}\left(\delta\mu^4\right)\right] \end{aligned} \tag{C.56}$$

It is easy to verify that the solitonic solution always has lower grand free energy (it is negative at leading order) than the RNAdS black hole (the free energy is positive at leading order) within the validity of perturbation theory. Consequently the system undergoes a first order phase transition from the RNAdS black hole to the solitonic phase as μ is raised above $\frac{4}{e}$

C.7.3 Hairy black hole

Hairy black holes exist whenever

$$\frac{\delta\mu}{\beta^2} \geq \frac{8\left(3e^2-32\right)^3}{9e^7} \tag{C.57}$$

Their thermodynamical quantities are given by

$$\begin{aligned} m =& \frac{4\left(252e^7\delta\mu + (32-3e^2)^2(27e^4-576e^2+5120)\beta^2\right)}{27e^6(9e^2-64)} + \mathcal{O}(\beta^4,\beta^2\delta\mu,\delta\mu^2) \\ q =& \frac{21e^7\delta\mu - 8(32-3e^2)^2(e^2-32)\beta^2}{3e^5(9e^2-64)} + \mathcal{O}(\beta^4,\beta^2\delta\mu,\delta\mu^2) \\ G(\beta,\delta\mu) =& \frac{3\pi}{8}\left[\frac{4(3e^2-32)^3\beta^2}{81e^6} + \frac{16(32-3e^2)^4(27e^6-696e^4+10752e^2-57344)\beta^4}{243e^{12}(9e^2-64)}\right. \\ & \left. - \frac{2\left(21e^7\delta\mu^2 - 16(32-3e^2)^2(e^2-32)\beta^2\delta\mu\right)}{9e^5(9e^2-64)} + \mathcal{O}\left(\beta^5,\beta^3\delta\mu,\delta\mu^2\beta\right)\right] \end{aligned} \tag{C.58}$$

It is easily verified that (within perturbation theory) hairy black holes are free energetically subdominant compared to solitons in their common domain of existence. On the other hand, it may be checked that they are free energetically dominant compared to RNAdS black holes, where the solutions coexist.

C.7.4 Phase diagrams

The phase diagram of our system is very simple when $e^2 \leq e_c^2$. Hairy black holes don't exist. The two phases that do exist - RNAdS black holes and the soliton - never coexist at the same μ and β. The RNAdS black hole dominates when it exists; the soliton dominates when it exists. The phase diagram is sketched in Fig. C.14 below.

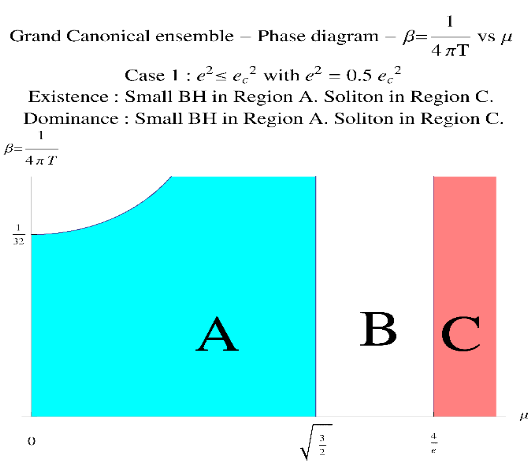

Figure C.14: Grand canonical ensemble: Case 1: $e^2 \leq e_c^2$.

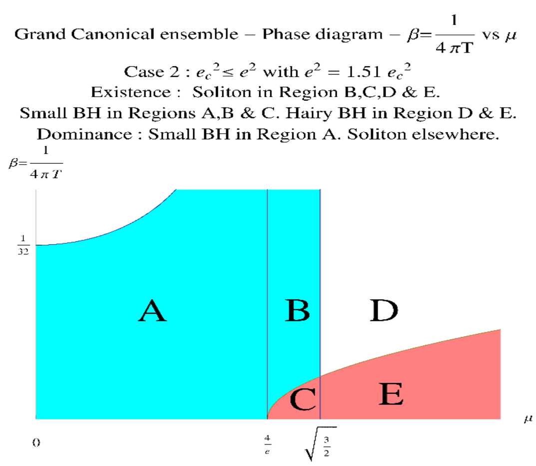

Figure C.15: Grand canonical ensemble: Case 2: $e_c^2 \leq e^2$.

The phase diagram is more interesting when $e^2 \geq e_c^2$. As we have mentioned above, the system undergoes a first order phase transition from the black hole to the solitonic phase as μ is raised above $\frac{4}{e}$. The hairy black hole phase is always subdominant compared to the soliton, but free energetically dominates the black hole when both exist. The phase diagram is depicted in Fig. C.15. Note that the black hole and hairy black holes phases are identical, where the hairy black hole is first created. In the absence of the solitonic solution, consequently, our system would have undergone a second order transition from the RNAdS black hole to the hairy black hole phase upon raising μ.

C.8 Notation

C.8.1 Basic setup

Throughout this chapter, we work in asymptotically (global) AdS_5 spacetimes with a bulk metric g, a bulk charged scalar field ϕ and a bulk gauge field A_μ with a Lagrangian

$$\begin{aligned} S &= \frac{1}{8\pi G_5}\int d^5x\sqrt{g}\left[\frac{1}{2}\left(\mathcal{R}[g]+12\right)-\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}-|D_\mu\phi|^2-m_\phi^2|\phi|^2\right] \\ \mathcal{F}_{\mu\nu} &\equiv \nabla_\mu A_\nu-\nabla_\nu A_\mu \qquad \text{and} \qquad D_\mu\phi\equiv\nabla_\mu\phi-ieA_\mu\phi \end{aligned} \tag{C.59}$$

where G_5 is the Newton's constant and the radius of AdS_5 is set to unity. This implies that the bulk cosmological constant is taken to be $\Lambda_5=-6$.

The radial co-ordinate of AdS_5 is denoted by r with the boundary of AdS_5 being at $r=\infty$. For solutions with the horizon, the outer horizon is taken to be at $r=R$. The Schwarzschild-like temporal co-ordinate is denoted by t. Sometimes, we find it convenient to work with rescaled co-ordinates $y\equiv r/R$ and $\tau\equiv t/R$, especially in the near-field expansion at small radius ($r<<1$). In appendix C.1, we shift to Eddington-Finkelstein like co-ordinates, with an Eddington-Finkelstein (EF) time co-ordinate denoted by v.

Since throughout this chapter we work with spherically symmetric solutions, we will leave the co-ordinates parametrising the S^3 implicit. Mostly, we work in a gauge where the bulk fields take a form

$$\begin{aligned} ds^2 &= -f(r)dt^2+g(r)dr^2+r^2d\Omega_3^2 \\ A_t &= A(r), \qquad A_r=A_i=0 \\ \phi &= \phi(r) \end{aligned} \tag{C.60}$$

In appendix C.1, we work with the EF co-ordinate metric for a small charged black hole (see below).

The charge of the scalar field ϕ is denoted by e and its mass by m_ϕ. We take $m_\phi=0$ for most of the work except in section 4.6.4 . By the standard rules of AdS/CFT, the dual boundary operator $\mathcal{O}_\phi$ has a scaling dimension

$$\Delta_0=\left[d/2+\sqrt{(d/2)^2+m_\phi^2}\,\right]_{d=4}=2+\sqrt{4+m_\phi^2}$$

This is also the energy of the lowest ϕ mode in vacuum AdS_5. For the case $m_\phi = 0$, $\Delta_0 = 4$. The other spherically symmetric modes of ϕ (dual to the descendants $\partial^{2n}\mathcal{O}_\phi$) have an energy

$$\Delta_n \equiv \Delta_0 + 2n = 2 + \sqrt{4 + m_\phi^2} + 2n$$

For the case $m_\phi = 0$, $\Delta_n = 4 + 2n$. The covariant derivative acting on ϕ is denoted by $D_\mu \equiv \nabla_\mu - ieA_\mu$. The symbol D^2 is used to denote the covariant Laplacian.

C.8.2 Thermodynamic quantities

We now turn to notations involving thermodynamic quantities. First, we omit a factor of G_5^{-1} from all our extensive quantities in order to simplify our expressions. With this understanding, we will denote the ADM mass by M, the charge of a solution by Q and its entropy by S. We often find it convenient to work with a rescaled mass m and a rescaled charge q which are related to the actual mass M and charge Q via the relations

$$Q \equiv \frac{\pi}{2} q \quad \text{and} \quad M \equiv \frac{3\pi}{8} m \tag{C.61}$$

We use $F \equiv M - TS$ to denote Helmholtz free-energy appropriate to the canonical ensemble and $G \equiv M - TS - \mu Q$ to denote the grand potential appropriate to the grand-canonical ensemble. Coming to the intensive variables μ represents the chemical potential and T represents the temperature both of which are defined by the first law

$$dM = TdS + \mu dQ$$

We sometimes find it convenient to work with the 'rationalised' inverse temperature $\beta \equiv (4\pi T)^{-1}$ and the chemical potential excess over the super-radiant bound $\delta\mu \equiv \mu - \Delta_n/e$.

C.8.3 Double expansions

Quantities in this chapter are often expressed as double expansions about two parameters - first is the parameter ϵ which is the amplitude of the leading normalisable mode in ϕ. Under AdS/CFT, roughly $\epsilon \sim <\mathcal{O}_\phi>$, the boundary expectation value of $\mathcal{O}_\phi$, the operator dual to ϕ. The second parameter is the outer horizon radius R of the small charged black hole at the core of the hairy black hole. Further, our solutions are often expressed in terms of a matched asymptotic expansion with a near-field expansion at small radius ($r << 1$) and a far field expansion far away from the horizon ($r >> R$) matched at their common domain of validity. We use the superscripts *in* and *out* to denote these two expansions respectively. Many formulae in this chapter involve the coefficients in this expansion which are defined via

$$f(r) = \begin{cases} \text{Near field}(r << 1) : f^{in} = \sum_{n=0}^{\infty} \epsilon^{2n} f_{2n}^{in} = \sum_{n=0}^{\infty} \epsilon^{2n} \sum_{k=0}^{\infty} R^{2k} f_{2n,2k}^{in} \\ \text{Far field}(r >> R) : f^{out} = \sum_{n=0}^{\infty} \epsilon^{2n} f_{2n}^{out} = \sum_{n=0}^{\infty} \epsilon^{2n} \sum_{k=0}^{\infty} R^{2k} f_{2n,2k}^{out} \end{cases}$$
$$g(r) = \begin{cases} \text{Near field}(r << 1) : g^{in} = \sum_{n=0}^{\infty} \epsilon^{2n} g_{2n}^{in} = \sum_{n=0}^{\infty} \epsilon^{2n} \sum_{k=0}^{\infty} R^{2k} g_{2n,2k}^{in} \\ \text{Far field}(r >> R) : g^{out} = \sum_{n=0}^{\infty} \epsilon^{2n} g_{2n}^{out} = \sum_{n=0}^{\infty} \epsilon^{2n} \sum_{k=0}^{\infty} R^{2k} g_{2n,2k}^{out} \end{cases}$$
$$A(r) = \begin{cases} \text{Near field}(r << 1) : A^{in} = \sum_{n=0}^{\infty} \epsilon^{2n} A_{2n}^{in} = \sum_{n=0}^{\infty} \epsilon^{2n} \sum_{k=0}^{\infty} R^{2k} A_{2n,2k}^{in} \\ \text{Far field}(r >> R) : A^{out} = \sum_{n=0}^{\infty} \epsilon^{2n} A_{2n}^{out} = \sum_{n=0}^{\infty} \epsilon^{2n} \sum_{k=0}^{\infty} R^{2k} A_{2n,2k}^{out} \end{cases}$$
$$\phi(r) = \begin{cases} \text{Near field}(r << 1) : \phi^{in} = \sum_{n=0}^{\infty} \epsilon^{2n+1} \phi_{2n}^{in} = \sum_{n=0}^{\infty} \epsilon^{2n+1} \sum_{k=0}^{\infty} R^{2k} \phi_{2n,2k}^{in} \\ \text{Far field}(r >> R) : \phi^{out} = \sum_{n=0}^{\infty} \epsilon^{2n+1} \phi_{2n}^{out} = \sum_{n=0}^{\infty} \epsilon^{2n+1} \sum_{k=0}^{\infty} R^{2k} \phi_{2n+,2k}^{out} \end{cases} \tag{C.62}$$

In a similar vein, one can define an expansion of the covariant Laplacian

$$D^2 = \begin{cases} \text{Near field}(r << 1) : (D^2)^{in} = \sum_{n=0}^{\infty} \epsilon^{2n} (D^2)_{2n}^{in} = \sum_{n=0}^{\infty} \epsilon^{2n} \sum_{k=0}^{\infty} R^{2k} (D^2)_{2n,2k}^{in} \\ \text{Far field}(r >> R) : (D^2)^{out} = \sum_{n=0}^{\infty} \epsilon^{2n} (D^2)_{2n}^{out} = \sum_{n=0}^{\infty} \epsilon^{2n} \sum_{k=0}^{\infty} R^{2k-2} (D^2)_{2n,2k}^{out} \end{cases}$$

Further, the boundary value of the gauge field or the chemical potential has the double expansion

$$\mu \equiv \lim_{r\to\infty} A(r) = \sum_{n=0}^{\infty} \epsilon^{2n} \mu_{2n}(R) = \sum_{n=0}^{\infty} \epsilon^{2n} \sum_{k=0}^{\infty} R^{2k} \mu_{2n,2k}$$

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