Applications of Non-critical String Theory To Non-perturbative Physics And Open-Closed String Duality

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by
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I dedicate this thesis to my loving parents and sister.

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Contents

Chapter 1

Introduction

1.1 A unified theory

The primary goal of string theory is to provide a unified description of the fundamental forces of nature. Three of the four fundamental forces, namely Electromagnetism, Weak and Strong nuclear forces have been accounted for by the Standard Model. The Standard Model is a relativistic quantum field theory of the constituents of matter, which has had phenomenal success in predicting the outcome of experiments. However, it suffers from two important shortcomings. The first is the notable omission of the fourth fundamental force, gravity. Also the Standard Model depends on a large number of parameters such as particle masses and coupling constants which cannot be derived from fundamental principles and can only be determined experimentally.

String theory attempts to resolve both these problems. The basic idea of strings is simple: it is a one-dimensional extended quantum object which propagates through a target spacetime. Just like a musical string, it has different vibrational modes. These modes manifest themselves as different particles. [1]. The starting point is a quantum relativistic theory of the string worldsheet, which has conformal symmetry, i.e. is invariant under scale transformations. Remarkably, string theory naturally includes gravity as a result of this symmetry. So, from

[1]For standard references on string theory see [1, 2, 3, 4].

the beginning, it is a quantum theory of gravity. String theory can be bosonic or supersymmetric. In order to be a consistent quantum theory the spacetime in which the bosonic string propagates must have the "critical" dimension of 26. Here each spacetime direction corresponds to a scalar with central charge 1, so the total central charge is 26. For the supersymmetric string the spacetime is 10 dimensional. In this case each spacetime dimension contributes $\frac{3}{2}$ to the central charge, so the total central charge is 15, which is the requirement for consistent quantization of a superstring. The superstring theory can still describe a 4 dimensional real world spacetime if we assume that the 6 extra dimensions are compact, and too small to be detected by present experiments.

The critical string theory has supersymmetry, and is a quantum theory of supergravity coupled to supersymmetric matter. Moreover, the interaction between the particle-like string excitations closely mimic those in the Standard Model. So it has many of the features which a unified theory of nature should have. However, in general it has proved difficult to perform exact calculations in a general string background. As a result, most of the results of string theory are derived to low orders in perturbation theory in the string coupling.

1.2 The non-critical string

The difficulty of finding exact solutions in string theory makes it very important to find backgrounds which admit an all-orders perturbative or non-perturbative solution, while retaining enough essential physics to be useful in deriving universal properties of string theory. It turns out that non-critical string theory provides us with such a background [1]. The name "non-critical" comes from the fact that the dimension of the target spacetime in this case is lower than the "critical" dimension (26 for bosonic, and 10 for the fermionic theory). In this thesis we will be concerned with a particular non-critical background which is 1+1 dimensional. The worldsheet theory consists of a scalar field X which is the time coordinate of the target spacetime and a Liouville field ϕ, which behaves like a space coordinate.

[1] See [5] and the references therein for an exhaustive review.

The presence of the Liouville coordinate ensures that the quantum theory of the string worldsheet is free of anomalies, even if the spacetime dimension is not critical.

There can be two kinds of non-critical theories with a 1+1 dimensional target spacetime. In the bosonic theory the scalar X has central charge $c = 1$, while the Liouville coordinate ϕ has central charge 25, so that the total central charge is 26, as required for consistent quantization. This is the $c = 1$ non-critical bosonic string. The spectrum of the $c = 1$ string has massless "tachyons"[1] which are closed string excitations. Note that since the target spacetime is two dimensional, gravity is non-dynamical. As we will see, the vacuum of the $c = 1$ string is non-perturbatively unstable. So the $c = 1$ string cannot be defined beyond perturbation theory.

It is possible to have supersymmetry on the string worldsheet, which leads to the Type 0 theories. In this case the central charge is $\frac{3}{2}$ for the scalar X and $\frac{27}{2}$ for the Liouville field ϕ so the total is 15. In the Type 0 theories it necessary to impose a GSO projection on the states in order to preserve unitarity and modular invariance of the theory. The GSO projection is non-chiral, so the spacetime theory does not have fermions. There can be two kinds of GSO projections, leading to the 0A and 0B theories:

$$\begin{aligned} \text{Type 0A}: &\quad (NS-, NS-) \oplus (NS+, NS+) \oplus (R+, R-) \oplus (R-, R+) \\ \text{Type 0B}: &\quad (NS-, NS-) \oplus (NS+, NS+) \oplus (R-, R-) \oplus (R+, R+) \end{aligned}$$

where NS and R refer to sectors with different boundary conditions. The $\pm$ sign gives the value of $e^{2\pi i F}$ where F is the worldsheet fermion number. The spectrum of the theory is as follows. In the $0A$ theory, there is a massless closed string "tachyon" T and two gauge fields F, $\tilde{F}$. These lead to two quantized fluxes $q, \tilde{q}$. In the $0B$ theory, there is also a "tachyon", but now there is an additional scalar from the RR sector, which consists of a self-dual and anti self-dual component. We will see that unlike the bosonic $c = 1$ string, the Type 0 theory is well defined

[1]The name tachyon is used because their mass squared becomes negative for $d > 2$. In this case these are just massless scalars.

non-perturbatively. In fact, the Type 0 string theories in two dimensions provides a perfect example of an exactly solved non-perturbative string background.

1.3 The matrix model description

The non-critical string has a remarkable alternative description in terms of a random matrix model (see [6, 7] for reviews). The matrix model provides a triangulation of the string worldsheet. This approximation becomes exact in the double scaling limit, when the matrix partition function describes the string theory exactly. The dynamics of the string theory is encoded in the double scaled matrix model as the excitations of free fermions moving in an inverse harmonic oscillator potential, which is unbounded below. The $c = 1$ bosonic string theory is obtained by filling up the energy levels on one side of this inverse harmonic oscillator potential, while the other side is empty, as shown in Figure 1.1. The

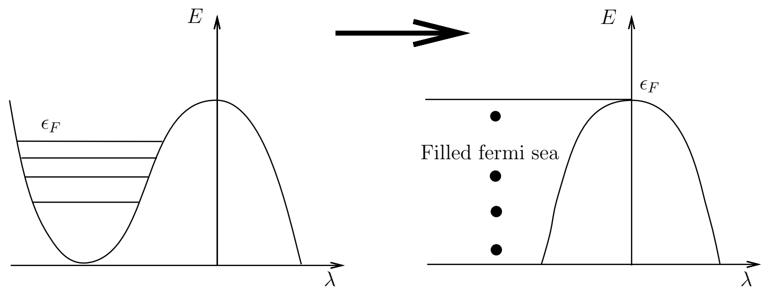

Figure 1.1: The free fermion picture of the $c = 1$ model

non-perturbative instability of the theory comes from the fact that the fermions can tunnel through the barrier and are lost on the other side. For the Type 0 string background, the energy levels on both sides of the barrier are filled up. In this case a fermion on one side cannot tunnel to the other side and decay because the lower energy levels on the other side are already occupied. So the Type 0 theory is stable non-perturbatively (see Figure 1.2). String theory correlation functions can be evaluated from the matrix model by computing scattering amplitudes of these fermions off the potential barrier.

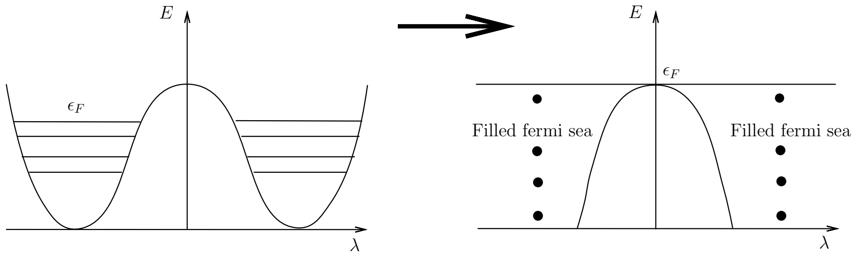

Figure 1.2: The free fermion picture for the Type 0 theory

The matrix model is a valuable tool because it can be solved exactly in perturbation theory or, in some cases, non-perturbatively. This property makes the 1+1 dimensional non-critical theory special. This has led to much interest in the subject. It was possible to calculate the partition function and correlators exactly for different non-critical string backgrounds. For instance the all-orders perturbative free energy for the $c = 1$ string was calculated by Gross and Klebanov [8]. In their solution the time direction is a circle of radius R and is Euclidean. This is a complete solution for the bosonic string as this background is not defined beyond the perturbative expansion. For the Type 0 case the full non-perturbative free energy was presented by Maldacena and Seiberg [9]. This work follows earlier studies of the Type 0 theory [10, 11, 12, 13].

In spite of the successes of non-critical string theories in computing correlators, it was not clear initially if the insights gained in this context can be applied directly to the critical string theories. However, as we will describe below, there were a number of remarkable developments in both critical and non-critical theories which showed that many of the important physical properties of critical string theory are realized in the non-critical context. Not only that, in the non-critical case the solutions are known exactly, either as an all-order perturbative expansion or non-perturbatively. This makes the study of non-critical string theory extremely relevant and interesting. In this thesis we will try to develop a better understanding of some of these phenomena from the non-critical string theory context.

1.4 D-branes

It was noticed that if one calculates the non-perturbative corrections to the non-critical string partition function at weak coupling using the matrix model picture, then these corrections go like e^{-1/g_s} where g_s is the string coupling constant. Based on this observation it was suggested by Shenker [14] that this e^{-1/g_s} dependence is a generic property of string theories and not unique to the non-critical string. It is known that instantons in field theories contribute non-perturbatively to the partition function, but such contributions are of the form e^{-1/g_s^2}. The contribution of string theory instantons is thus significantly larger. The realization of this phenomenon in critical string theories are the D-branes found by Polchinski, Dai, Leigh, Horava [15, 16]. D-branes are hypersurfaces in the target spacetime on which open strings can end, and they form boundaries on the string worldsheet. They are dynamical solitonic objects in string theory. The e^{-1/g_s} dependence was subsequently shown by Polchinski [17]. The D-branes he considered for this calculation were localized at a point in spacetime and hence were instantons. This verifies the proposal made by Shenker.

This series of developments came full-circle after the remarkable discovery of ZZ and FZZT branes in the non-critical string theories by Fateev, the Zamolodchikovs and Teschner [18, 19, 20]. The ZZ branes are unstable D0 branes and are localized at the strong coupling end of the Liouville direction ($\phi \to \infty$). The open string theory on the D0 brane has a tachyon[21], and the worldline theory of the D0 brane is described by the double scaled matrix model mentioned earlier. The FZZT branes on the other hand are stable D1 branes which extend along the Liouville direction. The corresponding $D0$ and $D1$ instantons contribute non-perturbatively to the partition function by terms like e^{-1/g_s}.

1.5 Open-closed duality for the $c = 1$ string

Open-closed duality is a generic phenomenon in string theory. The simplest example of open-closed duality can be found by considering a one-loop amplitude of an open string. This looks like a cylinder, with the boundaries of the cylinder being the endpoints of the string. It can be shown that the same amplitude can be derived by considering the tree level diagram of a closed string, which is expected since the cross-section of the cylinder resembles a closed string. This indicates that any open string theory also contains the closed string.

The fact that the closed string forms a sub-sector of the open string theory can be used to rewrite the boundary conditions on the open string endpoints as some closed string states (in our cylinder example, the closed string boundary states at the endpoints of the cylinder provides an equivalent description of the open string boundary conditions). Then in some proper limit we can replace the boundaries with an insertion of a closed string boundary state, also known as an Ishibashi state. Our cylinder then looks like a sphere with two punctures corresponding to the endpoints. If the above procedure is valid, then starting from an open string theory on a worldsheet with boundaries we end up with an equivalent (dual) closed string theory on a different background.

This idea of open-closed duality was realized for critical string theories by Maldacena [22]. The basic observation in this work is that the large N limit of some conformal field theories can describe closed string theory on a product of Anti de-Sitter space and a compact manifold. This is the AdS-CFT conjecture. The CFT is constructed as the low energy worldvolume gauge theory of N D3 branes in Type IIB string theory on a flat background, which is $SU(N)$ $\mathcal{N} = 4$ super Yang-Mills theory. It describes the dynamics of open strings on these D-branes. The dual closed string theory is the Type IIB string on $\mathrm{AdS}_5 \times \mathrm{S}_5$ space, which is the near-horizon geometry of these D3-branes.

In the non-critical theories, a similar open-closed duality can be realized in two different ways. The first way is through the the Gopakumar-Vafa correspondence [23, 24]. In this case the open string theory is an $SU(N)$ Chern-Simmons theory which lives on N topological A model 3-branes wrapped on an S^3 cycle of a deformed conifold space (see Chapter 5 for more details). The closed string theory is the topological A model on the resolved conifold space, where the conifold singularity is removed by blowing up an S^2 cycle. This change of the background from the deformed to the resolved conifold when one goes from the open to the closed description is known as the "geometric transition"[1].

The second setting in which open-closed duality is realized in the non-critical context is in the matrix model description of Liouville theory. This is the case which we consider in Chapter 2. We work with the $c = 1$ string with a compact Euclidean time direction of radius R. The target spactime thus looks like a cylinder of radius R. There are three different, but related matrix models which describe this background. The first of these models, known as Matrix Quantum Mechanics was already mentioned in Section 1.3. This describes the $c = 1$ string at radius R as a collection of free fermions. Starting from the Matrix Mechanics description, a new model describing the the $c = 1$ string at radius R was derived in Ref.[25]. This is the Normal Matrix Model. The Normal Matrix Model lagrangian does not explicitly depend on time, unlike the Matrix Mechanics. Similar to the Matrix Mechanics, the Normal Matrix Model depends only on closed string parameters, which are the couplings to momentum modes which perturb the vacuum. The third model which we consider is the Imbimbo-Mukhi model derived in Ref.[26]. It describes the $c = 1$ string at $R = 1$. This model depends on some closed string parameters and some open string parameters which are believed to be associated with FZZT branes. In the work described in Chapter 2 we are able to find an explicit map between the Normal Matrix Model and the Imbimbo-Mukhi model, both at $R = 1$. We argue that this correspondence between the two models encodes open-closed duality for the $c = 1$ string.

1.6 T-duality for the non-critical String

String theory compactified on a circle, such as in the case considered in Section 1.5 has a remarkable duality known as T-duality. The statement of T-duality is that a theory on a circle of radius R is dual to a theory on a circle of radius $\tilde{R} = \frac{\alpha'}{R}$, where $\frac{1}{2\pi\alpha'}$ is the fundamental string tension. Also, the duality maps the momentum modes of the string along the compact direction in the first theory to

[1]Note that this is not a dynamical process but a duality between the two different string backgrounds.

the winding modes along the circle in the second and vice versa. It relates the behaviour of the string at very small distances ($R \to 0$) to the behaviour at large distances $\tilde{R} \to \infty$. The T-dual string theories both have the same value of the string coupling constant, so T-duality is perturbative in nature.

T-duality a manifest symmetry of the worldsheet formulation of string theory. However, this is not so in the matrix model formulation, Matrix Quantum Mechanics. This is because winding and momentum operators which are related by T-duality are represented symmetrically in the worldsheet formulation, but the matrix model treats them differently. In the matrix model, momentum correlators are computed from the scattering amplitudes of free fermions in the singlet representation of the matrix, while winding perturbations have to be computed as expectation values of Wilson loops in a non-singlet representation.

In Chapter 3 we describe a new method which makes use of the Normal Matrix Model to compute arbitrary momentum correlation functions for the $c = 1$ string to all orders in perturbation theory. This allows us to obtain an exact expression for the $2n$-point function of unit momentum modes for the first time. This result can in principle be used to verify T-duality for the matrix model, once exact computations for the winding correlators are performed. This latter computation is an open problem at the moment, although there have been some attempts [27, 28].

1.7 The two dimensional black hole

Black holes are singular solutions of Einstein's equations which are formed from gravitational collapse of matter (for a review of black hole physics see [29, 30] and the references therein). Black holes can be produced at the end point of the life cycle of stars whose mass exceeds 1.4 solar masses (known as the Chandrasekhar limit). The gravitational force becomes so strong that it overcomes all other forces and the star collapses into a singularity to form a black hole.

String theory, being a quantum theory of gravity, should provide a microscopic quantum description of a black hole, which is only classically described by Einstein's equations. Some early work on the subject proposed a connection between the quantum states of the black hole and string excitations [31]. The low energy effective action of string theory describes classical general relativity and has black holes as the solutions of the equations of motion. In subsequent work it was found that it is possible to have a Schwarzchild-like black hole solution with heavy string states at strong coupling [32]. It was also seen that the degeneracy of perturbative string states can be used to compute the entropy of a certain class of black holes, thus indicating a relation between black holes and elementary string excitations [33]. The discovery of D-branes in string theory finally led to a microscopic description of black hole states. In Ref. [34] Strominger and Vafa provided a construction of a black hole state from string theory using D-branes wrapped over a compact manifold. They were able to show that the statistical entropy of this black hole computed from string theory reproduces the Bekenstein-Hawking entropy formula for a black hole of horizon area A: $S_{BH} = \frac{A}{4}$.

It turns out that the non-critical string also admits a two dimensional Euclidean black hole like solution. This background can be described by a conformal field theory (CFT) known as the gauged $SL(2,R)/U(1)$ Wess-Zumino-Witten model [35]. In Ref.[36] Fateev, Zamolodchikov and Zamolodchikov conjectured that this two dimensional black hole is generated by a condensate of unit winding modes or vortices in the non-critical string background. The unit winding mode is generated by a closed string which wraps once around the compact direction. This is known as FZZ duality. The presence of the vortex condensate pinches off the cylinder shaped spacetime at the location of the vortex, and the geometry then begins to resemble a cigar shape, which describes the black hole. Our computation of the $2n$-point function in Chapter 3 provides an exact expression for the partition function of this black hole after T-dualizing. In the work described in Chapter 4 we provide a new interpretation for the FZZ duality. We propose a new deformation of the $c = 1$ string background which makes clear how the black hole state originates from the non-critical string theory. We also propose a generalization of the FZZ duality in presence of a condensate of higher winding modes. The black hole background in this case should be a higher spin generalization of the two dimensional black hole solution. The existence of these solutions was proved in [37]. However, unlike in the unit winding case, the CFT corresponding to these multiple winding black hole states is not known yet.

1.8 The topological string

In Section 1.3 we introduced the matrix model description for non-critical strings, which proved to be very useful as a computational tool. It turns out that the two dimensional non-critical string theory has yet another alternative description in terms of topological string theory [38, 39, 40, 41, 42]. This correspondence presents an entirely new perspective on the properties of non-critical strings.

As already mentioned earlier, in order to get a four dimensional supersymmetric theory we need to compactify the ten dimensional superstring on a six dimensional manifold. In this setup supersymmetry is preserved only if the six dimensional compact manifold is a Calabi-Yau space. Topological string theory describes string propagation on this Calabi-Yau space. It provides a quantum theory of deformations of the Calabi-Yau. In the correspondence between non-critical string theory and topological strings, the relevant Calabi-Yau space is the conifold. It can be simply defined by its embedding in $\mathbb{C}^4$:

$$zw - px = 0, \tag{1.1}$$

where z, w, p, x are complex coordinates. There is a singularity at the origin. The singularity can be removed by deforming the conifold equation to:

$$zw - px = \mu \tag{1.2}$$

which blows up an S^3 of radius $\sqrt{|\mu|}$ at the origin. This space is the deformed conifold (DC), and μ is the complex deformation parameter.

The first demonstration of the correspondence was presented by Ghoshal and Vafa[43] who showed that non-critical $c = 1$ string theory at the self-dual radius is perturbatively equivalent to topological string theory on a deformed conifold. It can be generalized to integer radius [44, 45]. In this case the corresponding

Calabi-Yau space is a $\mathbb{Z}_p$ orbifold of the conifold[1]. It has p singularities, which can be removed by blowing p S^3 cycles as before, leading to:

$$zw - \prod_{k=1}^{p}(px - \mu_k) = 0 \tag{1.3}$$

where $\mu_i, i = 1, 2, ..., p$ are the sizes of the S^3's. This is the deformed orbifolded conifold (DOC).

Our work presented in Chapter 5 concerns the topological description of the Type 0 string. There have been some proposals for the topological correspondence at the self-dual[2] and integer radius $R = p$ [46, 47]. The topological string in this case lives on a $\mathbb{Z}_{2p}$ deformed orbifolded conifold. However these proposals only manage to reproduce the free energy of the Type 0 theory in a perturbative expansion in the string coupling constant, and not the exact answer derived for the Type 0 case in Ref. [9] mentioned in Section 1.3. In the first part of Chapter 5 we re-derive the existing perturbative correspondence using a more elegant and rigorous method. In the second part of this chapter we present a new construction with non-compact topological branes on the Calabi-Yau, which exactly reproduces the full non-perturbative free energy of the Type 0 string.

[1] In this chapter, as well as in Chapter 5, we use the symbol p to denote a complex coordinate of the conifold. This is to be distinguished from the integer p which denotes the order of the orbifold.

[2] Here self-dual means unit radius $R = 1$, which remains invariant under T-duality.

Chapter 2

$c = 1$ Matrix Models: Equivalences and Open-Closed String Duality

In this chapter we present an explicit demonstration of the equivalence between the Normal Matrix Model (NMM) of $c = 1$ string theory at selfdual radius and the Kontsevich-Penner (KP) model for the same string theory. We relate macroscopic loop expectation values in the NMM to condensates of the closed string tachyon, and discuss the implications for open-closed duality. As in $c < 1$, the Kontsevich-Miwa transform between the parameters of the two theories appears to encode open-closed string duality, though our results also exhibit some interesting differences with the $c < 1$ case. We also briefly comment on two different ways in which the Kontsevich model originates [48].

2.1 Introduction

In the last few years, enormous progress has been made in understanding noncritical string theory. One line of development started with the work of Refs.[49, 50, 51], in the context of D-branes of Liouville theory. These and subsequent works were inspired by the beautiful CFT computations that gave convincing evidence for the consistency of these branes[18, 19, 20], as well as Sen's picture of the decay of unstable D-branes via tachyon condensation[52]. Another independent line of development that has proved important was the attempt to formulate new matrix models to describe noncritical string theories and their deformations, including black hole deformations[25, 27, 53, 54].

Some of the important new results are related to nonperturbatively stable type 0 fermionic strings[10, 11], but even in the bosonic context, many old and new puzzles concerning matrix models as well as Liouville theory have been resolved. For $c < 1$ matter coupled to Liouville theory, a beautiful picture emerged of a Riemann surface governing the semiclassical dynamics of the model. Both ZZ and FZZT branes were identified as properties of this surface: the former are located at singularities while the latter arise as line integrals. This picture was obtained in Ref.[55] within the continuum Liouville approach and subsequently rederived in the matrix model formalism in Ref.[56] using earlier results of Ref.[57]. However, later it was realised[58] that the exact, as opposed to semiclassical, picture is considerably simpler: the Riemann surface disappears as a result of Stokes' phenomenon and is replaced by a single sheet. In the exact (quantum) case, correlation functions of macroscopic loop operators go from multiple-valued functions to the Baker-Akhiezer functions of the KP hierarchy, which are analytic functions of the boundary cosmological constant. Thus, for these models (and also their type 0 extensions) a rather complete picture now exists.

Another remarkable development in this context is an explicit proposal to understand open-closed string duality starting from open string field theory. This was presented in Ref.[59] and implemented there for the $(2, q)$ series of minimal models coupled to gravity (which can be thought of as perturbations of the "topological point" or $(2, 1)$ minimal model). The basic idea of Ref.[59] was to evaluate open string field theory on a collection of N FZZT branes in the $(2, 1)$ closed string background. This leads to the Kontsevich matrix model[60], which depends on a constant matrix A whose eigenvalues are the N independent boundary cosmological constants for this collection of branes. Now the Kontsevich model computes the correlators of *closed-string* observables in the same $(2, 1)$ background. So this relationship was interpreted as open-closed duality, following earlier ideas of Sen[61].

A different way of understanding what appears to be the same open-closed duality emerged in Ref.[58] for the $(2, 1)$ case. Extending some older observations in Ref.[62], the authors showed that if one inserts macroscopic loop operators $\det(x_i - \Phi)$, representing FZZT branes (each with its own boundary cosmological constant x_i) in the Gaussian matrix model, and takes a double-scaling limit, one obtains the Kontsevich matrix model. The constant matrix A in this model again arises as the boundary cosmological constants of the FZZT branes[1].

The situation is more complicated and less well-understood for $c = 1$ matter coupled to Liouville theory, namely the $c = 1$ string. The results of FZZT were derived for generic Liouville central charge c_L, but become singular as $c_L \to 25$, the limit that should give the $c = 1$ string. Attempts to understand FZZT branes at $c = 1$ (Refs.[66, 67]) rely on this limit from the $c < 1$ case which brings in divergences and can therefore be problematic. In particular, there is as yet no definite computation exhibiting open-closed duality at $c = 1$ starting from open string field theory in the $c = 1$ Liouville background. One should expect such a computation to give rise to the $c = 1$ analogue of the Kontsevich matrix model, namely the Kontsevich-Penner model[2] of Ref.[26].

In the present work we take a different approach to understand D-branes and open-closed duality in the $c = 1$ string, more closely tied to the approach of Refs.[58, 63]. The obvious point of departure at $c = 1$ would be to consider macroscopic loops in the Matrix Quantum Mechanics (MQM) and take a double-scaling limit. Indeed, FZZT branes at $c = 1$ have been investigated from this point of view, for example in Refs.[28, 68]. However, we will take an alternative route that makes use of the existence of the Normal Matrix Model (NMM)[25] for $c = 1$ string theory (in principle, at arbitrary radius R). This model is dual in a certain precise sense to the more familiar MQM, namely, the grand canonical

[1]This has been generalised[63] by starting with macroscopic loops in the double-scaled 2-matrix models that describe $(p, 1)$ minimal model strings. After double-scaling, one obtains the generalised Kontsevich models of Refs.[64, 65].

[2]This model is valid only at the selfdual radius $R = 1$.

partition function of MQM is the partition function of NMM in the large-N limit. Geometrically, the two theories correspond to different real sections of a single complex curve. More details about the interrelationship between MQM and NMM can be found in Ref.[25].

One good reason to start from the NMM is that it is a simpler model than MQM and does not require a double-scaling limit. Also, it has been a long-standing question whether the KP model and NMM are equivalent, given their structural similarities, and if so, what is the precise map between them. It is tempting to believe that open-closed duality underlies their mutual relationship. Indeed, the NMM does not have a parameter suggestive of a set of boundary cosmological constants, while the KP model has a Kontsevich-type constant matrix A. So another natural question is whether the eigenvalues of A are boundary cosmological constants for a set of FZZT branes/macroscopic loop operators of NMM.

In what follows we examine these questions and obtain the following results. First of all we find a precise map from the NMM (with arbitrary tachyon perturbations) to the KP model, thereby demonstrating their equivalence. While the former model depends on a non-Hermitian matrix Z constrained to obey $[Z, Z^{\dagger}] = 1$, the latter is defined in terms of a positive definite Hermitian matrix M. We find that the eigenvalues z_i and m_i are related by $m_i = z_i \bar{z}_i$. The role of the large-N limit in the two models is slightly different: in the KP model not only the random matrix but also the number of parameters (closed string couplings) is reduced at finite N. On the contrary, in the NMM the number of parameters is always infinite for any N, but one is required to take $N \to \infty$ to obtain the right theory (this was called "Model I" in Ref.[25]). The two models are therefore equivalent only on a subspace of the parameter space at finite N, with the limit $N \to \infty$ being required to obtain full equivalence. This is an important point to which we will return.

Next in § 2.5 we consider macroscopic loop operators of the form $\det(\xi - Z)$ in the NMM, and show that these operators when inserted into the NMM, *decrease* the value of the closed-string tachyon couplings in a precise way dictated by the

Kontsevich-Miwa transform. On the contrary, operators of the form $1/\det(\xi - Z)$ play the role of increasing, or turning on, the closed-string tachyon couplings. In particular, insertion of these *inverse determinant* operators in the (partially unperturbed) NMM leads to the Kontsevich-Penner model. (By partially unperturbed, we mean the couplings of the positive-momentum tachyons are switched off, while those of the negative-momentum tachyons are turned on at arbitrary values.) Calculationally, this result is a corollary of our derivation of the KP model from the perturbed NMM in § 2.4.

These results bear a rather strong analogy to the emergence of the Kontsevich model from the insertion of determinant operators at $c < 1$[58]. In both cases, the parameters of macroscopic loop operators turn into eigenvalues of a Kontsevich matrix. Recall that in Ref.[58], one inserts n determinant operators into the $N \times N$ Gaussian matrix model and then integrates out the Gaussian matrix. Taking $N \to \infty$ (as a double-scaling limit) we are then left with the Kontsevich model of rank n. In the $c = 1$ case, we insert n inverse determinant operators in the NMM. As we will see, $N - n$ of the normal matrix eigenvalues then decouple, and we are left with a Kontsevich-Penner model of rank n (here one does not have to take $N \to \infty$). We see that the two cases are rather closely analogous.

The main difference between our case at $c = 1$ and the $c < 1$ case of Ref.[58] is that we work with inverse determinant rather than determinant operators. However at infinite n we can remove even this difference: it is possible to replace the inverse determinant by the determinant of a different matrix, defining a natural pair of mutually "dual" Kontsevich matrices[1]. In terms of the dual matrix, one then recovers a relation between correlators of determinants (rather than inverse determinants) and the KP model.

In the concluding section we examine a peculiar property of the NMM, namely that it describes the $c = 1$ string even at finite N, if we set $N = \nu$, where ν is the analytically continued cosmological constant $\nu = -i\mu$. This was noted in Ref.[25], where this variant of the NMM was called "Model II". Now it was

[1]This dual pair is apparently unrelated to the dual pair of boundary cosmological constants at $c < 1$.

already observed in Ref.[26] that setting $N = \nu$ in the KP model (and giving a nonzero value to one of the deformation parameters) reduces the KP model to the original Kontsevich model that describes $(2, q)$ minimal strings. Thus we have a (two-step) process leading from the NMM to the Kontsevich model. However, we also know from Ref.[58] that the Kontsevich model arises from insertion of macroscopic loops in the double-scaled Gaussian matrix model. We will attempt to examine to what extent these two facts are related.

2.2 Normal Matrix Model

We start by describing the Normal Matrix Model (NMM) of $c = 1$ string theory[25] and making a number of observations about it. The model originates from some well-known considerations in the Matrix Quantum Mechanics (MQM) description of the Euclidean $c = 1$ string at radius R. Here, $R = 1$ is the selfdual radius, to which we will specialise later. The MQM theory has discrete "tachyons" T_k, of momentum $\frac{k}{R}$, where $k \in Z$. Let us divide this set into "positive tachyons" $T_k, k > 0$ and "negative tachyons" $T_k, k < 0$. (The zero-momentum tachyon is the cosmological operator and is treated separately). We now perturb the MQM by these tachyons, using coupling constants $t_k, k > 0$ for the positive tachyons and $\bar{t}_k, k > 0$ for the negative ones.

The grand canonical partition function of MQM is denoted $\mathcal{Z}(\mu, t_k, \bar{t}_k)$. At $t_k = \bar{t}_k = 0$, it can easily be shown to be:

$$\mathcal{Z}(\mu, t_k = 0, \bar{t}_k = 0) = \prod_{n \geq 0}^{\infty} \Gamma\left(-\frac{n + \frac{1}{2}}{R} - i\mu + \frac{1}{2}\right) \tag{2.1}$$

But this is also the partition function of the matrix integral:

$$\begin{aligned} \mathcal{Z}_{NMM} &= \int [dZ dZ^\dagger]\, e^{-\mathrm{tr} W(Z, Z^\dagger)} \\ &= \int [dZ dZ^\dagger]\, e^{\mathrm{tr}\left(-\nu (Z Z^\dagger)^R + \left[\frac{1}{2}(R-1) + (R\nu - N)\right] \log Z Z^\dagger\right)} \end{aligned} \tag{2.2}$$

where $\nu = -i\mu$ and $N \to \infty$. Here Z, $Z^\dagger$ are $N \times N$ matrices satisfying:

$$[Z, Z^\dagger] = 0 \tag{2.3}$$

Since the matrix Z commutes with its adjoint, the model defined by Eq. (2.2) is called the Normal Matrix Model (NMM)[1].

The equality above says that the unperturbed MQM and NMM theories are equivalent. The final step is to note that the tachyon perturbations correspond to infinitely many Toda "times" in the MQM partition function, which becomes a τ-function of the Toda integrable hierarchy. The same perturbations on the NMM side are obtained by adding to the matrix action the terms:

$$W(Z,Z^\dagger) \to W(Z,Z^\dagger) + \nu \sum_{k=1}^{\infty} \left(t_k Z^k + \bar{t}_k {Z^\dagger}^k \right) \tag{2.4}$$

It follows that the Normal Matrix Model, even after perturbations, is equivalent to MQM.

The equivalence of the full perturbed MQM and NMM gives an interesting interpretation of the perturbations in NMM in terms of the Fermi surface of the MQM. The unperturbed MQM Hamiltonian is given by:

$$H_0 = \frac{1}{2}\mathrm{tr}(-\hbar^2 \frac{\partial^2}{\partial X^2} - X^2) \tag{2.5}$$

where X is an $N \times N$ Hermitian matrix (here the compactification radius is R). In the $SU(N)$-singlet sector this system is described by N non-relativistic fermions moving in an inverted harmonic oscillator potential. The eigenvalues of X describe the positions of these fermions. In terms of eigenvalues the Hamiltonian can be written as:

$$H_0 = \frac{1}{2}\sum_{i=1}^{N}(\hat{p}_i^2 - \hat{x}_i^2), \tag{2.6}$$

[1]For the most part we follow the conventions of Ref.[25]. However we use the transcription $(1/i\hbar)_{\rm them} \to \nu_{\rm us}$ and $\mu_{\rm them} \to 1_{\rm us}$. The partition function depends on the ratio $(\mu/i\hbar)_{them} \to \nu_{us} = -i\mu_{\rm us}$. Our conventions for the NMM will be seen to match with the conventions of Ref.[26] for the KP model. Note that the integral is well-defined for all complex ν with a sufficiently large real part. It can then be extended by analytic continuation to all complex values of the parameter ν, other than those for which the argument of the Γ function is a negative integer. This is sufficient, since everything is ultimately evaluated at purely imaginary values of ν.

p_i being the momenta conjugate to x_i. We now want to consider perturbations of Eq. (2.6) by tachyon operators. For this it is convenient to change variables from $\hat{p}$, $\hat{x}$ to the "light cone" variables $\hat{x}_\pm$:

$$\hat{x}_\pm = \frac{\hat{x} \pm \hat{p}}{\sqrt{2}} \tag{2.7}$$

Since $[\hat{p}, \hat{x}] = -i\hbar$ it follows that $[\hat{x}_+, \hat{x}_-] = -i\hbar$ also. The MQM Hamiltonian in terms of the new variables is:

$$H_0 = -\sum_{i=1}^{N} \hat{x}_{+i}\hat{x}_{-i} - \frac{i\hbar N}{2} \tag{2.8}$$

In the phase space (x_+, x_-) the equation of the Fermi surface for the unperturbed MQM is given by:

$$x_+ x_- = \mu \tag{2.9}$$

The tachyon perturbations to the MQM Hamiltonian H_0 are given in terms of the new variables by:

$$H = H_0 - \sum_{k\geq 1}\sum_{i=1}^{N} \left(k\, t_{\pm k}\, x_{\pm i}^{\frac{k}{R}} + v_{\pm k}\, x_{\pm i}^{-\frac{k}{R}} \right) \tag{2.10}$$

In the above equation the v's are determined in terms of the t's from the orthonormality of the Fermion wavefunctions. The conventions chosen above simplifies the connection with NMM perturbations. The Fermi surface of the perturbed MQM is given by:

$$x_+ x_- = \mu + \sum_{k\geq 1} \left(k\, t_{\pm k}\, x_{\pm}^{\frac{k}{R}} + v_{\pm k}\, x_{\pm}^{-\frac{k}{R}} \right) \tag{2.11}$$

The equivalence between NMM and MQM relates the tachyon perturbations in Eq. (2.4) and Eq. (2.10) with the following identification between the tachyon operators of the two models:

$$\begin{aligned} \mathrm{tr} X_+^{\frac{n}{R}} &= \mathrm{tr} Z^n \\ \mathrm{tr} X_-^{\frac{n}{R}} &= \mathrm{tr} Z^{\dagger n} \end{aligned}$$

The coefficients $t_\pm$ are the same as $t, \bar{t}$ in the NMM. This means that any tachyon perturbation in the NMM is mapped directly to a deformation of the Fermi surface of MQM by Eq. (2.11).

At the selfdual radius R=1, the NMM simplifies and the full perturbed partition function can be written as:

$$\mathcal{Z}_{NMM}(t,\bar{t}) = \int [dZdZ^\dagger]\, e^{\mathrm{tr}\left(-\nu ZZ^\dagger + (\nu - N)\log ZZ^\dagger - \nu\sum_{k=1}^{\infty}\left(t_k Z^k + \bar{t}_k {Z^\dagger}^k\right)\right)} \tag{2.12}$$

We note several properties of this model.

(i) The unperturbed part depends only on the combination $ZZ^\dagger$ and not on $Z, Z^\dagger$ separately.

(ii) The model can be reduced to eigenvalues, leading to the partition function:

$$\mathcal{Z}_{NMM}(t,\bar{t}) = \int \prod_{i=1}^{N} dz_i d\bar{z}_i\ \Delta(z)\Delta(\bar{z})\, e^{\sum_{i=1}^{N}\left(-\nu z_i\bar{z}_i + (\nu - N)\log z_i\bar{z}_i - \nu\sum_{k=1}^{\infty}\left(t_k z_i^k + \bar{t}_k \bar{z}_i^k\right)\right)} \tag{2.13}$$

(iii) The model is symmetric under the interchange $t_k \leftrightarrow \bar{t}_k$, as can be seen by interchanging Z and $Z^\dagger$. In spacetime language this symmetry amounts to the transformation $X \to -X$ where X is the Euclidean time coordinate, which interchanges positive and negative momentum tachyons.

(iv) The correlator:

$$\langle \mathrm{tr}Z^{k_1}\mathrm{tr}Z^{k_2}\cdots\mathrm{tr}Z^{k_m}\mathrm{tr}{Z^\dagger}^{\ell_1}\mathrm{tr}{Z^\dagger}^{\ell_2}\cdots\mathrm{tr}{Z^\dagger}^{\ell_n}\rangle_{t_k=\bar{t}_k=0} \tag{2.14}$$

vanishes unless

$$\sum_m k_m = \sum_n \ell_n \tag{2.15}$$

This correlator is computed in the unperturbed theory. The above result follows by performing the transformation:

$$Z \to e^{i\theta} Z \tag{2.16}$$

for some arbitrary angle θ. The unperturbed theory is invariant under this transformation, therefore correlators that are not invariant must vanish. In spacetime language this amounts to the fact that tachyon momentum in the X direction is conserved.

(v) As a corollary, we see that if we set all $t_k = 0$, the partition function becomes independent of $\bar{t}_k$:

$$\mathcal{Z}_{NMM}(0, \bar{t}_k) = \mathcal{Z}_{NMM}(0, 0) \tag{2.17}$$

(vi) For computing correlators of a finite number of tachyons, it is enough to turn on a *finite* number of $t_k, \bar{t}_k$, i.e. we can always assume for such purposes that $t_k, \bar{t}_k = 0, k > k_{max}$ for some finite integer k_{max}. In that case, apart from the log term we have a polynomial matrix model.

(vii) We can tune away the log term by choosing $\nu = N$. This choice has been called Model II in Ref.[25]. In this case the model reduces to a Gaussian model (but of a normal, rather than Hermitian, matrix) with perturbations that are holomorphic + antiholomorphic in the matrix Z (i.e., in the eigenvalues z_i). If we assume that the couplings $t_k, \bar{t}_k$ vanish for $k > k_{max}$, as in the previous comment, then the perturbations are also polynomial. We will return to this case in a subsequent section.

2.3 The Kontsevich-Penner or W_∞ model

The Kontsevich-Penner or W_∞ model[26] (for a more detailed review, see Ref.[7]) is a model of a single positive-definite hermitian matrix, whose partition function is given by:

$$\mathcal{Z}_{KP}(A, \bar{t}) = (\det A)^\nu \int [dM]\, e^{\mathrm{tr}\left(-\nu MA + (\nu - N)\log M - \nu \sum_{k=1}^{\infty} \bar{t}_k M^k\right)} \tag{2.18}$$

where $\bar{t}_k$ are the couplings to negative-momentum tachyons, N is the dimensionality of the matrix M and A is a constant matrix. The eigenvalues of this matrix determine the couplings t_k to positive-momentum tachyons via the Kontsevich-Miwa (KM) transform:

$$t_k = -\frac{1}{\nu k}\mathrm{tr}(A^{-k}) \tag{2.19}$$

This model is derived by integrating the W_∞ equations found in Ref.[69]. The parameter ν appearing in the action above is related to the cosmological constant μ of the string theory by $\nu = -i\mu$. The model can also be obtained from the Penner matrix model[70, 71] after making a suitable change of variables (as explained in detail in Ref.[7]) and adding perturbations.

We now note some properties that are analogous to those of the NMM, as well as others that are quite different.

(i) By redefining $MA \to M$ we can rewrite the partition function without any factor in front, as:

$$\mathcal{Z}_{KP}(A, \bar{t}) = \int [dM]\, e^{\mathrm{tr}\left(-\nu M + (\nu - N)\log M - \nu \sum_{k=1}^{\infty} \bar{t}_k (MA^{-1})^k\right)} \tag{2.20}$$

(ii) This model has no radius deformation, and describes the $c = 1$ string theory directly at selfdual radius $R = 1$.

(iii) In view of the logarithmic term, the model is well-defined only if the integral over the eigenvalues m_i of the matrix M is restricted to the region $m_i > 0$.

(iv) The model can be reduced to eigenvalues, leading to the partition function:

$$\mathcal{Z}_{KP}(A, \bar{t}) = \left(\prod_{i=1}^{N} a_i\right)^{\nu} \int \prod_{i=1}^{N} dm_i\, \frac{\Delta(m)}{\Delta(a)}\, e^{\sum_{i=1}^{N}\left(-\nu m_i a_i + (\nu - N)\log m_i - \nu \sum_{k=1}^{\infty} \bar{t}_k m_i^k\right)} \tag{2.21}$$

(vi) In the representation Eq. (2.18), the operators $\mathrm{tr} M^k$ describe the negative-momentum tachyons. But there are no simple operators that directly correspond to positive-momentum tachyons. Nevertheless this model generates tachyon correlators of the $c = 1$ string as follows:

$$\langle T_{k_1} T_{k_2} \cdots T_{k_m} T_{-\ell_1} T_{-\ell_2} \cdots T_{-\ell_n} \rangle = \frac{\partial}{\partial t_{k_1}} \frac{\partial}{\partial t_{k_2}} \cdots \frac{\partial}{\partial t_{k_m}} \frac{\partial}{\partial \bar{t}_{\ell_1}} \frac{\partial}{\partial \bar{t}_{\ell_2}} \cdots \frac{\partial}{\partial \bar{t}_{\ell_n}} \log \mathcal{Z}_{KP} \tag{2.22}$$

where derivatives in t_k are computed using Eq. (2.19) and the chain rule.

(v) The symmetry of the partition function under the interchange of $t_k, \bar{t}_k$ is not manifest, since one set of parameters is encoded through the matrix A while the other appears explicitly.

(vi) The transformation

$$A \to \alpha A, \qquad \bar{t}_k \to \alpha^k\, \bar{t}_k \tag{2.23}$$

for arbitrary α, is a symmetry of the model (most obvious in the representation Eq. (2.20)). As a consequence, the tachyon correlators satisfy momentum conservation.

(vi) The partition function satisfies the "puncture equation":

$$\mathcal{Z}_{KP}(A-\epsilon, \bar{t}_k + \delta_{k,1}\,\epsilon) = e^{\epsilon\nu^2 t_1}\mathcal{Z}_{KP}(A, \bar{t}_k) \tag{2.24}$$

as can immediately be seen from Eq. (2.18).

2.4 Equivalence of the matrix models

2.4.1 $N = 1$ case

We start by choosing the selfdual radius $R = 1$, and will later comment on what happens at other values of R. As we have seen, in the perturbed NMM there are two (infinite) sets of parameters $t_k, \bar{t}_k$, all of which can be chosen independently. This is the case even at finite N, though the model describes $c = 1$ string theory only at infinite N (or at the special value $N = \nu$, as noted in Ref.[25], a point to which we will return later). In contrast, the Kontsevich-Penner model has one infinite set of parameters $\bar{t}_k$, as well as N additional parameters from the eigenvalues of the matrix A. The latter encode the t_k, as seen from Eq. (2.19) above. From this it is clear that at finite N, there can only be N independent parameters t_k ($k = 1, 2, \ldots N$) while the remaining ones ($t_k, k > N$) are dependent on these.

This makes the possible equivalence of the two models somewhat subtle. To understand the situation better, let us compare both models in the limit that is farthest away from $N \to \infty$, namely $N = 1$. While this is a "toy" example, we will see that it provides some useful lessons.

In this case the NMM partition function is:

$$\mathcal{Z}_{NMM,N=1}(t_k, \bar{t}_k) = \int dz\, d\bar{z}\; e^{-\nu z\bar{z} + (\nu-1)\log z\bar{z} - \nu\sum_{k=1}^{\infty}(t_k z^k + \bar{t}_k \bar{z}^k)} \tag{2.25}$$

while the Kontsevich-Penner partition function is:

$$\mathcal{Z}_{KP,N=1}(a, \bar{t}_k) = a^{\nu}\int dm\; e^{-\nu m a + (\nu-1)\log m - \nu\sum_{k=1}^{\infty}\bar{t}_k m^k} \tag{2.26}$$

We will now show that the two integrals above are equivalent if we assume that t_k in the NMM is given by:

$$t_k = -\frac{1}{\nu k}\, a^{-k} \tag{2.27}$$

which is the KM transform Eq. (2.19) in the special case where A is a 1×1 matrix, denoted by the single real number a. Note that this determines all the infinitely many t_k in terms of a.

To obtain the equivalence, insert the above relation and also perform the change of integration variable:

$$z = \sqrt{m}\, e^{i\theta} \tag{2.28}$$

in the NMM integral. Then we find that (up to a numerical constant):

$$\begin{aligned} \mathcal{Z}_{NMM,N=1}(a, \bar{t}_k) &= \int dm\, d\theta\ e^{-\nu m + (\nu-1)\log m + \sum_{k=1}^{\infty} \frac{1}{k} (\frac{\sqrt{m}}{a})^k e^{ik\theta} - \nu \sum_{k=1}^{\infty} \bar{t}_k (\sqrt{m})^k e^{-ik\theta}} \\ &= \int dm\, d\theta\ \frac{1}{1 - \frac{\sqrt{m}\, e^{i\theta}}{a}}\ e^{-\nu m + (\nu-1)\log m - \nu \sum_{k=1}^{\infty} \bar{t}_k (\sqrt{m})^k e^{-ik\theta}} \end{aligned} \tag{2.29}$$

Strictly speaking the last step is only valid for $\sqrt{m}/a < 1$, since otherwise the infinite sum fails to converge. Hence we fix m and a to satisfy this requirement and continue by evaluating the θ-integral. This can be evaluated by defining $e^{-i\theta} = w$ and treating it as a contour integral in w. We have

$$d\theta\, \frac{1}{1 - \frac{\sqrt{m}\, e^{i\theta}}{a}} \to dw\, \frac{1}{w - \frac{\sqrt{m}}{a}} \tag{2.30}$$

Since the rest of the integrand is well-defined and analytic near $w = 0$, we capture the simple pole at $w = \sqrt{m}/a$. That brings the integrand to the desired form. Now we can lift the restriction $\sqrt{m}/a < 1$, and treat the result as valid for all m by analytic continuation. Therefore we find:

$$\begin{aligned} \mathcal{Z}_{NMM,N=1}(a, \bar{t}_k) &= \int dm\ e^{-\nu m + (\nu-1)\log m - \nu \sum_{k=1}^{\infty} \bar{t}_k (m a^{-1})^k} \\ &= a^{\nu} \int dm\ e^{-\nu m a + (\nu-1)\log m - \nu \sum_{k=1}^{\infty} \bar{t}_k m^k} \\ &= \mathcal{Z}_{KP,N=1}(a, \bar{t}_k) \end{aligned} \tag{2.31}$$

Thus we have shown that the perturbed 1×1 Normal Matrix Model at $R = 1$ is equivalent to the 1×1 Kontsevich-Penner model. However, this equivalence only holds when we perform the 1×1 KM transform, which fixes all the perturbations t_k in terms of a single independent parameter a (while the $\bar{t}_k$ are left arbitrary).

An important point to note here is the sign chosen in Eq. (2.27). Changing the sign (independently of k) amounts to the transformation $t_k \to -t_k$. This is apparently harmless, leading to some sign changes in the correlation functions, but there is no way at $N = 1$ (or more generally at any finite N) to change a (or the corresponding matrix A) to compensate for this transformation. The sign we have chosen, given the signs in the original NMM action, is therefore the only one that gives the KP model. This point will become important later on.

Returning to the NMM-KP equivalence at $N = 1$, it is interesting to generalise it by starting with the NMM at an arbitrary radius R instead of $R = 1$ as was the case above. As seen from Eq. (2.2), the coupling of the log term is modified in this case as:

$$(\nu - 1) \to \frac{1}{2}(R-1) + (R\nu - 1) \tag{2.32}$$

and also the bilinear term $z\bar{z}$ is modified to $(z\bar{z})^R$. The above derivation goes through with only minor changes, and we end up with:

$$\mathcal{Z}_{NMM,N=1}(a, \bar{t}_k) = a^{\frac{1}{2}(R-1)+\nu} \int dm\ e^{-\nu(ma)^R + \left[\frac{1}{2}(R-1)+(R\nu-1)\right]\log m - \nu\sum_{k=1}^{\infty}\bar{t}_k m^k} \tag{2.33}$$

This appears to suggest that there is a variant of the Kontsevich-Penner model valid at arbitrary radius (or at least arbitrary integer radius, since otherwise it may become hard to define the integral). This would be somewhat surprising as such a model has not been found in the past. As we will see in the following subsection, the above result holds only for the $N = 1$ case. Once we go to $N \times N$ matrices, we will see that NMM leads to a KP matrix model only at $R = 1$, consistent with expectations.

Another generalisation of the above equivalence seems more interesting. In principle, even for the 1×1 matrix model, we can carry out a KM transform using an $n \times n$ matrix A where n is an arbitrary integer. Indeed, there is no logical reason why the dimension of the constant matrix A must be the same as that of the random matrices occurring in the integral. The most general example of this is to take $N \times N$ random matrices $Z, Z^\dagger$ in the NMM and then carry out a KM transform with A being an $n \times n$ matrix. The "usual" transform then emerges as the special case $n = N$. Of course all this makes sense only within the NMM and not in the KP model. If $n \neq N$ then the KP model, which has a $\mathrm{tr}MA$ term in its action, cannot even be defined. So we should not expect to find the KP model starting with the NMM unless $n = N$, but it is still interesting to see what we will find.

Here we will see what happens if we take $N = 1$ and $n > 1$. The full story will appear in a later subsection. Clearly the KM transform Eq. (2.19) permits more independent parameters t_k as n gets larger. Let us take the eigenvalues of A to be $a_1, a_2, \ldots, a_n$. Then it is easy to see that:

$$\begin{aligned} \mathcal{Z}_{NMM,N=1}(a_i, \bar{t}_k) &= \int dm\, d\theta\; e^{-\nu m+(\nu-1)\log m+\sum_{i=1}^{n}\sum_{k=1}^{\infty}\frac{1}{k}(\frac{\sqrt{m}}{a_i})^k e^{ik\theta}-\nu\sum_{k=1}^{\infty}\bar{t}_k(\sqrt{m})^k e^{-ik\theta}} \\ &= \int dm\, d\theta\; \frac{1}{\prod_{i=1}^{n}(1-\frac{\sqrt{m}\,e^{i\theta}}{a_i})}\; e^{-\nu m+(\nu-1)\log m-\nu\sum_{k=1}^{\infty}\bar{t}_k(\sqrt{m})^k e^{-ik\theta}} \end{aligned} \tag{2.34}$$

Converting to the w variable as before, we now encounter n poles. Picking up the residues, we get:

$$\mathcal{Z}_{NMM,N=1}(a_i, \bar{t}_k) = \int dm\; e^{-\nu m+(\nu-1)\log m}\sum_{l=1}^{n}\left(\frac{1}{\prod_{i\neq l}\left(1-\frac{a_l}{a_i}\right)}e^{-\nu\sum_{k=1}^{\infty}\bar{t}_k(\frac{m}{a_l})^k}\right) \tag{2.35}$$

This in turn can be expressed as a sum over n 1×1 Kontsevich-Penner models:

$$\mathcal{Z}_{NMM,N=1}(a_i, \bar{t}_k) = \sum_{l=1}^{n}\frac{1}{\prod_{i\neq l}\left(1-\frac{a_l}{a_i}\right)}\mathcal{Z}_{KP,N=1}(a_l, \bar{t}_k) \tag{2.36}$$

Note that if in this expression we take $a_n \to \infty$, one of the terms in the above equation (corresponding to $l = n$) decouples, and a_n also drops out from the remaining terms. Therefore we recover the same equation with $n \to n-1$. In this way we can successively decouple all but one of the a_i's.

To summarise, at the level of the 1×1 NMM, we have learned some interesting things: this model is equivalent to the 1×1 KP model if we specialise the parameters t_k to a 1-parameter family via the KM transform, while it is equivalent to a sum over n different 1×1 KP models if we specialise the parameters t_k to an n-parameter family. We also saw a 1×1 KP model arise when we are at a finite radius $R \neq 1$. In the next section we will see to what extent these lessons hold once we work with $N \times N$ random matrices.

2.4.2 General case

In this section we return to the $N \times N$ Normal Matrix Model. With the substitution Eq. (2.19) (where A is also an $N \times N$ matrix), its partition function becomes:

$$\mathcal{Z}_{NMM} = \int [dZ\, dZ^\dagger]\, e^{\mathrm{tr}\left(-\nu Z Z^\dagger + (\nu - N)\log Z Z^\dagger + \sum_{k=1}^\infty \frac{1}{k}\mathrm{tr}(A^{-k}) Z^k - \nu \sum_{k=1}^\infty \bar{t}_k Z^{\dagger k}\right)} \tag{2.37}$$

or, in terms of eigenvalues:

$$\begin{aligned} \mathcal{Z}_{NMM} &= \int \prod_{i=1}^N d^2 z_i\ \Delta(z)\Delta(\bar{z})\, e^{-\nu \sum_{i=1}^N z_i \bar{z}_i + (\nu - N)\sum_{i=1}^N \log z_i \bar{z}_i} \\ &\quad \times e^{\sum_{i,j=1}^N \sum_{k=1}^\infty \frac{1}{k}\left(\frac{z_i}{a_j}\right)^k} e^{-\nu \sum_{i=1}^N \sum_{k=1}^\infty \bar{t}_k \bar{z}_i^k} \end{aligned} \tag{2.38}$$

where $\Delta(z)$ is the Vandermonde determinant. Because of the normality constraint $[Z, Z^\dagger] = 0$ there is only one Vandermonde for z_i and one for $\bar{z}_i$.

The sum over k in the second line of Eq. (2.38) converges if $\frac{z_i}{a_j} < 1$ for all i, j, in which case it can be evaluated immediately giving:

$$\mathcal{Z}_{NMM} = \int \prod_{i=1}^N d^2 z_i\ |\Delta(z)|^2 \prod_{i,j=1}^N \frac{1}{1 - \frac{z_i}{a_j}}\ e^{\sum_{i=1}^N \left[-\nu z_i \bar{z}_i + (\nu - N)\log z_i \bar{z}_i - \nu \sum_{k=1}^\infty \bar{t}_k \bar{z}_i^k\right]} \tag{2.39}$$

To make contact with the Penner model, first change variables $z_i \to \sqrt{m_i}\, e^{i\theta_i}$ and then replace $e^{-i\theta_i}$ by w_i as before. Then we get $d^2 z_i \to dm_i \frac{dw_i}{w_i}$ and:

$$\begin{aligned} \mathcal{Z}_{NMM} &= \int \prod_{i=1}^N dm_i \oint \prod_{i=1}^N \frac{dw_i}{w_i} \prod_{i<j}^N \left(\frac{\sqrt{m_i}}{w_i} - \frac{\sqrt{m_j}}{w_j}\right) \left(\sqrt{m_i} w_i - \sqrt{m_j} w_j\right) \\ &\quad \times \prod_{i,j=1}^N \frac{1}{1 - \frac{\sqrt{m_i}}{w_i a_j}}\ e^{\sum_{i=1}^N \left[-\nu m_i + (\nu - N)\log m_i - \nu \sum_{k=1}^\infty \bar{t}_k (\sqrt{m_i} w_i)^k\right]} \end{aligned} \tag{2.40}$$

The contour integrals can be evaluated once this is rewritten in the more convenient form:

$$\mathcal{Z}_{NMM} = \int \prod_{i=1}^{N} dm_i \oint \prod_{i=1}^{N} dw_i \prod_{i<j}^{N} \left(\sqrt{m_i}w_j - \sqrt{m_j}w_i\right)\left(\sqrt{m_i}w_i - \sqrt{m_j}w_j\right)$$
$$\times \prod_{i,j=1}^{N} \frac{1}{w_i - \frac{\sqrt{m_i}}{a_j}} \; e^{\sum_{i=1}^{N}\left[-\nu m_i + (\nu - N)\log m_i - \nu \sum_{k=1}^{\infty} \bar{t}_k (\sqrt{m_i} w_i)^k\right]} \tag{2.41}$$

Next we pick up the residues at the poles. During the intermediate steps, we will assume that the eigenvalues a_i of the matrix A are non-degenerate. From the above expression, each integration variable w_i has a pole at each of the values:

$$w_i = \frac{\sqrt{m_i}}{a_j} \tag{2.42}$$

for all j. Thus the contributions can be classified by the set of poles:

$$(w_1, w_2, \ldots, w_N) = \left(\frac{\sqrt{m_1}}{a_{j_1}}, \frac{\sqrt{m_2}}{a_{j_2}}, \ldots, \frac{\sqrt{m_N}}{a_{j_N}}\right) \tag{2.43}$$

We now notice that the set $(j_1, j_2, \ldots, j_N)$ must consist of distinct elements, in other words it forms a permutation of $(1, 2, \ldots, N)$. This is because if two values of j_i coincide, one of the Vandermonde factors of the type $(\sqrt{m_i}w_j - \sqrt{m_j}w_i)$ vanishes and there is no contribution.

We start by considering the simplest permutation, the identity, namely:

$$(j_i, j_2, \ldots, j_N) = (1, 2, \ldots, N) \tag{2.44}$$

In this case the residues from the denominator and Vandermonde factors become:

$$\prod_{i<j}^{N} \left(\frac{\sqrt{m_i m_j}}{a_j} - \frac{\sqrt{m_i m_j}}{a_i}\right)\left(\frac{m_i}{a_i} - \frac{m_j}{a_j}\right) \prod_{j\neq i}^{N} \frac{1}{\frac{\sqrt{m_i}}{a_i} - \frac{\sqrt{m_i}}{a_j}} = \frac{\prod_{i<j}^{N}(m_i a_j - m_j a_i)}{\Delta(a)} \tag{2.45}$$

while the exponential measure factor becomes:

$$e^{\sum_{i=1}^{N}\left[-\nu m_i + (\nu - N)\log m_i - \nu \sum_{k=1}^{\infty} \bar{t}_k \left(\frac{m_i}{a_i}\right)^k\right]} \tag{2.46}$$

It is easy to check that for all the other possible permutations of $(j_1, j_2, \ldots, j_N)$ besides the identity permutation, a corresponding permutation of the integration variables m_i brings the above answer (exponential measure as well as prefactors) back to the same form as for the identity permutation. This means that (dropping a factor of $\frac{1}{N!}$) we have proved:

$$\begin{aligned}
\mathcal{Z}_{NMM} &= \int \prod_{i=1}^{N} dm_i \; \frac{\prod_{i<j}^{N}(m_i a_j - m_j a_i)}{\Delta(a)} e^{\sum_{i=1}^{N}\left[-\nu m_i + (\nu - N)\log m_i - \nu \sum_{k=1}^{\infty} \bar{t}_k \left(\frac{m_i}{a_i}\right)^k\right]} \\
&= \Big(\prod_{i=1}^{N} a_i\Big)^{\nu} \int \prod_{i=1}^{N} dm_i \; \frac{\Delta(m)}{\Delta(a)} \, e^{\sum_{i=1}^{N}\left[-\nu m_i a_i + (\nu - N)\log m_i - \nu \sum_{k=1}^{\infty} \bar{t}_k m_i^k\right]} \qquad (2.47)
\end{aligned}$$

where in the last step we have replaced $m_i \to m_i a_i$.

This is precisely the eigenvalue representation Eq. (2.21) of the KP matrix model Eq. (2.18). Thus we have provided a direct proof of equivalence of the perturbed Normal Matrix Model and the Kontsevich-Penner model. Notice that in performing the KM transform we reduced the independent t_k of the NMM to a finite number, namely N, so that eventually the $N \to \infty$ limit is required in order to encode all the independent parameters.

In the previous subsection we considered taking different ranks for the constant matrix A arising in the KM transform and the random matrix Z. The most general case is to take Z to be $N \times N$ and A to be $n \times n$. The computation is a simple extension of the one done above. We find the following results. When $n > N$ we again get a sum over Kontsevich-Penner models. The number of terms in the sum is the binomial coefficient ${}^nC_N$. This is a generalisation of the result given in Eq. (2.36) for $N = 1$, where we found n terms. In the general case let us denote by $a_{\{i,l\}}$ the i^{th} element of the set formed by one possible choice of N a_i's from a total of n, the index l labeling the particular choice. The complementary set, formed by the rest of the a_i's is denoted by $a_{\{\tilde{i},l\}}$, the index $\tilde{i}$ taking $n - N$ values. We then have:

$$\mathcal{Z}_{NMM}(a_i, \bar{t}_k) = \sum_{l=1}^{{}^nC_N} \prod_{i=1}^{N} \prod_{\tilde{i}=1}^{N-n} \frac{1}{\left(1 - \frac{a_{\{i,l\}}}{a_{\{\tilde{i},l\}}}\right)} \mathcal{Z}_{KP}(a_{\{l\}}, \bar{t}_k) \qquad (2.48)$$

so that the NMM is again expressed as a sum over KP models.

The other case, $n < N$, can be obtained by starting with $n = N$ and successively decoupling $N - n$ eigenvalues a_i by taking them to infinity. This is similar to what we observed in the $N = 1$ case following Eq. (2.36). In the present case one can easily show that $N - n$ matrix eigenvalues m_i also decouple in this limit (apart from a normalisation). In fact, it is straightforward to derive the formula:

$$\lim_{a_N \to \infty} Z_{KP}^{(N,\nu)}(A^{(N)}, \bar{t}_k) = \frac{\Gamma(\nu - N + 1)}{\nu^{\nu - N + 1}} Z_{KP}^{(N-1,\nu)}(A^{(N-1)}, \bar{t}_k) \tag{2.49}$$

which can then be iterated. Thus after $N-n$ eigenvalues a_i are decoupled, we find up to normalisation the KP model of rank n. As we remarked in the introduction, this exhibits a strong analogy to the insertion of n determinant operators in the Gaussian model, as described in Ref.[58], where the result is the $n \times n$ Kontsevich model.

2.4.3 Radius dependence

Finally, we can ask what happens to the radius-dependent NMM under the above procedure. Again the steps are quite straightforward and one arrives at the following generalisation of Eq. (2.47):

$$\begin{aligned} \mathcal{Z}_{NMM,R} &= \Big(\prod_{i=1}^{N} a_i\Big)^{\frac{1}{2}(R-1)+\nu} \int \prod_{i=1}^{N} dm_i \, \frac{\Delta(m)}{\Delta(a)} \\ &\quad \times e^{\sum_{i=1}^{N}\left[-\nu(m_i a_i)^R + \left[\frac{1}{2}(R-1)+(\nu R - N)\right]\log m_i - \nu \sum_{k=1}^{\infty} \bar{t}_k m_i^k\right]} \end{aligned} \tag{2.50}$$

The problem is that the above eigenvalue model cannot (as far as we can see) be converted back to a matrix model. The key to doing so in the $R = 1$ case was the linear term $\sum_i m_i a_i$ in the action, which (after absorbing the Vandermondes and using the inverse of the famous Harish Chandra formula) can be summed back into $\mathrm{tr} MA$. The quantity $\sum_i (m_i a_i)^R$ cannot be converted back into a matrix trace unless $R = 1$.

This clarifies a longstanding puzzle: while a KP model could only be found at $R = 1$, the NMM exists and describes the $c = 1$ string for any R. We see now that the correct extension of the KP model to $R \neq 1$ is the eigenvalue model given by Eq. (2.50) above, but unfortunately this does not correspond to a matrix model.

2.5 Loop operators in the NMM

In this section we will examine loop operators in the NMM. Our goal here is to understand whether correlation functions of these operators can be related to the Kontsevich-Penner model of Ref.[26], thereby providing the $c = 1$ analogue of the corresponding observations in Refs.[58, 63]. Though there are some similarities, we will also find some striking differences between this and the $c < 1$ case.

Macroscopic loops in a model of random matrices Φ are described by insertions of the operator:

$$W(x) = \operatorname{tr}\log(x - \Phi) \tag{2.51}$$

which creates a boundary in the world sheet. Here x is the boundary cosmological constant. The corresponding generating function for multiple boundaries is[58, 72, 73, 74, 75]:

$$e^{W(x)} = \det(x - \Phi) \tag{2.52}$$

Such operators have been studied extensively in $c < 1$ matrix models, describing (p, q) minimal models coupled to 2d gravity.

We will consider expectation values of operators of the form $\det(a - Z)$ in the NMM, where a is a real parameter. These operators create a hole in the dual graph in the Feynman diagram expansion of the matrix model. Since the NMM has vertices that are holomorphic/antiholomorphic in Z, the dual graph will have faces that are dual to Z or $Z^\dagger$. The loop operator $\det(a - Z)$ creates a hole in a Z-face, while its complex conjugate creates a hole in a $Z^\dagger$-face.

As we would expect, this means that the correlators are complex, but we have the identity[1]:

$$\left\langle \prod_i \det(a_i - Z) \right\rangle_{t_k, \bar{t}_k} = \left\langle \prod_i \det(a_i - Z^\dagger) \right\rangle_{\bar{t}_k, t_k} \tag{2.53}$$

[1] Here and in the rest of this section, all correlators are understood to be normalised correlators in the NMM.

where on the RHS the role of the deformations $t_k, \bar{t}_k$ has been interchanged. Therefore as long as we consider correlators only of $\det(a_i - Z)$ or $\det(a_i - Z^\dagger)$ the result is effectively the same. As we will see in a moment, a stronger statement is true: on the subspace of parameter space dictated by the KM transform, the unmixed correlators are individually real. Later we will also consider mixed correlators.

As a start, notice that in the 1×1 case,

$$\begin{aligned}
\mathcal{Z}_{NMM,N=1}(t_k = 0, \bar{t}_k) &= \int d^2z \; e^{-\nu z\bar{z} + (\nu-1)\log z\bar{z} - \nu \sum_{k=1}^{\infty} \bar{t}_k \bar{z}^k} \\
&= \int d^2z \; (a-z)\frac{1}{(a-z)} \, e^{-\nu z\bar{z} + (\nu-1)\log z\bar{z} - \nu \sum_{k=1}^{\infty} \bar{t}_k \bar{z}^k} \\
&= \frac{1}{a}\int d^2z \; (a-z) \, e^{-\nu z\bar{z} + (\nu-1)\log z\bar{z} - \nu \sum_{k=1}^{\infty} (t_k^0 z^k + \bar{t}_k \bar{z}^k)} \\
&= \frac{1}{a}\Big\langle (a-z) \Big\rangle_{t_k^0, \bar{t}_k} \; Z_{NMM,N=1}(t_k^0, \bar{t}_k)
\end{aligned} \tag{2.54}$$

where the expectation value in the last line is evaluated in the NMM with

$$t_k^0 = -\frac{1}{\nu k}\, a^{-k} \tag{2.55}$$

We see that the t_k^0 dependence drops out in the RHS because insertion of the loop operator cancels the dependence in the partition function. In fact, more is true: even the $\bar{t}_k$ dependence cancels out between the different factors on the RHS. This is a consequence of the property exhibited in Eq. (2.17).

A more general statement in the 1×1 case is:

$$\mathcal{Z}_{NMM,N=1}(t_k - t_k^0, \bar{t}_k) = \frac{1}{a}\Big\langle (a-z) \Big\rangle_{t_k, \bar{t}_k} \; Z_{NMM,N=1}(t_k, \bar{t}_k)$$

In other words, insertion of the macroscopic loop operator has the effect of decreasing the value of t_k, leaving $\bar{t}_k$ unchanged.

In the more general case of $N \times N$ random matrices, the corresponding result is as follows. The expectation value of a single exponentiated loop operator $\det(a - Z)$ is:

$$\Big\langle \det(a - Z) \Big\rangle_{t_k, \bar{t}_k} = \frac{\mathcal{Z}_{NMM}(t_k - t_k^0, \bar{t}_k)}{\mathcal{Z}_{NMM}(t_k, \bar{t}_k)}\, a^N \tag{2.56}$$

with t_k^0 again given by Eq. (2.55). Now we would like to consider multiple loop operators. Therefore consider the expectation value:

$$\left\langle \prod_{i=1}^{n} \det(a_i - Z) \right\rangle_{t_k,\bar{t}_k} \tag{2.57}$$

As noted in Ref.[58], this can be thought of as a single determinant in a larger space. Define the $n \times n$ matrix $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$ and extend it to an $nN \times nN$ matrix $A \otimes \mathbb{1}_{N\times N}$. Similarly, extend the $N \times N$ matrix Z to an $nN \times nN$ matrix $\mathbb{1}_{n\times n} \otimes Z$. Now we can write

$$\prod_{i=1}^{n} \det(a_i - Z) = \det(A \otimes \mathbb{1} - \mathbb{1} \otimes Z) = \prod_{i=1}^{n}\prod_{j=1}^{N}(a_i - z_j) \tag{2.58}$$

Rewriting this as:

$$\prod_{i=1}^{n} \det(a_i - Z) = (\det A)^N \det(\mathbb{1} \otimes \mathbb{1} - A^{-1} \otimes Z) \tag{2.59}$$

and expanding the second factor, we find:

$$\left\langle \prod_{i=1}^{n} \det(a_i - Z) \right\rangle_{t_k,\bar{t}_k} = \frac{\mathcal{Z}_{NMM}(t_k - t_k^0, \bar{t}_k)}{\mathcal{Z}_{NMM}(t_k, \bar{t}_k)} (\det A)^N \tag{2.60}$$

where now:

$$t_k^0 = -\frac{1}{\nu k}\,\mathrm{tr}A^{-k} \tag{2.61}$$

Thus we see that macroscopic loop correlators in this model are obtained by simply shifting the parameters t_k in the partition function, the shift being given by the KM transform.

The above considerations can be extended to mixed correlators as follows. Consider correlation functions of the form:

$$\left\langle \prod_{i=1}^{n} \det(a_i - Z) \prod_{j=1}^{m} \det(b_j - Z^\dagger) \right\rangle \tag{2.62}$$

Then, defining the $m \times m$ matrix $B = \mathrm{diag}(b_1, b_2, \ldots, b_m)$, the parameters t_k^0 as in Eq. (2.61), and the parameters $\bar{t}_k^0$ by:

$$\bar{t}_k^0 = -\frac{1}{\nu k}\,\mathrm{tr}B^{-k} \tag{2.63}$$

we find

$$\left\langle \prod_{i=1}^{n} \det(a_i - Z) \prod_{j=1}^{m} \det(b_j - Z^\dagger) \right\rangle_{t_k,\bar{t}_k} = \frac{\mathcal{Z}_{NMM}(t_k - t_k^0, \bar{t}_k - \bar{t}_k^0)}{\mathcal{Z}_{NMM}(t_k, \bar{t}_k)} (\det A \det B)^N \tag{2.64}$$

In the above we have seen how to re-express correlations of loop operators in terms of shifted closed-string parameters. This in itself is quite reminiscent of an open-closed duality. However we did not yet encounter the KP model. To do so, we note that besides the exponentiated loop operator $\det(a - Z)$, we can consider its inverse: $1/\det(a - Z)$. Just as insertion of $\det(a - Z)$ has the effect of decreasing each t_k by t_k^0 given by Eq. (2.55), insertion of the inverse operator *increases* t_k by the same amount.

Thus we may consider correlators like:

$$\left\langle \prod_{i=1}^{n} \frac{1}{\det(a_i - Z)} \right\rangle = \frac{1}{(\det A)^N} \left\langle \frac{1}{\det(\mathbb{1} \otimes \mathbb{1} - A^{-1} \otimes Z)} \right\rangle \tag{2.65}$$

As before, the two factors of the direct product in the above equation refer to $n \times n$ and $N \times N$ matrices. It is easy to see that the correlation function on the RHS has the effect of increasing the t_k by t_k^0 as given in Eq. (2.61).

Although in principle n and N are independent, here we will consider the case $n = N$. Now the inverse operator

$$\left\langle \frac{1}{\det(\mathbb{1} \otimes \mathbb{1} - A^{-1} \otimes Z)} \right\rangle \tag{2.66}$$

has already made an appearance in § 2.4, where one finds it in the eigenvalue basis (see for example Eq. (2.39)):

$$\prod_{i,j=1}^{N} \frac{1}{1 - \frac{z_i}{a_j}} \tag{2.67}$$

The interesting property of the inverse determinant operators is that they can be used to create the KP model starting from the *partially unperturbed* NMM (where $t_k = 0$ but $\bar{t}_k$ are arbitrary). Computationally this is similar to the derivation in § 2.4 of the KP model from the perturbed NMM. Thus we have:

$$\left\langle \frac{1}{\det(\mathbb{1} \otimes \mathbb{1} - A^{-1} \otimes Z)} \right\rangle_{0,\bar{t}_k} \mathcal{Z}_{NMM}(0, \bar{t}_k) = Z_{KP}(A, \bar{t}_k) \tag{2.68}$$

Here $\mathcal{Z}_{NMM}(0, \bar{t}_k)$ can be replaced by $\mathcal{Z}_{NMM}(0, 0)$ as we have noted previously. This equation then is the precise statement of one of our main observations, that inverse determinant expectation values in the (partially unperturbed) NMM give rise to the KP partition function.

It is clearly desirable to have a target space interpretation for these loop operators. Since the NMM is derived from correlators computed from matrix quantum mechanics, in principle one should be able to understand the loop operators of NMM starting from loop operators (or some other operators) in MQM. While that is beyond the scope of the present work, we will instead exhibit some suggestive properties of our loop operators and leave their precise interpretation for future work.

In matrix models for the $c < 1$ string, which are described by constant random matrices, exponentiated loop operators are determinants just like the ones discussed here for the NMM. In those models it has been argued that the loop operators represent FZZT branes. One striking observation is that in the Kontsevich/generalised Kontsevich description of $c < 1$ strings, the eigenvalues of the constant matrix A come from the boundary cosmological constants appearing in the loop operators. Moreover, Eq. (2.52) has been interpreted as evidence that the FZZT-ZZ open strings there are fermionic[58, 75].

In the present case, we see that the parameters a_i in the loop operators turn precisely into the eigenvalues of the constant matrix A of the Kontsevich-Penner model. We take this as evidence that our loop operators are likewise related in some way to FZZT branes. Indeed, one is tempted to call them FZZT branes of the NMM. Pursuing this analogy further, the role played by inverse determinants in the present discussion appears to suggest that the corresponding strings in the NMM are *bosonic* rather than fermionic. But the relationship of these operators to the "true" FZZT branes of matrix quantum mechanics remains to be understood, as we have noted above[1].

[1] In light of the discussions about the MQM Fermi surface in § 2.2 we can give an interpretation to both determinant and inverse determinant operators in the NMM. Since their insertions lead to opposite shifts in the t_k's, by virtue of the equivalence between MQM and NMM discussed above we can map each one directly to a corresponding deformation to the Fermi surface,

In the limit of infinite N, the inverse loop operators depending on a matrix A can be thought of as loop operators for a different matrix $\tilde{A}$. Thus, only in this limit, the inverse determinant operators can be replaced by more conventional determinants. This proceeds as follows. We have already seen that the KM transformation Eq. (2.19) encodes infinitely many parameters t_k via a constant $N \times N$ matrix A, in the limit $N \to \infty$. Now for fixed t_k, suppose we considered the (very similar) transform:

$$t_k = \frac{1}{\nu k}\mathrm{tr}\tilde{A}^{-k} \tag{2.69}$$

that differs only by a change of sign. The point is that this apparently harmless reversal of the t_k brings about a significant change in the matrix A. Moreover this is possible only in the infinite N limit, since we are trying to satisfy:

$$\mathrm{tr}A^{-k} = -\mathrm{tr}\tilde{A}^{-k} \tag{2.70}$$

for all k. Now it is easy to see that the matrices A and $\tilde{A}$ satisfy the following identity:

$$\det(\mathbb{1} \otimes \mathbb{1} - A^{-1} \otimes Z) = \frac{1}{\det(\mathbb{1} \otimes \mathbb{1} - \tilde{A}^{-1} \otimes Z)} \tag{2.71}$$

Therefore a correlator of inverse loop operators can be rewritten in terms of usual loop operators using:

$$\left\langle \frac{1}{\det(A \otimes \mathbb{1} - \mathbb{1} \otimes Z)} \right\rangle = \frac{1}{(\det A\tilde{A})^N} \left\langle \det(\tilde{A} \otimes \mathbb{1} - \mathbb{1} \otimes Z) \right\rangle \tag{2.72}$$

In the light of our previous observation that inverse determinant operators might indicate the bosonic nature of FZZT-ZZ strings at $c = 1$, it is tempting to think of Eq. (2.72) as a statement of fermi-bose equivalence!

In terms of the operator $\det(\tilde{A} \otimes \mathbb{1} - \mathbb{1} \otimes Z)$, we can make the statement that its insertion into the partially unperturbed NMM gives rise to the KP model depending on the "dual" Kontsevich matrix A.

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which can be read off from Eq. (2.11). This fact should facilitate direct comparison with the MQM.

2.6 Normal matrix model at finite N

The correspondence between NMM and KP model demonstrated in § 2.4 is valid for any N, as long as the parameters of the former are restricted to a subspace. The NMM itself is supposed to work at $N \to \infty$, in which case this restriction goes away. However, as noted in Ref.[25], there is another way to implement the NMM: by setting $N = \nu R$ (which amounts to $N = \nu$ for $R = 1$), which they labelled as "Model II". In other words, these authors argue that:

$$\lim_{N\to\infty} Z_{NMM}(N, t, \nu) = Z_{NMM}(N = \nu R, t, \nu) \tag{2.73}$$

Thus the NMM describes the $c = 1$ theory at this finite value of N, after analytically continuing the cosmological constant $\mu = i\nu$ to an imaginary value[1].

The key property of this choice is that the logarithmic term in the matrix potential of the NMM gets tuned away. Let us take $R = 1$ from now on. Suppose we evaluate the expectation value of the inverse determinant operator at this N (for the moment we assume that this special value is integral). For N insertions of the inverse determinant, it gives the KP model with $N = \nu$. Thus, as one would expect, the log term of the KP model is also tuned away. Now if we choose $\bar{t}_k = c\,\delta_{k3}$, with c some constant, then the KP model reduces to the Kontsevich model, as observed in Ref.[26]. This shows that the Kontsevich model is a particular deformation of the $c = 1$ string theory after analytic continuation to imaginary cosmological constant and condensation of a particular tachyon (T_3). Note that at the end of this procedure, the rank of the Kontsevich matrix is the same as that of the NMM matrix.

As mentioned earlier, there is a different route to the Kontsevich model starting from the Gaussian Matrix Model (GMM)[58]. Here one starts with a Gaussian matrix model of rank $\hat{N}$, with N insertions of the determinant operator, and takes $\hat{N} \to \infty$ as a double-scaling limit by focussing on the edge of the eigenvalue distribution. The result is the Kontsevich matrix model. This time the rank $\hat{N}$

[1] Whereas the authors of Ref.[25] presented this as the analytic continuation of N to the imaginary value $-i\mu$, we prefer to think of it as continuing the cosmological constant μ to the imaginary value iN.

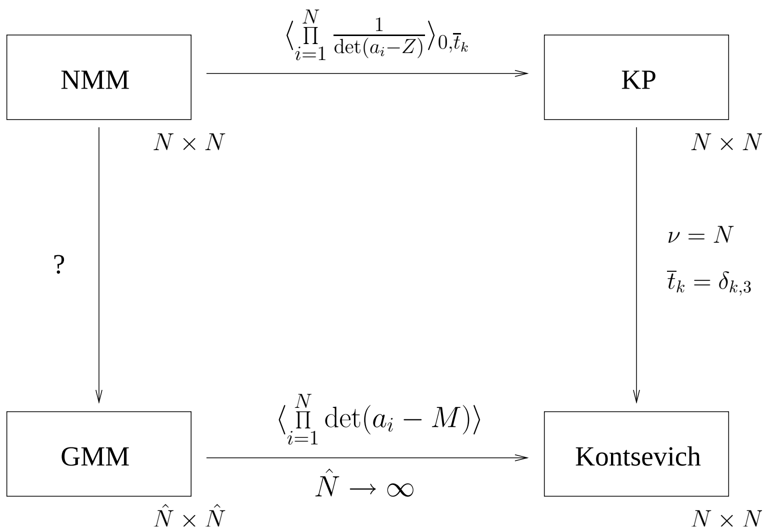

Figure 2.1: Two routes from NMM to Kontsevich

of the original matrix has disappeared from the picture (it was sent to infinity) while the Kontsevich matrix inherits its rank from the number of determinant insertions N.

A diagram of the situation is given in Fig.2.1. From the figure one sees that the diagram can be closed if we find a suitable relation of the NMM to the Gaussian matrix model. This is not hard to find at a qualitative level. In fact with $\nu = N$ and $t_k, \bar{t}_k = 0$ the NMM *is* a Gaussian matrix model. We choose the rank to be $\hat{N}$. The NMM eigenvalue distribution $\rho(z, \bar{z})$ is constant inside a disc in the z-plane (for $R = 1$)[25]. If we look at a contour along the real axis in the z-plane, then the effective eigenvalue distribution

$$\rho(x) = \int dy \; \rho(x, y) \tag{2.74}$$

is a semi-circle law, and we find the Gaussian matrix model. However, this picture of eigenvalue distributions is valid only at large $\hat{N}$. Inserting N determinant operators and taking $\hat{N} \to \infty$ as a double-scaling limit, one recovers the Kontsevich model. In this way of proceeding, the cubic coupling of the Kontsevich model is switched on automatically during the double-scaling limit. In the alternative route through the KP model, one has to switch on the coupling $\bar{t}_3$ by hand. A more detailed understanding of these two routes and their relationship should illuminate the question of how minimal model strings are embedded in $c = 1$. We leave this for future work.

2.7 Conclusions

We have established the equivalence between two matrix models of the $c = 1$ string (at selfdual radius): the Normal Matrix Model of Ref.[25] and the Kontsevich-Penner model of Ref.[26]. Both matrix models were initially found as solutions of a Toda hierarchy, so this equivalence is not very surprising. However, it is still helpful to have an explicit derivation, which also uncovered a few subtleties. Also we ended up showing why the KP matrix model does not exist at radius $R \neq 1$.

The more interesting aspect of this equivalence is that correlation functions of inverse determinant operators in the partially unperturbed NMM give rise to the KP model. This is analogous to corresponding results in Refs.[58, 63], with two important differences. In those cases, one considered determinants rather than inverse determinants, and their correlators were computed in a double-scaled matrix model. In the NMM there is no double-scaling as it already describes the grand canonical partition function of the double-scaled Matrix Quantum Mechanics. Another difference is that the N of the final (KP) model is equal to that of the NMM, and part of the matrix variables in NMM survive as the matrices of the KP model. All this suggests that, if one makes an analogy with the topological minimal models, the NMM occupies a position half-way between the original matrix model arising from dynamical triangulation of random surfaces (which requires a double-scaling limit to describe continuum surfaces) and the final "topological" model. If this is true, we may have only described half the story of open-closed duality at $c = 1$ while the correspondence between MQM and NMM constitutes the previous half. Further work may lead to a more coherent picture of the steps involved and thereby a deeper understanding of open-closed string duality at $c = 1$.

As we commented earlier, the inverse determinant operators seem to suggest bosonic statistics for FZZT-ZZ branes (at least in the NMM context) in contrast to fermionic statistics for $c < 1$. Another way to think of this is that both determinant and inverse determinant operators expand out to give the same set of macroscopic loops, the only difference being a minus sign for an odd number of loops in the latter case. Alternatively one can think of the basic loop operator as being changed by a sign to $-\mathrm{tr}\log(a - Z)$. Either way, the role of inverse determinant operators clearly calls for further investigation.

We commented earlier that trying to take the $c = 1$ limit of $c < 1$ FZZT correlators is problematic and therefore a derivation of the KP model from open-string field theory analogous to Ref.[59] has not been forthcoming. While this may yet be achieved, the situation recalls a historical parallel. In the 1990's, attempts to derive $c = 1$ closed string theory as a limit of the $c < 1$ theories were not very successful. Eventually it was found that at least at selfdual radius, the $c = 1$ string is a nonstandard case – rather than a limit – of the $c < 1$ models. This was understood by going over to the topological[39, 76, 77] rather than conventional, formulation of these string theories. It emerged that while the (p, q) minimal models for varying q were described by topological models labelled by an integer $k = p - 2 \geq 0$ (for example, $SU(2)_k/U(1)$ twisted Kazama-Suzuki models or twisted $\mathcal{N} = 2$ Landau-Ginzburg theories with superpotential X^{k+2}), the $c = 1$ string at selfdual radius was instead described by "continuations" of these models to $k = -3$[38, 39, 40, 41], rather than the more naive guess one might have made, namely $k \to \infty$. Therefore progress on FZZT branes at $c = 1$ in the continuum formulation might most naturally emerge in the context of topological D-branes in the twisted $SU(2)_{-3}/U(1)$ Kazama-Suzuki model or X^{-1} Landau-Ginzburg theory. Indeed, Ref.[78] represents important progress in this direction, and the Kontsevich model has been obtained there in the topological setup, predating the more recent derivations of Refs.[58, 59]. In fact, the KP model of Ref.[26] was also obtained in Ref.[78].

Extension of the NMM/KP models to include winding modes of the $c = 1$ string, as well as a better understanding of 2d black holes from matrix models[25, 27, 53, 54], remain open problems and perhaps the open-closed duality studied here will be helpful in this regard.

We have not pursued here an observation made in Ref.[7] that the KP model simplifies when we exponentiate the matrix variable via $M = e^{\Phi}$. The resulting model, which was named the "Liouville matrix model" there, is suggestive of N D-instantons moving in a Liouville plus linear potential. A similar exponentiation can be carried out in the NMM. In either case this is an almost trivial change of variables, therefore it does not seem important for the present considerations. However, in the light of the present work, these changes of variables might lead to new and more satisfying interpretations of the matrix models themselves.

As a final comment, we note that open-closed duality has in recent times been given a more fundamental basis in the Gopakumar programme[79, 80, 81] where closed string theory is proposed to be derived from quite general large-N field theories. Now this programme is expected to apply not just to noncritical strings but to all string theories. We know that the Kontsevich and Penner models compute topological invariants of the moduli space of Riemann surfaces, but the above works seem to suggest that these models play a role in more complicated string theories too. If so, equivalences and open-closed dualities such as we have discussed here may have more far-reaching implications than just providing examples in simplified string backgrounds.

2.8 Further Developments

In a subsequent paper [82] the authors provide a derivation of the Kontsevich-Penner integral Eq. (2.18) by inserting macroscopic loop operators of the form $\det(\Phi(t) - \mu_B)^{\pm 1}$ in the Matrix Quantum Mechanics path integral. We first define two quantities X_+, X_- by:

$$X_\pm = \frac{\Phi \pm P}{\sqrt{2}}, \tag{2.75}$$

where P is the matrix valued momentum conjugate to Φ. The quantities $X_\pm$ have a simple time evolution:

$$X_\pm(t) = e^{\pm it/R} X_\pm(t = 0). \tag{2.76}$$

The time dependent determinant operator is then given by:

$$e^{\pm W(t)} = \det\Big((e^{it/R}X_+(t=0) + e^{-it/R}X_-(t=0))/\sqrt{2} - \mu_B\Big)^{\pm 1}. \tag{2.77}$$

Naively inserting the above operator in the MQM action is not equivalent to generating a tachyon deformation in the Matrix Mechanics, because the dependence on the positive and negative momentum modes cannot be separated out. However, the authors of [82] show that this is possible in the scaling limit described below.

We consider the case $R = 1$. The first step is to analytically continue the time to imaginary values $t \to -it$. We have to also take the limit $t \to \infty$ and at the same time scale $\mu_B \to e^t \mu_B$. Dropping a normalization factor, the determinant operator Eq. (2.77) then becomes:

$$\det\left(1 - \frac{X_+}{\mu_B}\right)^{\pm}1 = e^{\mp \sum_n \frac{1}{n}\mu_B^{-n}\mathrm{tr}(X_+^n)}. \tag{2.78}$$

The above tachyon deformation is the same as that we obtained by inserting the operator $\det(1 - Z/\mu_B)^{\pm 1}$ in the Normal Matrix Model. If the matrix model time was analytically continued to $t \to it$ then the corresponding determinant in the Matrix mechanics is $\det(1 - X_-/\mu_B)^{\pm 1}$ which is equivalent to inserting the operator $\det(1 - Z^\dagger/\mu_B)^{\pm 1}$ in the NMM path integral.

It is now possible to compute the expectation value of the operator in Eq. (2.78), using the fact that the tachyon deformed partition function of the matrix model is a Baker-Akheizer function. Using the properties of this function the Kontsevich-Penner model Eq. (2.18) is obtained at $R = 1$. This work thus relates this model to Matrix Mechanics in the presence of FZZT branes.

The above argument was also extended to the Type 0 string. In this case the authors argued that the corresponding matrix integral similar to the one derived by Imbimbo and Mukhi [26] can be found at $R = 1/2$ by inserting the macroscopic loop operators in the Matrix Mechanics for the Type 0 theories. This also reproduces the time-independent matrix model for the Type 0A theory derived in [46, 83].

Chapter 3

Noncritical String Correlators, Finite-N Matrix Models and the Vortex Condensate

In this chapter we carry out a systematic study of correlation functions of momentum modes in the Euclidean $c = 1$ string, as a function of the radius and to all orders in perturbation theory. We obtain simple explicit expressions for several classes of correlators in terms of special functions. The Normal Matrix Model is found to be a powerful calculational tool that computes $c = 1$ string correlators even at finite N. This enables us to obtain a simple combinatoric formula for the $2n$-point function of unit momentum modes, which after T-duality determines the vortex condensate. We comment on possible applications of our results to T-duality at $c = 1$ and to the 2d black hole/vortex condensate problem [84].

3.1 Introduction

The $c = 1$ string (an excellent review is Ref.[6]) is a perturbatively consistent string theory in two spacetime dimensions. One of its attractive features is that it is solvable: from the powerful techniques of Matrix Quantum Mechanics (MQM), correlation functions of the momentum modes ("tachyons") can be determined to all orders in the string coupling (inverse cosmological constant). This holds true even in the Euclidean theory at finite radius R. Another feature is that its generalisation to the type 0 noncritical string, has similar properties in perturbation theory but is believed to also be non-perturbatively well-defined.

This makes the $c = 1$ string and its cousins a good laboratory to study various open questions in string theory. Two such questions that we would like to understand better in the noncritical context are the properties of string-scale black holes, and the nature of various dualities, including open-closed string duality[49][50][51]. Much work has been done on the former (some interesting recent studies can be found in Refs.[85][86][87]), while the latter question has also yielded some important illuminations[61][59][58][63][68][48][82].

It is known[36][27] that basic properties of black holes in noncritical string theory are controlled by condensates of winding tachyons in the Euclidean-continued background. These are thermal tachyons: strings winding around the compact time direction. It would therefore be useful to know the correlators of winding modes in Euclidean noncritical string theory to all orders in the string coupling (and even nonperturbatively in the stable type-0 case) as a function of μ and R, where μ^{-1} is the inverse string coupling and R is the radius of the Euclidean direction (inverse temperature). From the matrix model point of view, winding modes are related to the nonsinglet sector of the model, in which the eigenvalue fermions are no longer free but mutually coupled[88][89]. Computing correlators in this way is a harder task[28] and has raised some new puzzles involving leg factors which we will discuss in a later section. But one way to find the desired correlators is to assume that T-duality holds and perform it on the momentum correlators. This provides one of our motivations to study momentum correlators in the Euclidean theory in more explicit detail than has already been done.

As mentioned above, momentum correlators in the Euclidean $c = 1$ string are known in principle. They are summarised in the Toda hierarchy or W_∞ symmetries[69], or Hirota bilinear equations, or Normal Matrix Model (NMM)[25], all of which are supposed to be mutually equivalent. For the special case of self-dual radius $R = 1$ of the Euclidean time direction, they are encoded in a

Kontsevich-Penner matrix model[26][90] (see also [7][78]). We will summarise some relevant information about these solutions below. But while all these formal solutions allow us to extract the perturbation series for any specific correlator after a sufficient amount of work, we do not have many explicit answers in terms of special functions depending on the radius R and inverse string coupling μ.

At finite radius, correlators have been computed mostly at tree-level (corresponding to the dispersionless limit of the Toda hierarchy) or to a few low orders in perturbation theory. For example, while the $2n$-point function of n unit winding modes and n anti-winding modes is known as a function of n and R at tree level [91][27], an explicit expression for the same correlator to all orders in perturbation theory does not seem to exist in the literature[1]. To be more specific, denote by T_q the tachyons of momentum $q = n/R$, and by $\mathcal{T}_q$ the tachyons of n units of winding, where $q = nR$ is the value of $p_L = -p_R$ in vertex operator language. An explicit form is known for $\langle (T_{-1/R})^n (T_{1/R})^n \rangle$ at tree level. The T-dual of this expression was used in Ref.[27] to extract the critical behaviour of the Sine-Liouville theory defined by perturbing the original $c = 1$ string with $\mathcal{T}_{-R} + \mathcal{T}_R$ and then tuning the cosmological constant μ to zero. In particular, Ref.[27] showed that a sensible theory exists after this tuning, but only when the radius of the Euclidean direction lies in the range $1 < R < 2$.

One would like to know the structure of this correlator to all string loop orders. Accordingly, in what follows we will study $\langle (T_{-1/R})^n (T_{1/R})^n \rangle$ in detail, and one of our main results will be a simple formula for this correlator as a function of μ and R for every n. We expect this to lead to a better understanding of the exponentiated correlator $\langle \exp(T_{-1/R} + T_{1/R}) \rangle$, which in turn is T-dual to the vortex condensate $\langle \exp(\mathcal{T}_{-R} + \mathcal{T}_R) \rangle$ that relates directly to Euclidean 2d black holes.

Another motivation for our work is to understand T-duality of the $c = 1$ matrix quantum mechanics. This is established at the level of spectrum of states, since the partition function without perturbations is known to be T-dual[8]. Also,

[1]A differential equation for these correlators was written down in[27] together with an iterative solution to a few orders. Related work on Euclidean correlators can be found in Refs.[53],[54].

a formal argument has been given[27] that the winding correlators, like the momentum correlators, are given by a Toda hierarchy[1]. However, to our knowledge, beyond this result and a computation in [28], there has been no direct comparison of correlators in the momentum and winding sectors[2]. A convincing test of T-duality would consist of computing pure-momentum correlators in terms of free fermion eigenvalues, T-dualising the answers and comparing them with pure-winding correlators computed from the nonsinglet Hamiltonian. Ideally this should even be done beyond tree level. Although we will not be able to carry out such a test here, we have tried to systematise one side of the duality in a way that can be eventually compared with the other side when nonsinglet computations become more practicable.

In particular, the most direct way to check T-duality comes from comparing two-point functions. Accordingly we work out all two-point functions of momentum modes. In Ref.[28], the two-point function of unit-momentum modes was computed and an attempt made to match the leading result with a computation in the first nonsinglet sector of the matrix model, namely the adjoint sector. The comparison revealed the presence of unexplained normalisation factors. It was pointed out in Ref.[28] that if one could compute two-point functions of more general winding modes, namely $\langle \mathcal{T}_{-nR}\mathcal{T}_{nR}\rangle$, one might be able to shed some light on these normalisation factors. With this motivation we have performed this computation and obtained a simple explicit result, again as a function of μ and R and for all n. In a later section we discuss the relation to the non-singlet sectors.

Our initial computations have been performed using both the MQM and a model of constant matrices called the Normal Matrix Model (NMM)[25], with perfect agreement between the answers. In the former case we used the known infinite-radius correlators in the physical MQM (real and noncompact time)[93], and a formula which converts these to the correlators for Euclidean compact

[1]For a discussion of T-duality in type 0A,B matrix models, see Ref.[92].

[2]In addition to pure momentum or pure winding correlators, one would also like to know the correlators for a mixture of momentum and winding modes. In this case one has no choice but to tackle the difficult nonsinglet sector problem. The system is not expected to be integrable and the correlation functions are not known so far.

time[94]. In the latter case, we will describe how one performs computations after reducing the NMM to eigenvalues. One surprise emerging from comparison of the two approaches is that the NMM successfully computes correlators even when the matrices are of finite rank N, a stronger property than was claimed in Ref.[25], who did however suggest that the model contains some information even at finite N. We find that it actually contains *complete* information at finite N in the following sense: given a correlator, there is a minimum value N_{min} such that this correlator when computed in the NMM gives the correct result, to all orders in $1/\mu^2$, for all $N > N_{min}$. This makes the NMM a potentially powerful combinatoric tool. We then go on to demonstrate its power by deriving a combinatoric formula for the general correlation function $\langle (T_{-1/R})^n (T_{1/R})^n \rangle$ for any n.

We start in Section 3.2 by describing the two relevant matrix models, Matrix Quantum Mechanics and Normal Matrix Model. The former is too well-known to need a detailed discussion and we skip directly to the calculational techniques and answers. For the latter, we review the model in some detail, with special attention to the role of the matrix rank N. In Section 3.3 we work out some relevant correlators as a function of μ and R from MQM. In Section 3.4 we reproduce these correlators from the NMM, where we note the phenomenon that for a fixed correlator, the NMM at any N greater than a minimum value gives the complete answer. After a discussion of why this works, we use this property to derive a combinatoric formula for correlators of any number of unit momentum modes. In Section 3.5 we discuss applications of these results to some physically interesting problems, and conclude in Section 3.6. Several calculational details are presented in the appendices.

3.2 Matrix Quantum Mechanics and Normal Matrix Model

3.2.1 Matrix Quantum Mechanics

Matrix Quantum Mechanics is a model of a single $N \times N$ hermitian time-dependent matrix $M(t)$. In the absence of perturbations, the partition function of the model is given by:

$$\mathcal{Z}^{(N)}_{MQM} = \int [dM] \exp \left[-N \int dt \, \mathrm{tr} \left((D_t M)^2 + M^2 \right) \right] \tag{3.1}$$

where $D_t M \equiv \dot{M} + i[A_t, M]$ is the covariant derivative with respect to the time component of a gauge field.

The gauge field acts as a Lagrange multiplier and projects the model to the singlet sector, which is a system of N non-interacting non-relativistic fermions moving in an inverted harmonic oscillator potential. In the double-scaling limit, the fermi sea is filled nearly to the top and the number of fermions is taken to infinity. The scaled distance to the top of the potential, μ, is kept finite and corresponds to the cosmological constant. This model provides a description of 2D string theory, with μ^{-1} playing the role of the string coupling g_s.

The physical modes of 2D string theory can be constructed in terms of fermion eigenvalues. In [93] this model was used to calculate correlation functions of $c = 1$ string theory at infinite radius. One starts by computing correlators of free-fermion bilinears, which in turn can be used to extract correlators of the loop operators:

$$\mathcal{O}(k, \ell) = \int dt \, e^{ikt} \, \mathrm{tr} \, e^{-\ell M(t)} \tag{3.2}$$

Extracting the leading behaviour of these loops for small ℓ, one has

$$\mathcal{O}(k, \ell) \sim \ell^{|k|} \, T_k \tag{3.3}$$

The T_k are identified with the $c = 1$ string theory tachyons. When compared with the corresponding operators in Liouville theory, there is a change of normalisation:

$$T_k|_{MQM} = \Gamma(|k|) \; T_k|_{Liouville} \tag{3.4}$$

However this fact will not be relevant for us, since in what follows we will always work with the operators T_k in the MQM basis, i.e. the LHS of the above equation.

When the time direction is Euclidean and compact, we are in the finite temperature theory. Starting from the infinite-radius correlator, one can show[94] that correlators in the Euclidean theory at finite radius are obtained as:

$$\langle T_{q_1} T_{q_2} \cdots T_{q_n} \rangle_R = \frac{\frac{1}{2R}\partial_\mu}{\sin\left(\frac{1}{2R}\partial_\mu\right)} \langle T_{q_1} T_{q_2} \cdots T_{q_n} \rangle_\infty \tag{3.5}$$

In addition one must replace the momentum-conserving δ-function as:

$$\delta\Big(\sum_i q_i\Big) \to R\, \delta_{\sum_i q_i, 0} \tag{3.6}$$

The above prescriptions follow from the fact that the compact radial direction introduces an additional factor in the loop momentum integrals of the infinite-radius calculation, and this factor can now be taken out of the integrals whence it becomes a differential operator acting on the infinite-radius answer[1].

In the finite-temperature theory, the above modes can be thought of as carrying "momentum" in the time direction. In this situation one also expects to find winding modes corresponding to the thermal scalars of finite-temperature string theory. Many physical properties of string theory are encoded in these degrees of freedom, which are therefore quite important to study. To find them in the matrix model we must go beyond the singlet sector, in which the gauge field is topologically trivial and can be gauged away. Consider the gauge-invariant Wilson-Polyakov loop variable:

$$W_{\mathcal{R}} = \mathrm{tr}_{\mathcal{R}}\, P\, \exp(i \oint A_t dt) \tag{3.7}$$

where the trace is performed in the representation $\mathcal{R}$ of $SU(N)$. When $\mathcal{R}$ is the fundamental representation, this is to be associated with a unit winding mode:

$$\mathcal{W}_{\mathcal{R}=N} \sim \mathcal{T}_R \tag{3.8}$$

[1]It is also possible to calculate correlators directly at finite radius using the "reflection coefficient" formalism of Ref.[69]. Though we will not use this here, it would be interesting to know if our explicit results follow as easily in that approach.

Similarly the trace in the anti-fundamental will be $\mathcal{T}_{-R}$. One can also have loops where the trace continues to be in the fundamental but the contour winds multiple times over the Euclidean time direction. Computation of the correlation functions of all these Wilson-Polyakov loops is done by observing that in their presence, the matrix model receives contributions from definite non-singlet sectors. In these sectors it reduces to eigenvalue fermions but now with mutual interactions. For example, the two-point function of unit winding modes can be identified as follows:

$$\langle \mathcal{T}_{-R}\mathcal{T}_R\rangle\Big|_{\textbf{Liouville theory}} \sim \langle \mathcal{W}_{\bar{N}}\mathcal{W}_N\rangle\Big|_{\textbf{MQM}} \sim \langle \mathcal{W}_{\textbf{adjoint}}\rangle\Big|_{\textbf{MQM}} \tag{3.9}$$

Thus computing the partition function of MQM in the adjoint sector determines the two-point function of winding modes. Since in principle this is an independent computation from that of the momentum tachyon correlators, it can actually be used to check T-duality of the $c = 1$ string. We will return to this issue in a subsequent section.

3.2.2 Normal Matrix Model

The Normal Matrix Model (NMM)[25] is a relatively simple model of a complex matrix Z and its Hermitian adjoint, with the constraint that the two commute (hence Z is said to be "normal"). The potential is polynomial with an additional logarithmic piece. The matrix Z is constant rather than time-dependent, so in this sense it is more similar to the $c < 1$ string backgrounds which do not have a time direction[1].

The NMM is proposed to describe the correlators of the $c = 1$ string to all orders in perturbation theory, as follows. Let us introduce its partition function:

$$\mathcal{Z}^{(N)}_{NMM}(\nu, t, \bar{t}) = \int [dZdZ^\dagger]\; e^{\mathbf{tr}\left(-\nu(ZZ^\dagger)^R + \left(R\nu - N + \frac{R-1}{2}\right)\log ZZ^\dagger - \nu\sum_{k=1}^{\infty}(t_k Z^k + \bar{t}_k Z^{\dagger k})\right)} \tag{3.10}$$

[1]Perhaps this is the underlying reason why the NMM describes Euclidean $c = 1$ strings at an arbitrary radius R, but does not have a simple $R \to \infty$ limit where one might recover the Lorentzian theory.

Here R, ν are some (in general, complex) parameters, which will correspond to the compactification radius of Euclidean time and the cosmological constant respectively. The parameters t_k, $\bar{t}_k$ are couplings to the gauge-invariant operators $\mathrm{tr}Z^k$, $\mathrm{tr}{Z^\dagger}^k$ and Z, $Z^\dagger$ are $N \times N$ matrices satisfying:

$$[Z, Z^\dagger] = 0 \tag{3.11}$$

The operators $\mathrm{tr}Z^k, \mathrm{tr}{Z^\dagger}^k$ are identified with the tachyons $T_{k/R}, T_{-k/R}$ of momentum $\pm\frac{k}{R}$ respectively.

Since the matrix Z commutes with its adjoint, the two can be simultaneously diagonalised. The diagonalising matrices drop out of the action leaving behind Vandermonde factors. It turns out that one gets a single power of the Vandermonde for the eigenvalues $z_1, z_2, \ldots, z_N$ of Z, together with its complex conjugate corresponding to $Z^\dagger$. Thus, for example, the partition function at $t_k = \bar{t}_k = 0$ is:

$$\mathcal{Z}_{NMM} = \int \prod_{i=1}^{N} d^2 z_i \prod_{i<j} |z_i - z_j|^2 \; e^{-\nu \sum_{i=1}^{N} (z_i \bar{z}_i)^R + \left(R\nu - N + \frac{R-1}{2}\right) \sum_{i=1}^{N} \log z_i \bar{z}_i} \tag{3.12}$$

with an obvious generalisation to include the tachyon perturbations.

At $t_k = \bar{t}_k = 0$, it can be shown (though not directly from the action) that the NMM is invariant under the T-duality operation:

$$R \to \frac{1}{R}, \qquad \mu \to \mu R \tag{3.13}$$

This invariance is broken by the presence of momentum modes. Indeed, after T-duality, the tachyons $T_{\pm k/R}$ of the $c = 1$ string turn into winding modes of $\pm k$ units of winding, or equivalently (in vertex-operator language) of left/right momentum $(p_L, p_R) = \pm(kR, -kR)$. In what follows, these modes will be denoted $\mathcal{T}_{kR}, \mathcal{T}_{-kR}$.

In [25] two distinct equivalences between the NMM and the $c = 1$ string were proposed. The first, referred to as "Model I", requires us to take the large-N limit of the NMM. The result in this case was that:

$$\mathcal{Z}_{c=1}(\mu, t, \bar{t}) = \lim_{N\to\infty} \mathcal{Z}^{(N)}_{NMM}(\nu, t, \bar{t}), \tag{3.14}$$

after the analytic continuation $\nu = -i\mu$.

However, another equivalence, "Model II", was proposed which did not involve a large-N limit. It was argued that the $c = 1$ string theory can be obtained from the NMM at *finite* N, provided ν is set to the special value $\frac{N}{R}$ (note that this corresponds to an imaginary cosmological constant):

$$\mathcal{Z}_{c=1}\left(\mu = i\tfrac{N}{R}, t, \bar{t}\right) = \mathcal{Z}^{(N)}_{NMM}\left(\nu = \tfrac{N}{R}, t, \bar{t}\right), \tag{3.15}$$

In other words, the claim[1] is that an NMM calculation for a fixed integer value of N determines $\mathcal{Z}_{c=1}$ for a particular (imaginary) value of μ, namely

$$\mu = i\,\frac{N}{R} \tag{3.16}$$

If we T-dualise the above considerations so that $t, \bar{t}$ become couplings to winding tachyons, this relation becomes

$$\mu = iN \tag{3.17}$$

The above results seem to indicate that for finite N we can only generate the answer at a fixed μ, in which case we would never obtain the perturbative expansion in powers of $1/\mu^2$. However, below we will compute winding correlators using the NMM, and will see that it turns out much more powerful than expected. It actually does reproduce the entire perturbative correlators, as functions of μ and R, even at finite values of N. Evidence for this fact, as well as an explanation of it, will be provided in subsequent sections.

[1]The authors of Ref.[25] stated this a little differently: that one obtains $c = 1$ string amplitudes as a function of μ by computing NMM correlators as a function of N and μ, and then continuing N to the imaginary value $-i\mu R$. This procedure is less well-defined, as it requires us to make a discrete parameter continuous.

3.3 Correlators from Matrix Quantum Mechanics

3.3.1 Two-point functions

We start by presenting formulae for the two-point function $\langle T_{-n/R}T_{n/R}\rangle$ to all orders in $\frac{1}{\mu^2}$, from the Matrix Quantum Mechanics (MQM) approach. We will derive these formulae, valid at arbitary radius, starting from the infinite-radius formulae presented in [93]. We start by quoting the closed-form expression for the infinite-radius two-point function $\langle T_{-q}T_q\rangle$, or more precisely the first derivative of the two-point function with respect to the cosmological constant, which is actually more convenient for our purposes:

$$\partial_\mu \langle T_{-q}T_q\rangle_\infty = (\Gamma(-q))^2 \ \mathrm{Im}\, e^{i\pi q/2} \left(\frac{\Gamma\left(\frac{1}{2} - i\mu + q\right)}{\Gamma\left(\frac{1}{2} - i\mu\right)} - \frac{\Gamma\left(\frac{1}{2} - i\mu\right)}{\Gamma\left(\frac{1}{2} - i\mu - q\right)} \right), \quad (3.18)$$

where $q > 0$. For clarity of presentation we will drop the leg-pole factors $(\Gamma(-q))^2$ in what follows, keeping in mind that they can be restored whenever needed.

Now we obtain the corresponding amplitudes at a finite radius R, using Eqs.(3.5) and (3.6):

$$\langle T_{-q}T_q\rangle_R = R\, \frac{\frac{1}{2R}}{\sin\left(\frac{1}{2R}\partial_\mu\right)} \ \mathrm{Im}\, e^{i\pi q/2} \left(\frac{\Gamma\left(\frac{1}{2} - i\mu + q\right)}{\Gamma\left(\frac{1}{2} - i\mu\right)} - \frac{\Gamma\left(\frac{1}{2} - i\mu\right)}{\Gamma\left(\frac{1}{2} - i\mu - q\right)} \right)$$

where the first factor of R comes from the replacement of the δ-function by a Kronecker δ as in Eq. (3.6). The differential operator in front is real and acts only on functions of μ, so it can be moved inside and we thus need to evaluate

$$\frac{1}{2\sin\left(\frac{1}{2R}\partial_\mu\right)} \left(\frac{\Gamma\left(\frac{1}{2} - i\mu + q\right)}{\Gamma\left(\frac{1}{2} - i\mu\right)} - \frac{\Gamma\left(\frac{1}{2} - i\mu\right)}{\Gamma\left(\frac{1}{2} - i\mu - q\right)} \right)$$

This can be done very easily by expanding the operator as follows

$$\frac{1}{2\sin\left(\frac{1}{2R}\partial_\mu\right)} = -i \sum_{j=0}^{\infty} e^{i\left(j+\frac{1}{2}\right)\frac{1}{R}\partial_\mu}$$

Using this we get the required expression as

$$-i\sum_{j=0}^{\infty}\left(\frac{\Gamma\left(\frac{1}{2}-i\mu+q+\frac{j}{R}+\frac{1}{2R}\right)}{\Gamma\left(\frac{1}{2}-i\mu+\frac{j}{R}+\frac{1}{2R}\right)}-\frac{\Gamma\left(\frac{1}{2}-i\mu+\frac{j}{R}+\frac{1}{2R}\right)}{\Gamma\left(\frac{1}{2}-i\mu-q+\frac{j}{R}+\frac{1}{2R}\right)}\right) \tag{3.19}$$

Next, we choose $q = n/R$. We see that the j^{th} term from the first sum cancels the $(j+n)^{\text{th}}$ term from the second sum. So only the $j = 0, 1, \ldots, n-1$ terms from the second sum remain. Defining $r = n - j$, the above expression becomes[1]:

$$\langle T_{-n/R} T_{n/R} \rangle = \text{Re}\; e^{i\pi n/2R} \sum_{r=1}^{n} \frac{\Gamma\left(\frac{1}{2}-i\mu+(r-\frac{1}{2})\frac{1}{R}\right)}{\Gamma\left(\frac{1}{2}-i\mu+(r-n-\frac{1}{2})\frac{1}{R}\right)} \tag{3.20}$$

In order to obtain the expansion of this expression in powers of $1/\mu^2$, we can rewrite it in terms of the special functions:

$$\mathcal{F}^{\pm}(a,b;\mu) \equiv \frac{\Gamma(\frac{1}{2}-i\mu+a)}{\Gamma(\frac{1}{2}-i\mu+b)} \pm \frac{\Gamma(\frac{1}{2}-i\mu-b)}{\Gamma(\frac{1}{2}-i\mu-a)} \tag{3.21}$$

defined in Eq.(B.2) of Ref.[93]. We have:

$$\langle T_{-n/R} T_{n/R} \rangle = \text{Re}\; e^{i\pi n/2R} \sum_{r=1}^{n/2} \mathcal{F}^{+}\Big((r-\tfrac{1}{2})\tfrac{1}{R}, (r-n-\tfrac{1}{2})\tfrac{1}{R}; \mu\Big), \qquad n \text{ even} \tag{3.22}$$

$$= \text{Re}\; e^{i\pi n/2R}\left(\tfrac{1}{2}\mathcal{F}^{+}\left(\tfrac{n}{2R}, -\tfrac{n}{2R}; \mu\right) + \sum_{r=1}^{(n-1)/2} \mathcal{F}^{+}\Big((r-\tfrac{1}{2})\tfrac{1}{R}, (r-n-\tfrac{1}{2})\tfrac{1}{R}; \mu\Big)\right), \; n \text{ odd}$$

Next we use the asymptotics for large μ:

$$\mathcal{F}^{+}(a,b;\mu) = e^{-i\pi(a-b)/2}\, \mu^{a-b}\, f(a,b;\mu) \tag{3.23}$$

where $f(a,b;\mu)$ is a power series in $\frac{1}{\mu^2}$ with real coefficients and starting with a constant term:

$$f(a,b;\mu) = 2 - \tfrac{1}{12}(a-b)(a-b-1)\left(3(a+b)^2-(a-b)-1\right)\frac{1}{\mu^2} + \mathcal{O}\left(\frac{1}{\mu^4}\right) \tag{3.24}$$

[1]Here and in what follows, we drop the R subscript in the correlators wherever it is obvious that they are at finite R.

It follows that, for even n:

$$\begin{aligned}\langle T_{-n/R}T_{n/R}\rangle &= \text{Re}\ \mu^{n/R}\sum_{r=1}^{n/2} f\Big((r-\tfrac{1}{2})\tfrac{1}{R},(r-n-\tfrac{1}{2})\tfrac{1}{R};\mu\Big) \\ &= \mu^{n/R}\sum_{r=1}^{n/2} f\Big((r-\tfrac{1}{2})\tfrac{1}{R},(r-n-\tfrac{1}{2})\tfrac{1}{R};\mu\Big) \\ &= \left|\sum_{r=1}^{n}\frac{\Gamma(\frac{1}{2}-i\mu+(r-\frac{1}{2})\frac{1}{R})}{\Gamma(\frac{1}{2}-i\mu+(r-n-\frac{1}{2})\frac{1}{R})}\right| \end{aligned} \tag{3.25}$$

The first step above follows because the function f is real. The final equality is true for all n, and not just even values. This then is the complete answer for the perturbative expansion of two-point functions of momentum correlators at arbitrary radius.

Specialising to $n = 1$, we find the following expression, which will be useful later on:

$$\langle T_{-1/R}T_{1/R}\rangle = \left|\frac{\Gamma\left(\frac{1}{2}-i\mu+\frac{1}{2R}\right)}{\Gamma\left(\frac{1}{2}-i\mu-\frac{1}{2R}\right)}\right| \tag{3.26}$$

After a T-duality

$$R \to 1/R, \quad \mu \to \mu R \tag{3.27}$$

we get the unit-winding two-point function

$$\langle \mathcal{T}_{-R}\mathcal{T}_{R}\rangle = \left|\frac{\Gamma\left(\frac{1}{2}-i\mu R+\frac{R}{2}\right)}{\Gamma\left(\frac{1}{2}-i\mu R-\frac{R}{2}\right)}\right| \tag{3.28}$$

This expression was recently derived by Maldacena[28]. We should note that the above answer has to be multiplied by the leg pole factor $\big(\Gamma(-R)\big)^2$, which we dropped after Eq. (3.18).

3.3.2 Four-point functions

In this section we turn to the computation of higher point functions. In particular, we extend the results for the four-point function from MQM to finite R and then specialise to the case of unit winding modes. In this case we will be able to find an explicit all-orders result after summing an infinite series.

Upto leg pole factors (which can be unambiguously restored when needed) the connected four-point function at infinite radius is[93]:

$$\partial_\mu \langle (T_{-q}T_q)^2\rangle_\infty^{\rm conn} = {\rm Im}\; e^{i\pi q}\Bigg[\mathcal{F}^+(2q,0;\mu) - \mathcal{F}^+(q,-q;\mu) + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}2\left(\frac{\Gamma(-q+n)}{\Gamma(-q)}\right)^2 \times\left(\frac{\Gamma\left(2q-n+\frac{1}{2}-i\mu\right)}{\Gamma\left(\frac{1}{2}-i\mu\right)} - \frac{\Gamma\left(q-n+\frac{1}{2}-i\mu\right)}{\Gamma\left(-q+\frac{1}{2}-i\mu\right)}\right)\Bigg], \tag{3.29}$$

where $q > 0$ and the function $\mathcal{F}^+$ is defined in Eq. (3.21).

Substituting Eq. (3.21) in Eq. (3.29) we have

$$\partial_\mu \langle (T_{-q}T_q)^2\rangle_\infty^{\rm conn} = {\rm Im}\; e^{i\pi q}\Bigg[\frac{\Gamma\left(\frac{1}{2}-i\mu+2q\right)}{\Gamma\left(\frac{1}{2}-i\mu\right)} + \frac{\Gamma\left(\frac{1}{2}-i\mu\right)}{\Gamma\left(\frac{1}{2}-i\mu-2q\right)} - 2\frac{\Gamma\left(\frac{1}{2}-i\mu+q\right)}{\Gamma\left(\frac{1}{2}-i\mu-q\right)} +2\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left(\frac{\Gamma(-q+n)}{\Gamma(-q)}\right)^2\left(\frac{\Gamma\left(\frac{1}{2}-i\mu+2q-n\right)}{\Gamma\left(\frac{1}{2}-i\mu\right)} - \frac{\Gamma\left(\frac{1}{2}-i\mu+q-n\right)}{\Gamma\left(\frac{1}{2}-i\mu-q\right)}\right)\Bigg] \tag{3.30}$$

The connected finite-R amplitude is, therefore

$$\langle (T_{-q}T_q)^2\rangle_R^{\rm conn} = R\,\frac{\frac{1}{2R}\partial_\mu}{\sin\left(\frac{1}{2R}\partial_\mu\right)}\langle (T_{-q}T_q)^2\rangle_\infty^{\rm conn}$$

We use the expansion Eq. (3.3.1) of the differential operator and set $q = 1/R$ to get

$$\langle (T_{-1/R}T_{1/R})^2\rangle^{\rm conn} = {\rm Re}\; e^{i\pi/R}\Bigg(-\frac{\Gamma\left(\frac{1}{2}-i\mu+\frac{3}{2R}\right)}{\Gamma\left(\frac{1}{2}-i\mu-\frac{1}{2R}\right)} + \frac{\Gamma\left(\frac{1}{2}-i\mu+\frac{1}{2R}\right)}{\Gamma\left(\frac{1}{2}-i\mu-\frac{3}{2R}\right)} -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left(\frac{\Gamma(-\frac{1}{R}+n)}{\Gamma(-\frac{1}{R})}\right)^2\frac{\Gamma\left(\frac{1}{2}-i\mu+\frac{3}{2R}-n\right)}{\Gamma\left(\frac{1}{2}-i\mu-\frac{1}{2R}\right)}\Bigg) \tag{3.31}$$

It is convenient to add and subtract a term corresponding to $n = 0$ in the summation. This extends the sum from 0 to ∞, while the subtracted term changes the sign of the first term above, after which the first two terms combine into an $\mathcal{F}^+$. Thus we get:

$$\langle (T_{-1/R}T_{1/R})^2\rangle^{\rm conn} = {\rm Re}\; e^{i\pi/R}\Bigg(\mathcal{F}^+(\tfrac{3}{2R},-\tfrac{1}{2R};\mu) -2\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{\Gamma(-\frac{1}{R}+n)}{\Gamma(-\frac{1}{R})}\right)^2\frac{\Gamma\left(\frac{1}{2}-i\mu+\frac{3}{2R}-n\right)}{\Gamma\left(\frac{1}{2}-i\mu-\frac{1}{2R}\right)}\Bigg) \tag{3.32}$$

The sum is now easy to evaluate using the integral representations for the three Γ-functions in the numerator that depend on n (see Appendix 3.7.2). This finally leads to:

$$\begin{aligned}\langle (T_{-1/R}T_{1/R})^2\rangle^{\rm conn} &= \text{Re}\, e^{i\pi/R}\left(\mathcal{F}^+(\tfrac{3}{2R},-\tfrac{1}{2R};\mu)-2\Big(\tfrac{1}{2}\mathcal{F}^+(\tfrac{1}{2R},-\tfrac{1}{2R};\mu)\Big)^2\right)\\ &= \left|\mathcal{F}^+(\tfrac{3}{2R},-\tfrac{1}{2R};\mu)-2\Big(\tfrac{1}{2}\mathcal{F}^+(\tfrac{1}{2R},-\tfrac{1}{2R};\mu)\Big)^2\right| \qquad (3.33)\end{aligned}$$

One can verify that the two terms above are, respectively, the full (connected plus disconnected) correlator, and its disconnected part.

3.4 Correlators in the finite-N Normal Matrix Model

Having obtained explicit expressions for all two-point and a particular four-point function from the MQM, as a function of the cosmological constant μ and radius R, we now attempt to recover the same results from the NMM. This first of all provides a test of the NMM and its effectiveness. But once we explore the systematics it will become clear that we can compute much more. In fact, we will obtain a complete combinatorial formula for the $2n$-point functions of unit-momentum correlators. Via T-duality, this determines the corresponding winding correlators. We expect this to be useful in determining the full vortex condensate to all orders in perturbation theory.

As mentioned before, in the process of studying the NMM we will encounter a rather surprising result: for the purpose of computing correlators, one can actually take N to be a small finite value and yet obtain the correct answer *as a function of* μ. The finite value of N will be determined by the operators whose correlators we are calculating. For this purpose it is convenient to classify tachyon correlators into sectors labelled by an integer, the total positive momentum P flowing through that correlator, measured in units of $1/R$. For example in $\langle T_{-k_1/R}T_{-k_2/R}T_{m_1/R}T_{m_2/R}\rangle$, where k_1, k_2, m_1, m_2 are all positive, the total positive momentum is $P = m_1 + m_2 = k_1 + k_2$. This number will determine the minimum value of N required in the NMM to compute these correlators. In what follows we will first consider all correlators in the sectors $P = 1$ and $P = 2$. In the former case there is only a single two-point function, while in the latter case we have two, three and four-point functions. After presenting some examples we will discuss why the theory works in this way.

3.4.1 Two-point functions: examples

Example: $n = 1$

We begin by computing the two point function of the unit momentum operator. Since total momentum is conserved, this operator is paired with the one of negative unit momentum. So we will calculate the two point function $\langle T_{-1/R} T_{1/R} \rangle$ of unit momentum operators.

We first calculate the partition function of NMM at $N = 1$:

$$\mathcal{Z}^{N=1}_{NMM}(t=0) = \int dz d\bar{z}\, e^{-\nu(z\bar{z})^R + \left(R\nu - 1 + \frac{R-1}{2}\right)\log z\bar{z}} \tag{3.34}$$

Setting $z = \sqrt{m}\, e^{i\theta}$, $dzd\bar{z}\rangle dm\, d\theta$, we have:

$$\begin{aligned}
\mathcal{Z}^{N=1}_{NMM}(t=0) &= \int_0^\infty \int_0^{2\pi} dm\, d\theta\, e^{-\nu m^R + \left(R\nu - 1 + \frac{R-1}{2}\right)\log m} \\
&= 2\pi \int_0^\infty dm\, m^{\left(R\nu - 1 + \frac{R-1}{2}\right)} e^{-\nu m^R} \\
&= \frac{2\pi}{R} \nu^{-\left(\nu + \frac{1}{2} - \frac{1}{2R}\right)} \Gamma\left(\nu + \tfrac{1}{2} - \tfrac{1}{2R}\right)
\end{aligned} \tag{3.35}$$

As a function of ν, this is not the correct partition function of the $c = 1$ string, but it reduces to the correct partition function if in the above expression we set $\nu = \frac{1}{R}$ and compare this with $\mathcal{Z}_{c=1}(\frac{i}{R}, t = 0, \bar{t} = 0)$. This fact is a direct consequence of the claim in Ref.[25], see Eq. (3.15). It is also worth noting that the partition function at $N = 1$ is not invariant under T-duality. In fact, T-duality in the NMM partition function is recovered only in the limit $N \to \infty$. This makes it clear that the correct partition function, as a function of μ and R, can never be recovered at finite N.

For correlators, things are quite different, as we will now see. For the two-point function, we find:

$$\begin{aligned}\partial_{-1}\partial_1 \mathcal{Z}^{N=1}_{NMM}(t=0) &= \int_0^\infty \int_0^{2\pi} dm\, d\theta\, m\, e^{-\nu m^R + \left(R\nu - 1 + \frac{R-1}{2}\right)\log m} \\ &= \frac{2\pi}{R}\nu^{-\left(\nu+\frac{1}{2}+\frac{1}{2R}\right)}\Gamma\left(\nu+\tfrac{1}{2}+\tfrac{1}{2R}\right) \end{aligned} \tag{3.36}$$

From Eq. (3.35) and Eq. (3.36) we have:

$$\partial_{-1}\partial_1 \ln \mathcal{Z}^{N=1}_{NMM}(t=0) = \nu^{-\frac{1}{R}}\frac{\Gamma\left(\nu+\frac{1}{2}+\frac{1}{2R}\right)}{\Gamma\left(\nu+\frac{1}{2}-\frac{1}{2R}\right)} \tag{3.37}$$

Finally, we have to analytically continue $\nu = -i\mu$. The result is complex, but can easily be seen to have the form of an overall phase times a real power series in $1/\mu^2$. Dropping the phase is then equivalent to taking the modulus of the above expression. This gives:

$$\langle T_{-1/R} T_{1/R}\rangle^{N=1}_{NMM} = \mu^{-\frac{1}{R}}\left|\frac{\Gamma\left(\frac{1}{2}-i\mu+\frac{1}{2R}\right)}{\Gamma\left(\frac{1}{2}-i\mu-\frac{1}{2R}\right)}\right| \tag{3.38}$$

which agrees with Eq. (3.26) upto the prefactor, $\mu^{-1/R}$, which indicates that the "tachyons" of the NMM are normalised differently from those of MQM. Indeed we will argue later that the relationship is:

$$T_{n/R}|_{NMM} = \mu^{-n/2R}\, T_{n/R}|_{MQM} \tag{3.39}$$

We have discovered the surprising result that the exact two-point correlator of unit momentum tachyons is correctly calculated (as a function of μ and R) using only the 1×1 Normal Matrix Model! According to Eq. (3.15), we should have expected the result to be correct only for $\mu = i/R$. We will see that a similar feature holds for all two-point correlators, though the minimum required value of N depends on the correlator under consideration. Later we will extend this observation to higher-point correlators.

Example: $n = 2$

We consider another example, the correlator $\langle T_{-2/R} T_{2/R}\rangle$. In this case, according to the prediction in Eq. (3.15), we can perform a calculation at $N = 1$

and the result so obtained will be valid at the special value of the cosmological constant $\nu = 1$. However, we now face a puzzle. In the NMM at $N = 1$, one cannot distinguish the four correlators:

$$\langle T_{-2/R}T_{2/R}\rangle, \quad \langle T_{-2/R}T_{1/R}T_{1/R}\rangle, \quad \langle T_{-1/R}T_{-1/R}T_{2/R}\rangle, \quad \langle T_{-1/R}T_{-1/R}T_{1/R}T_{1/R}\rangle \tag{3.40}$$

because all of these are represented by the same NMM correlator $\langle z^2\bar{z}^2\rangle$. Therefore, assuming Eq. (3.15) continues to hold, either it has to be the case that all four correlators become the same at $\nu = \frac{1}{R}$, or else at best we can only hope to obtain some linear combination of them.

The calculation is straightforward and upon continuing to $\nu = -i\mu$ and taking the modulus, we find:

$$\langle T_{-2/R}T_{2/R}\rangle_{NMM}^{N=1} = \mu^{-2/R}\left|\frac{\Gamma\left(\frac{1}{2} - i\mu + \frac{3}{2R}\right)}{\Gamma\left(\frac{1}{2} - i\mu - \frac{1}{2R}\right)}\right| \tag{3.41}$$

This can be compared with the known result from Eq. (3.25). Specialising to the present case, and changing to the NMM normalisation via Eq. (3.39) gives us:

$$\langle T_{-2/R}T_{2/R}\rangle = \left|\frac{\Gamma(\frac{1}{2} - i\mu + \frac{1}{2R})}{\Gamma(\frac{1}{2} - i\mu - \frac{3}{2R})} + \frac{\Gamma(\frac{1}{2} - i\mu + \frac{3}{2R})}{\Gamma(\frac{1}{2} - i\mu - \frac{1}{2R})}\right| \tag{3.42}$$

Comparing Eqs.(3.41),(3.42), we see that the NMM result for this correlator at $N = 1$ is not correct. This is not a surprise. But now we see that it is incorrect even at the special value $\mu = i/R$, which appears to contradict Eq. (3.15). As we will see, this is due to the fact that the same NMM correlator can describe different tachyon correlation functions for low N. Indeed, one can check that the answer we have obtained at $N = 1$ in Eq. (3.41) is actually a linear combination of the correlators in Eq. (3.40) as calculated from matrix quantum mechanics.

Let us continue by evaluating the NMM correlator at $N = 2$. In this case the operator we are dealing with is $T_{2/R} \sim \mathrm{tr}Z^2$ which is linearly independent of $(T_{1/R})^2 \sim (\mathrm{tr}Z)^2$ once Z is a 2×2 matrix, so there is no longer a risk of mixing for the operators in Eq. (3.40). The computation is given in an Appendix, and leads to the answer Eq. (3.72), which after changing to the NMM normalisation is:

$$\langle T_{-2/R} T_{2/R} \rangle_{NMM}^{N=2} = \left| \frac{\Gamma\left(\frac{1}{2} - i\mu + \frac{3}{2R}\right)}{\Gamma\left(\frac{1}{2} - i\mu - \frac{1}{2R}\right)} + \frac{\Gamma\left(\frac{1}{2} - i\mu + \frac{1}{2R}\right)}{\Gamma\left(\frac{1}{2} - i\mu - \frac{3}{2R}\right)} \right| \tag{3.43}$$

Following Eq. (3.15) we would expect that this should give the correct answer for $\mu = 2i/R$. But now there is a surprise, since in fact it agrees perfectly with the MQM result Eq. (3.42) for *all* values of μ. Thus for the purposes of calculating $\langle T_{-2/R} T_{2/R} \rangle$ in $c = 1$ string theory, to all orders in the string coupling, a 2×2 matrix model is sufficient.

To summarise, we have found evidence that an NMM calculation of tachyon correlators at finite N (where the minimum required value of N depends on the correlator in question) gives the correct tachyon correlators for the $c = 1$ string, to all orders in perturbation theory. Below we will collect more evidence for this property, which appears to go far beyond the result of Ref.[25] as stated in Eq. (3.15) above.

3.4.2 Two-point functions: general case

Let us now consider the general case $\langle T_{-n/R} T_{n/R} \rangle$. and try to derive this result from the NMM. We will find that for this correlator, the NMM with $N = n$ is sufficient to give the correct result. Indeed, when we compute in the $N \times N$ NMM starting at $N = 1$ and increasing N in integer steps, we obtain the right $c = 1$ string correlator (as a function of μ) as long as $N \geq n$, though not for $N < n$. Thus the NMM calculation "stabilises" at a certain minimum value of N.

Since we will be computing normalised correlators, we start by computing the (unperturbed) partition function at a general value of N. This is given by

$$\mathcal{Z}_{NMM}^{N}(t = 0) = \int_0^\infty \prod_{r=1}^{N} dm_r \int_0^{2\pi} \prod_{r=1}^{N} d\theta_r \tag{3.44}$$

$$\times \prod_{j<k}^{N} \left(m_j + m_k - \sqrt{m_j m_k} (e^{i\theta_{jk}} + e^{-i\theta_{jk}}) \right) e^{-\nu\left(\sum_{r=1}^{N} m_r^R\right) + \left(R\nu - N + \frac{R-1}{2}\right)\left(\sum_{r=1}^{N} \log m_r\right)}$$

The next step is to perform the integration over the θ's. In general this will be quite tedious, because one has to pick out terms which are independent of θ

by expanding out the Vandermonde factor. However, we notice that since the above expression is invariant under permutations of m's, we can determine all terms surviving the θ integrals if we know just one of them, by permuting the m's among themselves.

The first such term is just the product of the first term from each of the Vandermonde factors, which is $m_1^{N-1} m_2^{N-2} \cdots m_{N-1}$. Thus we have, after evaluating the θ integrals

$$\begin{aligned} \mathcal{Z}_{NMM}^{N}(t=0) &= (2\pi)^N N! \int_0^\infty \prod_{r=1}^{N} dm_r \prod_{j=1}^{N-1} m_j^{N-j} \\ &\quad \times\, e^{-\nu\left(\sum_{r=1}^{N} m_r^R\right) + \left(R\nu - N + \frac{R-1}{2}\right)\left(\sum_{r=1}^{N} \log m_r\right)} \\ &= (2\pi)^N N! \prod_{r=1}^{N} \nu^{-\left(\nu + \frac{1}{2} - \left(r - \frac{1}{2}\right)\frac{1}{R}\right)} \Gamma\left(\nu + \tfrac{1}{2} - \left(r - \tfrac{1}{2}\right)\tfrac{1}{R}\right) \end{aligned} \tag{3.45}$$

¿From now on we will restrict to the case $N = n$.

The next step is to compute the two point function and then normalise by the above partition function. We have:

$$\begin{aligned} \partial_{-n}\partial_n \mathcal{Z}_{NMM}^{N=n}(t=0) &= \int \prod_{r=1}^{n} d^2 z_r \prod_{j<k}^{n} |z_j - z_k|^2 \left(\sum_{l=1}^{n} z_l^n\right)\left(\sum_{l=1}^{n} \bar{z}_l^n\right) \\ &\quad \times e^{-\nu\left(\sum_{r=1}^{n} (z_r \bar{z}_r)^R\right) + \left(R\nu - n + \frac{R-1}{2}\right)\left(\sum_{r=1}^{n} \log z_r \bar{z}_r\right)} \\ = \int_0^\infty \prod_{r=1}^{n} dm_r \int_0^{2\pi} \prod_{r=1}^{n} d\theta_r \prod_{j<k}^{n} &\left(m_j + m_k - \sqrt{m_j m_k}(e^{i\theta_{jk}} + e^{-i\theta_{jk}})\right) \\ \times \left(\sum_{r=1}^{n} \left(\sqrt{m_r}\right)^n e^{in\theta_r}\right) &\left(\sum_{r=1}^{n} \left(\sqrt{m_r}\right)^n e^{-in\theta_r}\right) \\ \times e^{-\nu\left(\sum_{r=1}^{n} m_r^R\right) + \left(R\nu - n + \frac{R-1}{2}\right)\left(\sum_{r=1}^{n} \log m_r\right)} & \end{aligned} \tag{3.46}$$

In this case also we can avoid tedious calculation by applying the permutation trick. The contribution to the first term from the Vandermonde is same as before, and the contribution from $\mathrm{tr} Z^n \, \mathrm{tr} Z^{\dagger n}$ is $\sum_{r=1}^{n} m_r^n$. The net contribution is then

$\left(\sum_{r=1}^n m_r^n\right) m_1^{n-1} m_2^{n-2} \cdots m_{n-1}$. Proceeding as before we have after the θ integrals

$$\begin{aligned}
\partial_{-n}\partial_n \mathcal{Z}_{NMM}^{N=n}(t=0) &= (2\pi)^n n! \int_0^\infty \prod_{r=1}^n dm_r \left(\sum_{r=1}^n m_r^n\right) \prod_{j=1}^{n-1} m_j^{n-j} \\
&\quad \times e^{-\nu\left(\sum_{r=1}^n m_r^R\right) + \left(R\nu - n + \frac{R-1}{2}\right)\left(\sum_{r=1}^n \log m_r\right)} \\
&= (2\pi)^n n! \sum_{j=1}^n \left[\nu^{-\left(\nu+\frac{1}{2}-\left(j-n-\frac{1}{2}\right)\frac{1}{R}\right)} \Gamma\left(\nu + \frac{1}{2} - \left(j - n - \frac{1}{2}\right)\frac{1}{R}\right) \right. \\
&\quad \left. \times \prod_{\substack{r=1 \\ r\neq j}}^n \nu^{-\left(\nu+\frac{1}{2}-\left(r-\frac{1}{2}\right)\frac{1}{R}\right)} \Gamma\left(\nu + \frac{1}{2} - \left(r - \frac{1}{2}\right)\frac{1}{R}\right)\right]
\end{aligned} \tag{3.47}$$

From Eq. (3.45) and Eq. (3.47) we find (after changing variables $j\rangle n+1-r$):

$$\langle T_{-n/R} T_{n/R}\rangle_{NMM}^{N=n} = \nu^{-n/R} \sum_{r=1}^n \frac{\Gamma\left(\frac{1}{2} - i\mu + \left(r - \frac{1}{2}\right)\frac{1}{R}\right)}{\Gamma\left(\frac{1}{2} - i\mu + \left(r - n - \frac{1}{2}\right)\frac{1}{R}\right)} \tag{3.48}$$

As before, we analytically continue $\nu = -i\mu$ and take the modulus to get:

$$\langle T_{-n/R} T_{n/R}\rangle_{NMM}^{N=n} = \mu^{-n/R} \left| \sum_{r=1}^n \frac{\Gamma\left(\frac{1}{2} - i\mu + \left(r - \frac{1}{2}\right)\frac{1}{R}\right)}{\Gamma\left(\frac{1}{2} - i\mu + \left(r - n - \frac{1}{2}\right)\frac{1}{R}\right)} \right| \tag{3.49}$$

After changing normalisation via Eq. (3.39), we see that this agrees perfectly with Eq. (3.25).

The above calculation was performed with matrices of rank $N = n$. It can easily be repeated for the other cases. When N is smaller than n, we find that the answer, as a function of μ, is not equal to the correct two-point function, and does not become the correct one even after choosing $\mu = in/R$. As before, this is due to "contamination" by correlators of higher point functions carrying the same total momentum, because for $N < n$ the corresponding correlators in the NMM are not all linearly independent. For $N > n$, instead, we actually get the *same* final answer as for $N = n$. The calculational procedure we described above seems to suggest that extra terms arise for $N > n$, but actually they are cancelled by contributions from the θ dependent terms in the Vandermonde factor. Thus when we take the ratio of $\partial_{-n}\partial_n\mathcal{Z}$ and $\mathcal{Z}$ we end up with the RHS of Eq. (3.48). Therefore as long as we take $N \geq n$, we get the right answer (independent of N) for every N. This is what we referred to as "stabilisation" above.

3.4.3 Four-point functions

Now we would like to compute the four-point function in the Normal Matrix Model. For $N = 1$, the calculation has already been performed, since as we noted above, it is the same as the corresponding calculation for the two-point function in Eq. (3.41) (more precisely the disconnected four-point function is the same as this two-point function). As we explained there, the result so obtained is a linear combination of the correct two, three and four-point functions of the $c = 1$ string, and to distinguish them we need to go to a higher value of N. Accordingly we have computed the above four-point function using the $N = 2$ NMM. The derivation can be found in Appendix 3.7.3, and the result is:

$$\langle (T_{-1/R}T_{1/R})^2\rangle_{NMM}^{N=2} = \mu^{-2/R}\left|\mathcal{F}^+(\tfrac{3}{2R},-\tfrac{1}{2R};\mu) - \tfrac{1}{2}\Big(\mathcal{F}^+(\tfrac{1}{2R},-\tfrac{1}{2R};\mu)\Big)^2\right| \qquad (3.50)$$

Changing from MQM to NMM normalisation using Eq. (3.39), and inserting the usual $1/R$ factor, we see that Eq. (3.50) above is identical to Eq. (3.33).

For completeness, let us briefly consider the two three-point functions

$$\langle \mathcal{T}_{-2R}\mathcal{T}_R\mathcal{T}_R\rangle_{NMM}^{N=2}, \qquad \langle \mathcal{T}_{2R}\mathcal{T}_{-R}\mathcal{T}_{-R}\rangle_{NMM}^{N=2} \qquad (3.51)$$

The two are actually equal to each other because of the symmetry $X \to -X$, where X is the Euclidean time direction. We have calculated these correlators both from MQM and NMM (at $N = 2$) and the agreement is exactly as for the cases considered above.

3.4.4 Why it works

As we reviewed in Section 3.2, the Normal Matrix Model determines every momentum correlator by differentiation with respect to the momentum couplings $t, \bar{t}$. However, the correlators so obtained should only be correct in the limit $N \to \infty$ ("Model I", Eq. (3.14)) or the special values $N = \nu R$ ("Model II", Eq. (3.15)). Now in the previous subsections we have shown in several examples (including the infinite set of two-point functions) that, given the total momentum P flowing in the correlator, the NMM with matrices of any rank $N \geq P$ suffices to compute the correlator *completely* as a function of μ and R. In view of this, the NMM appears to go beyond its expected range of validity. Here we will give an explanation as to how this comes about.

The basic observation is that the phenomenon we are observing is not to be viewed as an application of Model II, but rather of Model I. Indeed, using Model II and a definite value of N, it is clear from Eq. (3.15) that the answers obtained are correct only for a definite value of ν, namely $\nu = N/R$. This relation between N and ν defines a line in (N, ν) space, and the points on this line where N takes integer values are the ones where the procedure works. However, it is clear that in this way one can never recover the full ν dependence at a fixed N.

In contrast, in Model I one is supposed to compute correlators at an arbitrarily large value of N and in the limit $N \to \infty$, the correct answers are obtained as a function of ν. What we will now show is that, after computing a given correlator of total momentum P in this way, and then dividing by the partition function, infinitely many terms cancel out exactly in the ratio. The remaining terms, which actually contribute to the correlator of interest, are the same as one would compute for a finite value of N, namely $N = P$.

The argument goes as follows. From the derivation we have given in the previous subsections and the appendices, any correlator is generated (after θ_i integrations) by inserting an expression of the form $\prod_{i=1}^{N} m_i^{\alpha_i}$ into the m_i integrals, where $\{\alpha_i\}$ correspond to ordered partitions of P. Therefore we should first of all choose N large enough so that all such partitions can be realised and are distinguishable. This is possible for $N \geq P$. For $N < P$ we will miss some partitions, and thus the answer cannot be correct. But the case $N > P$ realises the same partitions as the case $N = P$ and thus gives the same answer. This causes what we earlier called "stabilisation", which amounts to saying that the result for $N = P$ is identical to the result for any $N > P$, and therefore for $N = \infty$. Invoking the converse of stabilisation, we can therefore start with the model defined at $N = \infty$ and "bring back" the value of N to any finite value $N \geq P$ without changing the result. This explains why a finite-N matrix model is sufficient to compute any momentum correlator.

3.4.5 Combinatorial result for $2n$-point functions

We have shown that the NMM is an effective tool by re-computing known correlators. Now that we understand how and why it works, we apply it to compute a new result: the full (connected plus disconnected) $2n$-point function $\langle (T_{-1/R}T_{1/R})^n\rangle$ for every n and to all orders in perturbation theory. The result, derived in Appendix 3.7.4, is the following:

$$\langle (T_{-1/R}T_{1/R})^n\rangle = \left| \sum_{\{k_i\}} C(\{k_i\})^2 \prod_{i=1}^{n} \frac{\Gamma\left(\frac{1}{2} - i\mu + (k_i - n + \frac{1}{2})\frac{1}{R}\right)}{\Gamma\left(\frac{1}{2} - i\mu - (i - \frac{1}{2})\frac{1}{R}\right)} \right| \tag{3.52}$$

with $C(\{k_i\})$ defined as:

$$C(\{k_i\}) = \sum_{\mathcal{P}} (-1)^{\mathcal{P}} \prod_{i=1}^{n} \begin{pmatrix} n - \sum_{j=1}^{i-1}(k_j - \mathcal{P}_j) \\ k_i - \mathcal{P}_i \end{pmatrix} \tag{3.53}$$

Here, $\{k_i\}$ are strictly ordered partitions of $n(n+1)/2$, namely:

$$k_1 > k_2 > \cdots > k_n, \qquad \sum_{i=1}^{n} k_i = \frac{n(n+1)}{2} \tag{3.54}$$

and $\mathcal{P}$ denote permutations of the n numbers $n-1, n-2, \cdots, 0$.

Let us examine this result more closely. In principle, for every n the answer is a sum of terms, each one being the ratio of n Γ-functions divided by n Γ-functions. However in practice, some of the numerator and denominator terms can cancel out. We can see this more explicitly if we list the first few special cases, of which the first two have already been noted above:

$$\begin{aligned}
\langle T_{-1/R}T_{1/R}\rangle &= \left| \frac{\Gamma\left(\frac{1}{2} - i\mu + \frac{1}{2R}\right)}{\Gamma\left(\frac{1}{2} - i\mu - \frac{1}{2R}\right)} \right| \\
\langle (T_{-1/R}T_{1/R})^2\rangle &= \left| \frac{\Gamma\left(\frac{1}{2} - i\mu + \frac{3}{2R}\right)}{\Gamma\left(\frac{1}{2} - i\mu - \frac{1}{2R}\right)} + \frac{\Gamma\left(\frac{1}{2} - i\mu + \frac{1}{2R}\right)}{\Gamma\left(\frac{1}{2} - i\mu - \frac{3}{2R}\right)} \right| \\
\langle (T_{-1/R}T_{1/R})^3\rangle &= \left| \frac{\Gamma(\frac{1}{2} - i\mu + \frac{5}{2R})}{\Gamma(\frac{1}{2} - i\mu - \frac{1}{2R})} + 4\frac{\Gamma(\frac{1}{2} - i\mu + \frac{3}{2R})}{\Gamma(\frac{1}{2} - i\mu - \frac{3}{2R})} + \frac{\Gamma(\frac{1}{2} - i\mu + \frac{1}{2R})}{\Gamma(\frac{1}{2} - i\mu - \frac{5}{2R})} \right|
\end{aligned} \tag{3.55}$$

The pattern emerging so far is misleadingly simple, as we see with the next example, the 8-point function:

$$\begin{aligned}\langle (T_{-1/R}T_{1/R})^4\rangle &= \left| \frac{\Gamma(\frac{1}{2}-i\mu+\frac{7}{2R})}{\Gamma(\frac{1}{2}-i\mu-\frac{1}{2R})} + 9\,\frac{\Gamma(\frac{1}{2}-i\mu+\frac{5}{2R})}{\Gamma(\frac{1}{2}-i\mu-\frac{3}{2R})} + 9\,\frac{\Gamma(\frac{1}{2}-i\mu+\frac{3}{2R})}{\Gamma(\frac{1}{2}-i\mu-\frac{5}{2R})}\right. \\ &+ \left. \frac{\Gamma(\frac{1}{2}-i\mu+\frac{1}{2R})}{\Gamma(\frac{1}{2}-i\mu-\frac{7}{2R})} + 4\,\frac{\Gamma(\frac{1}{2}-i\mu+\frac{1}{2R})}{\Gamma(\frac{1}{2}-i\mu-\frac{1}{2R})}\frac{\Gamma(\frac{1}{2}-i\mu+\frac{3}{2R})}{\Gamma(\frac{1}{2}-i\mu-\frac{3}{2R})}\right| \end{aligned} \quad (3.56)$$

We see that as the number of operators in the correlator grows, one gets products of more and more Γ-functions in the numerator and denominator. In this example we also see clearly that the coefficients are perfect squares.

Ideally one would like to know the connected part of the $2n$-point function. In principle this can of course be obtained by repeated application of Eq. (3.52), but one would like a more explicit and useful expression. However, for the most likely application, to the vortex condensate, we will not really need to make the distinction between connected and disconnected correlators. The vortex condensate corresponds to the partition function of a perturbed theory, and to find the connected component of that it suffices to take a logarithm. We will discuss this issue further in the following section.

3.5 Applications

3.5.1 T-duality at $c = 1$

In this subsection we discuss how our results can be applied to check T-duality of the $c = 1$ matrix model. As we have seen, in the Euclidean (finite-temperature) MQM, the momentum and winding modes with respect to the time direction are independently defined. The former arise from macroscopic loops defined in terms of fermion bilinears, while the latter are Wilson-Polyakov loops in the thermal direction, which project the theory onto nonsinglet sectors. From the continuum description we expect that there should be T-duality between these two sets of observables. Indeed, in Ref.[27] it has been formally argued that, like the momentum-perturbed matrix model, the winding-perturbed MQM also corresponds to the τ-function of a Toda hierarchy. To understand T-duality better, one would like to compare explicit correlation functions computed from the momentum and winding sides.

An attempt to directly check T-duality was made by Maldacena in [28], where the following two quantities were compared: (i) the two-point function of unit-momentum tachyons, after T-duality, and (ii) the partition function of MQM in the adjoint sector. From Eq. (3.28) we see that (i) is equal to:

$$\langle \mathcal{T}_{-R}\mathcal{T}_{R}\rangle = \left| \frac{\Gamma\left(\frac{1}{2} - i\mu R + \frac{R}{2}\right)}{\Gamma\left(\frac{1}{2} - i\mu R - \frac{R}{2}\right)} \right| \tag{3.57}$$

However, at this point we recall that leg-pole factors of $\Gamma(-|q|)$ were dropped after Eq. (3.18). Restoring them and taking the large-μ asymptotics of this correlator, we find[1]:

$$\langle \mathcal{T}_{-R}\mathcal{T}_{R}\rangle = \big(\Gamma(-R)\big)^2 (\mu R)^R \left(1 + \frac{1}{24}\Big(R - \frac{1}{R}\Big)\mu^{-2} + \mathcal{O}\left(\mu^{-4}\right)\right) \tag{3.58}$$

On the other hand, (ii) is obtained by solving MQM in the adjoint sector. In the large N limit, Maldacena obtained the leading (tree level) contribution to the partition function in this sector as:

$$\frac{Z_{\text{adj}}}{Z_{\text{sing}}} = \langle \mathcal{W}_{\text{adj}}\rangle = \frac{1}{4\sin^2 \pi R}\,\mu^R = \frac{1}{4\pi^2}\big(\Gamma(R+1)\Gamma(-R)\big)^2 \mu^R \tag{3.59}$$

The power of μ agrees with that in the leading term of Eq. (3.58). The remaining discrepancy can be assigned to the normalisation of the fundamental Wilson-Polyakov loop (or equivalently to the normalisation of the original momentum modes), and we see that Eqs.(3.59) and (3.58) agree to leading order if we change the normalisation of this loop variable to:

$$\mathcal{W}_N \to \frac{1}{2\pi}\,\frac{R^{\frac{R}{2}}}{\Gamma(R+1)}\mathcal{W}_N \tag{3.60}$$

This is a relatively simple change of normalisation[2], and appears to specify the basis in which T-duality holds in MQM.

[1] The factor R^R was not written in Ref.[28].

[2] Notice that the normalisation factor becomes trivial at the special radius $R = 1$.

It is not entirely surprising that one needs to change normalisation of the matrix model observables in order to implement T-duality. Indeed, this duality is most manifest in the worldsheet or Liouville approach, in which the momentum and winding vertex operators come with a natural normalisation and are related to each other by the simple change $(X_L, X_R) \to (X_L, -X_R)$. On the matrix model side, momentum operators in the MQM are related to the corresponding Liouville operators by a change of normalisation, Eq. (3.4). So one should expect that winding operators in MQM are also related to Liouville winding operators by a change of normalisation.

This is not to say we understand the nature of these normalisation factors in general. In fact, as stressed in Ref.[28], we need more examples in order to check the consistency of this picture. As an example, if one could compute the genus-1 correction to the adjoint sector partition function, this could be compared with the genus-1 term in Eq. (3.58). Similarly, if one could compute the leading term for those higher representations that correspond to $2n$-point functions of the winding tachyon, then one could match this with the asymptotics of the latter, which can be read off from our results in Section 3.4.5.

There will also be representations corresponding to the correlators of multiply wound tachyons $\mathcal{T}_{nR}$. These correlators can be found by T-dualising the relevant momentum correlators, for example the two-point functions are found by T-dualising Eq. (3.25), leading to:

$$\begin{aligned} \langle \mathcal{T}_{-nR}\mathcal{T}_{nR}\rangle &= \big(\Gamma(-nR)\big)^2 \left| \sum_{r=1}^{n} \frac{\Gamma(\frac{1}{2} - i\mu R + (r-\frac{1}{2})R)}{\Gamma(\frac{1}{2} - i\mu R + (r-n-\frac{1}{2})R)} \right| \qquad (3.61) \\ &= n(\mu R)^{nR}\big(\Gamma(-nR)\big)^2 \left(1 - \frac{nR(nR-1)\big((n^2-1)R^2 - nR - 1\big)}{24R^2}\mu^{-2} + \mathcal{O}\left(\mu^{-4}\right) \right) \end{aligned}$$

In the matrix model, this should correspond to the fundamental Wilson-Polyakov loop with a contour that winds n times over the time direction. In principle we are allowed an independent choice of normalisation for each winding number. In fact the momentum and winding modes have corresponding freedoms in normalisation, and the only thing relevant for T-duality is the relative normalisation between them. So when we consider the nonsinglet sector related to multiply wound loops, and the corresponding tachyons of n units of momentum, the leading-order comparison will be used to fix the normalisation and the loop corrections will constitute a genuine check of T-duality.

To summarise, we have not been able to address the problem of T-duality but only set up one side of it. Namely, we have exhibited the all-orders finite-radius correlators computed from the momentum side, after performing a T-duality transformation. This constitutes a prediction to be checked once it is properly understood how to perform nonsinglet computations for different representations and to higher orders in string perturbation theory.

There is one more intriguing point that we would like to mention. The correlators we have computed take very special values at the selfdual radius $R = 1$, the point of enhanced $SU(2)$ symmetry. In particular, all loop corrections to the two-point function of unit momentum tachyons vanish, as can be seen from Eq. (3.28). Thus the tree level answer is exact[1]. By T-duality the same property should hold for the two-point function of unit winding modes. It is plausible that one could extract this simple property just from the structure of the nonsinglet Hamiltonian – in this case it is the adjoint Hamiltonian that was studied in Ref.[28], specialised to $R = 1$. Similarly, at $R = 1$ the other two-point functions have perturbation series that terminate at a finite number of loops, as one can easily check from Eq. (3.61). So, for consistency this must also be a property of the antisymmetric-antisymmetric representations referred to above. It may be simpler to derive this kind of general result in the nonsinglet sector than to actually compute coefficients with precision.

3.5.2 Vortex condensate and black holes

It is believed that the Euclidean 2D black hole background, defined in the continuum by an $SL(2,R/U(1)$ CFT, is equivalent to the $c = 1$ matrix model perturbed by fundamental Wilson-Polyakov loops:

$$S_{MQM} \to S_{MQM} + \lambda\, \mathcal{W}_N + \bar{\lambda}\, \mathcal{W}_{\bar{N}} \tag{3.62}$$

[1] This was already known long ago, for example as the puncture equation in the Kontsevich-Penner model[26].

The basis for this belief is the FZZ conjecture[36], which relates the black hole background to Sine-Liouville theory[1]. Via the equivalence in Eq. (3.8), the latter is the same as the perturbed background above.

To be precise, the FZZ conjecture is not really an either/or statement wherein one uses either the black hole background or the Sine-Liouville perturbation. It has increasingly become clear that the backgrounds that one might call "black hole" or "Sine-Liouville" are the same, and both perturbations are turned on simultaneously. Depending on the value of the worldsheet coupling, one or the other of these perturbations is more dominant, but for example the exact correlation functions have poles corresponding to both perturbations[2]. In the present work we will not focus on these details, but will be content to treat the black hole story as a motivation to understand the vortex condensate:

$$\langle e^{\lambda\,\mathcal{W}_N+\bar{\lambda}\,\mathcal{W}_{\bar{N}}}\rangle\Big|_{MQM} \tag{3.63}$$

One way to compute this condensate would be to sum over an infinite set of nonsinglet sectors in the MQM with some definite weights. However, as we have seen, the technology to do this seems rather limited at present. An alternative is to assume T-duality to compute the correlator:

$$\langle e^{\lambda\,\mathcal{T}_R+\bar{\lambda}\,\mathcal{T}_{-R}}\rangle=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\lambda^n}{n!}\frac{\bar{\lambda}^m}{m!}\langle(\mathcal{T}_R)^n(\mathcal{T}_{-R})^m\rangle=\sum_{n=0}^{\infty}\frac{|\lambda|^{2n}}{(n!)^2}\langle(\mathcal{T}_R\mathcal{T}_{-R})^n\rangle \tag{3.64}$$

where the last equality follows from conservation of winding number.

Now from the computation in Appendix 3.7.4, we have the following result after T-duality:

$$\langle(\mathcal{T}_{-R}\mathcal{T}_R)^n\rangle=\left|\sum_{\{k_i\}}C(\{k_i\})^2\prod_{i=1}^{n}\frac{\Gamma\left(\frac{1}{2}-i\mu R-(i+k_i-\frac{1}{2})R\right)}{\Gamma\left(\frac{1}{2}-i\mu R-(i-\frac{1}{2})R\right)}\right| \tag{3.65}$$

where $\{k_i\}$ are strictly ordered partitions of $n(n+1)/2$, and $C(\{k_i\})$ are the combinatorial coefficients given in Eq. (3.93).

[1] This conjecture has been proved by Hori and Kapustin[95] in the $\mathcal{N}=2$ supersymmetric case. As Maldacena has argued[28], suitably orbifolding both sides of their argument leads to a proof for the bosonic case.

[2] See for example Ref.[86]. We are grateful to Ari Pakman for explaining this to us.

The above correlators contain both connected and disconnected contributions. We can now pass to the generating function:

$$\langle e^{\lambda\,\mathcal{T}_R+\bar{\lambda}\,\mathcal{T}_{-R}}\rangle = \sum_{n=0}^{\infty}\frac{|\lambda|^{2n}}{(n!)^2}\left|\sum_{\{k_i\}}C(\{k_i\})^2\prod_{i=1}^{n}\frac{\Gamma\left(\frac{1}{2}-i\mu R-(i+k_i-\frac{1}{2})R\right)}{\Gamma\left(\frac{1}{2}-i\mu R-(i-\frac{1}{2})R\right)}\right| \quad (3.66)$$

This is the partition function in the presence of a vortex condensate, and its logarithm is the free energy of the perturbed theory. So one does not need at any point to compute individual connected correlators.

The above expression is completely explicit and does not require integrating any equation or developing a recursion relation. We expect it will be useful to to extract physical quantities of interest related to the Euclidean 2d black hole. This is beyond the scope of the present work, however, and we hope to return to a more detailed analysis of this formula in the future.

Again it is worth pointing out that at the selfdual radius $R=1$ the vortex condensate is known exactly, though deriving it from the above expression would not be the easiest way. The puncture equation of Ref.[26] simply tells us that:

$$\langle e^{\lambda\,\mathcal{T}_R+\bar{\lambda}\,\mathcal{T}_{-R}}\rangle|_{R=1} = |e^{-i\mu\lambda\bar{\lambda}}| \quad (3.67)$$

and one can check easily that this agrees with the cases in Eq. (3.55) specialised to $R=1$.

The significance for the Euclidean 2d black hole of this simple result has not, to our knowledge, been explored. While it is true that the black hole CFT corresponds to a radius $R=\frac{3}{2}$, it is believed[27] to have a marginal deformation to other radii at least in the range $1<R<2$. So the physical consequences of the simple formula above at $R=1$ would be worth understanding better.

3.6 Conclusions

In this work we have examined the familiar $c=1$ bosonic noncritical string theory, or rather its Euclidean (finite temperature) version, from the perspective of correlation functions. Both old and new techniques were used to develop simple, elegant and explicit formulae as functions of two variables: the cosmological constant μ and the compactification radius or inverse temperature R. The key results are summarised in Eqs.(3.25),(3.33),(3.52). In addition we have shown that the Normal Matrix Model is a powerful computational tool.

An obvious extension of this work would be to the case of noncritical type 0 strings[11],[10]. In Ref.[9], explicit expressions are obtained for the partition functions of type 0A and 0B strings in the presence of fluxes. These expressions are richer than the corresponding ones for the bosonic noncritical string, both because of the flux dependence and because they are nonperturbative in μ. Our work should generalise quite straightforwardly, particularly to the Euclidean type 0B case, and the correlators so obtained will contain nonperturbative information about the theory.

A detailed investigation into the physical questions that motivated the present exercise, namely a better understanding of the 2d black hole background as well as of T-duality in the matrix model, is left for subsequent work. We also note that the physical origin of the Normal Matrix Model has not yet been understood. As it is clearly a correct and useful description of the $c = 1$ string, and moreover makes sense only in the Euclidean context, it would be worth trying to put it on a similar footing as MQM in terms of the dynamics of some appropriate (Euclidean) D-branes.

3.7 Useful formulae

In the following sections we list some useful formulae and some details of the computation described in this chapter.

3.7.1 Computation of two-point functions in the NMM

Here we present some of the details of how to compute two-point functions in the Normal Matrix Model. To start with, for the partition function we have

$$\begin{aligned} \mathcal{Z}_{NMM}^{N=2}(t=0) &= \int d^2z_1 d^2z_2\, |z_1 - z_2|^2 \\ &\quad \times e^{-\nu\left((z_1\bar{z}_1)^R + (z_2\bar{z}_2)^R\right) + \left(R\nu - 2 + \frac{R-1}{2}\right)(\log z_1\bar{z}_1 + \log z_2\bar{z}_2)} \end{aligned} \tag{3.68}$$

As before, we change variables $z_i = \sqrt{m_i}\, e^{i\theta_i}$, $d^2 z_i \rangle dm_i\, d\theta_i$ and we get

$$\begin{aligned}
\mathcal{Z}_{NMM}^{N=2}(t=0) &= \int_0^\infty dm_1 dm_2 \int_0^{2\pi} d\theta_1 d\theta_2 \left(m_1 + m_2 - \sqrt{m_1 m_2}(e^{i\theta_{12}} + e^{-i\theta_{12}})\right) \\
&\quad \times e^{-\nu(m_1^R + m_2^R) + \left(R\nu - 2 + \frac{R-1}{2}\right)(\log m_1 + \log m_2)} \\
&= 4\pi^2 \int_0^\infty dm_1 dm_2\, (m_1 + m_2)(m_1 m_2)^{\left(R\nu - 2 + \frac{R-1}{2}\right)} e^{-\nu\left(m_1^R + m_2^R\right)} \\
&= \tfrac{8\pi^2}{R^2} \nu^{-\left(\nu + \frac{1}{2} - \frac{1}{2R}\right)} \Gamma\left(\nu + \tfrac{1}{2} - \tfrac{1}{2R}\right) \times \nu^{-\left(\nu + \frac{1}{2} - \frac{3}{2R}\right)} \Gamma\left(\nu + \tfrac{1}{2} - \tfrac{3}{2R}\right),
\end{aligned} \tag{3.69}$$

where $\theta_{12} \equiv \theta_1 - \theta_2$. In a similar manner we have

$$\begin{aligned}
\partial_{-2}\partial_2 \mathcal{Z}_{NMM}^{N=2}(t=0) &= \int d^2 z_1 d^2 z_2\, |z_1 - z_2|^2 (z_1^2 + z_2^2)(\bar{z}_1^2 + \bar{z}_2^2) \\
&\quad \times e^{-\nu\left((z_1\bar{z}_1)^R + (z_2\bar{z}_2)^R\right) + \left(R\nu - 2 + \frac{R-1}{2}\right)(\log z_1\bar{z}_1 + \log z_2\bar{z}_2)} \\
&= \int_0^\infty dm_1 dm_2 \int_0^{2\pi} d\theta_1 d\theta_2 \left(m_1 + m_2 - \sqrt{m_1 m_2}(e^{i\theta_{12}} + e^{-i\theta_{12}})\right) \\
&\quad \times \left(m_1^2 + m_2^2 + m_1 m_2 e^{2i\theta_{12}} + m_1 m_2 e^{-2i\theta_{12}}\right) e^{-\nu(m_1^R + m_2^R) + \left(R\nu - 2 + \frac{R-1}{2}\right)(\log m_1 + \log m_2)} \\
&= 4\pi^2 \int_0^\infty dm_1 dm_2\, (m_1 + m_2)(m_1^2 + m_2^2)(m_1 m_2)^{\left(R\nu - 2 + \frac{R-1}{2}\right)} e^{-\nu\left(m_1^R + m_2^R\right)}
\end{aligned}$$

Evaluating the integrals on m_1, m_2 we get

$$\begin{aligned}
\partial_{-2}\partial_2 \mathcal{Z}_{NMM}^{N=2}(t=0) = &\ \tfrac{8\pi^2}{R^2} \nu^{-\left(\nu + \frac{1}{2} + \frac{3}{2R}\right)} \Gamma\left(\nu + \tfrac{1}{2} + \tfrac{3}{2R}\right) \times \nu^{-\left(\nu + \frac{1}{2} - \frac{3}{2R}\right)} \Gamma\left(\nu + \tfrac{1}{2} - \tfrac{3}{2R}\right) \\
&+ \tfrac{8\pi^2}{R^2} \nu^{-\left(\nu + \frac{1}{2} + \frac{1}{2R}\right)} \Gamma\left(\nu + \tfrac{1}{2} + \tfrac{1}{2R}\right) \times \nu^{-\left(\nu + \frac{1}{2} - \frac{1}{2R}\right)} \Gamma\left(\nu + \tfrac{1}{2} - \tfrac{1}{2R}\right)
\end{aligned} \tag{3.70}$$

From Eq. (3.69) and Eq. (3.70) we have

$$\langle T_{-2/R} T_{2/R} \rangle_{NMM}^{N=2} = \nu^{-2/R} \left(\frac{\Gamma\left(\nu + \frac{1}{2} + \frac{3}{2R}\right)}{\Gamma\left(\nu + \frac{1}{2} - \frac{1}{2R}\right)} + \frac{\Gamma\left(\nu + \frac{1}{2} + \frac{1}{2R}\right)}{\Gamma\left(\nu + \frac{1}{2} - \frac{3}{2R}\right)} \right) \tag{3.71}$$

As before, to get the correct two point function we have to analytically continue $\nu = -i\mu$ and take the modulus of the above expression. This gives:

$$\langle T_{-2/R} T_{2/R} \rangle_{NMM}^{N=2} = \mu^{-2/R} \left| \frac{\Gamma\left(\frac{1}{2} - i\mu + \frac{3}{2R}\right)}{\Gamma\left(\frac{1}{2} - i\mu - \frac{1}{2R}\right)} + \frac{\Gamma\left(\frac{1}{2} - i\mu + \frac{1}{2R}\right)}{\Gamma\left(\frac{1}{2} - i\mu - \frac{3}{2R}\right)} \right| \tag{3.72}$$

3.7.2 Evaluation of a summation in the MQM four-point function

In order to show the equivalence between Eqs. (3.32) and (3.33) we need to prove the following identity:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\Gamma(-\frac{1}{R}+n)}{\Gamma(-\frac{1}{R})} \right)^2 \frac{\Gamma\left(\frac{1}{2} - i\mu + \frac{3}{2R} - n\right)}{\Gamma\left(\frac{1}{2} - i\mu - \frac{1}{2R}\right)} = \left(\frac{\Gamma\left(\frac{1}{2} - i\mu + \frac{1}{2R}\right)}{\Gamma\left(\frac{1}{2} - i\mu - \frac{1}{2R}\right)} \right)^2 \tag{3.73}$$

Let us start with the expression:

$$\mathcal{E} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\Gamma(-\frac{1}{R}+n)}{\Gamma(-\frac{1}{R})} \right)^2 \Gamma\left(\tfrac{1}{2} - i\mu + \tfrac{3}{2R} - n\right) \tag{3.74}$$

Using the integral representation of the Γ function we write this as:

$$\mathcal{E} = \frac{1}{\left(\Gamma(-\frac{1}{R})\right)^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int d^3t \ (t_1 t_2)^{-\frac{1}{R}+n-1} t_3^{-n+\frac{1}{2}-i\mu+\frac{3}{2R}-1} e^{-t_1-t_2-t_3} \tag{3.75}$$

The sum over n can now be performed immediately and we have:

$$\begin{aligned} \mathcal{E} &= \frac{1}{\left(\Gamma(-\frac{1}{R})\right)^2} \int d^3t \ e^{-\frac{t_1 t_2}{t_3}} (t_1 t_2)^{-\frac{1}{R}-1} \, t_3^{\frac{1}{2}-i\mu+\frac{3}{2R}-1} \ e^{-t_1-t_2-t_3} \\ &= \frac{1}{\left(\Gamma(-\frac{1}{R})\right)^2} \int d^3t \ e^{-t_1(1+\frac{t_2}{t_3})} (t_1 t_2)^{-\frac{1}{R}-1} \, t_3^{\frac{1}{2}-i\mu+\frac{3}{2R}-1} \ e^{-t_2-t_3} \end{aligned} \tag{3.76}$$

Using the change of variables $t_1 \to t_1(1+\frac{t_2}{t_3})$ and performing the integral on t_1 we get:

$$\mathcal{E} = \frac{1}{\Gamma(-\frac{1}{R})} \int d^2t \ t_2^{-\frac{1}{R}-1} \, t_3^{\frac{1}{2}-i\mu+\frac{3}{2R}-1} \, t_3^{-\frac{1}{R}} \, (t_2+t_3)^{\frac{1}{R}} \, e^{-t_2-t_3} \tag{3.77}$$

We next introduce a parameter α which allows us to write the above equation as:

$$\mathcal{E} = \frac{1}{\Gamma(-\frac{1}{R})} \left(-\frac{\partial}{\partial\alpha} \right)^{\frac{1}{R}} \int d^2t \ t_2^{-\frac{1}{R}-1} \, t_3^{\frac{1}{2}-i\mu+\frac{1}{2R}-1} \ e^{-\alpha(t_2+t_3)} \Big|_{\alpha=1} \tag{3.78}$$

Changing variables $t_i \to \alpha t_i$ we have:

$$\begin{aligned} \mathcal{E} &= \frac{1}{\Gamma(-\frac{1}{R})} \left(-\frac{\partial}{\partial\alpha} \right)^{\frac{1}{R}} \alpha^{-\frac{1}{2}+i\mu+\frac{1}{2R}} \Big|_{\alpha=1} \int dt_2 \ t_2^{-\frac{1}{R}-1} \ e^{-t_2} \int dt_3 \ t_3^{\frac{1}{2}-i\mu+\frac{1}{2R}-1} \ e^{-t_3} \\ &= \Gamma(\tfrac{1}{2} - i\mu + \tfrac{1}{2R}) \left(-\frac{\partial}{\partial\alpha} \right)^{\frac{1}{R}} \alpha^{-\frac{1}{2}+i\mu+\frac{1}{2R}} \Big|_{\alpha=1} \end{aligned} \tag{3.79}$$

Using the relation:

$$\left(-\frac{\partial}{\partial\alpha}\right)^m \alpha^n\Big|_{\alpha=1} = \frac{\Gamma(-n+m)}{\Gamma(-n)} \tag{3.80}$$

we finally have:

$$\mathcal{E} = \frac{\left(\Gamma(\frac{1}{2} - i\mu + \frac{1}{2R})\right)^2}{\Gamma(\frac{1}{2} - i\mu - \frac{1}{2R})} \tag{3.81}$$

Using Eq. (3.74) and dividing both sides by $\Gamma(\frac{1}{2} - i\mu - \frac{1}{2R})$ we immediately get Eq. (3.73).

3.7.3 Four-point function in NMM

We now briefly describe the calculation of the connected four-point function of unit momentum modes in the NMM. This is obtained by differentiating the free energy $\mathcal{F}$ with respect to the couplings. We thus have:

$$\begin{aligned}\langle (T_{-1/R}T_{1/R})^2\rangle &= \partial_{-1}^2\partial_1^2\mathcal{F} \\ &= \langle (T_{-1/R}T_{1/R})^2\rangle^{\text{disconn}} - 2\langle T_{-1/R}T_{1/R}\rangle^2 \end{aligned} \tag{3.82}$$

where $\mathcal{F} = \ln \mathcal{Z}_{NMM}$. The second term in the above equation can be calculated from the NMM with $N=2$ and is given by:

$$\begin{aligned}\langle (T_{-1/R}T_{1/R})^2\rangle^{\text{disconn}} &= \langle (\text{tr}Z^\dagger)^2(\text{tr}Z)^2\rangle_{NMM}^{N=2} \\ &= \nu^{-2/R}\left(\frac{\Gamma\left(\nu+\frac{1}{2}+\frac{3}{2R}\right)}{\Gamma\left(\nu+\frac{1}{2}-\frac{1}{2R}\right)} + \frac{\Gamma\left(\nu+\frac{1}{2}+\frac{1}{2R}\right)}{\Gamma\left(\nu+\frac{1}{2}-\frac{3}{2R}\right)}\right)\end{aligned}$$

The explicit calculation is very similar to the calculation of $\langle T_{-2/R}T_{2/R}\rangle$ from the NMM. The disconnected piece is simply the square of the two-point function listed in Eq. (3.37). Putting everything together the connected four-point function is given by:

$$\begin{aligned}\langle (T_{-1/R}T_{1/R})^2\rangle_{NMM}^{\text{conn}} &= \nu^{-2/R}\left[\frac{\Gamma\left(\nu+\frac{1}{2}+\frac{3}{2R}\right)}{\Gamma\left(\nu+\frac{1}{2}-\frac{1}{2R}\right)} + \frac{\Gamma\left(\nu+\frac{1}{2}+\frac{1}{2R}\right)}{\Gamma\left(\nu+\frac{1}{2}-\frac{3}{2R}\right)}\right. \\ &\qquad \left. -2\left(\frac{\Gamma(n+\frac{1}{2}+\frac{1}{2R})}{\Gamma(n+\frac{1}{2}+\frac{1}{2R})}\right)^2\right]\end{aligned} \tag{3.83}$$

Analytically continuing $\nu = -i\mu$ and taking the modulus, and then using the definition of $\mathcal{F}^+$ in Eq. (3.21), we finally get:

$$\langle (T_{-1/R}T_{1/R})^2\rangle^{\text{conn}}_{NMM} = (\mu)^{-2/R}\left|\mathcal{F}^+(\tfrac{3}{2R},-\tfrac{1}{2R};\mu) - \tfrac{1}{2}\Big(\mathcal{F}^+(\tfrac{1}{2R},-\tfrac{1}{2R};\mu)\Big)^2\right| \quad (3.84)$$

3.7.4 $2n$-point functions in NMM

Here we present the detailed calculation of the $2n$-point functions from the NMM. In what follows we will take the rank of the matrix, N, to be equal to n. We have:

$$(\partial_{-1}\partial_1)^n \mathcal{Z}^{N=n}_{NMM}(t=0) = \int \prod_{i=1}^n d^2 z_i \prod_{i<j}^n |z_i - z_j|^2 \left(\sum_{i=1}^n z_i\right)^n \left(\sum_{i=1}^n \bar{z}_i\right)^n$$
$$\times e^{-\nu\sum_{r=1}^n (z_r\bar{z}_r)^R + \left(R\nu - n + \frac{R-1}{2}\right)\sum_{r=1}^n \log z_r\bar{z}_r} \quad (3.85)$$

We would now like to make the substitution $z_i = \sqrt{m_i}\, e^{i\theta_i}$ and perform the θ integrals. The remaining integrand will then be a function of the m_i and we will find that it has the form $\Big(\sum_{\{k_i\}} C(\{k_i\})^2 \prod_i m_i^{k_i} + \text{permutations}\Big)\, e^{-S_{NMM}}$. Here $\{k_i\}$ are positive integers corresponding to *strictly ordered* partitions of $n(n+1)/2$, i.e.:

$$\sum_{i=1}^n k_i = \tfrac{n(n+1)}{2}, \quad k_1 > k_2 > \cdots > k_n \geq 0 \quad (3.86)$$

The permutations referred to are of the m_i. Because the m_i are integration variables, summing over permutations simply amounts to multiplying by a factor of $n!$. The constant coefficients have been labelled $C(\{k_i\})^2$ in anticipation of the fact that they will turn out to be squares. After performing the integration over m_i and dividing by Z_{NMM} we get the final answer as a sum of ratios of products of gamma functions, with each term in the sum corresponding to a strictly ordered partition $\{k_i\}$ of $n(n+1)/2$.

We will first show that the coefficients are perfect squares $C(\{k_i\})^2$. After that we will turn to the calculation of the $C(\{k_i\})$. Consider the expression:

$$\mathcal{U} = \left(\sum_{i=1}^n z_i\right)^n \prod_{j<k}^n (z_j - z_k). \quad (3.87)$$

The full integrand is then $\mathcal{U}\bar{\mathcal{U}}$ times the exponential factor. Because the action is independent of the θ's, the entire θ-dependence of the integrand is in $\mathcal{U}\bar{\mathcal{U}}$. Note that $\mathcal{U}$ has only positive powers of $e^{i\theta_i}$ and $\bar{\mathcal{U}}$ has only negative powers. Only the θ-independent terms in the expansion of $\mathcal{U}\bar{\mathcal{U}}$ will survive the θ integrals.

It is easy to see that if we expand $\mathcal{U}, \bar{\mathcal{U}}$ then we get:

$$\begin{aligned} \mathcal{U} &= \sum_{\{\alpha_i\}} C(\{k_i\}) \prod_{i=1}^{n} z_i^{k_i} + \text{permutations} \\ \bar{\mathcal{U}} &= \sum_{\{\alpha_i\}} C(\{k_i\}) \prod_{i=1}^{n} \bar{z}_i^{k_i} + \text{permutations}, \end{aligned} \tag{3.88}$$

with $\{k_i\}$ defined as before. It is now clear that the coefficients of θ-independent terms in $\mathcal{T}\bar{\mathcal{T}}$ must be perfect squares, as the phase of a term in the first expression of Eq. (3.88) can only be cancelled by the complex conjugate term from the second expression, which has the same coefficient as the first term.

Let us now determine the coefficients $C(\{\alpha_i\})$. First we note the following property of the positive phase part of the Vandermonde:

$$\prod_{j<k}^{n} (z_j - z_k) = \sum_{\mathcal{P}} (-1)^{\mathcal{P}} \prod_{j=1}^{n} z_j^{\mathcal{P}_j}, \tag{3.89}$$

where $\mathcal{P}$ is a particular permutation of the n integers $(n-1, n-2, \cdots, 0)$ and $\mathcal{P}_j$ denotes the j^{th} element of the permutation $\mathcal{P}$[1]. The sign for the first permutation is positive by construction. Any other permutation can be arrived at by a series of interchanges $z_i \leftrightarrow z_j$. Each such interchange introduces a minus sign in the Vandermonde. Thus even permutations have a positive sign, while odd permutations have a negative sign, leading to Eq. (3.89). Expanding the first factor in Eq. (3.87) in a multinomial series and using Eq. (3.89) we get:

$$\begin{aligned} \mathcal{U} &= \left(\sum_{\{\beta_i\}} \prod_{i=1}^{n} \binom{n - \sum_{j=1}^{i-1} \beta_j}{\beta_i} z_i^{\beta_i} \right) \left(\sum_{\mathcal{P}} (-1)^{\mathcal{P}} \prod_{j=1}^{n} z_j^{\mathcal{P}_j} \right) \\ &= \sum_{\{\beta_i\}} \sum_{\mathcal{P}} (-1)^{\mathcal{P}} \prod_{i=1}^{n} \binom{n - \sum_{j=1}^{i-1} \beta_j}{\beta_i} z_i^{\beta_i + \mathcal{P}_i} \end{aligned} \tag{3.90}$$

[1]For example, $\mathcal{P}_j = n - j$ when $\mathcal{P}$ is the identity permutation.

where $\{\beta_i\}$ are the *unordered* partitions of n.

Let us examine the possible values of the exponent $k_i = \beta_i + \mathcal{P}_i$ in the above. If $k_i = k_j$ for some $i \neq j$ then the corresponding coefficient is zero. This can be traced back to the fact that the expression Eq. (3.87) is odd under pairwise interchange of the z's. Therefore we can rewrite the above as:

$$\mathcal{U} = \sum_{\substack{k_i \neq k_j \\ \sum_i k_i = n(n+1)/2}} \sum_{\mathcal{P}} (-1)^{\mathcal{P}} \prod_{i=1}^{n} \begin{pmatrix} n - \sum_{j=1}^{i-1}(k_j - \mathcal{P}_j) \\ k_i - \mathcal{P}_i \end{pmatrix} \prod_{i=1}^{n} z_i^{k_i} \tag{3.91}$$

Because the k_i are all distinct, we can limit ourselves to strictly ordered sets satisfying $k_1 > k_2 > \cdots k_n$. The other orderings are obtained by permuting these ones, or equivalently by permuting the z_i's. Thus we have:

$$\mathcal{U} = \sum_{\substack{k_1 > k_2 > \cdots > k_n \\ \sum_i k_i = n(n+1)/2}} C(\{k_i\}) \prod_{i=1}^{n} z_i^{k_i} + (\text{permutations of } z_i) \tag{3.92}$$

with

$$C(\{k_i\}) = \sum_{\mathcal{P}} (-1)^{\mathcal{P}} \prod_{i=1}^{n} \begin{pmatrix} n - \sum_{j=1}^{i-1}(k_j - \mathcal{P}_j) \\ k_i - \mathcal{P}_i \end{pmatrix} \tag{3.93}$$

Finally we combine $\mathcal{U}$ with $\bar{\mathcal{U}}$ and integrate over the angles to get:

$$\begin{aligned}(\partial_{-1}\partial_1)^n \mathcal{Z}_{NMM}^{N=n} &= (2\pi)^n \int \prod_{i=1}^{n} dm_i \sum_{\{k_i\}} C(\{k_i\})^2 \prod_{i=1}^{n} m_i^{k_i} e^{\sum_{i=1}^{n}(-\nu m_i^R + (R\nu - n + \frac{R-1}{2}) \log m_i)} \\ &\quad + \text{permutations} \qquad (3.94) \\ &= (2\pi)^n n! \sum_{\{k_i\}} C(\{k_i\})^2 \prod_{i=1}^{n} \nu^{-\left(\frac{1}{2} + \nu + (k_i - n + \frac{1}{2})\frac{1}{R}\right)} \Gamma\left(\tfrac{1}{2} + \nu + (k_i - n + \tfrac{1}{2})\tfrac{1}{R}\right)\end{aligned}$$

Using the expression for the partition function $\mathcal{Z}_{NMM}$ from Eq. (3.45) for $N = n$ we have:

$$\frac{(\partial_{-1}\partial_1)^n \mathcal{Z}_{NMM}^{N=n}}{\mathcal{Z}_{NMM}^{N=n}} = \nu^{-n/R} \sum_{\{k_i\}} C(\{k_i\})^2 \prod_{i=1}^{n} \frac{\Gamma\left(\frac{1}{2} + \nu + (k_i - n + \frac{1}{2})\frac{1}{R}\right)}{\Gamma\left(\frac{1}{2} + \nu - (i - \frac{1}{2})\frac{1}{R}\right)} \tag{3.95}$$

The $2n$-point function is given by analytically continuing $\nu = -i\mu$, changing to MQM normalisation using Eq. (3.39) (which amounts to removing the power of

ν in front), and finally taking the modulus:

$$\langle (T_{-1/R} T_{1/R})^n \rangle = \left| \sum_{\{k_i\}} C(\{k_i\})^2 \prod_{i=1}^{n} \frac{\Gamma\left(\frac{1}{2} - i\mu + (k_i - n + \frac{1}{2})\frac{1}{R}\right)}{\Gamma\left(\frac{1}{2} - i\mu - (i - \frac{1}{2})\frac{1}{R}\right)} \right| \tag{3.96}$$

with $C(\{k_i\})$ given by Eq. (3.93).

Chapter 4

FZZ Algebra

We have briefly discussed the connection between the vortex condensate of Sine-Liouville theory and the two dimensional black hole in Chapter 3. We revisit the duality in this chapter by considering the two possible Sine-Liouville dressings together. We show that this choice is consistent with the structure of correlation functions, and that the OPE of the two dressings yields the black hole deformation operator. As an application of this approach, we investigate the role of higher winding perturbations in the context of $c = 1$ strings, where we argue that they are related to higher-spin discrete states that generalize the 2d black hole operator [96].

4.1 Introduction

It has long been known that the bosonic string admits a two-dimensional black-hole like background, described as a gauged $SL(2, R)/U(1)$ WZW model[35] and can also be thought of for some values of the parameters as a solution of the lowest order (in α') effective action[97, 98]. Moreover, it was shown that when viewed as a perturbation of the $c = 1$ string theory, the leading term in this solution uniquely extends to a full solution of closed string field theory[37].

Some years ago, Fateev, Zamolodchikov and Zamolodchikov[36] proposed that the gauged $SL(2, R)/U(1)$ CFT has a dual description in terms of a free theory (with a linear dilaton) perturbed by a Sine-Liouville potential. This remarkable relation between two seemingly different models, the so called FZZ duality, has been explored and applied in several ways (see for example [27, 99, 100, 101, 102, 103]). The duality has an $N = 2$ supersymmetric version [95, 104, 105], as well as a realization on the boundary of the worldsheet [106, 107].

On the other hand, progress made during the last few years in the study of non-rational conformal field theories (see [5, 108, 109] for reviews) has shown that both dressings of Liouville-like perturbations in linear dilaton theories appear in the exact solutions [110]. The latter typically have two classical limits, and in each limit one of the two perturbations disappears. This suggests that the classically vanishing operator is a non-perturbative quantum effect generated by the backreaction of the first one.

Therefore it is natural to consider a Sine-Liouville theory where both dressings are taken into account and to ask how the FZZ duality fits in such a setting. In this work we propose an answer to this question which gives a new perspective on the FZZ duality. Our approach is based on the observation that the OPE of the two Sine-Liouville dressings yields the black hole perturbation. Therefore the latter operator closes a sort of algebra which we have dubbed the FZZ algebra. In order to preserve the exact marginality of the perturbations, the black hole operator should then be added to the action. Finally, a perturbative computation will show that the coefficient in front of the second sine-Liouville should be put to zero. In this way, the standard form of the FZZ duality is recovered, with just one Sine-Liouville perturbation along with that of the black hole.

This approach to the FZZ duality suggests in turn a natural generalization of the FZZ algebra in the $c = 1$ non-critical string context. This is motivated by the fact that in this case, the two-dimensional black hole is the first of an infinite family of solutions to the closed string field theory equations, each one corresponding to one of the discrete states of the $c = 1$ string [37]. We find that when Sine-Liouville perturbations with different winding numbers are turned on, all the discrete states of the $c = 1$ can be generated by multiple OPEs. This strongly points to the existence of an infinitely generalized FZZ duality in the

$c = 1$ string, which should be further investigated.

The organization of this work is as follows. In Section 2 we briefly review the two-dimensional black hole background and the FZZ duality. In Section 3 we introduce the FZZ algebra, we show that all the interactions involved are compatible with the parafermionic symmetry of the $SL(2,R)/U(1)$ coset and that the second Sine-Liouville dressing is consistent with the correlation functions of the theory. In Section 4, we briefly review the $c = 1$ string theory and present our proposal for the enlargement of the FZZ algebra in this model. Section 5 contains the conclusions.

4.2 Euclidean 2d Black Hole and FZZ Duality

4.2.1 2d Black Hole - Review

We start by reviewing some basic properties of the two dimensional cigar or black hole solution in noncritical string theory [35, 97, 98] This will also serve to establish some notation and conventions.

This black hole solution can be written as an exact conformal field theory (all orders in α'), namely an $SL(2,R)/U(1)$ WZW model [35], whose Euclidean version is a σ-model with metric:

$$\begin{aligned} ds^2 &= k\left((1-e^{2Q\phi})\,dt^2 + \frac{1}{1-e^{2Q\phi}}d\phi^2\right) , \\ \Phi - \Phi_0 &= Q\phi, \qquad -\infty < \phi < 0 \ . \end{aligned} \tag{4.1}$$

Here k is the level of the $SL(2,R)$ WZW model and $Q = \frac{1}{\sqrt{k-2}}$.

By a change of coordinates, this solution can also be written:

$$\begin{aligned} ds^2 &= k\left(dr^2 + \tanh^2 r\, d\theta^2\right) \\ \Phi - \Phi_0 &= -2\log\cosh r, \qquad -\infty < r < 0\,. \end{aligned} \tag{4.2}$$

The geometry of the Euclidean black hole is that of a cigar ending at $r = 0$. Its asymptotic radius as $r \to -\infty$ is

$$R = \sqrt{k}\,. \tag{4.3}$$

The value of the dilaton at the tip, Φ_0, can be identified with the mass of the black hole:

$$M \sim e^{-2\Phi_0}\,. \tag{4.4}$$

The 2d black hole can either be considered by itself as a string background, or adjoined to another "internal" CFT to form the total string background. In the former case, conformal invariance of the worldsheet theory requires:

$$c_{tot} = \frac{3k}{k-2} - 1 = 26 \quad \Rightarrow \quad k = \tfrac{9}{4} \quad \Rightarrow \quad R = \tfrac{3}{2}\,. \tag{4.5}$$

Therefore in this case one is in the regime of small k, and the spacetime solution is not very reliable. On the other hand, if we add an internal CFT then it is easy to see that k can be arbitrarily large and one expects the spacetime solution to be a reliable guide to the physics. In this and the next few sections we will assume the most general situation, with k arbitrary. Later we will specialise to the case where there is only a black hole and no internal CFT.

An important role will be played by the fact that the 2d black hole background has a parafermionic $SL(2,R)/U(1)$ symmetry. Note that a cosmological Liouville perturbation would spoil this symmetry. Hence we assume there is no cosmological perturbation, which is physically acceptable since the would-be strong coupling region is already cut off by the black hole geometry.

For large negative ϕ, the black hole metric can be written

$$\begin{aligned} ds^2 &= k\Big((1-e^{2Q\phi})\,dt^2 + (1+e^{2Q\phi})d\phi^2\Big)\,, \\ &= k\Big(dt^2 + d\phi^2 - (dt^2 - d\phi^2)e^{2Q\phi}\Big)\,. \end{aligned} \tag{4.6}$$

Thus, infinitesimally the black hole is generated by a perturbation

$$\Delta S = (\partial X \bar{\partial} X - \partial\phi\bar{\partial}\phi)e^{2Q\phi}\,. \tag{4.7}$$

The second term is a pure gauge in BRST cohomology. Therefore the black hole background is generated by the operator:

$$B = \partial X \bar{\partial} X e^{2Q\phi}\,. \tag{4.8}$$

It should be kept in mind that this operator only describes the 2d black hole far away from the horizon, in the weak-coupling region $\phi \to -\infty$. However, it unambiguously generates the full solution, in the sense that a CFT perturbed by B will flow to the CFT of the Euclidean 2d black hole[37]. In this process the spacetime gets cut off at the horizon, leading to the well-known property that winding number is violated: a string wrapped around the Euclidean time direction in the asymptotic region can be slipped off at the horizon. Violation of winding number is not, however, evident from inspection of the operator B, which by itself conserves winding number.

4.2.2 FZZ Duality

The FZZ duality [36] states that the Euclidean 2d black hole discussed above is "dual" to the Sine-Liouville perturbation of the linear dilaton theory.

The latter arises by coupling a compact "matter" coordinate X to the Liouville field. Since X is compact, it can be split into $X = X_L + X_R$. Let us normalize the holomorphic fields X_L and $\phi(z)$ as ($\alpha' = 1$)

$$X(z)X(w) \sim \phi(z)\phi(w) \sim -\frac{1}{2}\log(z-w)\,. \tag{4.9}$$

and similarly for the anti-holomorphic fields $X_R, \phi(\bar{z})$. The worldsheet stress tensor is

$$T = -(\partial X)^2 - (\partial\phi)^2 + Q\partial^2\phi, \tag{4.10}$$

with

$$Q = \frac{1}{\sqrt{k-2}}\,, \tag{4.11}$$

and the central charge is

$$c = 2 + 6Q^2\,, \tag{4.12}$$

which is the same as Eq. (4.5). The linear dilaton is given by

$$\Phi - \Phi_0 = Q\phi\,, \tag{4.13}$$

so that, with $g_s = e^{\Phi_0} e^{Q\phi}$, the theory is weakly coupled as $\phi \to -\infty$.

The vertex operators of this theory are written:

$$V_{\alpha,\beta} = e^{2i\alpha X} e^{2\beta\phi}\,, \tag{4.14}$$

and have conformal dimension

$$\Delta = \alpha^2 + \beta(Q - \beta)\,. \tag{4.15}$$

The wave function corresponding to these operators is obtained by multiplying by $g_s^{-1} \sim e^{-Q\phi}$. It follows that whenever $\beta < \frac{Q}{2}$ the wave function is non-normalizable, in that it is peaked about the weak-coupling region $\phi \to -\infty$. This is sometimes called the "allowed" dressing. Its insertion creates a local deformation of the worldsheet. For $\beta > \frac{Q}{2}$ the wave function decays at weak coupling and is normalizable, and its insertion creates a non-local deformation.

If the theory is perturbed by an operator that creates a "wall" at strong coupling, the situation is different. Only one linear combination of right-moving and left-moving waves survives. As a consequence, the corresponding Euclidean operator will be a linear combination of normalizable and non-normalizable ones.

Now let us introduce the Sine-Liouville perturbations:

$$\mathcal{T}^{+}_{\pm R} = e^{\pm iR(X_L - X_R)}\, e^{(Q - |Q - \frac{1}{Q}|)\phi}\,, \tag{4.16}$$

where as before, $R = \sqrt{k}$. The subscript labels the "winding momentum" for the matter part of the vertex operator, while the sign in the superscript labels the Liouville dressing. In particular, the above operators both have the "allowed" value of the Liouville dressing, so that the corresponding wave-functions grow at weak coupling and are non-normalizable. These operators carry winding number ± 1 around the Euclidean time direction.

The FZZ duality states that the 2d black hole theory is equivalent to Sine-Liouville. One of our goals in what follows will be to make this notion more precise. However first let us review the existing evidence for this duality. It comes from the knowledge of the exact two- and three-point functions (on the sphere) of the 2d black hole theory. For example, the two-point function, which we will re-obtain below, is:

$$R(j,m,\bar{m}) = \left(\frac{\mu\pi\Gamma(\frac{1}{k-2})}{\Gamma(1-\frac{1}{k-2})}\right)^{1-2j} \frac{\Gamma(2j-1)\Gamma(1+\frac{2j-1}{k-2})}{\Gamma(-2j+1)\Gamma(1-\frac{2j-1}{k-2})} \frac{\Gamma(-j+1+\bar{m})\Gamma(-j+1-m)}{\Gamma(j+\bar{m})\Gamma(j-m)} . \tag{4.17}$$

The poles in the first two Γ-functions of the numerator reflect the noncompact nature of the target space [111]. It can be shown that the positions of the poles of the first Γ-function, occurring at

$$j = 0, -\tfrac{1}{2}, -1, -\tfrac{3}{2}, \cdots \tag{4.18}$$

can be obtained using the black hole operator as a screening charge, and the residues at these poles can be computed using free field techniques. Together this determines the above correlator. On the other hand, the poles of the second Γ-function, at

$$1 + \frac{2j-1}{k-2} = 0, -1, -2, \cdots \tag{4.19}$$

can be obtained using the Sine-Liouville operator as the screening charge, and their residues again give the remaining factors in the correlator. The agreement has also been shown to hold for three point functions in [99, 101], for processes conserving and violating winding number.

It is intriguing that this duality works quite similarly to channel duality in critical string theory, where summing over the residues at the s-channel poles gives the same answer as summing over the residues at the t-channel poles. We are not aware if this similarity has any further implications.

4.3 FZZ Algebra

Let us now study the linear dilaton theory with a Sine-Liouville perturbation. In previous treatments it has been standard to add to the worldsheet action just one of the two "dressings" of the Sine-Liouville operator[1]. Here we will start

[1] In fact, in [27], the dressing is chosen to connect to the semiclassical limit valid for $Q \to \infty$. This is normalizable for $Q < 1$ and becomes non-normalizable for $Q > 1$.

by considering simultaneously both sine-Liouville dressings. This point of view has already been demonstrated to be useful in similar contexts (see for example [110]).

As we will see shortly, this approach provides a direct link between Sine-Liouville theory and the 2d black hole. We will demonstrate that the linear dilaton theory perturbed by both dressings of Sine-Liouville operators *requires* the black hole perturbation operator to be turned on for consistency, i.e. exact marginality of the perturbation. So it would seem that the true perturbed theory has both Sine-Liouville and black hole operators turned on at the same time. But this is not the end of the story. A perturbative quantum computation will show that the relative coefficients between all the perturbations are determined self-consistently, in such a way that the coefficient of the second Sine-Liouville perturbation should be set to zero. In other words, the second Sine-Liouville perturbation disappears after fulfilling the role of forcing the black hole perturbation to be present.

We will first describe the general arguments justifying this procedure, and will then show that using various different combinations of the Sine-Liouville operators (of both dressings) and black hole operator as screeners is consistent with the structure of the correlators of the theory and fixes the relative coefficients of the different perturbations. A certain parafermionic symmetry will prove useful in the discussion.

4.3.1 The Algebra of the Interactions

The perturbation to the action is as follows[1]

$$S \to S + \int d^2z \left(\mathcal{T}^+_R + \mathcal{T}^+_{-R} + \mathcal{T}^-_R + \mathcal{T}^-_{-R}\right) . \tag{4.20}$$

Between them, these four terms incorporate both signs of the matter momentum as well as both signs of the Liouville dressing. The operators $\mathcal{T}^+_{\pm R}$ were given in Eq. (4.16) while the other two are given by:

$$\mathcal{T}^-_{\pm R} = e^{\pm iR(X_L - X_R)}\, e^{(Q+|Q-\frac{1}{Q}|)\phi} , \tag{4.21}$$

[1]This is really Cosine-Liouville rather than Sine-Liouville.

As these operators are all of conformal dimension $(1,1)$, the above perturbation is marginal to first order. Now consider the requirements of exact marginality. The general rule is that the perturbations will be exactly marginal if their OPE does not produce another $(1,1)$ operator[2]. However, if they do produce such an operator then we are required to add that operator back into the Lagrangian to restore marginality.

Now it is easily seen that the following OPE holds between the mutually conjugate operators $\mathcal{T}^+_R$ and $\mathcal{T}^-_{-R}$ (a similar relation holds between $\mathcal{T}^-_R$ and $\mathcal{T}^+_{-R}$):

$$\mathcal{T}^+_R(z,\bar{z})\mathcal{T}^-_{-R}(w,\bar{w}) \sim \frac{1}{|z-w|^2}\partial X\bar{\partial}X\, e^{2Q\phi} + \cdots \tag{4.22}$$

Here we have exhibited only the $(1,1)$ operator appearing on the RHS. More singular terms correspond to operators that are BRST trivial. Even among the $(1,1)$ operators that can appear, we have dropped BRST-trivial contributions such as $\partial\phi\bar{\partial}\phi\, e^{2Q\phi}$.

On the right hand side of the above equation, we recognise the black hole perturbation operator. This tells us that the Sine-Liouville theory (when viewed as a perturbation of the original action by operators of both Liouville dressings) is not by itself exactly marginal, but marginality can be restored by including the black hole perturbation. In turn, it is known that the latter perturbation can be built up into a solution of closed string field theory which, being unique, must be equivalent to Witten's $SL(2,R)/U(1)$ CFT[1].

It is worth noting that, as in [112], the operators generated by requiring exact marginality to second order are not quite the physical operators, but rather some variants of them with an extra multiplicative factor of the Liouville field ϕ in front. In the present case the black-hole operator would be replaced by:

$$\partial X\bar{\partial}Xe^{2Q\phi} \to \phi\,\partial X\bar{\partial}Xe^{2Q\phi}\,. \tag{4.23}$$

This is reminiscent of the fact that, at $c=1$, the cosmological operator in the linear dilaton theory is not really $e^{Q\phi}$ but $\phi\, e^{Q\phi}$. As in that case, the distinction between the operator with and without a ϕ in front is expected to be unimportant for a large class of explicit computations.

[1]This was demonstrated for $k = 9/4$ in [37].

4.3.2 Parafermionic symmetry

To justify and spell out the above observations, we will now perform a more explicit study of the linear dilaton theory perturbed by Sine-Liouville and black hole operators. A useful tool in this study is the fact that when the dilaton slope of ϕ is $Q = \frac{1}{\sqrt{k-2}}$ and the radius of the compact direction is $R = \sqrt{k}$, the symmetry of the worldsheet is expanded from Virasoro to the parafermionic $SL(2,R)/U(1)$ algebra. A representation of this symmetry in terms of the ϕ and X bosons can be obtained by adding one free boson Z, normalized as in (4.9), and starting with the following free-field representation of the level k $SL(2,R)$ current algebra

$$\begin{aligned} J^3 &= -\sqrt{k}\partial Z\,, \\ J^\pm &= (i\sqrt{k}\partial X \mp \sqrt{k-2}\partial\phi)\, e^{\mp\frac{2}{\sqrt{k}}(iX-Z)}\,. \end{aligned} \tag{4.24}$$

Since J^3 corresponds to the direction gauged to obtain the $SL(2,R)/U(1)$ coset, the parafermionic generators can be obtained by dropping Z from the above expressions. This gives

$$\psi^\pm = (i\sqrt{k}\partial X \mp \sqrt{k-2}\partial\phi)e^{\mp\frac{2i}{\sqrt{k}}X}\,. \tag{4.25}$$

The currents (4.24) and (4.25) have similar anti-holomorphic copies. A generic primary of the coset can be written in terms of $SL(2,R)$ quantum numbers as

$$V_{j,m,\bar{m}} = e^{2jQ\phi}e^{-\frac{2im}{\sqrt{k}}X_L}e^{-\frac{2i\bar{m}}{\sqrt{k}}X_R}\,, \tag{4.26}$$

with

$$m = \frac{n+kw}{2} \qquad \bar{m} = \frac{n-kw}{2} \tag{4.27}$$

where n and w are the momentum and the winding of the X direction. This state has conformal dimensions

$$\Delta_{j,m} = -\frac{j(j-1)}{k-2} + \frac{m^2}{k}\,, \tag{4.28}$$

$$\bar{\Delta}_{j,\bar{m}} = -\frac{j(j-1)}{k-2} + \frac{\bar{m}^2}{k}\,, \tag{4.29}$$

and descends to the $SL(2,R)/U(1)$ coset from an $SL(2,R)$ primary with spin j and $J_0^3 = m$, $\bar{J}_0^3 = \bar{m}$.

We are interested in turning on marginal perturbations to the flat background which preserve the $SL(2,R)/U(1)$ symmetry. Natural candidates are exponentials of X and ϕ. Consider the OPE

$$\begin{aligned}\psi^{\pm}(z)\, e^{iaX_L+b\phi}(w) \;&\sim\; \frac{\mp b\sqrt{k-2}}{(z-w)^{1\pm\frac{a}{\sqrt{k}}}} e^{b\phi(w)+i(a\mp\frac{2}{\sqrt{k}})X_L(w)}\,(1+O(z-w)) \\ &\mp\; \frac{k}{2}\partial_z\left(\frac{e^{\mp i\frac{2}{\sqrt{k}}X_L(z)+iaX_L(w)}}{(z-w)^{\pm\frac{a}{\sqrt{k}}}}\right) e^{b\phi(w)}\,. \end{aligned} \tag{4.30}$$

Requiring mutual locality and no single pole fixes $a = w\sqrt{k}$, with w a non-zero integer. Thus the perturbation will be a winding mode belonging to the spectrum of the theory, if we combine left- and right-movers with opposite signs for w. For each w, there are two values of b which give an operator with $\Delta = \bar{\Delta} = 1$. For the case of one unit of winding, the Sine-Liouville operators are

$$S^1_{\pm} \equiv \lambda_1 \mathcal{T}^+_{\pm R} \;=\; \lambda_1 e^{\pm i\sqrt{k}(X_L-X_R)+\frac{1}{Q}\phi}\,, \tag{4.31}$$

$$S^2_{\pm} \equiv \lambda_2 \mathcal{T}^-_{\pm R} \;=\; \lambda_2 e^{\pm i\sqrt{k}(X_L-X_R)+(2Q-\frac{1}{Q})\phi}\,. \tag{4.32}$$

Turning on Liouville-like perturbations in linear dilaton theories has the effect of screening the strong coupling region $\phi \to +\infty$. This is indeed the case for $S^1_{\pm}$. For $S^2_{\pm}$, this happens only when $2Q > 1/Q$, i.e., $k < 4$. This region includes the $k = 9/4$ value corresponding to the pure two-dimensional black hole. Therefore, we will trust the Lagrangian description of the theory perturbed with $S^2_{\pm}$ in the region $k < 4$, and resort to the analytical continuation of the results otherwise.

The important point is that *both* operators are compatible with the $SL(2,R)/U(1)$ symmetry. Now, the chiral black hole perturbation $\partial X e^{2Q\phi}$ should also be compatible with the $SL(2,R)/U(1)$ symmetry. This is indeed the case, but happens only at a fixed point of the gauge orbit of the BRST trivial state $\partial\phi e^{2Q\phi}$. To find this point, consider

$$B = (\partial X + \alpha\partial\phi)\, e^{2Q\phi}\,. \tag{4.33}$$

Its OPE with ψ^+ is

$$\psi^+(z)B(w) \sim e^{-2i\frac{X(z)}{\sqrt{k}}+2Q\phi(w)} \left(-\frac{1}{2}\frac{\sqrt{k-2}}{(z-w)^2} + \frac{\partial\phi(w)}{z-w}\right) \times \left(\alpha + i\sqrt{\frac{k-2}{k}}\right) \tag{4.34}$$

and this fixes $\alpha = -i\sqrt{\frac{k-2}{k}}$. For this value of α, the OPE of ψ^- with B is

$$\psi^-(z)B(w) \sim -\frac{i}{\sqrt{k}Q^2}\partial_w \left(\frac{e^{2Q\phi(w)}}{z-w}\right) e^{-2i\frac{X(z)}{\sqrt{k}}} \tag{4.35}$$

so the integrated screening charge $\oint dwB(w)$ also commutes with ψ^-. In the following we rescale B by a constant and add the antichiral factor, so we will use

$$B = \mu \left(i\sqrt{k}\partial X_L + \frac{1}{Q}\partial\phi\right) \left(i\sqrt{k}\bar{\partial} X_R + \frac{1}{Q}\bar{\partial}\phi\right) e^{2Q\phi} \tag{4.36}$$

This, then, is the form of the black hole perturbation that is consistent with the parafermionic symmetry. We will make use of this, along with the Sine-Liouville operators of Eqs.(4.31),(4.32), as screening charges in the following subsections.

4.3.3 Correlation functions

Let us consider the two-point function of the interacting $SL(2,R)/U(1)$ theory with all the perturbations turned on. The vertex operators $V_{j,m,\bar{m}}$ and $V_{-j+1,m,\bar{m}}$ have the same conformal dimension and correspond to incoming and outgoing waves with the same momentum. We normalize them such that

$$\langle V_{-j+1,m,\bar{m}} V_{j,-m,-\bar{m}} \rangle = 1\,, \tag{4.37}$$

and we will consider the two-point function

$$R(j,m,\bar{m}) = \langle V_{j,m,\bar{m}} V_{j,-m,-\bar{m}} \rangle\,. \tag{4.38}$$

We ignore the divergent delta functions in both (4.38) and (4.37). Using the $SL(2,R)$ quantum numbers is useful because the coset theory inherits the structure of degenerate operators and fusion rules of the $SL(2,R)$ algebra. This in turn will allow us compute (4.38) by exploiting the trick of Teschner [19, 100, 113].

The affine $SL(2,R)$ algebra has degenerate primaries at spins [114]

$$j_{r,s} = -\frac{(r-1)}{2} - \frac{(s-1)}{2}k' \,, \tag{4.39}$$

where $k' \equiv k-2$ and r,s are integers with either $r,s>0$ or $r<0, s\leq 0$. The OPE of a primary with spin $j_{r,s}$ gives only a finite number of fields, according to fusion rules that were worked out in [115]. Below we will consider the constraints on the two-point function (4.38) which follow from the degenerate primaries with spins $j=-1/2$ and $j=-k'/2$.

The $j=-1/2$ degenerate field

The fusion of the degenerate primary $V_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}}$ with any other primary gives [115]

$$V_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}}V_{j,m,\bar{m}} \sim C^{+}_{j,m,\bar{m}}\left[V_{j-\frac{1}{2},m+\frac{1}{2},\bar{m}+\frac{1}{2}}\right] + C^{-}_{j,m,\bar{m}}\left[V_{j+\frac{1}{2},m+\frac{1}{2},\bar{m}+\frac{1}{2}}\right] . \tag{4.40}$$

Consider the auxiliary three-point function

$$\langle V_{j,m,\bar{m}}(x_1), V_{j+\frac{1}{2},-m-\frac{1}{2},-\bar{m}-\frac{1}{2}}(x_2) V_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}}(z)\rangle \,. \tag{4.41}$$

Taking $z\to x_1$ it is equal to

$$C^{-}_{j,m,\bar{m}}\, R\left(j+\frac{1}{2}, m+\frac{1}{2}, \bar{m}+\frac{1}{2}\right) . \tag{4.42}$$

Taking $z\to x_2$ it is equal to

$$C^{+}_{j,m,\bar{m}}\, R(j,m,\bar{m}) \,. \tag{4.43}$$

Equating the two expressions we get

$$\frac{R(j+\frac{1}{2}, m+\frac{1}{2}, \bar{m}+\frac{1}{2})}{R(j,m,\bar{m})} = \frac{C^{+}_{j,m,\bar{m}}}{C^{-}_{j,m,\bar{m}}} \,. \tag{4.44}$$

This is a functional equation for $R(j,m,\bar{m})$ that depends on the structure constants $C^{\pm}_{j,m,\bar{m}}$, which, from (4.37) and (4.40), are given by

$$C^{+}_{j,m,\bar{m}} = \langle V_{-j+\frac{3}{2},-m-\frac{1}{2},-\bar{m}-\frac{1}{2}}(\infty) V_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}}(1) V_{j,m,\bar{m}}(0)\rangle \,, \tag{4.45}$$

$$C^{-}_{j,m,\bar{m}} = \langle V_{-j+\frac{1}{2},-m-\frac{1}{2},-\bar{m}-\frac{1}{2}}(\infty) V_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}}(1) V_{j,m,\bar{m}}(0)\rangle \,, \tag{4.46}$$

where

$$V_{j,m,\bar{m}}(\infty) = \lim_{z,\bar{z}\to\infty} z^{2\Delta_{j,m}} \bar{z}^{2\bar{\Delta}_{j,\bar{m}}} V_{j,m,\bar{m}}(z,\bar{z}) \tag{4.47}$$

is the standard BPZ conjugate. In this approach, the computation of $C^{\pm}_{j,m,\bar{m}}$ (and similar constants associated to the second degenerate field below) is the only perturbative result needed, and allows to compare the role of the black hole/Sine-Liouville interactions. The presence of the background charge Q in the ϕ direction implies, for a correlator such as

$$\langle \prod_{i=1}^{n} e^{2\alpha_i Q\phi(z_i)} \rangle \,, \tag{4.48}$$

the anomalous conservation law

$$\sum_{i=1}^{n} \alpha_i = 1. \tag{4.49}$$

From (4.26) it follows that (4.49) is satisfied for $C^{+}_{j,m,\bar{m}}$ without any insertion of the interactions, so we have $C^{+}_{j,m,\bar{m}} = 1$. For $C^{-}_{j,m,\bar{m}}$, we can satisfy (4.49) by inserting one cigar screening charge (4.36). This gives

$$C^{-}_{j,m,\bar{m}} \;=\; \int d^2z \langle V_{-j+\frac{1}{2},-m-\frac{1}{2},-\bar{m}-\frac{1}{2}}(\infty) B(z) V_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}(1) V_{j,m,\bar{m}}(0)\rangle_{free} \tag{4.50}$$

To compute the integrand, we use the free field correlators

$$\begin{aligned}
\langle e^{i\frac{2m+1}{\sqrt{k}}X_L(\infty)} e^{-\frac{i}{\sqrt{k}}X_L(1)} e^{-\frac{2im}{\sqrt{k}}X_L(0)}\rangle &= 1\,,\\
i\sqrt{k}\langle e^{i\frac{2m+1}{\sqrt{k}}X_L(\infty)} \partial X_L(z) e^{-\frac{i}{\sqrt{k}}X_L(1)} e^{-\frac{2im}{\sqrt{k}}X_L(0)}\rangle &= -\frac{1/2}{z-1} - \frac{m}{z}\,,\\
\langle e^{(-2j+1)Q\phi(\infty)} e^{2Q\phi(z)} e^{-Q\phi(1)} e^{2jQ\phi(0)}\rangle &= z^{-2jQ^2}(z-1)^{Q^2}\,,\\
\frac{1}{Q}\langle e^{(-2j+1)Q\phi(\infty)} \partial\phi e^{2Q\phi(z)} e^{-Q\phi(1)} e^{2jQ\phi(0)}\rangle &= \left(\frac{1/2}{z-1} - \frac{j}{z}\right) z^{-2jQ^2}(z-1)^{Q^2},
\end{aligned}$$

and similar antiholomorphic expressions. This gives

$$\begin{aligned}
C^{-}_{j,m,\bar{m}} &= \mu(m+j)(\bar{m}+j)\int d^2z |z|^{-\frac{4j}{k-2}-2}|z-1|^{\frac{2}{k-2}}\\
&= -\mu\frac{\pi}{k'^2}(m+j)(\bar{m}+j)\gamma\left(-\frac{2j}{k'}\right)\gamma\left(\frac{2j-1}{k'}\right)\gamma\left(1/k'\right)\,,
\end{aligned} \tag{4.51}$$

where $\gamma(x) = \Gamma(x)/\Gamma(1-x)$ and we have used (4.93).

We now show that one obtains the same expression for $C^-_{j,m,\bar{m}}$ using the Sine-Liouville interactions of both dressings as screening charges. The conservation law (4.49) can also be satisfied by inserting one screening of type S^1 and one of type S^2, see (4.31)-(4.32). This gives

$$\begin{aligned} C^-_{j,m,\bar{m}} &= \int d^2z d^2w \langle V_{-j+\frac{1}{2},-m-\frac{1}{2},-\bar{m}-\frac{1}{2}}(\infty) S^1_+(w) S^2_-(z) V_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}}(1) V_{j,m,\bar{m}}(0)\rangle_{free} \\ &= \lambda_1\lambda_2 \int d^2z d^2w |z-w|^{-4}|z-1|^{\frac{2}{k-2}} z^{m+j-\frac{2j}{k-2}} \bar{z}^{\bar{m}+j-\frac{2j}{k-2}} w^{-m-j} \bar{w}^{-\bar{m}-j} \ . \end{aligned}$$

To compute this integral we can change variables from (z, w) to $(z, y = w/z)$, and we get

$$\begin{aligned} C^-_{j,m,\bar{m}} &= \lambda_1\lambda_2 \int d^2y |1-y|^{-4} y^{-m-j} \bar{y}^{-\bar{m}-j} \times \int d^2z |z|^{-2-\frac{4j}{k-2}} |1-z|^{\frac{2}{k-2}} \ , \quad (4.52) \\ &= -\lambda_1\lambda_2 \, \Gamma(-1) \frac{\pi^2}{k'^2} (m+j)(\bar{m}+j) \gamma\left(-\frac{2j}{k'}\right) \gamma\left(\frac{2j-1}{k'}\right) \gamma\left(1/k'\right) \ , \quad (4.53) \end{aligned}$$

where we have used twice eq.(4.93). Thus we have obtained precisely the same expression (4.51) for $C^-_{j,m,\bar{m}}$ using both Sine-Liouville screenings, and we can identify

$$\mu = \lambda_1 \tilde{\lambda}_2 \ , \quad (4.54)$$

where

$$\tilde{\lambda}_2 = \pi\Gamma(-1)\lambda_2 \quad (4.55)$$

is a renormalized value of λ_2. The reason to renormalize only λ_2 in the product $\lambda_1\lambda_2$ will become clear below.

The $j = -k'/2$ degenerate field

For this case, the fusion rules are [115]

$$\begin{aligned} V_{-\frac{k'}{2},\frac{k'}{2},\frac{k'}{2}} V_{j,m,\bar{m}} \sim \ & \tilde{C}^+_{j,m,\bar{m}} \left[V_{j-\frac{k'}{2},m+\frac{k'}{2},\bar{m}+\frac{k'}{2}}\right] + \tilde{C}^-_{j,m,\bar{m}} \left[V_{j+\frac{k'}{2},m+\frac{k'}{2},\bar{m}+\frac{k'}{2}}\right] \\ & + \tilde{C}^\times_{j,m,\bar{m}} \left[V_{\frac{k'}{2}-j+1,m+\frac{k'}{2},\bar{m}+\frac{k'}{2}}\right] \ . \quad (4.56) \end{aligned}$$

A similar reasoning as that used above leads leads to the functional equation

$$\frac{R(j+\frac{k'}{2}, m+\frac{k'}{2}, \bar{m}+\frac{k'}{2})}{R(j,m,\bar{m})} = \frac{\tilde{C}^{+}_{j,m,\bar{m}}}{\tilde{C}^{-}_{j,m,\bar{m}}} \,. \tag{4.57}$$

As before, we can set

$$\tilde{C}^{+}_{j,m,\bar{m}} = \langle V_{-j+\frac{k'}{2}+1,-m-\frac{k'}{2},-\bar{m}-\frac{k'}{2}}(\infty) V_{-\frac{k'}{2},\frac{k'}{2},\frac{k'}{2}}(1) V_{j,m,\bar{m}}(0)\rangle = 1\,, \tag{4.58}$$

since the conservation law (4.49) is satisfied without any perturbative insertion. As for

$$\tilde{C}^{-}_{j,m,\bar{m}} = \langle V_{-j-\frac{k'}{2}+1,-m-\frac{k'}{2},-\bar{m}-\frac{k'}{2}}(\infty) V_{-\frac{k'}{2},\frac{k'}{2},\frac{k'}{2}}(1) V_{j,m,\bar{m}}(0)\rangle\,, \tag{4.59}$$

we can satisfy (4.49) by inserting two Sine-Liouville S^1 interactions. This gives

$$\begin{aligned}\tilde{C}^{-}_{j,m,\bar{m}} &= \int d^2z d^2w \langle V_{-j-\frac{k'}{2}+1,-m-\frac{k'}{2},-\bar{m}-\frac{k'}{2}}(\infty) S^1_{-}(w) S^1_{+}(z) V_{-\frac{k'}{2},\frac{k'}{2},\frac{k'}{2}}(1) V_{j,m,\bar{m}}(0)\rangle_{free} \\ &= \lambda_1^2 \int d^2z d^2w |z-w|^{-2k+2} |z-1|^{2k'}\, z^{m-j}\, \bar{z}^{\bar{m}-j}\, w^{-m-j}\, \bar{w}^{-\bar{m}-j}\,.\end{aligned} \tag{4.60}$$

Changing now variables from (z,w) to $(z, y = w/z)$ we get

$$\begin{aligned}\tilde{C}^{-}_{j,m,\bar{m}} &= \lambda_1^2 \int d^2z |z|^{-4j-2k'} |z-1|^{2k'} \times \int d^2y^{-m-j} \bar{y}^{-\bar{m}-j} |1-y|^{-2(k-1)}\,, \\ &= \lambda_1^2 \pi^2 \gamma\left(-2j-k'+1\right) \gamma\left(2j-1\right) \frac{\Gamma(-m-j+1)}{\Gamma(-m-j-k'+1)} \frac{\Gamma(\bar{m}+j+k')}{\Gamma(\bar{m}+j)}\,,\end{aligned} \tag{4.61}$$

where we have used (4.93) twice.

Now that we have the structure constants, the solution to the functional equations (4.44) and (4.57) is

$$R(j,m,\bar{m}) = (\mu\pi\gamma(1/k'))^{1-2j} \frac{\Gamma(2j-1)\Gamma(1+\frac{2j-1}{k-2})}{\Gamma(-2j+1)\Gamma(1-\frac{2j-1}{k-2})} \frac{\Gamma(-j+1+\bar{m})\Gamma(-j+1-m)}{\Gamma(j+\bar{m})\Gamma(j-m)} \tag{4.62}$$

which is the expression we wrote above in (4.17). Also, we get also the relation

$$\lambda_1^2 \pi^2 = (\mu\pi\gamma(1/k'))^{k'} \tag{4.63}$$

which was first obtained in [100] by similar methods. Given this relation, it is clear that λ_2 rather than λ_1 is the coefficient which should absorb the divergence coming from $\Gamma(-1)$ in (4.53). But since from (4.54) we see that $\tilde{\lambda}_2$ is finite, it follows from (4.55) that λ_2 is effectively renormalized to zero, and therefore the second Sine-Liouville screening disappears from the theory.

Also note from (4.54) and (4.63) that only one of the three coefficients $\mu, \lambda_1, \tilde{\lambda}_2$ is independent. The expression (4.62) for $R(j, m, \bar{m})$ is symmetric under $m \leftrightarrow \bar{m}$ for $m - \bar{m} \in \mathbb{Z}$, using (4.90). It satisfies $R(-j+1, m, \bar{m}) = R^{-1}(j, m, \bar{m})$, and for delta normalizable states ($j = \frac{1}{2} + i\mathbb{R}$) it is a phase, namely, the phase shift between an incoming and an outgoing wave.

Using the Teschner trick one can also obtain the three-point function, and the same special structure constants we computed above enter similarly as an input for functional relations for the three-point function, which follow from crossing symmetry of an auxiliary four-point function [116].

Therefore, in this efficient approach to compute the correlators, the role of the second Sine-Liouville dressing is established for two and three point functions. It would be interesting to use the second Sine-Liouville screening to perform free-field computations similar to those in [99].

4.4 Generalized FZZ Algebra

As an application of the considerations detailed above, we will investigate a generalized class of Sine-Liouville models. The FZZ algebra procedure will then be employed to find the analogs of the dual black hole operators. However, such operators exist only in the special case of $c = 1$ matter coupled to Liouville theory. Therefore we will first give a brief survey of the relevant aspects of $c = 1$ string theory, including a listing of some interesting physical states in the cohomology, before proceeding to the model.

4.4.1 Cohomology of $c = 1$ strings

The $c = 1$ string is a special case of the linear dilaton background of Section 2.2 where we set $Q = 2$ to get a total central charge $c = 26$. With a cosmological perturbation to cut off the strong coupling region, the worldsheet action is:

$$S_{\rm c=1} = \int d^2z \left(-\partial X \bar{\partial} X + \partial\phi\bar{\partial}\phi + 2\hat{R}(z,\bar{z})\phi + 4\pi\mu\, e^{2\phi} \right) . \tag{4.64}$$

The string loop expansion in this theory is an expansion in $\frac{1}{\mu^2}$. The coordinate X has the interpretation of time, but in what follows we will consider its Euclidean continuation, corresponding to the case of finite temperature. Thus X is Euclidean (spacelike) and compactified:

$$X(z,\bar{z}) \sim X(z,\bar{z}) + 2\pi R\,. \tag{4.65}$$

The physical fields of the theory are defined by the BRST procedure, which is most tractable when the worldsheet theory is a free field theory. In the present case, the theory (at least in the given variables) is free only at $\mu = 0$, the limit in which the effective string coupling is infinite. For this case, the BRST cohomology has been worked out in [117, 118, 119, 120, 121, 122]. As observed in [123], at nonzero μ we can still use part of the previous results.

Let us therefore start by reviewing the cohomology at $\mu = 0$. One important class of physical operators[1] are the momentum "tachyons":

$$T^{\pm}_{\frac{n}{R}} = e^{i\frac{n}{R}X}\, e^{(2\mp\frac{n}{R})\phi}, \quad n \in \mathbb{Z} \tag{4.66}$$

with left and right conformal dimensions equal to 1. These are just the special cases of the operators already introduced in Eqs.(4.16),(4.21). As before, the superscripts $\pm$ refer to non-normalizable/normalizable operators respectively.

Another important class of observables are the winding modes. Writing $X = X_L + X_R$, we define $\tilde{X} = X_L - X_R$ in terms of which:

$$\mathcal{T}^{\pm}_{nR} = e^{i\,nR\,\tilde{X}}\, e^{(2\mp nR)\,\phi}, \quad n \in \mathbb{Z}\,. \tag{4.67}$$

[1] Here and in what follows, we refer to a dimension $(1,1)$ operator as a physical operator if it is BRST invariant after integration over the worldsheet. Typically such operators are also BRST invariant when multiplied by the ghost field combination $c\bar{c}$. We need to be more specific about the ghost dependence of a physical operator only if this dependence is nontrivial.

These are clearly also $(1,1)$ operators.

The operators $T_{\frac{n}{R}}$ and $\mathcal{T}_{nR}$ are dual to each other under (timelike) T-duality:

$$X_R \to -X_R, \quad \phi \to \phi - \log R \tag{4.68}$$

under which $X \to \tilde{X}$ and

$$R \to \frac{1}{R}, \quad \mu \to \mu R\,. \tag{4.69}$$

Note that $T_0 = \mathcal{T}_0 = e^{2\phi}$ is the cosmological operator.

There are other modes of dimension $(1,1)$. They are called "discrete states" [120, 121] and can be thought of as two-dimensional "remnants" of the higher-spin fields that exist in critical string theory. We start by writing the following chiral operators at the self-dual radius $R = 1$

$$W^{\pm}_{s,n}(z) = \mathcal{P}_{s,n}(\partial^j X)\, e^{2inX_L}\, e^{(2\mp 2s)\phi_L} \tag{4.70}$$

where $s = 0, \frac{1}{2}, 1, \ldots,$ and $n, n' = s, s-1, \ldots, 1-s, -s$ and $\mathcal{P}_{s,n}$ is a polynomial in derivatives of X_L with conformal dimension $s^2 - n^2$. In particular, $\mathcal{P}_{s,\pm s} = 1$.

Because the above operators depend only on the left-moving part of the Liouville field, which is a noncompact scalar field, they are not physical operators. The physical operators are the combinations:

$$Y^{\pm}_{s;n,n'}(z,\bar{z}) = W^{\pm}_{s,n}(z)\bar{W}^{\pm}_{s,n'}(\bar{z})\,. \tag{4.71}$$

For $n = n' = \pm s$ the above operators are the momentum modes $T_{\pm 2s}$, while for $n = -n' = \pm s$ they are the winding modes $\mathcal{T}_{\pm 2s}$. The remaining ones, with $n < s$ or $n' < s$ are the true discrete states. The time-independent discrete states are those with $n = n' = 0$. Simple examples are the ones with $s = 1, 2$ for which the relevant $c = 1$ primaries are:

$$\begin{aligned} \mathcal{P}_{1,0} &= \partial X\,, \\ \mathcal{P}_{2,0} &= (\partial X)^4 + \tfrac{3}{2}(\partial^2 X)^2 - 2\partial X \partial^3 X\,, \end{aligned} \tag{4.72}$$

and the corresponding non-normalizable discrete-state operators are:

$$Y^{+}_{1;0,0} = \partial X \bar{\partial} X\,, \tag{4.73}$$

$$Y^{+}_{2;0,0} = \mathcal{P}_{2,0}\bar{\mathcal{P}}_{2,0}\, e^{-2\phi}\,. \tag{4.74}$$

Although the above states have been tabulated for a specific radius $R = 1$, they will exist at other radii as long as $n + n'$ is an integer multiple of $1/R$ and $n - n'$ is an integer multiple of R. In particular this constraint is always satisfied for $n = n' = 0$, independent of the radius, hence the time-independent discrete states $Y^+_{s;0,0}$ exist for all radius. Of course s has to be an integer in order for $n = n' = 0$ to be allowed. Since we are working at general values of the radius, we will concentrate on this set of time-independent discrete states.

The first nontrivial state in this collection is just $Y^+_{1,0,0} = \partial X \bar{\partial} X$, the radius-changing operator. In the critical string this would have just been the zero-momentum mode of the graviton/dilaton. Here it is a "remnant" of those fields, and is forced to have zero momentum. The other discrete states are similar remnants of excited tensor states of the string, with fixed momenta.

Note that for the radius operator appearing in Eq. (4.73) there is a normalizable, or non-local, counterpart:

$$Y^-_{1,0,0} = \partial X \bar{\partial} X \, e^{4\phi} \,. \tag{4.75}$$

This is precisely the black hole perturbation of the previous sections, specialised to the case $Q = 2$. It has been shown [37] that starting from a perturbation of the $c = 1$ string by $Y^-_{1,0,0}$, there is no obstruction to finding a classical solution of closed string field theory (CSFT) to all orders in α', and moreover the solution so obtained is unique. Therefore, starting with Eq. (4.6) one generates an exact (at tree level) CFT describing a string background. It follows that this CFT must be the $SL(2,R)/U(1)$ black hole CFT. This closes the gap between the spacetime solution, valid only for large k, and the CFT, which lacks a direct spacetime interpretation.

But it also suggests a generalization. Observe that:

$$Y^-_{s;0,0} = \mathcal{P}_{s,0} \bar{\mathcal{P}}_{s,0}(\partial^j X, \bar{\partial}^j X) \, e^{(2+2s)\phi} \tag{4.76}$$

for $s = 0, 1, 2, \ldots$ defines an infinite family of normalizable operators, of which the first two ($s = 0, 1$) are the cosmological and black hole perturbations. Now the considerations in Ref.[37] were shown to be generally applicable to all these operators. Therefore each of them similarly generates a unique classical solution of CSFT, and so must correspond to some exact CFT. Unlike the first nontrivial case ($s = 1$, the usual 2d black hole) where the CFT is the $SL(2,R)/U(1)$ nonlinear σ-model, the CFT in the other cases is not explicitly known. The form of the states in Eq. (4.76) suggests that we are dealing with higher-spin generalizations of the 2d black hole. As we will now argue, these are related by a generalized FZZ duality to Sine-Liouville perturbations of higher winding number.

4.4.2 Higher Winding Sine-Liouville Perturbations

Supposing that instead of the unit winding perturbation $V = \mathcal{T}_1$, we perturb the action of the linear dilaton theory by Sine-Liouville operators of winding number 2:

$$\mathcal{T}^{\pm}_{\pm 2R} = e^{\pm 2iR(X_L - X_R)}\, e^{(2\mp 2R)\phi} \tag{4.77}$$

(recall that the $\pm$ sign in the subscript refers to the sign of the winding number while the one in the superscript refers to the dressing). It is easily checked that the OPE between mutually conjugate operators of this type is again the black hole perturbation:

$$\mathcal{T}^{+}_{2R}(z,\bar{z})\mathcal{T}^{-}_{-2R}(w,\bar{w}) \sim \frac{1}{|z-w|^2}\partial X\bar{\partial}X\, e^{4\phi} + \cdots \tag{4.78}$$

The same will be true for pairs of mutually conjugate operators of any winding number – in every case, the output of the OPE is the 2d black hole perturbation. One way to understand this is that we can orbifold the compact time direction to enhance the radius by an integer factor. The multiply wound Sine-Liouville perturbation of the original theory then become singly-wound perturbations in the orbifolded theory. But orbifolding in time does not affect the black hole perturbation operator, which is time-independent.

Things become more interesting if we perturb the theory simultaneously by operators of different winding numbers. As a first example, consider the theory perturbed by the single and double-winding operators:

$$S \to S + \int d^2z\, (\mathcal{T}^{+}_{R} + \mathcal{T}^{+}_{-R} + \mathcal{T}^{-}_{R} + \mathcal{T}^{-}_{-R} + \mathcal{T}^{+}_{2R} + \mathcal{T}^{+}_{-2R} + \mathcal{T}^{-}_{2R} + \mathcal{T}^{-}_{-2R}) \tag{4.79}$$

In this case, examining the OPE algebra, we find that the product of *three* of these operators can potentially produce a new $(1,1)$ operator on the RHS:

$$\mathcal{T}^{\pm}_{-2R}(z_1,\bar{z}_1)\mathcal{T}^{\mp}_{R}(z_2,\bar{z}_2)\mathcal{T}^{\mp}_{R}(z_3,\bar{z}_3) \;\sim\; \mathcal{P}_{2,0}(\partial X)\bar{\mathcal{P}}_{2,0}(\bar{\partial} X)\, e^{6\phi} = Y^{-}_{2,0,0} \tag{4.80}$$

where $\mathcal{T}^{\pm}_{nR}$ can be read off from Eq. (4.67) and $\mathcal{P}_{2,0}$ is given explicitly in Eq. (4.72).

Let us work this out in more detail. We have:

$$\begin{aligned}\mathcal{T}^{+}_{2R}(z_1,\bar{z}_1)\mathcal{T}^{-}_{-R}(z_2,\bar{z}_2)\mathcal{T}^{-}_{-R}(z_3,\bar{z}_3) &= :e^{2iRX_1}e^{(2-2R)\phi_1}::e^{-iRX_2}e^{(2+R)\phi_2}::e^{-iRX_3}e^{(2+R)\phi_3}: \\ &= \frac{1}{|z_{12}|^{4-2R}}\frac{1}{|z_{13}|^{4-2R}}\frac{1}{|z_{23}|^{4+4R}}:e^{iR(2X_1-X_2-X_3)}::e^{(2-2R)\phi_1+(2+R)\phi_2+(2+R)\phi_3}:\end{aligned} \tag{4.81}$$

where we have used the shorthand $X_i \equiv X(z_i,\bar{z}_i)$ and similarly for ϕ_i, as well as $z_{ij} \equiv z_i - z_j$.

After integration over the z_i, the RHS of the above may be written

$$\int \prod_{i=1}^{3} d^2 w_i \frac{1}{|w_1|^{4-2R}}\frac{1}{|w_2|^{4-2R}}\frac{1}{|w_1-w_2|^{4+4R}}\mathcal{O}(w_i,\bar{w}_i) \tag{4.82}$$

where we have defined $w_1 \equiv z_1 - z_2, w_2 \equiv z_1 - z_3, w_3 = z_3$, and

$$\mathcal{O}(w_i,\bar{w}_i) \equiv \big[:e^{iR(2X(w_2+w_3)-X(w_2-w_1+w_3)-X(w_3))}: \times\, (w\to\bar{w})\big] : e^{6\phi(w_3,\bar{w}_3)}: \tag{4.83}$$

Here we have moved all the Liouville fields ϕ_i to the location w_3 and dropped the new terms that arise in doing this. As we will see, at the end this will only lose us some terms that are trivial in the BRS cohomology.

Finally we expand the X-dependent vertex operator about the point w_3 as:

$$\big[:e^{iR(2X(w_2+w_3)-X(w_2-w_1+w_3)-X(w_3))}: \times\, (w\to\bar{w})\big] := \Big|\sum_{n_1,n_2=0}^{\infty} w_2^{n_1}(w_2-w_1)^{n_2}\mathcal{A}_{n_1,n_2}(w_3)\Big|^2 \tag{4.84}$$

where the operators $\mathcal{A}_{n_1,n_2}$ are built out of holomorphic derivatives of X, namely $\partial X, \partial^2 X, \cdots$ and have conformal dimension $(\Delta,\bar{\Delta}) = (n_1+n_2, 0)$. Their complex conjugates have dimension $(0, n_1+n_2)$. Anticipating that the final contribution can only come from physical operators in the cohomology, we keep only those operators in the sum which are of the form $|\mathcal{A}_{n_1,n_2}|^2$ with $n_1 + n_2 = 4$. Then the combined matter-Liouville operator in Eq. (4.83) can be replaced by:

$$|w_2|^{2n_1}|w_1 - w_2|^{2n_2}|\mathcal{A}_{n_1,n_2}(w_3)|^2 \ :e^{6\phi(w_3,\bar{w}_3)}: \tag{4.85}$$

Now we see that the composite operator

$$|\mathcal{A}_{n_1,n_2}(w_3)|^2 \ :e^{6\phi(w_3,\bar{w}_3)}: \tag{4.86}$$

has conformal dimension $(1,1)$ and is a local operator depending only on $(w_3, \bar{w}_3)$. The coefficient functions depend only on w_1, w_2 and combine under the integral sign into an expression of the form:

$$\int d^2w_1 d^2w_2 \frac{1}{|w_1|^\alpha} \frac{1}{|w_2|^\beta} \frac{1}{|w_1 - w_2|^\gamma} \tag{4.87}$$

for some α, β, γ satisfying $\alpha+\beta+\gamma = 4$. Thus the coefficient of the $(1,1)$ operator is logarithmically divergent, the sign of a nontrivial β-function.

At this stage it is clear without further computation that the operator $\mathcal{A}_{n_1,n_2}$ must be the Virasoro primary $\mathcal{P}_{2,0}$ defined in Eq. (4.72). The reason is that the three operators whose OPE we are computing are all in the cohomology and the output must therefore also be in the cohomology. Given the total matter and Liouville momenta of the fields on the LHS of the multiple OPE, there is a unique such operator that can appear on the RHS. Hence we have shown that the higher-spin black hole operator

$$Y^-_{2,0,0} = \mathcal{P}_{2,0}(\partial X)\bar{\mathcal{P}}_{2,0}(\bar{\partial} X)\, e^{6\phi} \tag{4.88}$$

appears in the β-function of the theory perturbed as in Eq. (4.79), thereby justifying Eq. (4.80).

The above result is quite general. For example, one can check that:

$$\mathcal{T}^{\pm}_{NR}(z_1,\bar{z}_1)\mathcal{T}^{\mp}_{-R}(z_2,\bar{z}_2)\cdots\mathcal{T}^{\mp}_{-R}(z_N,\bar{z}_N) \ \sim \ \mathcal{P}_{N,0}(\partial X)\bar{\mathcal{P}}_{N,0}(\bar{\partial} X)\, e^{(2+2N)\phi} = Y^-_{N,0,0} \tag{4.89}$$

Thus the higher-spin black hole operator of label N (i.e. Liouville momentum $2 + 2N$) arises when we perturb the linear dilaton theory with Sine-Liouville operators of windings 1 and N.

We see that, in a similar sense as for the FZZ algebra of the previous section, the multiply-wound Sine-Liouville operators are linked to higher-spin black holes. More precisely, perturbing by all Sine-Liouville operators of winding numbers $1, 2, \cdots, N$ gives rise to higher-spin black holes with all labels up to N (the spins realised in this way are $2k^2,\ k = 1, 2 \cdots, N$). This should be viewed as a generalization of the FZZ duality, and we expect that also here the coefficients of half of the Sine-Liouville operators get renormalized to zero.

To produce only a definite higher-spin black hole for $N \geq 2$, one must fine-tune the perturbation strengths so that the lower-spin operators are not produced.

4.5 Conclusions

In this work we presented a new approach to the FZZ duality between the two-dimensional black hole and the sine-Liouville conformal field theory. In this approach the duality is to be understood as coming from the fact that the Sine-Liouville perturbations of both dressings induce, via their mutual OPE, the operator representing a black hole deformation, and one the of the two Sine-Liouville perturbations then disappears because its coefficient gets renormalized to zero.

This approach has led us to propose a generalized FZZ duality for the $c = 1$ string. One side of this duality is a CFT generated by perturbing the linear dilaton background with higher-spin analogues of the black hole operator. Finding an exact description of this CFT would be very helpful in understanding this generalized duality better, though that appears to be a hard problem on which no progress has been made since the existence proof in Ref.[37]. One might instead try to use the holographic description in terms of double-scaled matrix models to get more insight into the nature of the theory that results from the fully back-reacted higher spin perturbation. Also, even partial progress in the worldsheet treatment of the higher spin perturbations (e.g. some correlation functions) could provide relevant tests for the generalized FZZ duality we have proposed.

4.6 Useful formulae

$$
\begin{aligned}
\Gamma(x)\Gamma(1-x) &= \frac{\pi}{\sin(\pi x)} && (4.90)\\
\gamma(x) &\equiv \frac{\Gamma(x)}{\Gamma(1-x)} && (4.91)\\
\gamma(x+1) &= -x^2\gamma(x) && (4.92)\\
\int_{\mathbb{R}^2} d^2x\, x^a \bar{x}^{\bar{a}} (1-x)^b (1-\bar{x})^{\bar{b}} &= \pi \frac{\Gamma(1+a)}{\Gamma(-\bar{a})}\frac{\Gamma(1+b)}{\Gamma(-\bar{b})}\frac{\Gamma(-\bar{a}-\bar{b}-1)}{\Gamma(a+b+2)} && (4.93)\\
&= (a, b \longleftrightarrow \bar{a}, \bar{b})
\end{aligned}
$$

The above integral is well defined only when $a-\bar{a}, b-\bar{b} \in Z$, and it is only then that the second line holds.

4.7 Further Developments

The FZZ duality was recently investigated from the point of view of the fully interacting $SL(2,R)/U(1)$ theory by Pakman [124]. In this work the author proposes a quantization of the Liouville field ϕ and finds the quantum versions of the vertex operators used earlier in this chapter. The FZZ duality is verified by matching the correlators computed from the Sine-Liouville theory and the black hole CFT. It is interesting that the reflection amplitudes $R(j,m,n)$ obtained from the quantized operators coincide with the ones obtained earlier in this chapter using the properties of degenerate operators of the parafermionic symmetry discussed in Section 4.3.2. In this chapter we have used the free field representations of the operators, which is valid when the $SL(2,R)/U(1)$ theory is non-interacting.

In another work by Giribet and Leoni [125] a twisted version of the FZZ duality was considered. On the Liouville side, instead of turning on unit ($n=\pm 1$) winding and anti-winding operators considered in Eq. (4.20) the authors use a "twisted" deformation where only the $n=1$ and $n=2$ winding modes are turned on:

$$
S \to S + \int d^2z \left(\mathcal{T}_R^+ + \mathcal{T}_{2R}^+\right) . \qquad (4.94)
$$

It was shown that using these operators as screening charges the FZZ conjecture can be proved in the twisted Sine-Liouville model. The authors also presented a general prescription to check the correspondence for any N-point tree level correlation functions of the $SL(2,R)/U(1)$ WZW model. Since this work also uses higher winding operators (namely two units of winding), it remains to be seen how our proposal for the generalized FZZ duality applies in this case.

Chapter 5

Noncritical-topological correspondence: Disc amplitudes and noncompact branes

In this chapter we examine the duality between type 0 noncritical strings and topological B-model strings, with special emphasis on the flux dependence. The former theory is known to exhibit holomorphic factorization up to a subtle flux-dependent disc term. We give a precise definition of the B-model dual and propose that it includes both compact and noncompact B-branes. The former give the factorized part of the free energy, while the latter violate holomorphic factorization and contribute the desired disc term. These observations are generalized to rational radii, for which we derive a non-perturbatively exact result. We also show that our picture extends to a proposed alternative topological-anti-topological picture of the correspondence for type 0 strings [126].

5.1 Introduction

It has been known for some time[38, 39, 40, 41, 42] that noncritical string theories in two spacetime dimensions have a topological description. Subsequently the actual correspondence between them and their topological duals on a Calabi-Yau manifold was found by Ghoshal and Vafa[43]. Noncritical $c = 1$ string theory at the self-dual radius is perturbatively equivalent to topological string theory on a deformed conifold. For integer multiples of the self-dual radius, the corresponding topological theory lives on a $\mathbb{Z}_n$ orbifold of the conifold geometry. The noncritical-topological equivalence was shown both via Landau-Ginsburg models and using the ground ring[121] construction.

The above correspondence has led to considerable illumination of the existence and properties of noncritical strings. But because the bosonic $c = 1$ string is nonperturbatively unstable, it has not been possible to extend the equivalence to the nonperturbative level. Such an extension can be explored in the case of the nonperturbatively stable Type 0 string theories in two dimensions. These too have a description in terms of topological string theory, first proposed in Ref.[46] and further explored in Refs.[47, 83, 127]. The Calabi-Yau dual to noncritical type 0 strings at a special radius is a $\mathbb{Z}_2$ orbifold of the conifold, while for integer multiples of this radius it is a $\mathbb{Z}_{2n}$ orbifold.

One of the most interesting aspects of noncritical type-0 strings is the possibility of turning on background RR fluxes: type 0A theory has two RR gauge fields and type 0B theory has an RR scalar whose equations of motion admit linear growth in space and time[10]. Thus in both cases there is a pair of independent RR fluxes q and $\tilde{q}$. At the level of the closed string perturbation expansion the theory depends only on $|q| + |\tilde{q}|$, but at the disc level there is a subtle and important additional term in the free energy which depends on $|q| - |\tilde{q}|$, as found by Maldacena and Seiberg in [9]. Only after this extra term is included in the free energy, one finds a satisfactory physical interpretation wherein one of the two fluxes is sourced by ZZ 0-branes while the other has no sources.

However, the dual topological B-model of Refs.[46] and [83] depends on the complex-structure moduli of the orbifolded conifold, which in turn depend on the fluxes only through the combination $|q| + |\tilde{q}|$. Thus it is not obvious how the correspondence can be extended to incorporate this effect. Our aim here will be to re-examine the duality with particular reference to flux dependence. We will make the existing proposal more precise and will then argue that the topological side must be extended to incorporate a new feature, namely noncompact branes[1]. When placed at appropriate locations, they contribute precisely the desired disc term in the free energy. Thereafter we generalise these observations to integer multiples of the special radius and to infinite radius.

One key difference between our analysis and previous ones is that we use the result of Ref.[9] which we consider to be rigorously true as a convergent integral representation of the full (nonperturbative) free energy of type 0 strings.

5.2 Noncritical-topological duality

5.2.1 Bosonic case

In this section we briefly review some relevant aspects of topological string theory on noncompact Calabi-Yau spaces. The simplest example is the conifold, described by the equation

$$zw - px = 0, \tag{5.1}$$

where z, w, p, x are complex coordinates of $\mathbb{C}^4$. This is therefore a three complex dimensional non-compact manifold. It has a singularity at the origin. The singularity can be removed by blowing up an S^3 cycle at the origin, after which the equation becomes:

$$zw - px = \mu, \tag{5.2}$$

where μ is in general complex and its modulus determines the size of the S^3. Eq. (5.2) is known as the deformed conifold (DC) and μ is its complex structure parameter. The singularity in Eq. (5.1) can alternatively be removed by blowing up an S^2 at the origin. The resulting manifold is the resolved conifold (RC).

The topological A model on any given Calabi-Yau is a theory of quantised deformations of the Calabi-Yau, sensitive only to the Kähler moduli. The B model is similar but depends only on the complex structure moduli. The noncritical-topological duality proposed by Ghoshal and Vafa[43] stems from the observation[121]

[1] A tentative indication of a role for noncompact branes in this duality was found in Ref.[46].

that the ground ring of $c = 1$ string theory at the self-dual radius has four generators z, w, p, x that are worldsheet operators of conformal dimension 0 in the BRST cohomology, satisfying the conifold relation Eq. (5.2) where

$$\mu = i g_s \mu_M \tag{5.3}$$

and μ_M is the cosmological constant on the worldsheet in the noncritical string theory[1]. Based on this and other evidence, they argued that the $c = 1$ string with cosmological constant μ_M at self-dual radius is equivalent to the topological B-model on the deformed conifold with deformation parameter μ. In particular their argument requires the genus-g partition functions of the two theories to coincide. Writing the genus expansions of the free energies of the $c = 1$ theory and the topological theory on the deformed conifold as:

$$\begin{aligned} \mathcal{F}^{c=1}(\mu_M) &= \sum_{g=0}^{\infty} \mathcal{F}_g^{c=1} \mu_M^{2-2g} \\ \mathcal{F}^{top,DC}(\mu) &= \sum_{g=0}^{\infty} \mathcal{F}_g^{top,DC} \mu^{2-2g} \end{aligned} \tag{5.4}$$

the claim then amounts to:

$$\mathcal{F}_g^{c=1} = (i g_s)^{2-2g} \mathcal{F}_g^{top,DC}, \quad \text{all } g \tag{5.5}$$

for which ample evidence has been found[128, 129]. There is also expected to be a 1-1 correspondence between the physical observables (tachyons in the $c = 1$ case and deformations of S^3 in the B-model case) and their correlators (for a recent discussion, see Ref.[130]).

Going beyond the self-dual radius, it has long been known[44] that the ground ring of the $c = 1$ string at integer multiples of the self-dual radius, $R = p$, is a $\mathbb{Z}_p$

[1] The factor of i exhibited here is often dropped in the literature, though it has been correctly placed in Refs.[47, 127]. It is important because the genus expansion of the topological string free energy is $F^{top} = \sum_g \chi_g \mu^{2-2g}$, with coefficients that alternate in sign (given by the virtual Euler characteristic of the moduli space of genus g Riemann surfaces), while that of the $c = 1$ string is $F_{c=1} = \sum_g |\chi_g| (g_s \mu_M)^{2-2g}$ and is therefore positive in every genus as befits a unitary theory. A discussion of this point may be found in Ref.[7].

orbifold of the conifold. This space has p singularities connected by (complex) lines. The deformed version of this space is described by the equation:

$$zw - \prod_{k=1}^{p}(px - \mu_k) = 0 \tag{5.6}$$

which has n homology 3-spheres of size $\mu_1, \mu_2, \ldots, \mu_p$, each concealing one of the singularities. The geometry develops a conifold singularity if any of the μ_i's become zero, and a line singularity if $\mu_i = \mu_j$ for $i \neq j$. If the μ_i's are all distinct and nonzero, the manifold is non-singular.

We expect the n deformation parameters to be in correspondence with p distinct (non-normalisable) deformations of the noncritical string theory[131]. If we only choose to perform the cosmological constant deformation μ_M then these p deformation parameters must be determined in terms of μ_M. It has been shown via a Schwinger computation[45] that for an integer radius the parameters μ_k are given by $ig_s\frac{\mu_M+ik}{p}$, and $k = -\frac{p-1}{2}, -\frac{p-1}{2}+1, \cdots, \frac{p-1}{2}$. Moreover, the free energy factorises[1] into a sum of contributions as follows:

$$\mathcal{F}_{c=1}^{R=p}(\mu) = \mathcal{F}^{top,DOC_p}(\{\mu_k\}) = \sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}} \mathcal{F}^{top,DC}(\mu_k) \tag{5.7}$$

This factorisation can be understood in the Riemann surface formulation of [78]. In this approach one thinks of the following class of noncompact Calabi-Yaus:

$$zw - H(p, x) = 0 \tag{5.8}$$

as a fibration described by the pair of equations:

$$zw = H, \quad H(p, x) = H \tag{5.9}$$

The fibre is $zw = H$, a complex hyperbola, and the base is the complexified p, x plane. Above points in the base satisfying $H(p, x) = 0$, the fibre degenerates to $zw = 0$, a pair of complex planes intersecting at the origin. Such points in the

[1]We use the word "factorise" even though the free energy splits into a sum, rather than a product, of terms. What factorises is of course the partition function.

base form a Riemann surface, and it is this surface that governs the physics of the topological string theory. Moreover the function $H(p,x)$ plays the role of a Hamiltonian and lends an integrable structure to the system.

In the present case of the orbifolded conifold of Eq. (5.6), the Hamiltonian is:

$$H(p,x) = \prod_{k=1}^{p} (px - \mu_k) \tag{5.10}$$

and hence the Riemann surface $H(p,x) = 0$ factorises into disjoint Riemann surfaces[83]. This is the physical reason for the factorisation of the free energy into a sum of contributions, one for each branch of the Riemann surface.

The above statements are meaningful only at the level of string perturbation theory, since the bosonic $c = 1$ is not well-defined nonperturbatively. Moreover, the computation of Ref.[45] is performed by manipulating a divergent series. Later we will discuss the analogous relation for the type 0A string, and will demonstrate factorisation of the free energy without ever using perturbation expansions or divergent series. In this way we will reliably show that it is nonperturbatively exact.

5.2.2 Type 0 case, $R = 1$

In Ref.[46] and subsequently Ref.[47, 83, 127], the above ideas were applied to the case of the type 0A string. Here it is convenient to choose units in which $\alpha' = 2$. A new feature of the type 0A string relative to the bosonic case (for more details, see Refs.[9, 10, 132] and references therein) is that it has two distinct quantised parameters q and $\tilde{q}$. In the Liouville description these arise as the fluxes of two distinct Ramond-Ramond 2-form field strengths, $F_{t\phi}, \tilde{F}_{t\phi}$. The theory has a symmetry, labelled S-duality, under which the cosmological constant μ_M changes sign and at the same time, $F \leftrightarrow \tilde{F}$. In the more powerful matrix quantum mechanics (MQM) description of the same string theory, the fluxes have quite an asymmetric origin. For $\mu_M < 0$, q is the difference in the number of $D0$ and $\bar{D}0$ branes, or the net number of $D0$ branes, in the MQM. On the other hand, $\tilde{q}$ is the coefficient of a Chern-Simons term involving gauge fields on the branes and anti-branes. For $\mu_M > 0$ the roles of $q, \tilde{q}$ are reversed. On reducing to eigenvalues, each of the integers q and $\tilde{q}$ can be interpreted as the quantised angular momentum of fermions moving in the complex plane. Moreover, if both are turned on there is an additional coupling term arising from projection to nonsinglet sectors, such that the Hamiltonian eventually depends only on $(q + \tilde{q})$.

The Euclidean type 0A theory has a special value of the radius, $R = 1$ in these units, at which the correspondence with the topological string is simplest. This radius is the analogue of the self-dual radius for the bosonic $c = 1$ string. For type 0A noncritical strings at the special radius, the corresponding dual geometry in the topological string has been proposed[46] to be a deformed $\mathbb{Z}_2$ orbifold of the conifold (DOC). The identification is again based on the analysis of the ground ring of the noncritical theory. The DOC dual to the type 0A string has two S^3's whose complex structure parameters are identified[1] with the type 0A parameters $\mu_M, \hat{q} = q + \tilde{q}$ as:

$$\begin{aligned} \mu = &\quad i g_s(\mu_M - \tfrac{i\hat{q}}{2}) &= \frac{g_s}{2} y \\ \mu' = &\quad -i g_s(\mu_M + \tfrac{i\hat{q}}{2}) &= \frac{g_s}{2} \bar{y} \end{aligned} \tag{5.11}$$

with:

$$y = \hat{q} + 2i\mu_M \tag{5.12}$$

Thus the equation of the DOC is:

$$zw + (px - \mu)(px - \mu') = 0 \tag{5.13}$$

Notice that complex conjugation exchanges the moduli of the two S^3's and acts as S-duality of the noncritical string. This is because both conjugation and S-duality act as $\hat{q} \to \hat{q}, \mu_M \to -\mu_M$. As a result the S-duality of type 0A noncritical strings is explicitly geometrised in the topological B-model dual.

We note at this point that a different point of view about noncritical-topological duality for type 0 strings is espoused in Ref.[47], according to which the topological string is defined on the "holomorphic square root" of the space we have

[1] Again, this identification substantially agrees with that in Refs.[47, 127] but differs from that in Refs.[46, 83] by factors of i.

been discussing, which is an ordinary conifold rather than an orbifolded one. The noncritical-topological correpondence then has to be reformulated by saying that we have to add topological and anti-topological free energies. While this seems to fit in with the picture of topological strings emerging from black hole studies[133, 134], it is not clear that in practical terms it differs from the older proposal of Ref.[46]. However we will see later that our proposal for a precise topological dual involving noncompact branes can also be phrased in topological-anti-topological language.

The manifold Eq. (5.13) exists and is nonsingular for all nonzero $\mu \neq \mu'$. However, the topological B-model on it is dual to type 0A noncritical string theory only in the special case $\mu' = \bar{\mu}$. With this restriction, the space is nonsingular as long as μ has an imaginary part. From Eq. (5.11), this will in turn be the case as long as the cosmological constant μ_M of the noncritical theory is nonzero, which is natural since μ_M cuts off the strong coupling end of the Liouville direction. Of course from the matrix model point of view there is still a sensible string theory when $\mu_M = 0$, but one in which the standard genus expansion of the continuum theory does not hold, and where the role of the string coupling is played by the inverse RR flux. The region where the RR flux is of the same order as, or larger than, the cosmological constant has received some discussion in the literature[12, 135].

The above identification leads to the following proposed equality between type 0A string and topological B model free energies:

$$\mathcal{F}_{0A}(\mu_M, q, \tilde{q}, R = 1) = \mathcal{F}^{top,DOC}\left(\mu = \frac{g_s}{2}y, \mu' = \frac{g_s}{2}\bar{y}\right) \qquad (5.14)$$

Using arguments analogous to those described above for the bosonic string, we also find that the RHS perturbatively factorises:

$$\mathcal{F}^{top,DOC}\left(\mu = \frac{g_s}{2}y, \mu' = \frac{g_s}{2}\bar{y}\right) = \mathcal{F}^{top,DC}\left(\frac{g_s}{2}y\right) + \mathcal{F}^{top,DC}\left(\frac{g_s}{2}\bar{y}\right) \qquad (5.15)$$

In principle we can now investigate the validity of the above correspondence beyond perturbation theory. This point was considered in Refs.[47, 127]. However, the methods used there involve manipulation of divergent series, and we will be able to derive all our correspondences using convergent integral representations of the relevant special functions.

Let us see how this works in some detail. As in Ref.[83], we consider the open-string dual of the DOC obtained from the Gopakumar-Vafa correspondence[23]. This theory lives on a resolved orbifolded conifold (ROC) with two P^1's whose (complex) size parameter is irrelevant in the B-model but which have respectively N_1, N_2 2-dimensional B-branes wrapped over them, where:

$$\begin{aligned} N_1 &= \frac{y}{2} = \frac{\hat{q}}{2} + i\mu_M \\ N_2 &= \frac{\bar{y}}{2} = \frac{\hat{q}}{2} - i\mu_M \end{aligned} \tag{5.16}$$

The number of branes in this correspondence is inevitably complex, and therefore a prescription is required to complexify starting from real integer values[1].

In the open string description, the partition function arises as follows. Using Eqs.(5.11) and (5.12), we find:

$$\begin{aligned} \mathcal{F}^{top,DOC}\left(\mu = \frac{g_s}{2}y, \mu' = \frac{g_s}{2}\bar{y}\right) &= \mathcal{F}^{top,ROC}\left(N_1 = \frac{y}{2}, N_2 = \frac{\bar{y}}{2}\right) \\ &= \mathcal{F}^{top,RC}\left(N = \frac{y}{2}\right) + \mathcal{F}^{top,RC}\left(N = \frac{\bar{y}}{2}\right) \end{aligned} \tag{5.17}$$

where in the last step, factorisation of the Hamiltonian $H(p,x)$ has been used.

On an ordinary resolved conifold, the free energy of N D-branes is given by the log of the matrix integral:

$$e^{-\mathcal{F}^{top,RC}(N)} = \frac{1}{\mathrm{vol}(U(N))}\int dM e^{-\frac{1}{2}\mathrm{tr}M^2} = \frac{(2\pi)^{\frac{N^2}{2}}}{\mathrm{vol}(U(N))} \tag{5.18}$$

Now we use[24]

$$\mathrm{vol}(U(N)) = \frac{(2\pi)^{\frac{1}{2}(N^2+N)}}{G_2(N+1)} \tag{5.19}$$

where $G_2(x)$ is the Barnes double-Γ function[136] defined by:

$$G_2(z+1) = \Gamma(z)G_2(z), \quad G_2(1) = 1 \tag{5.20}$$

[1]However, we see that the *total* number of branes in the background $N_1 + N_2 = \hat{q}$ is real and integer. This is striking, and somewhat reminiscent of fractional branes, though we do not have an explanation of this fact.

Thus we find

$$-\mathcal{F}^{top,RC}\left(N=\frac{y}{2}\right)-\mathcal{F}^{top,RC}\left(N=\frac{\bar{y}}{2}\right)=\left(\log G_2\left(\frac{y}{2}+1\right)-\frac{y}{4}\log 2\pi\right)+c.c. \tag{5.21}$$

Let us compare the above with what we know about the noncritical string starting from the matrix model. In Ref.[9] the authors have given a complete nonperturbative solution for the free energy of Type 0 noncritical strings at arbitrary radius R. The free energy of type 0A theory is given by:

$$-\mathcal{F}_{0A}(\mu_M,q,\tilde{q},R)=\Omega(y,R)+\Omega(\bar{y},R)+\frac{\pi\mu_M R}{2}(|q|-|\tilde{q}|) \tag{5.22}$$

where the function Ω is defined by the convergent (for $\mathrm{Re}\, y>-\left(1+\frac{1}{R}\right)$) integral:

$$\Omega(y,R)\equiv-\int_0^\infty\frac{dt}{t}\left[\frac{e^{-\frac{yt}{2}}}{4\sinh\frac{t}{2}\sinh\frac{t}{2R}}-\frac{R}{t^2}+\frac{Ry}{2t}+\left(\frac{1}{24}\left(R+\frac{1}{R}\right)-\frac{Ry^2}{8}\right)e^{-t}\right] \tag{5.23}$$

At the special radius $R=1$ it is easily shown from the integral form that:

$$\Omega(y,R=1)=\log G_2\left(\frac{y}{2}+1\right)-\frac{y}{4}\log 2\pi \tag{5.24}$$

where G_2 is the Barnes function discussed above.

If we temporarily ignore the last term in Eq. (5.22), we see that the free energy is the sum of holomorphic and antiholomorphic contributions. Moreover, each of these is known to be the (complexified) free energy of the bosonic $c=1$ string at radius R[6]. This is in agreement with Eqs.(5.14),(5.15).

However, the last term in Eq. (5.22) does not seem to come from the topological string. We will discuss this issue in the following section. First we will generalise the considerations of this subsection to the case where the radius of the time circle is different from $R=1$, in particular to integer radii. We will also comment on the case of rational radii $R=\frac{p}{p'}$.

5.2.3 Integer radius

We have seen that the $c=1$ bosonic string at $R=p$ (an integer multiple of the self-dual radius $R=1$) is dual to a topological string living on a Z_n orbifold of the conifold. An analogous result has been proposed for the type 0A string[47]. We will provide a simple and general derivation of this result using only properties of convergent integral representations.

Inserting the value $R = p$ into the expression for Ω, Eq. (5.23), we rewrite the first term in the integrand:

$$\frac{e^{-\frac{yt}{2}}}{4\sinh\frac{t}{2}\sinh\frac{t}{2p}} \to \frac{e^{-\frac{yt}{2}}}{4(\sinh\frac{t}{2})^2}\frac{\sinh\frac{t}{2}}{\sinh\frac{t}{2p}} \tag{5.25}$$

Next, use:

$$\frac{\sinh\frac{t}{2}}{\sinh\frac{t}{2p}} = \sum_{k=1}^{p} e^{\frac{t}{2p}(p-(2k-1))} \tag{5.26}$$

Now define:

$$y_k = y + \frac{-p+(2k-1)}{p}, \quad k = 1, 2, \ldots, p \tag{5.27}$$

Using the identities:

$$\begin{aligned}\sum_{k=1}^{p}\frac{1}{t^2} &= \frac{p}{t^2}\\ \sum_{k=1}^{p}\frac{y_k}{2t} &= \frac{py}{2t}\\ \sum_{k=1}^{p}\left(\frac{1}{12}-\frac{y_k^2}{8}\right) &= \frac{1}{24}\left(p+\frac{1}{p}\right)-\frac{py^2}{8}\end{aligned} \tag{5.28}$$

of which only the third one is not completely obvious, but nonetheless easy to prove. It follows that:

$$\Omega\Big(y, R = p\Big) = \sum_{k=1}^{p}\Omega(y_k, R = 1) \tag{5.29}$$

We see that the free energy at rational radius factorises into $2p$ distinct contributions, of which p are holomorphic in y and the remaining are anti-holomorphic. Each of the contributions corresponds to a theory at $R = 1$, or equivalently to the contribution of topological B-branes. The factorisation is exact.

Let us analyse this in some more detail. First, by definition $\mathrm{Re}\, y \geq 0$, which not only ensures convergence of the LHS of Eq. (5.29), but also ensures that the RHS is convergent since this implies that $\mathrm{Re}\, y_k > -\left(1+\frac{1}{R}\right) = -2$ for all k. Therefore the equality is between convergent integral representations as promised.

From the above result we can conclude that for type 0 string theory at every integer radius $R = p$, there is an exact noncritical-topological correspondence where the corresponding topological string lives on a Z_{2p} orbifold[44] of the conifold, whose deformed version is:

$$zw - \prod_{k=1}^{k=p}(px - \mu_k)\prod_{k=1}^{k=p}(px - \bar{\mu}_k) \tag{5.30}$$

where

$$\mu_k = \frac{g_s}{2} y_k \tag{5.31}$$

and y_k are defined in Eq. (5.27). This manifold has $2p$ independent 3-cycles that occur in complex conjugate pairs. The factorisation into contributions from these cycles is nonperturbatively exact upto non-universal terms, and even those terms vanish identically at integer 0A radius.

The resolved version of this correspondence would involve the same Z_{2p} orbifold of the conifold but now with the $2p$ singularities blown up into P^1's with N_k B-branes wrapped over each of the first p cycles, and the complex conjugate number of branes on the remaining p cycles, where:

$$N_k = \frac{y_k}{2} \tag{5.32}$$

As before, the partition function in this picture arises from the vol($U(N)$ factors associated to each set of N_k branes, giving the most direct derivation of the noncritical-topological correspondence.

This generalized correspondence too can be phrased in topological-anti-topological language. In this case the topological theory lives on a Z_p orbifold, with p cycles labelled by an integer k and N_k branes wrapped on each of them. The remaining contribution to the free energy arises on combining with the anti-topological version of this theory.

5.2.4 Rational radius

Let us now consider more general rational radii of the form $R = \frac{p}{p'}$, with p and p' co-prime. A similar derivation to the previous one goes through in this case, though the interpretation presents some subtleties that we will discuss.

Inserting the value of R into the expression for Ω, Eq. (5.23), we send $t \to \frac{t}{p'}$ and then rewrite the first term in the integrand:

$$\frac{e^{-\frac{yt}{2p'}}}{4\sinh\frac{t}{2p'}\sinh\frac{t}{2p}} \to \frac{e^{-\frac{yt}{2p'}}}{4(\sinh\frac{t}{2})^2}\frac{\sinh\frac{t}{2}}{\sinh\frac{t}{2p'}}\frac{\sinh\frac{t}{2}}{\sinh\frac{t}{2p}} \tag{5.33}$$

Using Eq. (5.26) and defining:

$$y_{k,k'} = \frac{y - p' + (2k'-1)}{p'} + \frac{-p + (2k-1)}{p}, \quad k = 1, 2, \ldots, p; \quad k' = 1, 2, \ldots, p' \tag{5.34}$$

we find the following identities, generalising Eq. (5.28):

$$\begin{aligned} \sum_{k=1}^{p}\sum_{k'=1}^{p'} \frac{1}{t^2} &= \frac{pp'}{t^2} \\ \sum_{k=1}^{p}\sum_{k'=1}^{p'} \frac{y_{k,k'}}{2t} &= \frac{py}{2t} \\ \sum_{k=1}^{p}\sum_{k'=1}^{p'} \left(\frac{1}{12} - \frac{y_{k,k'}^2}{8}\right) &= \frac{1}{24}\left(\frac{p}{p'} + \frac{p'}{p}\right) - \frac{py^2}{8p'} \end{aligned} \tag{5.35}$$

Thus we find:

$$\Omega\Big(y, R = \frac{p}{p'}\Big) = \sum_{k'=1}^{p'}\sum_{k=1}^{p} \Omega(y_{k,k'}, R = 1) - \left(\frac{1}{24}\Big(\frac{p}{p'} + \frac{p'}{p}\Big) - \frac{py^2}{8p'}\right)\log p' \tag{5.36}$$

Thus, at rational radius the free energy factorises into $2pp'$ distinct contributions, of which pp' are holomorphic in y and the remaining are anti-holomorphic. However, in general the factorisation is exact only upto an analytic and therefore non-universal term. If we consider the special case of $p' = 1$, corresponding to integer radius in the type 0A theory, then the non-universal term vanishes. On the other hand if we take $p = 1$, corresponding to even integer radius in the type 0B theory, then the non-universal term is present. Subtracting the two expressions (after scaling $y \to ym$ in one of them) we find:

$$\Omega\left(ym, R = \frac{1}{m}\right) - \Omega\left(y, R = m\right) = -\left(\frac{1}{24}(m + \frac{1}{m}) - \frac{y^2}{8m}\right)\log m \qquad (5.37)$$

which is precisely Eq.(A.39) of [9]. There, we see that the apparent violation of T-duality by the extra term is actually harmless and can be understand as due to the difference in natural cutoffs for type 0A and 0B. This explains the presence of the non-universal term, and confirms that its presence can be ignored.

We would now like to interpret the above factorisation property in terms of contributions from singularities. For the bosonic string, the original ground ring analysis of Ref.[44] tells us that the (singular) ring at $R = \frac{p}{p'}$ is a $Z_p \times Z_{p'}$ orbifold of the conifold. Assuming that in type 0 strings the parameters are complexified and occur in complex-conjugate pairs, we expect in this case to find a $Z_{2p} \times Z_{2p'}$ orbifold of the form:

$$\prod_{k'=1}^{p'}(zw - \alpha_{k'})\prod_{k'=1}^{p'}(zw - \bar{\alpha}_{k'}) = \prod_{k=1}^{p}(px - \beta_k)\prod_{k=1}^{p}(px - \bar{\beta}_k) \qquad (5.38)$$

for some set of $p + p'$ complex parameters $\alpha_{k'}, \beta_k$. Such a space no longer has an interpretation as a fibration over a Riemann surface and the analysis of its partition function is therefore more complicated. We expect that for some (not necessarily simple) choice of the parameters, the free energy on this space can be written as a sum of terms as in Eq. (5.36) but will not be able to show this here.

An alternate interpretation of the factorised free energy is that it corresponds to a $Z_{2pp'}$ orbifold of the conifold:

$$zw - \prod_{\substack{k=1\\k'=1}}^{\substack{k=p\\k'=p'}}(px - \mu_{k,k'})\prod_{\substack{k=1\\k'=1}}^{\substack{k=p\\k'=p'}}(px - \bar{\mu}_{k,k'}) \qquad (5.39)$$

where

$$\mu_{k,k'} = \frac{g_s}{2}y_{k,k'} \qquad (5.40)$$

and $y_{k,k'}$ are defined in Eq. (5.34). This manifold has $2pp'$ independent 3-cycles that occur in complex conjugate pairs. The resolved version of this space has the

$2pp'$ singularities blown up into P^1's with $N_{k,k'}$ B-branes wrapped over each of the first pp' cycles, and the complex conjugate number of branes on the remaining pp' cycles, where:

$$N_{k,k'} = \frac{y_{k,k'}}{2} \tag{5.41}$$

The advantage of this latter interpretation is that it preserves the fibred structure of the manifold with a Riemann surface as the base, and therefore all previous computations manifestly go through in the same way. Unfortunately this interpretation is at variance with the original proposal[43] that the variety occurring on the topological B-model side is in correspondence with the ground ring on the noncritical side.

5.3 Disc amplitudes and noncompact branes

5.3.1 $R = 1$

In the correspondence between noncritical type 0A strings and the B-model on the conifold Eq. (5.13) (and more generally Eq. (5.30)) that we have discussed above, there is right away a puzzle. The former depends on three parameters, $q, \tilde{q}, \mu_M$, which in the continuum Liouville description arise as the two independent RR fluxes and the cosmological constant (in the matrix model description these three parameters arise as a net D-brane number, a Chern-Simons term and the Fermi level respectively[9, 10]). However the topological dual only depends on the complex number $y = |q| + |\tilde{q}| + 2i\mu_M$, and therefore on only two of these three parameters. It reproduces most of the free energy, which indeed depends only on two parameters and is the sum of mutually complex conjugate terms. However, the extra term in the free energy:

$$\mathcal{F}^{disc,2} = -\frac{\pi R}{2}\mu_M(|q| - |\tilde{q}|) \tag{5.42}$$

is unaccounted for (the reason for the label on this contribution will become clear shortly).

This term is responsible for an important effect. From the factorised part of the free energy one extracts the following disc contribution in the limit of large

μ and fixed $\hat{q}$[9]:

$$\mathcal{F}^{disc,1} = +\frac{\pi R}{2}|\mu_M|(|q| + |\tilde{q}|) \tag{5.43}$$

Hence the total disc amplitude is:

$$\mathcal{F}^{disc} = \mathcal{F}^{disc,1} + \mathcal{F}^{disc,2} = \frac{\pi R}{2}\Big[\big(|\mu_M| - \mu_M\big)|q| + \big(|\mu_M| + \mu_M\big)|\tilde{q}|\Big] \tag{5.44}$$

This can be written as:

$$\begin{aligned} \mathcal{F}^{disc} &= (2\pi R)\frac{\mu_M}{2}|\tilde{q}|, \quad \mu_M > 0 \\ &= (2\pi R)\frac{|\mu_M|}{2}|q|, \quad \mu_M < 0 \end{aligned} \tag{5.45}$$

The physical interpretation is that for $\mu_M > 0$ the RR flux of $\tilde{q}$ units associated to the gauge field $\tilde{A}$ is supported by $|\tilde{q}|$ ZZ branes in the vacuum, with the contribution per brane to the free energy being given by the product of its extent in Euclidean time $(2\pi R)$ and its tension $(\frac{|\mu_M|}{2})$. The other flux of q units associated to the gauge field A has no source. Similarly for $\mu_M < 0$ the vacuum contains $|q|$ ZZ branes sourcing the first flux while the other flux of $\tilde{q}$ units is not supported by any source..

Note that in the absence of the term $\mathcal{F}^{disc,2}$ there is no satisfactory physical interpretation of the disc amplitude in terms of ZZ branes. This makes the term extremely important for a consistent noncritical string theory.

We now propose that the missing term is supplied, on the topological side, by noncompact B-branes wrapping a degenerate fibre of the Calabi-Yau over the Riemann surface $H(p, x) = 0$. Such branes have been extensively studied in Refs.[78, 137] where they have been shown to give rise to the Kontsevich parameters of topological matrix models. These branes are, in particular, fermionic. Since we are considering the free energy of the string theory, we work in the vacuum where such Kontsevich branes are absent. However, as we now explain, it is still possible to place noncompact branes at infinity on the Riemann surface and they can reproduce just the desired term in the free energy.

Consider the case $R = 1$. Suppose we place a single noncompact B-brane along one branch of the degenerate fibre over a point x on the Riemann surface.

We would like to isolate its contribution to the free energy compared with that of a brane at a fixed reference position x_*, or in other words we assume that the brane is asymptotically at x_* but its interior region has been moved to x. The action of such a brane has been shown[78, 137] to be[1]:

$$S(x) = \frac{1}{g_s}\int_{x_*}^{x} p(z)\, dz \tag{5.46}$$

As we have seen, for the case of interest to us the Riemann surface consists of two disjoint factors:

$$xp = \frac{g_s}{2}y, \quad xp = \frac{g_s}{2}\bar{y} \tag{5.47}$$

Thus a brane on the first branch contributes:

$$S(x) = \frac{\mu}{g_s}\ln\frac{x}{x_*} \tag{5.48}$$

Let us now place one noncompact brane above each of the two branches, and take their asymptotic positions to be at x_*, x'_* which will both be sent to infinity. Then their total contribution to the free energy is:

$$S(x,x') = \frac{1}{2}\left(y\,\ln\frac{x}{x_*} + \bar{y}\,\ln\frac{x'}{x'_*}\right) \tag{5.49}$$

Now we will choose our branes such that x, x' are also at infinity, but rotated by angles θ, θ' respectively along the circle at infinity relative to the original points x_*, x'_*. Namely:

$$x = x_*\, e^{i\theta}, \quad x' = x'_*\, e^{i\theta'} \tag{5.50}$$

It follows that:

$$\begin{aligned} S(x_1,x_2) &= \frac{i}{2}(y\,\theta + \bar{y}\,\theta') \\ &= -\mu_M(\theta-\theta') + i\frac{\hat{q}}{2}\,(\theta+\theta') \end{aligned} \tag{5.51}$$

The factors of g_s have conveniently cancelled out, and the real part of the above contribution is proportional to μ_M. Now if we choose:

$$\theta = -\theta' = \frac{\pi}{4}(|q| - |\tilde{q}|) \tag{5.52}$$

[1] In the language of Ref.[78], we place the branes in the "x-patch" and never move them to the "p-patch".

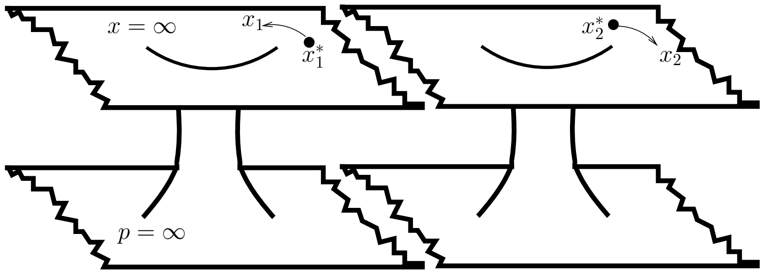

Figure 5.1: The Riemann surface with noncompact branes at infinity.

we find that the noncompact branes give a contribution:

$$S = -\frac{\pi}{2}\mu_M(|q| - |\tilde{q}|) \tag{5.53}$$

to the free energy, precisely equal to that in Eq. (5.42) at $R = 1$.

To summarise, we have shown that if we place a noncompact B-brane at $x \to \infty$ on each branch of the Riemann surface

$$H(p, x) = (px - \mu)(px - \mu') = 0 \tag{5.54}$$

and moreover require that the branes wind at infinity by the angles in Eq. (5.52), we precisely reproduce the disc contribution to the free energy of Eq. (5.42). This situation is depicted in Fig.5.1.

This then completes the definition of the topological dual to type 0A strings at the special radius.

The above system also has a description in topological-anti-topological language. As we have seen, the topological theory then lives on the pure conifold, having a Riemann surface with only one branch. Now we place a single noncompact brane on it with winding angle θ given by Eq. (5.52). Adding the anti-topological theory introduces the second noncompact brane with winding $-\theta$ and we recover the correct free energy.

5.3.2 Integer and rational radius

Let us now extend these considerations to other integer radii. At radius $R = p$, we have the possibility of placing noncompact branes at infinity on each of

$2n$ branches of the Riemann surface $H(x,p) = 0$ obtained from Eq. (5.30). Parametrising the angles by which these branes wind as:

$$x_i = x_{*i}\, e^{i\,\theta_i}, \quad x'_i = x'_{*i}\, e^{i\,\theta'_i}, \tag{5.55}$$

the contribution of these branes to the free energy is:

$$\begin{aligned} S(x_i, x'_i) &= \frac{i}{2}\sum_{j=1}^{n}\left(y_j\,\theta_j + \bar{y}_j\,\theta'_j\right) \\ &= -\mu_M \sum_{j=1}^{n}(\theta_j - \theta'_j) + i\sum_{j=1}^{n}\frac{\hat{q}_j}{2}(\theta_j + \theta'_j) \end{aligned} \tag{5.56}$$

where

$$\hat{q}_j = \hat{q} - 1 + \frac{2j-1}{n}, \quad j = 1, 2, \ldots, n \tag{5.57}$$

It is natural to take

$$\theta_j = -\theta'_j = \frac{\pi}{4}\left(|q| - |\tilde{q}|\right), \quad \text{all } j = 1, 2, \ldots, n \tag{5.58}$$

which leads to a contribution to the free energy:

$$S = -\frac{\pi n}{2}\mu_M(|q| - |\tilde{q}|) \tag{5.59}$$

in precise agreement with Eq. (5.42) for $R = n$.

It appears as if in this case the noncompact brane configuration is not unique. However, note that choosing $\theta_j = -\theta'_j$ for all j is essential to make the free energy real. After this, the choice we have made is the most symmetric one which gives the correct disc amplitude.

In the topological-anti-topological approach, we would instead have p branches in the Riemann surface and therefore p noncompact branes with associated angles θ_k. The remaining noncompact branes with angles $-\theta_k$ then arise on the anti-topological side.

It is quite nontrivial that we were able to reproduce the subtle disc term by a simple configuration of noncompact branes in every case. The scaling with g_s of the holomorphic Chern-Simons action and of the complex-structure moduli $\mu_{k,k'}$ defined in Eq. (5.40) exactly cancel out. Moreover, $\mu_{k,k'}$ all have a common imaginary part proportional to μ_M. These facts were important in allowing us to obtain the desired contribution from noncompact branes.

Now let us briefly consider rational radius. If we accept the $Z_{2p} \times Z_{2p'}$ orbifold interpretation of Eq. (5.38) then it is not clear how to extend the above considerations to radius $R = \frac{p}{p'}$. This is because the manifold is no longer of the form $zw = H(p, x)$ and therefore the Riemann surface interpretation itself needs to be generalized, which lies beyond the scope of the present work.

5.4 Discussion

One of our main results has been that the noncritical-topological correspondence for type 0 noncritical strings has to include noncompact branes on the topological side. This introduces a dependence on a new parameter which we interpret as $|q| - |\tilde{q}|$ on the noncritical side, and renders the duality consistent with the dependence of the noncritical theory on three parameters: μ_M, q and $\tilde{q}$.

The identification between the phases of noncompact branes and the parameter $|q| - |\tilde{q}|$, via Eq. (5.52), appears rather ad hoc. From Eq. (5.52) it is tempting to imagine that there could be a missing normalisation factor of 8 which changes $\frac{\pi}{4}$ to 2π. In that case one could have postulated that the noncompact branes have an integer winding at infinity and this integer gets identified with the integer $|q| - |\tilde{q}|$. This would make the identification a little less ad hoc. However we did not find such a missing normalisation factor.

Given that the subtle disc term is required in the noncritical string by consistency, one may ask if the presence of noncompact branes in the topological theory is also a consistency requirement. However, this seems not to be the case. On the noncritical side there is the possibility of ZZ branes in the vacuum, and it is only after including the subtle term that the vacuum has a definite intepretation as containing or not containing such branes. However ZZ branes do not (so far) have a direct analogue on the topological side and so it is possible that the topological theory without the subtle disc term, and hence with an exactly holomorphically factorised free energy, is consistent by itself. The only thing that would fail is its correspondence to the noncritical theory. Nevertheless it would be interesting if there were a way to understand ZZ branes from the noncritical side. It would be equally interesting to understand the presence[9] of $q\tilde{q}$ fundamental strings in the vacuum, for which we have not found a direct topological explanation.

We also found that the free energy of the full type 0A theory has a nonperturbatively exact factorisation into contributions from compact and noncompact branes. Apparently there is no room for any interactions between these different branes, or in other words the open strings stretched among any two of these branes (both compact, or both noncompact, or one of each) seem to decouple completely. This is somewhat puzzling but must be related in some way to the topological nature of the theory as well as to having distinct branches of the Riemann surface $H(p,x) = 0$.

It is amusing that compact and noncompact branes make use of different pieces of the holomorphic Chern-Simons theory restricted to a 2-cycle[138]:

$$S = \frac{1}{g_s}\int \mathrm{tr}(\Phi_1 \bar{D}\Phi_0 + W(\Phi_0)\omega) \tag{5.60}$$

For compact branes, the first term can be shown to be irrelevant while the second one gives a matrix-valued superpotential, which for our case is simply an independent quadratic for each branch of the Riemann surface. For noncompact branes it is the second term which is irrelevant (because the volume is infinite, we subtract the free energy of deformed compact branes from the undeformed ones[137]) while the first term leads to the expression $\int pdx$. In this case there is no matrix model, because we have placed only one brane on each branch.

It is clearly of interest to generalise our construction to include more noncompact branes that act as sources for incoming closed-string tachyons on the noncritical side[1], as well as non-normalisable deformations of the conifold which are associated to outgoing tachyons[78]. When carried out for the general orbifolded conifold Eq. (5.30), this will provide the analogue of the Normal Matrix Model[25, 48] for type 0 strings, valid for all rational radius. This is an important generalisation of the KP[26] model which has already been found using

[1] Our noncompact branes do not act as such sources precisely because they are located at $x \to \infty$.

the topological string construction for both bosonic $c = 1$ strings[78] and type 0 strings[46].

As is well known, topological descriptions of noncritical strings are simplest at $R = 1$ in appropriate units, and can then be generalized to integer multiples of this radius, as done before for bosonic strings and in this chapter for type 0 strings. In this way we can describe the Euclidean or finite-temperature version of the theory. To get to the zero-temperature case one then has to take the limit $R \to \infty$. This limit has been explored before, most recently in Ref.[139] where it was related to deconstruction. Our analysis in the present work can potentially add something to this story. Consider Eq. (5.34) at $p' = 1$ and take $p \to \infty$. In this limit we find that $y_{k,k'}$ varies continuously in the open interval $(y-1, y+1)$. From Eq. (5.12) this amounts to saying that the RR flux effectively varies continuously over the same interval. This suggests a higher-dimensional origin and may again link the topological theory to deconstruction in some way.

As we have seen, all our considerations extend to the topological-anti-topological picture of Ref.[47], which seems more natural in one sense. The Z_{2p} orbifolded conifold has in principle $2p$ independent complex structure parameters $\mu_k, \bar{\mu}_k$. The noncritical-topological correspondence requires half of them to be constrained to be complex conjugates of the other half, which is naturally achieved if we think of the system in the topological-anti-topological way. In that case only the p parameters μ_k can be independent. However, as we have seen, the μ_k are all determined in terms of two parameters embodied in y, and the topological-anti-topological picture does not seem to help in explaining this fact. Therefore, if it is to be genuinely useful, perhaps it needs to be extended to a generalized principle where the holomorphic part of the free energy further factorises into contributions from p independent theories.

Chapter 6

Conclusions and Open Questions

In the work presented so far we have investigated several aspects of non-critical string theory. We have found the following results:

i) In the work presented in Chapter 2 we have demonstrated an interesting correspondence valid at the the self dual radius $R = 1$ between two matrix model descriptions of the $c = 1$ non-critical bosonic string. The Normal Matrix model derived by Alexandrov, Kazakov and Kostov [25] depends only on closed string parameters which are the couplings to momentum deformations, while the Konsevich-Penner model derived by Imbimbo and Mukhi [26] depends on open string-like parameters, which we propose should have an interpretation as the boundary cosmological constants for FZZT-like branes in the matrix model. Our map between the two models thus seems to encode open-closed string duality for the $c = 1$ noncritical string.

ii) Using the Normal Matrix Model for the $c = 1$ string, we have developed a technique to easily calculate arbitrary correlation functions of momentum modes to all orders at a general radius R. In Chapter 3 we use this method to give an answer for the $2n$-point correlator of unit momentum modes. This particular correlator can be used to find the partition function for a condensate of unit winding modes, which is dual to the two dimensional black hole (known as the FZZ duality). Techniques for performing exact computations for winding mode correlators are not known at present. Our results can be used to explicitly check T-duality for the matrix model once computations for these winding correlators are available.

iii) In Chapter 4 we propose a new interpretation for the FZZ duality between the $c = 1$ non-critical string perturbed by Sine-Liouville operators and the two dimensional black hole. This is done by introducing a new term in the action which is a Sine-Liouville term with a different Liouville dressing. We show that this change makes the connection between the Sine-Liouville and the black hole CFT's more transparent. Developing this argument further we consider a generalized version of the FZZ duality where the non-critical string action is perturbed by higher winding modes. We conjecture that the multiply-wound Sine-Liouville operators are linked to higher-spin black holes.

iv) Topological string theory on a conifold space provides an alternative description of the non-critical string theory. We investigate this connection in context of the Type 0 non-critical string in Chapter 5. We present a complete non-perturbative topological dual to the Type 0A non-critical string. Non-compact topological branes wrapped over a degenerate fibre of the conifold space turn out to be an important ingredient which allows us to construct the topological dual.

Finally, we list some unsolved problems which are suitable for future work:

- The continuum counterparts of the inverse determinant operators that we encountered in the matrix models in Chapter 2 are not known. Knowledge of these operators would pin-point the corresponding open string degrees of freedom and thus clarify open/closed duality for the non-critical string.

- The equivalent of the Normal Matrix Model for general radius R for Type 0 theories is not known (however, there is a Kontsevich-like two matrix model valid at $R = 1$ [46]). Such a model has an immediate application in computing exact correlation functions in Type 0 theories, extending our work in Chapter 3.

- The vortex condensate for bosonic and Type 0 strings have not been studied in detail because of the technical difficulty of computing the infinite sum over $2n$-point functions. As mentioned earlier, it is also needed to show T-duality in the matrix model.

- The CFT dual to the generalized Sine-Liouville theory is not completely solved, as the continuum treatment of this CFT proves difficult. If a double-scaled matrix model dual is derived, we can use it to calculate the higher winding correlators and check our generalized FZZ conjecture.

It is hoped that some of these results will illuminate the more physically relevant critical string theories.

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