A Study of Five-Dimensional Small Black Rings and Closed String Tachyon Potential

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بسم الله الرحمن الرحيم

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اَلحَمدُ لِلّٰهِ رَبِّ العٰلَمِينَ

Whose hand and tongue is capable
To fulfil the obligations of thanks to Him? – Sa'di

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Ashik Iqubal

In my capacity as supervisor of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

Atish Dabholkar

Contents

Publications

- *Spinning Strings as Small Black Rings*, A. Iqubal, A. Dabholkar, N. Iizuka and M. Shigemori, hep-th/0611166
- *Precision Microstate Counting of Small Black Rings*, A. Iqubal, A. Dabholkar, N. Iizuka and M. Shigemori, Phys. Rev. Lett. **96**, 071601 (2006), hep-th/0511120
- *Off-shell interactions for closed-string tachyons*, A. Iqubal, A. Dabholkar, and J. Raeymaekers, JHEP **0405** (2004) 051, hep-th/0403238

A Study of 5-D Small Black Rings

A Study of 5-D Small Black Rings – Introduction

Black holes are a useful testing ground for the string theory approach to quantum gravity. One of the successes of string theory has been that for supersymmetric black holes with classically large area, Bekenstein-Hawking entropy can be explained in terms of statistical counting of microstates. The low energy string theory effective action gives higher order α' corrections to Einstein's equation which generate interesting stringy corrections to the classical geometry. Classically singular geometry maybe cloaked by stringy corrections. For example, small black objects (eg. small black holes) whose event horizons are classically zero can acquire a finite horizon via stringy corrections. Since, $\mathsf{entropy} \propto \mathsf{horizon\ size}$, they also acquire a non-zero entropy. These objects are very instructive examples of stringy black holes as *stringy* corrections generate a finite size which also correspond to *quantum* states giving a *non-zero* entropy which can be exactly counted using *stringy* techniques. In fact, for these small black holes, the macroscopic and miscrosopic contribution to the entropy can be computed and compared to all orders in perturbative expansion. Thus they are ideal toy objects for understanding the nature of black hole microstates, how they are generated, and for understanding the microscopic (quantum) properties of gravitating objects. They also suggests how Bekenstein-Hawking entropy formula needs to be modified to include quantum corrections.

So far our work has focused on how higher derivative terms (α' corrections) can desingularize the horizons of some spinning 4D/5D black holes and 5D black rings, and how to *exactly* calculate the microstate entropies of such black holes with stretched horizons.

Black Holes

Black holes are a fascinating research area for many reasons. They appear to be a very important constituent of our Universe. There are super-massive black holes at the centers of most galaxies. Small black holes are formed as remnants of certain supernovas.

Theoretically, black holes provide an arena to explore the challanges posed by the reconciliation of general relativity and quantum mechanics. Since string theory claims to be a consistent quantum theory of gravity, it should be able to address these challanges. Some of the most fascinating developements in string theory concern the quantum-mechanical aspects of black-hole physics.

The Einstein-Hilbert action for general relativity in D dimensions without any sources is given by

$$S = \frac{1}{16\pi G_D} \int d^D x \sqrt{-g} R \tag{2.1}$$

where G_D is the D-dimensional Newton constant. The classical equation of motion is

$$R_{\mu\nu} = 0 \tag{2.2}$$

whose solutions are Ricci-flat space-times. Generalizations include adding electromagnetic fields or tensor fields of various sorts. Some of the most interesting solutions describe black holes. They have singularities at which certain curvature invariants diverge. In most cases these singularities are cloaked by an *event horizon*, which is a hypersurface separating those space-time points connected to infinity by time-like path from those that are not. *Cosmic censorship conjecture* states that space-time singularities should always be cloaked by a horizon in physically allowed solutions. Classically, black holes are stable objects, whose mass increases only when matter or radiation crosses the horizon and becomes trapped forever. But quantum mechanically, black holes have thermodynamic properties, and they can decay by the emission of thermal radiation.

2.1 Einstein-Maxwell system

The Einstein-Hilbert action for gravity coupled to Maxwell system is given by:

$$S = \frac{1}{16\pi G}\int d^4x(-g)^{1/2}[R - GF_{\mu\nu}F^{\mu\nu}] \tag{2.3}$$

where,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.4}$$

The equation of motion for this system resulting from varying with respect to the metric $g_{\mu\nu}$ is:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{2.5}$$

where,

$$T_{\mu\nu} = \frac{1}{4\pi}(g^{\gamma\delta}F_{\mu\gamma}F_{\nu\delta} - \frac{1}{4}g_{\mu\nu}F_{\gamma\delta}F^{\gamma\delta}) \tag{2.6}$$

In 4-dimensions, for a spherically symmetric source of mass M and electric charge Q, the solution is given by the *Reissner-Nordstrom* metric,

$$ds^2 = -(1 - \frac{2MG}{r} + \frac{Q^2}{r^2})dt^2 + (1 - \frac{2MG}{r} + \frac{Q^2}{r^2})^{-1}dr^2 + r^2 d\Omega_2^2 \tag{2.7}$$

where $d\Omega_2^2 \equiv d\theta^2 + sin^2\theta d\phi^2$ is the metric of a S^2, and

$$F = \frac{Q^2}{r^2}dt \wedge dr \tag{2.8}$$

2.1.1 Schwarzschild Black Hole in D=4

In the case $Q = 0$, the empty-space solution of pure Einstein gravity is the Schwarzschild solution. It's metric is given by

$$ds^2 = -(1 - \frac{2MG}{r})dt^2 + (1 - \frac{2MG}{r})^{-1}dr^2 + r^2 d\Omega_2^2 \tag{2.9}$$

The radius $r_H = 2GM$ is known as the Schwarzschild radius. The Schwarzschild metric depends only on the total mass M, and it reduces to the Minkowski metric as $M \to 0$. Note that t is a time-like co-ordinate for $r \geq r_H$ and space-like cor-ordinate for $r \leq r_H$, while the reverse is true for r. The surface $r = r_H$ is called the *event horizon*. This metric is stationary because the metric components are independent of the time co-ordinate t.

The coefficients of the Schwarzschild metric becomes singular at $r = 0$ and also at the Schwarzschild radius $r = r_H$. To determine if the singularities are co-ordinate or physical singularities, we need to calculate the Ricci Scalar R. For Schwarzschild black hole in $D = 4$, the Ricci Scalar can be calculated to be

$$R = \frac{12r_H^2}{r^6} \tag{2.10}$$

This suggests that the singularity at $r = r_H$ is just a co-ordinate singularity, while there is a physical singularity located at $r = 0$.

2.1.2 Schwarzschild Black Hole in D dimensions

In D dimensions, the Schwarzschild metric takes the form

$$ds^2 = -[1 - (\frac{r_H}{r})^{D-3}]dt^2 + [1 - (\frac{r_H}{r})^{D-3}]^{-1}dr^2 + r^2 d\Omega_{D-2}^2 \tag{2.11}$$

where,

$$r_H = \frac{16\pi M G_D}{(D-2)\Omega_{D-2}} \tag{2.12}$$

and Ω_D is the volume of a unit D-sphere S^D, ie,

$$\Omega_D = \frac{2\pi^{(D+1)/2}}{\Gamma(\frac{D+1}{2})} \tag{2.13}$$

2.1.3 Reissner-Nordstrom Black Hole for D=4

The generalized black hole in the presence of electric charge Q, but without any angular momentum, is called *Reissner-Nordstrom* black hole. Charged black holes play an important role in string theory. In 4-dimensions the metric of a Reissner-Nordstrom black hole can be written in the form

$$ds^2 = -(1 - \frac{2MG}{r} + \frac{Q^2}{r^2})dt^2 + (1 - \frac{2MG}{r} + \frac{Q^2}{r^2})^{-1}dr^2 + r^2 d\Omega_2^2 \tag{2.14}$$

The metric components are singular for three values of r. There is a physical singularity at $r = 0$, which can be verfied by computing the Ricci Scalar R. The g_{tt} component of the metric vanishes for

$$r = r_\pm = MG_4 \pm \sqrt{(MG_4)^2 - Q^2 G_4} \tag{2.15}$$

which are referred to as the *inner horizon* and the *outer horizon*. The outer horizon r_+ is the event horizon for this black hole. These solutions exists only if the following bound is satisfied:

$$Q \leq M\sqrt{G_4} \tag{2.16}$$

If this bound is not satisfied, then the metric has a naked physical singularity at $r = 0$. According to *cosmic censorship conjecture*, such solutions are unphysical.

Extremal Reissner-Nordstrom Black Hole

In the limiting case when

$$Q = M\sqrt{G_4} \tag{2.17}$$

the inner horizon and the outer horizon conincide to

$$r_{\pm} = MG_4 \tag{2.18}$$

This limiting case is called an extremality condition and the black hole is called an *extremal black hole*. It has the maximum allowed charge for a given mass, as follows from the bound. When Reissner-Nordstrom black hole solution follows from a supersymmetric theory, the saturation of this extremality bound is often equivalent to the saturation of the BPS bound. This implies that the Reissner-Nordstrom black hole has some unbroken supersymmetry.

The metric of an extremal Reissner-Nordstrom black hole is given by

$$ds^2 = -(1 - \frac{MG_4}{r})^2 dt^2 + (1 - \frac{MG_4}{r})^{-2} dr^2 + r^2 d\Omega_2^2 \tag{2.19}$$

Black Holes in String Theory

Since Einstein's equations are obtained in the low energy supergravity limit of string theory, Black holes appear quite naturally in string theory. These black holes are supersymmetric extremal black holes, and include three-charge black hole solutions in five dimensions and four-charge black hole solutions in four dimensions.

3.1 Attractor Mechanism

Black holes are obtained when string theory or M-theory is compactified to lower dimensions and the branes are wrapped on non-trivial cycles of the compact manifold. The compactified theories have many moduli and appear as part of black hole solutions. These moduli can take any values at infinity (where the geometry is flat). As a result there is a dangerous possiblity that the black hole entropy depends on the continious values of the moduli fields. This is a problem because the number of microstates of a black hole with given charges is an integer and cannot depend on paraments which can be varied continiously. The black hole entropy should only depend on discrete quantities such as the charges of the black hole.

To resolve this puzzle, one has to realise that the black hole entropy depends on the behaviour of the solution at the horizon, and that the moduli fields takes on fixed values at the horizon (irrespective of their values at infinity) completely determined by black hole charges. The radial dependence of these moduli fields are determined by *attractor equations* whose solution flows to definite values at the horizon, regardless of their values at infinity. This is what is meant by *attractor mechanism.* The existence of attractor mechanism is necessary for a microscopic description of the black hole entropy.

3.1.1 Special Geometry and Prepotential Function

The action of $N = 2$ supergravity coupled to n vector multiplets is described by a prepotential function $F(X^I)$, which is a function of $n + 1$ complex scalars $X^I(I = 0, 1, .., n)$. $F(X^I)$ is a homogenous function of X^I's of degree 2, and X^I's are are projective co-ordinates. The bosonic degrees of freedom described by the prepotential are the metic $g_{\mu\nu}$, n complex scalars X^I/X^0, and $n + 1$ gauge fields A^I_μ.

Let us define the following:

$$F_I \equiv \frac{\partial F}{\partial X^I} \tag{3.1}$$

$$F_{IJ} \equiv \frac{\partial^2 F}{\partial X^I \partial X^J} \tag{3.2}$$

$$N_{IJ} \equiv \frac{1}{4}(F_{IJ} + \bar{F}_{IJ}) \tag{3.3}$$

$$\mathcal{N}_{IJ} \equiv \frac{1}{4}\bar{F}_{IJ} - \frac{N_{IK}X^K N_{JL}X^L}{NIJX^I X^J} \tag{3.4}$$

$$\mathcal{F}^I_{\mu\nu} \equiv \partial_\mu A^I_\nu - \partial_\nu A^I_\mu \tag{3.5}$$

$$\mathcal{F}^{I\pm}_{\mu\nu} = \frac{1}{2}(\mathcal{F}^I_{\mu\nu} \pm i\tilde{\mathcal{F}}^I_{\mu\nu}) \tag{3.6}$$

$$\text{where , } \tilde{\mathcal{F}}^I_{\mu\nu} \text{ is the hodge dual of } \mathcal{F}^I_{\mu\nu} \tag{3.7}$$

If we choose our gauge as

$$N_{IJ}X^I\bar{X}^J = \kappa^{-2} = 1 \tag{3.8}$$

and define $U(1)$ invariant fields as

$$Z^I = \frac{X^I}{X^0}, Z^0 = 1 \tag{3.9}$$

Then in this gauge, the Lagrangian for the scalars is given by

$$e^{-1}\mathcal{L}_{scalar} = \partial_\mu Z^A \partial^\mu \bar{Z}^B G_{A\bar{B}} \tag{3.10}$$

$$\text{where, } G_{A\bar{B}} = \frac{\partial}{\partial Z^A}\frac{\partial}{\partial \bar{Z}^B} logK \tag{3.11}$$

$$\text{and the Kahler potential, } K = -log[i(\bar{X}^I F_I - X^I \bar{F}_I)] \tag{3.12}$$

$$\tag{3.13}$$

And the Lagrangian for the guage fields is given by

$$e^{-1}\mathcal{L}_{gauge} = \frac{1}{4}(\mathcal{F}^{I+}_{\mu\nu}\mathcal{F}^{J+\mu\nu}\mathcal{N}_{IJ} + \mathcal{F}^{I-}_{\mu\nu}\mathcal{F}^{J-\mu\nu}\bar{\mathcal{N}}_{IJ}) \tag{3.14}$$

The part of effective Lagrangian that we are interested in is in the vector multiplet sector which is encoded in the function $F(X, \hat{A})$, where $\hat{A}$ is the lowest component of the chiral multiplet. By expansion of the function,

$$F(X, \hat{A}) = \sum_{g=0}^{\infty} F^{(g)}(X)\hat{A}^g \tag{3.15}$$

one finds the prepotential $F^0(X)$ which describes the minimal part of the Lagrangian and the coefficients $F^{(g\geq 1)}(X)$ of higher derivative couplings of the form C^2T^{2g-2}.

The derivation of both the macroscopic and microscopic black hole entropy consists of the most basic assumption of taking only terms which arise at the tree level in α'. In this case only the leading parts of $F^0(X)$ and $F^1(X)$ contribute

$$F^0(Z) = -i\frac{1}{6}C_{ABC}Z^AZ^BZ^C \tag{3.16}$$

$$F^1(Z) = -ic_{2A}Z^A \tag{3.17}$$

3.1.2 Attractor Equations and Black Holes

For the case of four-dimensional spherically symmetric supersymmetric extremal black holes, space-time metric is restricted to be of the form

$$ds^2 = -e^{-2U(r)}dt^2 + e^{2U(r)}d\vec{r}^2 \tag{3.18}$$

where the function $U(r)$ is related to the central charge or the so called BPS mass

$$Z = p^IF_I - q_IX^I \tag{3.19}$$

by

$$e^U(r) = \frac{e^{\kappa}|Z|^2}{r^2} \tag{3.20}$$

where κ is given by

$$e^{-\kappa} = i[\bar{X}^IF_I - \bar{F}_IX^I] \tag{3.21}$$

The values of the scalar fields X^I at the horizon are determined entirely in terms of the magnetic and electric charges (p^I, q_I) of the black hole. Their value is precisely determined by the *attractor equations*

$$\bar{Z}X^I - Z\bar{X}^I = ip^I \tag{3.22}$$

$$\bar{Z}F_I(X, \hat{A}) - Z\bar{F}_I(X, \hat{A}) = iq_I \tag{3.23}$$

In the presence of higher derivative corrections, Bekenstein-Hawking entropy formula is no longer applicable, and the correct entropy is given by the Wald's formula

$$\mathcal{S} = \pi[|Z|^2 - 256Im[F_{\hat{A}(X,\hat{A})}]] \tag{3.24}$$

where $\hat{A} = -64\bar{Z}^{-2}e^{-\kappa}$ and $F_{\hat{A}} = \frac{\partial F}{\partial \hat{A}}$

Small Black Holes

Recently, there has been great deal of progress in computing corrections to black hole entropy due to the effect of higher derivative terms in string theory effective action and comparing the answer to the statistical entropy of the same system [44–56]. A particularly interesting class of examples is provided by the stringy 'small' black holes. These are singular solutions of the classical supergravity equations of motion since they have vanishing area of the event horizon. However microscopically they are described by BPS states of the fundamental string and hence have non-zero degeneracies. A simple class of such examples is provided by the FP system or the so-called Dabholkar–Harvey (DH) system [57], which is obtained by winding a fundamental heterotic string $-w$ times around a circle S^1 and putting n units of momentum along the same circle. If the right-movers are in the ground state then such a state is BPS but it can carry arbitrary left-moving oscillations.[1] Its microscopic entropy is given by

$$S_{\text{micro}} = 4\pi\sqrt{nw}\,. \tag{4.0.1}$$

Although in the supergravity approximation the corresponding solution has zero horizon area and hence zero entropy, one expects that the result will be modified by the higher derivative corrections since close to the (singular) horizon the curvature and other field strengths become strong and the supergravity approximation is expected to break down. However one finds that the string coupling constant near the horizon is small and hence one need not worry about string loop corrections. By analyzing the behavior of the supergravity solution near the horizon and using various symmetries of the full string tree level effective action one finds that the α'-corrected entropy must

[1] In the literature one often finds two different sets of conventions. The first one uses the convention that for a BPS state the momentum and winding along S^1 will have the same sign. The second one uses the convention that for a BPS state the momentum and winding along S^1 will have opposite sign. Here we are using the second convention.

have the form

$$C\sqrt{nw}\,, \tag{4.0.2}$$

for some constant C [58–61]. However, the constant C cannot be calculated based on symmetry principles alone. Recently for the special case of four dimensional small black holes this constant was calculated in [52] using a special class of higher derivative terms in the effective action and was found to give the value 4π in precise agreement with the microscopic answer. Later it was demonstrated in [62–64] that other higher derivative corrections do not change the result. Furthermore, it was shown that using special ensembles to define entropy, the agreement between microscopic and macroscopic counting can be pushed beyond the leading order to all orders in the large charge expansion [52, 65–67].

The next natural question is what will happen if we add angular momentum J to the system. Studying microscopic states and making use of 4D-5D connection [68–73], it was argued in [74] that if large angular momentum is added to a small black hole in five dimensions with horizon topology S^3, it will turn into a small black ring whose horizon has topology of $S^1 \times S^2$. The size of S^2 is of the order of the string scale, and hence is small compared with the typical horizon scales of ordinary supersymmetric black ring solutions in [75–78] for which supergravity approximation is good. Furthermore, in [74], it was shown that the entropy of small black rings can be related to that of four dimensional small black holes,[2] and that the entropy calculation based on this argument matches the independent entropy calculation of the rotating DH system by Russo and Susskind [82] (see also [62, 68, 83–85]).

Since a ring-like horizon is expected to show up for large J, it is interesting to study the near horizon geometry of small black rings from a macroscopic, geometric point of view. In particular we would like to know the J dependence of the horizon geometry. One would also like to generalize the results to higher dimensions. These are the problems we shall address in this work. In particular we show that in arbitrary dimensions the supergravity equations of motion admit a singular black ring solution carrying the same charge quantum numbers as that of a rotating elementary string. This solution is characterized by the charges n and w introduced earlier, the angular momentum J and a dipole charge Q that represents how many times the fundamental string winds in the azimuthal directions. The microscopic entropy of such a system can be easily computed from the spectrum of elementary string states with spin and

[2] Using the fuzzball picture [79, 80] it was also argued in [81] that this system becomes a black ring.

gives the answer

$$S_{stat} = 4\pi\sqrt{nw - JQ}\,. \tag{4.0.3}$$

These states exist only for $nw \geq JQ$, which can be regarded as the Regge bound for BPS states. On the other hand since the supergravity solution is singular we cannot directly calculate the macroscopic entropy. Nevertheless by examining the solution in a region of low curvature, where supergravity approximation is expected to be valid, we find that the space-time associated with the solution develops closed time-like curves for $nw < JQ$ [86]. Thus absence of such curves requires $nw > JQ$, thereby providing a geometric derivation of the Regge bound.[3] Furthermore, by using a symmetry argument very similar to that in [58–61] we show that the entropy, if non-zero, is given by

$$S_{BH} = C\sqrt{nw - JQ}\,, \tag{4.0.4}$$

for some constant C. Furthermore C is the same constant that would appear in the expression (4.0.2) for the macroscopic entropy of a non-rotating black hole in one less dimension. Thus agreement between microscopic and macroscopic entropy of a $(d-1)$-dimensional non-rotating small black hole would imply a similar agreement for d-dimensional rotating small black rings. This analysis also determines the near horizon geometry in terms of some unknown constants of order unity. In particular we can estimate the sizes of various circles and spheres near the horizon.

[3]For oscillating string solutions describing string states with angular momentum, a derivation of the Regge bound was given earlier in [87] by requiring regularity of the source terms in the equation of motion.

Microstate Counting of Small Black Rings

5.1 Small black rings and 4D-5D connection

Consider heterotic string compactified on $\mathbf{T}^4 \times \mathbf{S}^1$ where $\mathbf{T}^4$ is a 4-torus in $\{6789\}$ directions and $\mathbf{S}^1$ is a circle along the $\{5\}$ direction. Consider now a string state with winding number w along the X^5 direction. In a given winding sector, there is a tower of BPS states each in the right-moving ground state but carrying arbitrary left-moving oscillations subject to the Virasoro constraint $N_L = 1 + nw$, where N_L is the left-moving oscillation number and n is the quantized momentum along X^5 [26, 57]. Note that N_L is positive and hence a BPS state that satisfies this constraint has positive n for positive w for large N_L. This state can carry angular momentum J say in the $\{34\}$ plane. The angular momentum operator J is given by $J = \sum_{n=1}^{\infty} \left[a_n^\dagger a_n - \bar{a}_n^\dagger \bar{a}_n\right]$, where a_n and $\bar{a}_n$ are the oscillator modes with frequency n of the coordinates $(X^3 + iX^4)$ and $(X^3 - iX^4)$ respectively, normalized as $[a_m, a_n^\dagger] = [\bar{a}_m, \bar{a}_n^\dagger] = \delta_{mn}$.

Following the 4D-5D connection explained in [74] we can map this state to a configuration in Type IIA compactified on $\mathbf{K3} \times \mathbf{T^2}$ with charges D2-D2-D0-D4 which in turn is dual to the DH states in heterotic string on $\mathbf{T^4} \times \mathbf{S^1} \times \tilde{\mathbf{S}}^\mathbf{1}$ with momentum and winding (n, w) and $(-\tilde{n}, \tilde{w})$ along the circles $\mathbf{S^1}$ and $\tilde{\mathbf{S}}^\mathbf{1}$ respectively, with all integers $n, w, \tilde{n}, \tilde{w}$ positive. It is useful to state the 4D-5D connection entirely in the heterotic language. The basic idea following [70–73] is to make use of the Taub-NUT geometry. For a Taub-NUT space with unit charge, the geometry near the origin is $\mathbf{R^4}$ whereas at asymptotic infinity it is $\mathbf{R^3} \times \tilde{\mathbf{S}}^\mathbf{1}$. Thus a contractible circle at the origin of Taub-NUT turns into a non-contractible circle $\tilde{\mathbf{S}}^\mathbf{1}$ at asymptotic infinity. The angular momentum J at the origin then turns into momentum $\tilde{n}$ along the circle at infinity [71].

Consider now DH states with spin in heterotic string theory on $\mathbf{T^4} \times \mathbf{S^1}$. Spinning strings that are wrapping along the circle $\mathbf{S^1}$ in $\{5\}$ direction and rotating in the $\{34\}$ plane have a helical profile [81, 87, 90, 91]. The helix goes around a contractible circle $\mathbf{S^1_\psi}$ of radius R_ψ along an angular coordinate ψ in the $\{34\}$ plane as the string wraps around the non-contractible circle $\mathbf{S^1}$. Let us denote the pitch of the helix by p, which is the winding number of the projection of the helix onto the contractible circle. Macroscopically it corresponds to a dipole charge. We can now embed this system in Taub-NUT space with very large Taub-NUT radius $R_{\text{TN}} \gg R_\psi$ and regard $\mathbf{S^1_\psi}$ in $\mathbf{R^4}$ as being situated at the origin of a Taub-NUT geometry. Varying the radius of Taub-NUT, which is a modulus, we can smoothly go to the regime $R_{\text{TN}} \ll R_\psi$. Then the contractible $\mathbf{S^1_\psi}$ effectively turns into the non-contractible circle $\tilde{\mathbf{S}}^\mathbf{1}$ at asymptotic infinity. The entropy of BPS states is not expected to change under such an adiabatic change of moduli. We can dimensionally reduce the system to 4D along $\tilde{\mathbf{S}}^\mathbf{1}$ and obtain a 4D DH state with four charges $(n, w, -\tilde{n}, \tilde{w})$ with the identification that $\tilde{n} = J$ and $\tilde{w} = p$. This system has a string scale horizon in 4D [8, 52–54, 58, 66, 67] which suggests that the original spinning DH system in 5D is a small black ring [74].

Since we have unit Taub-NUT charge to begin with, we do not have a purely electric configuration in 4D but instead have a Kaluza-Klein monopole of unit charge in addition to the 4-charge purely electric small black hole. However, since the helix is far away from the origin of Taub-NUT space in 5D before dimensional reduction, the KK-monopole is sitting far away from the 4-charge black hole in 4D. The separation is determined by Denef's constraint [37] and is determined by J and the asymptotic values of the moduli and can thus be made arbitrarily large. The local microscopic counting therefore does not depend on the addition of the KK-monopole and is given by the counting of DH states.

5.2 Four-dimensional counting

In the 4D description, the state is specified by the charge vector Q in the Narain charge lattice $\Gamma^{2,2}$ of the $\mathbf{S^1} \times \tilde{\mathbf{S}}^\mathbf{1}$ factor with four integer entries. The norm of this vector is

$$\frac{Q^2}{2} = \frac{1}{2} \begin{pmatrix} n & w & -\tilde{n} & \tilde{w} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} n \\ w \\ -\tilde{n} \\ \tilde{w} \end{pmatrix} \tag{5.2.1}$$

The degeneracy of these perturbative DH states can be computed exactly and the asymptotic degeneracy for large Q^2 is given as in [66, 67] by

$$\Omega_{\text{micro}}(n,w,\tilde{n},\tilde{w}) \sim \hat{I}_{13}(4\pi\sqrt{\frac{Q^2}{2}}) = \hat{I}_{13}(4\pi\sqrt{nw-\tilde{n}\tilde{w}})\,, \tag{5.2.2}$$

where $\hat{I}_{13}(z)$ is the modified Bessel function defined in [67].

Turning to the macroscopic degeneracy, we compute it using the OSV relation [17] between topological string partition function and the macroscopic degeneracy, for the non-spinning four-dimensional configuration. For this purpose we use the Type IIA description, viewing this state as a collection of D2-D2-D0-D4 branes. There are n D2-branes wrapping a 2-cycle α_1 in **K3** and w D2-branes wrapping a 2-cycle α_2 such that the intersection matrix of α_1 and α_2 is as in the upper-left 2×2 block of the matrix in (5.2.1). We therefore identify the charges as $(q_2, q_3) = (n, w)$. Similarly, we identify the D0-D4 charges as $(q_0, p^1) = (\tilde{n}, \tilde{w})$ so in the labelling of charges used in [67], we have $Q = (q_2, q_3, q_0, p^1)$ and all other charges zero. Using the formula (2.26) in [67] we then see that the macroscopic degeneracy is given by

$$\begin{aligned}\Omega_{\text{macro}}(n,w,J) &\sim (p^1)^2\hat{I}_{13}(4\pi\sqrt{q_2\,q_3 - p^1 q_0}) \\ &= (p^1)^2\hat{I}_{13}(4\pi\sqrt{nw-\tilde{n}\tilde{w}})\,.\end{aligned} \tag{5.2.3}$$

Therefore, up to the overall $(p^1)^2$ factor, the microscopic (5.2.2) and macroscopic (5.2.3) degeneracies match precisely to all orders in an asymptotic expansion for large Q^2.

5.3 Five-dimensional microscopic counting

We now would like to count the degeneracy of the spinning DH system from the 5D side. The nontrivial issue is to determine the correct ensemble. The relevant states correspond to quantum fluctuations around a specific coherent oscillating state which is essentially Bose-Einstein condensate on the worldsheet and describes the helical geometry with pitch p [74]. As we will argue below, the precise microstates turn out to be of the form

$$\underbrace{(a_p^\dagger)^J}_{\substack{\text{microscopic origin of}\\ \tilde{n}=J \text{ and } \tilde{w}=p \text{ of}\\ \text{the ring}}} \quad\times\quad \underbrace{\prod_{n=1}^{\infty}\left[\prod_{i=1,2,\pm,5\ldots 24}(\alpha^i_{-n})^{N_{ni}}\right]|0\rangle}_{\substack{\text{fluctuation: all possible states with}\\ \text{level } N_{\text{eff}}\equiv N-pJ. \text{ Angular mo-}\\ \text{mentum } J \text{ is not fixed.}}} \tag{5.3.1}$$

for $J, p > 0$. Namely, *we consider the states with the worldsheet energy* $N_{\text{eff}} \equiv N - pJ$ *and chemical potential* μ *conjugate to* J *set to zero, and multiply all those states by* $(a_p^\dagger)^J$. Note in particular that the $a_p^\dagger$ and $\bar{a}_p^\dagger$ oscillators are included in the fluctuation part. The degeneracy of the states (5.3.1) is the same as that of the DH system with $Q^2/2 = N_{\text{eff}}$ and proportional to $\hat{I}_{13}(4\pi\sqrt{N - pJ})$, in precise agreement with (5.2.2), (5.2.3). If $J, p < 0$, $(a_p^\dagger)^J$ in (5.3.1) must be replaced by $(\bar{a}_{|p|}^\dagger)^{|J|}$.

This separation between the classical coherent condensate that describes the large helix and the small quantum fluctuations around it that account for the entropy is similar to the one used in [62, 68, 83–85]. It is valid in the regime when R_ψ is much larger than the amplitude of fluctuation. Therefore, we conclude that in this regime, the states of the form (5.3.1) are the states that account for the microstates of the ring. This in turn agrees with (5.2.2) and (5.2.3) in 4D through the 4D-5D connection.

Note that (5.3.1) means that the microscopic counting in 5D must *not* be done for fixed angular momentum J. Fixing J would impose an additional constraint on the fluctuation part in (5.3.1). From the 4D point of view, it would correspond to imposing a constraint on the worldsheet oscillators of the DH system, which would lead to a result contradictory to the 4D degeneracy (5.2.2), (5.2.3). To demonstrate this, let us count the degeneracy of spinning DH states with *fixed* J. The degeneracy $\Omega(N, J)$ is summarized in the partition function

$$Z(\beta, \mu) = \sum_{N,J} \Omega(N, J)\, q^N c^J, \quad q = e^{-\beta}, \quad c = e^{\beta\mu}, \tag{5.3.2}$$

where $N \equiv n\,w = N_L - 1$. β can be thought as the inverse temperature on the worldsheet for a $1+1$ gas of left-moving 24 bosons conjugate to the total energy N and μ can be thought of as the chemical potential conjugate to the quantum number J of this gas. Since N_L is the oscillation number for the 24 left-moving transverse bosons, using the expression $J = \sum_{n=1}^{\infty} \left[a_n^\dagger a_n - \bar{a}_n^\dagger \bar{a}_n\right]$, the partition function can be readily evaluated [82] and is given by

$$\begin{aligned} Z(\beta, \mu) &= \left[q \prod_{n=1}^{\infty} (1 - q^n)^{22} (1 - c\, q^n)(1 - c^{-1} q^n)\right]^{-1} \\ &= \frac{1}{\eta^{21}(e^{-\beta})} \frac{2i \sinh(\beta\mu/2)}{\theta_{11}(\beta\mu/2\pi i, i\beta/2\pi)}, \end{aligned} \tag{5.3.3}$$

in terms of the standard Dedekind eta function and theta function with characteristics.

The number of states with given N and J is then given by the inverse Laplace transform: $\Omega(N, J) = \frac{1}{(2\pi i)^2} \int_{C_\beta} d\beta\, e^{\beta N} \int_{C_\mu} \beta\, d\mu\, e^{-\mu\beta J} Z(\beta, \mu)$, where the contour C_β

runs from $-i\pi+\gamma$ to $+i\pi+\gamma$ with $\gamma>0$ to avoid singularities on the imaginary axis. Similarly, C_μ goes from $-\pi i/\beta+\epsilon$ to $+\pi i/\beta+\epsilon$ with $-1<\epsilon<1$ to avoid poles. To find the asymptotic degeneracy at large N, we want to take the high temperature limit, or $\beta\to 0$. Using the modular properties of the Dedekind eta and the theta functions we can write the degeneracy at high temperature as in [82] as

$$\Omega(N,J)\sim\frac{1}{2\pi i}\int_{C_\beta}d\beta\, e^{\beta N+(2\pi)^2/\beta}\left(\frac{\beta}{2\pi}\right)^{12}I(\beta,J)\,, \tag{5.3.4}$$

where $I(\beta,J)$ is defined by

$$I(\beta,J)=\frac{1}{2\pi i}\int_{C_\mu}d\mu\, e^{-\beta\mu^2/2-\beta\mu J}\,\frac{\sinh(\beta\mu/2)}{\sin(\pi\mu)}\,. \tag{5.3.5}$$

To arrive at (5.3.4), we dropped terms that are exponentially suppressed for small β as $e^{-(2\pi)^2/\beta}$. This is justified although β is still to be integrated over, because the saddle point around $\beta\sim 1/\sqrt{N-J}\ll 1$ will make the leading contribution, as we will see below.

Now we evaluate (5.3.5) using the method of residues. Deform the contour C_μ into sum of three intervals $C_1=[-\pi i/\beta+\epsilon,-\pi i/\beta+K]$, $C_2=[-\pi i/\beta+K,\pi i/\beta+K]$, and $C_3=[\pi i/\beta+K,\pi i/\beta+\epsilon]$, with $K\gg 1$. One can readily show that the contour integral along $C_{1,2,3}$ vanishes due to the periodicity of the original integrand (5.3.3) in the small β and large K limit. In the process of deforming the contour, we pick up poles at $\mu=m$, $m=1,2,\ldots$. In the end, we obtain

$$\begin{aligned}I(\beta,J)&\sim-\operatorname*{Res}_{\mu=1}\Big[e^{-\beta\mu^2/2-\beta\mu J}\,\frac{\sinh(\beta\mu/2)}{\sin(\pi\mu)}\Big]+\mathcal{O}(e^{-2\beta J})\\&\sim(1-e^{-\beta})\,e^{-\beta J}+\mathcal{O}(e^{-2\beta J})\,.\end{aligned} \tag{5.3.6}$$

Here $\mathcal{O}(e^{-2\beta J})$ comes from the poles at $\mu=2,3,\cdots$ and negligible when $J=\mathcal{O}(N)$ since $\beta J=\mathcal{O}(N^{1/2})$. We interpret the term $\propto e^{-\beta J}$ as the contribution from the $p=1$ sector. Substituting this back into (5.3.4), we conclude that the degeneracy $\Omega(N,J)$ is $\sim\frac{1}{2\pi i}\int d\beta(\beta/2\pi)^{12}(\beta-\frac{\beta^2}{2}++\cdots)\,e^{(2\pi)^2/\beta+\beta(N-J)}$. Each term in the integral is of the Bessel type as discussed in [67], and thus the final result is $\Omega(N,J)\sim\hat{I}_{14}(4\pi\sqrt{N-J})-(2\pi/2)\hat{I}_{15}(4\pi\sqrt{N-J})+\cdots$, which agrees with (5.2.2) and (5.2.3) with $p=1$ only in the leading exponential but disagrees in the subleading corrections. This demonstrates that microscopic counting in 5D must be done not for fixed J but for the states (5.3.1). One can show that states with fixed $\mu\neq 0$ also lead to degeneracy in disagreement with (5.2.2), (5.2.3).

In general, subleading corrections to thermodynamic quantities depend on the choice of the statistical ensembles and are different for different ensembles. For example, even for non-spinning black holes the ensemble with fixed angular momentum $J = 0$ differs from the ensemble with fixed chemical potential $\mu = 0$ in subleading corrections. It was noted in [67] that the correct microscopic ensemble that is consistent with the OSV conjecture is the one with $\mu = 0$. In our case, the description of small black rings requires that we are also fixing the pitch of the helix as an additional requirement and that we are counting states around this classical coherent condensate on the worldsheet.

5.4 Five-dimensional macroscopic geometry

Finally we comment on the geometry of this 5D small black ring, which can be determined in the near ring limit by using the 4D-5D uplift. Exact uplift is possible near the horizon, since the near horizon geometry of 4D black hole can be determined precisely by using string-corrected attractor equations. Let us consider the case where $p\,J = \mathcal{O}(N)$. We consider a helix with general pitch p even though the contribution from the $p = 1$ subsector dominates entropy. In 4D heterotic string theory, the near-horizon black hole geometry is determined by the attractor equations to be $e^{2\phi_4} \sim 1/\sqrt{N - pJ}\,,\ \sqrt{g_{\psi\psi}} \sim \sqrt{J/p}$ and the horizon radius is $r_{\mathbf{S}^2} \sim l_s$. Using $e^{2\phi_5} \sim e^{2\phi_4}\sqrt{g_{\psi\psi}}$ and $l_s \sim e^{-2\phi_5/3} l_{pl}^{(5)}$, the scale of $\mathbf{S}^1_\psi$ along ring is given by

$$\begin{aligned} r_{\mathbf{S}^1} &\sim \sqrt{g_{\psi\psi}}\, l_s \sim p^{-1/3} J^{1/3} (N - pJ)^{1/6}\, l_{pl}^{(5)}\,, \\ r_{\mathbf{S}^2} &\sim l_s \sim p^{1/6} J^{-1/6} (N - pJ)^{1/6} l_{pl}^{(5)}\,, \end{aligned} \tag{5.4.1}$$

such that $S \sim A/4G_5 \sim r_{\mathbf{S}^1}(r_{\mathbf{S}^2})^2/(l_{pl}^{(5)})^3 \sim \sqrt{N - pJ}$. One important qualitative feature of the solution is that when pJ exceeds N, the solution develops closed timelike curves. When $N \geq pJ$ is saturated, $g_{\psi\psi} \geq 0$ is saturated at the ring horizon. Hence, the Regge bound $N \geq pJ$ on the angular momentum of the underlying microstates can be understood from the macroscopic solution as a consequence of the physical requirement that closed timelike curves be absent.

Scaling Analysis for Black Rings

In this section we shall study the geometry of a small black ring in arbitrary dimensions. Although in the supergravity approximation this solution is singular along the ring, by studying carefully the geometry near the singularity we shall be able to determine the dependence of the entropy on various charges *assuming that the higher derivative corrections modify the solution in a way such that it has a finite entropy.* This will essentially involve generalization of the scaling analysis of [58–61] to the case of rotating black rings.

6.1 Supergravity Small Black Rings in General Dimensions

In this section we shall construct small black ring solutions in heterotic string theory with d non-compact space-time dimensions, describing a rotating elementary string. The same solution in a different U-duality frame was derived in [86] as supergravity supertubes.

Consider heterotic string theory in $\mathbf{R}_t \times \mathbf{R}^{d-1} \times S^1 \times T^{9-d}$ with $4 \le d \le 9$. We denote the coordinates of $\mathbf{R}_t$, $\mathbf{R}^{d-1}$ and S^1 by t, $\mathbf{x} = (x^1, \ldots, x^{d-1})$ and x^d respectively, and the coordinate radius of the x^d direction by R_d. We would like to study the geometry of a small black ring sitting in the noncompact d-dimensional space $\mathbf{R}_t \times \mathbf{R}^{d-1}$. As in section 6, we shall regard this as a solution in the $(d+1)$-dimensional theory obtained by dimensional reduction of the ten dimensional heterotic string theory on T^{9-d}. Thus we shall use the $(d+1)$-dimensional fields to express the solution. This in particular will require us to take the solution to be independent of the coordinates of T^{9-d}; this is done by 'smearing' the ten dimensional solution along T^{9-d}.

In [87, 90], a large class of supergravity solutions were derived which correspond to a fundamental string with an arbitrary left-moving traveling wave on it, $\mathbf{x} = \mathbf{F}(t - x^d)$, where $\mathbf{F} = (F_1, \ldots, F_{d-1})$ are arbitrary functions. In [91] (see also [79]), the situation where a fundamental string is wrapping $-w$ ($w \gg 1$) times around the x^d direction and carrying n ($n \gg 1$) units of momentum along the x^d direction was considered in the particular case of $d = 5$. It was argued there that in such a situation the supergravity description is obtained by smearing the solution of [87] in the x^d direction. The smeared solution can be compactified on the x^d direction, giving a five-dimensional solution with an arbitrary profile of the fundamental string, $\mathbf{x} = \mathbf{F}(v)$, in the noncompact $\mathbf{R}^4$.

This construction of [91] can be straightforwardly generalized to arbitrary d. Namely, a solution of the $(d + 1)$-dimensional supergravity equations of motion, describing a fundamental string wrapping $-w$ ($w \gg 1$) times around the x^d direction, carrying n ($n \gg 1$) units of momentum along the x^d direction, and having an arbitrary shape in the noncompact $\mathbf{R}^{d-1}$ direction parametrized by the profile function $\mathbf{x} = \mathbf{F}(v)$ $(0 \le v \le L)$, is given by:

$$\begin{aligned} ds^2_{str,d+1} &= f_f^{-1}[-(dt - A_i dx^i)^2 + (dx^d - A_i dx^i)^2 + (f_p - 1)(dt - dx^d)^2] + d\mathbf{x}^2_{d-1} \\ e^{2\Phi_{d+1}} &= g^2 f_f^{-1}, \qquad B_{td} = -(f_f^{-1} - 1), \qquad B_{ti} = -B_{di} = f_f^{-1} A_i, \end{aligned} \tag{6.1.1}$$

where $i = 1, 2, \ldots, d-1$, and [1]

$$\begin{aligned} f_f(\mathbf{x}) &= 1 + \frac{Q_f}{L}\int_0^L \frac{dv}{|\mathbf{x} - \mathbf{F}(v)|^{d-3}}, \qquad f_p(\mathbf{x}) = 1 + \frac{Q_f}{L}\int_0^L \frac{|\dot{\mathbf{F}}(v)|^2 dv}{|\mathbf{x} - \mathbf{F}(v)|^{d-3}}, \\ A_i(\mathbf{x}) &= -\frac{Q_f}{L}\int_0^L \frac{\dot{F}_i(v) dv}{|\mathbf{x} - \mathbf{F}(v)|^{d-3}}. \end{aligned} \tag{6.1.2}$$

The dot denotes derivative with respect to v, and $L = 2\pi w R_d$. We also define

$$Q_p \equiv \frac{Q_f}{L}\int_0^L |\dot{\mathbf{F}}(v)|^2 dv\,. \tag{6.1.3}$$

For large $|\mathbf{x}|$, $f_f - 1$ and $f_p - 1$ fall off as $Q_f/|\mathbf{x}|^{d-3}$ and $Q_p/|\mathbf{x}|^{d-3}$ respectively. By computing the flux of the gauge fields associated with $G_{d\mu}$ and $B_{d\mu}$ at infinity, one finds that the relations between Q_f, Q_p and quantized charges n, w are

$$Q_f = \frac{16\pi G_d R_d}{(d-3)\Omega_{d-2}\alpha'} w, \qquad Q_p = \frac{16\pi G_d}{(d-3)\Omega_{d-2} R_d} n, \tag{6.1.4}$$

where Ω_D is the area of S^D and G_d is the d-dimensional Newton constant:

$$16\pi G_d = \frac{16\pi G_{d+1}}{2\pi R_d} = \frac{(2\pi)^{d-3} g^2 \alpha'^{(d-1)/2}}{R_d}. \tag{6.1.5}$$

[1]The relation to the harmonic functions in [79, 91] is $f_f = H^{-1}$, $f_p = K + 1$.

In order to arrive at (6.1.4), (6.1.5) we have used the fact that in the absence of higher derivative corrections, which are irrelevant in the asymptotic region, the action has the form given in (6.2.1)–(6.2.3). By examining the asymptotic form of the metric and the results of [101] one also sees that the angular momentum associated with the solution in the x^i-x^j plane is given by

$$J_{ij} = \frac{(d-3)\Omega_{d-2}}{16\pi G_d}\frac{Q_f}{L}\int_0^L (F_i\dot{F}_j - F_j\dot{F}_i)dv\,. \tag{6.1.6}$$

Before considering the small black ring solution, let us first consider the case with a circular profile:

$$\mathbf{F} = \mathbf{F}^{(0)}, \qquad \begin{cases} F_1^{(0)} + iF_2^{(0)} = Re^{i\omega v}, \\ F_3^{(0)} = \cdots = F_{d-1}^{(0)} = 0, \end{cases} \qquad \omega = \frac{2\pi Q}{L} = \frac{Q}{wR_d}. \tag{6.1.7}$$

This corresponds to a fundamental string which winds Q times along the ring of radius R in the x^1-x^2 plane [91, 102]. Introducing the coordinate system $(s, \psi, w, \vec{\xi})$ by

$$\begin{aligned} &x^1 = s\cos\psi, \quad x^2 = s\sin\psi, \\ &x^3 = w\,\xi^1, \quad x^4 = w\,\xi^2, \quad \ldots, \quad x^{d-1} = w\,\xi^{d-3} \qquad \sum_{a=1}^{d-3}(\xi^a)^2 = 1, \end{aligned} \tag{6.1.8}$$

the harmonic functions in (6.1.2) are computed as

$$\begin{aligned} f_f &= 1 + Q_f(s^2+w^2+R^2)^{-\frac{d-3}{2}}\,{}_2F_1\!\left(\tfrac{d-3}{4}, \tfrac{d-1}{4}; 1; \tfrac{4R^2s^2}{(s^2+w^2+R^2)^2}\right), \\ f_p &= 1 + Q_p(s^2+w^2+R^2)^{-\frac{d-3}{2}}\,{}_2F_1\!\left(\tfrac{d-3}{4}, \tfrac{d-1}{4}; 1; \tfrac{4R^2s^2}{(s^2+w^2+R^2)^2}\right), \\ A_\psi &= -\left(\tfrac{d-3}{2}\right)qR^2s^2(s^2+w^2+R^2)^{-\frac{d-1}{2}}\,{}_2F_1\!\left(\tfrac{d-1}{4}, \tfrac{d+1}{4}; 2; \tfrac{4R^2s^2}{(s^2+w^2+R^2)^2}\right), \end{aligned} \tag{6.1.9}$$

where we have defined

$$q \equiv Q_f\omega \tag{6.1.10}$$

and ${}_2F_1(\alpha, \beta; \gamma; z)$ denotes the hypergeometric function. For odd d, the hypergeometric functions in (6.1.9) can be written as rational functions, while for even d they involve elliptic integrals. Furthermore, from (6.1.3),

$$Q_p = Q_fR^2\omega^2. \tag{6.1.11}$$

Using (6.1.10) and the last equation of (6.1.7), one finds

$$q = \frac{16\pi G_d}{(d-3)\Omega_{d-2}\alpha'}Q\,. \tag{6.1.12}$$

Moreover, from (6.1.6) one finds that the solution carries an angular momentum $J = QR^2/\alpha'$ in the x^1-x^2 plane. This gives

$$R^2 = \alpha' \frac{J}{Q}. \tag{6.1.13}$$

From (6.1.4), (6.1.11), (6.1.13), and the last equation of (6.1.7), we obtain

$$JQ = nw, \tag{6.1.14}$$

i.e., the circular configuration (6.1.7) saturates the Regge bound.

Now let us proceed to construct the small black ring solution. This can be done by taking the profile function to be [92]

$$\mathbf{F} = \mathbf{F}^{(0)} + \delta\mathbf{F}, \tag{6.1.15}$$

where $\delta\mathbf{F}$ describes fluctuations around $\mathbf{F}^{(0)}$, whose detailed form is irrelevant as long as it satisfies certain conditions to be explained below. As the simplest example, take $\delta\mathbf{F}$ to be

$$\delta F_1 + i\,\delta F_2 = a\, e^{i(\nu v + b)}, \tag{6.1.16}$$

where we eventually take the limit

$$\frac{a}{R} \to 0, \qquad \frac{\nu}{\omega} \to \infty, \qquad a\nu :\ \text{fixed}. \tag{6.1.17}$$

In other words, $\delta\mathbf{F}$ of (6.1.16) represents a very small-amplitude ($a \ll R$), high-frequency ($\nu \gg \omega$) fluctuation. Because of the first condition in (6.1.17), $\mathbf{F}$ in the denominators in the integrand of (6.1.2) can be replaced by $\mathbf{F}^{(0)}$. Now expand this denominator as

$$\begin{aligned} |\mathbf{x} - \mathbf{F}^{(0)}|^{-(d-3)} &= [s^2 + w^2 + R^2 - 2sR\cos(\omega v - \psi)]^{-\frac{d-3}{2}} \\ &= (s^2 + w^2 + R^2)^{-\frac{d-3}{2}} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\Gamma(\frac{d-3}{2} + k)}{\Gamma(\frac{d-3}{2})} \left[\frac{2sR\cos(\omega v - \psi)}{s^2 + w^2 + R^2}\right]^k \end{aligned} \tag{6.1.18}$$

On the other hand, in the numerator, *e.g.* for f_p, we have

$$|\dot{\mathbf{F}}|^2 = |\dot{\mathbf{F}}^{(0)} + \delta\dot{\mathbf{F}}|^2 = R^2\omega^2 + a^2\nu^2 + 2R\omega a\nu \cos[(\omega - \nu)v - b]. \tag{6.1.19}$$

When we multiply (6.1.18) and (6.1.19), and integrate it over v, there will be nonvanishing contributions only when the frequencies of the cosines in (6.1.18) and (6.1.19) cancel each other, which happens only for $k \gtrsim \frac{\nu}{\omega}$. Since $\frac{\nu}{\omega} \to \infty$ in the limit (6.1.17), in fact there is no contribution from the last term in (6.1.19). Similarly, there is no contribution from $\delta\mathbf{F}$ to A_i in (6.1.2); only $\mathbf{F}^{(0)}$ contributes.

At the end of the day, the only effect of introducing the fluctuation (6.1.16) is to change eq. (6.1.11) to

$$Q_p = Q_f(R^2\omega^2 + a^2\nu^2), \tag{6.1.20}$$

whereas other expressions (6.1.9), (6.1.12) and (6.1.13) are unchanged. These give the supergravity small black ring solution we were after. Note that eqs. (6.1.4), (6.1.20), (6.1.13), and the last equation of (6.1.7) now imply the Regge bound,

$$JQ < nw. \tag{6.1.21}$$

Even if one considers more complicated fluctuations than (6.1.16) by taking linear combinations of many modes in all the x^i directions ($1 \leq i \leq (d-1)$), the above results remain unchanged as long as the condition (6.1.17) is met for each mode, except that the $a^2\nu^2$ term in (6.1.20) will be replaced by a sum over the contribution from all the modes. The fact that the resulting solution (6.1.9) is insensitive to the precise form of the fluctuation $\delta\mathbf{F}$ is the reflection of the fact that this supergravity small black ring represents all the underlying microstates whose entropy is given by (4.0.3).

Although in (6.1.9) we presented the small black ring solution in the $(s, \psi, w, \vec{\xi})$ coordinate system, it is more convenient for the analysis in the main text to go to the coordinate system $(y, \psi, x, \vec{\xi})$ defined by

$$s = \frac{\sqrt{y^2-1}}{x-y}R, \quad w = \frac{\sqrt{1-x^2}}{x-y}R, \qquad -1 \leq x \leq 1, \quad -\infty < y \leq -1. \tag{6.1.22}$$

In terms of these coordinates, the harmonic functions (6.1.9) become

$$\begin{aligned} f_f &= 1 + \tfrac{Q_f}{R^{d-3}}\left(\tfrac{x-y}{-2y}\right)^{(d-3)/2} {}_2F_1\left(\tfrac{d-3}{4}, \tfrac{d-1}{4}; 1; 1 - \tfrac{1}{y^2}\right), \\ f_p &= 1 + \tfrac{Q_p}{R^{d-3}}\left(\tfrac{x-y}{-2y}\right)^{(d-3)/2} {}_2F_1\left(\tfrac{d-3}{4}, \tfrac{d-1}{4}; 1; 1 - \tfrac{1}{y^2}\right), \\ A_\psi &= -\left(\tfrac{d-3}{2}\right)\tfrac{q}{R^{d-5}}\tfrac{(y^2-1)(x-y)^{(d-5)/2}}{(-2y)^{(d-1)/2}} {}_2F_1\left(\tfrac{d-1}{4}, \tfrac{d+1}{4}; 2; 1 - \tfrac{1}{y^2}\right), \end{aligned} \tag{6.1.23}$$

and the $(d-1)$-dimensional flat metric $d\mathbf{x}_{d-1}^2$ can be written as

$$d\mathbf{x}_{d-1}^2 = \frac{R^2}{(x-y)^2}\left[\frac{dy^2}{y^2-1} + (y^2-1)d\psi^2 + \frac{dx^2}{1-x^2} + (1-x^2)d\Omega_{d-4}^2\right]. \tag{6.1.24}$$

Eqs. (6.1.1), (6.1.23) and (6.1.24), together with the definitions of various parameters given in (6.1.4), (6.1.5), (6.1.12) and (6.1.13), describe the supergravity small black ring solution. Note that once the solution has been obtained this way, we can forget about how it was constructed, and simply analyze its properties by treating this as a singular solution of the supergravity equations of motion. This is the view point we have adopted in section 6.

6.2 Scaling Analysis

Consider heterotic string theory in $\mathbf{R}_t \times \mathbf{R}^{d-1} \times S^1 \times T^{9-d}$ with $4 \le d \le 9$. Since in our analysis the moduli associated with the torus T^{9-d} will be frozen completely, and furthermore all the mixed components of the metric and the anti-symmetric tensor field with one leg along T^{9-d} and another leg along one of the other directions will be set to zero, it will be more convenient to regard the theory as a theory in $(d+1)$ space-time dimensions.[2] The massless fields in $(d+1)$ space-time dimensions that are relevant for analyzing the black ring solution are the string metric $G_{\mu\nu}$, the anti-symmetric tensor field $B_{\mu\nu}$ and dilaton Φ_{d+1}. The string tree level action involving these fields has the form

$$\mathcal{S} = \int d^{d+1}x \sqrt{-detG}\, e^{-2\Phi_{d+1}} \mathcal{L}\,, \tag{6.2.1}$$

where $\mathcal{L}$ is a function of the metric, Riemann tensor, the 3-form field strength

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \text{cyclic permutations of } \mu,\nu,\rho + \Omega^{CS,L}_{\mu\nu\rho}(G)\,, \tag{6.2.2}$$

covariant derivatives of these quantities, as well as covariant derivatives of the dilaton Φ_{d+1} but not of Φ_{d+1} itself. Here $\Omega^{CS,L}_{\mu\nu\rho}(G)$ denotes the Lorentz Chern-Simons 3-form constructed out of the string metric $G_{\mu\nu}$ – more precisely out of the spin connection compatible with this metric – with some appropriate coefficient. Since we set all the gauge fields in $(d+1)$-dimensions to zero, there are no gauge Chern-Simons term in the definition of $H_{\mu\nu\rho}$. In the supergravity approximation where we have only two derivative terms, the Lorentz Chern-Simons term disappears from the expression for

[2]In other words we consider a solution for which the world-sheet theory of the fundamental string propagating in this background is a direct sum of two conformal field theories. The first one is a free field theory associated with the coordinates of T^{9-d} and the sixteen additional left-moving world-sheet bosons of the heterotic string theory. The second one is an interacting theory associated with the string propagation in the $(d+1)$-dimensional black ring solution. Our focus will be on the latter theory.

$H_{\mu\nu\rho}$ and $\mathcal{L}$ takes the form

$$\mathcal{L} = \frac{1}{(2\pi)^{d-2}(\alpha')^{(d-1)/2}} \left[R_G + 4G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{12}G^{\mu\mu'}G^{\nu\nu'}G^{\rho\rho'}H_{\mu\nu\rho}H_{\mu'\nu'\rho'} \right] . \quad (6.2.3)$$

We denote the coordinates of $\mathbf{R}_t$ and S^1 by t and x^d respectively. Let the coordinate radius of x^d direction be R_d. For $\mathbf{R}^{d-1}$ we use a special coordinate system in which the flat metric on $\mathbf{R}^{d-1}$ takes the form:

$$d\mathbf{x}_{d-1}^2 = \frac{R^2}{(x-y)^2} \left[\frac{dy^2}{y^2-1} + (y^2-1)d\psi^2 + \frac{dx^2}{1-x^2} + (1-x^2)d\Omega_{d-4}^2 \right] , \quad (6.2.4)$$

where $d\Omega_{d-4}$ denotes the line elements on the $(d-4)$-sphere, R is a constant whose value is given in (6.2.8), and x, y take values in the range

$$-1 \le x \le 1, \qquad -\infty < y \le -1 \, . \quad (6.2.5)$$

The relationship between these coordinates and the cartesian coordinates of $\mathbf{R}^{d-1}$ has been given in eqs. (6.1.8), (6.1.22).

A black ring solution in the supergravity approximation, describing a rotating fundamental string of the type described in section 5, has been constructed in Section 6.1 based on [87, 90–92] and takes the form:

$$\begin{aligned} ds^2_{str,d+1} &= f_f^{-1}[-(dt - A_i dx^i)^2 + (dx^d - A_i dx^i)^2 + (f_p - 1)(dt - dx^d)^2] + d\mathbf{x}^2_{d-1} \\ e^{2\Phi_{d+1}} &= g^2 \, f_f^{-1}, \qquad B_{td} = -(f_f^{-1} - 1), \qquad B_{ti} = -B_{di} = f_f^{-1} \, A_i, \end{aligned} \quad (6.2.6)$$

where $i = 1, 2, \ldots, d-1$,

$$\begin{aligned} f_f &= 1 + \frac{Q_f}{R^{d-3}} \left(\frac{x-y}{-2y} \right)^{(d-3)/2} {}_2F_1\left(\frac{d-3}{4}, \frac{d-1}{4}; 1; 1 - \frac{1}{y^2} \right), \\ f_p &= 1 + \frac{Q_p}{R^{d-3}} \left(\frac{x-y}{-2y} \right)^{(d-3)/2} {}_2F_1\left(\frac{d-3}{4}, \frac{d-1}{4}; 1; 1 - \frac{1}{y^2} \right), \\ A_i dx^i &= -\frac{d-3}{2} \frac{q}{R^{d-5}} \frac{(y^2-1)(x-y)^{(d-5)/2}}{(-2y)^{(d-1)/2}} \, {}_2F_1\left(\frac{d-1}{4}, \frac{d+1}{4}; 2; 1 - \frac{1}{y^2} \right) d\psi, \end{aligned} \quad (6.2.7)$$

and Q_f, Q_p, q and R are related to the quantized charges n, w, Q and the angular momentum J via the relations

$$q = \frac{16\pi G_d}{(d-3)\Omega_{d-2}\alpha'} Q, \qquad R^2 = \alpha' \frac{J}{Q}, \qquad Q_f = \frac{16\pi G_d R_d}{(d-3)\Omega_{d-2}\alpha'} w, \qquad Q_p = \frac{16\pi G_d}{(d-3)\Omega_{d-2}R_d} n \, . \quad (6.2.8)$$

Here Ω_D is the area of unit D-sphere and G_d is the d-dimensional Newton's constant obtained by regarding the S^1 direction as compact

$$16\pi G_d = \frac{16\pi G_{d+1}}{2\pi R_d} = \frac{(2\pi)^{d-3} g^2 \alpha'^{(d-1)/2}}{R_d}. \tag{6.2.9}$$

n and $-w$ denote respectively the number of units of momentum and winding charge along the S^1 direction labeled by x^d. Q represents the number of units of winding charge along the singular ring situated at $y = -\infty$. From the perspective of an asymptotic observer Q appears as a dipole charge and does not represent a conserved gauge charge. In order that the metric given in (6.2.6) has the standard signature, we require $R^2 > 0$, i.e.

$$JQ > 0\,. \tag{6.2.10}$$

For $d = 5$ the hypergeometric functions simplify and we have

$$f_f = 1 + \frac{Q_f(x-y)}{2R^2}, \qquad f_p = 1 + \frac{Q_p(x-y)}{2R^2}, \qquad A_i dx^i = -\frac{q}{2}(1+y)d\psi. \tag{6.2.11}$$

This $d = 5$ solution can be obtained by setting to zero one of the three charges and two of the three dipoles charges of the supersymmetric black ring solution of [75–78]. The solution with general d can also be found by U-dualizing the supergravity supertube solution of [86].

For reasons that will become clear later, we shall work with the following assignment of charges:

$$J \gg Q \gg 1, \qquad n \sim w, \qquad nw \sim JQ, \qquad 1 - \frac{JQ}{nw} \sim 1\,. \tag{6.2.12}$$

Since the ring singularity occurs as $y \to -\infty$ we shall study the geometry near the singularity by examining the large negative y region. For

$$\frac{R}{\sqrt{\alpha'}} \gg |y| \gg \left(\frac{1}{g^2\,Q}\right)^{\frac{1}{d-4}} \frac{R}{\sqrt{\alpha'}}\,, \left(\frac{R_d^2}{g^2\,Q}\right)^{\frac{1}{d-4}} \frac{R}{\sqrt{\alpha'}}\,, 1\,, \tag{6.2.13}$$

the functions f_f, f_p and A_ψ take the form:

$$f_f \simeq \begin{cases} c_d Q_f R^{3-d}|y|^{d-4} & \text{for } d > 4, \\ \frac{1}{\pi} Q_f R^{-1} \log|y| & \text{for } d = 4, \end{cases}$$

$$f_p \simeq \begin{cases} c_d Q_p R^{3-d}|y|^{d-4} & \text{for } d > 4, \\ \frac{1}{\pi} Q_p R^{-1} \log|y| & \text{for } d = 4, \end{cases}$$

$$A_\psi \simeq \begin{cases} -c_d q R^{5-d}|y|^{d-4} & \text{for } d > 4, \\ -\frac{1}{\pi} q R \log|y| & \text{for } d = 4, \end{cases} \tag{6.2.14}$$

where

$$c_d = \frac{\Gamma(\frac{d-4}{2})}{2\sqrt{\pi}\,\Gamma(\frac{d-3}{2})}\,. \tag{6.2.15}$$

Let us restrict to the case $d \geq 5$ and use the convention $\alpha' = 1$. Using eqs. (6.2.8) and (6.2.9) we see that in the region (6.2.14) the original solution (6.2.6) takes the form:

$$\begin{aligned}
ds^2_{str,d+1} &= \frac{n}{w}\frac{1}{R_d^2}\,(dx^d - dt)^2 + 2\,\frac{(d-3)\Omega_{d-2}}{c_d\,(2\pi)^{d-3}}\,\frac{1}{w}\,\frac{R}{g^2}\,\left(-\frac{R}{y}\right)^{d-4}\,dt\,(dx^d - dt) \\
&\quad +2\frac{J}{w\,R_d}\,d\psi\,(dx^d - dt) + R^2\,\frac{dy^2}{y^4} + R^2\,d\psi^2 + \left(-\frac{R}{y}\right)^2\,d\Omega^2_{d-3}\,, \\
\frac{1}{2}B_{\mu\nu}dx^\mu \wedge dx^\nu &= -\frac{(d-3)\Omega_{d-2}}{c_d\,(2\pi)^{d-3}}\,\frac{1}{w}\,\frac{R}{g^2}\,\left(-\frac{R}{y}\right)^{d-4}\,dt \wedge (dx^d - dt) \\
&\quad +\frac{J}{w\,R_d}d(x^d - t) \wedge d\psi + \text{constant}\,, \\
e^{2\Phi_{d+1}} &= \frac{(d-3)\Omega_{d-2}}{c_d\,(2\pi)^{d-3}}\,\frac{R}{w}\,\left(-\frac{R}{y}\right)^{d-4}\,,
\end{aligned} \tag{6.2.16}$$

where

$$d\Omega^2_{d-3} \equiv \frac{dx^2}{1-x^2} + (1-x^2)d\Omega^2_{d-4} \tag{6.2.17}$$

is the squared line element on a $(d-3)$-sphere and the constant in the expression for B represents a constant 2-form proportional to $dt \wedge dx^d$ which can be removed by gauge transformation. For this solution all scalars constructed out of curvatures and other field strengths are small in the region (6.2.13). For example the $(d+1)$-dimensional Ricci scalar in the string frame goes as $(y/R)^2$ in this region. As a result the supergravity approximation is still valid in this region and the form of the solution (6.2.16) can be trusted.

It is easy to see from (6.2.16) that the 2×2 matrix describing the metric in the ψ-x^d plane develops a negative eigenvalue for $JQ > nw$. Since the ψ-x^d plane is topologically a two dimensional torus, the corresponding space-time has closed time-like curves. Thus absence of closed time-like curve requires that

$$JQ \leq nw\,. \tag{6.2.18}$$

Thus is precisely the Regge bound. The fact that (6.2.18) can be derived by requiring absence of closed time-like curves was noted in [86] in a different U-duality frame. Here we see that this geometrical condition is identical to the Regge bound in the spectrum of elementary BPS string states that follows from the left-right level matching condition in the microscopic theory analyzed in section 5.

Note that both the conditions $JQ > 0$ given in (6.2.10) and the Regge bound (6.2.18) are expected from the profile of the microscopic string underlying the solution as discussed in the section 6.1. This construction led to a manifestly positive JQ and $(nw - JQ)$. The point however is that once we have obtained the solution, we can interpret it as a solution of the supergravity equations of motion parametrized by the charges J, Q, n and w without worrying about where it came from. Nevertheless regularity of the space-time geometry requires both the conditions (6.2.10) and (6.2.18) to be satisfied.

From now on we shall restrict our analysis to the case $JQ < nw$. By examining the solution (6.2.16) we see that this seems to depend on the various charges n, w, J, Q as well as the asymptotic values g and R_d of the moduli fields. We shall now show that by making a suitable coordinate transformation the solution can be made to be independent of the parameters g and R_d, and have simple dependence on the charges. We define

$$\sigma = \sqrt{\frac{n}{w} - \frac{JQ}{w^2}}\,\frac{1}{R_d}\,(x^d - t), \quad \rho = -\frac{R}{y}\,,$$
$$\tau = \frac{(d-3)\Omega_{d-2}}{c_d\,(2\pi)^{d-3}}\,\frac{R_d}{\sqrt{nw - JQ}}\,\frac{R}{g^2}\,t, \quad \chi = \sqrt{\frac{J}{Q}}\,\psi + \frac{\sqrt{JQ}}{w}\,\frac{1}{R_d}\,(x^d - t), \tag{6.2.19}$$

In this coordinate system the region (6.2.13) gets mapped to

$$1 \ll \rho \ll \left(g^2 Q\right)^{\frac{1}{d-4}}, \left(g^2 Q/R_d^2\right)^{\frac{1}{d-4}}, R\,, \tag{6.2.20}$$

and the coordinates χ and σ have the following periods:

$$(\sigma, \chi) \equiv \left(\sigma, \chi + 2\pi\sqrt{\frac{J}{Q}}\right) \equiv \left(\sigma + 2\pi\sqrt{\frac{n}{w}}\sqrt{1 - \frac{JQ}{nw}}, \chi + 2\pi\frac{\sqrt{JQ}}{w}\right). \tag{6.2.21}$$

The σ-χ plane at fixed values of the other coordinates is topologically a two dimensional torus of coordinate area

$$A_{\sigma\chi} = 4\pi^2\sqrt{\frac{Jn}{Qw}}\sqrt{1 - \frac{JQ}{nw}}. \tag{6.2.22}$$

In terms of the coordinates (6.2.19) the field configuration given in (6.2.16) in the region (6.2.20) takes the form:

$$ds^2_{str,d+1} = d\sigma^2 + d\chi^2 + 2\rho^{d-4}\,d\tau d\sigma + d\rho^2 + \rho^2 d\Omega^2_{d-3}$$

$$\frac{1}{2}B_{\mu\nu}dx^{\mu} \wedge dx^{\nu} = -\rho^{d-4}\, d\tau \wedge d\sigma + \left(\frac{nw}{JQ} - 1\right)^{-1/2} d\sigma \wedge d\chi$$
$$e^{2\Phi_{d+1}} = \frac{(d-3)\Omega_{d-2}}{c_d\,(2\pi)^{d-3}}\,\frac{1}{w}\sqrt{\frac{J}{Q}}\,\rho^{d-4}\,. \qquad (6.2.23)$$

By examining the solution we see that in this coordinate system the solution as well as the periodicities of the (σ, χ) coordinates are independent of the parameters g and R_d. Furthermore the dependence of the solution on the charges comes via some additive constants in the expressions for $B_{\sigma\chi}$ and Φ_{d+1}, and in the periodicities of the (σ, χ) plane. We shall make use of this observation later to determine how the α'-corrected solution depends on the charges.

We now note the following properties of this background:

- For $\rho \gg 1$ curvature and other field strengths associated with this configuration are small. Hence the higher derivative corrections to the equations of motion are negligible. Since $J \sim nw/Q \sim w^2/Q$ and $Q \gg 1$, the string coupling constant $e^{\Phi_{d+1}}$ is also small in this region showing that the string loop corrections are also negligible.

- For $\rho \sim 1$ the curvature and other field strengths become of order unity and hence the α' corrections become important. However $e^{\Phi_{d+1}}$ continues to be small since for the choice of charges of the form (6.2.12), $\frac{1}{w}\sqrt{\frac{J}{Q}} \sim \frac{1}{Q}$ is small. Thus string loop corrections are not important.

- If we naively put a 'stretched horizon' at $\rho \sim 1$, and calculate the naive entropy from the area of the stretched horizon, spanned by χ, σ and the coordinates of S^{d-3} we get an answer proportional to $\sqrt{nw - QJ}$. This seems to agree with the formula (5.2.3) for the statistical entropy. However we should keep in mind that at $\rho \sim 1$ higher derivative corrections are important and neither the form of the solution nor the Bekenstein-Hawking formula for the entropy can be trusted. This is the problem to which we shall now turn.

The analysis of higher derivative corrections to the solution given in (6.2.23) is facilitated by the following observations:

1. The solution given in (6.2.23) is independent of the coordinates τ, σ and χ. It also has a $SO(d-2)$ spherical symmetry acting on the coordinates of the unit

$(d-3)$-sphere whose line element has been denoted by $d\Omega_{d-3}$. We expect that the α'-corrected solution will also preserve these symmetries.

2. Since the string coupling constant at the horizon is small, we can ignore string loop corrections to the effective action. The tree level α' corrected theory can be described by an effective Lagrangian density in $(d+1)$ dimensions which does not depend on the periods of the σ and the χ coordinates. Since we are looking for solutions which are independent of the σ and χ coordinates, the independence of the Lagrangian density on their periodicities guarantees that there is no dependence of the solution on the periodicities of these variables.[3] In particular the solution will have precisely the same form even if the σ and the χ coordinates had been non-compact.

3. By examining the form of the solution (6.2.23) we see that except for an additive constant term $\left(\frac{nw}{JQ}-1\right)^{-1/2}$ in the expression for $B_{\sigma\chi}$ and an additive term of the form $\frac{1}{2}\ln\left(\frac{1}{w}\sqrt{\frac{J}{Q}}\right)$ in the expression for Φ_{d+1}, the solution is independent of the various charges and the asymptotic values of the various moduli *e.g.* g, R_d etc. Since the tree level effective Lagrangian density depends on Φ_{d+1} only via an overall multiplicative factor of $e^{-2\Phi_{d+1}}$ and terms involving derivatives of Φ_{d+1} and depends on $B_{\mu\nu}$ only through its field strength dB, $B_{\mu\nu}$ and Φ_{d+1} can be shifted by arbitrary constants without affecting the rest of the solution. Thus we shall expect that even after including the α' corrections the solution continues to be independent of the various charges and the parameters g, R_d etc. except for an additive factor of $\frac{1}{2}\ln\left(\frac{1}{w}\sqrt{\frac{J}{Q}}\right)$ in Φ_{d+1} and an additive factor of $\left(\frac{nw}{JQ}-1\right)^{-1/2}$ in $B_{\sigma\chi}$.

The general form of the modified solution subject to these requirements is given by

$$\begin{aligned}
ds^2_{str,d+1} &= g_{\alpha\beta}(\rho)\, d\zeta^\alpha d\zeta^\beta + f_1(\rho)\, d\Omega^2_{d-3} + d\rho^2 \\
\frac{1}{2}B_{\mu\nu}dx^\mu\wedge dx^\nu &= b_{\alpha\beta}(\rho)\, d\zeta^\alpha\wedge d\zeta^\beta + \left(\frac{nw}{JQ}-1\right)^{-1/2} d\sigma\wedge d\chi, \\
e^{2\Phi_{d+1}} &= \frac{1}{w}\sqrt{\frac{J}{Q}}\, f_2(\rho)\ ,
\end{aligned} \tag{6.2.24}$$

[3] One could worry about possible corrections to the effective action due to world-sheet instantons wrapping the χ-σ torus. However such contributions will be exponentially suppressed due to large area of this torus measured in the string metric.

where $\zeta \equiv (\zeta^0, \zeta^1, \zeta^2)$ stands collectively for the coordinates τ, σ and χ, and $g_{\alpha\beta}$, $b_{\alpha\beta}$, f_1 and f_2 are some universal functions of the coordinate ρ, independent of any other charges and parameters. The solution in the original coordinate system, if needed, can now be found by applying the inverse of the coordinate transformation (6.2.19) on (6.2.24).

A priori we do not know the form of the functions $g_{\alpha\beta}$, $b_{\alpha\beta}$, f_1 and f_2, but let us proceed with the assumption that α' corrections modify the near horizon geometry to that of an (extremal) black hole. In that case computation of the entropy requires us to integrate certain combinations of the fields over the horizon [93–96]. From the coordinate area of $4\pi^2\sqrt{\frac{Jn}{Qw}}\sqrt{1-\frac{JQ}{nw}}$ in the χ-σ plane we get a factor proportional to $\sqrt{\frac{Jn}{Qw}}\sqrt{1-\frac{JQ}{nw}}$ from integration along these coordinates. The multiplicative factor of $w\sqrt{\frac{Q}{J}}$ in $e^{-2\Phi_{d+1}}$ appears as an overall normalization factor in the α' corrected effective action and gives a factor proportional to $w\sqrt{\frac{Q}{J}}$ in the entropy. Besides these multiplicative factors the contribution to the entropy cannot depend on any other charges or parameters since the solution has no non-trivial dependence on any other parameter. This gives

$$S_{BH} = C\sqrt{nw - JQ} \tag{6.2.25}$$

for some constant C. This is in precise agreement with the answer (5.2.3) for the statistical entropy if we take $C = 4\pi$.

It is worth emphasizing the role of the scaling region (6.2.20) where the supergravity solution (6.2.23) is valid. As we have seen, in this region the dependence of the solution on the asymptotic moduli parameters R_d and g disappears completely. This is then used to argue that the α' corrected solutions near the horizon will also be independent of these parameters. Thus this scaling region acts as a shield which isolates the near horizon region from the asymptotic region. However, since for large but finite charges the scaling region has a large but finite size, we expect that this shielding will work only in the leading order, and could break down at the subleading order in an expansion in inverse powers of charges. In section 7 we shall see that if we assume that the near horizon geometry has an AdS_2 factor then the independence of the entropy of the asymptotic parameters holds to all orders since there is an infinite throat region of AdS_2 that separates the near horizon geometry from the asymptotic geometry by an infinite amount.

We note in passing that if we had tried to carry out a similar scaling analysis for the rotating black hole solutions of [97] or their generalization to higher dimension, we would be led to the conclusion that the result for the black hole entropy is of the form $\sqrt{nw}\, g(J^2/nw)$ for some function g [98]. This is in contradiction with the microscopic result which gives the entropy to be $\sqrt{nw}$ times a function of J/nw. This also shows that the ring geometry is the correct geometry for describing an elementary string state with spin.

Just based on the scaling analysis we cannot determine the value of this coefficient C. However since from the point of view of the near horizon geometry the coordinate ψ describes a non-contractible circle, we can regard this as a compact direction. In that case the solution (6.2.23) and its α' corrected version (6.2.24) can be regarded as one describing the near horizon geometry of a non-rotating black hole in a space-time with $(d-1)$ non-compact dimensions, carrying n units of momentum and $-w$ units of winding along the x^d directions and J units of momentum and Q units of winding along the ψ directions. Thus as long as non-rotating small black holes in $(d-1)$ dimensions have finite entropy, rotating d dimensional black holes also have finite entropy. Furthermore if the entropy of non-rotating small black holes in $(d-1)$ dimensions agrees with the microscopic entropy, the constant C is equal to 4π. This in turn will imply that the entropy of the rotating small black rings in d dimension also agrees with the corresponding statistical entropy.

Regarding the solution as a $(d-1)$-dimensional small black hole also gives us some insight into the form of eq. (6.2.25). The $(d-1)$-dimensional effective action is known to have a continuous $SO(2,2)$ symmetry to all orders in α' expansion due to the fact that we are examining a sector where the fields are independent of σ and χ directions. The only $SO(2,2)$ invariant combination of the charges n, w, J and Q is $nw - JQ$ or a function of this combination. Since (6.2.25) depends on this combination we see that this formula is consistent with the SO(2,2) invariance of the theory.

Entropy Function and Near Horizon Geometry

In section 6 we determined the geometry of the small black ring in terms of some unknown universal functions $g_{\alpha\beta}(\rho)$, $b_{\alpha\beta}(\rho)$, $f_1(\rho)$ and $f_2(\rho)$. Supergravity approximation to the effective action, which is valid for large ρ, determines the behavior of these functions at large ρ. In this section we shall describe a general procedure for determining the form of these functions at small ρ, assuming that the near horizon geometry in this region approaches that of an extremal black hole with an AdS_2 factor, and as a result possesses an enhanced isometry $SO(2,1)$. Our main tool in this analysis will be the entropy function method described in [88, 99].

As in section 6 we consider heterotic string theory compactified on $T^{9-d} \times S^1$ and consider an extremal black ring solution in this theory. From the analysis of section 6 we know that the geometry close to the horizon has two compact directions labeled by the angular coordinate ψ and the coordinate along S^1:

$$y^d = (x^d - t)/R_d \,, \tag{7.0.1}$$

each with period 2π. Thus we can analyze the near horizon geometry of such a black ring by analyzing the $(d-1)$-dimensional theory obtained via dimensional reduction of the original theory on these two circles. We parametrize these $(d-1)$ dimensions by the coordinates $\{\xi^m\}$ with $0 \leq m \leq (d-2)$, and introduce the following $(d-1)$-dimensional fields in terms of the original $(d+1)$-dimensional fields:[1]

$$ds^2_{str,d+1} = \widehat{G}_{mn}(\xi) d\xi^m d\xi^n + R(\xi)^2 (dy^d + A^{(1)}_m(\xi) d\xi^m)^2 + \widetilde{R}(\xi)^2 (d\psi + A^{(2)}_m(\xi) d\xi^m)^2$$

[1]Our definition of the $(d-1)$-dimensional antisymmetric tensor field $\widehat{B}_{mn}$ differs from the standard one, *e.g.* the one used in [54], by a term proportional to $(A^{(1)}_{[m} A^{(3)}_{n]} + A^{(2)}_{[m} A^{(4)}_{n]})$. As a consequence the expression for the gauge Chern-Simons term appearing in the expression for $\widehat{H}_{mnp}$ is also slightly different.

$$
\begin{aligned}
&\quad +2\, S(\xi)\, (dy^d + A^{(1)}_m(\xi) d\xi^m)\, (d\psi + A^{(2)}_n(\xi) d\xi^n)\,, \\
\frac{1}{2}\, B_{\mu\nu} dx^\mu \wedge dx^\nu &= \frac{1}{2}\, \widehat{B}_{mn}(\xi)\, d\xi^m \wedge d\xi^n + C(\xi)\, (dy^d + A^{(1)}_m(\xi) d\xi^m) \wedge (d\psi + A^{(2)}_m(\xi) d\xi^m) \\
&\quad + (dy^d + A^{(1)}_m(\xi) d\xi^m) \wedge A^{(3)}_n(\xi)\, d\xi^n + (d\psi + A^{(2)}_m(\xi) d\xi^m) \wedge A^{(4)}_n(\xi)\, d\xi^n\,.
\end{aligned}
\tag{7.0.2}
$$

Thus the fields in $(d-1)$ dimensions include a metric $\widehat{G}_{mn}$, an anti-symmetric tensor field $\widehat{B}_{mn}$, four gauge fields $A^{(i)}_m$ for $1 \le i \le 4$ and five scalar fields R, $\widetilde{R}$, S, C and Φ_{d+1}. The gauge invariant field strengths constructed out of the fields $A^{(i)}_m$ and $\widehat{B}_{mn}$ are:

$$
\begin{aligned}
F^{(i)}_{mn} &= \partial_m A^{(i)}_n - \partial_n A^{(i)}_m\,, \quad i = 1, 2, 3, 4 \\
\widehat{H}_{mnp} &= \Big(\partial_m \widehat{B}_{np} + \text{cyclic permutations of } m, n, p\Big) + \Omega^{CS}_{mnp}\,,
\end{aligned}
\tag{7.0.3}
$$

where

$$
\Omega^{CS}_{mnp} = \Big\{ (A^{(3)}_m F^{(1)}_{np} + A^{(4)}_m F^{(2)}_{np}) + \text{cyclic permutations of } m, n, p \Big\} + \Omega^{CS,L}_{mnp}(\widehat{G})\,. \tag{7.0.4}
$$

The action of the dimensionally reduced theory has the form:

$$
\mathcal{S} = \int d^{d-1}x\, \sqrt{-Det[{a \atop b}]\widehat{G}}\, e^{-2\Phi_{d+1}}\, \mathcal{L}_{d-1}\,, \tag{7.0.5}
$$

where $\mathcal{L}_{d-1}$ is a scalar function of the scalars R, $\widetilde{R}$, S and C, the metric $\widehat{G}_{mn}$, Riemann tensor, the field strengths $F^{(i)}_{mn}$ and $\widehat{H}_{mnp}$, $\partial_m \Phi_{d+1}$ and covariant derivatives of these quantities.

The presence of the Chern-Simons terms in the definition of $\widehat{H}_{mnp}$ makes this form of the action unsuitable for applying the entropy function formalism since the latter requires the Lagrangian density, when expressed in terms of the independent fields of the theory, to be a function of manifestly covariant quantities like the metric, Riemann tensor, gauge field strengths, scalar fields, covariant derivatives of these quantities etc., and not *e.g.* of non-covariant quantities like the gauge fields or the spin connection.[2] To get around this problem we need to dualize the theory [100]. For this note that $\widehat{H}_{mnp}$ satisfies the Bianchi identity

$$
\partial_{[m} \Big(\widehat{H}_{npq]} - \Omega^{CS}_{npq]}\Big) = 0\,. \tag{7.0.6}
$$

[2]The only exception to this rule are p-form gauge fields whose near horizon values themselves are manifestly invariant under all the isometries of the near horizon geometry. In this case we never need to explicitly use the gauge invariance associated with these fields and can regard them as ordinary tensor fields.

We now introduce a new $(d-5)$-form field $\mathcal{B}_{m_1\ldots m_{d-5}}$, define

$$\mathcal{H}_{m_1\ldots m_{d-4}} = \partial_{m_1}\mathcal{B}_{m_2\ldots m_{d-4}} + \text{cyclic permutations of } m_1\ldots m_{d-4} \text{ with sign}\,, \tag{7.0.7}$$

to be its field strength and consider a new action

$$\int d^{d-1}x\,\sqrt{-Det[{a \atop b}]\widehat{G}}\,\widetilde{\mathcal{L}}_{d-1}\,, \tag{7.0.8}$$

$$\sqrt{-Det[{a \atop b}]\widehat{G}}\,\widetilde{\mathcal{L}}_{d-1} = \left[\sqrt{-Det[{a \atop b}]\widehat{G}}\,e^{-2\Phi_{d+1}}\,\mathcal{L}_{d-1} + \epsilon^{m_1\ldots m_{d-1}}\left(\widehat{H}_{m_1m_2m_3} - \Omega^{CS}_{m_1m_2m_3}\right)\mathcal{H}_{m_4\ldots m_{d-1}}\right]\,, \tag{7.0.9}$$

regarding $\widehat{H}_{mnp}$ and $\mathcal{B}_{m_1\ldots m_{d-5}}$ as independent fields. Here $\epsilon^{m_1\ldots m_{d-1}}$ is totally antisymmetric in its indices, with $\epsilon^{01\ldots(d-2)} = 1$. Equations of motion of the $\mathcal{B}$ field gives the Bianchi identity (7.0.6). On the other hand the equations of motion of $\widehat{H}_{m_1m_2m_3}$ gives

$$\frac{\delta\mathcal{S}}{\delta\widehat{H}_{m_1m_2m_3}} + \epsilon^{m_1\ldots m_{d-1}}\,\mathcal{H}_{m_4\ldots m_{d-1}} = 0\,. \tag{7.0.10}$$

Together with the Bianchi identity $\partial_{[m_3}\mathcal{H}_{m_4\ldots m_{d-1}]} = 0$, (7.0.10) gives us the original equations of motion of the $\widehat{B}_{mn}$ field:

$$\partial_p\left(\frac{\delta\mathcal{S}}{\delta\widehat{H}_{mnp}}\right) = 0\,. \tag{7.0.11}$$

Thus classically (7.0.8) and (7.0.5) gives rise to the same theory and we can choose to work with the action (7.0.8).

The action (7.0.8) can now be brought into a manifestly gauge, local Lorentz and general coordinate invariant form by integrating the last term in (7.0.9) by parts. This gives a new Lagrangian density:

$$\begin{aligned}\sqrt{-Det[{a \atop b}]\widehat{G}}\,\widetilde{\mathcal{L}}'_{d-1} = {} & \sqrt{-Det[{a \atop b}]\widehat{G}}\,e^{-2\Phi_{d+1}}\,\mathcal{L}_{d-1} \\ & -(d-4)\epsilon^{m_1\ldots m_{d-1}}\,\partial_{m_4}\left(\widehat{H}_{m_1m_2m_3} - \Omega^{CS}_{m_1m_2m_3}\right)\mathcal{B}_{m_5\ldots m_{d-1}}\,.\end{aligned} \tag{7.0.12}$$

Since $d\Omega^{CS}$ can be expressed as a function of manifestly covariant quantities like the Riemann tensor and gauge field strengths $F^{(i)}_{mn}$, the Lagrangian density $\widetilde{\mathcal{L}}'_{d-1}$ is suitable for applying the entropy function formalism. Note however that $\widetilde{\mathcal{L}}'_{d-1}$ does not have manifest symmetry under the gauge transformation $\mathcal{B} \to \mathcal{B} + d\Lambda$, Λ being a $(d-6)$-form gauge transformation parameter. This will not affect our analysis since we shall be considering field configurations for which the field $\mathcal{B}$ (and not just its field strength

$\mathcal{H}$) has all the necessary symmetries. Thus, as discussed in footnote 2, we never need to make use of the gauge invariance associated with the field $\mathcal{B}$, and shall treat $\mathcal{B}$ as an independent tensor field.

We are now ready to apply the entropy function formalism. We begin with the basic postulate that in terms of the $(d-1)$-dimensional fields the near horizon metric of the extremal black ring has the structure of $AdS_2 \times S^{d-3}$, with all other field configurations respecting the $SO(2,1) \times SO(d-2)$ isometry of $AdS_2 \times S^{d-3}$. Then the general near horizon geometry of the black ring is of the form:[3]

$$\begin{aligned}
&\widehat{G}_{mn} d\xi^m d\xi^n = v_1 \left(-r^2 d\bar{t}^2 + \frac{dr^2}{r^2} \right) + v_2 \, d\Omega_{d-3}^2 \, , \\
&R = u_R, \quad \widetilde{R} = \widetilde{u}_R, \quad S = u_S \, , \quad C = u_C, \quad \Phi_{d+1} = u_\Phi \, , \\
&\frac{1}{2} F^{(i)}_{mn} dx^m \wedge dx^n = e_i \, dr \wedge d\bar{t} \, , \quad i = 1,2,3,4 \\
&\frac{1}{(d-5)!} \mathcal{B}_{m_1 \ldots m_{d-5}} d\xi^{m_1} \wedge \ldots \wedge d\xi^{m_{d-5}} = \begin{cases} b \text{ for } d=5 \\ b \, dr \wedge dt \text{ for } d=7 \\ 0 \text{ for } d \neq 5,7 \end{cases} ,
\end{aligned} \tag{7.0.13}$$

where v_1, v_2, u_R, $\widetilde{u}_R$, u_S, u_C, u_Φ, e_1, e_2, e_3, e_4 and b are constants and $d\Omega_{d-3}$ denotes the line element on a unit $(d-3)$-sphere. Note that we have not explicitly given the $\widehat{H}$ field background; we are implicitly assuming that $\widehat{H}$ has been eliminated from the action using its equation of motion. In any case, the only possible non-zero component of $\widehat{H}_{mnp}$ consistent with the symmetries of $AdS_2 \times S^{d-3}$ is a flux through S^{d-3} in the special case of $d = 6$. However since we are considering solutions without magnetic charge, — more specifically NS 5-brane charge — even for $d = 6$ this flux should vanish. Thus we can consistently set $\widehat{H}_{mnp}$ to zero. The entropy function for a black ring carrying n units of momentum and $-w$ units of winding along y^d and J units of momentum and Q units of winding along ψ, is now given by [88][4]

$$\mathcal{E}(n, J, w, Q, v_1, v_2, u_R, \widetilde{u}_R, u_S, u_C, u_\Phi, e_1, e_2, e_3, e_4, b)$$

[3] r is related to the coordinate ρ of section 6 by the relation $r = e^{\rho/\sqrt{v_1}}$. The coordinate $\bar{t}$ and τ are in general related by a scaling. However since a rescaling of the form $\bar{t} \to \lambda \bar{t}$, $r \to r/\lambda$, being an isometry of AdS_2, preserves the form of the solution, we can use this freedom to choose

$$\bar{t} = \tau, \quad r = c \, e^{\rho/\sqrt{v_1}} \, ,$$

for some constant c.

[4] According to the convention of section 6, n, J, w and $-Q$ represent the charges associated with the $A^{(1)}_m$, $A^{(2)}_m$, $A^{(3)}_m$ and $A^{(4)}_m$ fields respectively. This explains the signs in front of various charges in (7.0.14). Note however that we have chosen to call $-w$ and Q the winding charges along y^d and ψ respectively.

$$= 2\pi \left(ne_1 + Je_2 + we_3 - Qe_4 - \int_{S^{d-3}} \sqrt{-Det[{}^a_b]\widehat{G}}\, \widetilde{\mathcal{L}}'_{d-1} \right) , \qquad (7.0.14)$$

where $\sqrt{-Det[{}^a_b]\widehat{G}}\, \widetilde{\mathcal{L}}'_{d-1}$ has to be evaluated on the horizon. The entropy is given by $\mathcal{E}$ after extremizing it with respect to the near horizon parameters v_1, v_2, u_R, $\widetilde{u}_R$, u_S, u_C, u_Φ, e_1, e_2, e_3, e_4 and b.

This gives a general algebraic procedure for determining the near horizon geometry of a small black ring for a given action. We shall now show that the entropy calculated from this formalism has the same dependence on n, w, J and Q as was derived in the previous section. For this we need to use the known scaling properties of the α' corrected tree level effective action of the heterotic string theory. First of all note from (7.0.12) that

$$\widetilde{\mathcal{L}}'_{d-1} \to \lambda\, \widetilde{\mathcal{L}}'_{d-1} \quad \text{under} \quad e^{-2\Phi_{d+1}} \to \lambda\, e^{-2\Phi_{d+1}}, \quad \mathcal{B}_{m_1\ldots m_{d-5}} \to \lambda\, \mathcal{B}_{m_1\ldots m_{d-5}} . \qquad (7.0.15)$$

The freedom of changing the periodicity along the circle S^1 labeled by y^d and subsequently making a rescaling of the y^d coordinate to bring the period back to 2π gives another scaling property of the Lagrangian density:

$$\widetilde{\mathcal{L}}'_{d-1} \to \kappa\, \widetilde{\mathcal{L}}'_{d-1} \quad \text{under} \quad A^{(1)}_m \to \kappa^{-1}\, A^{(1)}_m, \quad A^{(3)}_m \to \kappa\, A^{(3)}_m, \quad R \to \kappa\, R, \quad S \to \kappa\, S, \quad C \to \kappa\, C . \qquad (7.0.16)$$

Finally there is a similar scaling property associated with the scaling of the ψ coordinate. This gives

$$\widetilde{\mathcal{L}}'_{d-1} \to \eta\, \widetilde{\mathcal{L}}'_{d-1} \quad \text{under} \quad A^{(2)}_m \to \eta^{-1} A^{(2)}_m, \quad A^{(4)}_m \to \eta A^{(4)}_m, \quad \widetilde{R} \to \eta\, \widetilde{R}, \quad S \to \eta\, S, \quad C \to \eta\, C . \qquad (7.0.17)$$

From (7.0.15), (7.0.16) and (7.0.17) we can derive the following properties of the entropy function:

$$\mathcal{E} \to \lambda\, \mathcal{E} \quad \text{under} \quad e^{-2u_\Phi} \to \lambda\, e^{-2u_\Phi}, \quad b \to \lambda\, b, \quad Q \to \lambda\, Q, \quad n \to \lambda\, n, \quad w \to \lambda\, w, \quad J \to \lambda\, J , \qquad (7.0.18)$$

$$\begin{aligned} \mathcal{E} \to \kappa \mathcal{E} \text{ under} \quad & e_1 \to \kappa^{-1}\, e_1, \quad n \to \kappa^2\, n, \quad J \to \kappa J, \quad e_3 \to \kappa e_3, \quad Q \to \kappa Q, \\ & u_R \to \kappa\, u_R, \quad u_S \to \kappa\, u_S, \quad u_C \to \kappa\, u_C , \end{aligned} \qquad (7.0.19)$$

and

$$\begin{aligned} \mathcal{E} \to \eta\, \mathcal{E} \text{ under} \quad & n \to \eta\, n, \quad e_2 \to \eta^{-1}\, e_2, \quad J \to \eta^2\, J, \quad w \to \eta\, w, \quad e_4 \to \eta\, e_4, \\ & \widetilde{u}_R \to \eta\, u_R, \quad u_S \to \eta\, u_S, \quad u_C \to \eta\, u_C . \end{aligned} \qquad (7.0.20)$$

From this it follows that after elimination of the various near horizon parameters by extremizing $\mathcal{E}$, the entropy $S_{BH}=\mathcal{E}$ has the property:

$$S_{BH}\to\lambda\,S_{BH}\quad\text{under}\quad n\to\lambda\,n,\quad w\to\lambda\,w,\quad J\to\lambda\,J\quad Q\to\lambda\,Q\,, \tag{7.0.21}$$

$$S_{BH}\to\kappa\,S_{BH}\quad\text{under}\quad n\to\kappa^2\,n,\quad Q\to\kappa\,Q,\quad J\to\kappa\,J\,, \tag{7.0.22}$$

and

$$S_{BH}\to\eta\,S_{BH}\quad\text{under}\quad n\to\eta\,n,\quad w\to\eta\,w,\quad J\to\eta^2\,J\,. \tag{7.0.23}$$

Eqs. (7.0.21), (7.0.22) and (7.0.23) give

$$S_{BH}=\sqrt{nw}\,f\left(\frac{JQ}{nw}\right)\,, \tag{7.0.24}$$

for some function f.

We can constrain the form of the function f by noting that the dimensionally reduced Lagrangian density has a further symmetry induced by a rotation in the $y^d-\psi$ plane by the matrix

$$U=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\,. \tag{7.0.25}$$

This induces a transformation

$$\begin{pmatrix}R^2 & S\\ S & R^2\end{pmatrix}\to U\begin{pmatrix}R^2 & S\\ S & \widetilde{R}^2\end{pmatrix}U^T,\quad \begin{pmatrix}A^{(1)}_m\\ A^{(2)}_m\end{pmatrix}\to U\begin{pmatrix}A^{(1)}_m\\ A^{(2)}_m\end{pmatrix},\quad \begin{pmatrix}A^{(3)}_m\\ A^{(4)}_m\end{pmatrix}\to U\begin{pmatrix}A^{(3)}_m\\ A^{(4)}_m\end{pmatrix}. \tag{7.0.26}$$

Thus the entropy function $\mathcal{E}$ is invariant under

$$\begin{pmatrix}e_1\\ e_2\end{pmatrix}\to U\begin{pmatrix}e_1\\ e_2\end{pmatrix},\quad \begin{pmatrix}e_3\\ e_4\end{pmatrix}\to U\begin{pmatrix}e_3\\ e_4\end{pmatrix},\quad \begin{pmatrix}n\\ J\end{pmatrix}\to U\begin{pmatrix}n\\ J\end{pmatrix},\quad \begin{pmatrix}w\\ -Q\end{pmatrix}\to U\begin{pmatrix}w\\ -Q\end{pmatrix},$$
$$\begin{pmatrix}u_R^2 & u_S\\ u_S & \widetilde{u}_R^2\end{pmatrix}\to U\begin{pmatrix}u_R^2 & u_S\\ u_S & \widetilde{u}_R^2\end{pmatrix}U^T\,. \tag{7.0.27}$$

As a result the black hole entropy S_{BH} is invariant under

$$\begin{pmatrix}w\\ -Q\end{pmatrix}\to U\begin{pmatrix}w\\ -Q\end{pmatrix},\quad \begin{pmatrix}n\\ J\end{pmatrix}\to U\begin{pmatrix}n\\ J\end{pmatrix}\,. \tag{7.0.28}$$

Together with (7.0.24) this uniquely fixes the form of S_{BH} to be

$$S_{BH}=C\,\sqrt{nw-JQ}\,, \tag{7.0.29}$$

for some constant C.

If the entropy function had no flat directions then we would also be able to determine the near horizon parameters labeling the solution by demanding that the solution remains invariant under the symmetry transformations described above. This however is not possible due to the existence of two flat directions of the entropy function.[5] Thus for a given set of charges (n, J, w, Q) the entropy function extremization condition gives a two parameter family of solutions. Which of these solutions actually appear as the near horizon configuration of the black ring will depend on the asymptotic data — in particular the asymptotic values of the moduli fields and the information that we are considering a black ring in d-dimensions rather than a black hole in $(d-1)$ dimensions where the ψ direction remains compact even asymptotically. Thus the entropy function itself cannot give complete information about the near horizon geometry even if we knew the full α' corrected action.

On the other hand, the analysis in section 6 does know about the asymptotic infinity since it is based on a solution of the supergravity equations of motion embedded in an asymptotically flat d-dimensional spacetime. Therefore the solution (6.2.23) in the scaling region (6.2.20) does not involve any unknown parameter, and in turn completely determines the general form (6.2.24) of the solution near the horizon. This suggests that we can combine the results of section 6 and this section to fix the form of the near horizon geometry. We first note that the solution (7.0.13) corresponds to the $(d+1)$-dimensional field configuration:

$$\begin{aligned}
ds^2_{str,d+1} &= v_1\left(-r^2 d\bar{t}^2 + \frac{dr^2}{r^2}\right) + v_2\, d\Omega^2_{d-3} + u_R^2(dy^d + e_1 r d\bar{t})^2 + \widetilde{u}_R^2(d\psi + e_2 r\, d\bar{t})^2 \\
&\quad + 2\, u_S\,(dy^d + e_1 r d\bar{t})\,(d\psi + e_2 r d\bar{t})\,, \\
\frac{1}{2}B_{\mu\nu}dx^\mu \wedge dx^\nu &= u_C\,(dy^d + e_1 r d\bar{t}) \wedge (d\psi + e_2 r d\bar{t}) + e_3 r\,(dy^d + e_1 r d\bar{t}) \wedge d\bar{t} \\
&\quad + e_4 r\,(d\psi + e_2 r d\bar{t}) \wedge d\bar{t}\,, \\
e^{-2\Phi_{d+1}} &= e^{-2u_\Phi}\,.
\end{aligned} \tag{7.0.30}$$

In writing (7.0.30) we have used the coordinate transformations $y^d \to y^d + a\bar{t}$, $\chi \to \chi + b\bar{t}$ to remove the constant terms in $A_t^{(1)}$ and $A_t^{(2)}$. On the other hand (6.2.19), (6.2.24)

[5] If we consider the case where the coordinates y^d and ψ are both compact and there are no fluxes then the vacuum moduli space, ignoring the T^{9-d} factor, is $SO(2,2)/(SO(2)\times SO(2))$ where $SO(2,2)$ represents the continuous T-duality symmetry associated with a T^2 compactification, and $SO(2)\times SO(2)$ is its maximal compact subgroup. Since the charge vector (n, J, w, Q) is invariant under an $SO(2,1)$ subgroup of this $SO(2,2)$ group, once we switch on a flux proportional to this charge vector the moduli space of solutions becomes the two dimensional space $SO(2,1)/SO(2)$. This would correspond to two flat directions of the entropy function. The only exceptions are light-like charge vectors in which case the little group will be SO(1,1), but this corresponds to the case $nw - JQ = 0$ which we are not considering here. In general a symmetry transformation of the type given in eqs. (7.0.15)-(7.0.17) and (7.0.26), instead of leaving the solution unchanged, will take one solution to another solution.

tells us that if we use the (σ, χ) coordinate system defined through

$$\sigma = \sqrt{\frac{n}{w} - \frac{JQ}{w^2}}\, y^d, \qquad \chi = \sqrt{\frac{J}{Q}}\, w + \frac{\sqrt{JQ}}{w}\, y^d\,, \tag{7.0.31}$$

then the solution has a universal form except for some additive constants in $B_{\sigma\chi}$ and $\ln \Phi_{d+1}$. Requiring (7.0.30) to satisfy this requirement we get the form of the solution to be:

$$\begin{aligned}
ds^2_{str,d+1} &= c_1 \left(-r^2 d\bar{t}^2 + \frac{dr^2}{r^2} \right) + c_2\, d\Omega^2_{d-3} + c_3\, (d\sigma + c_4\, r\, d\bar{t})^2 \\
&\quad + c_5\, (d\chi + c_6\, r\, d\bar{t})^2 + c_7\, (d\sigma + c_4\, r\, d\bar{t})\, (d\chi + c_6\, r\, d\bar{t}) \\
e^{2\Phi_{d+1}} &= c_8 \sqrt{\frac{J}{Q}}\, \frac{1}{w} \\
\frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu &= c_9\, r\, d\sigma \wedge d\bar{t} + c_{10} r\, d\chi \wedge d\bar{t} + c_{11} d\sigma \wedge d\chi + \left(\frac{nw}{JQ} - 1 \right)^{-1/2} d\sigma \wedge d\chi\,,
\end{aligned} \tag{7.0.32}$$

where $c_1, \ldots, c_{11}$ are some numerical constants independent of any charges or other asymptotic data, which can in principle be determined, up to the flat direction, by extremizing the entropy function. This gives the general form of the solution close to the horizon. Since the periodicities of the σ and χ directions, as given in (6.2.21), depends on the charges, the solution has some additional implicit dependence on the charges besides the ones shown in (7.0.32). This can be made explicit by rewriting the solution in terms of $(y^d, \psi, \bar{t}, r)$ coordinate system using (7.0.31) such that both the compact coordinates y^d and ψ have period 2π.

In the spirit of the discussion at the end of section 6 we note that from the point of view of the near horizon geometry the coordinate ψ can be regarded as a compact direction. In that case the entropy function $\mathcal{E}$ considered here can be regarded as that of a $(d-1)$ dimensional non-rotating black hole carrying n units of momentum and $-w$ units of winding along the y^d direction and J units of momentum and Q units of winding along the ψ direction. Thus as long as non-rotating small black holes in $(d-1)$-dimension have finite entropy, rotating d-dimensional black holes also have finite entropy.[6] Furthermore if the entropy of non-rotating small black holes in $(d-1)$ dimension agrees with the microscopic entropy, the constant C is equal to 4π. This in

[6]For example if we add to the action a $(d-1)$-dimensional Gauss-Bonnet term then the resulting non-rotating small black holes acquire a finite entropy [61].

turn will imply that the entropy of the rotating small black rings in d dimension also agrees with the corresponding statistical entropy.

In this context it is also worth emphasizing that if we were studying a small black hole with four charges in $(d-1)$ dimensions instead of a ring in d-dimensions, then the near horizon geometry does contain two arbitrary parameters whose values need to be determined from the knowledge of the asymptotic values of the moduli field. In particular if we carry out an analysis analogous to that of section 6, we shall find that even in the scaling region the supergravity solution continues to depend on a pair of asymptotic moduli.

Discussion

In this work we studied the near-horizon geometry of the small black ring carrying charge quantum numbers (n, J, w, Q) from two different viewpoints. First we examined the full singular black ring solution of the supergravity theory describing a rotating spinning string and identified a scaling region where, in a suitable coordinate system, the solution ceased to depend on the asymptotic moduli and its dependence on the various charges appear in a fashion such that higher derivative corrections are insensitive to the charges. This allowed us to express the α'-corrected solution in terms of a set of universal functions independent of any parameters. The entropy computed from the α'-corrected solution was found to have the form $C\sqrt{nw - JQ}$ where C is a numerical constant that cannot be computed in the absence of a complete knowledge of all the α' corrections. Even if the α' correction to the effective action is known, in this approach it would be a highly non-trivial task to actually find the solution to the equations of motion and calculate the coefficient C from the α'-corrected action. Nevertheless the result for the entropy found in this approach is consistent with the result for the statistical entropy of the same system, given by $4\pi\sqrt{nw - JQ}$.

In the second approach we focussed our attention on the near horizon geometry instead of examining the full solution. Assuming that the near horizon geometry has an enhanced $SO(2, 1)$ symmetry besides the manifest rotational isometries of the solution, we can write down the general form of the solution in terms of a set of constant parameters, and the entropy is obtained by extremizing the entropy function with respect to these parameters. Using the various known scaling properties of the α'-corrected effective action we then determined the dependence of the entropy obtained this way on the charges, and arrived at the same answer $C\sqrt{nw - JQ}$ for some constant C. Again computation of the constant C requires knowledge of the α'-corrected action, but in this case once we know the action there is a simple algorithm to compute the entropy without having to solve any differential equations.

In principle extremization of the entropy function also determines the parameters characterizing the near horizon geometry. In practice however this is plagued by the problem that the entropy function relevant for this problem has two flat directions, and hence the extremization condition does not determine the solution uniquely. Thus which member of this two parameter family appears as the actual near horizon geometry depends on the asymptotic data. Nevertheless by combining the information about the asymptotic data from the first approach with the requirement of enhanced $SO(2,1)$ symmetry of the near horizon geometry we can determine the near horizon geometry in terms of the charges and a few (presently unknown) numerical constants.

Clearly the most important open problem is to find the constant C. An insight into this problem can be gained from the observation that this constant is the same as what appears in the expression for the entropy of a non-rotating small black hole in one less dimension. Thus agreement between the microscopic and macroscopic entropy for non-rotating small black hole in $(d-1)$-dimension would also imply agreement between microscopic and macroscopic entropy of a rotating small black ring in d-dimensions. At present however concrete analysis of this constant C has been performed only in the case of four dimensional small black holes [74]. Initial studies of these black holes were based on keeping only a small subset of higher derivative corrections to the effective action, *e.g.* the F-type terms [52] or the Gauss-Bonnet term [61], yielding the same answer $C = 4\pi$. However later a general procedure for analyzing these black holes was developed by Kraus and Larsen [62] where, based on the assumption that the AdS_2 and the S^1 factor of the near horizon geometry combine to form an AdS_3 space, they were able to relate the coefficient C to the coefficients of the gauge and gravitational Chern-Simons terms in the action. Since these coefficients are known exactly, C also can be calculated exactly. This yielded the same answer $C = 4\pi$. In principle it should be possible to generalize the results of Kraus and Larsen to higher dimensional black holes, but so far this has not been done.

Closed String Tachyon Potential

Closed String Tachyon Potential – Introduction

There are a number of physical situations, for example in cosmology, where it is necessary to deal with unstable and time-dependent backgrounds. It is of interest to develop calculational tools within string theory that can describe such backgrounds in an essentially stringy way.

A useful laboratory for studying unstable or time-dependent backgrounds in string theory is provided by tachyons in open string theory. These tachyons correspond to the instabilities of various unstable brane configurations and their condensation is expected to describe the decay of these unstable branes to flat space. The static as well as time dependent aspects of such decays have been analyzed quite extensively.

By comparison, tachyons in closed string theory, even though more interesting physically, have proved to be less tractable. For these tachyons, in most cases there is no natural candidate for a stable minimum of the potential where the tachyon fields can acquire an expectation value. For a closed-string tachyon with a string-scale mass it is difficult to disentangle a well-defined potential from other gravitational effects. In some cases, as in the case of thermal tachyon which signifies the onset of Hagedorn transition, the mass can be fine-tuned to be very small [103, 104], but the tachyon has cubic couplings to the dilaton and other massless scalar fields. Consequently, it sources other massless fields which considerably complicates the dynamics and quickly drives the system into strong coupling or strong curvature region [105, 106]. Similarly, for the open string tachyon in the brane-antibrane system, the mass can be tuned to zero by adjusting the distance between the brane and the antibrane [107] to be the string scale. However, at the endpoint of condensation, the distance between brane and antibrane vanishes and the tachyon eventually has string scale mass and the effective field theory breaks down.

In this work we showed that for localized tachyons in the twisted sector of the $\mathbf{C}/\mathbf{Z}_N$ orbifold theories, some of these difficulties can be circumvented.

A Conjecture for the Height of the Tachyon Potential

We now pursue some of the analogies of closed-string localized tachyons and open-string tachyons with the aim of identifying a model where explicit computations are possible.

10.1 Analogies with Open-String Tachyons

There are three main simplifications which make the open-string tachyons tractable.

- The tachyons are localized on the worldvolume of an unstable brane. It is reasonable to assume that condensation of the tachyon corresponds to the annihilation of the brane and the system returns to empty flat space.
- Conservation of energy then implies Sen's conjecture [108, 109] that the height of the tachyon potential should equal the tension of the brane that is annihilated.
- The gravitational backreaction of D-branes can be made arbitrarily small by making the coupling very small because the tension of D-branes is inversely proportional to closed string coupling g_c whereas Newton's constant is proportional to g_c^2. This makes it possible to analyze the tachyon potential without including the backreaction of massless closed-string modes.

The twisted-sector tachyons in $\mathbf{C}/\mathbf{Z}_N$ string backgrounds are in many respects quite analogous.

- The $\mathbf{C}/\mathbf{Z}_N$ theory has the geometry of a cone with deficit angle $2\pi(1 - 1/N)$ and the twisted-sector tachyons are localized at the tip [110, 111]. There is considerable evidence now that condensation of these tachyons relaxes the cone to empty flat space and thus much like the open string tachyons, there is a natural candidate for the endpoint of the condensation [112].

- There is a precise conjecture for the effective height of the tachyon potential that is analogous to the Sen's conjecture in the open string case [113].
- At large N, some of the relevant tachyons are nearly massless. Therefore, one would expect that there is a natural separation between the string scale and the scale at which the dynamics of the tachyons takes place and thus higher order stringy corrections can be controlled. The required S-matrix elements are completely computable using orbifold CFT techniques.

Note that for the $\mathbf{C}/\mathbf{Z}_N$ tachyons we can talk about the height of the tachyon potential because the background is not Lorentz invariant. Quite generally, two closed-string CFT backgrounds which are both Lorentz invariant cannot be viewed as two critical points of a scalar potential at different heights. This is because a nonzero value of a scalar potential at a critical point with flat geometry would generate a tadpole for the dilaton and the string equations of motion would not be satisfied. By contrast, the $\mathbf{C}/\mathbf{Z}_N$ backgrounds are not flat because there is a curvature singularity at the tip of the cone. It is natural to assume that the tachyon potential provides the energy source for the curvature. There is not dilaton tadpole because, for a conical geometry, the change in the Einstein-Hilbert term in the action precisely cancels the change in the height of the potential. Hence the total bulk action which generates the dilaton tadpole vanishes on the solution. There is an important boundary contribution that is nonzero and as a result there is a net change in the classical action. This reasoning leads to a sensible conjecture for the height of the tachyon potential.

10.2 A Conjecture

String theory on the $\mathbf{C}/\mathbf{Z}_N$ orbifold background was first considered in [110, 111, 114] to model the physics of horizons in Euclidean space. Geometrically, $\mathbf{C}/\mathbf{Z}_N$ is a cone with deficit angle $2\pi(1-\frac{1}{N})$. The tip of the cone is a fixed point of the $\mathbf{Z}_N$ orbifold symmetry and there are tachyons in the twisted sectors that are localized at the tip signifying an instability of the background.

A physical interpretation of these tachyons was provided by Adams, Polchinski, and Silverstein [112]. They argued that giving expectation values to the tachyon fields would relax the cone to flat space. The most convincing evidence for this claim comes from the geometry seen by a D-brane probe in the sub-stringy regime. In the probe theory, one can identify operators with the right quantum numbers under the quantum $\hat{\mathbf{Z}}_N$ symmetry of the orbifold[1] that correspond to turning on tachyonic vevs. By selectively turning on specific tachyons, the quiver theory of the probe can be 'annealed' to successively go from the $\mathbf{Z}_N$ orbifold to a lower $\mathbf{Z}_M$ orbifold with $M < N$ all the way to flat space. The deficit angle seen by the probe in this case changes appropriately from $2\pi(1 - \frac{1}{N})$ to $2\pi(1 - \frac{1}{M})$.

Giving expectation value to the tachyon field in spacetime corresponds to turning on a relevant operator on the string worldsheet. Thus the condensation of tachyons to different CFT backgrounds is closely related to the renormalization group flows on the worldsheet between different fixed points upon turning on various relevant operators. An elegant description of the RG flows is provided using the gauged nonlinear sigma model [115] and mirror symmetry [116]. The worldsheet dynamics also supports the expectation that the cone will relax finally to flat space. Various aspects of localized tachyons and related systems have been analyzed in [117–136]. For a recent review see [137].

These results are consistent with the assumption that in the field space of tachyons there is a potential $V(\mathbf{T})$ where we collectively denote all tachyons by $\mathbf{T}$. The $\mathbf{Z}_N$ orbifold sits at the top of this potential, the various $\mathbf{Z}_M$ orbifolds with $M < N$ are the other critical points of this tachyonic potential, and flat space is at the bottom of this potential. Such a potential can also explain why a conformal field theory exists only for special values of deficit angles. We will be concerned here with the static properties such as the effective height of the potential and not so much with dynamical details of the process of condensation.

It may seem difficult to evaluate the change in the classical action in going from the $\mathbf{Z}_N$ orbifold to the $\mathbf{Z}_M$ orbifold but we are helped by the fact that the orbifold is exactly conformal. Hence the equations of motion for the dilaton and the graviton are satisfied exactly for both backgrounds. To calculate the change in the action, let us

[1] The quantum $\hat{\mathbf{Z}}_N$ symmetry is simply the selection rule that twists are conserved modulo N. The twist field in the k-th twisted sector has charge k under this symmetry.

consider the string effective action to leading order in α'

$$S = \frac{1}{2\kappa^2}\int_M \sqrt{-g}\, e^{-2\phi}\left[R + 4(\nabla_M\phi\nabla^M\phi) - 2\kappa^2\delta^2(x)(\nabla_\mu\mathbf{T}\nabla^\mu\bar{\mathbf{T}} + V(\mathbf{T}))\right] + \frac{1}{\kappa^2}\int_{\partial M}\sqrt{-g}\, e^{-2\phi} K\,, \qquad (10.2.1)$$

where K is the extrinsic curvature, $\frac{\kappa^2}{8\pi}$ is Newton's constant G, and $V(\mathbf{T})$ denotes the tachyon potential localized at the defect. Here $M = 0,\ldots,9$ run over all spacetime directions, the cone is along the 8, 9 directions, and $\mu = 0,\ldots,7$ are the directions longitudinal to the eight dimensional defect localized at the tip of the cone. The extrinsic curvature term is as usual necessary to ensure that the effective action reproduces the string equations of motion for variations $\delta\phi$ and δg that vanish at the boundary.

The action is very similar to the one for a cosmic string in four dimensions. For a cosmic string in four dimensions or equivalently for a 7-brane in ten dimensions, the deficit angle δ in the transverse two dimensions is given by $\delta = 8\pi G\rho \equiv \kappa^2\rho$ where ρ is the tension of the 7-brane. We are assuming that when the tachyon field $\mathbf{T}$ has an expectation value $\mathbf{T}_N$, its potential supplies an 7-brane source term for gravity such that the total spacetime is $\mathbf{M}_8 \times \mathbf{C}/\mathbf{Z}_N$ where $\mathbf{M}_8$ is the flat eight-dimensional Minkowski space. Einstein equations then imply $R = 2\kappa^2\delta^2(x)V(\mathbf{T})$ and a conical curvature singularity at $x = 0$. Because of this equality, there is no source term for the dilaton and as a result the dilaton equations are satisfied with a constant dilaton. We see that the bulk contribution to the action is precisely zero for the solution. The boundary has topology $\mathbf{R}^8 \times \mathbf{S}^1$. For a cone, the circle $\mathbf{S}^1$ has radius r but the angular variable will go from 0 to $\frac{2\pi}{N}$. The extrinsic curvature for the circle equals $1/r$ and thus the contribution to the action from the boundary term equals $\frac{2\pi A}{N\kappa^2}$. There is an arbitrary additive constant in the definition of the action that is determined by demanding that flat space should have vanishing action. In any case, we are concerned with only the differences and we conclude that in going from $\mathbf{C}/\mathbf{Z}_N$ to flat space the total change in action per unit area must precisely equal $\frac{2\pi}{\kappa^2}(1 - \frac{1}{N})$.

In the full string theory, we should worry about the higher order α' corrections to the effective action. These corrections are dependent on field redefinitions or equivalently on the renormalization scheme of the world-sheet sigma model. However, the total contribution of these corrections to the bulk action must nevertheless vanish for the orbifold because we know that the equations of motion of the dilaton are satisfied with a constant dilaton which implies no source terms for the dilaton in the bulk. Thus, the entire contribution to the action comes from the boundary term even when the α' corrections are taken into account and we can reliably calculate it in a scheme independent way using the conical geometry of the exact solution at the boundary.

One can convert this prediction for the change in action into a conjecture for the height of the tachyon potential. We expect that the tachyon potential should be identified with the source of energy that is creating the curvature singularity. Let us see how it works in some detail. Note that a cone has a topology of a disk and its Euler character χ equals one. Using the Gauss-Bonnet theorem, we then conclude $\chi = \frac{1}{4\pi}\int_{\mathbf{C}/\mathbf{Z}_N} R + \frac{1}{2\pi}\int_{\mathbf{S}^1} K = 1$. This implies that $R = 4\pi(1-\frac{1}{N})\delta^2(x)$ and we arrive at our conjecture that

$$V(\mathbf{T}_N) = \frac{2\pi}{\kappa^2}(1-\frac{1}{N}). \tag{10.2.2}$$

We are thus led to a plausible picture rather analogous to the open-string tachyons in which the tachyon potential $V(\mathbf{T})$ supplies the source of energy required to create a defect and flat space is the stable supersymmetric ground state. The landscape of the tachyon fields in the closed string case is, however, more intricate. There are several tachyonic modes and many critical points corresponding to cones with different deficit angles and thus a richer set of predictions to test.

10.3 A Model and a Strategy for Computing Off-Shell Interactions

The potential for open-string tachyons been analyzed using a number of different approaches. It has been possible to test Sen's conjecture within open string field theory in a number of different formalisms [138–141] both for the bosonic and the superstring. For a recent review and a more complete list of references see [142]. Some properties of the decay process have also been analyzed exactly in boundary conformal field theory [143] and in certain toy models exactly even nonperturbatively [144–146].

It would be interesting to similarly develop methods within closed-string theory to test the conjecture above for the potential of localized tachyons. For the bosonic string, the string field theory does not have the simple cubic form as in Witten's open string field theory [147]. Nevertheless, a well-developed formalism with non-polynomial interactions is available [148]. Okawa and Zwiebach have recently applied this formalism successfully in the level-truncation approximation [149] and have found more than 70% agreement with the conjectured answer which is quite encouraging. We will be interested here in the localized tachyon in the superstring. For superstrings, there is a string field theory formalism available only for the free theory [150] but not yet for the interacting theory so we need to approach the problem differently.

Corresponding to (10.2.2), there is a natural object in the worldsheet RG flows that can be identified with the tachyon potential [151]. For relevant flows, however, the relation between worldsheet quantities and spacetime physics is somewhat indirect given the fact that away from the conformal point the Liouville mode of the worldsheet no longer decouples. It is desirable to see, to what extent, (10.2.2) can be verified directly in spacetime.

To sidestep the use of string field theory, we work instead in the limit of large N and consider the decay process that takes $\mathbf{C}/\mathbf{Z}_N$ to $\mathbf{C}/\mathbf{Z}_k$ with $k = N - j$, for some small even integer $j = 2, 4, \ldots$. We assume that the tachyonic field T_k that connects these two critical points has a well-defined charge k under the quantum $\hat{\mathbf{Z}}_N$ symmetry with its mass given by $m_k^2 = -\frac{2(N-k)}{\alpha' N}$. This assumption is motivated from the worldsheet mirror description of this process [116] where the relevant operator that is turned on has a well-defined charge under the quantum symmetry. To justify this assumption further we will check that there are no cubic couplings between this tachyon and other nearly massless tachyons. Therefore, giving expectation value to this particular tachyon does not create a tadpole for other tachyons.

Because this tachyon is nearly massless we can consider its effective dynamics much below the string mass scale by integrating out the massive string modes and are justified in ignoring possible string scale corrections to the effective action. The simplest way to model the condensation process is to imagine an effective potential

$$\frac{2\pi}{\kappa^2}(1 - \frac{1}{N}) - (\frac{2}{\alpha'})\frac{(N-k)}{N}|T_k|^2 + \frac{\lambda_k}{4}|T_k|^4. \tag{10.3.1}$$

The potential has two extrema. At $T_k = 0$ it has a maximum and its value at the maximum is given by (10.2.2) which supplies the energy source required at the tip of the cone $\mathbf{C}/\mathbf{Z}_N$. There is a minimum at $|T_k|^2 = \frac{4(N-k)}{\alpha' \lambda_k N}$ and the energy at this minimum is lowered. We would like to identify this minimum with the cone $\mathbf{C}/\mathbf{Z}_k$ which implies the prediction

$$\frac{1}{\lambda_k}(\frac{2}{\alpha'})^2(\frac{N-k}{N})^2 = \frac{2\pi}{\kappa^2}(\frac{1}{k} - \frac{1}{N}), \tag{10.3.2}$$

so that the energy at the minimum is exactly what is needed to create the smaller deficit angle of the cone $\mathbf{C}/\mathbf{Z}_k$. We are implicitly working at large N because we have assumed that the tachyon is nearly massless and it is meaningful to talk about its potential ignoring the α' corrections. In the large N limit, the final prediction for the quartic term becomes

$$\lambda_k = \frac{\kappa^2}{2\pi}(\frac{2}{\alpha'})^2(N-k). \tag{10.3.3}$$

It would be natural to set $\alpha' = 2$ here as in most closed string calculations but we prefer to maintain α' throughout to keep track of dimensions and to allow for easy comparison with other conventions.

For the consistency for this picture it is essential that the tachyon T_k does not source any other nearly massless fields apart from the dilaton. We have already explained that the dilaton tadpole vanishes because the bulk action is zero for the cone. However, if there are tadpoles of any of the very large number of nearly massless tachyons in the system, it would ruin the simple picture above. Quantum $\hat{\mathbf{Z}}_N$ symmetry surely allows terms like $T_k T_k T_{N-2k}$ because the charge needs to be conserved only modulo N. If such a term is present then the tachyon T_{N-2k} will be sourced as soon as T_k acquires an expectation value and its equations of motion will also have to satisfied. We will then be forced to take into account the cubic and quartic interactions of all such fields. Fortunately, as we show in section 12.4, even though the cubic couplings of this type are allowed *a priori*, they actually vanish because of H-charge conservation. We can thus restrict our attention consistently to a single tachyon up to quartic order.

In what follows, we proceed with this simple ansatz. Note that $N-k=j$ is of order one and thus the required quartic term that is of order one. To extract the contact quartic term, we first need to calculate the four point tachyon scattering amplitude and subtract from it the massless exchanges.

There is a subtlety in this procedure that is worth pointing out. We are interested in the one-particle irreducible quartic interaction. To obtain it from the four particle scattering amplitude, we should subtract all one-particle reducible diagrams. Now, in string theory, an infinite number of particles of string-scale mass are exchanged along with the massless fields of supergravity and it would be impractical if we have to subtract all such exchanges. Note however, that the mass of the tachyon of interest is inversely proportional to N and there is a clear separation of energy scales. We are interested in the effective field theory at these much lower energies that are down by a factor of $1/N$ compared to the string scale. Therefore, massive string exchanges are to be integrated out. For tree-level diagrams, integrating out a massive field simply means that we *keep* all one-particle reducible diagrams in which the massive field is exchanged. This generates an effective quartic interaction in much the same way the four-fermi interaction is generated by integrating out the massive vector boson. For this reason, we do not need to subtract the exchanges of massive string modes.

The procedure would then be to compute the four-point amplitude, subtract from it all massless or nearly-massless exchanges, and then take the string scale to infinity and N to infinity keeping fixed the tachyon mass and the external momenta to focus on the energy scale of interest. To put it differently, if we subtract only the massless exchanges, we are automatically solving the equations of motion the massive fields [152–155]. This observation explains why the full formalism of string field theory is not needed in our case as it would be for a string scale tachyon and we can proceed consistently within effective field theory.

An analogous large-N approximation was used by Gava, Narain, and Sarmadi [156] to analyze the off-shell potential of an open string tachyon that arises in the D2-D0 system. This tachyon signals the instability of the system towards forming a lower energy bound state in which the D0-brane is dissolved into the D2-brane. These authors introduce a parameter N by considering a system of a single D2 brane with N D0-branes already dissolved in it and then introduce an additional D0-brane. The relevant tachyon is then nearly massless when N is large and one can consistently analyze the system in effective field theory in much the same way as we wish to do here. For the open string tachyon also, the mass-squared is inversely proportional to N and the quartic term turns out to be of order one. The potential then has a lower energy minimum corresponding precisely to the lowered energy of the additional D0-brane dissolved into the D2 brane.

One would hope that a similar story works for the nearly massless closed-string tachyons but there is no *a priori* way to determine the value of the quartic term without actually computing it. Unfortunately, our computation shows that the quartic term in this case is not of order one but much smaller, of order $1/N^3$. We discuss the results and implications in some detail in section §14.

It is now clear that for our purpose we require the S-matrix element for the scattering of four tachyons and the three-point couplings of these tachyons to massless or nearly massless fields. We in turn need the four-point and three-point correlation functions involving the twist fields of the bosonic and fermionic fields. The fermionic twist fields have a free field representation and their correlation function are straightforward to compute. The computations involving the bosonic twist fields are fairly involved and require the full machinery of orbifold CFT. For this reason in the next section we focus only on the CFT of a single complex twisted boson and determine the required correlators using CFT techniques and factorization.

Bosonic CFT on $\mathbf{C}/\mathbf{Z}_N$

The main thrust of this section will be the computation of various three-point functions involving the bosonic twist fields. These correlation functions enter into the cubic interactions of the tachyon with the untwisted massless fields carrying polarization and momenta along the cone directions. The orbifold CFT is fairly nontrivial and the correlations cannot be computed using free field theory. Fortunately, it turns out that all the three-point functions required for our purpose can be extracted from factorization of the four-twist correlation function which is already known in the literature. This fact considerably simplifies life.

Our starting point will be the four-twist correlation function which has been computed in [157–159]. We review some basic facts about the CFT of a single complex boson in the twisted and the untwisted sector in sections §11.1 and §11.2 and the relevant aspects of the four-twist correlation function in §11.3. We then calculate the various three-point functions in §11.4 up to a four-fold discrete ambiguity using factorization and symmetry arguments. The discrete ambiguity will be fixed later by demanding BRST invariance of the full string vertex.

11.1 Untwisted Sector

We begin with a review of the Hilbert space of the untwisted sector a complex scalar X taking values in $\mathbf{C}/\mathbf{Z}_N$. The purpose of this section is to keep track of factors of N and to collect some formulas on how the states in the oscillator basis split into primaries and descendants of the conformal algebra. This will be important later for factorization using conformal blocks.

11.1.1 States and Vertex Operators

States in the untwisted sector of the CFT on $\mathbf{C}/\mathbf{Z}_N$ are constructed by projecting the Hilbert space $\mathcal{H}_\mathbf{C}$ of CFT on the complex plane onto $\mathbf{Z}_N$ invariant states. The $\mathbf{Z}_N$ generator R satisfying $R^N = 1$ acts on $\mathcal{H}_\mathbf{C}$ as a unitary operator; $R^\dagger = R^{-1}$. From this one constructs the orthogonal projection operator

$$P = 1/N \sum_{k=0}^{N-1} R^k$$

satisfying $P^2 = P$; $P^\dagger = P$. The $\mathbf{C}/\mathbf{Z}_N$ Hilbert space is then $\mathcal{H}_{\mathbf{C}/\mathbf{Z}_N} = P\mathcal{H}_\mathbf{C}$. Defining complexified momenta

$$p = \frac{1}{\sqrt{2}}(p_8 - ip_9) \qquad \bar{p} = \frac{1}{\sqrt{2}}(p_8 + ip_9),$$

we start from momentum states $|p, \bar{p}\rangle$ in $\mathcal{H}_\mathbf{C}$ normalized as

$$\langle p', \bar{p}'|p, \bar{p}\rangle = (2\pi)^2 \delta(p - p')\delta(\bar{p} - \bar{p}').$$

The states

$$|p, \bar{p}\rangle_N \equiv P|p, \bar{p}\rangle = 1/N \sum_{k=0}^{N-1} |\eta^k p, \bar{\eta}^k \bar{p}\rangle$$

with $\eta = e^{2\pi i/N}$ form a continuous basis on $\mathbf{C}/\mathbf{Z}_N$ with normalization

$$_N\langle p', \bar{p}'|p, \bar{p}\rangle_N = (2\pi)^2 \delta_N(\vec{p}, \vec{p}') \tag{11.1.1}$$

where we have defined

$$\delta_N(\vec{p}, \vec{p}') \equiv 1/N \sum_{k=0}^{N-1} \delta(p - \eta^k p')\delta(\bar{p} - \bar{\eta}^k \bar{p}').$$

The completeness relation on $\mathcal{H}_{\mathbf{C}/\mathbf{Z_N}}$ reads

$$\mathbf{1} = \int_\mathbf{C} \frac{dp d\bar{p}}{(2\pi)^2} |p, \bar{p}\rangle_N \; {}_N\langle p, \bar{p}|.$$

For an arbitrary vertex operator $\mathcal{O}$, we denote its projection onto the $\mathbf{Z}_N$ invariant subspace by $[\mathcal{O}]_N$ defined by

$$[\mathcal{O}]_N \equiv 1/N \sum_{k=0}^{N-1} R^k \mathcal{O} R^{-k}. \tag{11.1.2}$$

The vertex operators corresponding to the states $|p,\bar{p}\rangle_N$ are $[e^{i(pX+\bar{p}\bar{X})}]_N$. Their BPZ inner product is related to the Hermitian inner product (11.1.1) on the Hilbert space by an overall normalization constant A and a sign change on one of the momenta

$$\langle [e^{i(p'X+\bar{p}'\bar{X})}]'_N(\infty)[e^{i(pX+\bar{p}\bar{X})}]_N(0)\rangle = A(2\pi)^2\delta^2_N(\vec{p},-\vec{p}') \tag{11.1.3}$$

where the prime means $\mathcal{O}'(\infty) = \lim_{z\to\infty} z^{2h}\bar{z}^{2\tilde{h}}\mathcal{O}(z)$ [160]. By construction, the overall normalization A should be the same as for the CFT on $\mathbf{C}$. We will later explicitly check this from unitarity.

The full Hilbert space in the untwisted sector is built up by acting on the momentum eigenstates with creation operators $\alpha_{-\{m\}},\ \bar{\alpha}_{-\{m\}},\ \tilde{\alpha}_{-\{m\}},\ \bar{\tilde{\alpha}}_{-\{m\}}$ and then taking the $\mathbf{Z}_N$ invariant combinations. A general state can be written as

$$\mathcal{O}^{\{\bar{m}\}\{\bar{\tilde{m}}\}}_{\{m\}\{\tilde{m}\}}(p,\bar{p}) = [\prod_i(\alpha_{-i})^{m_i}\prod_j(\bar{\alpha}_{-j})^{\bar{m}_j}\prod_k(\tilde{\alpha}_{-k})^{m_k}\prod_l(\bar{\tilde{\alpha}}_{-l})^{m_l}\cdot e^{i(pX+\bar{p}\bar{X})}]_N.$$

The *level* of a state is defined as the pair (M,N) with $M = \sum(m_i+\bar{m}_i)$, $N = \sum(\tilde{m}_i+\bar{\tilde{m}}_i)$. Of particular interest to us are the lowest level operators which are part of the vertex operators for massless string states:

$$\begin{aligned}
&\text{level } (0,0): && \mathcal{O}^{00}_{00}\\
&\text{level } (1,0): && \mathcal{O}^{00}_{10}, \mathcal{O}^{10}_{00}\\
&\text{level } (0,1): && \mathcal{O}^{00}_{01}, \mathcal{O}^{01}_{00}\\
&\text{level } (1,1): && \mathcal{O}^{00}_{11},\ \mathcal{O}^{11}_{00},\ \mathcal{O}^{01}_{10}\ , \mathcal{O}^{10}_{01}.
\end{aligned} \tag{11.1.4}$$

Their explicit definition is

$$\begin{aligned}
\mathcal{O}^{00}_{00} &= [e^{i(pX+\bar{p}\bar{X})}]_N\\
\mathcal{O}^{00}_{10} &= [\alpha_{-1}\cdot e^{i(pX+\bar{p}\bar{X})}]_N,\ \ \mathcal{O}^{00}_{01} = [\tilde{\alpha}_{-1}\cdot e^{i(pX+\bar{p}\bar{X})}]_N\\
\mathcal{O}^{10}_{00} &= [\bar{\alpha}_{-1}\cdot e^{i(pX+\bar{p}\bar{X})}]_N,\ \ \mathcal{O}^{01}_{00} = [\bar{\tilde{\alpha}}_{-1}\cdot e^{i(pX+\bar{p}\bar{X})}]_N\\
\mathcal{O}^{00}_{11} &= [\alpha_{-1}\tilde{\alpha}_{-1}\cdot e^{i(pX+\bar{p}\bar{X})}]_N,\ \ \mathcal{O}^{11}_{00} = [\bar{\alpha}_{-1}\bar{\tilde{\alpha}}_{-1}\cdot e^{i(pX+\bar{p}\bar{X})}]_N\\
\mathcal{O}^{01}_{10} &= [\alpha_{-1}\bar{\tilde{\alpha}}_{-1}\cdot e^{i(pX+\bar{p}\bar{X})}]_N,\ \ \mathcal{O}^{10}_{01} = [\bar{\alpha}_{-1}\tilde{\alpha}_{-1}\cdot e^{i(pX+\bar{p}\bar{X})}]_N
\end{aligned}$$

For example, the vertex operator for $\mathcal{O}^{00}_{01}$ is given by

$$\mathcal{O}^{00}_{01}(p,\bar{p}) = i\sqrt{\frac{2}{\alpha'}}[\bar{\partial}Xe^{i(pX+\bar{p}\bar{X})}]_N.$$

11.1.2 Primaries and descendants

For computing cubic vertices, We need to write the operators (11.1.4) in terms of primaries and descendants of the Virasoro algebra. For a primary field at level (M, N) we will use the notation $\mathcal{V}^{(M,N)}(p, \bar{p})$[1].

The state at level $(0, 0)$ is obviously a primary,

$$\mathcal{V}^{(0,0)} = \mathcal{O}^{00}_{00} = [e^{i(pX+\bar{p}\bar{X})}]_N \tag{11.1.5}$$

At level $(1, 0)$, there is a primary $\mathcal{V}^{(1,0)}$ and a descendant $L_{-1} \cdot \mathcal{V}_{(0,0)}$:

$$\begin{aligned} \mathcal{V}^{(1,0)} &= \frac{1}{\sqrt{2p\bar{p}}} \left(\bar{p}\mathcal{O}^{00}_{10} - p\mathcal{O}^{10}_{00}\right) \\ L_{-1} \cdot \mathcal{V}^{(0,0)} &= \sqrt{\frac{\alpha'}{2}} \left(\bar{p}\mathcal{O}^{00}_{10} + p\mathcal{O}^{10}_{00}\right). \end{aligned} \tag{11.1.6}$$

where primary fields are delta-function normalized under the BPZ inner product as in (11.1.3). Similarly, at level $(0, 1)$ we have:

$$\begin{aligned} \mathcal{V}^{(0,1)} &= \frac{1}{\sqrt{2p\bar{p}}} \left(p\mathcal{O}^{00}_{01} - \bar{p}\mathcal{O}^{01}_{00}\right) \\ \tilde{L}_{-1} \cdot \mathcal{V}^{(0,0)} &= \sqrt{\frac{\alpha'}{2}} \left(p\mathcal{O}^{00}_{01} + \bar{p}\mathcal{O}^{01}_{00}\right). \end{aligned} \tag{11.1.7}$$

At level $(1, 1)$, we find a primary and three descendants:

$$\begin{aligned} \mathcal{V}^{(1,1)} &= \frac{1}{2p\bar{p}} \left(p^2\mathcal{O}^{00}_{11} - p\bar{p}\mathcal{O}^{01}_{10} - p\bar{p}\mathcal{O}^{10}_{01} + \bar{p}^2\mathcal{O}^{11}_{00}\right) \\ \tilde{L}_{-1}L_{-1} \cdot \mathcal{V}^{(0,0)} &= \frac{\alpha'}{2} \left(p^2\mathcal{O}^{00}_{11} + p\bar{p}\mathcal{O}^{01}_{10} + p\bar{p}\mathcal{O}^{10}_{01} + \bar{p}^2\mathcal{O}^{11}_{00}\right) \\ \tilde{L}_{-1} \cdot \mathcal{V}^{(1,0)} &= \sqrt{\frac{\alpha'}{2p\bar{p}}} \left(p^2\mathcal{O}^{00}_{11} + p\bar{p}\mathcal{O}^{01}_{10} - p\bar{p}\mathcal{O}^{10}_{01} - \bar{p}^2\mathcal{O}^{11}_{00}\right) \\ L_{-1} \cdot \mathcal{V}^{(0,1)} &= \sqrt{\frac{\alpha'}{2p\bar{p}}} \left(p^2\mathcal{O}^{00}_{11} - p\bar{p}\mathcal{O}^{01}_{10} + p\bar{p}\mathcal{O}^{10}_{01} - \bar{p}^2\mathcal{O}^{11}_{00}\right) \end{aligned} \tag{11.1.8}$$

11.2 Twisted Sector

Twisted sector states are created by the insertion of *twist fields*. The bosonic twist field that creates a state in the k-th twisted sector is denoted by σ_k. The σ_k are primary

[1] For the levels that we need to consider, there is only one primary field at each level so for our purposes there is no ambiguity in this notation.

fields of weight $h = \tilde{h} = \frac{1}{2}k/N(1 - k/N)$. Their OPE's are given by

$$\begin{aligned} \partial X(z)\sigma_k(0) &\sim z^{-1+k/N}\tau_k(0) + \dots \\ \partial \bar{X}(z)\sigma_k(0) &\sim z^{-k/N}\tau'_k(0) + \dots \end{aligned} \tag{11.2.1}$$

where τ_k, τ'_k are *excited twist fields*. In the presence of a twist field, the mode numbers of ∂X, $\partial \bar{X}$ get shifted:

$$\begin{aligned} \partial X &= -i\sqrt{\frac{\alpha'}{2}}\sum_m \alpha_{m-k/N} z^{-m-1+k/N} \\ \partial \bar{X} &= -i\sqrt{\frac{\alpha'}{2}}\sum_m \bar{\alpha}_{m+k/N} z^{-m-1-k/N}. \end{aligned} \tag{11.2.2}$$

The state $|\sigma_k\rangle \equiv \sigma_k(0)|0\rangle$ is annihilated by all positive frequency modes. The commutation relations are

$$[\alpha_{n-k/N}, \bar{\alpha}_{m+k/N}] = (m + k/N)\delta_{m,-n}.$$

We take the BPZ inner product between twist fields to be normalized as

$$\langle \sigma'_{N-k}(\infty)\sigma_k(0)\rangle = A \tag{11.2.3}$$

where A is the same constant that appears in (11.1.3). This is just a convenient choice – a different normalization of the twist fields can be absorbed in the proportionality constant multiplying the vertex operators for strings in the twisted sectors. The latter is ultimately determined by unitarity as we discuss in §12.

11.3 Four-Twist Correlation Function

The four-twist amplitude is given by [157–159]

$$Z_4(z,\bar{z}) \equiv \langle \sigma'_{N-k}(\infty)\sigma_k(1)\sigma_{N-k}(z,\bar{z})\sigma_k(0)\rangle = AB|z(1-z)|^{-2\frac{k}{N}(1-\frac{k}{N})} I(z,\bar{z})^{-1} \tag{11.3.1}$$

where

$$I(z,\bar{z}) = F(z)\bar{F}(1-\bar{z}) + \bar{F}(\bar{z})F(1-z),$$

$$F(z) = {}_2F_1(k/N, 1-k/N, 1, z).$$

The overall normalization A is expressible as a functional determinant and is common to all amplitudes of the X, $\bar{X}$ CFT on the sphere, while B is a numerical factor which we will relate, through factorization, to the normalization of the two-twist correlator. We will frequently need the asymptotics for the hypergeometric function F:

$$\begin{aligned}
F(z) &\sim 1, \qquad z \to 0, \\
F(1-z) &\sim -\frac{1}{\pi}\sin\frac{\pi k}{N}\ln\frac{z}{\delta}, \qquad z \to 0, \\
F(z) &\sim e^{\pi i k/N}\frac{\Gamma(1-2k/N)}{\Gamma^2(1-k/N)}z^{-k/N} - e^{-\pi i k/N}\frac{\Gamma(2k/N-1)}{\Gamma^2(k/N)}z^{-(1-k/N)}, \ z \to \infty, \\
F(1-z) &\sim \frac{\Gamma(1-2k/N)}{\Gamma^2(1-k/N)}z^{-k/N} + \frac{\Gamma(2k/N-1)}{\Gamma^2(k/N)}z^{-(1-k/N)}, \ z \to \infty,
\end{aligned} \tag{11.3.2}$$

where δ is defined by

$$\ln\delta = 2\psi(1) - \psi(1-k/N) - \psi(k/N) \tag{11.3.3}$$

with $\psi(z) \equiv \frac{d}{dz}\ln\Gamma(z)$. For $1-k/N$ small, $\ln\delta$ behaves as

$$\ln\delta = \frac{N}{N-k} + \mathcal{O}(\frac{N-k}{N})^2.$$

11.4 Three-point Correlation Functions

We are now ready to calculate various three-point functions of two twist fields and an untwisted field by factorizing the four-twist amplitude[2]. In a general CFT, the four-point amplitude can be expanded as

$$Z_4(z,\bar z) = \sum_p (C^p_{-+})^2\mathcal{F}(p|z)\bar{\mathcal{F}}(p|\bar z) \tag{11.4.1}$$

where the sum runs over primary fields, C^p_{-+} are coefficients in the $\sigma_{N-k}\sigma_k$ OPE, and $\mathcal{F}(p|z)$, $\bar{\mathcal{F}}(p|\bar z)$ are the conformal blocks. The conformal blocks in turn can be expanded for small z as

$$\begin{aligned}
\mathcal{F}(p|z) &= z^{h_p - k/N(1-k/N)}(1 + \frac{1}{2}h_p z + \mathcal{O}(z^2)) \\
\bar{\mathcal{F}}(p|\bar z) &= \bar z^{h_p - k/N(1-k/N)}(1 + \frac{1}{2}\tilde h_p \bar z + \mathcal{O}(\bar z^2)).
\end{aligned} \tag{11.4.2}$$

For a discussion and derivation of this formula see for example [161].

In our case, the sum over primaries in (11.4.1) is in fact a discrete sum over primaries at different levels as well as an integral over 'momenta' $(p,\bar p)$. Therefore, to order $|z|^2$

[2]Some of the correlation functions have been computed independently in [149].

we have

$$
\begin{aligned}
Z_4(z,\bar{z}) = \int_{\mathbf{C}} \frac{dp d\bar{p}}{(2\pi)^2 A} |z|^{\alpha' p\bar{p} - 2k/N(1-k/N)} \Big[& (C_{-+}^{(0,0)}(p,\bar{p}))^2 \\
& + z \left(\frac{\alpha'}{4} p\bar{p} (C_{-+}^{(0,0)}(p,\bar{p}))^2 + (C_{-+}^{(1,0)}(p,\bar{p}))^2 \right) \\
& + \bar{z} \left(\frac{\alpha'}{4} p\bar{p} (C_{-+}^{(0,0)}(p,\bar{p}))^2 + (C_{-+}^{(0,1)}(p,\bar{p}))^2 \right) \\
& + |z|^2 \Big((\frac{\alpha' p\bar{p}}{4})^2 (C_{-+}^{(0,0)}(p,\bar{p}))^2 + (C_{-+}^{(1,1)}(p,\bar{p}))^2 \\
& + \frac{\alpha' p\bar{p}}{4} ((C_{-+}^{(1,0)}(p,\bar{p}))^2 + (C_{-+}^{(0,1)}(p,\bar{p}))^2) \Big) + \ldots \Big]
\end{aligned}
\qquad (11.4.3)
$$

The coefficients $C_{-+}^{(M,N)}$ are the three-point functions

$$
C_{-+}^{(M,N)}(p,\bar{p}) = \langle \sigma'_{N-k}(\infty) \sigma_k(1) \mathcal{V}^{(M,N)}(p,\bar{p})(0) \rangle
$$

that we are interested in.

Factorization implies that the above expression for Z_4 should equal the expansion of (11.3.1) for small z which is given by

$$
\begin{aligned}
Z_4(z,\bar{z}) &= \frac{AB\pi}{4\sin(\frac{\pi k}{N})} |z|^{-2k/N(1-k/N)} \left(-\frac{1}{\log\frac{|z|}{\delta}} + \frac{a(z+\bar{z})}{(\log\frac{|z|}{\delta})^2} - \frac{2a^2|z|^2}{(\log\frac{|z|}{\delta})^3} + \ldots \right) \\
&= \frac{\alpha' AB}{2\sin\pi k/N} \int_{\mathbf{C}} dp d\bar{p} |z|^{\alpha' p\bar{p} - 2k/N(1-k/N)} \delta^{-\alpha' p\bar{p}} \\
&\quad \left[1 + a\alpha' p\bar{p}(z+\bar{z}) + (a\alpha' p\bar{p})^2 |z|^2 + \ldots \right]
\end{aligned}
\qquad (11.4.4)
$$

where

$$
a = \frac{1}{2} \left((k/N)^2 + (1-k/N)^2 \right). \qquad (11.4.5)
$$

In the second line of (11.4.4), we have used the identity

$$
\left(\log\frac{|z|}{\delta} \right)^{-(n+1)} = \frac{(-\alpha')^{n+1}}{2\pi n!} \int_{\mathbf{C}} dp d\bar{p} (p\bar{p})^n \left(\frac{|z|}{\delta} \right)^{\alpha' p\bar{p}}.
$$

Comparing 11.4.3 and 11.4.4, we can now read off various operator product coefficients. For the first coefficient we find

$$
(C_{-+}^{(0,0)}(p,\bar{p}))^2 = \frac{\pi^2 \alpha' A^2 B}{\sin\pi k/N} \delta^{-\alpha' p\bar{p}}.
$$

Using the fact that

$$
C_{-+}^{(0,0)}(0,\bar{0}) = A
$$

as in (11.1.5) and (11.2.3) we can determine the numerical constant B that appears in 11.3.1,

$$B = \frac{\sin\frac{\pi k}{N}}{\pi^2\alpha'}. \tag{11.4.6}$$

Further comparing (11.4.3) and (11.4.4) determines the higher operator product coefficients up to signs:

$$\begin{aligned} C_{-+}^{(0,0)}(p,\bar{p}) &= A\delta^{-\alpha' p\bar{p}/2} \\ C_{-+}^{(1,0)}(p,\bar{p}) = -C_{-+}^{(0,1)}(p,\bar{p}) &= \epsilon_1 A\frac{\sqrt{\alpha' p\bar{p}}}{2}(1-2k/N)\delta^{-\alpha' p\bar{p}/2} \\ C_{-+}^{(1,1)}(p,\bar{p}) &= \epsilon_2 A\frac{\alpha' p\bar{p}}{4}(1-4k/N(1-k/N))\delta^{-\alpha' p\bar{p}/2}. \end{aligned} \tag{11.4.7}$$

where $\epsilon_{1,2} = \pm 1$ are the sign ambiguities that arise from taking square roots. In the second line, we have taken the opposite sign for the coefficients as required by world-sheet parity, which takes $\mathcal{V}_{(1,0)} \to \mathcal{V}_{(0,1)}$ and $\sigma_k \to \sigma_{N-k}$. The three-point functions involving descendants are easily calculated from the ones involving primaries using the fact that L_{-1} and $\tilde{L}_{-1}$ act on vertex operators as ∂ and $\bar{\partial}$ respectively.

These results are easily transformed to the α-oscillator basis. The operators we need were denoted by $\mathcal{O}_{m\tilde{m}}^{\bar{m}\bar{\tilde{m}}}$ in §11.1. Using the formulas (11.1.5-11.1.8) to transform to the α-oscillator basis one finds the required three-point functions. Using the notation

$$D_{m\tilde{m}}^{\bar{m}\bar{\tilde{m}}}(p,\bar{p},k) \equiv \langle \sigma'_{N-k}(\infty)\sigma_k(1)\mathcal{O}_{m\tilde{m}}^{\bar{m}\bar{\tilde{m}}}(p,\bar{p})(0)\rangle$$

we find

$$\begin{aligned} D_{00}^{00} &= A\delta^{-\alpha' p\bar{p}/2} \\ D_{10}^{00} &= \sqrt{\frac{\alpha'}{2}}A\bar{p}\delta^{-\alpha' p\bar{p}/2}\left(\frac{1}{2}(1+\epsilon_1) - \epsilon_1 k/N\right) \\ D_{00}^{10} &= \sqrt{\frac{\alpha'}{2}}Ap\delta^{-\alpha' p\bar{p}/2}\left(\frac{1}{2}(1-\epsilon_1) + \epsilon_1 k/N\right) \\ D_{01}^{00} &= \sqrt{\frac{\alpha'}{2}}A\bar{p}\delta^{-\alpha' p\bar{p}/2}\left(\frac{1}{2}(1-\epsilon_1) + \epsilon_1 k/N\right) \\ D_{00}^{01} &= \sqrt{\frac{\alpha'}{2}}Ap\delta^{-\alpha' p\bar{p}/2}\left(\frac{1}{2}(1+\epsilon_1) - \epsilon_1 k/N\right) \\ D_{11}^{00} &= \frac{\alpha'}{8}A\bar{p}^2\delta^{-\alpha' p\bar{p}/2}\left(\epsilon_2(1-4k/N(1-k/N))+1\right) \\ D_{00}^{11} &= \frac{\alpha'}{8}Ap^2\delta^{-\alpha' p\bar{p}/2}\left(\epsilon_2(1-4k/N(1-k/N))+1\right) \\ D_{10}^{01} &= \frac{\alpha'}{8}Ap\bar{p}\delta^{-\alpha' p\bar{p}/2}\left(-\epsilon_2(1-4k/N(1-k/N))+1+2\epsilon_1(1-2k/N)\right) \end{aligned}$$

$$D_{01}^{10} = \frac{\alpha'}{8} Ap\bar{p}\delta^{-\alpha' p\bar{p}/2} \left(-\epsilon_2(1 - 4k/N(1 - k/N)) + 1 - 2\epsilon_1(1 - 2k/N)\right) \quad (11.4.8)$$

We later argue that the correct choice for our purpose is $\epsilon_1 = \epsilon_2 = -1$ which makes the total cubic string vertex BRST-invariant.

Note that this method to calculate the three-point function from factorization works only as long as there is only one primary field at each level. At level $(2, 0)$, for example, one finds that there are two primaries and their correlators cannot be determined from factorization only. Computation of these higher correlators is substantially more difficult but fortunately we do not require them here. We need to subtract only massless exchanges and the higher level primary fields do not enter into the massless vertex operators. Equipped with the four and three point correlation functions we are thus ready to discuss the tachyon interactions.

Strings on $\mathbf{C}/\mathbf{Z}_N$

We now consider type IIA/B strings on $\mathbf{M}^8 \times \mathbf{C}/\mathbf{Z}_N$ with $\mathbf{M}^8$ representing $7+1$ dimensional Minkowski space. After reviewing the conventions and the vertex operators, we write the four point scattering amplitude. We then write the gauge invariant cubic interaction between the tachyon and the massless fields and finally determine the one-particle irreducible effective quartic interaction.

12.1 Conventions

We use the coordinates $X^M = (X^\mu, X, \bar{X})$ where $\mu = 0, \ldots, 7$ and $X = \frac{1}{\sqrt{2}}(X^8 + iX^9)$. Similarly, the world-sheet fermions are $\psi^M = (\psi^\mu, \psi, \bar{\psi})$ with $\psi = \frac{1}{\sqrt{2}}(\psi^8 + i\psi^9)$. Our metric signature is $(- + + \ldots +)$. The orbifold $\mathbf{C}/\mathbf{Z}_N$ represents a cone with opening angle $2\pi/N$. The $\mathbf{Z}_N$ generator is given by

$$R = \exp(2\pi i \frac{N+1}{N} J_{89}).$$

where J_{89} generates rotations in the X^8, X^9 plane. We take N to be odd so that $R^N = 1$ on spacetime fermions and the bulk tachyon is projected out by GSO projection [112].

In the untwisted sector, one has to project onto $\mathbf{Z}_N$ invariant states, for which we use the notation $[\ldots]_N$:

$$[\mathcal{O}]_N \equiv 1/N \sum_{k=0}^{N-1} R^k \mathcal{O} (R^{-1})^k. \qquad (12.1.1)$$

The vertex operators in the sector twisted by R^k will contain bosonic twist fields σ_k and fermionic ones s_k. Their role is to create branch cuts in the OPEs with X, $\bar{X}$ and ψ, $\bar{\psi}$ respectively. In the presence of a twist field, mode numbers get shifted by an amount k/N.

The bosonic twisted sector has been discussed in detail in the previous section. The fermionic twist fields are denoted by s_k. Their OPEs are

$$\begin{aligned} \psi(z)s_k(0) &\sim z^{k/N}t_k + \dots \\ \bar{\psi}(z)s_k(0) &\sim z^{-k/N}t'_k + \dots \end{aligned} \tag{12.1.2}$$

The mode expansions in the NS sector are

$$\begin{aligned} \psi(z) &= \sum_{r\in\mathbf{Z}+\frac{1}{2}} \psi_{r-k/N} z^{-r-\frac{1}{2}+k/N} \\ \bar{\psi}(z) &= \sum_{r\in\mathbf{Z}+\frac{1}{2}} \bar{\psi}_{r+k/N} z^{-r-\frac{1}{2}-k/N} \end{aligned} \tag{12.1.3}$$

with commutation relations

$$\{\psi_{r-k/N}, \bar{\psi}_{s+k/N}\} = \delta_{r,-s}.$$

These twist fields have a free field representation in terms of bosonized fields H, $\tilde{H}$. The latter are defined by

$$\begin{aligned} \psi &= e^{iH}, & \bar{\psi} &= e^{-iH} \\ \tilde{\psi} &= e^{i\tilde{H}}, & \bar{\tilde{\psi}} &= e^{-i\tilde{H}} \end{aligned} \tag{12.1.4}$$

The twist fields are then represented by

$$s_k = e^{i\frac{k}{N}H}, \qquad \tilde{s}_k = e^{-i\frac{k}{N}\tilde{H}} \tag{12.1.5}$$

In this representation, the only computationally nontrivial CFT correlators are the ones involving the bosonic twist fields σ_k that were calculated in the previous section. General amplitudes are restricted by *quantum symmetry* $\hat{Z}_N$ and *charge conservation* for the $H, \tilde{H}$ fields.

12.2 Tachyon Spectrum and Vertex Operators

Let us review the GSO projection and the tachyonic spectrum [110, 112]. The vertex operator for the k-th twisted sector ground state, in the -1 picture, is

$$\sigma_k e^{i\frac{k}{N}(H-\tilde{H})} e^{-\phi-\tilde{\phi}}$$

For k odd, this ground state is not projected out by the GSO projection and the lowest lying mode is a tachyon with mass $m^2 = -2/\alpha'(1-\frac{k}{N})$. Its vertex operator is

$$T_k^{(-1,-1)}(z,\bar{z},p) = g_c' e^{ip\cdot X} c\tilde{c}\sigma_k e^{i\frac{k}{N}(H-\tilde{H})} e^{-\phi-\tilde{\phi}}(z,\bar{z}) \qquad k\ odd \tag{12.2.1}$$

The normalization constant g_c' will be determined in terms of the closed string coupling g_c from factorization and the requirement that the tachyon vertex operator represents a canonically normalized field in $7+1$ dimensions.

For k even, the ground state is projected out by GSO projection and the tachyon is an excited state $\bar{\psi}_{-\frac{1}{2}+k/N}\tilde{\bar{\psi}}_{-\frac{1}{2}+k/N}|0,p\rangle$ with vertex operator

$$T_k^{(-1,-1)}(z,\bar{z},p) = g_c' e^{ip\cdot X} c\tilde{c}\sigma_k e^{i(k/N-1)(H-\tilde{H})} e^{-\phi-\tilde{\phi}}(z,\bar{z}) \qquad k\ even.$$

Its mass-shell condition is $\frac{-\alpha' p^2}{4} = -\frac{1}{2}\frac{k}{N}$. This vertex operator is the complex conjugate of T_{N-k} and we denote it by $\bar{T}_{N-k}$. Henceforth, we take k to be odd and describe all tachyons by the vertex operators T_k, $\bar{T}_k$. The most marginal tachyon has $k = N-2$ and $m^2 = -\frac{4}{N\alpha'}$.

We will also need the tachyon vertex operators in the 0 picture

$$T_k^{(0,0)}(z,\bar{z},p) = -\frac{\alpha'}{2} g_c' e^{ip\cdot X} p\cdot\psi\ p\cdot\tilde{\psi} c\tilde{c}\sigma_k e^{i\frac{k}{N}(H-\tilde{H})}(z,\bar{z}) + \ldots$$

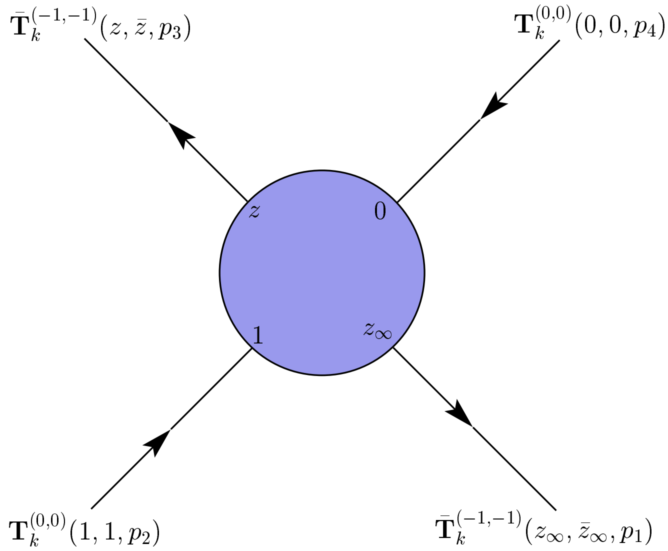
The omission stands for terms that arise from the part of the picture-changing operator involving the $\mathbf{C}/\mathbf{Z}_N$ fields. These terms have different $(H,\tilde{H})$ charges from the one displayed. In the amplitudes we consider it is possible to choose pictures such that the omitted terms do not contribute because of H-charge conservation.

12.3 Scattering Amplitude for Four Tachyons

We are now ready to write down the four-tachyon scattering amplitude. Using (11.3.1) it is given by

$$\begin{aligned} V_4 &= \int d^2z \langle \bar{T}_k^{(-1,-1)}(z_\infty,\bar{z}_\infty,p_1) T_k^{(0,0)}(1,1,p_2) \bar{T}_k^{(-1,-1)}(z,\bar{z},p_3) T_k^{(0,0)}(0,0,p_4)\rangle \\ &= \frac{g_c'^4 C}{\pi^2\alpha'} \sin(\frac{\pi k}{N}) \Delta(\alpha' p_1\cdot p_3/2)^2 \int d^2z |1-z|^{-\frac{\alpha' t}{2}-2} |z|^{-\frac{\alpha' s}{2}-2} I(z,\bar{z})^{-1}, \end{aligned} \tag{12.3.1}$$

where C represents the product of A with similar functional determinants from the X^μ, the fermions and the ghosts. The assignment of pictures and worldsheet positions



$$\bar{T}_k^{(-1,-1)}(z,\bar{z},p) = g_c'\sigma_{N-k}e^{-i\frac{k}{N}(H-\tilde{H})}c\tilde{c}e^{-\phi-\tilde{\phi}}e^{ip\cdot X(z,\bar{z})}$$

$$T_k^{(0,0)}(z,\bar{z},p) = -\frac{\alpha'}{2}g_c'\sigma_k e^{i\frac{k}{N}(H-\tilde{H})}c\tilde{c}p\cdot\psi(z,\bar{z})p\cdot\tilde{\psi}(z,\bar{z})e^{ip\cdot X(z,\bar{z})} + \ldots$$

Fig 12.1: Four-tachyon scattering amplitude.

of the vertex operators are shown in figure 1. The Mandelstam variables $s,\ t,\ u$ are defined as

$$s = -(p_1+p_2)^2,\ t = -(p_2+p_3)^2,\ u = -(p_1+p_3)^2$$

and the symbol Δ to denotes the 8-dimensional momentum-conserving delta-function

$$\Delta \equiv i(2\pi)^8\delta^8(\sum p_i).$$

All momenta are incoming and the arrows indicate the flow of quantum $\hat{\mathbf{Z}}_N$ charge.

The asymptotics of $I(z,\bar{z})$ as $z\to 0$ or $z\to 1$ imply that V_4 does not have the usual pole structure in the s and t channels. This is a consequence of the noncompactness of the cone. In the s and t channels, the exchanged states are untwisted states which have momentum along the cone directions $(X,\bar{X})$ and a continuous mass spectrum from eight-dimensional point of view. These states can contribute because there is no translation invariance in those directions and hence momentum is not conserved. As a result, the poles are replaced by softer logarithmic divergences. The leading s-channel behavior comes from integrating near $z=0$,

$$V_4^s \approx -\frac{{g_c'}^4 C}{2\pi\alpha'}\Delta(\alpha' p_1\cdot p_3/2)^2\int_0 d^2z\frac{|z|^{-\frac{\alpha' s}{2}-2}}{\log\frac{|z|}{\delta}}. \tag{12.3.2}$$

Integrating near $z = 1$ gives the leading t-channel contribution

$$V_4^t \approx -\frac{g_c'^4 C}{2\pi\alpha'}\Delta(\alpha' p_1 \cdot p_3/2)^2 \int_0 d^2z \frac{|z|^{-\frac{\alpha' t}{2}-2}}{\log\frac{|z|}{\delta}}. \tag{12.3.3}$$

In the u channel, which comes from the $z \to \infty$ region of the integral, the exchanged states are localized twisted sector states and there one gets a sum over pole terms. There are no massless (or nearly massless) poles in the u-channel; the first contribution comes from a massive exchange

$$V_4^u \approx -\frac{2g_c'^4 C}{\pi\alpha'}\tan(\frac{\pi k}{N})\Delta(\alpha' p_1 \cdot p_3/2)^2 \frac{\Gamma^4(\frac{k}{N})}{\Gamma^2(\frac{2k}{N}-1)} \frac{1}{\alpha' u/2 + 2(2-3k/N)}. \tag{12.3.4}$$

12.4 Cubic Couplings to Other Twisted States

We now show that, at least in the quartic approximation to the tachyon potential, a constant vev for one of the nearly-marginal tachyons does not generate a tadpole for any of the other nearly-massless fields in the twisted sectors. It is therefore consistent to neglect these states in the analysis to quartic order.

The couplings between twisted sector states are severely restricted by quantum $\hat{\mathbf{Z}}_N$ symmetry and H-charge conservation. Let us start with the three-point couplings. Possible tadpoles come from couplings $\langle \bar{T}_k^{(-1,-1)} \bar{T}_k^{(-1,-1)} \Phi^{(0,0)} \rangle$ where the superscript denotes the picture. Quantum symmetry and H-charge conservation imply that the zero-picture state $\Phi^{(0,0)}$ has to be proportional to $\sigma_{2k-N} e^{i2k/N(H-\tilde{H})}$. The lowest state with these quantum numbers has vertex operator

$$\Phi^{(0,0)} = e^{ip_\mu X^\mu} \sigma_{2k-N} e^{i2k/N(H-\tilde{H})} c\tilde{c}$$

which has mass-squared $m^2 = 4/\alpha'(-2 + 3k/N)$ which is of order of the string scale if k/N is close to one for example when $k = N - 2$. This is consistent with what we find in equation (12.3.4). It shows that the lowest lying exchanged state in the u-channel has precisely this mass, and there are no poles from exchanging tachyonic or nearly massless fields.

There are four-point couplings of the form $\langle \bar{T}_k^{(-1,-1)} T_k^{(0,0)} \bar{T}_k^{(-1,-1)} \Phi^{(0,0)} \rangle$ which could also source other nearly-massless tachyons. Again using quantum symmetry and H-charge conservation one sees that $\Phi^{(0,0)}$ is either equal to $T_k^{(0,0)}$ or proportional to $\sigma_k e^{i(k/N+1)(H-\tilde{H})}$. The lowest mass state with the latter quantum numbers has vertex operator

$$\Phi^{(0,0)} = e^{ip_\mu X^\mu}\sigma_k e^{i(k/N+1)(H-\tilde{H})}c\tilde{c}$$

with $m^2 = 2/\alpha'(-1+3k/N)$. This is again a massive state with string scale mass for $k/N \approx 1$.

It seems possible to generalize this argument for at least a large class of higher point functions but we restrict ourselves only up to quartic order.

12.5 Cubic Coupling to Untwisted Massless Fields

We now calculate the cubic vertex for two tachyons and one massless field from the untwisted sector. The vertex operator for a massless state with polarization tensor e_{MN} in the zero picture is

$$H^{(0,0)}(z,\bar{z},p,e) \equiv -g_c e_{MN}\frac{2}{\alpha'}\left[(i\partial X^M + \frac{\alpha'}{2}p\cdot\psi\psi^M)(i\bar{\partial}X^N + \frac{\alpha'}{2}p\cdot\tilde{\psi}\tilde{\psi}^N)c\tilde{c}e^{ip\cdot X}(z,\bar{z})\right]_N . \quad (12.5.1)$$

Gravitons are described by a symmetric, traceless polarization tensor and B-field fluctuations correspond to an antisymmetric polarization tensor. The dilaton vertex operator requires a bit more care. In the (-1) picture, it is given by [162]

$$\frac{1}{\sqrt{8}}\left[(\psi\cdot\tilde{\psi}e^{-\phi-\tilde{\phi}} - \partial\xi\tilde{\eta}e^{-2\phi} - \tilde{\partial}\tilde{\xi}\eta e^{-2\tilde{\phi}})c\tilde{c}e^{ip\cdot X}\right]_N$$

Applying the picture changing operator to this, we find that the zero-picture dilaton vertex operator is given by (12.5.1) with $e_{MN} = \frac{1}{\sqrt{8}}\eta_{MN}$ plus terms with either ϕ-charge different from zero or ghost number different from one. Such terms don't contribute to the three-point amplitude.

Using the three-point functions from (11.4.8) with $\epsilon_1 = \epsilon_2 = -1$ one finds the cubic coupling

$$\begin{aligned} V_3(p_1,p_2;p_3,e) &= \langle T_k^{(-1,-1)}(z_\infty,\bar{z}_\infty,p_1)\bar{T}_k^{(-1,-1)}(1,1,p_2)H^{(0,0)}(0,0,p_3,e)\rangle \\ &= -\frac{\alpha' {g'_c}^2 g_c C}{2}\delta^{\frac{-\alpha' p_3\bar{p}_3}{2}}\Delta \\ &\quad \left(e_{\mu\nu}p_2^\mu p_2^\nu - e_{\mu X}p_2^\mu\bar{p}_3 - e_{\bar{X}\mu}p_2^\mu p_3 + e_{\bar{X}X}p_3\bar{p}_3\right) \end{aligned} \quad (12.5.2)$$

Note that this expression is not symmetric in the polarization indices; this means that there is a coupling to the B-field as well as to the graviton and dilaton. One way to motivate the choice $\epsilon_1 = \epsilon_2 = -1$ in (11.4.8) is that it is the only one that leads to a BRST-invariant amplitude; indeed one easily checks that (12.5.2) is invariant under

$$e_{MN} \to e_{MN} + p_{3M} a_N + p_{3N} b_M \tag{12.5.3}$$

upon using $p_1^2 = p_2^2 = -m^2$, $p_3^2 = 0$. In (12.5.2), the graviton is in the transverse-traceless gauge. In order to compute the graviton exchange diagram we would like to know the correct vertex to use in the harmonic gauge $p^M e_{(MN)} - \frac{1}{2} p_N e_M^M = 0$ for the graviton. In this gauge there is more residual gauge invariance and one would typically expect to have to add terms proportional to $p^M e_{(MN)}$ and e_M^M until the vertex is invariant under the larger set of gauge transformations. In our case however we saw that (12.5.2) is already invariant under (12.5.3) without imposing $a \cdot p_3 = b \cdot p_3 = 0$. Hence (12.5.2) is the correct vertex to use in the harmonic gauge. Another proof of the validity of (12.5.2) in the harmonic gauge will be given in [163].

The final form of the cubic coupling (12.5.2) shows that the tachyon has a Gaussian form factor in its coupling to the massless untwisted fields. At large N, the width of the Gaussian in position space scales as $\sqrt{N}$. The coupling is thus not point-like but spread over a very large radius of order $\sqrt{N}$ times the string length. Because the opening angle of the cone is also getting smaller as $1/N$, the total area over which this interaction takes place is still of order one in string units.

Unitarity and Massless Exchanges

In this section, we first determine the over-all normalization from unitarity in §13.1, and then compute the massless exchange diagrams in §13.2. By subtracting these exchanges from the four tachyon scattering amplitude we can extract a possible quartic contact term. The Feynman diagram is shown in figure 2. We find that in the u-channel only massive particles of string-scale mass are exchanged consistent with (12.3.4). Hence we need to subtract only the s and t channel exchanges. As explained in §10.3, the quartic contact term on the right hand side of figure 2 is the effective quartic term at low energy and includes the exchanges of particles of string scale mass.

13.1 Determination of Normalization Constants

So far we have introduced a number of undetermined normalization constants:

C: overall normalization of the path integral on the sphere.

g_c: normalization of the graviton vertex operator or the 'closed string coupling'.

g'_c: normalization of the tachyon vertex operator.

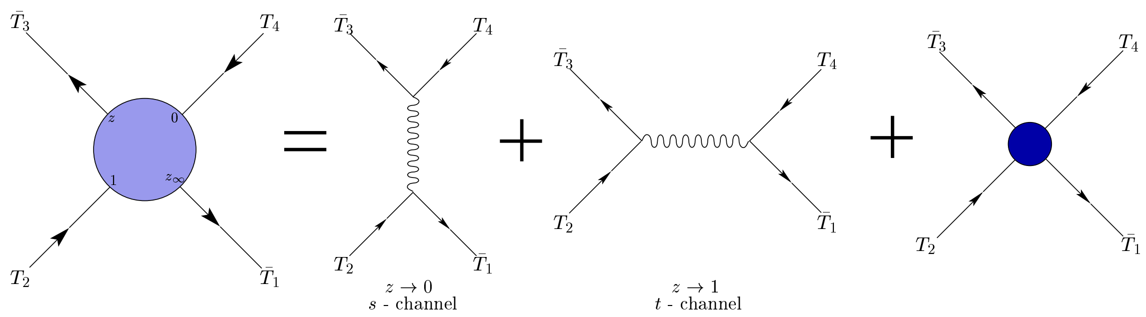


Fig 13.1: Factorization and the quartic contact interaction.

Note that the constant A that was introduced for the $X\bar{X}$ CFT in (11.2.3) is absorbed in C along with other functional determinants and the constant B is already determined in (11.4.6).

Unitarity of the S-matrix allows us to express all constants in terms of α' and g_c. The latter is in turn proportional to the gravitational constant κ. We now work out these relations keeping track of possible factors of N.

The constant C can be expressed in terms of α' and g_c by factoring the four-tachyon amplitude on graviton exchange. From (12.5.2) we calculate the contribution to the 4-point function coming from the exchange of longitudinal gravitons,

$$V_4^{exch} = -i \int \frac{d^8 p^\mu}{(2\pi)^8} \int_{\mathbf{C}} \frac{dp d\bar{p}}{(2\pi)^2} \frac{V^3_{\mu\nu}(p_1, p_2; p) V^{3\mu\nu}(p_3, p_4; p)}{p^\mu p_\mu + 2p\bar{p}} \tag{13.1.1}$$

$$= -g_c'^4 g_c^2 C^2 \frac{1}{16\pi^2} \Delta(\alpha' p_1 \cdot p_3/2)^2 \int_0 d^2 z \frac{|z|^{-\frac{\alpha' s}{2}-2}}{\log \frac{|z|}{\delta}} . \tag{13.1.2}$$

In writing the momentum space propagator in the first line, we have assumed a specific normalization for the spacetime field created by the vertex operator with normalization g_c. The normalization we used is appropriate for a canonically normalized field on the covering space $\mathbf{M}^8 \times \mathbf{C}$ which is periodic under $x \to e^{2\pi i/N} x$. Comparing (13.1.2) with (12.3.2) we find the overall normalization

$$C = \frac{8\pi}{\alpha' g_c^2} . \tag{13.1.3}$$

This is the familiar flat-space value as it should be by construction.

The normalization of the tachyon vertex operator g_c' can be determined in terms of g_c by factoring the 2 graviton-2 tachyon amplitude on the pole coming from tachyon exchange:

$$W_4 \equiv \int d^2 z \langle T_k^{(0,0)}(z_\infty, \bar{z}_\infty, p_1) H^{(-1,-1)}(1, 1, p_2, e_2) \bar{T}_k^{(0,0)}(z, \bar{z}, p_3) H^{(-1,-1)}(0, 0, p_4, e_4) \rangle \tag{13.1.4}$$

For simplicity we can take the graviton to have polarization along the longitudinal X^μ directions. For $z \to 0$, there is a pole at $s = -2/\alpha'(1 - k/N)$ coming from tachyon exchange. We can find the coefficient at the pole from the OPE of $\bar{T}_k^{(0,0)}$ with $H^{(-1,-1)}$. From the 3-point function (11.4.8) we know that

$$\sigma_{N-k}(z)[e^{i(px+\bar{p}\bar{x})}]_N(0) \sim |z|^{-\alpha' p\bar{p}} \delta^{-\alpha' p\bar{p}/2} \sigma_{N-k}$$

Using this we find the OPE

$$\bar{T}_k^{(0,0)}(z,\bar{z},p_3)H^{(-1,-1)}(0,0,p_4,e_4)$$

$$\sim \frac{\alpha'}{2} g_c g_c' e_{4\mu\nu} p_3^\mu p_3^\nu |z|^{-s-m^2-2} \delta^{-\alpha' p_4 \bar{p}_4} e^{i(p_3+p_4)_\mu X^\mu} \sigma_{N-k} e^{-i/N(H-\tilde{H})} c\tilde{c} e^{-\phi-\tilde{\phi}}(0)$$

Substituting in (13.1.4) and integrating around $z=0$ gives the pole term

$$W_4 \sim \frac{-2\pi\alpha' g_c^2 {g_c'}^2 C \delta^{-\alpha'/2(p_3\bar{p}_3+p_4\bar{p}_4)}(e_{2\mu\nu}p_1^\mu p_1^\nu)(e_{4\rho\sigma}p_3^\rho p_3^\sigma)}{s+2/\alpha'(1-k/N)}\Delta$$

Comparing with (12.5.2) we get

$$ {g_c'}^2 C = \frac{8\pi}{\alpha'}$$

and hence

$$g_c' = g_c. \tag{13.1.5}$$

We have not yet determined the proportionality constant between the vertex operator normalization g_c and the gravitational coupling κ. To compare with the prediction (10.2.2) we should use κ which is the cubic coupling for gravitons canonically normalized on $\mathbf{M}^8 \times \mathbf{C}$. Hence, it is related to the closed string coupling as usual by

$$\kappa = 2\pi g_c. \tag{13.1.6}$$

13.2 Massless Exchange Diagrams

Having obtained the cubic vertex for two tachyons and a massless field in (12.5.2), we can calculate the contribution of massless exchange diagrams to the 4-tachyon amplitude. These diagrams will contain integrals over the momentum along the cone; they will be of the form

$$\begin{aligned} I_n(s) &\equiv \int_{\mathbf{C}} \frac{dp d\bar{p}}{(2\pi)^2} \frac{(p\bar{p})^{n-1}\delta^{-\alpha' p\bar{p}}}{-s+2p\bar{p}} \\ &= -\frac{(-\alpha')^{1-n}(n-1)!}{16\pi^2}\int_{D_1} d^2 z \frac{|z|^{-\frac{\alpha' s}{2}-2}}{(\log\frac{|z|}{\delta})^n}. \end{aligned} \tag{13.2.1}$$

The second form is useful for comparing with the string amplitude (12.3.1). The domain D_1 is the unit disc.

Dilaton exchange

The vertex is

$$V_3^{dil}(p_1, p_2; p_3) = -\frac{\kappa}{\sqrt{2}}(p_3\bar{p}_3 - m^2)\delta^{-\alpha' p_3 \bar{p}_3/2}$$

and the dilaton propagator is given by:

$$-\frac{i}{p^M p_M}.$$

Hence we find the exchange amplitude

$$V_4^{exch,\ dil} = \frac{4\kappa^2}{8}\Delta(m^4 I_1 - 2m^2 I_2 + I_3)$$

B-field exchange

The vertex is

$$V_3^B(p_1, p_2; p_3, e) = 2\kappa\Delta\left(e_{[\mu x]}p_2^\mu \bar{p}_3 + e_{[\bar{x}\mu]}p_2^\mu p_3 + e_{[x\bar{x}]}p_3\bar{p}_3\right)\delta^{-\alpha' p_3\bar{p}_3/2}$$

The propagator is, in the Feynman gauge $p^M e_{[MN]} = 0$,

$$-\frac{i}{p^M p_M}\left(\frac{1}{2}\eta_{MR}\eta_{NS} - \frac{1}{2}\eta_{MS}\eta_{NR}\right).$$

This gives the exchange amplitude

$$V_4^{exch,\ B} = -4\kappa^2\Delta(p_2 \cdot p_4 I_2 + \frac{1}{2}I_3).$$

Graviton exchange

The vertex is given by

$$\begin{aligned} V_3^{grav}(p_1, p_2; p_3, e) = -2\kappa\Delta\Big(& e_{(\mu\nu)}p_2^\mu p_2^\nu - e_{(\mu x)}p_2^\mu \bar{p}_3 - e_{(\bar{x}\mu)}p_2^\mu p_3 \\ & + \frac{1}{2}e_{(\bar{x}x)}p_3\bar{p}_3\Big)\delta^{-\alpha' p_3\bar{p}_3/2}. \end{aligned} \quad (13.2.2)$$

The propagator in the harmonic gauge $p^M e_{(MN)} - \frac{1}{2}p_N e^M_M = 0$ reads

$$-\frac{i}{p^M p_M}\left(\frac{1}{2}\eta_{MR}\eta_{NS} + \frac{1}{2}\eta_{MS}\eta_{NR} - \frac{1}{8}\eta_{MN}\eta_{RS}\right).$$

This gives the exchange amplitude

$$\begin{aligned} V_4^{exch,\ grav} = 4\kappa^2\Delta\Big(& ((p_2 \cdot p_4)^2 - \frac{m^4}{8})I_1 \\ & + \frac{m^2}{4}I_2 + p_2 \cdot p_4 I_2 + \frac{1}{2}I_3 - \frac{1}{8}I_3\Big). \end{aligned} \quad (13.2.3)$$

13.3 Subtractions and quartic term for the tachyon

Summing these contributions, we see that many terms cancel and we are left with

$$V_4^{exch,\ total} = 4\kappa^2\Delta(p_1\cdot p_3)^2 I_1(s). \qquad (13.3.1)$$

A similar term comes from the massless t-channel exchanges. These contributions yield precisely the asymptotics of the string amplitude (12.3.2) without extra terms finite at zero momentum. Such terms, if present, would have contributed to the quartic contact term for the tachyon. In fact, for the open string system studied in [156], the massless subtractions do yield such extra terms and, in that case, they give the leading contribution to the quartic tachyon potential.

The quartic tachyon coupling is given by integral (12.3.1) with the massless exchanges subtracted. The coefficient in front of the integral reads

$$\frac{g_c'^4 C\alpha'}{4\pi^2}\sin\frac{\pi k}{N}(p_1\cdot p_3)^2 = \frac{\kappa^2}{2\pi^3}\sin\frac{\pi k}{N}(p_1\cdot p_3)^2$$

which is of order $1/N^3$. We shall now show that the remaining integral is of order one. It is given by

$$J = \int_{\mathbf{C}} d^2z|1-z|^{-\frac{\alpha' t}{2}-2}|z|^{-\frac{\alpha' s}{2}-2}I(z,\bar{z})^{-1} + \frac{\pi}{2\sin(\pi k/N)}\int_{D_1}\frac{|z|^{-\frac{\alpha' s}{2}-2}+|z|^{-\frac{\alpha' t}{2}-2}}{\ln\frac{|z|}{\delta}}.$$

The second term comes from subtracting the massless exchanges (13.3.1). In evaluating the first term in J for large N, one should be careful because the hypergeometric function I does not converge uniformly. The function I approaches the value 2 everywhere except in the points $z = 0, 1$ (see (11.3.2)). In evaluating the integral numerically, one finds numerical convergence problems in the regions around $z = 0, 1$. A similar situation was encountered in [164]. We therefore split the integral in three parts: we cut out two small discs of radius ϵ around $z = 0, 1$ where we approximate the integrand by its asymptotics (11.3.2) which we can integrate analytically. The integral over the remainder of the complex plane will be easy to evaluate numerically using Mathematica. After summing the three integrals and subtracting the massless exchanges we take ϵ to zero.

Let us start with the integral near zero with the s-channel exchanges subtracted. The result is an integral over the unit disc with a small disc around $z = 0$ removed:

$$J_{z=0} = \frac{2\pi^2}{\sin(\pi k/N)}\int_\epsilon^1 dr\frac{r^{-\frac{\alpha' s}{2}-1}}{\ln\frac{|z|}{\delta}}$$

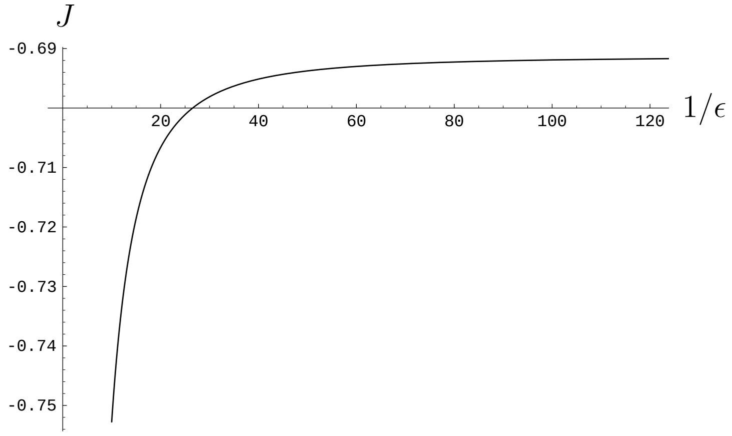


Fig 13.2: The integral J for $s,t=0$ and $N=\infty$ as a function of $1/\epsilon$.

$$\begin{aligned} &= \frac{2\pi^2}{\sin(\pi k/N)}\delta^{-\frac{\alpha' s}{2}}\left(E_1\left(\frac{\alpha' s}{2}\ln\frac{\epsilon}{\delta}\right) - E_1\left(-\frac{\alpha' s}{2}\ln\delta\right)\right) \\ &\approx 2\pi\ln\epsilon + \mathcal{O}(1/N^2) \end{aligned} \qquad (13.3.2)$$

Here, E_1 is the exponential integral $E_1(x) = \int_x^\infty dt e^{-t}/t$ and, in the last line, we have displayed the leading term at large N. The integral around $z=1$ with the t-channel exchanges subtracted has the same leading behavior. So the leading term for J is

$$J \approx 4\pi\ln\epsilon + \frac{1}{2}\int_{\mathbf{C}\backslash\mathrm{discs}} |1-z|^{-2}|z|^{-2}$$

The integral runs over the complex plane with small discs around $z=0,1$ removed. We have used that, in this integration region, the function I uniformly approaches the value of 2 at large N. We have also taken $s=t=0$. The result of the numerical integration is plotted as a function of $1/\epsilon$ in figure 13.2. As ϵ goes to zero, J converges to

$$J \approx -0.691.$$

Conclusions and Comments

We have seen that the system of localized tachyons in $\mathbf{C}/\mathbf{Z}_N$ backgrounds provides a tractable system to study aspects of off-shell closed string theory. Condensation of these tachyons connects all $\mathbf{C}/\mathbf{Z}_N$ backgrounds to each other and to flat space. There is a well-defined conjecture for the height of the potential that is rather analogous to the open-string case. Moreover, considerable computational control is possible using orbifold CFT techniques.

We have analyzed the system in the large-N approximation where it is possible to read off the off-shell action from the S-matrix. We have been able to compute all three-point correlation functions required to describe the interaction of the tachyons with massless untwisted fields. Motivated by a simple model of the tachyon potential we compute the quartic contact term. If the quartic contact term is of order one, then the minimum can occur very close to the origin and higher point interactions can be consistently ignored. Our calculation however yields a quartic term that is much too small and goes instead as $1/N^3$. This implies that our simple model is not valid for describing the potential.

There are a number of possible ways to get around this problem. One possibility is that by going beyond quartic order one can find the new minimum of the potential of the desired depth. It is not clear however how one can obtain a very shallow minimum if higher order terms are important. Another more likely possibility is that the direction that we have chosen in the field space of tachyons is not the correct one for finding the minimum. Our choice of this specific tachyon was guided by the analysis of [115, 116] which indicates that to go from $\mathbf{C}/\mathbf{Z}_N$ orbifold to the $\mathbf{C}/\mathbf{Z}_k$ orbifold by RG-flow, one needs to turn on a specific relevant operator of definite charge k under the quantum symmetry. Our tachyon corresponds precisely to this relevant operator near the conformal point. This assumption is further supported by our finding in §12.4

that turning on the tachyon of charge k does not source other tachyons and thus is a consistent approximation.

It is possible however that other excited tachyons are also involved in this process. There are a number excited states in the twisted sectors that are nearly massless or massless. The analysis of [116] is not sensitive to excited tachyons because it deals with only the chiral primaries. In particular, the analysis of [112] shows that to go from $\mathbf{C}/\mathbf{Z}_N$ to $\mathbf{C}/\mathbf{Z}_k$, the operator that is turned on in the quiver theory does not have definite charge under the quantum symmetry. It would be important to understand how these two pictures – the D-probe analysis and the RG-flows – are consistent with each other. This might help in identifying all the fields that are involved in the condensation [163].

A computation of the height of the potential for the $\mathbf{C}/\mathbf{Z}_N$ tachyons was attempted earlier in [165] where a large-N approximation was made inside the integral over the worldsheet coordinate z. As we have seen, this approximation is, however, not uniform over the region of integration and breaks down near $z = 0, 1$. As a result one obtains poles for massless exchanges instead of the correct softer logarithmic behavior that we have found. Moreover, the subtractions made in [165] were based on a postulated effective action which differs substantially from the actual cubic interactions that result from our computations.

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