Indices for superconformal field theories in 3 dimension

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To the memories of my childhood
and to all those who made them sweet.

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Synopsis

The study of symmetries has proved to be very useful in our comprehension of Nature. Knowledge of symmetries combined with the framework of Quantum Field Theories (QFT) has led to a deep understanding of processes at very small length scales (or equivalently very high energies). However a description of nature at even smaller length scales (at or bellow the Plank-scale) is yet ill understood, the framework of String Theory being a prominent step in that direction. QFTs with a new kind of symmetry namely supersymmetry (a symmetry between bosons and fermions) has attracted attention due to several positive features. It is believed that at low or high energies such nontrivial supersymmetric QFTs has an enhanced superconformal symmetry. This is due to the presence of a IR/UV superconformal fixed point into which these theories flow under renormalization group evolution. Such an enhanced symmetry is characterized by scale invariance. The Hilbert space of such superconformal field theories would bear a representation of the superconformal conformal algebra (SCA). In our present work we study the SCA in 3 space-time dimensions.

Superconformal algebras (like conformal algebras) has certain special representation which are called BPS or short representations. As the name suggests they have fewer states compared to generic long representations. They occur at special values of energies which is determined by the rest of the charges. At energies infinitesimally away from these special values we have representations with discontinuously larger number of states. Thus if we consider a slow variation of a parameter of the theory (keeping the symmetries intact) under which the spectrum evolves continuously, it is impossible for a single short representation to evolve into a long representation. Therefore naively the number of such short representations should not change under such continuous evolution of the spectrum. However it is possible for more than one short representations to combine into a long representation. Thus by studying such combination of short representations into long representations it is possible to construct all the protected quantities that can be inferred from the Superconformal algebra alone. Then we go further to construct a single quantity (similar to the Witten index) which captures all these protected information. In this thesis we present the construction of this Witten index for d = 3.

Thus under ordinary circumstances the Witten index constructed here is the most general protected quantity. However with some extra dynamical input from the theory it may be possible to extract more information about supersymmetric states. Also the protection of the index is not always guaranteed. For instance if we have a theory with a continuous spectrum then the arguments for the protection of the index fail and the index is no longer protected.

Further we go on to compute our index over the world volume theory of N M2 branes (in the large N limit) and compare it with the partition function computed over supersymmetric states. The growth in the density of states for the index was much slower ($S \sim E^{\frac{2}{3}}$) compared to that for the actual partition function (computed over the supersymmetric states) ($S \sim E^{\frac{5}{6}}$). The behavior of the partion function was same as a six dimensional gas of photons. The reason for this is that there is four supersymmetric scalars and one supersymmetric derivative on the world volume of M2 branes. We also compute the index over the world volume theory of a single M2 brane. Even in this case we find that the partition function grows much slowly for the index compared to the partition function. The partition function grows exponentially with temperature while the index grows only as a power law. Also we compute our index for the recently constructed Chern-Simons theories in the large N limit. We find that with c (≥ 3) matter fields in the adjoint representation the index undergoes a phase transition. Also unlike the partition function due to large cancellations between supersymmetric states the index remains well defined at strict infinite temperature. Also if we consider N_f matter fields in the fundamental representation of $U(N_c)$, we find that in the Veneziano limit ($N_c \to \infty$ with $c = \frac{N_f}{N_c}$ fixed) with $c \geq 3$ the index undergoes a phase transition.

This thesis is primarily based on [1]. Also for the discussion of conformal and superconformal algebra primary reference has been made to [5].

Contents

Chapter 1

Introduction

1.1 Introduction

Supersymmetric fixed points of the renormalization group equations are believed to be always either free or superconformally invariant. Thus the IR/UV behavior of any supersymmetric field theory, if nontrivial, is governed by a superconformal fixed point. Consequently, the study of superconformal dynamics has a special place in the study of supersymmetric field theories.

In radial quantization, the Hilbert space of any unitary superconformal field theory may be decomposed into a direct sum over irreducible unitary, lowest energy representation of the superconformal algebra. Such representations have been classified in every dimension (see [2, 3, 4, 5, 7] and references therein); the list of these representations turn out to include a special set of BPS representations. These representations are called 'short' because they have fewer states than generic representations (we explain this more precisely below); they also have the property that the energies of all states they host are determined by the other conserved charges that label the representation.

Consider any fixed line of superconformal field theories labeled by some continuous 'coupling constant' λ. Suppose that, at any given value of λ, the Hilbert space of the theory possesses some states that transform in short representations of the superconformal algebra. Under an infinitesimal variation of λ the energies of the corresponding states can only change if some of these representations jump from being short to long. However short representation always contain fewer states than long representations with (almost) the same quantum numbers. As a consequence, the jump of a single BPS representation from short to long is inconsistent with the continuity of the spectrum of the theory as a function of λ. Indeed such jumps are consistent with continuity only when they occur simultaneously for a group of short representations that have the property that their state content is identical to the content of a long representation. Such a bunch of BPS representations can continuously be transmuted into a long representation, after which the energies of its constituent states can be renormalized.

Consequently, a detailed study of all possible ways in which short representations can combine up into long representations permits the the classification of superconformal indices for superconformal field theories.[1] In this thesis we perform this study for superconformal algebras in $d = 3$ and use our results to provide a complete classification of all superconformal indices in this dimensions. We also provide a trace formula that, when evaluated in a superconformal field theory, may be used to extract all these superconformal indices. This is the analogue of the trace formula described in [8] for the Witten index. Thus the Witten index we define in this thesis constitutes the most general superconformal index in $d = 3$.[2]

We then proceed to compute our superconformal Witten index for specific superconformal field theories. We first perform this computation for the superconformal field theories on the world volume of N M2 branes, at $N = 1$ (using field theory) and at $N = \infty$ (using the dual supergravity description). We find that our index has significant cancellations compared to the simple partition function over supersymmetric states. In each case, the density of states in the index grows slower in comparison to the supersymmetric entropy. We also compute our index for some of the Chern Simons superconformal field theories recently analyzed by Gaiotto and Yin [12]; and find that, in some cases, this index undergoes a large N phase transition as a function of chemical potentials.

[1] By a superconformal index we mean any function of the spectrum that is forced by the superconformal algebra to remain constant under continuous variations of the spectrum.

[2] The corresponding results are known in $d = 4, 5, 6$ [1, 9]. In 2 dimensions the analogue of the indices we will study here is the famous 'elliptic genus' [10, 11] while superconformal algebras do not exist in $d > 6$.

Finally, we wish to mention a subtlety that we have avoided in our discussion above. Indices may fail to be protected if the spectrum of the theory contains a continuum [8, 13] or is singular for some parameters. Lately, this issue has attracted interest in the context of 2 dimensional conformal field theories and we direct the reader to [15, 16, 17] for some recent discussions.

This thesis is organized as follows. In chapter 2 we study conformal and superconformal algebras in general and consider the $d = 3$ superconformal algebra in detail. In chapter 3 we exhaustively construct all possible protected quantities (i.e. the vector space of indices) that can be inferred from the $d = 3$ SCA alone. In chapter 4 we define the Witten index and demonstrate that that it captures all the information present in the entire vector space of indices. In chapter 5 we evaluate the witten index over M theory multi-gravitons in $AdS_4 \otimes S^7$. In Chapter 6 we again evaluate our index for some recently studied Chern-Simons matter theories. Finally we conclude by a discussion of the future applications of our index.

Chapter 2

The Algebra

Before engaging in a detailed discussion of the superconformal algebra, let us briefly discuss the conformal algbra in arbitrary dimensions. This will help us to understand the essential features of restrictions imposed by unitarity in a comaparitively simpler setting.

2.1 The conformal algebra

The generators of the conformal algebra in d space-time dimensions with lorenzian signature comprises of $\frac{d(d-1)}{2}$ Lorentz generators ($M_{\mu\nu}$), d momntum P_μ, d special conformal generators and a dilatation D. which satisfy the following commutation relations,

$$\begin{aligned}
[M_{\mu\nu}, M_{\alpha\beta}] &= (-i)\left(\eta_{\mu\beta}M_{\nu\alpha} + \eta_{\nu\alpha}M_{\mu\beta} - \eta_{\mu\alpha}M_{\nu\beta} - \eta_{\nu\beta}M_{\mu\alpha}\right) \\
[M_{\mu\nu}, P_\alpha] &= (-i)\left(\eta_{\nu\alpha}P_\mu - \eta_{\mu\alpha}P_\nu\right) \\
[D, M_{\mu\nu}] &= 0 \\
[M_{\mu\nu}, K_\alpha] &= (-i)\left(\eta_{\nu\alpha}K_\mu - \eta_{\mu\alpha}K_\nu\right) \\
[D, P_\mu] &= -iP_\mu \\
[D, K_\mu] &= -i(-K_\mu) \\
[P_\mu, K_\nu] &= (-i)\left(2\eta_{\mu\nu}D + 2M_{\mu\nu}\right)
\end{aligned} \tag{2.1}$$

The algebra generated by the above genetators is isomorphic to the $SO(d,2)$ algebra. If S_{ab} are the generators of the $SO(d,2)$ algebra (where $a,\ b$ runs form

-1 to d, the -1 and 0 being associated with the -1 in the metric) then the isomorphism with the conformal algebra is realized by,

$$\begin{aligned} S_{\mu\nu} =& M_{\mu\nu} \\ S_{-1d} =& D \\ S_{\mu-1} =& \frac{1}{2}[P_\mu + K_\mu] \\ S_{\mu d} =& \frac{1}{2}[P_\mu - K_\mu] \end{aligned} \tag{2.2}$$

where the greek indices runs froms 0 to d. In a QFT on d dimensionsal Minkowski space-time it might be possible to define suitable action of the conformal killing vectors (tensors) on the fields[1] so as to obtain a conformal field theory (CFT). Then the Hilbert space of such a theory will consists of a representation (reducible or irreducible) of this $SO(d,2)$ conformal algebra. So we would be interested in unitary representations of the $SO(d,2)$ algebra and hence the generators are hermitian operators in such representations. Now we consider a Wick rotation of this CFT. We then quantize this Wick rotated theory radially (i.e. we consider the concentric spheres centred at the origin as constant time surfaces). The resultant Euclidian QFT has invariance under the Eucledian conformal algebra $SO(d+1,1)$. The generators of the Eucledian conformal algebra (distnguished from the corresponding generators of the Minkowski version by a prime) are related to that of $SO(d,2)$ by the following relations,

$$\begin{aligned} M'_{\mu\nu} &= S_{\mu\nu} \\ D' &= (i)S_{-10} \\ P'_\mu &= S_{\mu-1} + iS_{\mu 0} \\ K'_\mu &= S_{\mu-1} - iS_{\mu\ 0} \end{aligned} \tag{2.3}$$

Now since we want all the S_{ab} to be hermitian therefore from (2.3) it follows that (due to the presence of the is) the primed generators are not all hermitian.

[1]the conformal generators in (2.1) may be represented by the differential operators $M_{\mu\nu} = -i(x_\mu\partial_\nu - x_\nu\partial_\mu)$, $P_\mu = -i\partial_\mu$, $K_\mu = i(2x_\mu x.\partial - x^2\partial_\mu)$ and $H = x.\partial$ which constitutes the killing vectors corresponding to infinitesimal conformal transformations

In fact they obey the following conjugation relations,

$$\begin{aligned} M'^{\dagger} =& M' \\ D'^{\dagger} =& -D' \\ P'^{\dagger} =& K' \\ K'^{\dagger} =& P'. \end{aligned} \tag{2.4}$$

In order to understand these hermiticity relations physically we note that the surfaces of constant time are spheres and thus the M' operators remain hermitian. However the constant x surfaces are not constant time surfaces. Hence the P' operators corresponding to x translations are not hermitian. Also the scale transformaitions operator D' is the Eucledian hamiltonian and is therefore naturally anti-hermitian. Thus for our purpose of studying Hilbert space of CFTs we may consider unitary representations of $SO(d, 2)$ or eqivalently study the corresponding *non*-unitary representations of $SO(d+1, 1)$.

We would like to study the restrictions imposed by unitarity on the eigenvalues of scaling dimension D in the Lorenzian theory. However it turns out that such a study (and also the study of the Hilbert space of CFTs) is more convenient in the Eucledian theory. Thus for most of our purposes we consider the primed generators. We study the restrictions on eigenvalues of D' to be interpreted as the scaling dimension of an operator only after a Eucledian continuation. For future reference we note that the P' and K' operators are eigenstates of D' with eigen value $+1$ and -1 respectively. This can be seen by working out the commutators between these operators using their defination and (2.1).

The structure of unitary representations

The unitary representation of the conformal algebra $SO(d, 2)$ consists of a direct sum of representations of the maximal compact sub-algebra $SO(d) \otimes SO(2)$. The $SO(d)$ is generated by the M' generators while the $SO(2)$ part is generated by the D'. Thus we can write,

$$R_{SO(d,2)} = \sum_{\oplus} R_{S0(d)\otimes SO(2)}. \tag{2.5}$$

For physically acceptable representations of the conformal algebra there is a lower bound on the scaling dimension of operators (eigen values of D', the Hamiltonian) which we denote by ϵ_0. A representation of the conformal algebra is said to be irreducible if only one irreducible representation of $SO(d)$ (denoted by highest weights $h_1, h_2, \ldots, h_{[\frac{d}{2}]}$ in a conveniently chosen basis) has ϵ_0 as the eigen value of D'. These states with the lowest scaling dimension is said to be the primary states of the representation. As noted earlier K' has scaling dimension -1. Therefore as ϵ_0 is the lowest scaling dimension all the primary states are annihilated by K'. All the other states in the representation are obtained by the action of P' on the primary states. As the scaling dimension of P' is $+1$ hence that of all the states in the representation are of the form $\epsilon_0 + n$ with n being a positive integer. We call the states with scaling dimension $\epsilon_0 + n$ to be the nth level states. Here we recall that P' s also transform in a representation of $SO(d)$ namely the vector representation. So these higher level states (characterised by higher values of scaling dimension) are simply obtained by the tensor product of vector of $SO(d)$ with the $SO(d)$ representations present in one lower level. Thus a unitary infinite dimensional irreducible representation of the conformal algebra is completely specified by the lowest scaling dimension ϵ_0 and the $SO(d)$ highest weights $h_1, h_2, \ldots, h_{[\frac{d}{2}]}$ with this scaling dimension.

The restrictions imposed by unitarity

To uderstand the restrictions imposed by unitarity let us consider a irreducible representation of the conformal algebra with lowest scaling dimension ϵ_0 and the $SO(d)$ representation (R_p) with this scaling dimension is denoted by the highest weight $h_1, h_2, \ldots, h_{[\frac{d}{2}]}$. Let us denote the primary states in this representation of $SO(d)$ by $|s_1, s_2, \ldots, s_{[\frac{d}{2}]}\rangle$ (which we may write as $|\{s\}\rangle$) where $s_1, s_2, \ldots, s_{[\frac{d}{2}]}$ are the weights of these states.

Now as explained earlier the states at level 1 is obtained by the action of P'_μ on the primary states $|\{s\}\rangle$ which ammounts to taking a tensor product of the vector with R_p (we denote the level one states by $|S^{(1)}_{\mu\otimes\{s\}}\rangle$). All these states have scaling dimension $\epsilon_0 + 1$. Then the *norm-matrix* at level 1 (i.e. the matrix of inner product of all the states at level one) is given by,

$$\begin{aligned} A_{\nu\otimes\{t\},\ \mu\otimes\{s\}} =& \langle T^{(1)}_{\mu\otimes\{t\}} | S^{(1)}_{\mu\otimes\{s\}} \rangle \\ =& \langle \{t\} | K'_\nu\ P'_\mu | \{s\} \rangle \end{aligned} \tag{2.6}$$

where we have used the fact that K'_ν is the hermitian conjugate of P'_μ. Now uitarity demands that there are no negetive norm[1] states in the entire representation and hence the matrix A should not have any negetive eigenvalues.

Since K'_ν annihilates all the primary states hence we can write,

$$A_{\nu\otimes\{t\},\ \mu\otimes\{s\}} = \langle \{t\} |\ [K'_\nu, P'_\mu]\ | \{s\} \rangle \tag{2.7}$$

Using (2.1) and the defination of the primed operators we can easily work out this commutaion. Further using the fact $D' = (-i)\epsilon_0$ for all the primary states the above expression for matrix A reduce to,

$$A_{\nu\otimes\{t\},\ \mu\otimes\{s\}} = 2\ \langle \{t\} | \epsilon_0 + (-i) M_{\mu\nu} | \{s\} \rangle \tag{2.8}$$

Now the non-negetivity of the eigen values of A as demanded by unitarity, implies that ϵ_0 must be greater than the highest eigen value of the matrix B given by,

$$B_{\nu\otimes\{t\},\ \mu\otimes\{s\}} = -\langle \{t\} |\ (-i) M_{\mu\nu}\ | \{s\} \rangle. \tag{2.9}$$

This is the restriction imposed on the lowest scaling dimension by unitarity. These restrictics may be calculated explicitly. However since they are not directly related to our purpose therefore we do not present it here. For further details we refer the reader to [5].

We note that unitarity puts a lower bound on the lowest scaling dimension. This is consistent with our expectation on physical grounds. Recal we had already assumed this while discussing the structure of unitary representation. Here we

[1] There may be zero norm states in the representation. It turns out to be consistent to delete these states and the states obtained from them by the action of P', from the representation (becasuse they do not mix with the rest of the states under conformal transformation). Such representations are called *short* representations and they occur when the unitarity bound is saturated.

have shown how the unitarity at level one puts restrictions on ϵ_0. However it is possible that restrictions on ϵ_0 from similar calculation at some higher level is more stingent than that obtained at level one. In other words the lower bound on ϵ_0 demanding positivity of a *norm-matrix* at some higher level is lower than that at level one. In such a case the condition obtained at level one is necessary but *not* sufficient for unitarity. We should be careful to take such cases into account.

2.2 The Superconformal Algebra

The superconformal algebra has been discussed in detail by several authors [5, 6]. Here we present only a relevent and brief discussion. As the name suggests superconformal algebra is an extension of the conformal algebra where in addition to the bosonic generators (of the conformal algebra) we have fermionic generators Q and S as the supersymmetry generators and the special superconformal generators respectively. As demanded by the spin statistic theorem, the Qs being fermionic objects transform in the spinor of $SO(d-1,1)$ while together with the Ss they form the spinor of $SO(d,2)$ (the conformal group).

This is possible because the spinors of $SO(d,2)$ and $SO(d-1,1)$ have identical reality properties. If γ_μ, γ_{d+1}[1] are the $SO(d-1,1)$ gamma matrixes then in terms of these matrices the gamma matrices of $SO(d,2)$ are given by,

$$\begin{array}{ll} \Gamma_\mu = \begin{pmatrix} \gamma_\mu & 0 \\ 0 & -\gamma_\mu \end{pmatrix} & \Gamma_{-1} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \\ \Gamma_d = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} & \Gamma_{d+1} = \begin{pmatrix} \gamma_{d+1} & 0 \\ 0 & -\gamma_{d+1} \end{pmatrix} \end{array} \quad (2.10)$$

[2]

Now recal that in case of the conformal algebra it was convenient to define primed (non-hermitian) operators that obeyed the commutaion of the Eucledian conformal algebra. These operators found physical interpretation in the Wick

[1] γ_{d+1} exists only in even dimensions

[2] here again the Γ_{d+1} exists independantly only if d is even.

rotated CFT. Similarly it would be convenient to define such primed versions of the Q and S that have similar physical interpretation. Thus we define,

$$\begin{aligned} Q' =& \frac{1}{\sqrt{2}}(Q - i\gamma_0 S) \\ S' =& \frac{1}{\sqrt{2}}(Q + i\gamma_0 S) \end{aligned} \tag{2.11}$$

The commutation of Q' and S' with other (primed) generators of the (Eucledian) conformal algebra is given by,

$$\begin{aligned} [M'_{pq}, Q'_\alpha] =& (i/4)[\Gamma_p, \Gamma_q]_\alpha^\beta Q'_\beta \\ [M'_{pq}, S'_{\tilde{\alpha}}] =& (i/4)[\tilde{\Gamma}_p, \tilde{\Gamma}_q]_{\tilde{\alpha}}^{\tilde{\beta}} Q'_{\tilde{\beta}} \\ [D', Q'_\alpha] =& (-i/2)Q'_\alpha \\ [D', S'_{\tilde{\alpha}}] =& (-i/2) - S'_{\tilde{\alpha}} \\ [P'_p, Q'_\alpha] =& 0 \\ [K'_p, S'_{\tilde{\alpha}}] =& 0 \\ [P'_p, S'_{\tilde{\alpha}}] =& -(\tilde{\Gamma}_p\gamma_0)_{\tilde{\alpha}}^\beta Q_\beta \\ [K'_p, Q'_\alpha] =& +(\Gamma_p\gamma_0)_\alpha^{\tilde{\beta}} S_{\tilde{\beta}} \end{aligned} \tag{2.12}$$

Where the Γ and the $\tilde{\Gamma}$ are defined as follows,

$$\Gamma_i = \gamma_i, \quad \Gamma_d = -i\gamma_0, \quad \tilde{\Gamma}_i = \gamma_i, \quad \tilde{\Gamma}_d = i\gamma_0. \tag{2.13}$$

Now in order to complete the (anti)commutation relation for superconformal algebra we also need to specify that between Q' and S'. Often we would be interested in extended supersymmetry and therefore take into consideration the R-symmetry generators (which we denote by R say). Now unlike ordinary supersymmetry algebra (where the R-symmetry generators decouple from the rest of the generators) in case of supersonformal algebra the R-symmetry generators enters nontrivially. Infact it enters the Q' and S' anti-commutator which[1] is central to our discussion of the unitary representations of superconformal algebra (SCA).

[1] just like P', K' commutator for the conformal algebra

Table 2.1: R-Symmetry algebra in various dimension in which SCA exists

Space-time dimension	Conformal Algegra	R-Symmetry Algebra
3	$SO(3,1)$	$SO(n)$
4	$SO(4,2)$	$U(n)$
5	$SO(5,2)$	$SU(2)$
6	$SO(6,2)$	$Sp(n)$

Now given the QQ and RQ (anti)commutators, certain Jacobi identitities must be satisfied (from which other (anti)commutators like SK, SR, SP etc may also be derived). Now these Jacobi identities require certain relations between the gamma matrices to be satisfied. This is true only in lower space-time dimensions. In fact these constrains turns out to be so restrictive that SCA exixts only in $d \leq 6$. Now a SCA must be a classical Lie super algebra (CLSA). By studying how SCA fits into the structure of CLSA, it is also possible to understand why SCA does not exists in higher dimensions. For further discussions on this refer to appendix D.

The possible R-symmetry algebra in various dimensions is summarized in *table*:2.2. Note that the R-symmetry and the reality properties of spinors is different in different dimentions.

d=3: The case of our present interest

In this case the Qs and Ss are in the spinor of $SO(2,1)$. As we know this representation is a real representaiton.[1] Since Q and S are real hence from (2.11) it follows that

$$Q'^{\dagger} = S' \; ; \quad S'^{\dagger} = Q' \tag{2.14}$$

In case of three dimensions the R-Symmetry algebra is $SO(n)$. Therefore other than spinor index (denoted by greek letters α, β etc), the Q and S bear a $SO(n)$

[1] We can choose a basis in which all the γ matrices are hermtian except γ_0 which is anitihermitian. The charge conjugation matrix must satisfy $C^{-1}\gamma C = -\gamma^T$, which clearly is satisfied by γ_0

index (remember the Q and S are charged under R-symmetry algebra) which we denote by roman indices i, j etc. Then the commutation relations involving the R-symmetry generators is given by

$$\begin{aligned}
[I_{ij}, I_{mn}] =& (-i)[I_{in}\delta_{jm} + I_{jm}\delta_{ij} - I_{im}\delta_{jn} - I_{jn}\delta_{im}] \\
[I_{ij}, Q_m] =& (-i)[Q_i\delta_{jm} - Q_j\delta_{im}] \\
[I_{ij}, Q'_m] =& (-i)[Q'_i\delta_{jm} - Q'_j\delta_{im}] \\
[I_{ij}, S_m] =& (-i)[S_i\delta_{jm} - S_j\delta_{im}] \\
[I_{ij}, S'_m] =& (-i)[S'_i\delta_{jm} - S'_j\delta_{im}] \\
[I_{ij}, M_{pq}] =& 0
\end{aligned} \tag{2.15}$$

Note that the first equation is simply the commutation relation of the $SO(n)$ algebra. Now the anticommutation relations involving the susy generators $Q_{\alpha,i}$ and the special supconformal generators $S_{\alpha,i}$ is given by

$$\begin{aligned}
\{Q_{i\alpha}, Q_{j\beta}\} &= (\not{P}C)_{\alpha\beta}\delta_{ij} \\
\{S_{i\alpha}, S_{j\beta}\} &= (\not{K}C)_{\alpha\beta}\delta_{ij} \\
\{Q_{i\alpha}, S_{j\beta}\} &= \frac{\delta_{ij}}{2}[(M_{\mu\nu}\Gamma_\mu\Gamma_\nu C)_{\alpha\beta} + 2DC_{\alpha\beta}] - C_{\alpha\beta}I_{ij}
\end{aligned} \tag{2.16}$$

In terms of the primed odd generators the equations (2.16) reduce to

$$\begin{aligned}
\{Q'_{i\alpha}, Q'_{j\beta}\} &= (\not{P}'C)_{\alpha\beta}\delta_{ij} \\
\{S'_{i\tilde{\alpha}}, S'_{j\tilde{\beta}}\} &= (\tilde{\not{K}}'C)_{\tilde{\alpha}\tilde{\beta}}\delta_{ij} \\
\{Q'_{i\alpha}, S'_{j\tilde{\beta}}\} &= i\frac{\delta_{ij}}{2}[(M'_{\mu\nu}\Gamma_\mu\Gamma_\nu C)_{\alpha\tilde{\beta}} + 2D'\delta_{\alpha\tilde{\beta}}] - (i)\delta_{\alpha\tilde{\beta}}I_{ij}
\end{aligned} \tag{2.17}$$

The last of the relation is very crucial to our analysis of restrictions imposed by unitarity. Note that this relation involves the R-symmetry generators. Thus unlike ordinary supersymmetry the R-symmetry does not entirely decouple from the rest of the algebra in case of SCA. This feature is commonto all the SCA irrespective of the space time dimensions. Note here the primed P and K are to be interpreted as the eucledian version of the momentun and superconformal generators as described in the preveous section.

Structure of unitary representations of SCA

The Unitary irreducible representation of a superconformal algebra can be written as the direct sum of irreducible representations of the its conformal subalgebra ($SO(d, 2)$). Here of course the maximal compact part of the bosonic subalgebra is $SO(2) \otimes SO(d) \otimes R - symmetry$. In this case there is one more additional subtlety; besides the bosonic or even generators we have to take into account the odd generators. The primary states are specified by the highest weight state of this compact bosonic subalgebra. These states are annihilated by the K' and S' oprerators; for an irrep of the SCA these primary states are irreps of this compact subalgera. Now just as P' acts on these states to build the conformal tower, they may also be acted upon by the Q's. However unlike the P's the Q's being fermionic generate only a finite number of states by acting on the primary states. Let us refer to these states as the *Q-generated* states. Since the Q's are charged under this maximal compact subalgebra thefore these *Q-generated* states are obtained by taking a tensor product of the representation borne by the Q's and that of the primary states (and states derived form them). Here we must be careful to account for the fact that Q's squares to zero. Now each of these *Q-generated* states may be taken to be a conformal primary and we can build on them the conformal tower by acting with the P's. Just like conformal algera a representation of the superconformal algebra can also have zero norm or null states. Again here it is consistent to delete these states(and states derived from them) from the representation as they do not mix with the non-null states under any superconformal transformation.

In all the subsequent discussions we omit the prime for $Q,\ S,\ P$ and K; whenever we refer to these operators we actually refer to their primed counterparts.

2.3 The unitary representations of the $d = 3$ superconformal algebra

The bosonic subalgebra of the $d = 3$ superconformal algebra is $SO(3, 2) \times SO(n)$ (the conformal algebra times the R symmetry algebra). The anticommuting generators in this algebra may be divided into the generators of supersymmetry (Q) and the generators of superconformal symmetries (S). Supersymmetry generators transform in the vector representation of the R-symmetry group $SO(n)$,[1] have charge half under dilatations (the $SO(2)$ factor of the compact $SO(3) \times SO(2) \in SO(3,2)$) and are spinors under the $SO(3)$ factor of the same decomposition. Superconformal generators $S_i^\mu = (Q_\mu^i)^\dagger$ transform in the spinor representation of $SO(3)$, have scaling dimension (dilatation charge) $(-\frac{1}{2})$, and also transform in the vector representation of the R-symmetry group. In our notation for supersymmetry generators i is an $SO(3)$ spinor index while μ is an R symmetry vector index.

We pause to remind the reader of the structure of the commutation relations and irreducible unitary representations of the $d = 3$ superconformal algebra (see [5] and references therein). As usual, the anticommutator between two supersymmetries is proportional to momentum times an R symmetry delta function, and the anticommutator between two superconformal generators is obtained by taking the Hermitian conjugate of these relations. The most interesting relationship in the algebra is the anticommutator between Q and S. Schematically

$$\{S_i^\mu, Q_\nu^j\} \sim \delta_\nu^\mu T_i^j - \delta_i^j M_\nu^\mu$$

Here T_i^j are the $U(2) \sim SO(3) \times SO(2)$ generators and M_ν^μ are the $SO(n)$ generators.

Irreducible unitary lowest energy representations of this algebra possess a distinguished set of lowest energy states called primary states. Primary states have the lowest value of ϵ_0 – the eigenvalue of the dilatation (or energy) operator – of all states in their representation. They transform in irreducible representation of $SO(3) \times SO(n)$, and are annihilated by all special superconformal generators and special conformal generators.[2]

Primary states are special because all other states in the unitary (always infinite dimensional) representation may be obtained by acting on the primary

[1] In the literature on the worldvolume theory of the $M2$ brane, the supercharges are taken to transform in a spinor of $SO(8)$. This is consistent with the statement above, because for $n = 8$, the vector and spinor representations are related by a triality flip and a change of basis takes one to the other.

[2] i.e. all generators of negative scaling dimension.

with the generators of supersymmetry and momentum. For a primary with energy ϵ_0, a state obtained by the action of k different Q s on the primary has energy $\epsilon_0 + \frac{k}{2}$, and is said to be a state at the k^{th} level in the representation. The energy, $SO(3)$ highest weight (denoted by $j = 0, \frac{1}{2}, 1 \ldots$) and the R-symmetry highest weights $(h_1, h_2 \ldots h_{[n/2]})$ [1] of primary states form a complete set of labels for the entire representation in question.

Any irreducible representation of the superconformal algebra may be decomposed into a finite number of distinct irreducible representations of the conformal algebra. The latter are labeled by their own primary states, which have a definite lowest energy and transform in a given irreducible representation of $SO(3)$. The state content of an irreducible representation of the superconformal algebra is completely specified by the quantum numbers of its constituent conformal primaries.

As we have mentioned in the introduction, the superconformal algebra possesses special short or BPS representations which we will now explore in more detail. Consider a representation of the algebra, whose primary transforms in the spin j representation of $SO(3)$ and in the $SO(n)$ representation labeled by highest weights $\{h_i\}$ $i = 1, \cdots, \left[\frac{n}{2}\right]$. We normalize primary states to have unit norm. The superconformal algebra – plus the Hermiticity relation $(Q^i_\mu)^\dagger = S^\mu_i$ – completely determines the inner products between any two states in the representation. All states in an unitary representation must have positive norm: however this requirement is not algebraically automatic, and, in fact imposes a nontrivial restriction on the quantum numbers of primary states. This restriction takes the form $\epsilon_0 \geq f(j, h_i)$ as we will now explain.[2]

Let us first consider descendant states, at level one, of a representation whose primary has $SO(3)$ and $SO(n)$ quantum numbers $j, (h_1 \ldots h_{[n/2]})$. It is easy to compute the norm of all such states by using the commutation relations of the algebra. As long as $j \neq 0$ it turns out that the level one states with lowest norm transform in the spin $j - \frac{1}{2}$ representation of the conformal group and in the

[1] h_i are eigenvalues under rotations in orthogonal 2 planes in R^n. Thus, for instance, $\{h_i\} = (1, 0, 0, ..0)$ in the vector representation

[2] These techniques have been used in the investigation of unitarity bounds for conformal and superconformal algebras in [2, 3, 4, 5, 18, 19].

$(h_1 + 1, \{h_i\})$ $i = 2, \cdots, \left[\frac{n}{2}\right]$ representation of $SO(n)$ [5]. The highest weight state in this representation may be written explicitly as (see [19])

$$|Zn_1\rangle = A_1^-|h.w\rangle \equiv \left(Q_1^{-\frac{1}{2}} - Q_1^{\frac{1}{2}} J_- \left(\frac{1}{2J_z}\right)\right)|h.w\rangle \tag{2.18}$$

where J_- denotes the spin lowering operator of $SO(3)$ and $Q_1^{\pm\frac{1}{2}}$ are supersymmetry operators with $j = \pm\frac{1}{2}$ and $(h_1, h_2, \ldots h_{[n/2]}) = (1, 0, \ldots, 0)$. Here $|h.w\rangle$ is a highest weight state with energy ϵ_0, $SU(2)$ charge j and $SO(n)$ charge $(h_1, h_2, \ldots, h_{[n/2]})$. The norm of this state is easily computed and is given by,

$$\langle Zn_1|Zn_1\rangle = \left(1 + \frac{1}{2j}\right)(\epsilon_0 - j - h_1 - 1) \tag{2.19}$$

It follows that the non negativity of norms of states at level one (and so the unitarity of the representation) requires that the charges of the primary should satisfy

$$\epsilon_0 \geq j + h_1 + 1 \tag{2.20}$$

For $j \neq 0$ this inequality turns out to be the necessary and sufficient condition for a representation to be unitary.

When the primary saturates the bound (2.20) the representation possess zero norm states: however it turns out to be consistent to define a truncated representation by simply deleting all zero norm states. This procedure yields a physically acceptable representation whose quantum numbers saturate (2.20). This truncated representation is unitary (has only positive norms) but has fewer states than the generic representation, and so is said to be 'short' or BPS.

The set of zero norm states we had to delete, in order to obtain the BPS representation described above, themselves transform in a representation of the superconformal algebra. This representation is labeled by the primary state $|Zn_1\rangle$ (see (2.18)).

Let us now turn to the special case $j = 0$. In this case $|Zn_1\rangle$ is ill defined and does not exist; no states with its quantum numbers occur at level one. The states of lowest norm at level one transform in the spin half $SO(3)$ representation, and have $SO(n)$ highest weights $h'_1 = h_1 + 1, \{h_i\}$ $i = 2, \cdots, \frac{n}{2}$. The highest weight state in this representation is $|Zn_2\rangle = A_1^+|h.w.\rangle \equiv Q_1^{\frac{1}{2}}|h.w\rangle$. The norm of this state is $(\epsilon_0 - h_1)$. Unitarity thus imposes the constraint $\epsilon_0 \geq h_1$. However, in this case, this condition is necessary but not sufficient to ensure unitarity, as we now explain.

As we have remarked above, the state $|Zn_1\rangle = A_1^-|h.w\rangle$ is ill defined when $j = 0$. However $|s_2\rangle = \left(A_1^+ A_1^-\right)|h.w\rangle = Q_1^{\frac{1}{2}} Q_1^{-\frac{1}{2}}|h.w\rangle$ is well defined even in this situation (when $j = 0$). The norm of this state is easily computed and is given by,[1]

$$\langle s_2|s_2\rangle = (\epsilon_0 + j - h_1)(\epsilon_0 - j - h_1 - 1). \tag{2.21}$$

It follows that, at $j = 0$, the positivity of norm of all states requires either that $\epsilon_0 \geq h_1 + 1$ or that $\epsilon_0 = h_1$. This turns out to be the complete set of necessary and sufficient conditions for the existence of unitary representations. Representations with $j = 0$ and $\epsilon_0 = h_1 + 1$ or $\epsilon_0 = h_1$ are both short. The representation at $\epsilon_0 = h_1$ is an isolated short representation since there is no representation in the energy gap $h_1 \leq \epsilon_0 \leq (h_1 + 1)$; its first zero norm state occurs at level one. The first zero norm state in the $j = 0$ representation at $\epsilon_0 = h_1 + 1$ occurs at level 2 and is given by $|s_2\rangle$.

In summary, short representations occur when the highest weights of the primary state satisfy one of the following conditions [5].

$$\begin{aligned} \epsilon_0 &= j + h_1 + 1 \quad \text{when j} \geq 0, \\ \epsilon_0 &= h_1 \quad \text{when j} = 0. \end{aligned} \tag{2.22}$$

The last condition gives an isolated short representation.

[1] When $j \neq 0$, the norm of $|s_2\rangle$ had to be proportional to $(\epsilon_0 - j - h_1 - 1)$ simply because the norm of $|s_2\rangle$ must vanish whenever $|Zn_1\rangle$ is of zero norm. The algebra that leads to this result is correct even at $j = 0$ (i.e. when $|Zn_1\rangle$ is ill defined).

Chapter 3

The constuction of all possible indices using the d=3 SCA

In this chapter we study the way in which short representation combine into long representation of $d = 3$ SCA. Then we proceed to construct all possible protected quantities (i.e. indices) that can be inferred from the algebra alone.

3.1 Null Vectors and Character Decomposition of a Long Representation at the Unitarity Threshold

As we have explained in the previous subsection, short representations of the $d = 3$ superconformal algebra are of two sorts. The energy of a 'regular' short representation is given by $\epsilon_0 = j + h_1 + 1$. The null states of this representation transform in an irreducible representation of the algebra. When $j \neq 0$ the highest weights of the primary at the head of this null irreducible representation is given in terms of the highest weights of the representation itself by $\epsilon_0' = \epsilon_0 + \frac{1}{2}, \quad j' = j - \frac{1}{2}, \quad h_1' = h_1 + 1, \quad h_i' = h_i$. Note that $\epsilon_0' - j' - h_1' - 1 = \epsilon_0 - j - h_1 - 1 = 0$, so that the null states also transform in a regular short representation. As the union of the ordinary and null states of such a short representation is identical to the state content of a long representation at the edge of the unitarity bound, we conclude that

$$\lim_{\delta\to 0}\chi[j+h_1+1+\delta,j,h_1,h_j]=\chi[j+h_1+1,j,h_1,h_j]+\chi[j+h_1+3/2,j-\frac{1}{2},h_1+1,h_j] \tag{3.1}$$

where $\chi[\epsilon_0, j, h_i]$ denotes the supercharacter of the irreducible representation with energy ϵ_0, $SO(3)$ highest weight j and $SO(n)$ highest weights $\{h_i\}$. Note that the χ s appearing on the RHS of (3.1) are the supercharacters corresponding to short representations.

On the other hand when $j = 0$ the null states of the regular short representation occur at level 2 and are labelled by a primary with highest weights $\epsilon'_0 = \epsilon_0 + 1, \quad j' = 0, h'_1 = h_1 + 2, \quad h'_i = h_i$. Note in particular that $j' = 0$ and $\epsilon'_0 - h'_1 = \epsilon_0 - h_1 - 1 = 0$. It follows that the null states of this representation transform in an isolated short representation, and we conclude

$$\lim_{\delta\to 0}\chi[h_1+1+\delta,j=0,h_1,h_j]=\chi[h_1+1,j=0,h_1,h_j]+\chi[h_1+2,j=0,h_1+2,h_j] \tag{3.2}$$

Recall that isolated short representations are separated from all other representations with the same $SO(3)$ and $SO(n)$ quantum numbers by a gap in energy. As a consequence it is not possible to 'approach' such representations with long representations; consequently we have no equivalent of (3.2) at energies equal to $h_1 + \delta$.

For use below we define some notation. We will use $c(j, h_i)$ (with $i = 1, 2, \ldots, [\frac{n}{2}]$) to denote a regular short representation with $SO(3)$ and $SO(n)$ highest weights j, h_i respectively and $\epsilon_0 = j + h_1 + 1$ (when $j \geq 0$). We will also use the symbol $c(-\frac{1}{2}, h_1, h_j)$ (with $h_1 \geq h_2 - 1$) to denote the isolated short representation with $SO(3)$ quantum number 0, $SO(n)$ quantum numbers $h_1 + 1, h_j$ (here $j = 2, 3, \ldots, [\frac{n}{2}]$) respectively and $\epsilon_0 = h_1 + 1$. The utility of this notation will become apparent below.

3.2 Indices

The state content of any unitary superconformal quantum field theory may be decomposed into a sum of an (in general infinite number of) irreducible representations of the superconformal algebra. This state content is completely determined by specifying the number of times any given representation occurs in this decomposition. Consider any linear combination of the multiplicities of short representations. If this linear combination evaluates to zero on every collection of representations that appears on the RHS of each of (3.1) and (3.2) (for all values of parameters), it is said to be an index. It follows from this definition that indices are unaffected by all possible pairing up of short representations into long representations, and so are invariant under any deformation of superconformal Hilbert space under which the spectrum evolves continuously. We now proceed to list these indices.

1. The simplest indices are simply given by the multiplicities of representations in the spectrum that never appear on the RHS of (3.2) and (3.1) (for any values of the quantum numbers of the long representations on the LHS of those equations). All such representations are easy to list; they are $SO(3)$ singlet isolated representations whose $SO(n)$ quantum number $h_1 - |h_2| \leq 1$ where h_1 and h_2 are both either integers or half integers, and $h_1 \geq |h_2| \geq 0$.

2. We can also construct indices from linear combinations of the multiplicities of representations that do appear on the RHS of (3.2) and (3.1). The complete list of such linear combinations is given by

$$I_{M,\{h_j\}} = \sum_{p=-1}^{M-|h_2|} (-1)^{p+1} n\{c(\frac{p}{2}, M-p, h_j\}, \tag{3.3}$$

   where $n[R]$ denotes the multiplicities of representations of type R and the index label M is the value of $h_1 + 2j$ for every regular short representation that appears in the sum above. Thus $M \geq |h_2|$ and both M and h_2 are either integers or half-integers.Also the set $\{h_j\}$ must satisfy the condition $h_2 \geq h_3..... \geq |h_{[\frac{n}{2}]}|$ where all the h_i are either integers or all are half-integers.

Chapter 4

The constuction of the Witten index

In this chapter we shall present a trace formula for the Witten index and demonstate that it is the most general index that can be inferred from the algebra alone.

4.1 Minimally BPS states: distinguished supercharge and commuting superalgebra

We will now describe states that are annihilated by at least one supercharge and its conjugate. Consider the special supercharge Q with charges $(j = -\frac{1}{2}, h_1 = 1, h_i = 0, \epsilon_0 = \frac{1}{2})$. Let $S = Q^\dagger$; it is easily verified that

$$\{S, Q\} = \Delta = \epsilon_0 - (h_1 + j) \tag{4.1}$$

Below we will be interested in a partition function over states annihilated by Q. Clearly all such states transform in irreducible representations of that subalgebra of the superconformal algebra that commutes with Q, S and hence Δ. This subalgebra is easily determined to be a real form of the supergroup $D(\frac{n-2}{2}, 1)$ or $B(\frac{n-3}{2}, 1)$, depending on whether n is even or odd. We follow the same notation as [5].

The bosonic subgroup of this commuting superalgebra is $SO(2,1) \times SO(n-2)$. The usual Cartan charge of $SO(2,1)$ (the $SO(2)$ rotation) and the Cartan charges of $SO(n-2)$ are given in terms of the Cartan elements of the parent superconformal algebra by

$$E = \epsilon_0 + j, \quad H_i = h_{i+1} \quad \left(\text{with i} = 1, 2, \ldots, [\frac{\text{n}-2}{2}]\right). \tag{4.2}$$

4.2 A Trace formula for the general index and its Character Decomposition

Let us define the Witten index

$$I^W = \text{Tr}_R[(-1)^F \exp(-\zeta\Delta + G)], \tag{4.3}$$

where the trace is evaluated over any Hilbert space R that hosts a representation (not necessarily irreducible) of the superconformal algebra. Here F is the Fermion number operator; by the spin statistics theorem $F = 2j$ in any quantum field theory. G is any element of the subalgebra that commutes with $\{S, Q, \Delta\}$; by a similarity transformation, G may be rotated into a linear combination of the Cartan generators of the subalgebra.

The Witten index (A.7) receives contributions only from states that are annihilated by both Q and S (all other states yield contributions that cancel in pairs) and have $\Delta = 0$. So, it is independent of ζ. The usual arguments (as given in appendix A and also see [8]) ensure that I^W is an index; consequently it must be possible to expand I^W as a linear sum over the indices defined in the previous section. In fact it is easy to check that for any representation A(reducible or irreducible),

$$I^W(A) = \sum_{M,\{h_i\}} I_{M,\{h_i\}} \chi_{sub}(M+2, h_i) + \sum_{\{h_j\}, h_1-|h_2|=0,1} n\{c(-\frac{1}{2}, h_1-1, h_i)\} \chi_{sub}(h_1, h_i). \tag{4.4}$$

where $\chi_{sub}(E, H_i)$ (with $i = 1, 2, \ldots, [\frac{n-2}{2}]$) is the supercharacter of the subalgebra[1] with E and H_i being the highest weights of a representation of the

[1]The supercharacter of a representation R is defined as $\chi_{sub}(R) = \text{tr}_R(-1)^F \text{tr}\ e^{\mu\cdot\mathbf{H}}$, where

subalgebra in the convention defined by (4.2). In the first term on the RHS of (4.4) the sum runs over all the values of $M, \{h_j\}$ for which $I_{M,\{h_j\}}$ is defined (see below (3.3)). In the second term the sum runs over all the values of the set $\{h_j\}$ such that $h_2 \geq h_3..... \geq |h_{[\frac{n}{2}]}|$. In order to obtain (4.5) we have used

$$I^W(c(j, h_1, h_j)) = (-1)^{2j+1}\chi_{sub}(2j + h_1 + 2, h_i) \tag{4.5}$$

$$I^W(c(j = -\frac{1}{2}, h_1, h_j)) = \chi_{sub}(h_1 + 1, h_j) \tag{4.6}$$

Equation (4.5) asserts that the set of $\Delta = 0$ states (the only states that contribute to the Witten index) in any short irreducible representation of the superconformal algebra transform in a single irreducible representation of the commuting subalgebra. In the case of regular short representations, the primary of the full representation has $\Delta = 1$. The primary of the subalgebra is obtained by acting on the primary of the full representation with a supercharge with quantum numbers $(j = \frac{1}{2}, h_1 = 1, h_i = 0, \epsilon_0 = \frac{1}{2}, \Delta = -1)$. On the other hand the highest weight of an isolated superconformal short primary itself has $\Delta = 0$, and so is also a primary of the commuting sub super algebra. Equation (4.4) follows immediately from these facts.

Note that every index that appears in the list of subsection 2.3 appears as the coefficient of a distinct subalgebra supercharacter in (4.4). As supercharacters of distinct irreducible representations are linearly independent, it follows that knowledge of I^W is sufficient to reconstruct all superconformal indices of the algebra. In this sense (4.4) is the most general index that is possible to construct from the superconformal algebra alone.

$\mu \cdot \mathbf{H}$ is some linear combination of the Cartan generators specified by a chemical potential vector μ. F is defined to anticommute with Q and commute with the bosonic part of the algebra. The highest weight state is *always taken* to have $F = 0$.

Chapter 5

The index for the M2 brane

In this chapter we compute our index over the worldvolume theory of N M2 branes in the large N limit using the M theory graviton spectrum in $AdS_4 \times S^7$. Then we go onto compute the same quantity for the worldvolume theory of a single M2 brane.

5.1 The index over M theory multi gravitons in $AdS_4 \times S^7$

We will now compute the Witten index defined above in specific examples of three dimensional superconformal field theories. In this subsection we focus on the world volume theory of the M2 brane in the large N limit. The corresponding theory has supersymmetries and 16 superconformal symmetries. The bosonic subgroup of the relevant superconformal algebra is $SO(3,2) \times SO(8)$. We take the supercharges to transform in the vector representation of $SO(8)$; this convention is related to the one used in much of literature on this theory by a triality flip.

In the strict large N limit, the index over the M2 brane conformal field theory is simply the index over the Fock space of supergravitons for M theory on $AdS_4 \times S^7$ [20, 21]. In order to compute this quantity we first compute the index over single graviton states; the index over multi gravitons is given by the appropriate Bose- Fermi exponentiation (sometimes called the plethystic exponential).[1]

[1]The index we will calculate is sensitive to $\frac{1}{16}$ BPS states. However, the $\frac{1}{8}$ BPS partition

Single particle supergravitons in $AdS_4 \times S^7$ transform in an infinite class of representations of the superconformal algebra. The primaries for this spectrum have charges (see [23, 24]) ($\epsilon_0 = \frac{n}{2}$,$j = 0$,$h_1 = \frac{n}{2}$,$h_2 = \frac{n}{2}$, $h_3 = \frac{n}{2}$,$h_4 = -\frac{n}{2}$) (h_1, h_2, h_3 and h_4 denote $SO(8)$ highest weights in the orthogonal basis; recall Qs here are taken to transform in the vector rather than the spinor of $SO(8)$) where n runs from 1 to ∞ (we are working with the '$U(N)$ theory; $n = 1$ would be omitted for the $SU(N)$ theory).

It is not difficult to decompose each of these irreducible representations of the superconformal algebra into representations of the conformal algebra, and thereby compute the partition function and the index over each of these representations. The necessary decompositions were performed in [23], and we have verified their results independently by means a procedure described in appendix B. The results are listed in Table 5.1 below.[1]

It is now a simple matter to compute the index over single gravitons. The Witten index for the n^{th} graviton representation (R_n) is given by

$$\begin{aligned} I^W_{R_n} &= \mathrm{Tr}_{\Delta=0}\Big[(-1)^F x^{\epsilon_0+j} y_1^{H_1} y_2^{H_2} y_3^{H_3}\Big] \\ &= \sum_q \frac{(-1)^{2j_q} x^{(\epsilon_0+j)_q} \chi_q^{SO(6)}(y_1, y_2, y_3)}{1-x^2}, \end{aligned} \tag{5.1}$$

where q runs over all conformal representations with $\Delta = 0$ that appear in the decomposition of R_n in table 5.1. H_1, H_2, H_3 are the Cartan charges of $SO(6)$ in the 'orthogonal' basis that we always use in this paper. $\chi^{SO(6)}$, the $SO(6)$ character, may be computed using the Weyl character formula. The full index over single gravitons is

$$I_{sp} = \sum_{n=2}^{\infty} I^W_{R_n} + I^W_{R_1}, \tag{5.2}$$

function has been calculated, even at finite N, in [22]

[1] Some of the conformal representations obtained in this decomposition are short (as conformal representations) when n is either 1 or 2; the negative contributions in table 1 represent the charges of the null states, which physically are operators set to zero by the equations of motion. See [25]

Table 5.1: d=3 graviton spectrum

range of n	$\epsilon_0[SO(2)]$	$SO(3)$	$SO(8)$[orth.(Qs in vector)]	Δ	contribution
$n \geq 1$	$\frac{n}{2}$	0	$(\frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{-n}{2})$	0	+
$n \geq 1$	$\frac{n+1}{2}$	$\frac{1}{2}$	$(\frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{-(n-2)}{2})$	0	+
$n \geq 2$	$\frac{n+2}{2}$	1	$(\frac{n}{2}, \frac{n}{2}, \frac{(n-2)}{2}, \frac{-(n-2)}{2})$	0	+
$n \geq 2$	$\frac{n+3}{2}$	$\frac{3}{2}$	$(\frac{n}{2}, \frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{-(n-2)}{2})$	0	+
$n \geq 2$	$\frac{n+4}{2}$	2	$(\frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{-(n-2)}{2})$	1	+
$n \geq 2$	$\frac{n+2}{2}$	0	$(\frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{-(n-4)}{2})$	1	+
$n \geq 3$	$\frac{n+3}{2}$	$\frac{1}{2}$	$(\frac{n}{2}, \frac{n}{2}, \frac{(n-2)}{2}, \frac{-(n-4)}{2})$	1	+
$n \geq 3$	$\frac{n+4}{2}$	1	$(\frac{n}{2}, \frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{-(n-4)}{2})$	1	+
$n \geq 3$	$\frac{n+5}{2}$	$\frac{3}{2}$	$(\frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{-(n-4)}{2})$	2	+
$n \geq 4$	$\frac{n+5}{2}$	$\frac{1}{2}$	$(\frac{n}{2}, \frac{(n-2)}{2}, \frac{(n-4)}{2}, \frac{-(n-4)}{2})$	2	+
$n \geq 4$	$\frac{n+7}{2}$	$\frac{1}{2}$	$(\frac{(n-2)}{2}, \frac{(n-4)}{2}, \frac{(n-4)}{2}, \frac{-(n-4)}{2})$	4	+
$n \geq 4$	$\frac{n+6}{2}$	1	$(\frac{(n-2)}{2}, \frac{(n-2)}{2}, \frac{(n-4)}{2}, \frac{-(n-4)}{2})$	3	+
$n \geq 4$	$\frac{n+4}{2}$	0	$(\frac{n}{2}, \frac{n}{2}, \frac{(n-4)}{2}, \frac{-(n-4)}{2})$	2	+
$n \geq 4$	$\frac{n+6}{2}$	0	$(\frac{n}{2}, \frac{(n-4)}{2}, \frac{(n-4)}{2}, \frac{-(n-4)}{2})$	3	+
$n \geq 4$	$\frac{n+8}{2}$	0	$(\frac{(n-4)}{2}, \frac{(n-4)}{2}, \frac{(n-4)}{2}, \frac{-(n-4)}{2})$	6	+
$n = 1$	2	$\frac{1}{2}$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	1	−
$n = 1$	$\frac{5}{2}$	0	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$	2	−
$n = 2$	3	0	$(1, 1, 0, 0)$	2	−
$n = 2$	$\frac{7}{2}$	$\frac{1}{2}$	$(1, 0, 0, 0)$	2	−
$n = 2$	4	1	$(0, 0, 0, 0)$	3	−

After some algebra we find

$$I_{sp} = \Big[-x\left(x^2-1\right) y_1 y_2 y_3^2 + \sqrt{x}\sqrt{y_1}\sqrt{y_2}\left(x^3 - y_2 + y_1\left(x^3 y_2 - 1\right)\right) y_3^{3/2} \\ - x\left(x^2-1\right)\left(y_1+y_2\right)\left(y_1 y_2 + 1\right) y_3 + \sqrt{x}\sqrt{y_1}\sqrt{y_2}\left(y_2 x^3 + y_1\left(x^3 - y_2\right) - 1\right) \\ \sqrt{y_3} - x\left(x^2-1\right) y_1 y_2 \Big] / \Big[\left(x^2-1\right)\left(\sqrt{x}\sqrt{y_1}\sqrt{y_2} - \sqrt{y_3}\right) \\ \left(\sqrt{x}\sqrt{y_1}\sqrt{y_3} - \sqrt{y_2}\right)\left(\sqrt{x}\sqrt{y_2}\sqrt{y_3} - \sqrt{y_1}\right)\left(\sqrt{x} - \sqrt{y_1}\sqrt{y_2}\sqrt{y_3}\right) \Big] \tag{5.3}$$

The index over the Fock-space of gravitons may now be obtained from the above single particle index using

$$I_{fock} = \exp\left(\sum_n \frac{1}{n} I_{sp}(x^n, y_1^n, y_2^n, y_3^n) \right). \tag{5.4}$$

In order to get a feel for this result, let us set $y_i = 1$. The single graviton index reduces to

$$I_{sp} = \frac{2\sqrt{x}\left(2x + \sqrt{x} + 2\right)}{\left(\sqrt{x}-1\right)^2 (x+1)} \tag{5.5}$$

In the high energy limit, $x \equiv e^{-\beta} \to 1$, this expression simplifies to $I_{sp} \approx \frac{20}{\beta^2}$ In this limit the expression for the full Witten index I_{fock} in (5.4) reduces to,

$$I_{fock} \approx \exp \frac{20\zeta(3)}{\beta^2} \tag{5.6}$$

It follows that the thermodynamic expectation value of $\epsilon_0 + j$ (which we denote by $E_{\rm av}^{\rm ind}$) is given by

$$E_{\rm av}^{\rm ind} = -\frac{\partial \ln I_{fock}}{\partial \beta} = \frac{40\zeta(3)}{\beta^3}. \tag{5.7}$$

The index 'entropy' defined by

$$I_{fock} = \int dy \exp\{(-\beta y) + S_{\rm ind}(y)\}, \tag{5.8}$$

evaluates to

$$S_{\rm ind}(E) = \frac{60\zeta(3)}{(40\zeta(3))^{\frac{2}{3}}} E^{\frac{2}{3}}. \tag{5.9}$$

It is instructive to compare this result with the relation between entropy and E computed from the supersymmetric partition function, obtained by summing over all supersymmetric states with no $(-1)^F$ – once again in the gravity approximation. The single particle partition function evaluated on the $\Delta = 0$ states with all the other chemical potentials except the one corresponding to $E = \epsilon_0 + j$ set to zero is given by,

$$Z_{sp}(x) = \mathrm{tr}_{\Delta=0} x^E = \frac{2\sqrt{x}(x+1)\left(x^{5/2} - 2x^2 + 2x^{3/2} + 2x - 3\sqrt{x} + 2\right)}{\left(\sqrt{x}-1\right)^4 \left(x^2-1\right)}, \quad (5.10)$$

where once again $x \equiv e^{-\beta}$, with β being the chemical potential corresponding to $E = \epsilon_0 + j$. The bosonic and fermionic contributions to the partition function in (5.10) are respectively given by,

$$Z_{sp}^{\mathrm{bose}}(x) = \mathrm{tr}_{\Delta=0 \text{ bosons}} x^E = \frac{-\left(-x^4 + 4x^{7/2} - 6x^3 + x^2 - 4x^{3/2} + 6x - 4\sqrt{x}\right)}{\left(1-\sqrt{x}\right)^5 \left(\sqrt{x}+1\right)(x+1)} \quad (5.11)$$

$$Z_{sp}^{fermi}(x) = \mathrm{tr}_{\Delta=0 \text{ fermions}} x^E = \frac{-\left(-x^4 + x^2 - 4x^{3/2}\right)}{\left(1-\sqrt{x}\right)^5 \left(\sqrt{x}+1\right)(x+1)} \quad (5.12)$$

To obtain the index on the Fock space, we need to multi-particle the partition function above with the correct Bose-Fermi statistics. This leads to

$$Z_{fock} = \exp \sum_n \frac{Z_{sp}^{bose}(x^n) + (-1)^{n+1} Z_{sp}^{fermi}(x^n)}{n}. \quad (5.13)$$

We find, that for $\beta << 1$

$$\ln Z_{fock} = \frac{63\zeta(6)}{\beta^5}, \quad (5.14)$$

and a calculation similar to the one done above yields

$$S(E) = \frac{378\zeta(6)}{(315\zeta(6))^{\frac{5}{6}}} E^{\frac{5}{6}}. \quad (5.15)$$

which is the growth of states with energy of a six dimensional gas, an answer that could have been predicted on qualitative grounds. Recall that the theory of the worldvolume of the $M2$ brane has 4 supersymmetric transverse fluctuations and one supersymmetric derivative. Bosonic supersymmetric gravitons are in one to one correspondence with 'words' formed by acting on symmetric combinations of these scalars with an arbitrary number of derivatives. Consequently, supersymmetric gravitons are labelled by 5 integers n_i, n_d (the number of occurrences of each of these four scalars $i = 1 \dots 4$ and the derivative n_d) and the energy of these gravitons is $E = \frac{1}{2}(\sum_i n_i) + n_d$. This is the same as the formula for the energy of massless photons in a five spatial dimensional rectangular box, four of whose sides are of length two and whose remaining side is of unit length, explaining the effective six dimensional growth.

We conclude that the growth of states in the effective index entropy is slower than the growth of supersymmetric states in the system; this is a consequence of partial Bose-Fermi cancellations (due to the $(-1)^F$).

5.2 The index on the worldvolume theory of a single $M2$ brane

We will now compute our index over the worldvolume theory of a single $M2$ brane. For this free theory, the single particle state content is just the representation corresponding to $n = 1$ in Table 5.1 of the previous subsection. This means that it corresponds to the representation of the $d = 3$ superconformal group with the primary having charges $\epsilon_0 = \frac{1}{2}, j = 0$ and $SO(8)$ highest weights (in the convention described above) $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}]$.

For the reader's convenience, we reproduce the conformal multiplets that appear in this representation in the Table below. Physically, these multiplets correspond to the 8 transverse scalars, their fermionic superpartners and the equations of motion for each of these fields.[1]

letter	ϵ_0	j	$[h_1, h_2, h_3, h_4]$	$\Delta = \epsilon_0 - j - h_1$
ϕ^a	$\frac{1}{2}$	0	$[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}]$	0
ψ^a	1	$\frac{1}{2}$	$[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$	0
$\partial\!\!\!/ \psi^a = 0$	2	$\frac{1}{2}$	$[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$	1
$\partial^2 \phi^a = 0$	$\frac{5}{2}$	0	$[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}]$	2

(5.16)

[1] Please see [26, 27] and references therein for more details on this worldvolume theory and [40] for some recent work.

The index over these states is

$$\begin{aligned} I^{\rm sp}_{M_2}(x,y_i) &= {\rm Tr}\left[(-1)^F x^{\epsilon_0+j} y_1^{H_1} y_2^{H_2} y_3^{H_3}\right] \\ &= \frac{x^{\frac{1}{2}}\left(1+y_1y_2+y_1y_3+y_2y_3\right) - x^{\frac{3}{2}}\left(y_1+y_2+y_3+y_1y_2y_3\right)}{\left(y_1y_2y_3\right)^{\frac{1}{2}}\left(1-x^2\right)} \end{aligned} \tag{5.17}$$

For simplicity, let us set $y_i \to 1$. Then, we find

$$I^{\rm sp}_{M_2}(x, y_i = 1) = \frac{4x^{\frac{1}{2}}}{1+x} \tag{5.18}$$

Multiparticling this index, to get the index over the Fock space on the M_2 brane, we find that

$$\begin{aligned} I_{M_2}(x, y_i = 1) &= \exp \sum_{n\geq 1} \frac{I_{M_2}(x^n, y_i = 1)}{n} \\ &= \left(\prod_{n\geq 0} \frac{1-x^{2n+\frac{3}{2}}}{1-x^{2n+\frac{1}{2}}}\right)^4 \end{aligned} \tag{5.19}$$

At high temperatures $x \equiv e^{-\beta} \to 1$, the index grows as

$$I_{M_2}|_{x\to 1, y_i=1} = \left(\frac{\beta}{2}\right)^{-2} \tag{5.20}$$

The single particle supersymmetric partition function, obtained by summing over all $\Delta = 0$ single particle states with no $(-1)^F$ is,

$$\begin{aligned} Z^{\rm susy,sp}_{M_2}(x,y_i) &= {\rm Tr}_{\Delta=0}\left[x^{\epsilon_0+j} y_1^{H_1} y_2^{H_2} y_3^{H_3}\right] \\ &= \frac{x^{\frac{1}{2}}\left(1+y_1y_2+y_1y_3+y_2y_3\right) + x^{\frac{3}{2}}\left(y_1+y_2+y_3+y_1y_2y_3\right)}{\left(y_1y_2y_3\right)^{\frac{1}{2}}\left(1-x^2\right)} \end{aligned} \tag{5.21}$$

Setting $y_i \to 1$,

$$Z^{\rm susy,sp}_{M_2}(x, y_i = 1) = \frac{4x^{\frac{1}{2}}}{1-x} \tag{5.22}$$

with individual contributions from bosons and fermions being

$$\begin{aligned} Z^{\rm susy,sp,bose}_{M_2}(x) &= {\rm tr}_{\Delta=0\ \rm bosons} x^E = \frac{4x^{\frac{1}{2}}}{(1-x^2)} \\ Z^{\rm susy,sp,fermi}_{M_2}(x) &= {\rm tr}_{\Delta=0\ \rm fermions} x^E = \frac{4x^{\frac{3}{2}}}{(1-x^2)} \end{aligned} \tag{5.23}$$

Finally, multi-particling this partition function with the appropriate bose-fermi statistics, we find that

$$Z_{M_2}(x, y_i = 1) = \left(\prod_{n \geq 0} \frac{1 + x^{2n+\frac{3}{2}}}{1 - x^{2n+\frac{1}{2}}} \right)^4 \tag{5.24}$$

At high temperatures $x \to 1$, the supersymmetric partition function grows as

$$Z_{M_2}(x \to 1, y_i = 1) \approx \exp\left\{ \frac{\pi^2}{2\beta} \right\} \tag{5.25}$$

Note, that this partition function grows significantly faster at high temperatures than the index (5.19).

Chapter 6

The index for d=3 Chern-Simons's matter theories

In this chapter, we will calculate the Witten index described above for a class of the superconformal Chern Simons matter theories recently studied by Gaiotto and Yin [12]. The theories studied by these authors are three dimensional Chern Simons gauge theories coupled to matter fields; we will focus on examples that enjoy invariance under a superalgebra consisting of 4 Qs and 4 Ss (i.e. the R symmetry of these theories is $SO(2)$). The matter fields, which may thought of as dimensionally reduced $d = 4$ chiral multiplets, carry the only propagating degrees of freedom. The general constructions of Gaiotto and Yin allow the possibility of nonzero superpotentials with a coupling α that flows in the infra-red to a fixed point of order $\frac{1}{k}$ where k is the level of the Chern Simons theory. In our analysis below we will focus on the limit of large k. In this limit, the theory is 'free' and moreover we may treat $\frac{1}{k}$ as a continuous parameter. The arguments above then indicate index that we compute below for the free theory will be invariant under small deformations of $\frac{1}{k}$.

Consider this free conformal 3 dimensional theory on S^2. We are interested in calculating the letter partition function (i.e. the single particle partition function) for the propagating fields which comprise a complex scalar ϕ and its fermionic superpartner ψ. This may be done by enumerating all operators, linear in these fields, modulo those operators that are set to zero by the equations of motion. We will be interested in keeping track of several charges: the energy ϵ_0, $SO(3)$

angular momentum j, $SO(2)$ R-charge h and $\Delta = \epsilon_0 - h - j$ of our states. The following table (which lists these charges) is useful for that purpose

letter	ϵ_0	j	h	$\Delta = \epsilon_0 - j - h$
ϕ	$\frac{1}{2}$	0	$\frac{1}{2}$	0
$\phi*$	$\frac{1}{2}$	0	$\frac{-1}{2}$	1
ψ	1	$\frac{1}{2}$	$\frac{-1}{2}$	1
$\psi*$	1	$\frac{1}{2}$	$\frac{1}{2}$	0
∂_μ	1	$\{\pm 1, 0\}$	0	$\{0, 2, 1\}$
$\partial_\mu \sigma^\mu \psi = 0$	2	$\frac{1}{2}$	$\frac{-1}{2}$	2
$\partial_\mu \sigma^\mu \psi^* = 0$	2	$\frac{1}{2}$	$\frac{1}{2}$	1
$\partial^2 \phi = 0$	$\frac{5}{2}$	0	$\frac{1}{2}$	2
$\partial^2 \phi^* = 0$	$\frac{5}{2}$	0	$\frac{-1}{2}$	3

(6.1)

The last four lines, with equations of motion count with minus signs in the partition function. The list above comprises two separate irreducible representations of the superconformal algebra. ϕ, ψ and derivatives on these letters make up one representation. The other representation consists of the conjugate fields.

Let the partition functions over these two representations be denoted by z_1 and z_2. We find

$$\begin{aligned} z_1[x, y, t] &= \mathrm{tr}_{\phi,\psi,\ldots}(x^{2\epsilon_0} y^{2j} t^h) = \frac{t^{\frac{1}{2}} x(1 + x^2) + t^{\frac{-1}{2}} x^2 (y + 1/y)}{(1 - x^2 y^2)(1 - x^2/y^2)} \\ z_2[x, y, t] &= \mathrm{tr}_{\phi^*,\psi^*,\ldots}(x^{2\epsilon_0} y^{2j} t^{2h}) = \frac{t^{\frac{-1}{2}} x(1 + x^2) + t^{\frac{1}{2}} x^2 (y + 1/y)}{(1 - x^2 y^2)(1 - x^2/y^2)} \end{aligned} \tag{6.2}$$

The index (A.7) over single particle states is obtained by setting $t \to 1/x, y \to -1$

$$\begin{aligned} I_1[x] &= z_1[x, -1, 1/x] = \mathrm{tr}((-1)^F (x)^{2\epsilon_0 - h}) = \frac{x^{\frac{1}{2}}}{1 - x^2} \\ I_2[x] &= z_2[x, -1, 1/x] = \mathrm{tr}((-1)^F x^{2\epsilon_0 - h}) = \frac{-x^{\frac{3}{2}}}{1 - x^2} \\ I[x] &= I_1[x] + I_2[x] = \frac{x^{\frac{1}{2}}}{1 + x} \end{aligned} \tag{6.3}$$

In terms of these quantities, the index of the full theory is given by[28, 29]

$$I^W = \int DU \exp\left[\sum_{n=1}^{\infty} \sum_m \frac{I(x^n)}{n} Tr_{R_m}(U^n)\right] \tag{6.4}$$

where m run over the chiral multiplets of the theory, which are taken to transform in the R_m representation of $U(N)$, and Tr_{R_m} is the trace of the group element in the R_m^{th} representation of $U(N)$.

In the large N limit the integral over U in (6.4) may be converted into an integral over the eigenvalue distribution of U, $\rho(\theta)$, which, in turn, may be computed via saddle points.[1] The Fourier coefficients of this eigenvalue density function are given by:

$$\rho_n = \int_{-\pi}^{\pi} \rho(\theta) \cos(n\theta) \tag{6.5}$$

6.1 Adjoint Matter

In order to get a feel for this formula, we specialize to a particular choice of matter field content. We consider a theory with c matter fields all in the adjoint representation. In the large N limit the index is given by

$$\begin{aligned} \mathcal{I}(x) &= Tr_{\text{coloursinglets}}(-1)^F x^{2\epsilon_0 - h} \\ &= \int d\rho_n \exp\left(-N^2 \sum_{n=1}^{\infty} \frac{1}{n}(1 - cI[x^n])\rho_n^2 \right) \end{aligned} \tag{6.6}$$

The behaviour of this index as a function of x is dramatically different for $c \leq 2$ and $c \geq 3$. In order to see this note that at any given value of x, the saddle point occurs at $\rho(\theta) = \frac{1}{2\pi}$ i.e $\rho_0 = 1, \rho_n = 0, n > 0$ provided that[28, 29]

$$1 - cI[x^n] > 0, \forall n \tag{6.7}$$

In this case the saddle point contribution to the index vanishes; the leading contribution to the integral is then from the Gaussian fluctuations about this saddle point. Under these conditions the logarithm of the index or the 'free-energy' [2] is then of order 1 in the $\frac{1}{N}$ expansion.

It is easy to check that (6.7) is satisfied at all values of x (which must lie between zero and one in order for (A.7) to be well defined) when $c \leq 2$. On the

[1] Note that $N\rho(\theta)d\theta$ gives the number of eigenvalues between $e^{i\theta}$ and $e^{i(\theta+d\theta)}$ and $\int_{-\pi}^{\pi} \rho(\theta)d\theta = 1, \quad \rho(\theta) \geq 0$

[2] We use this term somewhat loosely, since we are referring here to an index and not a partition function

other hand, if $c \geq 3$ this condition is only met for

$$x < \left(\frac{1}{2}\left(c - \sqrt{c^2-4}\right)\right)^2 \tag{6.8}$$

At this value of x the coefficient of ρ_1^2 in (6.6) switches sign and the saddle point above with a uniform eigenvalue distribution is no longer valid. The new saddle point that dominates this integral above this value of x, has a Gross-Witten type gap in the eigenvalue distribution. The index undergoes a large N first order phase transition at the critical temperature listed in (6.8). At and above this temperature the 'free-energy' is of order N^2.

Note that $I(1) = \frac{1}{2}$. It follows that the index is well defined even at strictly infinite temperature This is unlike the logarithm of the actual partition function of the same theory, whose $x \to 1$ limit scales like $N^2/(1-x)^2$ as $x \to 1$ (for all values of c) reflecting the T^2 dependence of a 2+1 dimensional field theory. This difference between the high temperature limits of the index and the partition function reflects the large cancellations of supersymmetric states in their contribution to the index.

6.2 Fundamental Matter

As another special example, let us consider a theory whose N_f matter fields all transform in the fundamental representation of $U(N)$. We take the Veneziano limit: $N_c \to \infty, c = \frac{N_f}{N_c}$ fixed. The index for the theory is now given by

$$\begin{aligned} \mathcal{I}(x) &= Tr_{\text{coloursinglets}}(-1)^F x^{2\epsilon_0 - h} \\ &= \int d\rho_n \exp(-N^2 \sum_{n=1}^{\infty} \frac{(\rho_n - cI[x^n])^2 - c^2 I[x^n]^2}{n}) \end{aligned} \tag{6.9}$$

At low temperatures the integral in (6.9) is dominated by the saddle point

$$\rho_n = cI(x^n) \tag{6.10}$$

As the temperature is raised the integral in (6.9) undergoes a Gross-Witten type phase transition when c is large enough. This is easiest to appreciate in the limit $c \gg 1$. In this limit $\rho_1 = \frac{1}{2}$ in the low temperature phase when at $x \approx \frac{1}{4c^2}$, and $\rho_n = \frac{1}{2^n c^{n-1}} \ll 1$. At approximately this value of x the low temperature eigenvalue distribution $\rho(\theta)$ formally turns negative at $\theta = \pi$. This is physically unacceptable (as an eigenvalue density is, by definition, intrinsically positive). In actual fact the system undergoes a phase transition at this value of x. At large c this phase transition is very similar to the one described by Gross and Witen in [30] and in a more closely related context by [31]. The high temperature eigenvalue distribution is 'gapped' i.e. it has support on only a subset (centered about zero) of the interval $(-\pi, \pi)$.

For this phase transition to occur, we need $c \geq 3$. To arrive at this result, we notice that the distribution (6.10) implies

$$\lim_{x \to 1-} \rho(\pi) = \lim_{x \to 1-} \rho(-\pi) = \frac{1}{\pi}\left(\frac{1}{2} - \frac{c}{4}\right) \tag{6.11}$$

So, for $c \geq 3$, $\rho(\pi)$ would always turn negative for some value of x. Beyond this temperature the saddle point (6.10) is no longer valid.

Chapter 7

Discussions

In this thesis we have presented formulae for the most general superconformal index for superconformal algebras in 3 dimensions. This is done following the analogous construction of an index for four dimensional conformal field theories presented in [9]. Similar construction for 5 and 6 dimensions may be found in [1].

We hope that our work will find eventual use in the study of the space of superconformal field theories in 3 dimensions. It has recently become clear that the space of superconformal field theories in four dimensions is much richer than previously suspected [33]. The space of superconformal field theories in $d = 3$ may be equally intricate, although this question has been less studied. As our index is constant on any connected component in the space of superconformal field theories, it may play a useful role in the study of this space.

In this thesis we have also demonstrated that the most general superconformal index is captured by a simple trace formula. This observation may turn out to be useful as traces may easily be reformulated as path integrals, which in turn can sometimes be evaluated, using either perturbative techniques or localization arguments.

Aharony, Bergman, Jaffiers and Maldacena (ABJM) have recently proposed that a class of d = 3, $U(N) \otimes U(N)$ $\mathcal{N}$=6 superconformal Chern Simons field theories (at level k) admit a dual description in terms of M theory compactified on $AdS_4 \otimes S^7/Z_k$ [40]. This theory has the discrete parameters N and k which can take integral values. Now in the 't Hooft's limit ($N \to \infty$ with $\lambda = N/k$ fixed) $\lambda = N/k$ effectively becomes a continuous parameter. Then in this 'tHooft scaling limit it is possible to verify the ABJM proposal (see [41] for more details) by computing the index, constructed in this thesis, at $\lambda = 0$ using the field description (by evaluating a path integral using techniques of [14]) and at large λ using the spectrum of gravitons in $AdS_4 \otimes S^7/Z_k$. The result at large λ may also be obtained by taking a particular limit (see [41]) of the index computed over the graviton spectrum in $AdS_4 \otimes S^7$ presented in chapter 5. This amounts to a direct and important application of the index constructed in this thesis. It would be interesting to study this proposed equivalence away from this 't Hooft scaling limit at finite k (specially for $k = 1, 2$ where enhanced symmetries are expected). Again the index computed in this thesis might be useful in such a study.

Appendix A

Arguments for Witten Index to be a protected quantity

In all the dimensions we have chosen a special Q. Also we have S such that $Q^\dagger = S$. Then we defined a quantity Δ through the following relations,

$$\{S, Q\} = \Delta. \tag{A.1}$$

Now consider a unitary representation(R) of the concerned superconformal algebra. We then consider the various subspaces(R_Δ) of this representation characterized by their value under Δ.

All $\Delta \neq 0$ subspaces can be split into a direct sum of two subspaces one which is annihilated by Q (but not by S; we denote it by R_Δ^Q) and the other annihilated by S(but not by Q; we denote it by R_Δ^S).

proof

States in a $\Delta \neq 0$ subspace can be of four kind; two of these are states that lie in R_Δ^S and R_Δ^Q. The other two cases include states that are annihilated by both Q and S, and states that are annihilated by neither of them.

Let us consider a state $|\psi\rangle$ such that it satisfies simultaneously,

$$\begin{aligned} Q|\psi\rangle &\neq 0, \\ S|\psi\rangle &\neq 0, \\ \Delta|\psi\rangle &= \Delta_0|\psi\rangle, \quad \text{with} \quad \Delta_0 \neq 0. \end{aligned} \tag{A.2}$$

However the states $SQ|\psi\rangle$ and $QS|\psi\rangle$ clearly belong to R^S_Δ and R^Q_Δ respectively.

But,

$$\Delta|\psi\rangle = SQ|\psi\rangle + QS|\psi\rangle, \tag{A.3}$$

which implies,

$$|\psi\rangle = \frac{1}{\Delta_0}\left(SQ|\psi\rangle + QS|\psi\rangle\right), \tag{A.4}$$

which clearly shows that whenever $|\psi\rangle$ satisfies (A.2) it can be written as a linear combination of states in R^S_Δ and R^Q_Δ.

Now if possible let there exist a state $|\psi\rangle$ such that it satisfies simultaneously all the following conditions,

$$\begin{aligned} Q|\psi\rangle &= 0, \\ S|\psi\rangle &= 0, \\ \Delta|\psi\rangle &= \Delta_0|\psi\rangle, \quad \text{with} \quad \Delta_0 \neq 0. \end{aligned} \tag{A.5}$$

Then since,

$$\Delta|\psi\rangle = (SQ + QS)|\psi\rangle, \tag{A.6}$$

the above scenario (i.e. (A.5)) is clearly impossible.

Thefore a basis in R^S_Δ and R^Q_Δ spans the entire $\Delta = \Delta_0$ subspace. QED

We note that for $\Delta \neq 0$ there is a one-one correspondence between the states in R^Q_Δ and R^S_Δ with S providing the map (and $\frac{Q}{\Delta_0}$ proving the inverse map).

Now we define a quantity,

$$I^W = \mathrm{Tr}_R[(-1)^F \exp(-\zeta\Delta + G)], \tag{A.7}$$

where G is any element of the subalgebra that commutes with the set $\{Q, S, \Delta\}$. Then for $\Delta \neq 0$ the states in R_Δ do not contribute to I^W as the contributions from R^S_Δ and R^Q_Δ cancel. Thus I^W receives contribution only from the $\Delta = 0$ subspace and hence it is independent of ζ. Since long representations do not contain $\Delta = 0$ states therefore I^W evaluated over long representations is zero. Then by continuity we conclude I^W must evaluate to zero on any combination of short representations that combine to form a long representation. Thus we conclude I^W is a protected quantity i.e. an index.

Appendix B

The Racah Speiser Algorithm

In this appendix, we describe the Racah-Speiser algorithm, that may be used to determine the state content of the supergraviton representations described in Tables 5.1 . This appendix is out of the main line of this paper, since this state content may also be found in [23, 24, 32]

First, we remind the reader how irreducible representations of Lie Algebras, and affine Lie Algebras may be constructed using Verma modules [34, 35]. A nice description that is particularly applicable to our situation is provided in [36].

One starts by decomposing the algebra ($\mathcal{G}$) as :

$$\mathcal{G} = \mathcal{G}^{+} \oplus \mathcal{H} \oplus \mathcal{G}^{-} \tag{B.1}$$

where $\mathcal{G}^{+}(\mathcal{G}^{-})$ corresponds to the positive (negative) roots of $\mathcal{G}$ and $\mathcal{H}$ is the Cartan subalgebra.

To construct the Verma module $\mathcal{V}$ corresponding to a lowest weight $|\Omega\rangle$, one considers the linear space made up of the states $P(\mathcal{G}^{+})|\Omega\rangle$ where P is any polynomial of the positive generators.

One may calculate the character of this module,

$$\chi_{\mathcal{V}}(\mu) = \mathrm{tr}_{\mathcal{V}} e^{\mu \cdot \mathcal{H}} \tag{B.2}$$

where μ is a vector in the dual space of $\mathcal{H}$. The Weyl group $\mathcal{W}$ of the algebra has a natural action on $\mathcal{H}$ and this induces a natural action on μ. Finally, to obtain the character of the irreducibe representation $R(\Omega)$, one symmetrizes $\chi_{\mathcal{V}}$

with respect to $\mathcal{W}$.

$$\chi_{R(\Omega)} = \sum_{w \in \mathcal{W}} \chi_{\mathcal{V}}(w(\mu)). \tag{B.3}$$

One may now read off the list of states in R using $\chi_{R(\Omega)}$.

Let us elucidate the method above by constructing the character of a representation of $SU(2)$ of weight j. If $J_{\pm}$ denote the raising and lowering operators and J_3 be the Cartan, then the Verma module corresponding to a lowest weight state of weight $|-j\rangle$ is spanned by the states $(J_+)^l|-j\rangle$ with $l = 0, 1, 2, \ldots$. The character for this Verma module is given by,

$$(\chi_{\mathcal{V}})_j(x) = \mathrm{tr} x^{J_3} = \sum_{l=0}^{\infty} x^{-j+l} = \frac{x^{-j+1}}{1-x}, \tag{B.4}$$

The Weyl group of $SU(2)$ is $\mathbb{Z}_2$ which has two elements. One is just the identity. The other takes $x \to x^{-1}$. So the character of the irreducible representation corresponding to the highest weight j is given by

$$\chi_j(x) = \frac{x^{j+1}}{x-1} + \frac{x^{-j-1}}{x^{-1}-1} = \frac{x^{j+\frac{1}{2}} - x^{-j-\frac{1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}. \tag{B.5}$$

The corresponding theory for superalgebras is not as well known but was developed following the work of Kac [37]. Its application to the 4 dimensional superconformal algebra may be found in [19, 36]. Here, although we do not have a proof of this algorithm from first principles, we have followed the natural generalization of the procedure described in [19, 38, 39] for superconformal algebras in $d = 4$.

Starting with a lowest weight state one acts on this state with all the 'raising' operators of the algebra(which includes the supersymmetry generators). Then, one discards null states and all their descendants as explained in the sections above. This process results in a Verma module.

The character of this Verma module is particularly easy to construct. Although the exact structure of null vectors may be quite complicated, the charges characterizing the null state (which is all that is important for the character) are always obtained by adding the charges of a particular supercharge (or combination of supercharges) to the charges of the primary. So, the character of the

Verma module may be obtained by counting all possible actions of supercharges except for the specific combinations that lead to null states or their descendants.

One now symmetrizes this character over the Weyl group of the *maximal compact subgroup* to obtain the character of the irreducible representation corresponding to our highest weight.

Appendix C

Charges

In this appendix, explicitly list the charges of the supersymmetry generators in the worldvolume theory of the $M2$ and $M5$ branes and also for the superconformal algebra in $d = 5$. For the $M2$ brane, we have 16 supersymmetry generators 'Q'. We use the notation $[\epsilon_0, j, h_1, h_2, h_3, h_4]$, where ϵ_0 is the energy, j the $SO(3)$ charge and h_1, h_2, h_3, h_4 are the $SO(8)$ charges in the orthogonal basis (with a choice of Cartans in which the Qs are in the vector). With this notation, the Qs have charges

$$\begin{aligned}
Q_1 &= [\frac{1}{2}, \frac{1}{2}, 1, 0, 0, 0] \; ; \; Q_2 = [\frac{1}{2}, \frac{1}{2}, -1, 0, 0, 0], \\
Q_3 &= [\frac{1}{2}, \frac{1}{2}, 0, 1, 0, 0] \; ; \; Q_4 = [\frac{1}{2}, \frac{1}{2}, 0, -1, 0, 0], \\
Q_5 &= [\frac{1}{2}, \frac{1}{2}, 0, 0, 1, 0] \; ; \; Q_6 = [\frac{1}{2}, \frac{1}{2}, 0, 0, -1, 0], \\
Q_7 &= [\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 1] \; ; \; Q_8 = [\frac{1}{2}, \frac{1}{2}, 0, 0, 0, -1], \\
Q_9 &= [\frac{1}{2}, -\frac{1}{2}, 1, 0, 0, 0] \; ; \; Q_{10} = [\frac{1}{2}, -\frac{1}{2}, -1, 0, 0, 0], \\
Q_{11} &= [\frac{1}{2}, -\frac{1}{2}, 0, 1, 0, 0] \; ; \; Q_{12} = [\frac{1}{2}, -\frac{1}{2}, 0, -1, 0, 0], \\
Q_{13} &= [\frac{1}{2}, -\frac{1}{2}, 0, 0, 1, 0] \; ; \; Q_{14} = [\frac{1}{2}, -\frac{1}{2}, 0, 0, -1, 0], \\
Q_{15} &= [\frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 1] \; ; \; Q_{16} = [\frac{1}{2}, -\frac{1}{2}, 0, 0, 0, -1].
\end{aligned} , \tag{C.1}$$

Appendix D

Classical Lie Super Algebras and Super Conformal Algebras

The simple finite dimensional Lie super Algebra consists of an even part G_0 and an odd part G_1. If G_1 bears a reducible represenaion (type-I) or irreducible representation (type-II) of G_0 then the Lie Super algebra is said to be classical.

The type-I Classical Lie Super Algebras can be classified as shown in table D.1. Note that the even part (G_0) of the algebra consists of ordinary semisiple algebras.

Table D.1: Classification of type-II classial lie super algebra

The Classical Lie Super Algebra ($G_0 + G_1$)	The even part (G_0)	Representaion of G_0 on G_1
$B(m, n)$	$B_m + C_n$	$vector \otimes vector$
$D(m, n)$	$D_m + C_n$	$vector \otimes vector$
$D(2, 1, \alpha)$	$A_1 + A_1 + A_1$	$vector \otimes vector \otimes vector$
$F(4)$	$B_3 + A_1$	$spinor \otimes vector$
$G(3)$	$G_2 + A_1$	$spinor \otimes vector$
$Q(n)$	A_n	$adjoint$

The more common notation for the ordinary lie algebras are tabulated in table D.2.

For type-I Classial Lie Super Algebras(CLSA) the representation of G_0 on G_1 is reducible. However G_1 consists of two pieces $G_{\bar{1}}$ and $G_{-\bar{1}}$ such that they individually bear irreps of G_0. For basic type-I CLSA these two representations

Table D.2: Common notations for ordinary lie algebras

Cartan Notation	common Notation
A_n	$SU(n+1)$
B_n	$SO(2n+1)$
C_n	$Sp(n)$
D_n	$SO(2n)$

are contragradient of each other (i.e. weights are negetive of each other). The classification of type-I Classical Lie Super Alegbra is given in table D.3.

The Classical Lie Super Algebra $(G_0 + G_1)$	The even part (G_0)	Representaion of G_0 on G_1
$A(m,n)$	$A_m + A_n +$ C	$vector \otimes vector \otimes C$
$A(m,m)$	$A_m + A_m$	$vector \otimes vector$
$C(n)$	$C_{n-1} +$ C	$vector \otimes C$
$P(n)$	A_n	Anti-Symmetric tensor (on $G_{\bar{1}}$ A_n is represented as the symmetric tensor)

Table D.3: Classification of type-II classial lie super algebra

Note that the $P(n)$ type of CLSA is not a basic type-I. This is because in this case the representation of A_n on $G_{\bar{1}}$ is not contragradient to that on $G_{-\bar{1}}$. Also in table D.3 the C denotes the algebra of Complex numbers and C denotes its one dimensional representation.

Having enlisted the classification of CLSA we now discuss how the SCA that we have discussed in chapter2 fits into this classification. Here we will find that the requirement that SCA must be one of these CLSA turns out to be very restrictive. In fact this criterion makes it impossible for SCAs to occur in dimensions greater than six.

As we have seen in a super conformal algebra the conformal algebra $SO(d,2)$ is represented spinorially on the odd generators by the spin statistics theorem. Among the CLSA the ones which have $SO(d+2)$ (whic we loosely take to be $SO(d+2)$) as a subalgebra in the even sector are $B(m,n)$, $D(m,n)$ and $F(4)$ (refer to table D.1). However for the first two the $SO(d,2)$ is represented vectorically on the odd part. Hence they *do not* serve our purpose. $F(4)$ on the other hand has $SO(7)$ ($\sim SO(5,2)$) represented as spinors on the odd part. Thus $F(4)$ serves as the super conformal algebra in five space-time dimensions. Here the other part of the bosonic subalgebra namely $SU(2)$ (A_1) serves as the R-symmetry group and the fermionic generators lie in the vector of this $SU(2)$.

Futher certain isomorphism between algebras makes it possile to have SCA in lower dimensions. $SO(5)$ ($\sim SO(3,2)$) is isomorphic to $Sp(2)$ or C_2. Hence B(m,2) and D(m,2) are SCA in three dimensions with R-symmetry identified to be $SO(2m)$ or $SO(2m+1)$. Again the $SO(6)$ ($\sim SO(4,2)$) is isomorphic to $SU(4)$. The vector of $SU(4)$ is the spinor of $SO(6)$; hence A(3,m) is the SCA in four dimensions. Here the R-symmetry have to be A_n + C ($\sim U(n)$). Finally $SO(8)$ ($\sim SO(6,2)$) have triality due to whic the vector and the spinor representatins are identical. Hence B(4,n) with this triality transformation can serve as the SCA in six dimensions. Here the R-symmetry is then C_n (or $Sp(n)$).

The SCA in any higher dimension *does not* fit into this classification. Which is consistent with the fact that if we try to write down SCA in seven or higher dimensins then the Jacobi identities arising from the prescribed commutaitoin relations are *not* satisfied.

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