Non Supersymmetric Attractors

A thesis submitted to the
Tata Institute of Fundamental Research, Mumbai
for the degree of
PhD, in Physics

by
Rudra Pratap Jena
Department of Theoretical Physics, School of Natural Sciences
Tata Institute of Fundamental Research, Mumbai
Aug, 2007

I dedicate this thesis to my parents.

Acknowledgements

It is a pleasure to thank the many people who made this thesis possible.

It is difficult to overstate my gratitude to my Ph.D. supervisor, Sandip Trivedi. With his enthusiasm, his inspiration, and his great efforts to explain things clearly and simply, he helped to make physics fun for me. Throughout my PhD period, he provided encouragement, sound advice, good teaching, good company, and lots of good ideas. I would have been lost without him. I could not have imagined having a better advisor and mentor for my PhD, and without his knowledge, perceptiveness and cracking-of-the-whip I would never have finished.

I would like to mention special thanks to Norihiro Iizuka, Kevin Goldstein, Ashoke Sen and Gautam Mandal who have taught me lot of physics, during my work with them.

I thank all my teachers for inspiring me to do physics. I have really benefitted from various stimulating discussions in TIFR string theory group. It was exciting to attend all those string lunches and string theory seminars. I have learnt a lot of physics during these, thanks to Sunil Mukhi, Shiraz Minwalla, Atish Dabholkar, Spenta Wadia and Avinash Dhar.

I am indebted to my many student colleagues for providing a stimulating and fun environment in which to learn and grow. I am especially grateful to Pallab Basu, Basudeb Dasgupta, Aniket Basu, Rahul Nigam, Anindya Mukherji, Shamik Gupta, Sashideep Gutti, Suresh Nampuri, Loganayagam, Sayantani, Partha, Debasish, Jyotirmoy from TIFR, Suvrat Raju, Lars Grant, Subhaniel Lahiri, Joe

Marsano and Kyriakos from Harvard, Liam Mcallister from Stanford and Matthew Buican from Princeton. I would also like to thank my batchmates at TIFR, who made my stay really memorable, Sourin, Ravi, Subhendu and Kelkar.

I am grateful to the secretaries in the Department of Theoretical Physics, for helping the department to run smoothly and for assisting me in many different ways. Raju Bathija, Girish Ogale, Pawar and Shinde, all deserve special mention.

I wish to thank my entire extended family for providing a loving environment for me. My sister and some first cousins were particularly supportive. I also wish to thank Farheen for her invaluable support at times.

Lastly, and most importantly, I wish to thank my parents. They raised me, supported me, taught me, and loved me. To them I dedicate this thesis.

Contents

Chapter 1

Introduction

Black holes provide an interesting area to explore the challenges which arise in attempts to reconcile general relativity and quantum mechanics. Since string theory includes within its framework a quantum theory of gravity, it should be able to address these challenges. In fact, some of the most fascinating developments in string theory concerns black hole physics. Classically, black holes are solutions to Einstein's equations of general relativity, which have an event horizon. Due to the intense gravitational pull of the black hole, nothing can escape from inside this horizon.

Let us begin with the simplest black hole solutions of Einstein's equations in four dimensions, which are the Schwarzschild and Reissner-Nordstrom black holes.

In four dimensions the Schwarzschild black hole metric, in Schwarzschild coordinates is,

$$ds^2 = -(1 - \frac{r_H}{r})dt^2 + (1 - \frac{r_H}{r})^{-1}dr^2 + r^2 d\Omega_2^2. \tag{1.1}$$

where $r_H = 2G_4M$.

Here r_H is known as Schwarzschild radius and G_4 is Newton's constant and M is the mass of the black hole. Here t is a timelike coordinate for $r > r_H$ and spacelike for $r < r_H$, while reverse is true for r. The surface $r = r_H$ is called the event horizon, which seperates these two regions.

The generalisation of the Schwarzschild black hole to one with electric charge Q, is called the Reissner-Nordstrom black hole. In four dimensions, the metric of

Reissner-Nordstrom black hole takes the form,

$$s^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega_2^2, \tag{1.2}$$

where $\Delta = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$.
The g_{tt} component of the metric vanishes for,

$$r = r_\pm = M \pm \sqrt{M^2 - Q^2} \tag{1.3}$$

which are the inner and outer horizons respectively. The outer horizon $r = r_+$ is the event horizon. It is only present if

$$M \geq |Q| \tag{1.4}$$

In the limiting case

$$r_\pm = M \quad or \quad M = |Q| \tag{1.5}$$

the black hole is called extremal, and it has minimum mass allowed given its charge, as follows from the bound (1.4). For a general charge configuration, an extremal black hole configuration is one with lowest mass allowed by the charges. The metric of the extremal Reissner-Nordstrom black hole takes the form,

$$ds^2 = -(1 - \frac{r_0}{r})^2 dt^2 + (1 - \frac{r_0}{r})^{-2} dr^2 + r^2 d\Omega_2^2. \tag{1.6}$$

where $r_0 = M$. In the near horizon limit, where $r \approx 0$, the geometry approaches $AdS_2 \times S_2$.
The mass M and charge Q of the black hole are defined in terms of appropriate surface integrals at spatial infinity. Two important quantities associated with the black hole horizon are its area A and the surface gravity κ_s. The surface gravity, which is constant on the horizon, is related to the force(measured at spatial infinity) that holds a unit test mass in place.
Classical black holes behave like thermodynamic systems characterised by temperature and entropy. Let us first consider the thermodynamic description, i.e., the macroscopic description of black holes. The first law for thermodynamics states that the variation of the total energy is equal to the temperature times the variation of the entropy plus work terms. The corresponding formula for black holes states that the variation of black hole mass is related to the variation of

the horizon area plus the work terms proportional to the variation of the angular momentum and a term proportional to a variation of the charge, multiplied by the electric/magnetic potential μ at the horizon.

$$\delta M = \frac{\kappa_s}{2\pi}\frac{\delta A}{4} + \mu\delta Q + \Omega\delta J. \tag{1.7}$$

It was found by Hawking [1] that any black hole emits radiation with the spectrum of a black body at temperature $\kappa_s/2\pi$. This leads to the identification of the black hole entropy in terms of the horizon area,

$$S_{macro} = \frac{1}{4}A \tag{1.8}$$

This relation, known as Bekenstein-Hawking entropy, appears to be universally valid, atleast when A is sufficiently large.

This identification raises the question what the black hole microstates are, and where they are located. We want to know what the statistical mechanics of black holes is, and express (1.8) as the logarithm of a number of microstates that are compatible with a given macrostate, i.e., a black hole with a given set (M, J, Q). String theory provides a microscopic quantum description of some supersymmetric black holes. For these black holes, the logarithm of the degeneracy of states as a function of charge Q, $d(Q)$, has been shown to exactly match the Bekenstein-Hawking entropy in the limit of large charges. The first microstate counting for black holes in string theory that reproduced the right numerical coefficent of the Bekenstein Hawking entropy was done by Strominger and Vafa . They considered supersymmetric (and thus necessarily extremal and charged) black holes. In the presence of supersymmetry, there exists non renormalization theorems, which essentially say that the weak coupling results are protected from quantum corrections. This means that the number of states one counts at weak coupling cannot change as one increases the coupling, i.e., when a black hole forms. These microstates correspond to configuration of wrapped branes. More recently, it has been shown that the Bekenstein Hawking Wald entropy for a number of black hole examples, agree to all orders in preturbation theory in inverse charges going well beyond the perturbation limit.

Supersymmetric black holes are well known to exhibit a striking phenomenon called the attractor behavior [2, 3, 4]. Moduli fields in these black hole backgrounds take value at the horizon independent of their asymptotic values at infinity, determined by the charges only. As a result, the macroscopic entropy depends only on the charges. This is consistent with the fact that the microscopic entropy, more precisely, an index is also independent of the asymptotic moduli due to the BPS property of the states.
So far the attractor mechanism has been studied almost exclusively in the context of $\mathcal{N} = 2$ supergravity. The question which interests us in this thesis is if this mechanism is more general. We want to study whether it works for non supersymmetric black holes as well. The extremal black holes have some properties which are similar to supersymmetric black holes, for example they have vanishing surface gravity and have the minimal possible mass compatible with the given charges and angular momentum. This motivates the search of attractor behavior in extremal configurations. We investigate the attractor mechanism for extremal black holes and black rings in two derivative actions describing gravity coupled to scalar fields and abelian gauge fields. These black holes might be solutions in theories which have no supersymmetry or might be non supersymmetric solutions in supersymmetric theories.

1.1 Outline of the Thesis

The theories we consider consist of gravity, gauge fields and scalar fields with the action

$$S = \frac{1}{\kappa^2}\int d^4x\sqrt{-G}(R-2(\partial\phi_i)^2-f_{ab}(\phi_i)F^a_{\ \mu\nu}F^{b\,\mu\nu}-\tfrac{1}{2}\tilde{f}_{ab}(\phi_i)F^a_{\ \mu\nu}F^b_{\ \rho\sigma}\epsilon^{\mu\nu\rho\sigma}). \quad (1.9)$$

Here the index i denotes the different scalars and a, b the different gauge fields and $F^a_{\mu\nu}$ stands for the field strength of the gauge field. $f_{ab}(\phi_i)$ determines the gauge couplings and $\tilde{f}_{ab}(\phi_i)$ determines the axionic couplings.
The scalars determine the gauge couplings and thereby couple to the gauge fields.

It is important that the scalars do not have a potential of their own that gives them in particular a mass. Such a potential would mean that the scalars are no longer moduli.

The theories described by (1.9), are four dimensional and do not have a cosmological constant. This action can be generalized to higher dimensions as well as theories with cosmological constant. In chapter 2, we explore the attractor mechanism in spherical symmetric extremal black holes for theories mentioned in (1.9) [5]. The moduli fields in a black hole background vary radially and get attracted to specific values at the horizon, which depends only on the charges. The attractor behavior also refers to the fact that the near horizon geometry approaches $AdS_2 \times S^2$ and the AdS_2 and S^2 radii depend only on the charges. We show that the attractor mechanism works quite generally in such theories provided two conditions are met. These conditions are succinctly stated in terms of an "effective potential" V_{eff} for the scalar fields, ϕ_i.
With both electric and magnetic charges the gauge fields take the form,

$$F^a = f^{ab}(\phi_i)(Q_{eb} - \tilde{f}_{bc}Q^c_m)\frac{1}{b^2}dt \wedge dr + Q^a_m sin\theta d\theta \wedge d\phi, \tag{1.10}$$

where Q^a_m, Q_{ea} are constants that determine the magnetic and electric charges carried by the gauge field F^a, and f^{ab} is the inverse of f_{ab}.
The effective potential is then given by,

$$V_{eff}(\phi_i) = f^{ab}(Q_{ea} - \tilde{f}_{ac}Q^c_m)(Q_{eb} - \tilde{f}_{bd}Q^d_m) + f_{ab}Q^a_m Q^b_m. \tag{1.11}$$

The effective potential is proportional to the energy density in the electromagnetic field and arises after solving for the gauge fields in terms of the charges carried by the black hole. The two conditions that need to be met are the following. First, as a function of the moduli fields V_{eff} must have a critical point, $\partial_i V_{eff}(\phi_{i0}) = 0$. And second, the matrix of second derivatives of the effective potential at the critical point, $\partial_{ij}V_{eff}(\phi_{i0})$, must be have only positive eigenvalues. The resulting attractor values for the moduli are the critical values, ϕ_{i0}. And the entropy of the black hole is proportional to $V(\phi_{i0})$, and is thus independent of the asymptotic values for the moduli. It is worth noting that the two conditions stated above are met by BPS black hole attractors in an $\mathcal{N} = 2$ supersymmetric theory.

The analysis for BPS attractors simplifies greatly due to the use of the first order equations of motion. In the non-supersymmetric context one has to work with the second order equations directly and this complicates the analysis. The central idea in establishing the attractor behavior lies in the perturbation analysis. Although the equations are second order, in perturbation theory they are linear, and this makes them tractable. For the static spherically symmetric case, we can consistently set the scalars to their critical values for all values of r. This gives rise to an extremal Reissner Nordstrom black hole. If the scalar field takes value at asymptotic infinity which are small deviations from their attractor values, we show that the extremal black hole solution continues to exist. The scalar takes the attractor value at the horizon. The analysis can be carried out quite generally for any effective potential for the scalars and shows that the two conditions stated above are sufficient for the attractor phenomenon to hold.

We also carry out numerical analysis to go beyond the perturbation results. This requires a specific form of the effective potential. The numerical analysis corroborates the perturbation theory results mentioned above. In simple cases we have explored so far, we have found evidence for a only single basin of attraction, although multiple basins must exist in general as is already known from the supersymmetric(SUSY) cases.

We generalize these results to other settings. We find that the attractor phenomenon continues to hold in higher dimensions and Anti-de Sitter space (AdS), as long as the two conditions mentioned above are valid for a suitable defined effective potential.

For a completely general class of gravity actions including higher derivative interactions, assuming an extremal near horizon geometry of form $AdS_2 \times S^2$ but without assuming supersymmetry, the attractor values of the moduli can be obtained by extremizing an "entropy function"[6]. For the two derivative theories, the entropy function and the effective potential are different quantities, but the extremisation results give the same attractor values.

In chapter 2, we construct an effective potential for extremal black holes, which is minimized with respect to the moduli to get the attractor values of the scalars. For supersymmetric black holes, there is a function of the moduli and the charges carried by the black hole, called the central charge, which plays an important

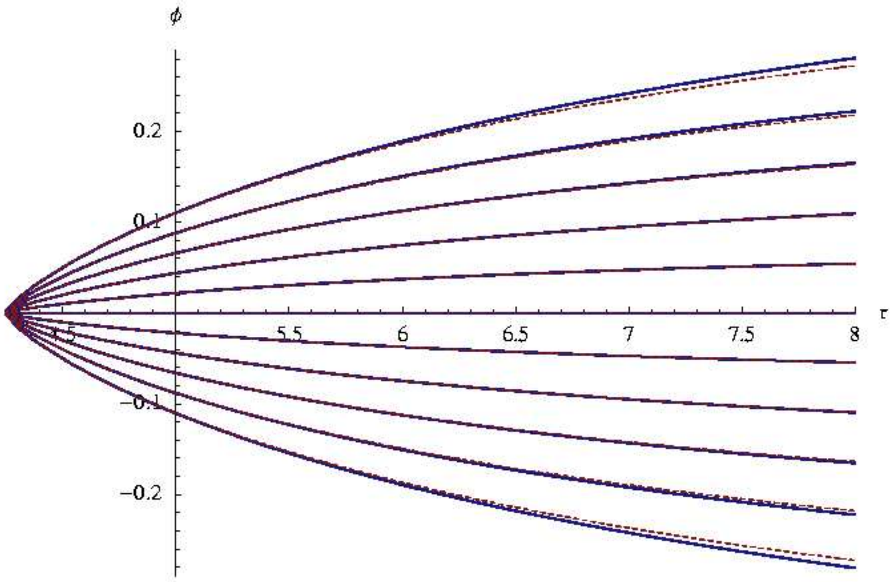


Figure 1.1: Variation of the scalar moduli as we move in from infinity into the horizon.

role in the discussion of the attractor. The attractor values of the scalars, which are obtained at the horizon of the black hole, are given by minimizing the central charge with respect to the moduli.

There is a another sense in which the central charge is also minimized at the horizon of a supersymmetric attractor. One finds that the central charge, now regarded as a function of the position coordinate, evolves monotonically from asymptotic infinity to the horizon and obtains its minimum value at the horizon of the black hole. It is natural to ask whether there is an analogous quantity in the non-supersymmetric case and in particular if the effective potential is also monotonic and minimized in this sense for non-supersymmetric attractors.

In chapter 3, we propose a c-function for non-supersymmetric spherically symmetric attractors. We first study the four dimensional case. The c-function has a simple geometrical and physical interpretation in this case. We are interested in spherically symmetric and static configurations in which all fields are functions of only one variable - the radial coordinate. The c-function, $c(r)$, is given by

$$c(r) = \frac{1}{4}A(r), \tag{1.12}$$

where $A(r)$ is the area of the two-sphere, of the $SO(3)$ isometry group orbit, as a function of the radial coordinate [1]. For any asymptotically flat solution we show that the area function satisfies a c-theorem and monotonically decreases as one moves towards the horizon, which is the direction of increasing redshift from infinity. For a black hole solution, the static region ends at the horizon, so in the static region the c-function attains its minimum value at the horizon. This horizon value of the c-function equals the entropy of the black hole. While the horizon value of the c-function is also proportional to the minimum value of the effective potential, more generally, away from the horizon, the two are different. In fact we find that the effective potential need not vary monotonically in a non-supersymmetric attractor. The c-theorem we prove is applicable for supersymmetric black holes as well. In the supersymmetric case, there are three quantities of interest, the c-function, the effective potential and the square of the central charge. At the horizon these are all equal, up to a constant of proportionality. But away from the horizon they are in general different.

We work directly with the second order equations of motion in our analysis and it might seem puzzling at first that one can prove a c-theorem at all. The answer to the puzzle lies in boundary conditions. For black hole solutions we require that the solutions are asymptotically flat. This is enough to ensure that going inwards to the horizon from asymptotic infinity the c-function decreases monotonically.

We generalize the c-function to higher dimensions. We analyze a system of rank q gauge fields and moduli coupled to gravity and once again find a c-function that satisfies a c-theorem. In $D = p + q + 1$ dimensions this system has extremal black brane solutions whose near horizon geometry is $AdS_{p+1} \times S^q$. We show that the c-function is non-increasing from infinity up to the near horizon region. It's minimum value in the $AdS_{p+1} \times S^q$ region agrees with the conformal anomaly in the dual boundary theory for p even [7].

[1] We have set $G_N = 1$.

In the first two chapters, we analyse the spherically symmetric configurations. In chapter 4 we extend the study of the attractor mechanism to rotating black hole solutions. Our starting point is an observation made in [8] that the near horizon geometries of extremal Kerr and Kerr-Newman black holes have SO(2,1)×U(1) isometry. Armed with this observation we prove a general result that is as powerful as its non-rotating counterpart. In the context of 3+1 dimensional theories, our analysis shows that in an arbitrary theory of gravity coupled to abelian gauge fields and neutral scalar fields with a gauge and general coordinate invariant local Lagrangian density, the entropy of a rotating extremal black hole remains invariant, except for occasional jumps, under continuous deformation of the asymptotic data for the moduli fields if an extremal black hole is defined to be the one whose near horizon field configuration has $SO(2,1) \times U(1)$ isometry.

The strategy for obtaining this result is to use the entropy function formalism [6]. We find that the near horizon background of a rotating extremal black hole is obtained by extremizing a functional of the background fields on the horizon, and that Bekenstein Hawking entropy is given by precisely the same functional evaluated at its extremum. Thus if this functional has a unique extremum with no flat directions then the near horizon field configuration is determined completely in terms of the charges and angular momentum, with no possibility of any dependence on the asymptotic data on the moduli fields. On the other hand if the functional has flat directions so that the extremization equations do not determine the near horizon background completely, then there can be some dependence of this background on the asymptotic data, but the entropy, being equal to the value of the functional at the extremum, is still independent of this data. Finally, if the functional has several extrema at which it takes different values, then for different ranges of asymptotic values of the moduli fields the near horizon geometry could correspond to different extrema. In this case as we move in the space of asymptotic data the entropy would change discontinuously as we cross the boundary between two different domains of attraction, although within a given domain it stays fixed. The stability analysis of the rotating attractors has not been done.

We explore this formalism in detail in the context of an arbitrary two derivative theory of gravity coupled to scalar and abelian vector fields. The extremization conditions now reduce to a set of second order differential equations with parameters and boundary conditions which depend only on the charges and the angular momentum. Thus the only ambiguity in the solution to these differential equations arise from undetermined integration constants. We prove explicitly that in a generic situation all the integration constants are fixed once we impose the appropriate boundary conditions and smoothness requirement on the solutions. We also show that even in a non-generic situation where some of the integration constants are not fixed (and hence could depend on the asymptotic data on the moduli fields), the value of the entropy is independent of these undetermined integration constants.

We test our general results for some known examples. Here we take some of the known extremal rotating black hole solutions in two derivative theories of gravity coupled to matter, and study their near horizon geometry to determine if they exhibit attractor behavior. We focus on two particular classes of examples — the Kaluza-Klein black holes studied in [9, 10] and black holes in toroidally compactified heterotic string theory studied in [11]. In both these examples, we find two kinds of extremal limits. One of these branches does not have an ergo-sphere . We call this the ergo-free branch. The other branch does have an ergo-sphere. We call this the ergo-branch. On both branches the entropy turns out to be independent of the asymptotic values of the moduli fields, in accordance with our general arguments. We find however that while on the ergo-free branch the scalar and all other background fields at the horizon are independent of the asymptotic data on the moduli fields, this is not the case for the ergo-branch. Thus on the ergo-free branch we have the full attractor behavior, whereas on the ergo-branch only the entropy is attracted to a fixed value independent of the asymptotic data.

In five dimensions, one has a black hole with an event horizon of topology $S^1 \times S^2$, which is called a black ring. Ref[12] obtained an explicit solution of five dimensional vacuum general relativity describing such an object.

By examining the BPS equations for black rings, [13], found the attractor equations for supersymmetric extremal black rings. Motivated by the results of our analysis for black holes, which demonstrate the attractor mechanism is independent of supersymmetry, we extend our analysis to extremal black rings [14] in chapter 5.

We start with five dimensional Lagrangian consisting of gravity, abelian gauge fields, neutral massless scalars and a Chern-Simons term. We dimensionally reduce it to four dimensions. Then we apply the entropy function formalism. We study extremal black rings whose near horizon geometry have $AdS_2 \times S^2 \times U(1)$ symmetries. After dimensional reduction, we get an $AdS_2 \times S^2$ near horizon geometry. This analysis is extended to the five dimensional extremal rotating black holes. In all these cases, we demonstrate the attractor mechanism for black rings with out recourse to supersymmetry by using the entropy function formalism.

Chapter 2

Non Supersymmetric Attractors

In this chapter, we consider theories with gravity, gauge fields and scalars in four-dimensional asymptotically flat space-time. By studying the equations of motion directly we show that the attractor mechanism can work for non-supersymmetric extremal black holes. Two conditions are sufficient for this, they are conveniently stated in terms of an effective potential involving the scalars and the charges carried by the black hole. Our analysis applies to black holes in theories with $\mathcal{N} \leqslant 1$ supersymmetry, as well as non-supersymmetric black holes in theories with $\mathcal{N} = 2$ supersymmetry. Similar results are also obtained for extremal black holes in asymptotically Anti-de Sitter space and in higher dimensions.

2.1 Attractor in Four-Dimensional Asymptotically Flat Space

2.1.1 Equations of Motion

In this section we consider gravity in four dimensions with $U(1)$ gauge fields and scalars. The scalars are coupled to gauge fields with dilaton-like couplings. It is important for the discussion below that the scalars do not have a potential so that there is a moduli space obtained by varying their values.

The action we start with has the form,

$$S = \frac{1}{\kappa^2}\int d^4x\sqrt{-G}(R - 2(\partial\phi_i)^2 - f_{ab}(\phi_i)F^a_{\mu\nu}F^{b\ \mu\nu}) \tag{2.1}$$

Here the index i denotes the different scalars and a, b the different gauge fields and $F^a_{\mu\nu}$ stands for the field strength of the gauge field. $f_{ab}(\phi_i)$ determines the gauge couplings, we can take it to be symmetric in a, b without loss of generality.

The Lagrangian is

$$\mathcal{L} = (R - 2(\partial\phi_i)^2 - f_{ab}(\phi_i)F^a_{\mu\nu}F^{b\ \mu\nu}) \tag{2.2}$$

Varying the metric gives [1],

$$R_{\mu\nu} - 2\partial_\mu\phi_i\partial_\nu\phi_i = 2f_{ab}(\phi_i)F^a{}_{\mu\lambda}F^b{}_\nu{}^\lambda + \frac{1}{2}G_{\mu\nu}\mathcal{L} \tag{2.3}$$

The trace of the above equation implies

$$R - 2(\partial\phi_i)^2 = 0 \tag{2.4}$$

The equations of motion corresponding to the metric, dilaton and the gauge fields are then given by,

$$\begin{aligned}
R_{\mu\nu} - 2\partial_\mu\phi_i\partial_\nu\phi_i &= f_{ab}(\phi_i)\left(2F^a{}_{\mu\lambda}F^b{}_\nu{}^\lambda - \tfrac{1}{2}G_{\mu\nu}F^a{}_{\kappa\lambda}F^{b\kappa\lambda}\right) && (2.5)\\
\frac{1}{\sqrt{-G}}\partial_\mu(\sqrt{-G}\partial^\mu\phi_i) &= \frac{1}{4}\partial_i(f_{ab})F^a{}_{\mu\nu}F^{b\mu\nu} && (2.6)\\
\partial_\mu(\sqrt{-G}f_{ab}(\phi_i)F^{b\mu\nu}) &= 0.
\end{aligned}$$

The Bianchi identity for the gauge field is,

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0. \tag{2.7}$$

We now assume all quantities to be function of r. To begin, let us also consider the case where the gauge fields have only magnetic charge, generalisations to both electrically and magnetically charged cases will be discussed shortly. The metric and gauge fields can then be written as,

$$\begin{aligned}
ds^2 &= -a(r)^2dt^2 + a(r)^{-2}dr^2 + b(r)^2d\Omega^2 && (2.8)\\
F^a &= Q^a_m sin\theta d\theta \wedge d\phi && (2.9)
\end{aligned}$$

[1] In our notation $G_{\mu\nu}$ refers to the components of the metric.

Using the equations of motion we then get,

$$R_{tt} = \frac{a^2}{b^4} V_{eff}(\phi_i) \tag{2.10}$$

$$R_{\theta\theta} = \frac{1}{b^2} V_{eff}(\phi_i) \tag{2.11}$$

where,

$$V_{eff}(\phi_i) \equiv f_{ab}(\phi_i) Q_m^a Q_m^b. \tag{2.12}$$

This function, V_{eff}, will play an important role in the subsequent discussion. We see from eq.(2.10) that up to an overall factor it is the energy density in the electromagnetic field. Note that $V_{eff}(\phi_i)$ is actually a function of both the scalars and the charges carried by the black hole.

The relation, $R_{tt} = \frac{a^2}{b^2} R_{\theta\theta}$, after substituting the metric ansatz implies that,

$$(a^2(r)b^2(r))'' = 2. \tag{2.13}$$

The $R_{rr} - \frac{G^{tt}}{G^{rr}} R_{tt}$ component of the Einstein equation gives

$$\frac{b''}{b} = -(\partial_r \phi)^2. \tag{2.14}$$

Also the R_{rr} component itself yields a first order "energy" constraint,

$$-1 + a^2 b'^2 + \frac{a^{2'} b^{2'}}{2} = \frac{-1}{b^2}(V_{eff}(\phi_i)) + a^2 b^2 (\phi')^2 \tag{2.15}$$

Finally, the equation of motion for the scalar ϕ_i takes the form,

$$\partial_r (a^2 b^2 \partial_r \phi_i) = \frac{\partial_i V_{eff}}{2b^2}. \tag{2.16}$$

We see that $V_{eff}(\phi_i)$ plays the role of an "effective potential " for the scalar fields.

Let us now comment on the case of both electric and magnetic charges. In this case one should also include "axion" type couplings and the action takes the form,

$$S = \frac{1}{\kappa^2} \int d^4x \sqrt{-G} (R - 2(\partial\phi_i)^2 - f_{ab}(\phi_i) F^a_{\ \mu\nu} F^{b\,\mu\nu} - \tfrac{1}{2} \tilde{f}_{ab}(\phi_i) F^a_{\ \mu\nu} F^b_{\ \rho\sigma} \epsilon^{\mu\nu\rho\sigma}). \tag{2.17}$$

We note that $\tilde{f}_{ab}(\phi_i)$ is a function independent of $f_{ab}(\phi_i)$, it can also be taken to be symmetric in a, b without loss of generality.

The equation of motion for the metric which follows from this action is unchanged from eq.(2.5). While the equations of motion for the dilaton and the gauge field now take the form,

$$\frac{1}{\sqrt{-G}}\partial_\mu(\sqrt{-G}\partial^\mu\phi_i) = \tfrac{1}{4}\partial_i(f_{ab})F^a_{\ \mu\nu}F^{b\,\mu\nu} + \tfrac{1}{8}\partial_i(\tilde{f}_{ab})F^a_{\ \mu\nu}F^b_{\ \rho\sigma}\epsilon^{\mu\nu\rho\sigma} \tag{2.18}$$

$$\partial_\mu\left(\sqrt{-G}\left(f_{ab}(\phi_i)F^{b\,\mu\nu} + \tfrac{1}{2}\tilde{f}_{ab}F^b_{\ \rho\sigma}\epsilon^{\mu\nu\rho\sigma}\right)\right) = 0. \tag{2.19}$$

With both electric and magnetic charges the gauge fields take the form,

$$F^a = f^{ab}(\phi_i)(Q_{eb} - \tilde{f}_{bc}Q^c_m)\frac{1}{b^2}dt\wedge dr + Q^a_m sin\theta d\theta\wedge d\phi, \tag{2.20}$$

where Q^a_m, Q_{ea} are constants that determine the magnetic and electric charges carried by the gauge field F^a, and f^{ab} is the inverse of f_{ab} [1]. It is easy to see that this solves the Bianchi identity eq.(2.7), and the equation of motion for the gauge fields eq.(2.19).

A little straightforward algebra shows that the Einstein equations for the metric and the equations of motion for the scalars take the same form as before, eq.(2.13, 2.14, 2.15, 2.16), with V_{eff} now being given by,

$$V_{eff}(\phi_i) = f^{ab}(Q_{ea} - \tilde{f}_{ac}Q^c_m)(Q_{eb} - \tilde{f}_{bd}Q^d_m) + f_{ab}Q^a_mQ^b_m. \tag{2.21}$$

As was already noted in the special case of only magnetic charges, V_{eff} is proportional to the energy density in the electromagnetic field and therefore has an immediate physical significance. It is invariant under duality transformations which transform the electric and magnetic fields to one-another.

Our discussion below will use (2.13, 2.14, 2.15, 2.16) and will apply to the general case of a black hole carrying both electric and magnetic charges.

It is also worth mentioning that the equations of motion, eq.(2.13, 2.14, 2.16) above can be derived from a one-dimensional action,

$$S = \frac{2}{\kappa^2}\int dr\left((a^2b)'b' - a^2b^2(\phi')^2 - \frac{V_{eff}(\phi_i)}{b^2}\right). \tag{2.22}$$

[1] We assume that f_{ab} is invertible. Since it is symmetric it is always diagonalisable. Zero eigenvalues correspond to gauge fields with vanishing kinetic energy terms, these can be omitted from the Lagrangian.

The constraint, eq.(2.15) must be imposed in addition.

One final comment before we proceed. The eq.(2.17) can be further generalised to include non-trivial kinetic energy terms for the scalars of the form,

$$\int d^4x\sqrt{-G}\left(-g_{ij}(\phi_k)\partial\phi^i\partial\phi^j\right). \tag{2.23}$$

The resulting equations are easily determined from the discussion above by now contracting the scalar derivative terms with the metric g_{ij}. The two conditions we obtain in the next section for the existence of an attractor are not altered due to these more general kinetic energy terms.

2.1.2 Conditions for an Attractor

We can now state the two conditions which are sufficient for the existence of an attractor. First, the charges should be such that the resulting effective potential, V_{eff}, given by eq.(2.21), has a critical point. We denote the critical values for the scalars as $\phi_i = \phi_{i0}$. So that,

$$\partial_i V_{eff}(\phi_{i0}) = 0 \tag{2.24}$$

Second, the matrix of second derivatives of the potential at the critical point,

$$M_{ij} = \frac{1}{2}\partial_i\partial_j V_{eff}(\phi_{i0}) \tag{2.25}$$

should have positive eigenvalues. Schematically we write,

$$M_{ij} > 0 \tag{2.26}$$

Once these two conditions hold, we show below that the attractor phenomenon results. The attractor values for the scalars are [1] $\phi_i = \phi_{i0}$.

The resulting horizon radius is given by,

$$b_H^2 = V_{eff}(\phi_{i0}) \tag{2.27}$$

and the entropy is

$$S_{BH} = \frac{1}{4}A = \pi b_H^2. \tag{2.28}$$

[1] Scalars which do not enter in V_{eff} are not fixed by the requirement eq.(2.24). The entropy of the extremal black hole is also independent of these scalars.

There is one special solution which plays an important role in the discussion below. From eq.(2.16) we see that one can consistently set $\phi_i = \phi_{i0}$ for all values of r. The resulting solution is an extremal Reissner Nordstrom (ERN) Black hole. It has a double-zero horizon. In this solution $\partial_r \phi_i = 0$, and a, b are

$$a_0(r) = \left(1 - \frac{r_H}{r}\right), \quad b_0(r) = r \tag{2.29}$$

where r_H is the horizon radius. We see that $a_0^2, (a_0^2)'$ vanish at the horizon while b_0, b_0' are finite there. From eq.(2.15) it follows then that the horizon radius b_H is indeed given by

$$r_H^2 = b_H^2 = V_{eff}(\phi_{i0}), \tag{2.30}$$

and the black hole entropy is eq.(2.28).

If the scalar fields take values at asymptotic infinity which are small deviations from their attractor values we show below that a double-zero horizon black hole solution continues to exist. In this solution the scalars take the attractor values at the horizon, and $a^2, (a^2)'$ vanish while b, b' continue to be finite there. From eq.(2.15) it then follows that for this whole family of solutions the entropy is given by eq.(2.28) and in particular is independent of the asymptotic values of the scalars.

For simple potentials V_{eff} we find only one critical point. In more complicated cases there can be multiple critical points which are attractors, each of these has a basin of attraction.

One comment is worth making before moving on. A simple example of a system which exhibits the attractor behaviour consists of one scalar field ϕ coupled to two gauge fields with field strengths, $F^a, a = 1, 2$. The scalar couples to the gauge fields with dilaton-like couplings,

$$f_{ab}(\phi) = e^{\alpha_a \phi} \delta_{ab}. \tag{2.31}$$

If only magnetic charges are turned on,

$$V_{eff} = e^{\alpha_1 \phi}(Q_1)^2 + e^{\alpha_2 \phi}(Q_2)^2. \tag{2.32}$$

(We have suppressed the subscript "m" on the charges). For a critical point to exist α_1 and α_2 must have opposite sign. The resulting critical value of ϕ is given by,

$$e^{\phi_0} = \left(-\frac{\alpha_2(Q_2)^2}{\alpha_1(Q_1)^2}\right)^{\frac{1}{\alpha_1-\alpha_2}} \tag{2.33}$$

The second derivative, eq.(2.25) now is given by

$$\frac{\partial^2 V_{eff}}{\partial \phi^2} = -2\alpha^1\alpha^2 \tag{2.34}$$

and is positive if α_1, α_2 have opposite sign.

This example will be useful for studying the behaviour of perturbation theory to higher orders and in the subsequent numerical analysis.

As we will discuss further in section 7, a Lagrangian with dilaton-like couplings of the type in eq.(2.31), and additional axionic terms (which can be consistently set to zero if only magnetic charges are turned on), can always be embedded in a theory with $\mathcal{N} = 1$ supersymmetry. But for generic values of α we do not expect to be able to embed it in an $\mathcal{N} = 2$ theory. The resulting extremal black hole, for generic α, will also then not be a BPS state.

2.1.3 Comparison with the $\mathcal{N} = 2$ Case

It is useful to compare the discussion above with the special case of a BPS black hole in an $\mathcal{N} = 2$ theory. The role of the effective potential, V_{eff} for this case was emphasised by Denef, [15]. It can be expressed in terms of a superpotential W and a Kahler potential K as follows:

$$V_{eff} = e^K [K^{i\bar{j}} D_i W (D_j W)^* + |W|^2], \tag{2.35}$$

where $D_i W \equiv \partial_i W + \partial_i K W$. The attractor equations take the form,

$$D_i W = 0 \tag{2.36}$$

And the resulting entropy is given by

$$S_{BH} = \pi |W|^2 e^K. \tag{2.37}$$

with the superpotential evaluated at the attractor values.

It is easy to see that if eq.(2.36) is met then the potential is also at a critical point, $\partial_i V_{eff} = 0$. A little more work also shows that all eigenvalues of the second derivative matrix, eq.(2.25) are also positive in this case. Thus the BPS attractor meets the two conditions mentioned above. We also note that from eq.(2.35) the value of V_{eff} at the attractor point is $V_{eff} = e^K|W|^2$. The resulting black hole entropy eq.(2.27, 2.28) then agrees with eq.(2.37).

We now turn to a more detailed analysis of the attractor conditions below.

2.1.4 Perturbative Analysis

2.1.4.1 A Summary

The essential idea in the perturbative analysis is to start with the extremal RN black hole solution described above, obtained by setting the asymptotic values of the scalars equal to their critical values, and then examine what happens when the scalars take values at asymptotic infinity which are somewhat different from their attractor values, $\phi_i = \phi_{i0}$.

We first study the scalar field equations to first order in the perturbation, in the ERN geometry without including backreaction. Let ϕ_i be a eigenmode of the second derivative matrix eq.(2.25) [1]. Then denoting, $\delta\phi_i \equiv \phi_i - \phi_{i0}$, neglecting the gravitational backreaction, and working to first order in $\delta\phi_i$, we find that eq.(2.16) takes the form,

$$\partial_r \left((r - r_H)^2 \partial_r(\delta\phi_i)\right) = \frac{\beta_i^2 \delta\phi_i}{r^2}, \tag{2.38}$$

where β_i^2 is the relevant eigenvalue of $\frac{1}{2}\partial_i\partial_j V(\phi_{i0})$. In the vicinity of the horizon, we can replace the factor $1/r^2$ on the r.h.s by a constant and as we will see below, eq.(2.38), has one solution that is well behaved and vanishes at the horizon provided $\beta_i^2 \geqslant 0$. Asymptotically, as $r \to \infty$, the effects of the gauge fields die away and eq.(2.38) reduces to that of a free field in flat space. This has

[1] More generally if the kinetic energy terms are more complicated, eq.(2.23), these eigenmodes are obtained as follows. First, one uses the metric at the attractor point, $g_{ij}(\phi_{i0})$, and calculates the kinetic energy terms. Then by diagonalising and rescaling one obtains a basis of canonically normalised scalars. The second derivatives of V_{eff} are calculated in this basis and gives rise to a symmetric matrix, eq.(2.25). This is then diagonalised by an orthogonal transformation that keeps the kinetic energy terms in canonical form. The resulting eigenmodes are the ones of relevance here.

two expected solutions, $\delta\phi_i \sim \textit{constant}$, and $\delta\phi_i \sim 1/r$, both of which are well behaved. It is also easy to see that the second order differential equation is regular at all points in between the horizon and infinity. So once we choose the non-singular solution in the vicinity of the horizon it can be continued to infinity without blowing up.

Next, we include the gravitational backreaction. The first order perturbations in the scalars source a second order change in the metric. The resulting equations for metric perturbations are regular between the horizon and infinity and the analysis near the horizon and at infinity shows that a double-zero horizon black hole solution continues to exist which is asymptotically flat after including the perturbations.

In short the two conditions, eq.(2.24), eq.(2.26), are enough to establish the attractor phenomenon to first non-trivial order in perturbation theory.

In 4-dimensions, for an effective potential which can be expanded in a power series about its minimum, one can in principle solve for the perturbations analytically to all orders in perturbation theory. We illustrate this below for the simple case of dilaton-like couplings, eq.(2.31), where the coefficients that appear in the perturbation theory can be determined easily. One finds that the attractor mechanism works to all orders without conditions other than eq.(2.24), eq.(2.26) [1].

When we turn to other cases later, higher dimensional or AdS space etc., we will sometimes not have explicit solutions, but an analysis along the above lines in the near horizon and asymptotic regions and showing regularity in-between will suffice to show that a smoothly interpolating solution exists which connects the asymptotically flat region to the attractor geometry at horizon.

To conclude, the key feature that leads to the attractor is the fact that both solutions to the linearised equation for $\delta\phi$ are well behaved as $r \to \infty$, and one solution near the horizon is well behaved and vanishes. If one of these features fails the attractor mechanism typically does not work. For example, adding a mass term for the scalars results in one of the two solutions at infinity diverging. Now it is typically not possible to match the well behaved solution near the

[1] For some specific values of the exponent γ_i, eq.(2.41), though, we find that there can be an obstruction which prevents the solution from being extended to all orders.

horizon to the well behaved one at infinity and this makes it impossible to turn on the dilaton perturbation in a non-singular fashion.

We turn to a more detailed description of perturbation theory below.

2.1.4.2 First Order Solution

We start with first order perturbation theory. We can write,

$$\delta\phi_i \equiv \phi_i - \phi_{i0} = \epsilon\phi_{i1}, \tag{2.39}$$

where ϵ is the small parameter we use to organise the perturbation theory. The scalars ϕ_i are chosen to be eigenvectors of the second derivative matrix, eq.(2.25).

From, eq.(2.13), eq.(2.14), eq.(2.15), we see that there are no first order corrections to the metric components, a, b. These receive a correction starting at second order in ϵ. The first order correction to the scalars ϕ_i satisfies the equation,

$$\partial_r(a_0^2 b_0^2 \partial_r \phi_{i1}) = \frac{\beta_i^2}{b_0^2}\phi_{i1}. \tag{2.40}$$

where, β_i^2 is the eigenvalue for the matrix eq.(2.25) corresponding to the mode ϕ_i. Substituting for a_0, b_0, from eq.(2.29) we find,

$$\phi_{i1} = c_{1i}\left(\frac{r - r_H}{r}\right)^{\frac{1}{2}(\pm\sqrt{1+4\beta_i^2/r_H^2}-1)} \tag{2.41}$$

We are interested in a solution which does not blow up at the horizon, $r = r_H$. This gives,

$$\phi_{i1} = c_{1i}\left(\frac{r - r_H}{r}\right)^{\gamma_i}, \tag{2.42}$$

where

$$\gamma_i \;=\; \tfrac{1}{2}\left(\sqrt{1+\tfrac{4\beta_i^2}{r_H^2}} - 1\right). \tag{2.43}$$

Asymptotically, as $r \to \infty$, $\phi_{i1} \to c_{1i}$, so the value of the scalars vary at infinity as c_{1i} is changed. However, since $\gamma_i > 0$, we see from eq.(2.42) that ϕ_{i1} vanishes at the horizon and the value of the dilaton is fixed at ϕ_{i0} regardless of its value at infinity. This shows that the attractor mechanism works to first order in perturbation theory.

It is worth commenting that the attractor behaviour arises because the solution to eq.(2.40) which is non-singular at $r = r_H$, also vanishes there. To examine this further we write eq.(2.40) in standard form, [?],

$$\frac{d^2y}{dx^2} + P(x)y + Q(x)y = 0, \tag{2.44}$$

with $x = r - r_H$, $y = \phi_{i1}$. The vanishing non-singular solution arises because eq.(2.40) has a single and double pole respectively for $P(x)$ and $Q(x)$, as $x \to 0$. This results in (2.44) having a scaling symmetry as $x \to 0$ and the solution goes like x^{γ_i} near the horizon. The residues at these poles are such that the resulting indical equation has one solution with exponent $\gamma_i > 0$. In contrast, in a non-extremal black hole background, the horizon is still a regular singular point for the first order perturbation equation, but $Q(x)$ has only a single pole. It turns out that the resulting non-singular solution can go to any constant value at the horizon and does not vanish in general.

2.1.4.3 Second Order Solution

The first order perturbation of the dilaton sources a second order correction in the metric. We turn to calculating this correction next.

Let us write,

$$\begin{aligned} b &= b_0 + \epsilon^2 b_2 \\ a^2 &= a_0^2 + \epsilon^2 a_2 \\ b^2 &= b_0^2 + 2\epsilon^2 b_2 b_0, \end{aligned} \tag{2.45}$$

where b_0 and a_0 are the zeroth order extremal Reissner Nordstrom solution eq.(2.29).

Equation (2.13) gives,

$$a^2b^2 = (r - r_H)^2 + d_1 r + d_2. \tag{2.46}$$

The two integration constants, d_1, d_2 can be determined by imposing boundary conditions. We are interested in extremal black hole solutions with vanishing surface gravity. These should have a horizon where b is finite and a^2 has a "double-zero", i.e., both a^2 and its derivative $(a^2)'$ vanish. By a gauge choice we can always take the horizon to be at $r = r_H$. Both d_1 and d_2 then vanish. Substituting eq.(2.45) in the equation(2.13) we get to second order in ϵ,

$$2a_0^2 b_0 b_2 + b_0^2 a_2 = 0. \tag{2.47}$$

Substituting for a_0, b_0 then determines, a_2 in terms of b_2,

$$a_2 = -2\left(1 - \frac{r_H}{r}\right)^2 \frac{b_2}{r}. \tag{2.48}$$

From eq.(2.14) we find next that,

$$b_2(r) = -\sum_i \frac{c_{1i}^2 \gamma}{2(2\gamma_i - 1)} r \left(\frac{r - r_H}{r}\right)^{2\gamma_i} + A_1 r + A_2 r_H \tag{2.49}$$

A_1, A_2 are two integration constants. The two terms proportional to these integration constant solve the equations of motion for b_2 in the absence of the $O(\epsilon)^2$ source terms from the dilaton. This shows that the freedom associated with varying these constants is a gauge degree of freedom. We will set $A_1 = A_2 = 0$ below. Then, b_2 is,

$$b_2(r) = -\sum_i \frac{c_{1i}^2 \gamma_i}{2(2\gamma_i - 1)} r \left(\frac{r - r_H}{r}\right)^{2\gamma_i} \tag{2.50}$$

It is easy to check that this solves the constraint eq.(2.15) as well.

To summarise, the metric components to second order in ϵ are given by eq.(2.45) with a_0, b_0 being the extremal Reissner Nordstrom solution and the second order corrections being given in eq.(2.48) and eq.(2.50). Asymptotically, as $r \to \infty$, $b_2 \to c \times r$, and, $a_2 \to -2 \times c$, so the solution continues to be asymptotically flat to this order. Since $\gamma_i > 0$ we see from eq.(2.48, 2.50) that the second order corrections are well defined at the horizon. In fact since b_2 goes to zero at the horizon, a_2 vanishes at the horizon even faster than a double-zero. Thus the second order solution continues to be a double-zero horizon black hole with vanishing surface gravity. Since b_2 vanishes the horizon area does not change to second order in perturbation theory and is therefore independent of the asymptotic value of the dilaton.

The scalars also gets a correction to second order in ϵ. This can be calculated in a way similar to the above analysis. We will discuss this correction along with higher order corrections, in one simple example, in the next subsection.

Before proceeding let us calculate the mass of the black hole to second order in ϵ. It is convenient to define a new coordinate,

$$y \equiv b(r) \tag{2.51}$$

Expressing a^2 in terms of y one can read off the mass from the coefficient of the $1/y$ term as $y \to \infty$, as is discussed in more detail in section 2.9. This gives,

$$M = r_H + \epsilon^2 \sum_i \frac{r_H c_{i1}^2 \gamma_i}{2} \tag{2.52}$$

where r_H is the horizon radius given by (2.30). Since γ_i is positive, eq.(2.43), we see that as ϵ increases, with fixed charge, the mass of the black hole increases. The minimum mass black hole is the extremal RN black hole solution, eq.(2.29), obtained by setting the asymptotic values of the scalars equal to their critical values.

2.1.4.4 An Ansatz to All Orders

Going to higher orders in perturbation theory is in principle straightforward. For concreteness we discuss the simple example, eq.(2.31), below. We show in this example that the form of the metric and dilaton can be obtained to all orders in perturbation theory analytically. We have not analysed the coefficients and resulting convergence of the perturbation theory in great detail. In a subsequent section we will numerically analyse this example and find that even the leading order in perturbation theory approximates the exact answer quite well for a wide range of charges. This discussion can be generalised to other more complicated cases in a straightforward way, although we will not do so here.

Let us begin by noting that eq.(2.13) can be solved in general to give,

$$a^2 b^2 = (r - r_H)^2 + d_1 r + d_2 \tag{2.53}$$

As in the discussion after eq(2.46) we set $d_1 = d_2 = 0$, since we are interested in extremal black holes. This gives,

$$a^2 b^2 = (r - r_H)^2, \tag{2.54}$$

where r_H is the horizon radius given by eq.(2.30). This can be used to determine a in terms of b.

Next we expand b, ϕ and a^2 in a power series in ϵ,

$$b = b_0 + \sum_{n=1}^{\infty} \epsilon^n b_n \tag{2.55}$$

$$\phi = \phi_0 + \sum_{n=1}^{\infty} \epsilon^n \phi_n \tag{2.56}$$

$$a^2 = a_0^2 + \sum_{n=1}^{\infty} \epsilon^n a_n \tag{2.57}$$

where b_0, a_0 are given by eq.(2.29) and ϕ_0 is given by eq.(2.33).

The ansatz which works to all orders is that the n^{th} order terms in the above two equations take the form,

$$\phi_n(r) = c_n \left(\frac{r - r_H}{r}\right)^{n\gamma} \tag{2.58}$$

$$b_n(r) = d_n r \left(\frac{r - r_H}{r}\right)^{n\gamma}, \tag{2.59}$$

and,

$$a_n = e_n \left(\frac{r - r_H}{r}\right)^{n\gamma+2}, \tag{2.60}$$

where γ is given by eqs.(2.43) and in this case takes the value,

$$\gamma = \frac{1}{2}\left(\sqrt{1 - 2\alpha_1\alpha_2} - 1\right). \tag{2.61}$$

The discussion in the previous two subsections is in agreement with this ansatz. We found $b_1 = 0$, and from eq.(2.50) we see that b_2 is of the form eq.(2.59). Also, we found $a_1 = 0$ and from eq.(2.48) a_2 is of the form eq.(2.60). And from eq.(2.42) we see that ϕ_1 is of form eq.(2.58). We will now verify that this ansatz consistently solves the equations of motion to all orders in ϵ. The important point is that with the ansatz eq.(2.58, 2.59) each term in the equations of motion of order ϵ^n has a functional dependence $(\frac{(r-r_H)}{r})^{2\gamma n}$. This allows the equations to be solved consistently and the coefficients c_n, d_n to be determined.

Let us illustrate this by calculating c_2. From eq.(2.14) and eq.(2.54) we see that the equation of motion for ϕ can be written in the form,

$$2b(r)^2\partial_r((r-r_H)^2\partial_r\phi) = e^{\alpha_i\phi}Q_i^2\alpha_i \tag{2.62}$$

To $O(\epsilon^2)$ this gives,

$$\left(\frac{r-r_H}{r}\right)^{2\gamma}\left(2c_2(e^{\alpha_i\phi_0}Q_i^2\alpha_i^2-4r_H^2\gamma(1+2\gamma))+e^{\alpha_i\phi_0}Q_i^2\alpha_i^3c_1^2\right)=0 \tag{2.63}$$

Notice that the term $(\frac{r-r_H}{r})^{2\gamma}$ has factored out. Solving eq.(2.63) for c_2 we now get,

$$c_2 = \frac{1}{2}c_1^2(\alpha_1+\alpha_2)\frac{(\gamma+1)}{(3\gamma+1)} \tag{2.64}$$

More generally, as discussed in section 2.9, working to the required order in ϵ we can recursively find, c_n, d_n, e_n.

One more comment is worth making here. We see from eq.(2.50) that b_2 blows up when when $\gamma=1/2$. Similarly we can see from eq.(2.165) that b_n blows up when $\gamma=\frac{1}{n}$ for b_n. So for the values, $\gamma=\frac{1}{n}$, where n is an integer, our perturbative solution does not work.

Let us summarise. We see in the simple example studied here that a solution to all orders in perturbation theory can be found. b, ϕ and a^2 are given by eq.(2.59), eq.(2.58) and eq.(2.60) with coefficients that can be determined as discussed in section 2.9. In the solution, a^2 vanishes at r_H so it is the horizon of the black hole. Moreover a^2 has a double-zero at r_H, so the solution is an extremal black hole with vanishing surface gravity. One can also see that b_n goes linearly with r as $r\to\infty$ so the solution is asymptotically flat to all orders. It is also easy to see that the solution is non-singular for $r\geqslant r_H$. Finally, from eq.(2.58) we see that $\phi_n=0$, for all $n>0$, so all corrections to the dilaton vanish at the horizon. Thus the attractor mechanism works to all orders in perturbation theory. Since all corrections to b also vanish at the horizon we see that the entropy is uncorrected in perturbation theory. This is in agreement with the general argument given after eq.(2.28). Note that no additional conditions had to be imposed, beyond eq.(2.24, 2.26), which were already appeared in the lower order discussion, to ensure the attractor behaviour [1].

2.2 Numerical Results

There are two purposes behind the numerical work we describe in this section. First, to check how well perturbation theory works. Second, to see if the attractor behaviour persists, even when ϵ, eq.(2.39), is order unity or bigger so that the deviations at asymptotic infinity from the attractor values are big. We will confine ourselves here to the simple example introduced near eq.(2.31), which was also discussed in the higher orders analysis in the previous subsection.

In the numerical analysis it is important to impose the boundary conditions carefully. As was discussed above, the scalar has an unstable mode near the horizon. Generic boundary conditions imposed at $r \to \infty$ will therefore not be numerically stable and will lead to a divergence. To avoid this problem we start the numerical integration from a point r_i near the horizon. We see from eq.(2.58, 2.59) that sufficiently close to the horizon the leading order perturbative corrections [2] becomes a good approximation. We use these leading order corrections to impose the boundary conditions near the horizon and then numerically integrate the exact equations, eq.(2.13,2.14), to obtain the solution for larger values of the radial coordinate.

The numerical integration is done using the Runge-Kutta method. We characterise the nearness to the horizon by the parameter

$$\delta r = \frac{r_i - r_H}{r_i} \tag{2.67}$$

[1] In our discussion of exact solutions in section 4 we will be interested in the case, $\alpha_1 = -\alpha_2$. From eq.(2.64, 2.165) we see that the expressions for c_2 and d_3 become,

$$c_2 = 0 \tag{2.65}$$
$$d_3 = 0 \tag{2.66}$$

It follows that in the perturbation series for ϕ and b only the c_{2n+1}(odd) terms and d_{2n}(even) terms are non-vanishing respectively.

[2] We take the $O(\epsilon)$ correction in the dilaton, eq.(2.42), and the $O(\epsilon^2)$ correction in b, a^2, eq.(2.48, 2.49). This consistently meets the constraint eq.(2.15) to $O(\epsilon^2)$.

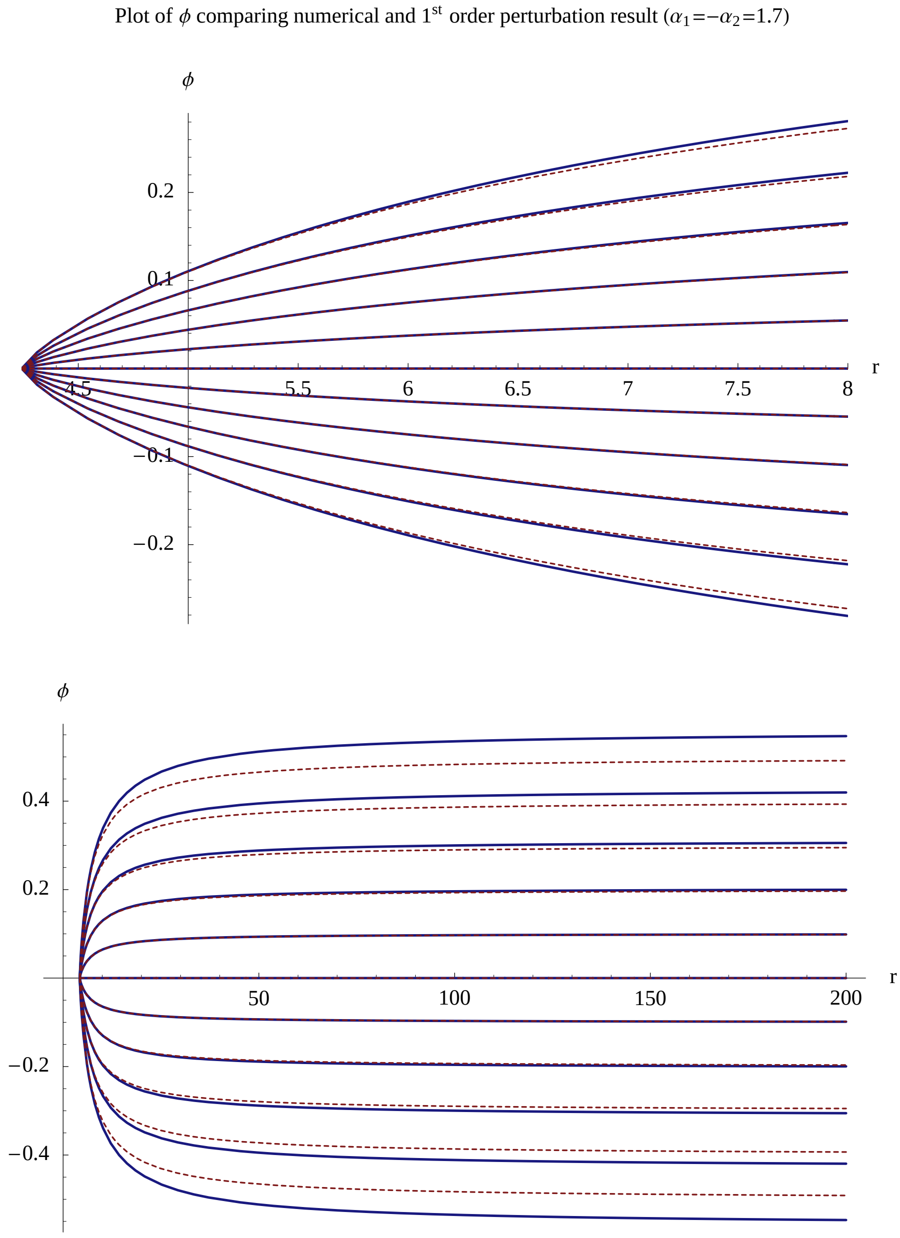
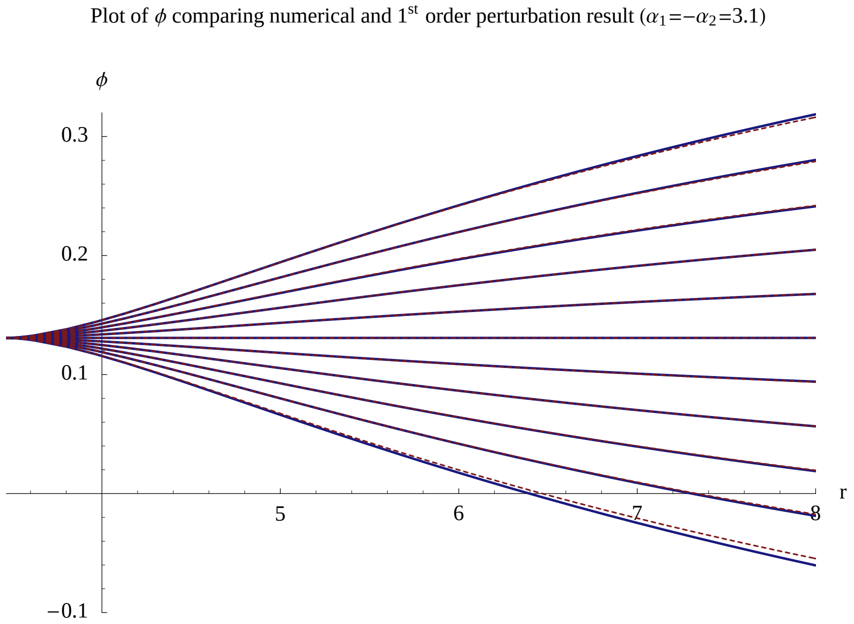


Figure 2.1: Comparison of numerical integration of ϕ with 1st order perturbation result. The upper graph is a close up of the lower one near the horizon. The perturbation result is denoted by a dashed line. We chose $\alpha_1, -\alpha_2 = 1.7$, $Q_1 = 3$, $Q_2 = 3$, $\delta r = 2.3 \times 10^{-8}$ and c_1 in the range $[-\frac{1}{2}, \frac{1}{2}]$
. $\phi_0 = 0$.



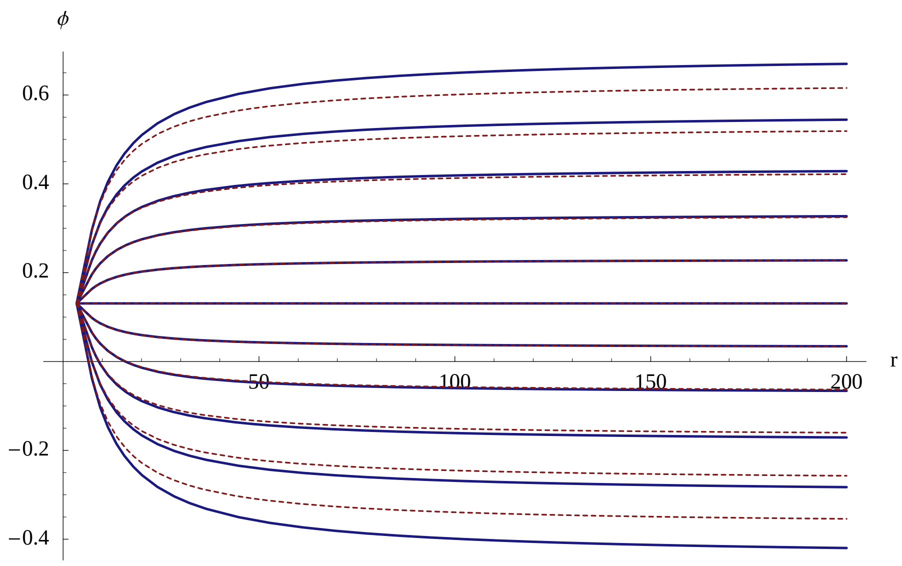


Figure 2.2: Comparison of numerical integration of ϕ with 1st order perturbation result. The upper graph is a close up of the lower one near the horizon. The perturbation result is denoted by a dashed line. We chose $\alpha_1, -\alpha_2 = 3.1$, $Q_1 = 2$, $Q_2 = 3$, $\delta r = 2.9 \times 10^{-8}$ and c_1 is in the range $[-\frac{1}{2}, \frac{1}{2}]$
. $\phi_0 = 0.13$.

where r_i is the point at which we start the integration. c_1 refers to the asymptotic value for the scalar, eq.(2.42).

In figs. (2.1,2.2) we compare the numerical and 1st order correction. The numerical and perturbation results are denoted by solid and dashed lines respectively. We see good agreement even for large r. As expected, as we increase the asymptotic value of ϕ, which was the small parameter in our perturbation series, the agreement decreases.

Note also that the resulting solutions turn out to be singularity free and asymptotically flat for a wide range of initial conditions. In this simple example there is only one critical point, eq.(2.33). This however does not guarantee that the attractor mechanism works. It could have been for example that as the asymptotic value of the scalar becomes significantly different from the attractor value no double-zero horizon black hole is allowed and instead one obtains a singularity. We have found no evidence for this. Instead, at least for the range of asymptotic values for the scalars we scanned in the numerical work, we find that the attractor mechanism works with attractor value, eq.(2.33).

It will be interesting to analyse this more completely, extending this work to cases where the effective potential is more complicated and several critical points are allowed. This should lead to multiple basins of attraction as has already been discussed in the supersymmetric context in e.g., [15, 16].

2.3 Exact Solutions

In certain cases the equation of motion can be solved exactly [17]. In this section, we shall look at some solvable cases and confirm that the extremal solutions display attractor behaviour. In particular, we shall work in 4 dimensions with one scalar and two gauge fields, taking V_{eff} to be given by eq.(2.32),

$$V_{eff} = e^{\alpha_1 \phi}(Q_1)^2 + e^{\alpha_2 \phi}(Q_2)^2. \tag{2.68}$$

We find that at the horizon the scalar field relaxes to the attractor value (2.33)

$$e^{(\alpha_1 - \alpha_2)\phi_0} = -\frac{\alpha_2 Q_2^2}{\alpha_1 Q_1^2} \tag{2.69}$$

which is the critical point of V_{eff} and independent of the asymptotic value, ϕ_∞. Furthermore, the horizon area is also independent of ϕ_∞ and, as predicted in section 2.1.2, it is proportional to the effective potential evaluated at the attractor point. It is given by

$$\begin{aligned} \text{Area} &= 4\pi b_H^2 = 4\pi V_{eff}(\phi_0) && (2.70) \\ &= 4\pi\spadesuit(Q_1)^{2\frac{-\alpha_2}{\alpha_1-\alpha_2}}(Q_2)^{2\frac{\alpha_1}{\alpha_1-\alpha_2}} && (2.71) \end{aligned}$$

where

$$\spadesuit = \left(-\tfrac{\alpha_2}{\alpha_1}\right)^{\frac{\alpha_1}{\alpha_1-\alpha_2}} + \left(-\tfrac{\alpha_2}{\alpha_1}\right)^{\frac{-\alpha_2}{\alpha_1-\alpha_2}} \tag{2.72}$$

is a numerical factor. It is worth noting that when $\alpha_1 = -\alpha_2$, one just has

$$\tfrac{1}{4}\text{Area} = 2\pi|Q_1 Q_2| \tag{2.73}$$

Interestingly, the solvable cases we know correspond to $\gamma = 1, 2, 3$ where γ is given by (2.43). The known solutions for $\gamma = 1, 2$ are discussed in [17] and references therein (although they fixed $\phi_\infty = 0$). We found a solution for $\gamma = 3$ and it appears as though one can find exact solutions as long as γ is a positive integer. Details of how these solutions are obtained can be found in the references and section 2.10.

For the cases we consider, the extremal solutions can be written in the following form

$$\begin{aligned} e^{(\alpha_1-\alpha_2)\phi} &= \left(-\frac{\alpha_2}{\alpha_1}\right)\left(\frac{Q_2}{Q_1}\right)^2\left(\frac{f_2}{f_1}\right)^{-\frac{1}{2}\alpha_1\alpha_2} && (2.74) \\ b^2 &= \spadesuit\left((Q_1 f_1)^{-\alpha_2}(Q_2 f_2)^{\alpha_1}\right)^{\frac{2}{\alpha_1-\alpha_2}} && (2.75) \\ a^2 &= \rho^2/b^2 && (2.76) \end{aligned}$$

where $\rho = r - r_H$ and the f_i are polynomials in ρ to some fractional power. In general the f_i depend on ϕ_∞ but they have the property

$$f_i|_{\text{Horizon}} = 1. \tag{2.77}$$

Substituting (2.77) into (2.74,2.75), one sees that that at the horizon the scalar field takes on the attractor value (2.69) and the horizon area is given by (2.71).

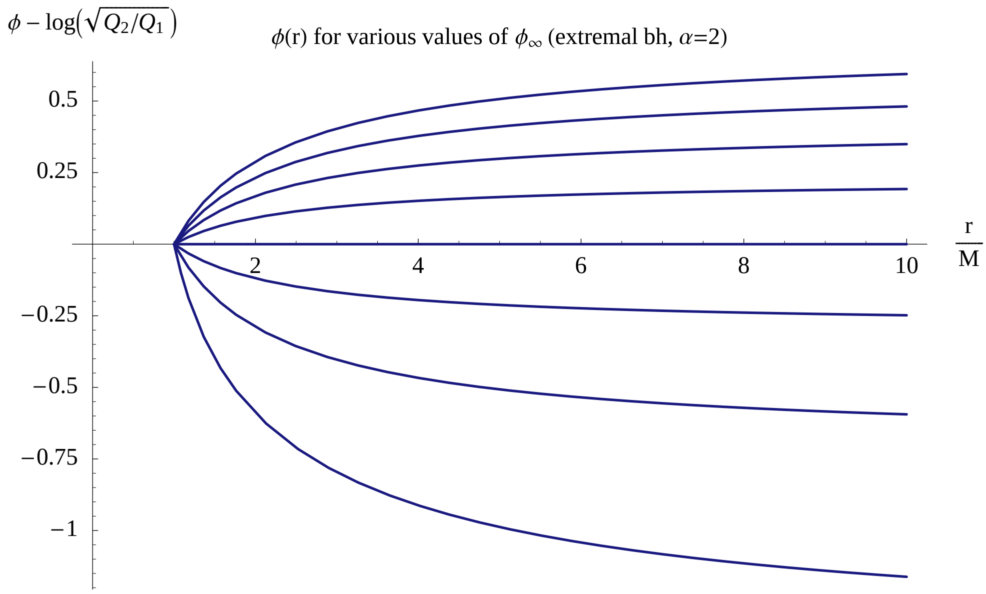


Figure 2.3: Attractor behaviour for the case $\gamma = 1$; $\alpha_1, -\alpha_2 = 2$

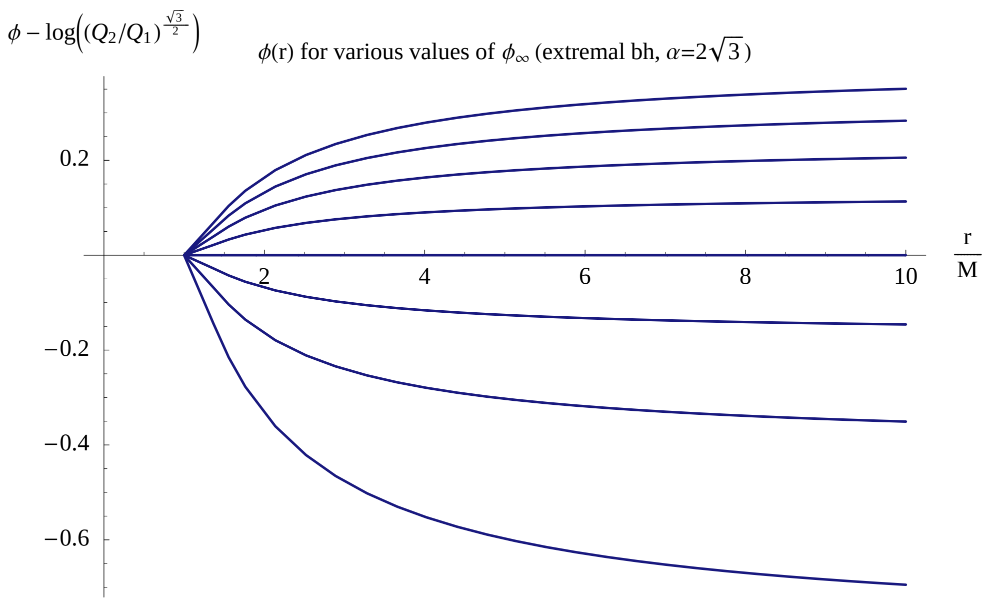


Figure 2.4: Attractor behaviour for the case $\gamma = 2$; $\alpha_1, -\alpha_2 = 2\sqrt{3}$

Notice that, when $\alpha = |\alpha_i|$, (2.74,2.75) simplify to

$$e^{\alpha\phi} = \frac{|Q_2|}{|Q_1|}\left(\frac{f_2}{f_1}\right)^{\frac{1}{4}\alpha^2} \tag{2.78}$$

$$b^2 = 2|Q_1||Q_2|\,(f_1 f_2) \tag{2.79}$$

2.3.1 Explicit Form of the f_i

In this section we present the form of the functions f_i mainly to show that, although they depend on ϕ_∞ in a non trivial way, they all satisfy (2.77) which ensures that the attractor mechanism works. It is convenient to define

$$\bar{Q}_i^2 = e^{\alpha_i \phi_\infty} Q_i^2 \qquad \text{(no summation)} \tag{2.80}$$

which are the effective $U(1)$ charges as seen by an asymptotic observer. For the simplest case, $\gamma = 1$, we have

$$f_i = 1 + \left(\bar{Q}_i^{-1}|\alpha_i|(4+\alpha_i^2)^{-\frac{1}{2}}\right)\rho \tag{2.81}$$

Taking $\gamma = 2$ and $\alpha_1 = -\alpha_2 = 2\sqrt{3}$ one finds

$$f_i = \left(1 + (\bar{Q}_1\bar{Q}_2)^{-\frac{2}{3}}(\bar{Q}_1^{\frac{2}{3}} + \bar{Q}_2^{\frac{2}{3}})^{\frac{1}{2}}\rho + \frac{1}{2}(\bar{Q}_i\bar{Q}_1\bar{Q}_2)^{-\frac{2}{3}}\rho^2\right)^{\frac{1}{2}} \tag{2.82}$$

Finally for $\gamma = 3$ and $\alpha_1 = 4$, $\alpha_2 = -6$ we have

$$f_1 = \left(1 - 6a_2\rho + 12a_2^2\rho^2 - 6a_0\rho^3\right)^{\frac{1}{3}} \tag{2.83}$$

$$f_2 = \left(1 - \tfrac{24}{3}a_2\rho + 24a_2\rho^2 - (48a_2^3 - 12a_0)\rho^3 + \left(48a_2^4 - 24a_0a_2\right)\rho^4\right)^{\frac{1}{4}} \tag{2.84}$$

where a_0 and a_2 are non-trivial functions of $\bar{Q}_i$. Further details are discussed in section 2.8 and section 2.10. The scalar field solutions for $\gamma = 1$ and 2 are illustrated in figs. 2.3 and 2.4 respectively.

2.3.2 Supersymmetry and the Exact Solutions

As mentioned above, the first two cases ($\gamma = 1, 2$) have been extensively studied in the literature.

The SUSY of the extremal $\alpha_1 = -\alpha_2 = 2$ solution is discussed in [18]. They show that it is supersymmetric in the context of $\mathcal{N} = 4$ SUGRA. It saturates the BPS bound and preserves $\frac{1}{4}$ of the supersymmetry - ie. it has $\mathcal{N} = 1$ SUSY. There are BPS black-holes in this context which carry only one $U(1)$ charge and preserve $\frac{1}{2}$ of the supersymmetry. The non-extremal blackholes are of course non-BPS.

On the other hand, the extremal $\alpha_1 = -\alpha_2 = 2\sqrt{3}$ blackhole is non-BPS [19]. It arises in the context of dimensionally reduced $5D$ Kaluza-Klein gravity [20] and is embeddable in $\mathcal{N} = 2$ SUGRA. There however are BPS black-holes in this context which carry only one $U(1)$ charge and once again preserve $\frac{1}{2}$ of the supersymmetry [21].

We have not investigated the supersymmetry of the $\gamma = 3$ solution, we expect that it is not a BPS solution in a supersymmetric theory.

2.4 General Higher Dimensional Analysis

2.4.1 The Set-Up

It is straightforward to generalise our results above to higher dimensions. We start with an action of the form,

$$S = \frac{1}{\kappa^2} \int d^d x \sqrt{-G} (R - 2(\partial\phi_i)^2 - f_{ab}(\phi_i) F^a F^b) \tag{2.85}$$

Here the field strengths, F_a are $(d-2)$ forms which are magnetic dual to 2-form fields.

We will be interested in solution which preserve a $SO(d-2)$ rotation symmetry. Assuming all quantities to be function of r, and taking the charges to be purely magnetic, the ansatz for the metric and gauge fields is [1]

$$ds^2 = -a(r)^2 dt^2 + a(r)^{-2} dr^2 + b(r)^2 d\Omega_{d-2}^2 \tag{2.86}$$

$$F^a = Q^a \sin^{d-3}\theta \sin^{d-4}\phi \cdots d\theta \wedge d\phi \wedge \cdots \tag{2.87}$$

[1] Black hole which carry both electric and magnetic charges do not have an $SO(d-2)$ symmetry for general d and we only consider the magnetically charged case here. The analogue of the two-form in 4 dimensions is the $d/2$ form in d dimensions. In this case one can turn on both electric and magnetic charges consistent with $SO(d/2)$ symmetry. We leave a discussion of this case and the more general case of p-forms in d dimensions for the future.

$$\tilde{F}^a = Q^a \sin^{d-3}\theta \sin^{d-4}\phi \cdots d\theta \wedge d\phi \wedge \cdots \tag{2.88}$$

The equation of motion for the scalars is

$$\partial_r(a^2 b^{d-2}\partial_r\phi_i) = \frac{(d-2)!\partial_i V_{eff}}{4b^{d-2}}. \tag{2.89}$$

Here V_{eff}, the effective potential for the scalars, is given by

$$V_{eff} = f_{ab}(\phi_i)Q^a Q^b. \tag{2.90}$$

From the $(R_{rr} - \frac{G^{tt}}{G^{rr}}R_{tt})$ component of the Einstein equation we get,

$$\sum_i (\phi_i')^2 = -\frac{(d-2)b''(r)}{2b(r)}. \tag{2.91}$$

The R_{rr} component gives the constraint,

$$-(d-2)\{(d-3) - ab'(2a'b + (d-3)ab')\} = 2\phi_i'^2 a^2 b^2 - \tfrac{(d-2)!}{b^{2(d-3)}}V_{eff}(\phi_i) \tag{2.92}$$

In the analysis below we will use eq.(2.89) to solve for the scalars and then eq.(2.91) to solve for b. The constraint eq.(2.92) will be used in solving for a along with one extra relation, $R_{tt} = (d-3)\frac{a^2}{b^2}R_{\theta\theta}$, as is explained in section 2.11. These equations (aside from the constraint) can be derived from a one-dimensional action

$$\begin{aligned} S \;=\; \tfrac{1}{\kappa^2}\int dr \quad \Big(&(d-3)(d-2)b^{d-4}(1+a^2b'^2) + (d-2)b^{d-3}(a^2)'b' \\ &-2a^2b^{d-2}(\partial_r\phi)^2 - \tfrac{(d-2)!}{b^{d-2}}V_{eff}\Big) \end{aligned} \tag{2.93}$$

As the analysis below shows if the potential has a critical point at $\phi_i = \phi_{i0}$ and all the eigenvalues of the second derivative matrix $\partial_{ij}V(\phi_{i0})$ are positive then the attractor mechanism works in higher dimensions as well.

2.4.2 Zeroth and First Order Analysis

Our starting point is the case where the scalars take asymptotic values equal to their critical value, $\phi_i = \phi_{i0}$. In this case it is consistent to set the scalars to be a constant, independent of r. The extremal Reissner Nordstrom black hole in d dimensions is then a solution of the resulting equations. This takes the form,

$$a_0(r) = \left(1 - \frac{r_H^{d-3}}{r^{d-3}}\right) \qquad b_0(r) = r \tag{2.94}$$

where r_H is the horizon radius. From the eq.(2.92) evaluated at r_H we obtain the relation,

$$r_H^{2(d-3)} = (d-4)! V_{eff}(\phi_{i0}) \tag{2.95}$$

Thus the area of the horizon and the entropy of the black hole are determined by the value of $V_{eff}(\phi_{i0})$, as in the four-dimensional case.

Now, let us set up the first order perturbation in the scalar fields,

$$\phi_i = \phi_{i0} + \epsilon \phi_{i1} \tag{2.96}$$

The first order correction satisfies,

$$\partial_r(a_0^2 b_0^{d-2} \partial_r \phi_{i1}) = \frac{\beta_i^2}{b^{d-2}} \phi_{i1} \tag{2.97}$$

where, β_i^2 is the eigenvalue of the second derivative matrix $\frac{(d-2)!}{4} \partial_{ij} V_{eff}(\phi_{i0})$ corresponding to the mode ϕ_i. This equation has two solutions. If $\beta_i^2 > 0$ one of these solutions blows up while the other is well defined and goes to zero at the horizon. This second solution is the one we will be interested in. It is given by,

$$\phi_{i1} = c_{i1}(1 - r_H^{d-3}/r^{d-3})^{\gamma_i} \tag{2.98}$$

where γ is given by

$$\gamma_i = \frac{1}{2}\left(-1 + \sqrt{1 + 4\beta_i^2 r_H^{6-2d}/(d-3)^2}\right) \tag{2.99}$$

2.4.2.1 Second order calculations (Effects of backreaction)

The first order perturbation in the scalars gives rise to a second order correction for the metric components, a, b. We write,

$$b(r) = b_0(r) + \epsilon^2 b_2(r) \tag{2.100}$$

$$a(r)^2 = a_0(r)^2 + \epsilon^2 a_2(r) \tag{2.101}$$

$$b(r)^2 = b_0(r)^2 + 2\epsilon^2 b_2(r) b_0(r) \tag{2.102}$$

where a_0, b_0 are given in eq.(2.94).

From (2.91) one can solve for the second order perturbation $b_2(r)$. For simplicity we consider the case of a single scalar field, ϕ. The solution is given by double-integration form,

$$\begin{aligned}
\partial_r^2 b_2(r) &= -\frac{2}{(d-2)} r(\partial_r\phi_1)^2 = -c_1' \frac{1}{r^{2d-5}} \left(\frac{r^{d-3} - r_H^{d-3}}{r^{d-3}}\right)^{2\gamma-2} \\
\Rightarrow b_2(r) &= d_1 r + d_2 - \frac{c_1' r}{2(d-3)(d-4)\gamma(2\gamma-1) r_H^{2d}} \times \\
&\Big(-(d-4)F\left[\tfrac{1}{3-d}, 1-2\gamma, \tfrac{d-4}{d-3}; (\tfrac{r_H}{r})^{d-3}\right] \\
&+(2\gamma-1)\left(\tfrac{r_H}{r}\right)^{d-3} F\left[\tfrac{d-4}{d-3}, 1-2\gamma, \tfrac{2d-7}{d-3}; (\tfrac{r_H}{r})^{d-3}\right]\Big), \quad (2.103)
\end{aligned}$$

where $c_1' \equiv 2(d-3)^2 c_1^2 \gamma^2 r_H^d/(d-2)$, a positive definite constant, and F is Gauss's Hypergeometric function. More generally, for several scalar fields, b_2 is obtained by summing over the contributions from each scalar field. The integration constants d_1, d_2, in eq.(2.103), can be fixed by coordinate transformations and requiring a double-zero horizon solution. We will choose coordinate so that the horizon is at $r = r_H$, then as we will see shortly the extremality condition requires both d_1, d_2 to vanish. As $r \to r_H$ we have from eq.(2.103) that

$$b_2(r) \propto -\left(\frac{r^{d-3} - r_H^{d-3}}{r^{d-3}}\right)^{2\gamma} \quad (2.104)$$

Since $\gamma > 0$, we see that b_2 vanishes at the horizon and thus the area and the entropy are uncorrected to second order. At large r, $b_2(r) \propto \mathcal{O}(r) + \mathcal{O}(1) + \mathcal{O}(r^{7-2d})$ so asymptotic behaviour is consistent with asymptotic flatness of the solution.

The analysis for a_2 is discussed in more detail in section 2.11. In the vicinity of the horizon one finds that there is one non-singular solution which goes like, $a_2(r) \to C(r - r_H)^{(2\gamma+2)}$. This solution smoothly extends to $r \to \infty$ and asymptotically, as $r \to \infty$, goes to a constant which is consistent with asymptotic flatness.

Thus we see that the backreaction of the metric is finite and well behaved. A double-zero horizon black hole continues to exist to second order in perturbation theory. It is asymptotically flat. The scalars in this solution at the horizon take their attractor values irrespective of their values at infinity..

Finally, the analysis in principle can be extended to higher orders. Unlike four dimensions though an explicit solution for the higher order perturbations is not possible and we will not present such an higher order analysis here.

We end with Fig 5. which illustrates the attractor behaviour in asymptotically flat $4+1$ dimensional space. This figure has been obtained for the example, eq.(2.31, 2.32). The parameter δr is defined in eq.(2.67).

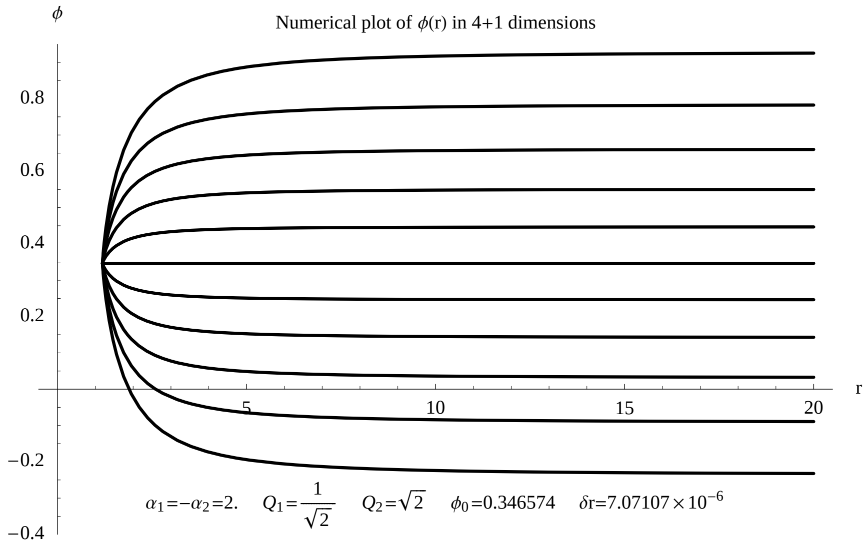


Figure 2.5: Numerical plot of $\phi(r)$ with $\alpha_1 = -\alpha_2 = 2$ for the an extremal black hole in $4+1$ dimensions displaying attractor behaviour.

2.5 Attractor in AdS_4

Next we turn to the case of Anti-de Sitter space in four dimensions. Our analysis will be completely analogous to the discussion above for the four and higher dimensional case and so we can afford to be somewhat brief below.

The action in 4-dim. has the form

$$S = \frac{1}{\kappa^2}\int d^4x\sqrt{-G}(R - 2\Lambda - 2(\partial\phi_i)^2 - f_{ab}(\phi_i)F^aF^b - \tfrac{1}{2}\tilde{f}_{ab}(\phi_i)\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^b_{\rho\sigma}) \tag{2.105}$$

where $\Lambda = -3/L^2$ is the cosmological constant. For simplicity we will discuss the case with only one scalar field here. The generalisation to many scalars is immediate and along the lines of the discussion for asymptotically flat four-dimensional case. Also we take the coefficient of the scalar kinetic energy term to be field independent.

For spherically symmetric solutions the metric takes the form, eq.(2.8). The field strengths are given by eq.(2.20). This gives rise to a one dimensional action

$$S = \frac{1}{\kappa^2}\int dr \left(2 - (a^2b^2)'' - 2a^2bb'' - 2a^2b^2(\partial_r\phi)^2 - 2\frac{V_{eff}}{b^2} + \frac{3b^2}{L^2}\right), \tag{2.106}$$

where V_{eff} is given by eq.(2.21). The equations of motion, which can be derived either from eq.(2.106) or directly from the action, eq.(2.105) are now given by,

$$\partial_r(a^2b^2\partial_r\phi) = \frac{\partial_\phi V_{eff}(\phi)}{2b^2} \tag{2.107}$$

$$\frac{b''}{b} = -(\partial_r\phi)^2, \tag{2.108}$$

which are unchanged from the flat four-dimensional case, and,

$$(a^2(r)b^2(r))'' = 2(1 - 2\Lambda b^2), \tag{2.109}$$

$$-1 + a^2b'^2 + \frac{a^{2'}b^{2'}}{2} = \frac{-1}{b^2}(V_{eff}(\phi)) + a^2b^2(\partial_r\phi)^2 + \frac{3b^2}{L^2}, \tag{2.110}$$

where the last equation is the first order constraint.

2.5.1 Zeroth and First Order Analysis for V

The zeroth order solution is obtained by taking the asymptotic values of the scalar field to be its critical values, ϕ_0 such that $\partial_i V_{eff}(\phi_0) = 0$.

The resulting metric is now the extremal Reissner Nordstrom black hole in AdS space, [22], given by,

$$a_0(r)^2 = \frac{(r - r_H)^2(L^2 + 3r_H^2 + 2r_H r + r^2)}{L^2r^2} \tag{2.111}$$

$$b_0(r) = r \tag{2.112}$$

The horizon radius r_H is given by evaluating the constraint eq.(2.110) at the horizon,

$$\frac{(L^2 r_H^2 + 2r_H^4)}{L^2} = V_{eff}(\phi_0). \tag{2.113}$$

The first order perturbation for the scalar satisfies the equation,

$$\partial_r(a_0^2 b_0^2 \partial_r \phi_1) = \frac{\beta^2}{b^2}\phi_1 \tag{2.114}$$

where,

$$\beta^2 = \frac{1}{2}\partial_\phi^2 V_{eff}(\phi_0). \tag{2.115}$$

This is difficult to solve explicitly.

In the vicinity of the horizon the two solutions are given by

$$\phi_1 = C_\pm (r - r_H)^{t_\pm} \tag{2.116}$$

If $V''_{eff}(\phi_0) > 0$ one of the two solutions vanishes at the horizon. We are interested in this solution. It corresponds to the choice,

$$\phi_1 = C(r - r_H)^\gamma, \tag{2.117}$$

where,

$$\gamma = \frac{\sqrt{1 + \frac{4\beta^2}{\delta r_H^2}} - 1}{2}, \tag{2.118}$$

and, $\delta = \frac{(L^2 + 6r_H^2)}{L^2}$. As discussed in the section 2.12 this solution behaves at $r \to \infty$ as $\phi_1 \to C_1 + C_2/r^3$. Also, all other values of r, besides the horizon and ∞, are ordinary points of the second order equation eq.(2.114). All this establishes that there is one well-behaved solution for the first order scalar perturbation. In the vicinity of the horizon it takes the form eq.(2.116) with eq.(2.118), and vanishes at the horizon. It is non-singular everywhere between the horizon and infinity and it goes to a constant asymptotically at $r \to \infty$.

We consider metric corrections next. These arise at second order. We define the second order perturbations as in eq.(2.45). The equation for b_2 from the second order terms in eq.(2.108) takes the form,

$$b_2'' = -r(\phi_1'(r))^2, \tag{2.119}$$

and can be solved to give,

$$b_2(r) = -\int_{r_H}\int_{r_H}[r(\phi_1'(r))^2] \tag{2.120}$$

We fix the integration constants by taking take the lower limit of both integrals to be the horizon. We will see that this choice gives rise to an double-zero horizon solution. Since ϕ_1 is well behaved for all $r_H \le r \le \infty$ the integrand above is well behaved as well. Using eq.(2.116) we find that in the near horizon region

$$b_2 \sim (r - r_H)^{(2\gamma)} \tag{2.121}$$

At $r \to \infty$ using the fact that $\phi_1 \to C_1 + C_2/r^3$ we find

$$b_2 \sim D_1 r + D_2 + D_3/r^6. \tag{2.122}$$

This is consistent with an asymptotically AdS solution.

Finally we turn to a_2. As we show in section 2.12 a solution can be found for a_2 with the following properties. In the vicinity of the horizon it goes like,

$$a_2 \propto (r - r_H)^{(2\gamma+2)}, \tag{2.123}$$

and vanishes faster than a double-zero. As $r \to \infty$, $a_2 \to d_1 r$ and grows more slowly than a_0^2. And for $r_H < r < \infty$ it is well-behaved and non-singular.

This establishes that after including the backreaction of the metric we have a non-singular, double-zero horizon black hole which is asymptotically AdS. The scalar takes a fixed value at the horizon of the black hole and the entropy of the black hole is unchanged as the asymptotic value of the scalar is varied.

Let us end with two remarks. In the AdS case one can hope that there is a dual description for the attractor phenomenon. Since the asymptotic value of the scalar is changing we are turning on a operator in the dual theory with a varying value for the coupling constant. The fact that the entropy, for fixed charge, does not change means that the number of ground states in the resulting family of dual theories is the same. This would be worth understanding in the dual description better. Finally, we expect this analysis to generalise in a straightforward manner to the AdS space in higher dimensions as well.

Fig. 6 illustrates the attractor mechanism in asymptotically AdS_4 space. This Figure is for the example, eq.(2.31, 2.32). The cosmological constant is taken to be, $\Lambda = -2.91723$, in $\kappa = 1$ units.

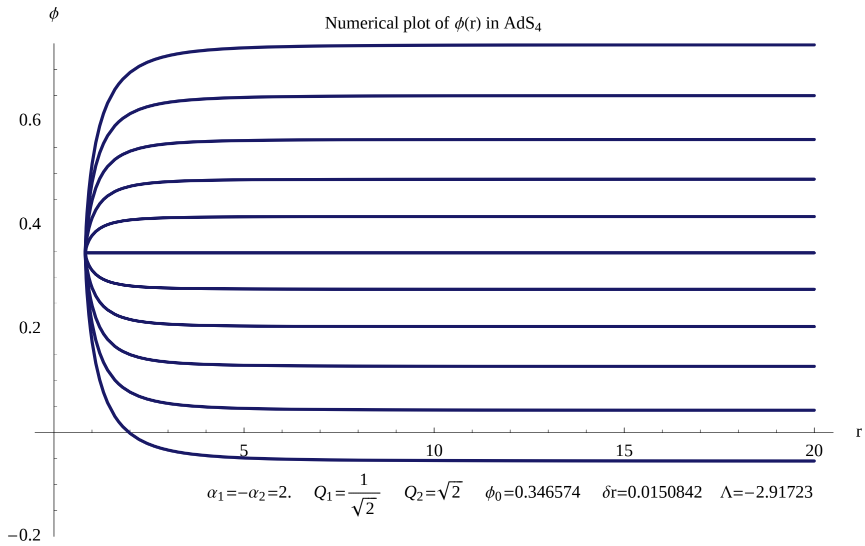


Figure 2.6: Numerical plot of $\phi(r)$ with $\alpha_1 = -\alpha_2 = 2$ for the an extremal black hole in AdS_4 displaying attractor behaviour.

2.6 Additional Comments

The theories we considered in the discussion of asymptotically flat space-times and AdS spacetimes have no potential for the scalars. We comment on this further here.

Let us consider a theory with $\mathcal{N} = 1$ supersymmetry containing chiral superfields whose lowest component scalars are,

$$S_i = \phi_i + ia_i \tag{2.124}$$

We take these scalars to be uncharged under the gauge symmetries. These can be coupled to the superfields W^a_α by a coupling

$$L_{gaugekinetic} = \int d^2\theta f_{ab}(S_i) W^a_\alpha W^b_\alpha \tag{2.125}$$

Such a coupling reproduces the gauge kinetic energy terms in and eq.(2.105), eq.(2.106), (we now include both ϕ_i, a_i in the set of scalar fields which we denoted by ϕ_i in the previous sections).

An additional potential for the scalars would arise due to F-term contributions from a superpotential. If the superpotential is absent we get the required feature of no potential for these scalar. Setting the superpotential to be zero is at least technically natural due to its non-renormalisability.

In a theory with no supersymmetry there is no natural way to suppress a potential for the scalars and it would arise due to quantum effects even if it is absent at tree-level. In this case we have no good argument for not including a potential for the scalar and our analysis is more in the nature of a mathematical investigation.

The absence of a potential is important also for avoiding no-hair theorems which often forbid any scalar fields from being excited in black hole backgrounds [**?**]. In the presence of a mass m in asymptotically flat four dimensional space the two solutions for first order perturbation at asymptotic infinity go like,

$$\phi \sim C_1 e^{mr}/r, \phi \sim C_2 e^{-mr}/r. \tag{2.126}$$

We see that one of the solutions blows up as $r \to \infty$. Since one solution to the equation of motion also blows up in the vicinity of the horizon, as discussed in section 2, there will generically be no non-singular solution in first order perturbation theory. This argument is a simple-minded way of understanding the absence of scalar hair for extremal black holes under discussion here. In the absence of mass terms, as was discussed in section 2, the two solutions at asymptotic infinity go like $\phi \sim$ const and $\phi \sim 1/r$ respectively and are both acceptable. This is why one can turn on scalar hair. The possibility of scalar hair for a massless scalar is of course well known. See [23], [17], for some early examples of solutions with scalar hair, [24, 25, 26, 27], for theorems on uniqueness in the presence of such hair, and [28] for a discussion of resulting thermodynamics.

In asymptotic AdS space the analysis is different. Now the $(mass)^2$ for scalars can be negative as long as it is bigger than the BF bound. In this case both solutions at asymptotic infinity decay and are acceptable. Thus, as for the massless case, it should be possible to turn on scalar fields even in the presence of these mass terms and study the resulting black holes solutions. Unfortunately, the resulting equations are quite intractable. For small $(mass)^2$ we expect the attractor mechanism to continue to work.

If the $(mass)^2$ is positive one of the solutions in the asymptotic region blows up and the situation is analogous to the case of a massive scalar in flat space discussed above. In this case one could work with AdS space which is cut off at large r (in the infrared) and study the attractor phenomenon. Alternatively, after incorporating back reaction, one might get a non-singular geometry which departs from AdS in the IR and then analyse black holes in this resulting geometry. In the dual field theory a positive $(mass)^2$ corresponds to an irrelevant operator. The growing mode in the bulk is the non-normalisable one and corresponds to turning on a operator in the dual theory which grows in the UV. Cutting off AdS space means working with a cut-off effective theory. Incorporating the back-reaction means finding a UV completion of the cut-off theory. And the attractor mechanism means that the number of ground states at fixed charge is the same regardless of the value of the coupling constant for this operator.

2.7 Asymptotic de Sitter Space

In de Sitter space the simplest way to obtain a double-zero horizon is to take a Schwarzschild black hole and adjust the mass so that the de Sitter horizon and the Schwarzschild horizon coincide. The resulting black hole is the extreme Schwarzschild-de Sitter spacetime [29]. We will analyse the attractor behaviour of this black hole below. The analysis simplifies in 5-dimensions and we will consider that case, a similar analysis can be carried out in other dimensions as well. Since no charges are needed we set all the gauge fields to zero and work only with a theory of gravity and scalars. Of course by turning on gauge charges one can get other double-zero horizon black holes in dS, their analysis is left for the future.

We start with the action of the form,

$$S = \frac{1}{\kappa^2}\int d^5x\sqrt{-G}(R - 2(\partial\phi)^2 - V(\phi)) \tag{2.127}$$

Notice that the action now includes a potential for the scalar, $V(\phi)$, it will play the role of V_{eff} in our discussion of asymptotic flat space and AdS space. The required conditions for an attractor in the dS case will be stated in terms of V. A concrete example of a potential meeting the required conditions will be given at the end of the section. For simplicity we have taken only one scalar, the analysis is easily extended for additional scalars.

The first condition on V is that it has a critical point, $V'(\phi_0) = 0$. We will also require that $V(\phi_0) > 0$. Now if the asymptotic value of the scalar is equal to its critical value, ϕ_0, we can consistently set it to this value for all times t. The resulting equations have a extremal black hole solution mentioned above. This takes the form

$$ds^2 = -\frac{t^2}{(t^2/L - L/2)^2}dt^2 + \frac{(t^2/L - L/2)^2}{t^2}dr^2 + t^2 d\Omega_3^2 \tag{2.128}$$

Notice that it is explicitly time dependent. L is a length related to $V(\phi_0)$ by , $V(\phi_0) = \frac{20}{L^2}$. And $t = \pm\frac{L}{\sqrt{2}}$ is the location of the double-zero horizon. A suitable near-horizon limit of this geometry is called the Nariai solution, [30].

2.7.1 Perturbation Theory

Starting from this solution we vary the asymptotic value of the scalar. We take the boundary at $t \to -\infty$ as the initial data slice and investigate what happens when the scalar takes a value different from ϕ_0 as $t \to -\infty$. Our discussion will involve part of the space-time, covered by the coordinates in eq.(2.128), with $-\infty \leq t \leq t_H = -\frac{L}{\sqrt{2}}$. We carry out the analysis in perturbation theory below.

Define the first order perturbation for the scalar by,

$$\phi = \phi_0 + \epsilon\phi_1$$

This satisfies the equation,

$$\partial_t(a_0^2 b_0^3 \partial_t \phi_1) = \frac{b^3}{4}V''(\phi_0)\phi_1 \tag{2.129}$$

where $a_0 = \frac{(t^2/L - L/2)}{t}$, $b_0 = t$. This equation is difficult to solve in general.

In the vicinity of the horizon $t = t_H$, we have two solutions which go like,

$$\phi_1 = C_\pm (t - t_H)^{\frac{-1+\sqrt{1+\kappa^2}}{2}} \tag{2.130}$$

where

$$\kappa^2 = -\frac{1}{4}V''(\phi_0) \tag{2.131}$$

We see that one of the two solutions in eq.(2.130) is non-divergent and in fact vanishes at the horizon if

$$V''(\phi_0) < 0. \tag{2.132}$$

We will henceforth assume that the potential meets this condition. Notice this condition has a sign opposite to what was obtained for the asymptotically flat or AdS cases. This reversal of sign is due to the exchange of space and time in the dS case.

In the vicinity of $t \to -\infty$ there are two solutions to eq.(2.129) which go like,

$$\phi_1 = \tilde{C}_{\pm}|t|^{p_{\pm}} \tag{2.133}$$

where

$$p_{\pm} = 2(-1 \pm \sqrt{1 + \kappa^2/4}). \tag{2.134}$$

If the potential meets the condition, eq.(2.132) then $\kappa^2 > 0$ and we see that one of the modes blows up at $t \to -\infty$.

2.7.2 Some Speculative Remarks

In view of the diverging mode at large $|t|$ one needs to work with a cutoff version of dS space [1]. With such a cutoff at large negative t we see that there is a one parameter family of solutions in which the scalar takes a fixed value at the horizon. The one parameter family is obtained by starting with the appropriate linear combination of the two solutions at $t \to -\infty$ which match to the well behaved solution in the vicinity of the horizon. While we will not discuss the metric perturbations and scalar perturbations at second order these too have a non-singular solution which preserves the double-zero nature of the horizon. The metric perturbations also grow at the boundary in response to the growing scalar mode and again the cut-off is necessary to regulate this growth. This suggests that in the cut-off version of dS space one has an attractor phenomenon. Whether such a cut-off makes physical sense and can be implemented appropriately are question we will not explore further here.

[1] This is related to some comments made in the previous section in the positive $(mass)^2$ case in AdS space.

One intriguing possibility is that quantum effects implement such a cut-off and cure the infra-red divergence. The condition on the potential eq.(2.132) means that the scalar has a negative $(mass)^2$ and is tachyonic. In dS space we know that a tachyonic scalar can have its behaviour drastically altered due to quantum effects if it has a $(mass)^2 < H^2$ where H is the Hubble scale of dS space. This can certainly be arranged consistent with the other conditions on the potential as we will see below. In this case the tachyon can be prevented from "falling down" at large $|t|$ due to quantum effects and the infrared divergences can be arrested by the finite temperature fluctuations of dS space. It is unclear though if any version of of the attractor phenomenon survives once these quantum effects became important.

We end by discussing one example of a potential which meets the various conditions imposed above. Consider a potential for the scalar,

$$V = \Lambda_1 e^{\alpha_1 \phi} + \Lambda_2 e^{\alpha_2 \phi}. \tag{2.135}$$

We require that it has a critical point at $\phi = \phi_0$ and that the value of the potential at the critical point is positive. The critical point for the potential eq.(2.135) is at,

$$e^{\phi_0} = -\left(\frac{\alpha_2 \Lambda_2}{\alpha_1 \Lambda_1}\right)^{\frac{1}{\alpha_1 - \alpha_2}} \tag{2.136}$$

Requiring that $V(\phi_0) > 0$ tells us that

$$V(\phi_0) = \Lambda_2 e^{\alpha_2 \phi_0} \left(1 - \frac{\alpha_2}{\alpha_1}\right) > 0 \tag{2.137}$$

Finally we need that $V''(\phi_0) < 0$ this leads to the condition,

$$V''(\phi_0) = \Lambda_2 e^{\alpha_2 \phi_0} \alpha_2 (\alpha_2 - \alpha_1) < 0 \tag{2.138}$$

These conditions can all be met by taking both $\alpha_1, \alpha_2 > 0$, $\alpha_2 < \alpha_1$, $\Lambda_2 > 0$ and $\Lambda_1 < 0$. In addition if $\alpha_2 \alpha_1 \gg 1$ the resulting $-(mass)^2 \gg H^2$.

2.8 Non-Extremal = Unattractive

We end this chapter by examining the case of an non-extremal black hole which has a single-zero horizon. As we will see there is no attractor mechanism in this case. Thus the existence of a double-zero horizon is crucial for the attractor mechanism to work.

Our starting point is the four dimensional theory considered in section 2 with action eq.(2.17). For simplicity we consider only one scalar field. We again start by consistently setting this scalar equal to its critical value, ϕ_0, for all values of r, but now do not consider the extremal Reissner Nordstrom black hole. Instead we consider the non-extremal black hole which also solves the resulting equations. This is given by a metric of the form, eq.(2.8), with

$$a^2(r) = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right), \qquad b(r) = r \tag{2.139}$$

where $r_\pm$ are not equal. We take $r_+ > r_-$ so that r_+ is the outer horizon which will be of interest to us.

The first order perturbation of the scalar field satisfies the equation,

$$\partial_r(a^2 b^2 \partial_r \phi_i) = \frac{V''_{eff}(\phi_0)}{4b^2}\phi_1 \tag{2.140}$$

In the vicinity of the horizon $r = r_+$ this takes the form,

$$\partial_y(y\partial_y\phi_1) = \alpha\phi_1 \tag{2.141}$$

where α is a constant dependent on $V''(\phi_0), r_+, r_-$, and $y \equiv r - r_+$.

This equation has one non-singular solution which goes like,

$$\phi_1 = C_0 + C_1 y + \cdots \tag{2.142}$$

where the ellipses indicate higher terms in the power series expansion of ϕ_1 around $y = 0$. The coefficients $C_1, C_2, \cdots$ are all determined in terms of C_0 which can take any value. Thus we see that unlike the case of the double-horizon extremal black hole, here the solution which is well-behaved in the vicinity of the horizon does not vanish.

Asymptotically, as $r \to \infty$ both solutions to eq.(2.140) are well defined and go like $1/r, constant$ respectively. It is then straightforward to see that one can choose an appropriate linear combination of the two solutions at infinity and match to the solution, eq.(2.142) in the vicinity of the horizon. The important difference here is that the value of the constant C_0 in eq.(2.142) depends on the asymptotic values of the scalar at infinity and therefore the value of ϕ does not go to a fixed value at the horizon. The metric perturbations sourced by the scalar perturbation can also be analysed and are non-singular. In summary, we find a family of non-singular black hole solutions for which the scalar field takes varying values at infinity. The crucial difference is that here the scalar takes a value at the horizon which depends on its value at asymptotic infinity. The entropy and mass for these solutions also depends on the asymptotic value of the scalar [1].

It is also worth examining this issue in a non-extremal black holes for an exactly solvable case.

If we consider the case $|\alpha_i| = 2$, section 2.3, the non-extremal solution takes on a relatively simple form. It can be written[18]

$$\begin{aligned} \exp(2\phi) &= e^{2\phi_\infty}\frac{(r+\Sigma)}{(r-\Sigma)} \\ a^2 &= \frac{(r-r_+)(r-r_-)}{(r^2-\Sigma^2)} \\ b^2 &= (r^2-\Sigma^2) \end{aligned} \tag{2.143}$$

where[2]

$$r_\pm = M \pm r_0 \qquad r_0 = \sqrt{M^2+\Sigma^2-\bar{Q}_2^2-\bar{Q}_1^2}. \tag{2.144}$$

and the Hamiltonian constraint becomes

$$\Sigma^2 + M^2 - \bar{Q}_1^2 - \bar{Q}_2^2 = \frac{1}{4}(r_+ - r_-)^2. \tag{2.145}$$

The scalar charge, Σ, defined by $\phi \sim \phi_\infty + \frac{\Sigma}{r}$, is not an independent parameter. It is given by

$$\Sigma = \frac{\bar{Q}_2^2 - \bar{Q}_1^2}{2M}. \tag{2.146}$$

There are horizons at $r = r_\pm$, the curvature singularity occurs at $r = \Sigma$ and r_0 characterises the deviation from extremality. We see that the non-extremal solution does not display attractor behaviour.

Fig. 2.7 shows the behaviour of the scalar field [3] as we vary ϕ_∞ keeping M

[1] An intuitive argument was given in the introduction in support of the attractor mechanism. Namely, that the degeneracy of states cannot vary continuously. This argument only applies to the ground states. A non-extremal black hole corresponds to excited states. Changing the asymptotic values of the scalars also changes the total mass and hence the entropy in this case.

[2] The radial coordinate r in eq.(2.143) is related to our previous one by a constant shift.

[3] for $\alpha_1 = -\alpha_2 = 2$

and Q_i fixed. The location of the horizon as a function of r depends on ϕ_∞, eq.(2.144). The horizon as a function of ϕ_∞ is denoted by the dotted line. The plot is terminated at the horizon.

In contrast, for the extremal black hole,

$$M = \frac{|\bar{Q}_2| + |\bar{Q}_1|}{\sqrt{2}} \qquad \Sigma = \frac{|\bar{Q}_2| - |\bar{Q}_1|}{\sqrt{2}}, \tag{2.147}$$

so (2.143) gives

$$e^{2\phi_0} = e^{2\phi_\infty}\frac{M+\Sigma}{M-\Sigma} = \frac{|Q_2|}{|Q_1|}, \tag{2.148}$$

which is indeed the attractor value.

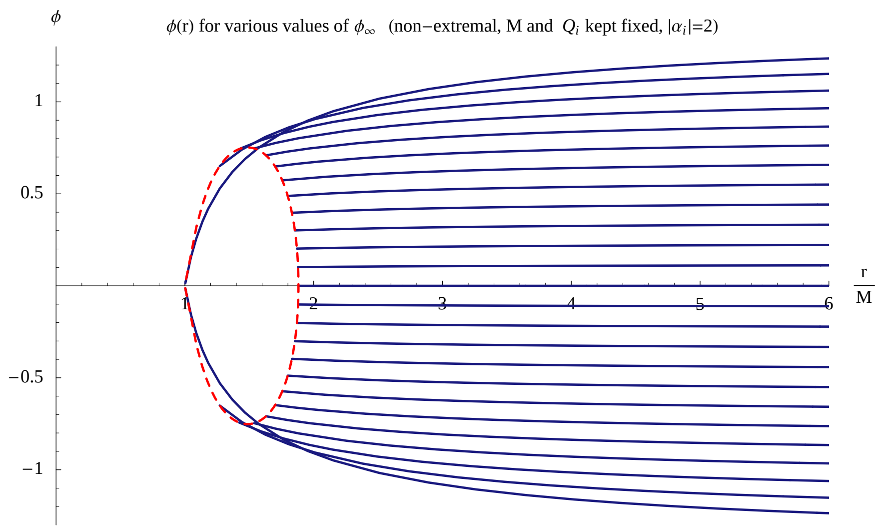


Figure 2.7: Plot $\phi(r)$ with $\alpha_1 = -\alpha_2 = 2$ for the non-extremal black hole with M, Q_i held fixed while varying ϕ_∞. The dotted line denotes the outer horizon at which we terminate the plot. It is clearly unattractive.

2.9 Perturbation Analysis

2.9.1 Mass

Here, we first calculate the mass of the extremal black hole discussed in section 2.2. From eq.(2.50), for large r, b_2 is given by,

$$b_2 = cr + d \tag{2.149}$$

where

$$\begin{aligned} c &= -\frac{c_1^2\gamma}{2(2\gamma-1)} && (2.150)\\ d &= \frac{r_H c_1^2\gamma^2}{(2\gamma-1)} && (2.151) \end{aligned}$$

Now, we can easily write down the expression for a_2 using eq.(2.48). We choose coordinate y as introduced in eq.(2.51) such that at large r,

$$\begin{aligned} r^2 + 2\epsilon^2(cr^2 + dr) &= y^2 && (2.152)\\ \frac{1}{r} &= \frac{1}{y}(1+\epsilon^2(c+\frac{d}{y})) && (2.153) \end{aligned}$$

We use the extremality condition (2.54) to find,

$$a(r) = \left(\frac{r - r_H}{y}\right) \tag{2.154}$$

Using, eq.(2.149, 2.154) one finds that asymptotically, as $r \to \infty$ the metric takes the form,

$$ds^2 = -\left(1 - \frac{2(r_H + \epsilon^2(cr_H + d))}{y}\right)d\tilde{t}^2 + \frac{1}{(1 - \frac{2(r_H+\epsilon^2(cr_H+d))}{y})}dy^2 + y^2 d\Omega^2 \tag{2.155}$$

where $\tilde{t}$ is obtained by rescaling t and $d\Omega^2$ denotes the metric of S^2. The mass M of the black hole is then given by the $1/y$ term in the g_{yy} component of the metric. This gives,

$$M = r_H + \epsilon^2 \frac{r_H c_1^2 \gamma}{2}. \tag{2.156}$$

2.9.2 Perturbation Series to All Orders

Next we go on to discuss the perturbation series to all orders, Using (2.55) for b and (2.56) for ϕ in eq.(2.14)and eq.(2.62), we get,

$$b_k'' = -\sum_{i=0}^{k}\sum_{j=0}^{k-i} b_i \phi_j' \phi_{k-i-j}' \tag{2.157}$$

$$\sum_{i+j\leqslant k} 2b_j b_{k-i-j}((r-r_H)^2\phi_i')' = Q_i^2 e^{\alpha_i\phi_0}\alpha_i \mathcal{V}_{ik} \tag{2.158}$$

where

$$\mathcal{V}_{ik} = \sum_{\{n_1,n_2\ldots n_k\}} \frac{\phi_1^{n_1}\phi_2^{n_2}\ldots\phi_k^{n_k}}{n_1!n_2!\ldots n_k!}\alpha_i^{n_1+n_2+\ldots+n_k}. \tag{2.159}$$

$$\tag{2.160}$$

After substituting our ansatz (2.58)and (2.59), the above equations give,

$$k(k\gamma-1)d_k = -\gamma\sum_{i+j<k} j(k-i-j)d_i c_j c_{k-i-j} \tag{2.161}$$

and

$$k(k\gamma+1)c_k + T_k = (\gamma+1)(c_k+S_k) \tag{2.162}$$

where S_k and T_k are given by

$$S_k = \sum_{\{n_1,n_2\ldots n_{k-1}\}} \frac{c_1^{n_1}c_2^{n_2}\ldots c_{k-1}^{n_{k-1}}}{n_1!n_2!\ldots n_{k-1}!}\left(\alpha_1^{\sum n_l-1}+\alpha_2^{\sum n_l-1}\right) \tag{2.163}$$

and

$$T_k = \sum_{\substack{j+l\leqslant k\\ l<k}} l(l\gamma+1)d_j d_{k-l-j}c_l. \tag{2.164}$$

Then solving for d_k and c_k gives

$$d_k = -\frac{\gamma}{k(k\gamma-1)}\sum_{i+j<k} j(k-i-j)d_i e_j e_{k-i-j} \tag{2.165}$$

$$c_k = \frac{(\gamma+1)S_k - T_k}{((k+1)\gamma+1)(k-1)} \tag{2.166}$$

Finally, e_k can be obtained using eq.(2.54), eq.(2.59). It can be verified that the ansatz, eq.(2.58, 2.59, 2.60) with the coefficients eq.(2.165, 2.166) also solves the constraint eq.(2.15).

2.10 Exact Analysis

Exact solutions can be found by writing the equations of motion as generalised Toda equations [31], which may, in certain special cases, be solved exactly [17] - we rederive this result in slightly different notation below. As noted in [32], in a marginally different context, the extremal solutions, are, in appropriate variables, polynomial solutions of the Toda equations. The polynomial solutions are much easier to find and are related to the functions f_i mentioned in section 2.3. For ease of comparison we occasionally use notation similar to [32].

2.10.1 New Variables

To recast the equations of motion into a generalised Toda equation we define the following new variables

$$u_1 = \phi \qquad u_2 = \log a \qquad z = \log ab \qquad \cdot = \partial_\tau = a^2 b^2 \partial_r \tag{2.167}$$

In terms of r, τ is given by

$$\tau = \int \frac{dr}{a^2 b^2} = \frac{1}{(r_+ - r_-)} \log\left(\frac{r - r_+}{r - r_-}\right) \tag{2.168}$$

where $r_\pm$ are the integration constants of (2.13). In general (2.13) implies

$$a^2 b^2 = (r - r_+)(r - r_-). \tag{2.169}$$

Notice that

$$\tau \to 0 \quad \text{as} \quad r \to \infty \tag{2.170}$$

$$\tau \to -\infty \quad \text{as} \quad r \to r_+ \tag{2.171}$$

When we have a double-zero horizon, $r_H = r_\pm$, τ takes the simple form

$$\tau^{-1} = -(r - r_H). \tag{2.172}$$

Since we are mainly interested in solutions with double-zero horizons, in what follows it will be convenient to work with a new radial coordinate, ρ, defined by

$$\rho = -\tau^{-1}. \tag{2.173}$$

which has the convenient property that $\rho_H = 0$.

2.10.2 Equivalent Toda System

In terms of these new variables the equations of motion become

$$\begin{aligned}
\ddot{u}_1 &= \tfrac{1}{2}\alpha_1 e^{2u_2+\alpha_1 u_1} Q_1^2 + \tfrac{1}{2}\alpha_2 e^{2u_2+\alpha_2 u_1} Q_2^2 && (2.174)\\
\ddot{u}_2 &= e^{2u_2+\alpha_1 u_1} Q_1^2 + e^{2u_2+\alpha_2 u_1} Q_2^2 && (2.175)\\
\ddot{z} &= e^{2z} && (2.176)
\end{aligned}$$

$$\dot{u_1}^2 + \dot{u_2}^2 - \dot{z}^2 + e^{2z} - e^{2u_2+\alpha_1 u_1} Q_1^2 - e^{2u_2+\alpha_2 u_1} Q_2^2 = 0 \tag{2.177}$$

(2.176) decouples from the other equations and is equivalent to (2.13). Finally making the coordinate change

$$X_i = n_{ij}^{-1} u_j + m_{ij}^{-1} \log\left((\alpha_1 - \alpha_2) Q_j^2\right) \tag{2.178}$$

where

$$n^{-1} = \begin{pmatrix} 2 & -\alpha_2 \\ -2 & \alpha_1 \end{pmatrix} \tag{2.179}$$

and

$$m_{ij} = \frac{1}{2\left(\alpha_1 - \alpha_2\right)} \left(4 + \alpha_i \alpha_j\right) \tag{2.180}$$

we obtain the generalised 2 body Toda equation

$$\ddot{X}_i = e^{m_{ij} X_j}, \tag{2.181}$$

together with

$$\sum_{ij} \left(\tfrac{1}{2}\dot{X}_i m_{ij} \dot{X}_j - e^{m_{ij} X_j}\right) = (\alpha_1 - \alpha_2)\mathcal{E} \tag{2.182}$$

where $\mathcal{E} = \frac{1}{4}(r_+ - r_-)^2$. After solving the above, the original fields will be given by

$$\begin{aligned}
e^{(\alpha_1-\alpha_2)\phi} &= \frac{Q_2^2}{Q_1^2} e^{\frac{1}{2}(\alpha_1 X_1 + \alpha_2 X_2)} && (2.183)\\
a^2 &= e^{\frac{2}{\alpha_1-\alpha_2}(X_1+X_2)} / \diamondsuit && (2.184)\\
b^2 &= (r - r_+)(r - r_-)/a^2 && (2.185)
\end{aligned}$$

where

$$\diamondsuit = (\alpha_1 - \alpha_2) Q_1^{2\frac{-\alpha_2}{\alpha_1-\alpha_2}} Q_2^{2\frac{\alpha_1}{\alpha_1-\alpha_2}} \tag{2.186}$$

2.10.3 Solutions

2.10.3.1 Case I: $\gamma = 1 \Leftrightarrow \alpha_1\alpha_2 = -4$

In this case, m_{ij} is diagonal

$$m = \mathrm{diag}(\alpha_1/2, -\alpha_2/2), \tag{2.187}$$

so the equations of motion decouple:

$$\ddot{X}_i = e^{\frac{|\alpha_i|}{2}X_i}. \tag{2.188}$$

(2.188) has solutions

$$X_i = \frac{2}{|\alpha_i|}\log\left(\frac{4c_i^2}{|\alpha_i|\sinh^2(c_i(\tau - d_i))}\right) \tag{2.189}$$

The integration constants are fixed by imposing asymptotic boundary conditions and requiring that the solution is finite at the horizon. Letting

$$F_i = \sinh(c_i\,(\tau - d_i)) \tag{2.190}$$

in terms of ϕ and a we get

$$\begin{aligned}
e^{(\alpha_1-\alpha_2)\phi} &= \frac{Q_2^2}{Q_1^2}e^{\frac{1}{2}(\alpha_1X_1+\alpha_2X_2)} \\
&= \left(-\frac{\alpha_2}{\alpha_1}\right)\left(\frac{Q_2F_2c_1}{Q_1F_1c_2}\right)^2 \\
a^2 &= e^{\frac{2}{\alpha_1-\alpha_2}(X_1+X_2)} \,/\, \diamondsuit \\
&= \left(\frac{c_1}{Q_1F_1}\right)^{2\frac{-\alpha_2}{\alpha_1-\alpha_2}}\left(\frac{c_2}{Q_2F_2}\right)^{2\frac{\alpha}{\alpha-\tilde{\alpha}}} \,/\, \spadesuit
\end{aligned} \tag{2.191}$$

As $r \to r_+$(ie. $\tau \to -\infty$) the scalar field goes like

$$e^{(\alpha_1-\alpha_2)\phi} \sim e^{2(c_1-c_2)\tau} \tag{2.192}$$

so we require

$$c := c_1 = c_2 \tag{2.193}$$

for a finite solution at the horizon. Also at the horizon

$$b^2 \sim (r - r_+)/a^2 \sim e^{((r_+-r_-)-2c)\tau} \tag{2.194}$$

which necessitates

$$(r_+ - r_-) = 2c \tag{2.195}$$

To find the extremal solutions we take the limit $c \to 0$ which gives

$$e^{(\alpha_1-\alpha_2)\phi} = \left(-\frac{\alpha_2}{\alpha_1}\right)\frac{(Q_2 f_2)^2}{(Q_1 f_1)^2} \tag{2.196}$$

$$b^2 = \spadesuit\, (Q_1 f_1)^{\frac{-2\alpha_2}{\alpha_1-\alpha_2}} (Q_2 p_2)^{\frac{2\alpha_1}{\alpha_1-\alpha_2}} \tag{2.197}$$

$$a^2 = \rho^2/b^2 \tag{2.198}$$

where

$$f_i = 1 + d_i\rho. \tag{2.199}$$

Requiring $\phi \to \phi_\infty$ and $a \to 1$ as $r \to \infty$ fixes

$$d_i = \bar{Q}_i^{-1}\sqrt{\frac{|\alpha_i|}{\alpha_1-\alpha_2}} \tag{2.200}$$

where as before

$$\bar{Q}_i^2 = e^{\alpha_i\phi_\infty} Q_i^2 \qquad \text{(no summation)}. \tag{2.201}$$

For comparison with the non-extremal solution in this case see section 2.8.

2.10.3.2 Case II: $\gamma = 2$ and $\alpha_1 = -\alpha_2 = 2\sqrt{3}$

In this case, m_{ij} becomes

$$m = \begin{pmatrix} \frac{2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} \end{pmatrix}. \tag{2.202}$$

It is convenient to use the coordinates

$$q_i = \frac{1}{\sqrt{3}} X_i - \sqrt{3}\log\sqrt{3} \tag{2.203}$$

so the equations of motion are the two particle Toda equations

$$\ddot{q}_1 = e^{2q_1-q_2} \tag{2.204}$$

$$\ddot{q}_2 = e^{2q_2-q_1}. \tag{2.205}$$

These maybe integrated exactly but the explicit form is, in general, a little complicated. Fortunately we are mainly interested in extremal solutions which have a simpler form [32]. As in, [32], taking the ansatz that e^{-q_i} is a second order polynomial one finds

$$\begin{aligned} e^{-q_1} &= a_0 + a_1\tau + \frac{1}{2}\tau^2 &(2.206)\\ e^{-q_2} &= a_1^2 - a_0 + a_1\tau + \frac{1}{2}\tau^2 &(2.207)\end{aligned}$$

Finally, returning to the original variables and imposing the asymptotic boundary conditions gives the solution

$$\begin{aligned} e^{4\sqrt{3}\phi} &= \left(\frac{Q_2}{Q_1}\right)^2\left(\frac{f_2}{f_1}\right)^6 &(2.208)\\ b^2 &= 2Q_1Q_2f_1f_2 &(2.209)\\ a^2 &= \rho^2/b^2 &(2.210)\end{aligned}$$

where

$$f_i = \left(1 + (\bar{Q}_1\bar{Q}_2)^{-\frac{2}{3}}(\bar{Q}_1^{\frac{2}{3}} + \bar{Q}_2^{\frac{2}{3}})^{\frac{1}{2}}\rho + \tfrac{1}{2}(\bar{Q}_i\bar{Q}_1\bar{Q}_2)^{-\frac{2}{3}}\rho^2\right)^{\frac{1}{2}} \qquad (2.211)$$

as quoted in section 2.3.

For completeness we note that the general, non-extremal solution of [20, 23], modified for a non-zero asymptotic value of ϕ, is

$$\begin{aligned} \exp(4\phi/\sqrt{3}) &= e^{4\phi_\infty/\sqrt{3}}\ \frac{p_2}{p_1} &(2.212)\\ a^2 &= \frac{(r-r_+)(r-r_-)}{\sqrt{p_1p_2}} &(2.213)\\ b^2 &= \sqrt{p_1p_2} &(2.214)\end{aligned}$$

where

$$p_i = (r - r_{i+})(r - r_{i-}) \qquad (2.215)$$

$$r_{i\pm} = \frac{2}{(-\alpha_i)}\Sigma \pm \bar{Q}_i\sqrt{\frac{4\Sigma}{2\Sigma + \alpha_i M}} \qquad (2.216)$$

and scalar charge, Σ, which is again not an independent parameter, is given by

$$\frac{1}{\sqrt{3}}\Sigma = \frac{\bar{Q}_2^2}{2M(\lambda-1)} + \frac{\bar{Q}_1^2}{2M(\lambda+1)} \qquad \lambda = \frac{\Sigma}{\sqrt{3}M} \qquad (2.217)$$

2.10.3.3 Case III: $\gamma = 3$ and $\alpha_1 = 4 \qquad \alpha_2 = -6$

In this case, m_{ij} becomes

$$m = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \tag{2.218}$$

Making the coordinate change

$$q_1 = \tfrac{1}{2}X_1 - \log 2 \tag{2.219}$$

$$q_2 = X_2 - \log 2 \tag{2.220}$$

The equations of motion are

$$\ddot{q}_1 = e^{2q_1 - q_2} \tag{2.221}$$

$$\ddot{q}_2 = e^{2q_2 - 2q_2} \tag{2.222}$$

Now consider the three particle Toda system

$$\ddot{q}_1 = e^{2q_1 - q_2} \tag{2.223}$$

$$\ddot{q}_2 = e^{2q_2 - q_1 - q_3} \tag{2.224}$$

$$\ddot{q}_3 = e^{2q_3 - q_2} \tag{2.225}$$

which may be integrated exactly. Notice that by identifying q_1 and q_3 we obtain (2.223-2.225). Once again the general solution is slightly complicated but taking the ansatz that e^{-q_i} is a polynomial one finds

$$e^{-q_1} = a_0 + 2a_2^2\tau + a_2\tau^2 + \frac{1}{6}\tau^3 \tag{2.226}$$

$$e^{-q_2} = 4a_2^4 - 2a_0a_2 + (4a_2^3 - a_0)\tau + 2a_2^2\tau^2 + \frac{2}{3}a_2\tau^3 + \frac{1}{12}\tau^4 \tag{2.227}$$

Rewriting in terms of the original fields we get

$$e^{10\phi} = \left(\frac{Q_2}{Q_1}\right)^2 \exp\left(2X_1 - 3X_2\right) \tag{2.228}$$

$$= \frac{6}{4}\left(\frac{Q_2}{Q_1}\right)^2 \left(\frac{f_2}{f_1}\right)^{12} \tag{2.229}$$

$$b^2 = \rho^2 10 Q_1^{\frac{6}{5}} Q_2^{\frac{4}{5}} \exp\left(-\frac{1}{5}X_1 - \frac{1}{5}X_2\right) \tag{2.230}$$

$$= \frac{5}{2}\left(\frac{2}{3}\right)^{\frac{3}{5}} Q_1^{\frac{6}{5}} Q_2^{\frac{4}{5}} f_1 f_2 \tag{2.231}$$

$$a^2 = \rho^2 / b^2 \tag{2.232}$$

where

$$f_1 = \left(1 - 6a_2\rho + 12a_2^2\rho^2 - 6a_0\rho^3\right)^{\frac{1}{3}} \tag{2.233}$$

$$f_2 = \left(1 - \tfrac{24}{3}a_2\rho + 24a_2\rho^2 - (48a_2^3 - 12a_0)\rho^3 + \left(48a_2^4 - 24a_0a_2\right)\rho^4\right)^{\frac{1}{4}} \tag{2.234}$$

At the horizon we do indeed have ϕ at the critical point of V_{eff}:

$$e^{10\phi_0} = \frac{3}{2}\frac{Q_2^2}{Q_1^2} \tag{2.235}$$

and b^2 given by $V_{eff}(\phi_0)$:

$$b_H^2 = \frac{5}{2}\left(\frac{2}{3}\right)^{\frac{3}{5}} Q_1Q_2\left(\frac{Q_2}{Q_1}\right)^{\frac{1}{5}}. \tag{2.236}$$

Imposing the asymptotic boundary conditions we get

$$a_0 = \pm\frac{2^{\frac{5}{7}}}{\bar{Q}_1^{\frac{10}{7}}\bar{Q}_2^{\frac{5}{7}}} \qquad \left(4a_2^4 - 2a_0a_2\right) = \frac{2^{\frac{11}{7}}}{\bar{Q}_1^{\frac{22}{7}}\bar{Q}_2^{\frac{18}{7}}} \tag{2.237}$$

so letting

$$\clubsuit = \frac{2^{\frac{11}{7}}}{Q_1^{\frac{22}{7}}Q_2^{\frac{18}{7}}} \tag{2.238}$$

$$\Delta_1 = 3\sqrt[3]{a_0^3 + \sqrt{a_0^6 + \frac{64}{3}a_0^3\clubsuit^3}} \tag{2.239}$$

$$\Delta_2 = \sqrt{\frac{3^{\frac{1}{3}}\Delta_1}{a_0} - \frac{3^{\frac{2}{3}}4}{\Delta_1}} \tag{2.240}$$

we may write a_2 as

$$a_2 = \pm\frac{1}{2\sqrt{6}}\Delta_2 \pm \frac{1}{2}\sqrt{\frac{2\clubsuit}{3^{\frac{1}{3}}\Delta_1} - \frac{\Delta_1}{2\,3^{\frac{2}{3}}} + \frac{\sqrt{6}}{\Delta_2}} \tag{2.241}$$

Despite the non-trivial form of the solution we see that it still takes on the attractor value at the horizon.

In terms of the $U(1)$ charges (written implicitly in terms of a_0 and a_2), the mass and scalar charge are expressed below

$$\Sigma = \frac{3a_0^2 - 28a_0a_2^3 + 32a_2^6}{40a_0a_2^4 - 20a_0^2a_2} \tag{2.242}$$

$$M = \frac{(a_0^2 + 4a_0 a_2^3 - 16a_2^6)}{2^{\frac{3}{5}} 50 a_0^{\frac{7}{5}} a_2 (a_0 - 2a_2^3)(2a_2^4 - a_0 a_2)^{\frac{1}{5}} Q_1^{\frac{6}{5}} Q_2^{\frac{4}{5}}} \tag{2.243}$$

This solution is related to a 3 charge p-brane solution found in [32] - in this case we have identified two of the degrees of freedom.

2.11 Higher Dimensions

Here we give some more details related to our discussion of the higher dimensional attractor in section 5. The Ricci components calculated from the metric, eq.(2.86) are,

$$R_{tt} = a^2 \left(a'^2 + \frac{(d-2)aa'b'}{b} + aa'' \right) \tag{2.244}$$

$$R_{rr} = -\{b\left(a'^2 + aa''\right) + (d-2)a\left(a'b' + ab''\right)\}/a^2 b \tag{2.245}$$

$$R_{\theta\theta} = (d-3) - 2aba'b' - a^2\left((d-3)b'^2 + bb''\right) \tag{2.246}$$

The Einstein equations from the action eq.(2.85), take the form,

$$R_{tt} = \frac{(d-3)(d-3)!a(r)^2}{b(r)^{2(d-2)}} V_{eff}(\phi_i) \tag{2.247}$$

$$R_{rr} = 2(\partial_r \phi)^2 - \frac{(d-3)(d-3)!}{b(r)^{2(d-2)} a(r)^2} V_{eff}(\phi_i) \tag{2.248}$$

$$R_{\theta\theta} = \frac{(d-3)!}{b^{2(d-3)}} V_{eff}(\phi_i), \tag{2.249}$$

where V_{eff} is given by eq.(2.90).

Taking the combination, $\frac{1}{2}(R_{rr} - \frac{G_{rr}}{G_{tt}} R_{tt})$ gives, eq.(2.91). Similarly we have,

$$\begin{aligned} & \frac{b(r)^2}{a(r)^2} R_{tt} + a(r)^2 b(r)^2 R_{rr} - (d-2) R_{\theta\theta} \\ = & -(d-2)\{d - 3 - a(r)b'(r)(2a'(r)b(r) + (d-3)a(r)b'(r))\} \\ = & \; 2(\partial_r \phi_i)^2 a(r)^2 b(r)^2 - \frac{(d-2)(d-3)!}{b^{2(d-3)}} V_{eff}(\phi_i) \end{aligned} \tag{2.250}$$

This gives eq.(2.92). Finally the relation, $R_{tt} = (d-3)\frac{a^2}{b^2} R_{\theta\theta}$ yields,

$$\begin{aligned} & (d-3)^2(-1 + a(r)^2 b'(r)^2) + b(r)^2(a'(r)^2 + a(r)a''(r)) \\ & + a(r)b(r)((-8+3d)a'(r)b'(r) + (d-3)a(r)b''(r)) = 0. \end{aligned} \tag{2.251}$$

We now discuss solving for a_2, the second order perturbation in the metric component a, in some more detail. We restrict ourselves to the case of one scalar field, ϕ. The constraint, eq.(2.92), to $O(\epsilon^2)$ is,

$$\begin{aligned}(d-2)ra_2' + (d-2)(d-3)a_2 - 2(\phi_1')^2 r^2 (1-(\frac{r_H}{r})^{d-3})^2 \\ -2(d-2)(d-3)^2 \frac{r_H^{2(d-3)}}{r^{2(d-3)+1}} b_2 + 2(d-3)^2 \frac{\gamma(\gamma+1)\phi_1^2}{r^{2(d-3)} r_H^{6-2d}} \\ +2(d-2)\frac{(d-3)(r_H^3 r^d - r_h^d r^3)}{r_H^6 r^{2d}} \left\{ r_H^d r^2 b_2 + r_h^3 r^d b_2' \right\} = 0\end{aligned} \tag{2.252}$$

This is a first order equation for a_2 of the form,

$$f_1 a_2' + f_2 a_2 + f_3 = 0, \tag{2.253}$$

where,

$$\begin{aligned} f_1 &= (d-2)r \\ f_2 &= (d-2)(d-3) \\ f_3 &= -2(\phi_1')^2 r^2 (1-(\frac{r_H}{r})^{d-3})^2 - 2(d-2)(d-3)^2 \frac{r_H^{2(d-3)}}{r^{2(d-3)+1}} b_2 \\ &\quad +2(d-3)^2 \frac{t(t+1)\phi_1^2}{r^{2(d-3)} r_H^{6-2d}} \\ &\quad +2(d-2)\frac{(d-3)(r_H^3 r^d - r_H^d r^3)}{r_H^6 r^{2d}} \left\{ r_H^d r^2 b_2 + r_H^3 r^d b_2' \right\} \end{aligned} \tag{2.254}$$

The solution to this equation is given by,

$$a_2(r) = Ce^{\mathfrak{F}} - e^{\mathfrak{F}} \int e^{-\mathfrak{F}} \frac{f_3}{f_1} dr \tag{2.255}$$

where $\mathfrak{F} = -\int \frac{f_2}{f_1} dr$. It is helpful to note that $e^{\mathfrak{F}} = \frac{1}{r^{(d-3)}}$ and, $\frac{e^{-\mathfrak{F}}}{f_1} = \frac{r^{d-4}}{(d-2)}$.

Now the first term in eq.(2.255), proportional to C, blows up at the horizon. We will omit some details but it is easy to see that the second term in eq.(2.255) goes to zero. Thus for a non-singular solution we must set $C = 0$. One can then extract the leading behaviour near the horizon of a_2 from eq.(2.255), however it is slightly more convenient to use eq.(2.251) for this purpose instead. From the behaviour of the scalar perturbation ϕ_1, and metric perturbation, b_2, in the vicinity of the horizon, as discussed in the section on attractors in higher dimensions, it is easy to see that

$$a_2(r) = A_2(r^{d-3} - r_H^{d-3})^{2\gamma+2} \tag{2.256}$$

where, A_2 is an appropriately determined constant. Thus we see that the non-singular solution in the vicinity of the horizon vanishes like $(r - r_H)^{(2\gamma+2)}$ and the double-zero nature of the horizon persists after including back-reaction to this order.

Finally, expanding eq.(2.255) near $r \to \infty$ (with $C = 0$) we get that $a_2 \to \mathrm{Const} + \mathcal{O}(1/r^{d-3})$. The value of the constant term is related to the coefficient in the linear term for b_2 at large r in a manner consistent with asymptotic flatness.

In summary we have established here that the metric perturbation a_2 vanishes fast enough at the horizon so that the black hole continues to have a double-zero horizon, and it goes to a constant at infinity so that the black hole continues to be asymptotically flat.

2.12 More Details on Asymptotic AdS Space

We begin by considering the asymptotic behaviour at large r of ϕ_1, eq.(2.114). One can show that this is given by

$$\phi_1(r) \to c_+ \frac{1}{r^{3/2}} I_{3/4}\left(\frac{\beta L}{2r^2}\right) + c_- \frac{1}{r^{3/2}} I_{-3/4}\left(\frac{\beta L}{2r^2}\right) \tag{2.257}$$

Here $I_{3/4}$ stands for a modified Bessel function [1] Asymptotically, $I_\nu \propto r^{-2\nu}$. Thus ϕ_r has two solutions which go asymptotically to a constant and as $1/r^3$ respectively.

Next, we consider values of r, $r_H < r < \infty$. These are all ordinary points of the differential equation eq.(2.114). Thus the solution we are interested is

[1] Modified Bessel function $I_\nu(r), K_\nu(r)$ does satisfy following differential eq.

$$z^2 I_\nu''(z) + z I_\nu'(z) - (z^2 + \nu^2) I(z) = 0. \tag{2.258}$$

well-behaved at these points. For a differential equation of the form,

$$\mathcal{L}(\psi) = \frac{d^2\psi}{dz^2} + p(z)\frac{d\psi}{dz} + q(z)\psi = 0, \tag{2.259}$$

all values of z where $p(z), q(z)$ are analytic are ordinary points. About any ordinary point the solutions to the equation can be expanded in a power series, with a radius of convergence determined by the nearest singular point [?].

We turn now to discussing the solution for a_2. The constraint eq.(2.110) takes the form,

$$2a_0^2 b_2' + a_2 + (a_0^2)'(rb_2)' + ra_2' = \frac{-1}{r^2}\beta^2\phi_1^2 + a_0^2 r^2(\partial_r\phi_1)^2 + \frac{2b_2}{r^3}(r_H^2 + \frac{2r_H^4}{L^2}) + \frac{6rb_2}{L^2} \tag{2.260}$$

The solution to this equation is given by,

$$a_2(r) = \frac{c_2}{r} - \frac{1}{r}\int_{r_H} f_3 dr \tag{2.261}$$

where

$$f_3 = 2a_0^2 b_2' + (a_0^2)'(rb_2)' + \frac{1}{r^2}\beta^2\phi_1^2 - a_0^2 r^2(\partial_r\phi_1)^2 - \frac{2b_2}{r^3}(r_H^2 + \frac{2r_H^4}{L^2}) - \frac{6rb_2}{L^2} \tag{2.262}$$

. We have set the lower limit of integration in the second term at r_H. We want a solution the preserves the double-zero structure of the horizon. This means c_2 must be set to zero.

To find an explicit form for a_2 in the near horizon region it is slightly simpler to use the equation, eq.(2.109). In the near horizon region this can easily be solved and we find the solution,

$$a_2 \propto (r - r_H)^{(2\gamma+2)}. \tag{2.263}$$

At asymptotic infinity one can use the integral expression, eq.(2.261) (with $c_2 = 0$). One finds that $f_3 \to r$ as $r \to \infty$. Thus $a_2 \to d_2 r$. This is consistent with the asymptotically AdS geometry.

In summary we see that that there is an attractor solution to the metric equations at second order in which the double-zero nature of the horizon and the asymptotically AdS nature of the geometry both persist.

Chapter 3

C- function for Non-Supersymmetric Attractors

In this chapter, we present a c-function for spherically symmetric, static and asymptotically flat solutions in theories of four-dimensional gravity coupled to gauge fields and moduli. The c-function is valid for both extremal and non-extremal black holes. It monotonically decreases from infinity and in the static region acquires its minimum value at the horizon, where it equals the entropy of the black hole. Higher dimensional cases, involving p-form gauge fields, and other generalisations are also discussed.

3.1 Background

We begin with some background related to the discussion of non-supersymmetric attractors.

Consider a theory consisting of four dimensional gravity coupled to $U(1)$ gauge fields and moduli, whose bosonic terms have the form,

$$S = \frac{1}{\kappa^2}\int d^4x\sqrt{-G}(R - 2g_{ij}(\partial\phi^i)(\partial\phi^j) - f_{ab}(\phi^i)F^a_{\mu\nu}F^{b\ \mu\nu} - \tfrac{1}{2}\tilde{f}_{ab}(\phi^i)F^a_{\mu\nu}F^b_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}). \tag{3.1}$$

$F^a_{\mu\nu}, a = 0, \cdots N$ are gauge fields. $\phi^i, i = 1, \cdots n$ are scalar fields. The scalars have no potential term but determine the gauge coupling constants. We note that

g_{ij} refers to the metric in the moduli space, this is different from the spacetime metric, $G_{\mu\nu}$.

A spherically symmetric space-time metric in $3+1$ dimensions takes the form,

$$ds^2 \;=\; -a(r)^2dt^2 + a(r)^{-2}dr^2 + b(r)^2d\Omega^2 \tag{3.2}$$

The Bianchi identity and equation of motion for the gauge fields can be solved by a field strength of the form,

$$F^a = f^{ab}(Q_{eb} - \tilde{f}_{bc}Q^c_m)\frac{1}{b^2}dt \wedge dr + Q^a_m sin\theta d\theta \wedge d\phi, \tag{3.3}$$

where Q^a_m, Q_{ea} are constants that determine the magnetic and electric charges carried by the gauge field F^a, and f^{ab} is the inverse of f_{ab}.

The effective potential V_{eff} is then given by,

$$V_{eff}(\phi_i) = f^{ab}(Q_{ea} - \tilde{f}_{ac}Q^c_m)(Q_{eb} - \tilde{f}_{bd}Q^d_m) + f_{ab}Q^a_mQ^b_m. \tag{3.4}$$

For the attractor mechanism it is sufficient that two conditions to be met. First, for fixed charges, as a function of the moduli, V_{eff} must have a critical point. Denoting the critical values for the scalars as $\phi^i = \phi^i_0$ we have,

$$\partial_i V_{eff}(\phi^i_0) = 0. \tag{3.5}$$

Second, the effective potential must be a minimum at this critical point. I.e. the matrix of second derivatives of the potential at the critical point,

$$M_{ij} = \frac{1}{2}\partial_i\partial_j V_{eff}(\phi^i_0) \tag{3.6}$$

should have positive eigenvalues. Schematically we can write,

$$M_{ij} > 0. \tag{3.7}$$

As discussed in [33], it is possible that some eigenvalues of M_{ij} vanish. In this case the leading correction to the effective potential along the zero mode directions should be such that the critical point is a minimum. Thus, an attractor would result if the leading correction is a quartic term, $V_{eff} = V_{eff}(\phi^i_0) + \lambda(\phi - \phi_H)^4$, with $\lambda > 0$ but not if it is a cubic term, $V_{eff} = V_{eff}(\phi^i_0) + \lambda(\phi - \phi_H)^3$.

Once the two conditions mentioned above are met it was argued in [5] that the attractor mechanism works. There is an extremal Reissner Nordstrom black hole solution in the theory, where the black hole carries the charges specified by the parameters, Q^a_m, Q_{ea} and the moduli take the critical values, ϕ_0 at infinity. For small enough deviations at infinity of the moduli from these values, a double-horizon extremal black hole solution continues to exist. In this extremal black hole the scalars take the same fixed values, ϕ_0, at the horizon independent of their values at infinity. The resulting horizon radius is given by,

$$b_H^2 = V_{eff}(\phi_0^i) \tag{3.8}$$

and the entropy is

$$S_{BH} = \frac{1}{4}A = \pi b_H^2. \tag{3.9}$$

In $\mathcal{N} = 2$ supersymmetric theory, V_{eff} can be expressed, [34], in terms of a Kahler potential, K and a superpotential, W as,

$$V_{eff} = e^K[g^{i\bar{j}}\nabla_i W(\nabla_j W)^* + |W|^2], \tag{3.10}$$

where $\nabla_i W \equiv \partial_i W + \partial_i K W$. The Kahler potential and Superpotential in turn can be expressed in terms of a prepotential F, as,

$$K = -\ln Im(\sum_{a=0}^{N} X^{a*}\partial_a F(X)), \tag{3.11}$$

and,

$$W = q_a X^a - p^a \partial_a F, \tag{3.12}$$

respectively. Here, $X^a, a = 0, \cdots N$ are special coordinates to describe the special geometry of the vector multiplet moduli space. And q_a, p^a are the electric and magnetic charges carried by the black hole [1].

For a BPS black hole, the central charge given by,

$$Z = e^{K/2}W, \tag{3.13}$$

is minimised, i.e., $\nabla_i Z = \partial_i Z + \frac{1}{2}\partial_i K Z = 0$. This condition is equivalent to,

$$\nabla_i W = 0. \tag{3.14}$$

[1] These can be related to Q_{ea}, Q^a_m, using eq.(3.3).

The resulting entropy is given by

$$S_{BH} = \pi e^{K}|W|^2. \tag{3.15}$$

with the Kahler potential and superpotential evaluated at the attractor values.

3.2 The c-function in 4 Dimensions.

3.2.1 The c-function

The equations of motion which follow from eq.(3.1) take the form,

$$\begin{aligned}
R_{\mu\nu} - 2g_{ij}\partial_\mu\phi^i\partial_\nu\phi^j &= f_{ab}\left(2F^a{}_{\mu\lambda}F^b{}_\nu{}^\lambda - \tfrac{1}{2}G_{\mu\nu}F^a{}_{\kappa\lambda}F^{b\kappa\lambda}\right) \\
\tfrac{1}{\sqrt{-G}}\partial_\mu\left(\sqrt{-G}g_{ij}\partial^\mu\phi^j\right) &= \tfrac{1}{4}\partial_i(f_{ab})F^a{}_{\mu\nu}F^{b\mu\nu} \\
&\quad + \tfrac{1}{8}\partial_i(\tilde{f}_{ab})F^a_{\mu\nu}F^b_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} \\
\partial_\mu\left(\sqrt{-G}(f_{ab}F^{b\mu\nu} + \tfrac{1}{2}\tilde{f}_{ab}F^b_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma})\right) &= 0.
\end{aligned} \tag{3.16}$$

We are interested in static, spherically symmetric solutions to the equations of motion. The metric and gauge fields in such a solution take the form, eq.(3.2), eq.(3.3). We will be interested in asymptotically flat solutions below. For these the radial coordinate r in eq.(3.2) can be chosen so that $r \to \infty$ is the asymptotically flat region.

The scalar fields are a function of the radial coordinate alone, and substituting for the gauge fields from, eq.(3.3), the equation of motion for the scalar fields take the form,

$$\partial_r(a^2b^2g_{ij}\partial_r\phi^j) = \frac{\partial_i V_{eff}}{2b^2}, \tag{3.17}$$

where V_{eff} is defined in eq.(3.4).

The Einstein equation for the rr component takes the form of an "energy constraint",

$$-1 + a^2b'^2 + \frac{a^{2'}b^{2'}}{2} = \frac{-1}{b^2}(V_{eff}(\phi_i)) + a^2b^2g_{ij}(\partial_r\phi^i)\partial_r\phi^j \tag{3.18}$$

Of particular relevance for the present discussion is the equation obtained for $R_{rr} - \frac{G^{tt}}{G^{rr}}R_{tt}$ component of the Einstein equation. From eq.(3.16), this is,

$$\frac{b(r)''}{b(r)} = -g_{ij}\partial_r\phi^i\partial_r\phi^j. \tag{3.19}$$

Here prime denotes derivative with respect to the radial coordinate r.

Our claim is that the c-function is given by,

$$c = \frac{1}{4}A(r), \tag{3.20}$$

where $A(r)$ is the area of the two-sphere defined by constant t and r,

$$A(r) = \pi b^2(r). \tag{3.21}$$

We show below that in any static, spherically symmetric, asymptotically flat solution, c decreases monotonically as we move inwards along the radial direction from infinity. We assume that the spacetime in the region of interest has no singularities and the scalar fields lie in a singularity free region of moduli space with a metric which is positive, i.e., all eigenvalues of the moduli space metric, g_{ij}, are positive. For a black hole we show that the minimum value of c, in the static region, equals the entropy at the horizon.

To prove monotonicity of c it is enough to prove monotonicity of b. Let us define a coordinate $y = -r$ which increases as we move inwards from the asymptotically flat region. We see from eq.(3.19), since the eigenvalues of $g_{ij} > 0$, that $d^2b/dy^2 \leqslant 0$ and so db/dy must be non-increasing as y increases. Now for an asymptotically flat solution, at infinity as $r \to \infty$, $b(r) \to r$. This means $db/dy = -1$. Since db/dy is non-increasing as y increases this means that for all $y > -\infty$, $db/dy < 0$ and thus b is monotonic. This proves the c-theorem.

3.2.2 Some Comments

A few comments are worth making at this stage.

It is important to emphasise that our proof of the c-theorem applies to any spherically symmetric, static solution which is asymptotically flat. This includes both extremal and non-extremal black holes. The boundary of the static region of spacetime, where the killing vector $\frac{\partial}{\partial t}$ is time-like, is the horizon where $a^2 \to 0$. The c function is monotonically decreasing in the static region, and obtains its minimum value on the boundary at the horizon. We see that this minimum value of c is the entropy of the black hole. We will comment on what happens to c when one goes inside the horizon towards the end of this section.

For extremal black holes it is worth noting that the c-function is not V_{eff} itself. At the horizon, where c obtains its minimum value, the two are indeed equal (up to a constant of proportionality). This follows from the constraint, eq.(3.18), after noting that at a double horizon where a^2 and $a^{2'}$ both vanish, $V_{eff}(\phi_0^i) = b_H^2$. But more generally, away from the horizon, c and V_{eff} are different. In particular, we will consider an explicit example in section 3.5 of a flow from infinity to the horizon where V_{eff} does not evolve monotonically.

In the supersymmetric case it is worth commenting that the c-function discussed above and the square of the central charge agree, up to a proportionality constant, at the horizon of a black hole. But in general, away from the horizon, they are different. For example in a BPS extremal Reissner Nordstrom black hole, obtained by setting the scalars equal to their attractor values at infinity, the central charge is constant, while the Area is infinite asymptotically and monotonically decreases to its minimum at the horizon.

It is also worth commenting that c' can vanish identically only in a Robinson-Bertotti spacetime [1]. If c is constant, b is constant. From, eq.(3.19) then ϕ^i are constant. Thus V_{eff} is extremised. It follows from the other Einstein equations then that $a(r) = r/b$ leading to the Robinson-Bertotti spacetime. From this we learn that a flow from one asymptotically (in the sense that c' and all its derivatives vanish) $AdS_2 \times S^2$ where the scalars are at one critical point of V_{eff} to an asymptotically $AdS_2 \times S^2$ spacetime where the scalars are at another critical point is not possible. Once the scalars begin evolving c' will became negative and cannot return to zero.

The c-theorem discussed above is valid more generally than the specific system consisting of gravity, gauge fields and scalars we have considered here. Consider any four-dimensional theory with gravity coupled to matter which satisfies the null energy condition. By this we mean that the stress-energy satisfies the condition,

$$T_{\mu\nu}\zeta^\mu\zeta^\nu \geqslant 0, \tag{3.22}$$

where ζ^a is an arbitrary null vector. One can show that in such a system the c-theorem is valid for all static, spherically symmetric, asymptotically flat, solutions

[1] By c' vanishing identically we mean that c' and all its derivatives vanish in some region of spacetime.

of the equations of motion. To see this, note that from the metric eq.(3.2), it follows that,

$$-R_{tt}G^{tt} + R_{rr}G^{rr} = -2a^2\frac{b''}{b}. \tag{3.23}$$

From Einstein's equations and the null energy condition we learn that the l.h.s above is positive, since

$$-R_{tt}G^{tt} + R_{rr}G^{rr} = T_{\mu\nu}\zeta^\mu\zeta^\nu > 0 \tag{3.24}$$

where $\zeta^\mu = (\zeta^t, \zeta^r)$ are components of a null vector, satisfying the relations, $(\zeta^t)^2 = -G^{tt}, (\zeta^r)^2 = G^{rr}$. Thus as long as we are outside the horizon, and $a^2 > 0$, i.e. in any region of space-time where the Killing vector related to time translations is time-like, $b'' < 0$ [1]. This is enough to then prove the monotonicity of b and thus c. The importance of the null energy condition for a c-theorem was emphasised in [7] [2].

In fact the c-theorem follows simply from the Raychaudhuri equation and the null energy condition. Consider a congruence of null geodesics, where each geodesic has (θ, ϕ) coordinates fixed, with, (t, r), being functions of the affine parameter, λ. The expansion parameter of this congruence is

$$\vartheta = \frac{d\ln A}{d\lambda}, \tag{3.25}$$

where A is the area, eq.(3.21). Choosing in going null geodesics for which $dr/d\lambda < 0$ we see that $\vartheta < 0$ at $r \to \infty$, for an asymptotically flat space-time. Now, Raychaudhuri's equation tells us that $\frac{d\vartheta}{d\lambda} < 0$ if the energy condition, eq.(3.22), is met. Then it follows that $\vartheta < 0$ for all $r < \infty$ and thus the area A must monotonically decrease. The comments in this paragraph provides a more coordinate independent proof of the c-theorem. Although the focus of this chapter is time independent, spherically symmetric configurations, these comments also suggest that a similar c-theorem might be valid more generally. The connection between c-theorems and the Raychaudhuri equation was emphasised in [35], [36].

[1] In fact the same conclusion also holds inside the horizon. Now t is space-like and r time-like and $T_{\mu\nu}\zeta^\mu\zeta^\nu = 2a^2\frac{b''}{b} \geqslant 0$. Since $a^2 < 0$, we conclude that $b'' < 0$. We will return to this point at the end of the section.

[2] In [7] this condition is referred to as the weaker energy condition.

In the higher dimensional discussion which follows we will see that the c function is directly expressed in terms of the expansion parameter ϑ for radial null geodesics. The reader might wonder why we have not considered an analogous c function in four-dimensions. From the discussion of the previous paragraph we see that any function of the form, $1/\vartheta^p$, where p is a positive power, is monotonically increasing in r. However, in an $AdS_2 \times S^2$ spacetime, $\vartheta \to 0$ and thus such a function will blow up and not equal the entropy of the corresponding extremal black hole.

It seems puzzling at first that a c-function could arise from the analysis of second order equations of motion. As mentioned in the introduction, the answer to this puzzle lies in the fact that we were considering solutions which satisfy asymptotically flat boundary conditions. Without imposing any boundary conditions, we cannot prove monotonicity of c. But one can use the arguments above to show that there is at most one critical point of c as long as the region of spacetime under consideration has no spacetime singularities and also the scalar fields take non-singular values in moduli space. If the critical point occurs at $r = r_*$, c monotonically decreases for all $r < r_*$ and cannot have another critical point. Similarly, for $r > r_*$. From the Raychaudhuri equation it follows that the critical point, at r_*, is a maximum.

Usually the discussion of supersymmetric attractors involves the regions from the horizon to asymptotic infinity. But we can also ask what happens if we go inside the horizon. This is particularly interesting in the non-extremal case where the inside is a time dependent cosmology. In the supersymmetric case one finds that the central charge (and its square) has a minimum at the horizon and increases as one goes away from it towards the outside and also towards the inside. This can be seen as follows. Using continuity at the horizon a modulus take the form in an attractor solution,

$$\phi(r) - \phi_0 \sim |r - r_H|^\alpha \tag{3.26}$$

where α is a positive coefficient and ϕ_0 is the attractor value for the modulus [1].

[1] We are working in the coordinates, eq.(3.2). These breakdown at the horizon but are valid for $r > r_H$ and also $r < r_H$ (where $a^2 < 0$). The solution written here is valid in both these regions; for $r = r_H$ we need to take the limiting value.

Since the central charge is minimised by ϕ_0, one finds by expanding in the vicinity of $r = r_H$, that the central charge is also minimised as a function of r [1]. In contrast, the c-function we have considered here, monotonically decreases inside the horizon till we reach the singularity. In fact it follows from the Raychaudhuri equation that the expansion parameter ϑ monotonically decreases and becomes $-\infty$ at the singularity.

3.3 The c-function In Higher Dimensions

We analyse higher dimensional generalisations in this section. Consider a system consisting of gravity, gauge fields with rank q field strengths, $F^a_{m_1\cdots m_q}, a = 1,\cdots N$, and moduli $\phi^i, i = 1,\cdots n$, in $p+q+1$ dimensions, with action,

$$S = \frac{1}{\kappa^2}\int d^Dx\sqrt{-G}\left(R - 2g_{ij}(\partial\phi^i)\partial\phi^j - f_{ab}(\phi^i)\frac{1}{q!}F^a_{\mu\nu....}F^{b\ \mu\nu......}\right). \qquad (3.27)$$

Take a metric and field strengths of form,

$$ds^2 = a(r)^2\left(-dt^2 + \sum_{i=1}^{p-1} dy_i^2\right) + a(r)^{-2}dr^2 + b(r)^2 d\Omega_q^2, \qquad (3.28)$$

$$F^a = Q^a_m \omega_q. \qquad (3.29)$$

Here $d\Omega_q^2$ and ω_q are the volume element and volume form of a unit q dimensional sphere sphere. Note that the metric has Poincare invariance in p direction, t, y_i, and has $SO(q)$ rotational symmetry. The field strengths thread the q sphere and the configuration carries magnetic charge. Other generalisations, which we do not discuss here include, forms of different rank, and also field strengths carrying both electric and magnetic charge.

Define an effective potential,

$$V_{eff} = f_{ab}(\phi^i)Q^a_m Q^b_m. \qquad (3.30)$$

Now, as we discuss further in section 3.7, it is easy to see that if V_{eff} has a critical point where $\partial_{\phi^i}V_{eff}$ vanishes, then by setting the scalars to be at their

[1] The effective potential V_{eff} in the non-supersymmetric case is similar. As a function of r it attains a local minimum at the horizon.

critical values, $\phi^i = \phi^i_0$, one has extremal and non extremal black brane solutions in this system with metric, eq.(3.79). For extremal solutions, the near horizon limit is $AdS_{p+1} \times S^q$, with metric given by eq.(3.83),

$$ds^2 = \frac{r^2}{R^2}\left(-dt^2 + dy_i^2\right) + \frac{R^2}{r^2}dr^2 + b_H^2 d\Omega_q^2 \tag{3.31}$$

where

$$R = \left(\frac{p}{q-1}\right) b_H \tag{3.32}$$

$$(b_H)^{2(q-1)} = \frac{p}{(p+q-1)(q-1)} V_{eff}(\phi^i_0). \tag{3.33}$$

In the extremal case, using arguments analogous to [5] one can show that the $AdS_{p+1} \times S^q$ solution is an attractor if the effective potential is minimised at the critical point ϕ^i_0. That is, for small deviations from the attractor values for the moduli at infinity, there is an extremal solution in which the moduli are drawn to their critical values at the horizon and the geometry in the near-horizon region is $AdS_{p+1} \times S^q$.

We now turn to discussing the c-function in this system. The discussion is motivated by the analysis in [7] of a c-theorem in AdS space. Our claim is that a c-function for the system under consideration is given by,

$$c = c_0 \frac{1}{\tilde{A}^{(p-1)}}. \tag{3.34}$$

Here, c_0 is a constant of proportionality chosen so that $c > 0$. $\tilde{A}$ is defined by

$$\tilde{A} = A' \left(\frac{a}{b^{\frac{q}{p-1}}}\right) \tag{3.35}$$

where A is defined to be,

$$A = \ln(a b^{\frac{q}{p-1}}), \tag{3.36}$$

and prime denotes derivative with respect to r. We show below that for any static, asymptotically flat solution of the form, eq.(3.28), c, eq.(3.34), is a monotonic function of the radial coordinate.

The key is once again to use the null energy condition. Consider the $R_{tt}G^{tt} - R_{rr}G^{rr}$ component of the Einstein equation. For the metric, eq.(3.28), we get,

$$-R_{tt}G^{tt} + R_{rr}G^{rr} = a^2\left[-(p-1)\frac{a''}{a} - q\frac{b''}{b}\right] = T_{\mu\nu}\zeta^\mu\zeta^\nu, \tag{3.37}$$

where (ζ^t, ζ^r) are the components of a null vector which satisfy the relation, $(\zeta^t)^2 = -G^{tt}, (\zeta^r)^2 = G^{rr}$. The null energy condition tells us that the r.h.s cannot be negative. For the system under consideration the r.h.s can be calculated giving,

$$-(p-1)\frac{a''}{a} - q\frac{b''}{b} = 2g_{ij}\partial_r\phi^i\partial_r\phi^j. \tag{3.38}$$

It is indeed positive, as would be expected since the matter fields we include satisfy the null energy condition.

From eq.(3.38) we find that

$$\frac{d\tilde{A}}{dr} = -\frac{a}{b^{\frac{q}{p-1}}}\left[\frac{2}{p-1}g_{ij}\phi^i\phi^j + \left(\frac{q}{p-1} + \frac{q^2}{(p-1)^2}\right)\left(\frac{b'}{b}\right)^2\right], \tag{3.39}$$

and thus, $\frac{d\tilde{A}}{dr} \leqslant 0$.

Now we turn to the monotonicity of c. Consider a solution which becomes asymptotically flat as $r \to \infty$. Then, $a \to 1, b \to r$, as $r \to \infty$. It follows then that $\tilde{A} \to 0^+$ asymptotically. Since, we learn from eq.(3.39) that $\tilde{A}$ is a non-increasing function of r it then follows that for all $r < \infty$, $\tilde{A} > 0$. Since, $a, b > 0$, we then also learn from, eq.(3.35), that $A' > 0$ for all finite r.

Next choose a coordinate $y = -r$ which increases as we go in from asymptotic infinity. We have just learned that $dA/dy = -A' < 0$, for finite r. It is now easy to see that

$$\frac{dc}{dy} = -(p-1)\frac{a}{b^{\frac{q}{p-1}}}c\frac{dA}{dy}\frac{1}{\tilde{A}^2}\frac{d\tilde{A}}{dr}. \tag{3.40}$$

Then given that $a, b > 0$, $c > 0$, and $dA/dy < 0$, $\frac{d\tilde{A}}{dr} \leqslant 0$, it follows that $dc/dy \leqslant 0$, so that the c-function is a non-increasing function along the direction of increasing y. This completes our proof of the c-theorem.

For a black brane solution the static region of spacetime ends at a horizon, where a^2 vanishes. The c-function monotonically decreases from infinity and in the static region obtains its minimum value at the horizon. For the extremal black brane the near horizon geometry is $AdS_{p+1} \times S^q$. We now verify that for p even the c function evaluated in the $AdS_{p+1} \times S^q$ geometry agrees with the conformal anomaly in the boundary Conformal Field Theory. From eq.(3.31) we see that in $AdS_{p+1} \times S^q$,

$$a' = 1/R \tag{3.41}$$

$$b = \frac{q-1}{p} R. \tag{3.42}$$

where R is the radius of the AdS_{p+1}. Then

$$c \propto \frac{R^{p+q-1}}{G_N^{p+q+1}} \propto \frac{R^{p-1}}{G_N^{p+1}} \tag{3.43}$$

where G_N^{p+q+1}, G_N^{p+1} refer to Newton's constant in the $p+q+1$ dimensional spacetime and the $p+1$ dimensional spacetime obtained after KK reduction on the S^q respectively. The right hand side in eq.(3.43) is indeed proportional to the value of the conformal anomaly in the boundary theory when p is even [37]. By choosing c_0, eq.(3.34), appropriately, they can be made equal. Let us also comment that c in the near horizon region can be expressed in terms of the minimum value of the effective potential. One finds that $c \propto (V_{eff}(\phi_0^i))^{\frac{(p+q-1)}{2(q-1)}}$, where the critical values for the moduli are $\phi^i = \phi_0^i$.

A few comments are worth making at this stage. We have only considered asymptotically flat spacetimes here. But our proof of the c-theorem holds for other cases as well. Of particular interest are asymptotically $AdS_{p+1} \times S^q$ spacetime. The metric in this case takes the form, eq.(3.31), as $r \to \infty$. The proof is very similar to the asymptotically flat case. Once again one can argue that $A' > 0$ for $r < \infty$ and then defining a coordinate $y = -r$ it follows that dc/dy is a non-increasing function of y. The c-theorem allows for flows which terminate in another asymptotic $AdS_{p+1} \times S^q$ spacetime. The second $AdS_{p+1} \times S^q$ space-time, which lies at larger y, must have smaller c. Such flows can arise if V_{eff} has more than one critical point. It is also worth commenting that requiring that c is a constant in some region of spacetime leads to the unique solution (subject to the conditions of a metric which satisfies the ansatz, eq.(3.28)) of $AdS_{p+1} \times S^q$ with the scalars being constant and equal to a critical value of V_{eff}.

We mentioned above that our definition of the c function is motivated by [7]. Let us make the connection clearer. The c-function in[1] [7],[39] is defined for a

[1]Another c-function has been defined in [38].

spacetime of the form,

$$ds^2 = e^{2A} \sum_{\mu,\nu=0,\cdots p} \eta_{\mu\nu} dy^\mu dy^\nu + dz^2, \tag{3.44}$$

and is given by

$$c = \frac{c_0}{(dA/dz)^{p-1}}. \tag{3.45}$$

Note that eq.(3.44) is the Einstein frame metric in $p+1$ dimensions. Starting with the metric, eq.(3.28), and Kaluza-Klein reducing over the Q sphere shows that A defined in eq.(3.36) agrees with the definition eq.(3.44) above and dA/dz agrees with $\tilde{A}$ in eq.(3.35). This shows that the c-function eq.(3.34) and eq.(3.45)are the same.

The monotonicity of c follows from that of $\tilde{A}$, eq.(3.35). One can show that for a congruence of null geodesics moving in the radial direction, with constant (θ, ϕ), the expansion parameter ϑ is given by,

$$\vartheta = \left(\frac{a'}{a} + \frac{q}{p-1}\frac{b'}{b}\right). \tag{3.46}$$

Raychaudhuri's equation and the null energy condition then tells us that $\frac{d\vartheta}{dr} < 0$. However, in an $AdS_{p+1} \times S^q$ spacetime ϑ diverges, this behaviour is not appropriate for a c-function. From eq.(3.35) we see that $\tilde{A}$ differs from ϑ by an additional multiplicative factor, $a/b^{\frac{q}{p-1}}$. This factor is chosen to preserve monotonicity and now ensures that c goes to a finite constant in $AdS_{p+1} \times S^q$ spacetimes. A similar comment also applies to the c-function discussed in [7].

3.4 Concluding Comments

In two-dimensional field theories it has been suggested sometime ago [40, 41, 42] that the c function plays the role of a potential, so that the RG equations take the form of a gradient flow,

$$\beta_i = -\frac{\partial c}{\partial g^i},$$

where c is the Zamolodchikov c-function [43]. This phenomenon has a close analogy in the case of supersymmetric black holes, where the radial evolution of the moduli is determined by the gradient of the central charge in a first order equation. In contrast, the c-function we propose does not satisfy this property in either the supersymmetric or the non-supersymmetric case. In particular, in the non-supersymmetric case the scalar fields satisfy a second order equation and in particular the gradient of the c-function does not directly determine their radial evolution.

It might seem confusing at first that our derivation of the c-theorem followed from the second order equations of motion. The following simple mechanically model is useful in understanding this. Consider a particle moving under the force of gravity. The c-function in this case is the height x which satisfies the condition

$$\ddot{x} = -g, \tag{3.47}$$

where g is the acceleration due to gravity. Now, if the initial conditions are such that $\dot{x} < 0$ then going forwards in time x will monotonically decrease. However, if the direction of time is chosen so that $\dot{x} > 0$, going forward in time there will be a critical point for x and thus x will not be a monotonic function of time. In this case though there can be at most one such critical point.

While the equations of motion that govern radial evolution are second order, the attractor boundary conditions restrict the allowed initial conditions and in effect make the equations first order. This suggests a close analogy between radial evolution and RG flow. The existence of a c-function which we have discussed in this chapter adds additional weight to the analogy. In the near-horizon region, where the geometry is $AdS_{p+1} \times S^q$, the relation between radial evolution and RG flow is quite precise and well known. The attractor behaviour in the near horizon region can be viewed from the dual CFT perspective. It corresponds to turning on operators which are irrelevant in the infra-red. These operators are dual to the moduli fields in the bulk, and their being irrelevant in the IR follows from the fact that the mass matrix, eq.(3.6), has only positive eigenvalues.

It is also worth commenting that the attractor phenomenon in the context of black holes is quite different from the usual attractor phenomenon in dynamical systems. In the latter case the attractor phenomenon refers to the fact that there is a universal solution that governs the long time behaviour of the system, regardless of initial conditions. In the black hole context a generic choice of initial conditions at asymptotic infinity does not lead to the attractor phenomenon. Rather there is one well behaved mode near the horizon and choosing an appropriate combination of the two solutions to the second order equations at infinity allows us to match on to this well behaved solution at the horizon. Choosing generic initial conditions at infinity would also lead to triggering the second mode near the horizon which is ill behaved and typically would lead to a singularity.

Finally, we end with some comments about attractors in cosmology. Scalar fields exhibit a late time attractor behaviour in FRW cosmologies with growing scale factor (positive Hubble constant H). Hubble expansion leads to a friction term in the scalar field equations,

$$\ddot{\phi} + 3H\dot{\phi} + \partial_\phi V = 0. \tag{3.48}$$

As a result at late times the scalar fields tend to settle down at the minimum of the potential generically without any precise tuning of initial conditions. This is quite different from the attractor behaviour for black holes and more akin to the attractor in dynamical systems mentioned above.

Actually in AdS space there is an analogy to the cosmological attractor. Take a scalar field which has a negative $(mass)^2$ in AdS space (above the BF bound). This field is dual to a relevant operator. Going to the boundary of AdS space a perturbation in such a field will generically die away. This is the analogue of the late time behaviour in cosmology mentioned above. Similarly there is an analogue to the black hole attractor in cosmology. Consider dS space in Poincare coordinates,

$$ds^2 = -\frac{dt^2}{t^2} + t^2 dx_i^2, \tag{3.49}$$

and a scalar field with potential V propagating in this background. Notice that $t \to 0$ is a double horizon. For the scalar field to be well behaved at the horizon, as $t \to 0$, it must go to a critical point of V, and moreover this critical point will be stable in the sense that small perturbations of the scalar about the critical point will bring it back, if $V'' < 0$ at the critical point, i.e., if the critical point is a maximum. This is the analogue of requiring that V_{eff} is at a minimum for attractor behaviour in black hole [1]. It is amusing to note that a cosmology in which scalars are at the maximum of their potential, early on in the history of the universe, could have other virtues as well in the context of inflation.

3.5 V_{eff} Need Not Be Monotonic

In this section we construct an explicit example showing that V_{eff} as a function of the radial coordinate need not be monotonic. The basic point in our example is simple. The scalar field ϕ is a monotonic function of the radial coordinate, r, eq.(3.2) . But the effective potential is not a monotonic function of ϕ, and as a result is not monotonic in r.

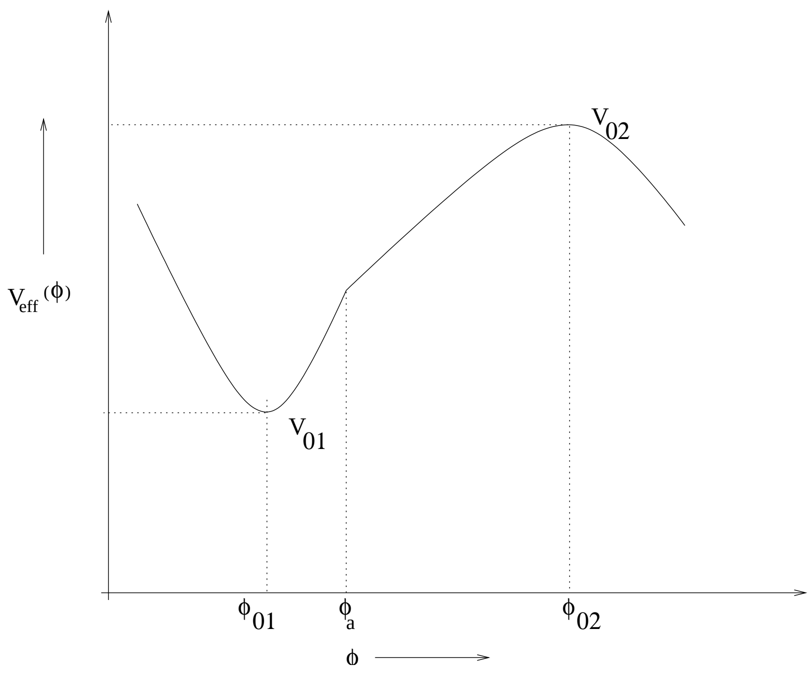


Figure 3.1: The effective potential V_{eff} as a function of ϕ

We work with the following simple V_{eff} to construct such a solution,

$$V_{eff} = V_{01} + \frac{1}{2}m^2(\phi - \phi_{01})^2, \ \phi \leqslant \phi_a \tag{3.50}$$

[1] The sign reversal is due to the interchange of a space and time directions.

$$V_{eff} = V_{02} - \frac{1}{2}m^2(\phi - \phi_{02})^2, \ \phi \geqslant \phi_a. \tag{3.51}$$

At ϕ_a, the potential is continuous, giving the relation,

$$V_{02} = V_{01} + \frac{1}{2}m^2(\phi - \phi_{01})^2 + \frac{1}{2}m^2(\phi - \phi_{02})^2. \tag{3.52}$$

We will take the potential as being specified by $V_{01}, \phi_{01}, \phi_{02}, \phi_a, m^2$ with V_{02} being determined by eq.(3.52). The effective potential is given in fig. 1. Note that with a minimum at ϕ_{01} and a maximum at ϕ_{02}, V_{eff}, is a non-monotonic function of ϕ. Note also that the the first derivative of the potential has a finite jump at $\phi = \phi_a$. Since the equations of motion are second order this means the scalar fields and the metric components, a, b, and their first derivatives will be continuous across ϕ_a. The finite jump is thus mild enough for our purposes.

The attractor value for the scalar is ϕ_{01}. By setting $\phi = \phi_{01}$, independent of r, we get an extremal Reissner Nordstrom black hole solution. The radius of the horizon, r_H in this solution is given by

$$r_H^2 = V_{01}. \tag{3.53}$$

This solution is our starting point. We now construct the solution of interest in perturbation theory, following the analysis in [5], whose conventions we also adopt. For the validity of perturbation theory, we take, $\phi_a - \phi_{01} \ll 1$, and also $\phi_{02} - \phi_{01} \ll 1$. The non-monotonicity of the potential then comes into play even when the scalar field makes only small excursions around the minimum ϕ_{01}. In addition we will also take, $\frac{4m^2}{r_H^2} < 1$, it then follows that $\frac{V_{02}-V_{01}}{V_{01}} \ll 1$.

We construct the solution for the scalar field to first order in perturbation theory below. In the solution the scalar field is a monotonic function of r. This allows the solution to be described in two regions. In region I, $\phi_{01} \leqslant \phi \leqslant \phi_a$, it is given by,

$$\phi = \phi_{01} + A(r - r_H)^\alpha, \tag{3.54}$$

$$\alpha = \frac{1}{2}\left(\sqrt{1 + \frac{4m^2}{r_H^2}} - 1\right). \tag{3.55}$$

And in region II, $\phi > \phi_a$, it is given by,

$$\phi = \phi_{02} + B_1(r - r_H)^{(-\gamma_1)} + B_2(r - r_H)^{(-\gamma_2)}, \tag{3.56}$$

$$\gamma_1 = \frac{1}{2}\left(1 + \sqrt{1 - \frac{4m^2}{r_H^2}}\right), \tag{3.57}$$

$$\gamma_2 = \frac{1}{2}\left(1 - \sqrt{1 - \frac{4m^2}{r_H^2}}\right). \tag{3.58}$$

The boundary between the two is at r_a, where $\phi = \phi_a$, and ϕ and its first derivative with respect to r are continuous. The continuity conditions allow us to solve for B_1, B_2, in terms of A, and also determine r_a in terms of A. The solution is thus completely specified by the constant, A. r_a satisfies the relation,

$$\left(1 - \frac{r_h}{r_a}\right)^{\alpha} = \frac{(\phi_a - \phi_{01})}{A}. \tag{3.59}$$

We will omit some details of the subsequent analysis. One finds that as long as

$$(\phi_a - \phi_{01}) < A < \left(\frac{\gamma_1}{\gamma_2}\right)^{\frac{\alpha}{\gamma_1 - \gamma_2}} (\phi_a - \phi_{01}), \tag{3.60}$$

the scalar field monotonically evolves with r and transits from region I to region II as r increases. Now we see from eq.(3.56) that if $B_1 + B_2 > 0$, $\phi(r \to \infty) > \phi_{02}$. This ensures that V_{eff} is not a monotonic function of r. It will first increases and then decreases as r decreases from ∞ to r_H. The condition, $B_1 + B_2 > 0$, gives rise to the condition,

$$(\phi_{02} - \phi_a) < \alpha \frac{[1 - (1 - \frac{r_H}{r_a})^{\gamma_1 - \gamma_2}]}{[\gamma_1 - \gamma_2(1 - \frac{r_H}{r_a})^{\gamma_1 - \gamma_2}]} (\phi_a - \phi_{01}). \tag{3.61}$$

Having picked a value of A that lies in the range, eq.(3.60), we can then determine r_a from eq.(3.59). As long as ϕ_{02} is small enough and satisfies condition eq.(3.61) we see that the asymptotic value of $\phi(r \to \infty) > \phi_{02}$. It then follows, as argued above, that in the resulting solution V_{eff} is not a monotonic function of r.

We end with three comments. First, we have not obtained the the corrections to the metric components a, b in perturbation theory here. But this can be done following the analysis in [5]. One finds that the corrections are small. Second, the c-function is of course monotonic as a function of the radial coordinate in this example too. The area of the extremal Reissner Nordstrom black hole monotonically decreases and this is true even after including the small corrections in perturbation theory. Finally, we have not obtained the effective potential above starting with gauge fields coupled to moduli. In fact, for dilaton-like couplings, the simplest example we have been able to construct, where V_{eff} has multicritical points with some minimal and maxima, involves two moduli, a dilaton and axion, and two gauge fields. Our discussion above has a close parallel in this case as well (with both dilaton and axion excited) and we expect, by dialling the charges and couplings, that the analogue of condition eq.(3.61) can be met leading to solutions where V_{eff} evolves non-monotonically with the radial coordinate r.

3.6 More Details in Higher Dimensional Case

The equations of motion that follow from the action, eq.(3.27), are,

$$\begin{array}{rcl} R_{\mu\nu} - 2\partial_\mu\phi_i\partial_\nu\phi_i & = & \frac{q}{q!}f_{ab}(\phi_i)F^a_{\mu\lambda....}F^{b\lambda....}_{\nu} - \frac{q-1}{(p+q-1)q!}G_{\mu\nu}f_{ab}(\phi_i)F^a_{\mu\nu....}F^{b\ \mu\nu......} \\ \frac{1}{\sqrt{-G}}\partial_\mu(\sqrt{-G}\partial^\mu\phi_i) & = & \frac{1}{4q!}\partial_i f_{ab}(\phi_i)F^a_{\mu\nu....}F^{b\ \mu\nu......} \\ \partial_\mu(\sqrt{-G}f_{ab}(\phi_i)F^{b\mu\nu}) & = & 0. \end{array} \tag{3.62}$$

Substituting for the gauge fields from eq.(3.29) we learn that $R_{tt} = \frac{a^2}{b^2}(\frac{q-1}{p})R_{\theta\theta}$, which yields the equation,

$$pb^2\left(pa^{'2} + \frac{qaa^{'}b^{'}}{b} + aa^{''}\right) = (q-1)\left((q-1) - (p+1)aba^{'}b^{'} - a^2\left((q-1)b^{'2} + bb^{''}\right)\right) \tag{3.63}$$

where we have computed the curvature components using the metric, eq.(3.28). The $R_{rr} - \frac{G^{tt}}{G^{rr}}R_{tt}$ component of the Einstein equation gives

$$(p-1)\frac{a^{''}}{a} + \frac{qb^{''}}{b} = -2g_{ij}\partial_r\phi^i\partial_r\phi^j. \tag{3.64}$$

Also the R_{rr} component itself yields a first order "energy" constraint,

$$(p(p-1)b^2a^{'2} + 2pqaba^{'}b^{'} + q(q-1)(-1+a^2b^{'2})) = 2a^2b^2g_{ij}\partial_r\phi^i\partial_r\phi^j - V_{eff}(\phi_i)b^{-2(q-1)} \tag{3.65}$$

where V_{eff} is defined in eq.(3.30).

The equation of motion of the scalar field is given by,

$$\partial_r(a^{p+1}b^q\partial_r g_{ij}\phi^j) = \frac{a^{p-1}\partial_i V_{eff}}{4b^q}. \tag{3.66}$$

Setting $\phi^i = \phi_0^i$, where ϕ_0^i is a critical point of V_{eff} one finds that $AdS_{p+1} \times S^q$ is a solution of these equations with metric, eq.(3.31).

3.7 Higher Dimensional p-Brane Solutions

Fixing the scalars at their attractor values, as described in section 4, we are left with the action

$$S = \frac{1}{\kappa^2}\int d^Dx\sqrt{-G}\left\{R - \frac{1}{q!}\sum_a F_{(q)}^{a\ 2}\right\} \tag{3.67}$$

where f_{ab} has been diagonalised and the attractor values of the scalars have been absorbed into the a redefinition of the gauge charges, Q^a. We denote the new charges as $\bar{Q}^a$.

To find solutions, we can dimensionally reduce this action along the brane and use known blackhole solutions. To this end take the metric

$$ds^2 = \underbrace{e^{\lambda\rho}d\hat{s}^2}_{t,r,\omega_1,\ldots,\omega_q} + \underbrace{e^{-\left(\frac{q}{p-1}\right)\lambda\rho}dy^2}_{i_1\ldots i_{p-1}} \tag{3.68}$$

where

$$\lambda = \pm\sqrt{\frac{2(p-1)}{q(p+q-1)}} \tag{3.69}$$

then

$$R \;=\; e^{-\lambda\rho}\left(\hat{R} - \lambda^2\hat{\nabla}^2\rho - \frac{1}{2}(\hat{\nabla}\rho)^2\right) \tag{3.70}$$

where $\hat{R}$ and $\hat{\nabla}$ are respectively the Ricci scalar and covariant derivative for $d\hat{s}^2$. The coefficient, λ, has been fixed by requiring that, we remain in the Einstein frame, and that the kinetic term for ρ has canonical normalisation. Upon neglecting the boundary term, the action becomes

$$S = \frac{V_{(p-1)}}{\kappa^2}\int d^{(q+2)}x\sqrt{-\hat{G}}\left\{\hat{R} - \frac{1}{2}\left(\hat{\nabla}\rho\right)^2 - \frac{1}{q!}e^{\beta\rho}\sum_a\left(\hat{F}_{(q)}^a\right)^2\right\} \tag{3.71}$$

where

$$\beta = -(q-1)\lambda. \tag{3.72}$$

The black hole solution to eq.(3.71) is [17, 44]:

$$
\begin{aligned}
d\hat{s}^2 &= -(f_+)(f_-)^{1-\hat{\gamma}(q-1)}dt^2 + (f_+)(f_-)^{\hat{\gamma}-1}du^2 + (f_-)^{\hat{\gamma}}u^2 d\Omega_q^2 & (3.73)\\
e^{\lambda\rho} &= (f_-)^{-\hat{\gamma}} & (3.74)\\
f_\pm &= \left(1 - \left(\frac{u_\pm}{u}\right)^{q-1}\right) & (3.75)
\end{aligned}
$$

where

$$\hat{\gamma} = \frac{2(p-1)}{(q-1)p} \tag{3.76}$$

with

$$
\begin{aligned}
\hat{F}^a &= \bar{Q}^a \omega_q & (3.77)\\
\sum_a (\bar{Q}^a)^2 &= \frac{\hat{\gamma}(q-1)^3(u_+u_-)^{q-1}}{\beta^2}. & (3.78)
\end{aligned}
$$

Using eq.(3.68) we find the solution to the original action, eq.(3.67), is

$$ds^2 = (f_-)^{\frac{2}{p}}\left(-\left(\frac{f_+}{f_-}\right)dt^2 + dy^2\right) + (f_+f_-)^{-1}du^2 + u^2 d\Omega_q^2. \tag{3.79}$$

So finally, the extremal solution is

$$
\begin{aligned}
ds^2 &= (f)^{\frac{2}{p}}\left(-dt^2 + dy^2\right) + (f)^{-2}du^2 + u^2 d\Omega_q^2 & (3.80)\\
f &= \left(1 - \left(\frac{b_H}{u}\right)^{q-1}\right) & (3.81)
\end{aligned}
$$

where $b_H = u_\pm$. Now we take the near horizon limit,

$$u \longrightarrow b_H + \epsilon R\left(\frac{r}{R}\right)^p, \tag{3.82}$$

with t and y rescaled appropriately, which indeed gives the near horizon geometry $AdS_{p+1} \times S^q$:

$$ds^2 = \frac{r^2}{R^2}\left(-dt^2 + dy^2\right) + \frac{R^2}{r^2}dr^2 + b_H^2 d\Omega_q^2 \tag{3.83}$$

where

$$R = \left(\frac{p}{q-1}\right) b_H \tag{3.84}$$

and

$$V_{eff} = \frac{(p+q-1)(q-1)}{p}(b_H)^{2(q-1)}. \tag{3.85}$$

Chapter 4

Rotating Attractors

In this chapter, we prove that, in a general higher derivative theory of gravity coupled to abelian gauge fields and neutral scalar fields, the entropy and the near horizon background of a rotating extremal black hole is obtained by extremizing an entropy function which depends only on the parameters labeling the near horizon background and the electric and magnetic charges and angular momentum carried by the black hole. If the entropy function has a unique extremum then this extremum must be independent of the asymptotic values of the moduli scalar fields and the solution exhibits attractor behaviour. If the entropy function has flat directions then the near horizon background is not uniquely determined by the extremization equations and could depend on the asymptotic data on the moduli fields, but the value of the entropy is still independent of this asymptotic data. We illustrate these results in the context of two derivative theories of gravity in several examples. These include Kerr black hole, Kerr-Newman black hole, black holes in Kaluza-Klein theory, and black holes in toroidally compactified heterotic string theory.

4.1 General Analysis

We begin by considering a general four dimensional theory of gravity coupled to a set of abelian gauge fields $A_\mu^{(i)}$ and neutral scalar fields $\{\phi_s\}$ with action

$$\mathcal{S} = \int d^4x \, \sqrt{-\det g} \, \mathcal{L} \, , \tag{4.1}$$

where $\sqrt{-\det g}\,\mathcal{L}$ is the lagrangian density, expressed as a function of the metric $g_{\mu\nu}$, the scalar fields $\{\Phi_s\}$, the gauge field strengths $F^{(i)}_{\mu\nu} = \partial_\mu A^{(i)}_\nu - \partial_\nu A^{(i)}_\mu$, and covariant derivatives of these fields. In general $\mathcal{L}$ will contain terms with more than two derivatives. We consider a rotating extremal black hole solution whose near horizon geometry has the symmetries of $AdS_2 \times S^1$. The most general field configuration consistent with the $SO(2,1) \times U(1)$ symmetry of $AdS_2 \times S^1$ is of the form:

$$
\begin{aligned}
& ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = v_1(\theta)\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + \beta^2\, d\theta^2 + \beta^2\, v_2(\theta)(d\phi - \alpha r dt)^2 \\
& \Phi_s = u_s(\theta) \\
& \frac{1}{2}\, F^{(i)}_{\mu\nu} dx^\mu \wedge dx^\nu = (e_i - \alpha b_i(\theta)) dr \wedge dt + \partial_\theta b_i(\theta) d\theta \wedge (d\phi - \alpha r dt)\,,
\end{aligned} \tag{4.2}
$$

where α, β and e_i are constants, and v_1, v_2, u_s and b_i are functions of θ. Here ϕ is a periodic coordinate with period 2π and θ takes value in the range $0 \le \theta \le \pi$. The SO(2,1) isometry of AdS_2 is generated by the Killing vectors[8]:

$$
L_1 = \partial_t, \qquad L_0 = t\partial_t - r\partial_r, \qquad L_{-1} = (1/2)(1/r^2 + t^2)\partial_t - (tr)\partial_r + (\alpha/r)\partial_\phi\,. \tag{4.3}
$$

The form of the metric given in (4.2) implies that the black hole has zero temperature.

We shall assume that the deformed horizon, labelled by the coordinates θ and ϕ, is a smooth deformation of the sphere.[1] This requires

$$
\begin{aligned}
v_2(\theta) &= \theta^2 + \mathcal{O}(\theta^4) \quad \text{for } \theta \simeq 0 \\
&= (\pi - \theta)^2 + \mathcal{O}((\pi - \theta)^4) \quad \text{for } \theta \simeq \pi\,.
\end{aligned} \tag{4.4}
$$

[1]Although in two derivative theories the horizon of a four dimensional black hole is known to have spherical topology, once higher derivative terms are added to the action there may be other possibilities. Our analysis can be easily generalized to the case where the horizon has the topology of a torus rather than a sphere. All we need is to take the θ coordinate to be a periodic variable with period 2π and expand the various functions in the basis of periodic functions of θ. However if the near horizon geometry is invariant under both ϕ and θ translations, then in the expression for L_{-1} given in (4.3) we could add a term of the form $-(\gamma/r)\partial_\theta$, and the entropy could have an additional dependence on the charge conjugate to the variable γ. This represents the Noether charge associated with θ translation, but does not correspond to a physical charge from the point of view of the asymptotic observer since the full solution is not invariant under θ translation.

For the configuration given in (4.2) the magnetic charge associated with the ith gauge field is given by

$$p_i = \int d\theta d\phi F^{(i)}_{\theta\phi} = 2\pi(b_i(\pi) - b_i(0))\,. \tag{4.5}$$

Since an additive constant in b_i can be absorbed into the parameters e_i, we can set $b_i(0) = -p_i/4\pi$. This, together with (4.5), now gives

$$b_i(0) = -\frac{p_i}{4\pi}, \qquad b_i(\pi) = \frac{p_i}{4\pi}\,. \tag{4.6}$$

Requiring that the gauge field strength is smooth at the north and the south poles we get

$$\begin{aligned} b_i(\theta) &= -\frac{p_i}{4\pi} + \mathcal{O}(\theta^2) \quad \text{for } \theta \simeq 0 \\ &= \frac{p_i}{4\pi} + \mathcal{O}((\pi-\theta)^2) \quad \text{for } \theta \simeq \pi\,. \end{aligned} \tag{4.7}$$

Finally requiring that the near horizon scalar fields are smooth at the poles gives

$$\begin{aligned} u_s(\theta) &= u_s(0) + \mathcal{O}(\theta^2) \quad \text{for } \theta \simeq 0 \\ &= u_s(\pi) + \mathcal{O}((\pi-\theta)^2) \quad \text{for } \theta \simeq \pi\,. \end{aligned} \tag{4.8}$$

Note that the smoothness of the background requires the Taylor series expansion around $\theta = 0, \pi$ to contain only even powers of θ and $(\pi - \theta)$ respectively.

A simple way to see the $SO(2,1) \times U(1)$ symmetry of the configuration (4.2) is as follows. The $U(1)$ transformation acts as a translation of ϕ and is clearly a symmetry of this configuration. In order to see the SO(2,1) symmetry of this background we regard ϕ as a compact direction and interprete this as a theory in three dimensions labelled by coordinates $\{x^m\} \equiv (r, \theta, t)$ with metric $\hat{g}_{mn}$, vectors $a^{(i)}_m$ and a_m (coming from the ϕ-m component of the metric) and scalar fields Φ_s, $\psi \equiv g_{\phi\phi}$ and $\chi_i \equiv A^{(i)}_\phi$. If we denote by $f^{(i)}_{mn}$ and f_{mn} the field strengths associated with the three dimensional gauge fields $a^{(i)}_m$ and a_m respectively, then the background (4.2) can be interpreted as the following three dimensional background:

$$\begin{aligned} &\widehat{ds}^2 \equiv \hat{g}_{mn} dx^m dx^n = v_1(\theta)\left(-r^2 dt^2 + \frac{dr^2}{r^2}\right) + \beta^2\, d\theta^2 \\ &\Phi_s = u_s(\theta), \qquad \psi = \beta^2\, v_2(\theta), \quad \chi_i = b_i(\theta)\,, \\ &\frac{1}{2}\, f^{(i)}_{mn} dx^m \wedge dx^n = e_i\, dr \wedge dt, \qquad \frac{1}{2}\, f_{mn} dx^m \wedge dx^n = -\alpha dr \wedge dt\,. \end{aligned} \tag{4.9}$$

The (r, t) coordinates now describe an AdS$_2$ space and this background is manifestly $SO(2,1)$ invariant. In this description the Killing vectors take the standard form

$$L_1 = \partial_t, \qquad L_0 = t\partial_t - r\partial_r, \qquad L_{-1} = (1/2)(1/r^2 + t^2)\partial_t - (tr)\partial_r\,. \tag{4.10}$$

Eq.(4.9) and hence (4.2) describes the most general field configuration consistent with the $SO(2,1) \times U(1)$ symmetry. Thus in order to derive the equations of motion we can evaluate the action on this background and then extremize the resulting expression with respect to the parameters labelling the background (4.2). The only exception to this are the parameters e_i and α labelling the field strengths. The variation of the action with respect to these parameters do not vanish, but give the corresponding conserved electric charges q_i and the angular momentum J (which can be regarded as the electric charge associated with the three dimensional gauge field a_m.)

To implement this procedure we define:

$$f[\alpha, \beta, \vec{e}, v_1(\theta), v_2(\theta), \vec{u}(\theta), \vec{b}(\theta)] = \int d\theta d\phi \sqrt{-\det g}\, \mathcal{L}\,. \tag{4.11}$$

Note that f is a function of α, β, e_i and a functional of $v_1(\theta)$, $v_2(\theta)$, $u_s(\theta)$ and $b_i(\theta)$. The equations of motion now correspond to[1]

$$\frac{\partial f}{\partial \alpha} = J, \quad \frac{\partial f}{\partial \beta} = 0, \quad \frac{\partial f}{\partial e_i} = q_i\,, \quad \frac{\delta f}{\delta v_1(\theta)} = 0\,, \quad \frac{\delta f}{\delta v_2(\theta)} = 0, \quad \frac{\delta f}{\delta u_s(\theta)} = 0, \quad \frac{\delta f}{\delta b_i(\theta)} = 0\,. \tag{4.12}$$

Equivalently, if we define:

$$\mathcal{E}[J, \vec{q}, \alpha, \beta, \vec{e}, v_1(\theta), v_2(\theta), \vec{u}(\theta), \vec{b}(\theta)] = 2\pi \left(J\alpha + \vec{q}\cdot\vec{e} - f[\alpha, \beta, \vec{e}, v_1(\theta), v_2(\theta), \vec{u}(\theta), \vec{b}(\theta)] \right)\,, \tag{4.13}$$

then the equations of motion take the form:

$$\frac{\partial \mathcal{E}}{\partial \alpha} = 0, \quad \frac{\partial \mathcal{E}}{\partial \beta} = 0, \quad \frac{\partial \mathcal{E}}{\partial e_i} = 0\,, \quad \frac{\delta \mathcal{E}}{\delta v_1(\theta)} = 0\,, \quad \frac{\delta \mathcal{E}}{\delta v_2(\theta)} = 0, \quad \frac{\delta \mathcal{E}}{\delta u_s(\theta)} = 0, \quad \frac{\delta \mathcal{E}}{\delta b_i(\theta)} = 0\,. \tag{4.14}$$

These equations are subject to the boundary conditions (4.4), (4.7), (4.8). For formal arguments it will be useful to express the various functions of θ appearing

[1] Our definition of the angular momentum differs from the standard one by a $-$ sign.

here by expanding them as a linear combination of appropriate basis states which make the constraints (4.4), (4.7) manifest, and then varying $\mathcal{E}$ with respect to the coefficients appearing in this expansion. The natural functions in terms of which we can expand an arbitrary ϕ-independent function on a sphere are the Legendre polynomials $P_l(\cos\theta)$. We take

$$v_1(\theta) = \sum_{l=0}^{\infty} \widetilde{v}_1(l)\, P_l(\cos\theta)\,, \quad v_2(\theta) = \sin^2\theta + \sin^4\theta \sum_{l=0}^{\infty} \widetilde{v}_2(l)\, P_l(\cos\theta)\,,$$
$$u_s(\theta) = \sum_{l=0}^{\infty} \widetilde{u}_s(l)\, P_l(\cos\theta)\,, \quad b_i(\theta) = -\frac{p_i}{4\pi}\cos\theta + \sin^2\theta \sum_{l=0}^{\infty} \widetilde{b}_i(l)\, P_l(\cos\theta)\,. \tag{4.15}$$

This expansion explicitly implements the constraints (4.4), (4.7) and (4.8). Substituting this into (4.13) gives $\mathcal{E}$ as a function of J, q_i, α, β, e_i, $\widetilde{v}_1(l)$, $\widetilde{v}_2(l)$, $\widetilde{u}_s(l)$ and $\widetilde{b}_i(l)$. Thus the equations (4.14) may now be reexpressed as

$$\frac{\partial\mathcal{E}}{\partial\alpha} = 0, \quad \frac{\partial\mathcal{E}}{\partial\beta} = 0, \quad \frac{\partial\mathcal{E}}{\partial e_i} = 0\,, \quad \frac{\partial\mathcal{E}}{\partial\widetilde{v}_1(l)} = 0\,, \quad \frac{\partial\mathcal{E}}{\partial\widetilde{v}_2(l)} = 0, \quad \frac{\partial\mathcal{E}}{\partial\widetilde{u}_s(l)} = 0, \quad \frac{\partial\mathcal{E}}{\partial\widetilde{b}_i(l)} = 0\,. \tag{4.16}$$

Let us now turn to the analysis of the entropy associated with this black hole. For this it will be most convenient to regard this configuration as a two dimensional extremal black hole by regarding the θ and ϕ directions as compact. In this interpretation the zero mode of the metric $\widehat{g}_{\alpha\beta}$ given in (4.9), with $\alpha, \beta = r, t$, is interpreted as the two dimensional metric $h_{\alpha\beta}$:

$$h_{\alpha\beta} = \frac{1}{2}\int_0^{\pi} d\theta\, \sin\theta\, \widehat{g}_{\alpha\beta}\,, \tag{4.17}$$

whereas all the non-zero modes of $\widehat{g}_{\alpha\beta}$ are interpreted as massive symmetric rank two tensor fields. This gives

$$h_{\alpha\beta}dx^\alpha dx^\beta = v_1(-r^2dt^2 + dr^2/r^2)\,, \qquad v_1 = \widetilde{v}_1(0)\,. \tag{4.18}$$

Thus the near horizon configuration, regarded from two dimensions, involves AdS_2 metric, accompanied by background electric fields $f^{(i)}_{\alpha\beta}$ and $f_{\alpha\beta}$, a set of massless and massive scalar fields originating from the fields $u_s(\theta)$, $v_2(\theta)$ and $b_i(\theta)$, and a set of massive symmetric rank two tensor fields originating from $v_1(\theta)$. According to the general results derived in [45, 46, 47, 48], the entropy of this black hole is given by:

$$S_{BH} = -8\pi \, \frac{\delta \mathcal{S}^{(2)}}{\delta R^{(2)}_{rtrt}} \sqrt{-h_{rr}\, h_{tt}}\,, \tag{4.19}$$

where $R^{(2)}_{\alpha\beta\gamma\delta}$ is the two dimensional Riemann tensor associated with the metric $h_{\alpha\beta}$, and $\mathcal{S}^{(2)}$ is the general coordinate invariant action of this two dimensional field theory. In taking the functional derivative with respect to $R_{\alpha\beta\gamma\delta}$ in (4.19) we need to express all multiple covariant derivatives in terms of symmetrized covariant derivatives and the Riemann tensor, and then regard the components of the Riemann tensor as independent variables.

We now note that for this two dimensional configuration that we have, the electric field strengths $f^{(i)}_{\alpha\beta}$ and $f_{\alpha\beta}$ are proportional to the volume form on AdS_2, the scalar fields are constants and the tensor fields are proportional to the AdS_2 metric. Thus the covariant derivatives of all gauge and generally covariant tensors which one can construct out of these two dimensional fields vanish. In this case (4.19) simplifies to:

$$S_{BH} = -8\pi \sqrt{-\det h}\, \frac{\partial \mathcal{L}^{(2)}}{\partial R^{(2)}_{rtrt}} \sqrt{-h_{rr}\, h_{tt}} \tag{4.20}$$

where $\sqrt{-\det h}\,\mathcal{L}^{(2)}$ is the two dimensional Lagrangian density, related to the four dimensional Lagrangian density via the formula:

$$\sqrt{-\det h}\,\mathcal{L}^{(2)} = \int d\theta d\phi \sqrt{-\det g}\,\mathcal{L}\,. \tag{4.21}$$

Also while computing (4.20) we set to zero all terms in $\mathcal{L}^{(2)}$ which involve covariant derivatives of the Riemann tensor and other gauge and general coordinate covariant combinations of fields.

We can now proceed in a manner identical to that in [6] to show that the right hand side of (4.20) is the entropy function at its extremum. First of all from (4.18) it follows that

$$R^{(2)}_{rtrt} = v_1 = \sqrt{-h_{rr} h_{tt}}\,. \tag{4.22}$$

Using this we can express (4.20) as

$$S_{BH} = -8\pi \sqrt{-\det h}\frac{\partial \mathcal{L}^{(2)}}{\partial R^{(2)}_{rtrt}} R^{(2)}_{rtrt}\,. \tag{4.23}$$

Let us denote by $\mathcal{L}_\lambda^{(2)}$ a deformation of $\mathcal{L}^{(2)}$ in which we replace all factors of $R^{(2)}_{\alpha\beta\gamma\delta}$ for $\alpha, \beta, \gamma, \delta = r, t$ by $\lambda R^{(2)}_{\alpha\beta\gamma\delta}$, and define

$$f_\lambda^{(2)} \equiv \sqrt{-\det h}\, \mathcal{L}_\lambda^{(2)} \,, \tag{4.24}$$

evaluated on the near horizon geometry. Then

$$\lambda \frac{\partial f_\lambda^{(2)}}{\partial \lambda} = \sqrt{-\det h}\, R^{(2)}_{\alpha\beta\gamma\delta} \frac{\partial \mathcal{L}^{(2)}}{\delta R^{(2)}_{\alpha\beta\gamma\delta}} = 4 \sqrt{-\det h}\, R^{(2)}_{rtrt} \frac{\partial \mathcal{L}^{(2)}}{\partial R^{(2)}_{rtrt}} \,. \tag{4.25}$$

Using this (4.23) may be rewritten as

$$S_{BH} = -2\pi\lambda \frac{\partial f_\lambda^{(2)}}{\partial \lambda}\Big|_{\lambda=1} \,. \tag{4.26}$$

Let us now consider the effect of the scaling

$$\lambda \to s\lambda, \quad e_i \to s e_i, \quad \alpha \to s\alpha, \quad \widetilde{v}_1(l) \to s\widetilde{v}_1(l) \quad \text{for} \quad 0 \le l < \infty \,, \tag{4.27}$$

under which $\lambda\, R^{(2)}_{\alpha\beta\gamma\delta} \to s^2\, \lambda\, R^{(2)}_{\alpha\beta\gamma\delta}$. Now since $\mathcal{L}^{(2)}$ does not involve any explicit covariant derivatives, all indices of $h^{\alpha\beta}$ must contract with the indices in $f^{(i)}_{\alpha\beta}$, $f_{\alpha\beta}$, $R^{(2)}_{\alpha\beta\gamma\delta}$ or the indices of the rank two symmetric tensor fields whose near horizon values are given by the parameters $\widetilde{v}_1(l)$. From this and the definition of the parameters e_i, $\widetilde{v}_1(l)$, and α it follows that $\mathcal{L}_\lambda^{(2)}$ remains invariant under this scaling, and hence $f_\lambda^{(2)}$ transforms to $s f_\lambda^{(2)}$, with the overall factor of s coming from the $\sqrt{-\det h}$ factor in the definition of $f_\lambda^{(2)}$. Thus we have:

$$\lambda \frac{\partial f_\lambda^{(2)}}{\partial \lambda} + e_i \frac{\partial f_\lambda^{(2)}}{\partial e_i} + \alpha \frac{\partial f_\lambda^{(2)}}{\partial \alpha} + \sum_{l=0}^{\infty} \widetilde{v}_1(l) \frac{\partial f_\lambda^{(2)}}{\partial \widetilde{v}_1(l)} = f_\lambda^{(2)} \,. \tag{4.28}$$

Now it follows from (4.11), (4.21) and (4.24) that

$$f[\alpha, \beta, \vec{e}, v_1(\theta), v_2(\theta), \vec{u}(\theta), \vec{b}(\theta)] = f^{(2)}_{\lambda=1} \,. \tag{4.29}$$

Thus the extremization equations (4.12) implies that

$$\frac{\partial f_\lambda^{(2)}}{\partial e_i} = q_i, \quad \frac{\partial f_\lambda^{(2)}}{\partial \alpha} = J, \quad \frac{\partial f_\lambda^{(2)}}{\partial \widetilde{v}_1(l)} = 0 \,, \qquad \text{at } \lambda = 1 \,. \tag{4.30}$$

Hence setting $\lambda = 1$ in (4.28) we get

$$\lambda \frac{\partial f_\lambda^{(2)}}{\partial \lambda}\Big|_{\lambda=1} = -e_i q_i - J\alpha + f_{\lambda=1}^{(2)} = -e_i q_i - J\alpha + f[\alpha, \beta, \vec{e}, v_1(\theta), v_2(\theta), \vec{u}(\theta), \vec{b}(\theta)]\,. \tag{4.31}$$

Eqs.(4.26) and the definition (4.13) of the entropy function now gives

$$S_{BH} = \mathcal{E} \tag{4.32}$$

at its extremum.

Using the fact that the black hole entropy is equal to the value of the entropy function at its extremum, we can derive some useful results following the analysis of [6, 49]. If the entropy function has a unique extremum with no flat directions then the extremization equations (4.16) determine the near horizon field configuration completely and the entropy as well as the near horizon field configuration is independent of the asymptotic moduli since the entropy function depends only on the near horizon quantities. On the other hand if the entropy function has flat directions then the extremization equations do not determine all the near horizon parameters, and these undetermined parameters could depend on the asymptotic values of the moduli fields. However even in this case the entropy, being independent of the flat directions, will be independent of the asymptotic values of the moduli fields.

Although expanding various θ-dependent functions in the basis of Legendre polynomials is useful for general argument leading to attractor behaviour, for practical computation it is often more convenient to directly solve the differential equation in θ. For this we shall need to carefully take into account the effect of the boundary terms. We shall see this while studying explicit examples.

4.2 Extremal Rotating Black Hole in General Two Derivative Theory

We now consider a four dimensional theory of gravity coupled to a set of scalar fields $\{\Phi_s\}$ and gauge fields $A_\mu^{(i)}$ with a general two derivative action of the form:[1]

$$\mathcal{S} = \int d^4x \sqrt{-\det g}\,\mathcal{L}\,, \tag{4.1}$$

$$\mathcal{L} = R - h_{rs}(\vec{\Phi}) g^{\mu\nu}\partial_\mu\Phi_s\partial_\nu\Phi_r - f_{ij}(\vec{\Phi}) g^{\mu\rho}g^{\nu\sigma}F^{(i)}_{\mu\nu}F^{(j)}_{\rho\sigma} - \frac{1}{2}\widetilde{f}_{ij}(\vec{\Phi})\,(\sqrt{-\det g})^{-1}\epsilon^{\mu\nu\rho\sigma}F^{(i)}_{\mu\nu}F^{(j)}_{\rho\sigma}\,, \tag{4.2}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the totally anti-symmetric symbol with $\epsilon^{tr\theta\phi} = 1$ and h_{rs}, f_{ij} and $\widetilde{f}_{ij}$ are fixed functions of the scalar fields $\{\Phi_s\}$. We use the following ansatz for the near horizon configuration of the scalar and gauge fields[2]

$$\begin{aligned} ds^2 &= \Omega(\theta)^2 e^{2\psi(\theta)}(-r^2dt^2 + dr^2/r^2 + \beta^2 d\theta^2) + e^{-2\psi(\theta)}(d\phi - \alpha r dt)^2 \\ \Phi_s &= u_s(\theta) \\ \frac{1}{2}F^{(i)}_{\mu\nu}dx^\mu \wedge dx^\nu &= (e_i - \alpha b_i(\theta))dr\wedge dt + \partial_\theta b_i(\theta)d\theta\wedge(d\phi - \alpha r dt)\,, \end{aligned} \tag{4.3}$$

with $0 \le \phi < 2\pi$, $0 \le \theta \le \pi$. Regularity at $\theta = 0$ and $\theta = \pi$ requires that

$$\Omega(\theta)e^{\psi(\theta)} \to \text{constant as } \theta \to 0, \pi\,, \tag{4.4}$$

and

$$\beta\Omega(\theta)e^{2\psi(\theta)}\sin\theta \to 1 \qquad \text{as } \theta \to 0, \pi\,. \tag{4.5}$$

This gives

$$\begin{aligned} \Omega(\theta) &\to a_0\sin\theta, \quad e^{\psi(\theta)} \to \frac{1}{\sqrt{\beta a_0}\sin\theta}, \quad \text{as } \theta \to 0, \\ \Omega(\theta) &\to a_\pi\sin\theta, \quad e^{\psi(\theta)} \to \frac{1}{\sqrt{\beta a_\pi}\sin\theta}, \quad \text{as } \theta \to \pi\,, \end{aligned} \tag{4.6}$$

where a_0 and a_π are arbitrary constants. In the next two sections we shall describe examples of rotating extremal black holes in various two derivative theories of

[1] In the rest of the chapter we shall be using the normalization of the Einstein-Hilbert term as given in eq.(4.2). This corresponds to choosing the Newton's constant G_N to be $1/16\pi$.

[2] This is related to the ansatz (4.2) by a reparametrization of the θ coordinate.

gravity with near horizon geometry of the form described above. However none of these black holes will be supersymmetric even though many of them will be found in supersymmetric theories.

Using (4.2), (4.3) and (4.5) we get

$$
\begin{aligned}
\mathcal{E} \; &\equiv \; 2\pi(J\alpha + \vec{q}\cdot\vec{e} - \int d\theta d\phi \sqrt{-\det g}\,\mathcal{L}) \\
&= \; 2\pi J\alpha + 2\pi\vec{q}\cdot\vec{e} - 4\pi^2 \int d\theta \Big[2\Omega(\theta)^{-1}\beta^{-1}(\Omega'(\theta))^2 - 2\Omega(\theta)\beta - 2\Omega(\theta)\beta^{-1}(\psi'(\theta))^2 \\
&\quad + \frac{1}{2}\alpha^2\Omega(\theta)^{-1}\beta e^{-4\psi(\theta)} - \beta^{-1}\Omega(\theta)h_{rs}(\vec{u}(\theta))u_r'(\theta)u_s'(\theta) + 4\widetilde{f}_{ij}(\vec{u}(\theta))(e_i - \alpha b_i(\theta))b_j'(\theta) \\
&\quad + 2f_{ij}(\vec{u}(\theta))\left\{\beta\Omega(\theta)^{-1}e^{-2\psi(\theta)}(e_i - \alpha b_i(\theta))(e_j - \alpha b_j(\theta)) - \beta^{-1}\Omega(\theta)e^{2\psi(\theta)}b_i'(\theta)b_j'(\theta)\right\}\Big] \\
&\quad + 8\pi^2 \left[\Omega(\theta)^2 e^{2\psi(\theta)} \sin\theta(\psi'(\theta) + 2\Omega'(\theta)/\Omega(\theta))\right]_{\theta=0}^{\theta=\pi} .
\end{aligned}
\tag{4.7}
$$

The boundary terms in the last line of (4.7) arise from integration by parts in $\int \sqrt{-\det g}\mathcal{L}$. Eq.(4.7) has the property that under a variation of Ω for which $\delta\Omega/\Omega$ does not vanish at the boundary and/or a variation of ψ for which $\delta\psi$ does not vanish at the boundary, the boundary terms in $\delta\mathcal{E}$ cancel if (4.6) is satisfied. This ensures that once the $\mathcal{E}$ is extremized under variations of ψ and Ω for which $\delta\psi$ and $\delta\Omega$ vanish at the boundary, it is also extremized with respect to the constants a_0 and a_π appearing in (4.6) which changes the boundary values of Ω and ψ. Also due to this property we can now extremize the entropy function with respect to β without worrying about the constraint (4.5) since the additional term that comes from the compensating variation in Ω and/or ψ will vanish due to Ω and/or ψ equations of motion.

The equations of motion of various fields may now be obtained by extremizing the entropy function $\mathcal{E}$ with respect to the functions $\Omega(\theta)$, $\psi(\theta)$, $u_s(\theta)$, $b_i(\theta)$ and the parameters e_i, α, β labelling the near horizon geometry. This gives

$$
\begin{aligned}
& -4\beta^{-1}\Omega''(\theta)/\Omega(\theta) + 2\beta^{-1}(\Omega'(\theta)/\Omega(\theta))^2 - 2\beta - 2\beta^{-1}(\psi'(\theta))^2 - \frac{1}{2}\alpha^2\Omega(\theta)^{-2}\beta e^{-4\psi(\theta)} \\
& -\beta^{-1}h_{rs}(\vec{u}(\theta))u_r'(\theta)u_s'(\theta) \\
& +2f_{ij}(\vec{u}(\theta))\left\{-\beta\Omega(\theta)^{-2}e^{-2\psi(\theta)}(e_i - \alpha b_i(\theta))(e_j - \alpha b_j(\theta)) - \beta^{-1}e^{2\psi(\theta)}b_i'(\theta)b_j'(\theta)\right\} \\
& = 0 ,
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
&4\beta^{-1}\Omega(\theta)\psi''(\theta) + 4\beta^{-1}\Omega'(\theta)\psi'(\theta) - 2\alpha^2\Omega(\theta)^{-1}\beta e^{-4\psi(\theta)} \\
&+2f_{ij}(\vec{u}(\theta))\left\{-2\beta\Omega(\theta)^{-1}e^{-2\psi(\theta)}(e_i-\alpha b_i(\theta))(e_j-\alpha b_j(\theta)) \; -2\beta^{-1}\Omega(\theta)e^{2\psi(\theta)}b_i'(\theta)b_j'(\theta)\right\} \\
&= 0\,,
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
&2\left(\beta^{-1}\Omega(\theta)h_{rs}(\vec{u}(\theta))u_s'(\theta)\right)' - \beta^{-1}\Omega(\theta)\partial_r h_{ts}(\vec{u}(\theta))u_t'(\theta)u_s'(\theta) \\
&+2\partial_r f_{ij}(\vec{u}(\theta))\left\{\beta\Omega(\theta)^{-1}e^{-2\psi(\theta)}(e_i-\alpha b_i(\theta))(e_j-\alpha b_j(\theta)) - \beta^{-1}\Omega(\theta)e^{2\psi(\theta)}b_i'(\theta)b_j'(\theta)\right\} \\
&+4\partial_r\widetilde{f}_{ij}(\vec{u}(\theta))(e_i-\alpha b_i(\theta))b_j'(\theta) \\
&= 0\,,
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
&-4\alpha\beta f_{ij}(\vec{u}(\theta))\Omega(\theta)^{-1}e^{-2\psi(\theta)}(e_j-\alpha b_j(\theta)) + 4\beta^{-1}\left(f_{ij}(\vec{u}(\theta))\Omega(\theta)e^{2\psi(\theta)}b_j'(\theta)\right)' \\
&\quad -4\partial_r\widetilde{f}_{ij}(\vec{u}(\theta))u_r'(\theta)(e_j-\alpha b_j(\theta)) = 0\,,
\end{aligned}
\tag{4.11}
$$

$$
q_i = 8\pi\int d\theta\,\left[f_{ij}(\vec{u}(\theta))\beta\Omega(\theta)^{-1}e^{-2\psi(\theta)}(e_j-\alpha b_j(\theta)) + \widetilde{f}_{ij}(\vec{u}(\theta))b_j'(\theta)\right]\,, \tag{4.12}
$$

$$
\begin{aligned}
J \;=\; 2\pi\int_0^\pi d\theta\,&\Big[\alpha\Omega(\theta)^{-1}\beta e^{-4\psi(\theta)} - 4\beta f_{ij}(\vec{u}(\theta))\Omega(\theta)^{-1}e^{-2\psi(\theta)}(e_i-\alpha b_i(\theta))b_j(\theta) \\
&-4\widetilde{f}_{ij}(\vec{u}(\theta))b_i(\theta)b_j'(\theta)\Big]\,,
\end{aligned}
\tag{4.13}
$$

$$
\int d\theta\, I(\theta) = 0\,, \tag{4.14}
$$

$$
\begin{aligned}
I(\theta) \;\equiv\; &-2\Omega(\theta)^{-1}\beta^{-2}(\Omega'(\theta))^2 - 2\Omega(\theta) + 2\Omega(\theta)\beta^{-2}(\psi'(\theta))^2 + \frac{1}{2}\alpha^2\Omega(\theta)^{-1}e^{-4\psi(\theta)} \\
&+\beta^{-2}\Omega(\theta)h_{rs}(\vec{u}(\theta))u_r'(\theta)u_s'(\theta) \\
&+2f_{ij}(\vec{u}(\theta))\left\{\Omega(\theta)^{-1}e^{-2\psi(\theta)}(e_i-\alpha b_i(\theta))(e_j-\alpha b_j(\theta)) + \beta^{-2}\Omega(\theta)e^{2\psi(\theta)}b_i'(\theta)b_j'(\theta)\right\}\,.
\end{aligned}
\tag{4.15}
$$

Here $'$ denotes derivative with respect to θ. The required boundary conditions, following from the requirement of the regularity of the solution at $\theta = 0$, π, and that the magnetic charge vector be $\vec{p}$, are:

$$
b_i(0) = -\frac{p_i}{4\pi}, \qquad b_i(\pi) = \frac{p_i}{4\pi}\,, \tag{4.16}
$$

$$
\Omega(\theta)e^{\psi(\theta)} \to \text{constant as } \theta \to 0, \pi\,, \tag{4.17}
$$

$$\beta\Omega(\theta)e^{2\psi(\theta)}\sin\theta \to 1 \qquad \text{as } \theta \to 0, \pi\,. \tag{4.18}$$

$$u_s(\theta) \to \text{constant as } \theta \to 0, \pi\,. \tag{4.19}$$

Using eqs.(4.8)-(4.11) one can show that

$$I'(\theta) = 0\,. \tag{4.20}$$

Thus $I(\theta)$ is independent of θ. As a consequence of eq.(4.14) we now have

$$I(\theta) = 0\,. \tag{4.21}$$

Combining eqs.(4.8) and (4.21) we get

$$\Omega'' + \beta^2\Omega = 0\,. \tag{4.22}$$

A general solution to this equation is of the form

$$\Omega = a\sin(\beta\theta + b)\,, \tag{4.23}$$

where a and b are integration constants. In order that Ω has the behaviour given in (4.6) for θ near 0 and π, and not vanish at any other value of θ, we must have

$$b = 0, \qquad \beta = 1\,, \tag{4.24}$$

and hence

$$\Omega(\theta) = a\sin\theta\,. \tag{4.25}$$

In order to analyze the rest of the equations, it will be useful to consider the Taylor series expansion of $u_r(\theta)$ and $b_i(\theta)$ around $\theta = 0, \pi$

$$\begin{aligned}
u_r(\theta) &= u_r(0) + \frac{1}{2}\theta^2 u_r''(0) + \cdots \\
u_r(\theta) &= u_r(\pi) + \frac{1}{2}(\theta-\pi)^2 u_r''(\pi) + \cdots \\
b_i(\theta) &= b_i(0) + \frac{1}{2}\theta^2 b_i''(0) + \cdots \\
b_i(\theta) &= b_i(\pi) + \frac{1}{2}(\theta-\pi)^2 b_i''(\pi) + \cdots\,,
\end{aligned} \tag{4.26}$$

where we have made use of (4.7), (4.8). We now substitute (4.26) into (4.11) and study the equation near $\theta = 0$ by expanding the left hand side of the equation in powers of θ and using the boundary conditions (4.6). Only odd powers of θ are non-zero. The first non-trivial equation, appearing as the coefficient of the order θ term, involves $b_i(0)$, $b_i''(0)$ and $b_i''''(0)$ and can be used to determine $b_i''''(0)$ in terms of $b_i(0)$ and $b_i''(0)$. Higher order terms determine higher derivatives of b_i at $\theta = 0$ in terms of $b_i(0)$ and $b_i''(0)$. As a result $b_i''(0)$ is not determined in terms of $b_i(0)$ by solving the equations of motion near $\theta = 0$ and we can choose $b_i(0)$ and $b_i''(0)$ as the two independent integration constants of this equation. Of these $b_i(0)$ is determined directly from (4.16). On the other hand for a given configuration of the other fields, $b_i''(0)$ is also determined from (4.16) indirectly by requiring that $b_i(\pi)$ be $p_i/4\pi$. Thus we expect that generically the integration constants associated with the solutions to eqs.(4.11) are fixed by the boundary conditions (4.16).

Let us now analyze eqs.(4.10) and (4.21) together, – eq.(4.9) holds automatically when the other equations are satisfied. For this it will be useful to introduce a new variable

$$\tau = \ln \tan \frac{\theta}{2}\,, \tag{4.27}$$

satisfying

$$\frac{d\tau}{d\theta} = \frac{1}{\sin\theta}\,. \tag{4.28}$$

As θ varies from 0 to π, τ varies from $-\infty$ to ∞. We denote by $\cdot$ derivative with respect to τ and rewrite eqs.(4.10) and (4.21) in this variable. This gives

$$\begin{aligned} 2a^2(h_{rs}(\vec{u})\dot{u}_s)^{\cdot} - a^2\partial_t h_{rs}(\vec{u})\dot{u}_t\dot{u}_s + 4a\partial_r \widetilde{f}_{ij}(\vec{u})(e_i - \alpha b_i)\dot{b}_j \\ +2\partial_r f_{ij}(\vec{u})\left\{e^{-2\psi}(e_i-\alpha b_i)(e_j-\alpha b_j) - a^2 e^{2\psi}\dot{b}_i\dot{b}_j\right\} = 0\,, \end{aligned} \tag{4.29}$$

and

$$-2a^2+2a^2\dot{\psi}^2+\frac{1}{2}\alpha^2 e^{-4\psi}+a^2 h_{rs}(\vec{u})\dot{u}_r\dot{u}_s+2f_{ij}(\vec{u})\left\{e^{-2\psi}(e_i-\alpha b_i)(e_j-\alpha b_j)+a^2e^{2\psi}\dot{b}_i\dot{b}_j\right\}=0\,. \tag{4.30}$$

If we denote by m the number of scalars then we have a set of m second order differential equations and one first order differential equation, giving altogether $2m+1$ constants of integration. We want to see in a generic situation how many of these constants are fixed by the required boundary conditions on $\vec{u}$ and ψ. We shall do this by requiring that the equations and the boundary conditions are consistent. Thus for example if ψ, $\{b_i\}$ and $\{u_s\}$ satisfy their required boundary conditions then we can express the equations near $\theta = 0$ (or $\theta = \pi$) as:

$$2a^2(\hat{h}_{rs}\dot{u}_s)^{\cdot} \simeq 0\,, \tag{4.31}$$

and

$$-2a^2 + a^2\hat{h}_{rs}\dot{u}_r\dot{u}_s + 2a^2\dot{\psi}^2 \simeq 0\,. \tag{4.32}$$

Here $\hat{h}_{rs}$ are constants giving the value of $h_{rs}(\vec{u})$ at $\vec{u} = \vec{u}(0)$ (or $\vec{u} = \vec{u}(\pi)$). Note that we have used the boundary conditions to set some of the terms to zero but have kept the terms containing highest derivatives of ψ and u_r even if they are required to vanish due to the boundary conditions. The general solutions to these equations near $\theta = 0$ are

$$u_s(\theta) \simeq c_s + v_s\tau\,, \quad \psi(\theta) \simeq c - \tau\sqrt{1 - \frac{1}{2}\hat{h}_{rs}v_sv_s}\,. \tag{4.33}$$

where c_s, v_s and c are the $2m+1$ integration constants. Since $\tau \to -\infty$ as $\theta \to 0$, in order that u_s approaches a constant value $u_s(0)$ as $\theta \to 0$, we must require all the v_s to vanish. On the other hand requiring that ψ satisfies the boundary condition (4.18) determines c to be $-\ln(2\sqrt{a})$. This gives altogether $m+1$ conditions on the $(2m+1)$ integration constants. Carrying out the same analysis near $\theta = \pi$ gives another $(m+1)$ conditions among the integration constants. Thus the boundary conditions on $\vec{u}$ and ψ not only determine all $(2m+1)$ integration constants of (4.29), (4.30), but give an additional condition among the as yet unknown parameters a, α and e_i entering the equations.

This constraint, together with the remaining equations (4.12) and (4.13), gives altogether $n+2$ constraints on the $n+2$ variables e_i, a and α, where n is the number of $U(1)$ gauge fields. Since generically $(n+2)$ equations in $(n+2)$ variables have only a discrete number of solutions we expect that generically the solution to eqs.(4.8)-(4.19) has no continuous parameters.

In special cases however some of the integration constants may remain undetermined, reflecting a family of solutions corresponding to the same set of charges. As discussed in section 4.1, these represent flat directions of the entropy function and hence the entropy associated with all members of this family will have identical values. We shall now give a more direct argument to this effect. Suppose as we go from one member of the family to a neighbouring member, each scalar field changes to

$$u_r(\theta) \to u_r(\theta) + \delta u_r(\theta), \tag{4.34}$$

and suppose all the other fields and parameters change in response, keeping the electric charges q_i, magnetic charges p_i and angular momentum fixed:

$$\begin{aligned} \Omega \to \Omega + \delta\Omega, \quad \psi \to \psi + \delta\psi, \quad b_i \to b_i + \delta b_i, \\ e_i \to e_i + \delta e_i, \quad \alpha \to \alpha + \delta\alpha, \quad \beta \to \beta + \delta\beta\,. \end{aligned} \tag{4.35}$$

Let us calculate the resulting change in the entropy $\mathcal{E}$. The changes in e_i, α, β do not contribute to any change in $\mathcal{E}$, since $\partial_{e_i}\mathcal{E} = 0$, $\partial_\alpha \mathcal{E} = 0$ and $\partial_\beta \mathcal{E} = 0$. The only possible contributions from varying Ω, ψ, b_i, u_r can come from boundary terms, since the bulk equations are satisfied. Varying $\mathcal{E}$ subject to the equations of motion, one finds the following boundary terms at the poles:

$$\begin{aligned} \delta\mathcal{E} &= 8\pi^2 \Big[\beta^{-1}\Omega h_{rs} u_r' \delta u_s - 2\widetilde{f}_{ij}(e_i - \alpha b_i)\delta b_j + 2 f_{ij}\left\{\beta^{-1}\Omega e^{2\psi} b_i'\right\}\delta b_j \\ &\quad + \beta^{-1}\left(-2\Omega^{-1}\Omega'\delta\Omega + 2\Omega\psi'\delta\psi + \delta(\Omega\psi' + 2\Omega')\right)\Big]_{\theta=0}^{\theta=\pi}\,. \end{aligned} \tag{4.36}$$

Terms involving δb_i at the boundary vanish since the boundary conditions (4.16), (4.26) imply that for fixed magnetic charges δb_i and b_i' must vanish at $\theta = 0$ and $\theta = \pi$. Our boundary conditions imply that variations of Ω and ψ at the poles are not independent. From the boundary condition (4.5) it follows that

$$\delta\Omega = -2\Omega\delta\psi \tag{4.37}$$

at $\theta = 0$, π, while from (4.6) one can see that at the poles

$$\delta\psi' = 0\,. \tag{4.38}$$

Combining the previous two equations gives

$$\delta\Omega' = -2\Omega'\delta\psi \tag{4.39}$$

at the poles. If we vary just Ω and ψ one finds

$$\begin{aligned} \delta_{\{\Omega,\psi\}}\mathcal{E} &= 8\pi^2\beta^{-1}\left[-2\Omega^{-1}\Omega'\delta\Omega + 2\Omega\psi'\delta\psi + \delta(\Omega\psi' + 2\Omega')\right]_{\theta=0}^{\theta=\pi} \\ &= 8\pi^2\beta^{-1}\left[4\Omega'\delta\psi + 2\Omega\psi'\delta\psi + \psi'\delta\Omega + 2\delta\Omega'\right]_{\theta=0}^{\theta=\pi} \\ &= 0\,. \end{aligned} \tag{4.40}$$

Finally, the boundary terms proportional to δu_r go like,

$$\delta_{\vec{u}}\mathcal{E} \propto \Big[\Omega h_{rs} u_r' \delta u_s\Big]_0^\pi . \tag{4.41}$$

Since $\Omega \to 0$ as $\theta \to 0, \pi$, these too vanish. Thus we learn that the entropy is independent of any undetermined constant of integration.

Before concluding this section we would like to note that using the equations of motion for various fields we can express the charges q_i, the angular momentum J as well as the black hole entropy, i.e. the value of the entropy function at its extremum, as boundary terms evaluated at $\theta = 0$ and $\theta = \pi$. For example using (4.11) we can express (4.12) as

$$q_i = \frac{8\pi}{\alpha}\Big[f_{ij}\Omega e^{2\psi} b_j' - \tilde{f}_{ij}(e_j - \alpha b_j)\Big]_{\theta=0}^{\theta=\pi} \tag{4.42}$$

Similarly using (4.9) and (4.11) we can express (4.13) as

$$J = \frac{4\pi}{\alpha}\Big[\Omega\psi' - \Omega f_{ij} e^{2\psi} b_i b_j' + \tilde{f}_{ij} b_i (e_j - \alpha b_j)\Big]_{\theta=0}^{\theta=\pi} - \frac{q_i e_i}{2\alpha} \tag{4.43}$$

Finally using (4.8), (4.9) we can express the entropy function $\mathcal{E}$ given in (4.7) as

$$\mathcal{E} = 8\pi^2\left[-2\Omega' + \Omega^2 e^{2\psi}\sin\theta\left(\psi' + 2\frac{\Omega'}{\Omega}\right)\right]_{\theta=0}^{\theta=\pi} \tag{4.44}$$

Using eq.(4.25) and the boundary conditions (4.6) this gives,

$$\mathcal{E} = 16\pi^2 a \tag{4.45}$$

Using eqs. (4.3) and (4.25) it is easy to see that $\mathcal{E} = A/4G_N$ where A is the area of the event horizon. (Note that in our conventions $G_N = 1/16\pi$). This is the expected result for theories with two derivative action.

4.3 Solutions with Constant Scalars

In this section we shall solve the equations derived in section 4.2 in special cases where there are no scalars or where the scalars $u_s(\theta)$ are constants:

$$\vec{u}(\theta) = \vec{u}_0 . \tag{4.1}$$

In this case we can combine (4.9), (4.21), (4.24) and (4.25) to get

$$\sin^2\theta(\psi''+(\psi')^2)+\sin\theta\cos\theta\psi'-\frac{\alpha^2}{4a^2}e^{-4\psi}-1=0\,. \tag{4.2}$$

The unique solution to this equation subject to the boundary conditions (4.18) is:

$$e^{-2\psi(\theta)} = \frac{2a\sin^2\theta}{2-(1-\sqrt{1-\alpha^2})\sin^2\theta}\,. \tag{4.3}$$

We now define the coordinate ξ through the relation:

$$\xi=-\frac{2}{\alpha}\tan^{-1}\left(\frac{\alpha}{1+\sqrt{1-\alpha^2}}\cos\theta\right)\,, \tag{4.4}$$

so that

$$d\xi=\frac{d\theta}{\Omega(\theta)e^{2\psi(\theta)}}\,. \tag{4.5}$$

As θ varies from 0 to π, ξ varies from $-\xi_0$ to ξ_0, with ξ_0 given by

$$\xi_0=\frac{1}{\alpha}\sin^{-1}\alpha\,. \tag{4.6}$$

In terms of this new coordinate ξ, (4.11) takes the form:

$$\frac{d^2}{d\xi^2}(e_i-\alpha b_i(\theta))+\alpha^2(e_i-\alpha b_i(\theta))=0\,. \tag{4.7}$$

This has solution:

$$(e_i-\alpha b_i(\theta))=A_i\sin\left(\alpha\xi+B_i\right)\,, \tag{4.8}$$

where A_i and B_i are integration constants. These can be determined using the boundary condition (4.16):

$$A_i\sin(-\alpha\xi_0+B_i)=e_i+\alpha\frac{p_i}{4\pi}\,,\qquad A_i\sin(\alpha\xi_0+B_i)=e_i-\alpha\frac{p_i}{4\pi}\,. \tag{4.9}$$

This gives

$$\begin{aligned} B_i &= \tan^{-1}\left(-\frac{4\pi e_i}{\alpha p_i}\tan(\alpha\xi_0)\right)=\tan^{-1}\left(-\frac{4\pi e_i}{p_i\sqrt{1-\alpha^2}}\right)\,,\\ A_i &= \left(\frac{e_i^2}{\cos^2(\alpha\xi_0)}+\frac{\alpha^2p_i^2}{16\pi^2\sin^2(\alpha\xi_0)}\right)^{1/2}=\left(\frac{e_i^2}{1-\alpha^2}+\frac{p_i^2}{16\pi^2}\right)^{1/2}\,. \end{aligned} \tag{4.10}$$

Using (4.42) we now get:

$$q_i = 16\pi \sum_j \left(f_{ij}(\vec{u}_0)\sin B_j - \widetilde{f}_{ij}(\vec{u}_0)\cos B_j \right) A_j = 16\,\pi \sum_j \left\{ f_{ij}(\vec{u}_0)\frac{e_j}{\sqrt{1-\alpha^2}} + \widetilde{f}_{ij}(\vec{u}_0)\,\frac{p_j}{4\pi} \right\}. \tag{4.11}$$

This gives A_i, B_i and e_i in terms of a, α, $\vec{u}_0$ and the charges $\vec{q}$, $\vec{p}$, J.

Substituting the known solutions for $\Omega(\theta)$, $\psi(\theta)$ and $b_i(\theta)$ into eq.(4.21) and evaluating the left hand side of this equation at $\theta = \pi/2$ we get

$$a\sqrt{1-\alpha^2} = \sum_{i,j} f_{ij}(\vec{u}_0) A_i A_j \cos(B_i - B_j) = \sum_{i,j} f_{ij}(\vec{u}_0) \left\{ \frac{p_i p_j}{16\pi^2} + \frac{e_i e_j}{1-\alpha^2} \right\}. \tag{4.12}$$

On the other hand (4.43) gives

$$J = 8\pi a\alpha\,. \tag{4.13}$$

Since A_i, B_i and e_i are known in terms of a, α, $\vec{u}_0$ and $\vec{q}$, $\vec{p}$, J, we can use (4.12) and (4.13) to solve for α and a in terms of $\vec{u}_0$, $\vec{q}$, $\vec{p}$ and J. (4.45) then gives the black hole entropy in terms of $\vec{u}_0$, $\vec{q}$, $\vec{p}$ and J. The final results are:

$$\alpha = \frac{J}{\sqrt{J^2 + V_{eff}(\vec{u}_0,\vec{q},\vec{p})^2}}, \qquad a = \frac{\sqrt{J^2 + V_{eff}(\vec{u}_0,\vec{q},\vec{p})^2}}{8\pi}, \tag{4.14}$$

and

$$S_{BH} = 2\pi\sqrt{J^2 + V_{eff}(\vec{u}_0,\vec{q},\vec{p})^2}\,, \tag{4.15}$$

where

$$V_{eff}(\vec{u}_0,\vec{q},\vec{p}) = \frac{1}{32\pi} f^{ij}(\vec{u}_0)\widehat{q}_i\widehat{q}_j + \frac{1}{2\pi} f_{ij}(\vec{u}_0) p_i p_j \tag{4.16}$$

is the effective potential introduced in [5]. Here $f^{ij}(\vec{u}_0)$ is the matrix inverse of $f_{ij}(\vec{u}_0)$ and

$$\widehat{q}_i \equiv q_i - 4\,\widetilde{f}_{ij}(\vec{u}_0)\,p_j\,. \tag{4.17}$$

Finally we turn to the determination of $\vec{u}_0$. If there are no scalars present in the theory then of course there are no further equations to be solved. In the presence of scalars we need to solve the remaining set of equations (4.10). In the special case when all the f_{ij} and $\widetilde{f}_{ij}$ are independent of $\vec{u}$ these equations are satisfied by any constant $\vec{u} = \vec{u}_0$. Thus $\vec{u}_0$ is undetermined and represent flat directions of the entropy function. However if f_{ij} and $\widetilde{f}_{ij}$ depend on $\vec{u}$ then there will be constraints on $\vec{u}_0$. First of all note that since the entropy must be extremized with respect to all possible deformations consistent with the $SO(2,1)\times U(1)$ symmetry, it must be extremized with respect to $\vec{u}_0$. This in turn requires that $\vec{u}_0$ be an extremum of $V_{eff}(\vec{u}_0,\vec{q},\vec{p})$ as in [5]. In this case however there are further conditions coming from (4.10) since the entropy function must also be extremized with respect to variations for which the scalar fields are not constant on the horizon. In fact in the generic situation it is almost impossible to satisfy (4.10) with constant $\vec{u}(\theta)$. We shall now discuss a special case where it is possible to satisfy these equations, – this happens for purely electrically charged black holes when there are no $F\widetilde{F}$ coupling in the theory (i.e. when $\widetilde{f}_{ij}(\vec{u})=0$).[1] In this case (4.10) gives

$$B_i=\frac{\pi}{2}\,,\qquad A_i=\frac{e_i}{\cos(\alpha\xi_0)}=\frac{e_i}{\sqrt{1-\alpha^2}}\,, \tag{4.18}$$

and eqs.(4.11), (4.8) give, respectively,

$$A_i=\frac{1}{16\pi}f^{ij}(\vec{u}_0)q_j,\qquad e_i=\frac{\sqrt{1-\alpha^2}}{16\pi}f^{ij}(\vec{u}_0)q_j\,, \tag{4.19}$$

$$\begin{aligned}(e_i-\alpha b_i(\theta)) &= A_i\cos(\alpha\xi)=\frac{1}{16\pi}f^{ij}(\vec{u}_0)q_j\cos(\alpha\xi)\\ &= \frac{1}{16\pi}f^{ij}(\vec{u}_0)q_j\,\frac{2\sqrt{1-\alpha^2}+(1-\sqrt{1-\alpha^2})\sin^2\theta}{2-(1-\sqrt{1-\alpha^2})\sin^2\theta}\,.\end{aligned} \tag{4.20}$$

If following (4.16) we now define:

$$V_{eff}(\vec{u},\vec{q})=\frac{1}{32\pi}\,f^{ij}(\vec{u})q_iq_j\,, \tag{4.21}$$

then substituting the known solutions for Ω and ψ into eq.(4.10) and using (4.20) we can see that (4.10) is satisfied if the scalars are at an extremum $\vec{u}_0$ of V_{eff}, i.e.

$$\partial_r V_{eff}(\vec{u}_0,\vec{q})=0\,. \tag{4.22}$$

With the help of (4.19), eq.(4.12) now takes the form:

$$a\sqrt{1-\alpha^2}=\frac{1}{256\pi^2}f^{ij}(\vec{u}_0,\vec{q})q_iq_j=\frac{1}{8\pi}\,V_{eff}(\vec{u}_0,\vec{q})\,, \tag{4.23}$$

[1] Clearly there are other examples with non-vanishing p_i and/or $\widetilde{f}_{ij}$ related to this one by electric-magnetic duality rotation.

Using (4.13), (4.23) we get

$$\alpha = \frac{J}{\sqrt{J^2 + (V_{eff}(\vec{u}_0, \vec{q}))^2}}\,, \qquad a = \frac{\sqrt{J^2 + (V_{eff}(\vec{u}_0, \vec{q}))^2}}{8\pi}\,, \tag{4.24}$$

$$\begin{aligned} \Omega &= \frac{\sqrt{J^2 + (V_{eff}(\vec{u}_0, \vec{q}))^2}}{8\pi} \sin\theta, \\ e^{-2\psi} &= \frac{1}{4\pi} \frac{\left(J^2 + (V_{eff}(\vec{u}_0, \vec{q}))^2\right) \sin^2\theta}{(1+\cos^2\theta)\sqrt{J^2 + (V_{eff}(\vec{u}_0, \vec{q}))^2} + V_{eff}(\vec{u}_0, \vec{q}) \sin^2\theta}\,, \end{aligned} \tag{4.25}$$

$$(e_i - b_i(\theta)) = \frac{1}{16\pi} f^{ij}(\vec{u}_0) q_j \, \frac{2V_{eff} + (\sqrt{J^2 + V_{eff}^2} - V_{eff}) \sin^2\theta}{2\sqrt{J^2 + V_{eff}^2} - (\sqrt{J^2 + V_{eff}^2} - V_{eff}) \sin^2\theta} \tag{4.26}$$

Eq.(4.45) now gives the black hole entropy to be

$$S_{BH} = 2\pi \sqrt{J^2 + (V_{eff}(\vec{u}_0, \vec{q}))^2}\,. \tag{4.27}$$

We shall now illustrate the results using explicit examples of extremal Kerr black hole and extremal Kerr-Newman black hole.

4.3.1 Extremal Kerr Black Hole in Einstein Gravity

We consider ordinary Einstein gravity in four dimensions with action

$$\mathcal{S} = \int d^4x \sqrt{-\det g}\, \mathcal{L}, \qquad \mathcal{L} = R\,. \tag{4.28}$$

In this case since there are no matter fields we have $V_{eff}(\vec{u}_0, \vec{q}) = 0$. Let us for definiteness consider the case where $J > 0$. It then follows from the general results derived earlier that

$$\alpha = 1, \qquad a = \frac{J}{8\pi}\,, \tag{4.29}$$

$$\Omega = \frac{J}{8\pi} \sin\theta, \qquad e^{-2\psi} = \frac{J}{4\pi} \frac{\sin^2\theta}{1+\cos^2\theta}\,, \tag{4.30}$$

and

$$S_{BH} = 2\pi J\,. \tag{4.31}$$

Thus determines the near horizon geometry and the entropy of an extremal Kerr black hole and agrees with the results of [8].

4.3.2 Extremal Kerr-Newman Black Hole in Einstein-Maxwell Theory

Here we consider Einstein gravity in four dimensions coupled to a single Maxwell field:

$$\mathcal{S} = \int d^4x \sqrt{-\det g}\,\mathcal{L}, \qquad \mathcal{L} = R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \,. \tag{4.32}$$

In this case we have $f_{11} = \frac{1}{4}$. Hence $f^{11} = 4$ and

$$V_{eff}(\vec{u}_0, \vec{q}) = \frac{q^2}{8\pi} \,. \tag{4.33}$$

Thus we have

$$\alpha = \frac{J}{\sqrt{J^2 + (q^2/8\pi)^2}}\,, \qquad a = \frac{\sqrt{J^2 + (q^2/8\pi)^2}}{8\pi}\,. \tag{4.34}$$

$$\Omega = a \sin\theta, \qquad e^{-2\psi} = \frac{2a \sin^2\theta}{1 + \cos^2\theta + q^2 \sin^2\theta \Big/ \left(8\pi\sqrt{J^2 + (q^2/8\pi)^2}\right)}\,, \tag{4.35}$$

and

$$S_{BH} = 2\pi \sqrt{J^2 + (q^2/8\pi)^2}\,. \tag{4.36}$$

The near horizon geometry given in (4.34), (4.35) agrees with the results of [8].

Comparing (4.24)-(4.27) with (4.34)-(4.36) we see that the results for the general case of constant scalar field background is obtained from the results for extremal Kerr-Newman black hole carrying electric charge q via the replacement of q by q_{eff} where

$$q_{eff} = \sqrt{8\pi\, V_{eff}(\vec{u}_0, \vec{q})}\,. \tag{4.37}$$

4.4 Examples of Attractor Behaviour in Full Black Hole Solutions

The set of equations (4.8)-(4.13) and (4.21) are difficult to solve explicitly in the general case. However there are many known examples of rotating extremal black hole solutions in a variety of two derivative theories of gravity. In this section we shall examine the near horizon geometry of these solutions and check that they obey the consequences of the generalized attractor mechanism discussed in sections 4.1 and 4.2.

4.4.1 Rotating Kaluza-Klein Black Holes

In this section we consider the four dimensional theory obtained by dimensional reduction of the five dimensional pure gravity theory on a circle. The relevant four dimensional fields include the metric $g_{\mu\nu}$, a scalar field Φ associated with the radius of the fifth dimension and a U(1) gauge field A_μ. The lagrangian density is given by

$$\mathcal{L} = R - 2g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - e^{2\sqrt{3}\Phi}g^{\mu\rho}g^{\nu\sigma}\,F_{\mu\nu}F_{\rho\sigma}\,. \tag{4.1}$$

Identifying Φ as Φ_1 and A_μ as $A_\mu^{(1)}$ and comparing (4.2) and (4.1) we see that we have in this example

$$h_{11} = 2, \quad f_{11} = e^{2\sqrt{3}\Phi}\,. \tag{4.2}$$

Suppose we have an extremal rotating black hole solution in this theory with near horizon geometry of the form given in (4.3). Let us define $\tau = \ln\tan(\theta/2)$ as in (4.27), denote by $\cdot$ derivative with respect to τ and define

$$\chi(\theta) = e - \alpha b(\theta)\,. \tag{4.3}$$

Using (4.24) and (4.25) we can now express appropriate linear combinations of eqs.(4.9) - (4.11) and (4.21) as

$$\ddot{\psi} = \frac{\alpha^2}{4a^2}e^{-4\psi} + 1 - \dot{\psi}^2 - \dot{\Phi}^2 \tag{4.4}$$

$$\ddot{\Phi} + \sqrt{3}e^{2\sqrt{3}\Phi}\left\{e^{-2\psi}a^{-2}\chi^2 - \alpha^{-2}e^{2\psi}\dot{\chi}^2\right\} = 0 \tag{4.5}$$

$$\alpha^2a^{-2}e^{2\sqrt{3}\Phi-2\psi}\chi + \left(e^{2\sqrt{3}\Phi+2\psi}\dot{\chi}\right)^{\cdot} = 0\,. \tag{4.6}$$

$$-2a^2+2a^2\dot{\psi}^2+\frac{1}{2}\alpha^2e^{-4\psi}+2a^2\dot{\Phi}^2+2\left\{e^{2\sqrt{3}\Phi-2\psi}\chi^2 + a^2\alpha^{-2}e^{2\sqrt{3}\Phi+2\psi}\dot{\chi}^2\right\} = 0\,. \tag{4.7}$$

Refs.[9, 10, 50] explicitly constructed rotating charged black hole solutions in this theory. Later we shall analyze the near horizon geometry of these black holes in extremal limit and verify that they satisfy eqs.(4.4)-(4.7).

Next we note that the lagrangian density (4.1) has a scaling symmetry:

$$\Phi \to \Phi + \lambda, \qquad F_{\mu\nu} \to e^{-\sqrt{3}\lambda} F_{\mu\nu} \,. \tag{4.8}$$

Since the magnetic and electric charges p and q are proportional to $F_{\theta\phi}$ and $\partial\mathcal{L}/\partial F_{rt}$ respectively, we see that under the transformation (4.8), q and p transforms to $e^{\sqrt{3}\lambda}q$ and $e^{-\sqrt{3}\lambda}p$ respectively. Thus if we want to keep the electric and the magnetic charges fixed, we need to make a compensating transformation of the parameters labelling the electric and magnetic charges of the solution. This shows that we can generate a one parameter family of solutions carrying fixed electric and magnetic charges by using the transformation:

$$\Phi \to \Phi + \lambda, \quad F_{\mu\nu} \to e^{-\sqrt{3}\lambda} F_{\mu\nu}, \quad Q \to e^{-\sqrt{3}\lambda} Q, \quad P \to e^{\sqrt{3}\lambda} P \,, \tag{4.9}$$

where Q and P are electric and magnetic charges labelling the original solution. This transformation will change the asymptotic value of the scalar field Φ leaving the electric and magnetic charges fixed. Thus according to the general arguments given in section 4.1, the entropy associated with the solution should not change under the deformation (4.9). On the other hand since (4.8) is a symmetry of the theory, the entropy is also invariant under this transformation. Combining these two results we see that the entropy must be invariant under

$$Q \to e^{-\sqrt{3}\lambda} Q, \quad P \to e^{\sqrt{3}\lambda} P \,. \tag{4.10}$$

Furthermore if the entropy function has no flat direction so that the near horizon geometry is fixed completely by extremizing the entropy function then the near horizon geometry, including the scalar field configuration, should be invariant under the transformation (4.9).

4.4.1.1 The black hole solution

We now turn to the black hole solution described in [9, 10, 50]. The metric associated with this solution is given by

$$ds^2 = -\frac{\widetilde{\Delta}}{\sqrt{f_p f_q}}(dt - w d\phi)^2 + \frac{\sqrt{f_p f_q}}{\Delta} dr^2 + \sqrt{f_p f_q}\, d\theta^2 + \frac{\Delta\sqrt{f_p f_q}}{\widetilde{\Delta}} \sin^2\theta\, d\phi^2 \tag{4.11}$$

where

$$\begin{aligned} f_p &= r^2 + a_K^2 \cos^2\theta + r(\tilde{p} - 2M_K) + \frac{\tilde{p}}{\tilde{p}+\tilde{q}} \frac{(\tilde{p}-2M_K)(\tilde{q}-2M_K)}{2} \\ &\quad - \frac{\tilde{p}\sqrt{(\tilde{p}^2-4M_K^2)(\tilde{q}^2-4M_K^2)}}{2(\tilde{p}+\tilde{q})} \frac{a_K}{M_K} \cos\theta \qquad (4.12) \\ f_q &= r^2 + a_K^2 \cos^2\theta + r(\tilde{q} - 2M_K) + \frac{\tilde{q}}{\tilde{p}+\tilde{q}} \frac{(\tilde{p}-2M_K)(\tilde{q}-2M_K)}{2} \\ &\quad + \frac{\tilde{q}\sqrt{(\tilde{p}^2-4M_K^2)(\tilde{q}^2-4M_K^2)}}{2(\tilde{p}+\tilde{q})} \frac{a_K}{M_K} \cos\theta \qquad (4.13) \\ w &= \sqrt{\tilde{p}\tilde{q}} \frac{(\tilde{p}\tilde{q}+4M_K^2)r - M_K(\tilde{p}-2M_K)(\tilde{q}-2M_K)}{2(\tilde{p}+\tilde{q})\widetilde{\Delta}} \frac{a_K}{M_K} \sin^2\theta \qquad (4.14) \\ \Delta &= r^2 - 2M_K r + a_K^2 \qquad (4.15) \\ \widetilde{\Delta} &= r^2 - 2M_K r + a_K^2 \cos^2\theta\,. \qquad (4.16) \end{aligned}$$

M_K, a_K, $\tilde{p}$ and $\tilde{q}$ are four parameters labelling the solution. The solution for the dilaton is of the form

$$\exp(-4\Phi/\sqrt{3}) = \frac{f_p}{f_q}\,. \qquad (4.17)$$

The dilaton has been asymptotically set to 0, but this can be changed using the transformation (4.9). Finally, the gauge field is given by

$$A_t = -f_q^{-1}\left(\frac{Q}{4\sqrt{\pi}}\left(r + \frac{\tilde{p}-2M_K}{2}\right) + \frac{1}{2}\frac{a_K}{M_K}\sqrt{\frac{\tilde{q}^3(\tilde{p}^2-4M_K^2)}{4(\tilde{p}+\tilde{q})}}\cos\theta\right) \qquad (4.18)$$

$$\begin{aligned} A_\phi &= -\frac{P}{4\sqrt{\pi}}\cos\theta - f_q^{-1}\frac{P}{4\sqrt{\pi}} a_K^2 \sin^2\theta\cos\theta \\ &\quad -\frac{1}{2} f_q^{-1} \sin^2\theta \frac{a_K}{M_K}\sqrt{\frac{\tilde{p}(\tilde{q}^2-4M_K^2)}{4(\tilde{p}+\tilde{q})^3}}\left[(\tilde{p}+\tilde{q})(\tilde{p}r - M_K(\tilde{p}-2M_K)) + \tilde{q}(\tilde{p}^2-4M_K^2)\right] \end{aligned} \qquad (4.19)$$

where Q and P, labelling the electric and magnetic charges of the black hole, are given by,

$$\begin{aligned} Q^2 &= 4\pi \frac{\tilde{q}(\tilde{q}^2-4M_K^2)}{(\tilde{p}+\tilde{q})} \qquad (4.20) \\ P^2 &= 4\pi \frac{\tilde{p}(\tilde{p}^2-4M_K^2)}{(\tilde{p}+\tilde{q})}\,. \qquad (4.21) \end{aligned}$$

The mass and angular momentum of the black hole can be expressed in terms of M_K, a_K, $\tilde{p}$ and $\tilde{q}$ as follows:[1]

$$M = 4\pi\,(\tilde{q}+\tilde{p}) \tag{4.22}$$

$$J = 4\pi\, a_K\, (\tilde{p}\tilde{q})^{1/2}\, \frac{\tilde{p}\tilde{q}+4M_K^2}{M_K(\tilde{p}+\tilde{q})}\,. \tag{4.23}$$

4.4.1.2 Extremal limit: The ergo-free branch

As first discussed in [10], in this case the moduli space of extremal black holes consist of two branches. Let us first concentrate on one of these branches corresponding to the surface W in [10]. We consider the limit: $M_K, a_K \to 0$ with a_K/M_K, $\tilde{q}$ and $\tilde{p}$ held finite. In this limit $\tilde{q}$, $\tilde{p}$ and a_K/M_K can be taken as the independent parameters labelling the solution. Then (4.20-4.23) become

$$\begin{aligned} M &= 4\pi\,(\tilde{q}+\tilde{p}) && (4.24)\\ Q^2 &= 4\pi\,\frac{\tilde{q}^3}{(\tilde{q}+\tilde{p})} && (4.25)\\ P^2 &= 4\pi\,\frac{\tilde{p}^3}{(\tilde{q}+\tilde{p})} && (4.26)\end{aligned}$$

$$J = 4\pi\,\frac{a_K}{M_K}\,\frac{(\tilde{p}\tilde{q})^{3/2}}{\tilde{p}+\tilde{q}} = \frac{a_K}{M_K}|PQ|\,. \tag{4.27}$$

For definiteness we shall take P and Q to be positive from now on.

In this limit Δ, $\widetilde{\Delta}$, f_p, f_q, w and A_μ become

$$\Delta = \widetilde{\Delta} = r^2 \tag{4.28}$$

$$\begin{aligned} f_p &= r^2+\tilde{p}r+\frac{\tilde{p}^2\tilde{q}}{2(\tilde{p}+\tilde{q})}\left(1-\frac{a_K}{M_K}\cos\theta\right) && (4.29)\\ f_q &= r^2+\tilde{q}r+\frac{\tilde{q}^2\tilde{p}}{2(\tilde{p}+\tilde{q})}\left(1+\frac{a_K}{M_K}\cos\theta\right) && (4.30)\\ w &= \frac{(\tilde{p}\tilde{q})^{\frac{3}{2}}}{2(\tilde{p}+\tilde{q})}\frac{a_K}{M_K}\frac{\sin^2\theta}{r} = \frac{J}{8\pi}\frac{\sin^2\theta}{r} && (4.31)\end{aligned}$$

[1] In defining the mass and angular momentum we have taken into account the fact that we have $G_N = 1/16\pi$. At present the normalization of the charges Q and P have been chosen arbitrarily, but later we shall relate them to the charges q and p introduced in section 4.2.

$$
\begin{aligned}
A_t &= -\frac{Q}{4\sqrt{\pi}} f_q^{-1} \left(\left(r + \frac{\tilde{p}}{2} \right) + \frac{1}{2} \left(\frac{a_K}{M_K} \right) \tilde{p} \cos\theta \right) && (4.32) \\
A_\phi &= -\frac{P}{4\sqrt{\pi}} \left[\cos\theta + \frac{1}{2} f_q^{-1} \sin^2\theta \left(\frac{a_K}{M_K} \right) \frac{\tilde{q}}{(\tilde{p}+\tilde{q})} \left((\tilde{p}+\tilde{q}) r + \tilde{q}\tilde{p} \right) \right] && (4.33)
\end{aligned}
$$

In order that the scalar field configuration is well defined everywhere outside the horizon, we need f_p/f_q to be positive in this region. This gives

$$
a_K \le M_K \,. \tag{4.34}
$$

This in turn implies that the coefficient of g_{tt}, being proportional to $\widetilde{\Delta}/\sqrt{f_p f_q}$ remains positive everywhere outside the horizon. Thus there is no ergo-sphere for this black hole. We call this branch of solutions the ergo-free branch.

4.4.1.3 Near horizon behaviour

In our coordinate system the horizon is at $r = 0$. To find the near horizon geometry, we consider the limit

$$
r \to sr, \qquad t \to s^{-1} t \qquad s \to 0 \,. \tag{4.35}
$$

Metric The near horizon behaviour of the metric is given by:

$$
ds^2 = -\frac{r^2}{v_1(\theta)} (dt - \frac{b}{r} d\phi)^2 + v_1(\theta) \left(\frac{dr^2}{r^2} + d\theta^2 + \sin^2\theta d\phi^2 \right) \tag{4.36}
$$

with

$$
v_1(\theta) = \lim_{r\to 0} \sqrt{f_p f_q} = \frac{1}{8\pi} \sqrt{P^2 Q^2 - J^2 \cos^2\theta}, \qquad b = \frac{J}{8\pi} \sin^2\theta \,. \tag{4.37}
$$

By straightforward algebraic manipulation this metric can be rewritten as

$$
ds^2 = \frac{a^2 \sin^2\theta}{v_1(\theta)} \left(d\phi - \alpha r dt' \right)^2 + v_1(\theta) \left(-r^2 dt'^2 + \frac{dr^2}{r^2} + d\theta^2 \right) \tag{4.38}
$$

with

$$
\begin{aligned}
t' &= t/a \,, && (4.39) \\
a &= \frac{1}{8\pi} \sqrt{P^2 Q^2 - J^2} \,, && (4.40) \\
\alpha &= -J/\sqrt{P^2 Q^2 - J^2} \,. && (4.41)
\end{aligned}
$$

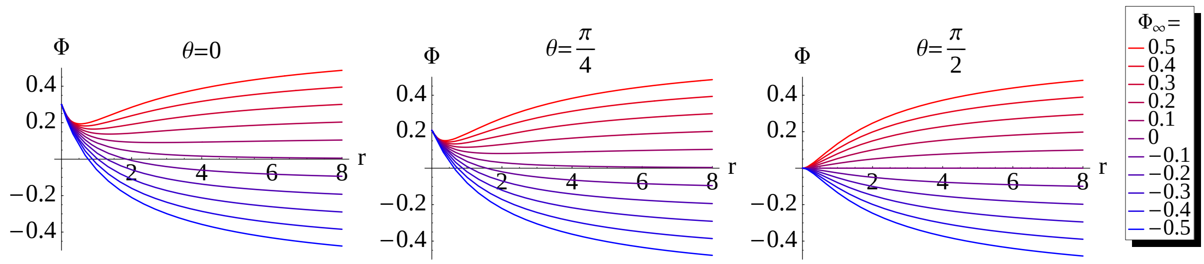


Figure 4.1: Radial evolution of the scalar field starting with different asymptotic values at three different values of θ. We take $P = Q = 4\sqrt{\pi}$, $J = 16\pi/3$ for $\Phi_\infty = 0$, and then change Φ_∞ and P, Q using the transformation (4.9).

Gauge fields Near the horizon the gauge fields behave like

$$\frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu = \left[\frac{2a\sqrt{\pi}}{Q}\frac{1}{(1+\mu\cos\theta)}dr\wedge dt' + \frac{1}{4\sqrt{\pi}}P\sin\theta\frac{(1-\mu^2)}{(1+\mu\cos\theta)^2}d\theta\wedge(d\phi-\alpha r dt')\right], \tag{4.42}$$

where

$$\mu = \frac{J}{PQ}\,. \tag{4.43}$$

Scalar Field In the near horizon limit the scalar field becomes

$$e^{-4\Phi/\sqrt{3}}\Big|_{r=M} = \left(\frac{P}{Q}\right)^{\frac{2}{3}}\frac{PQ - J\cos\theta}{PQ + J\cos\theta} \tag{4.44}$$

Entropy Finally the entropy associated with this solution is given by

$$S_{BH} = 4\pi\int d\theta d\phi\sqrt{g_{\theta\theta}\,g_{\phi\phi}} = 16\pi^2 a = 2\pi\sqrt{P^2Q^2 - J^2}\,. \tag{4.45}$$

We now see that the entropy is invariant under (4.10) and the near horizon background, including the scalar field configuration given in (4.44), is invariant

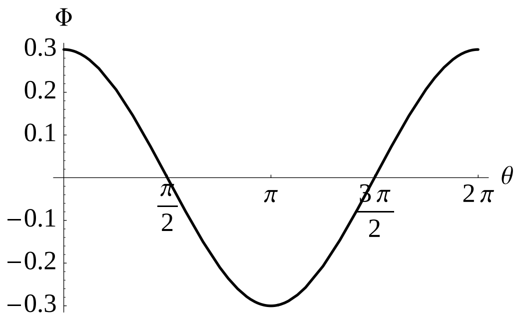


Figure 4.2: Scalar field profile at the horizon of the Kaluza-Klein black hole. We take $P = Q = 4\sqrt{\pi}$, $J = 16\pi/3$ for $\Phi_\infty = 0$, and then change Φ_∞ and P, Q using the transformation (4.9). The figure shows that the scalar field profile at the horizon is independent of Φ_∞.

under the transformation (4.9).[1] This shows that the near horizon field configuration is independent of the asymptotic value of the modulus field Φ. This can also be seen explicitly by studying the radial evolution of Φ for various asymptotic values of Φ; numerical results for this evolution have been plotted in fig.4.1. Fig.4.2 shows the plot of $\Phi(\theta)$ vs. θ at the horizon of the black hole.

4.4.1.4 Entropy function analysis

The analysis of section 4.4.1.3 shows that the near horizon field configuration is precisely of the form described in eq.(4.3) with

$$\Omega(\theta) = a\sin\theta, \quad e^{-2\psi(\theta)} = \frac{8\pi\, a^2 \sin^2\theta}{\sqrt{P^2Q^2 - J^2\cos^2\theta}}, \quad e - \alpha b(\theta) = \frac{2\sqrt{\pi}\, a}{Q}\frac{1}{(1+\mu\cos\theta)},$$
$$e^{-4\Phi/\sqrt{3}} = \left(\frac{P}{Q}\right)^{\frac{2}{3}} \frac{PQ - J\cos\theta}{PQ + J\cos\theta}, \quad a = \frac{1}{8\pi}\sqrt{P^2Q^2 - J^2}, \quad \alpha = -\frac{J}{\sqrt{P^2Q^2 - J^2}}. \tag{4.46}$$

[1] As described in eqs.(4.48), (4.49), the charges q, p are related to the parameters Q, P by some normalization factors. These factors do not affect the transformation laws of the charges given in (4.9), (4.10).

We can easily verify that this configuration satisfies eqs.(4.4)-(4.7) obtained by extremizing the entropy function.

Using eq.(4.16) with values of h_{11} and f_{11} given in (4.2) we get

$$e = \frac{1}{2}\left[(e-\alpha b(\pi)) + (e-\alpha b(0))\right] = \frac{P^2 Q}{4\sqrt{\pi}\sqrt{P^2Q^2 - J^2}}\,, \tag{4.47}$$

and

$$p = -\frac{2\pi}{\alpha}\left[(e-\alpha b(\pi)) - (e-\alpha b(0))\right] = \sqrt{\pi}\,P\,. \tag{4.48}$$

Eq.(4.42) now gives

$$q = \frac{8\pi}{\alpha}\left[\frac{e^{2\sqrt{3}\Phi} b'}{\sin\theta}\right]_0^{\pi} = 4\sqrt{\pi}\,Q\,. \tag{4.49}$$

Finally the right hand side of eq.(4.43) evaluated for the background (4.46) gives the answer J showing that we have correctly identified the parameter J as the angular momentum carried by the black hole.

4.4.1.5 The ergo-branch

The extremal limit on this branch, corresponding to the surface S in [10], amounts to taking

$$a_K = M_K \tag{4.50}$$

in the black hole solution. Thus we have the relations

$$Q^2 = 4\pi\,\frac{\tilde{q}(\tilde{q}^2 - 4M_K^2)}{(\tilde{p}+\tilde{q})}, \quad P^2 = 4\pi\,\frac{\tilde{p}(\tilde{p}^2 - 4M_K^2)}{(\tilde{p}+\tilde{q})}, \quad J = 4\pi\,\sqrt{\tilde{p}\tilde{q}}\,\frac{\tilde{p}\tilde{q} + 4M_K^2}{(\tilde{p}+\tilde{q})}\,. \tag{4.51}$$

In order to take the near horizon limit of this solution we first let

$$r \to r + M_K \tag{4.52}$$

which shifts the horizon to $r = 0$. Near the horizon Δ, $\tilde{\Delta}$ and w become

$$\Delta = r^2 \tag{4.53}$$

$$\tilde{\Delta} = -M_K^2 \sin^2\theta + \mathcal{O}(r^2) \tag{4.54}$$

$$w = -\sqrt{\tilde{q}\tilde{p}}\,(1 + \bar{w} r) + \mathcal{O}(r^2) \tag{4.55}$$

with

$$\bar{w} = \frac{\tilde{p}\tilde{q} + 4M_K^2}{2(\tilde{p} + \tilde{q})M_K^2}\,. \tag{4.56}$$

Note that $\widetilde{\Delta}$ changes from being positive at large distance to negative at the horizon. Thus g_{tt} changes sign as we go from the asymptotic region to the horizon and the solution has an ergo-sphere. We call this branch of solutions the ergo-branch. Using eqs.(4.53)-(4.56) we can write the metric as

$$ds^2 = \frac{M_K^2 \sin^2\theta}{\sqrt{f_p f_q}}\left(dt + \sqrt{\tilde{q}\tilde{p}}(1+\bar{w}r)d\phi\right)^2 + \sqrt{f_p f_q}\left(\frac{dr^2}{r^2} + d\theta^2 - \frac{r^2}{M_K^2}d\phi^2\right) + \cdots \tag{4.57}$$

where $\cdots$ denote terms which will eventually vanish in the near horizon limit that we are going to describe below. After letting

$$\phi \to \phi - t/\sqrt{\tilde{q}\tilde{p}} \tag{4.58}$$

and taking the near horizon limit

$$r \to s\,r, \qquad t \to s^{-1}\,t, \qquad s \to 0\,, \tag{4.59}$$

the metric becomes

$$ds^2 = \frac{M_K^2 \sin^2\theta}{v_1(\theta)}(\sqrt{\tilde{q}\tilde{p}}d\phi - \bar{w}r dt)^2 + v_1(\theta)\left(\frac{dr^2}{r^2} + d\theta^2 - \frac{r^2}{M_K^2\,\tilde{q}\tilde{p}}dt^2\right) \tag{4.60}$$

where

$$v_1(\theta) = \lim_{r\to 0}\sqrt{f_p\,f_q}\,. \tag{4.61}$$

Finally rescaling

$$t \to M_K\sqrt{\tilde{q}\tilde{p}}\;t \tag{4.62}$$

the metric becomes of the form given in (4.3) with

$$\Omega = M_K\sqrt{\tilde{p}\tilde{q}}\,\sin\theta, \quad e^{-2\psi} = \frac{M_K^2\tilde{p}\tilde{q}\sin^2\theta}{v_1(\theta)}, \quad \alpha = M_K\;\bar{w}\,. \tag{4.63}$$

Using eqs.(4.56) and (4.51) we find that

$$\alpha = \frac{J}{\sqrt{J^2 - P^2Q^2}}, \quad \Omega = \frac{1}{8\pi}\sqrt{J^2 - P^2Q^2}\,\sin\theta, \quad e^{-2\psi} = \frac{(J^2 - P^2Q^2)\sin^2\theta}{64\pi^2 v_1(\theta)}\,. \tag{4.64}$$

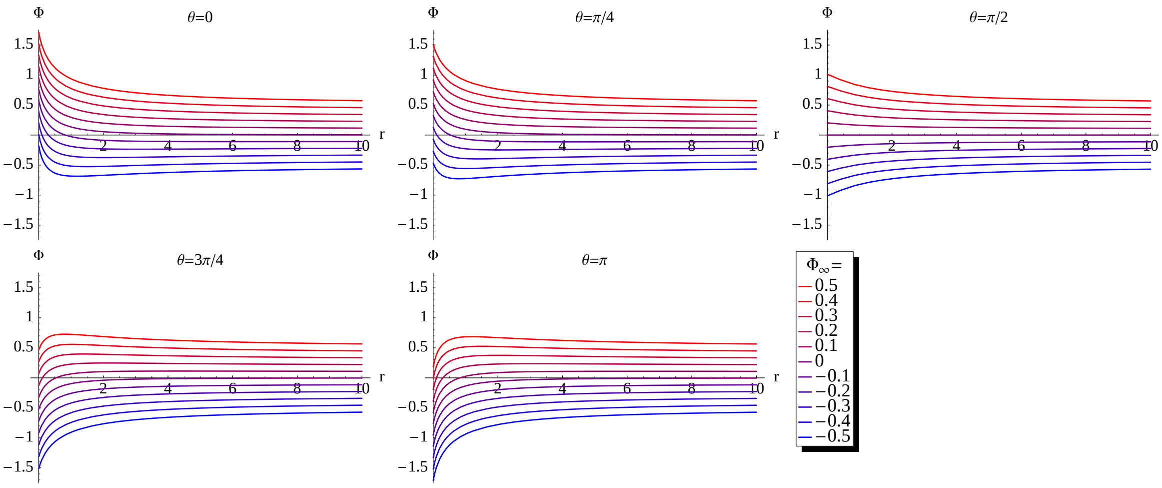


Figure 4.3: Radial evolution of the scalar field for an ergo-branch black hole starting with different asymptotic values at five different values of θ. We take $P = Q = 2\sqrt{\pi}$ and $J = 4\pi\sqrt{2}$ for $\Phi_\infty = 0$, and then change Φ_∞ and P, Q using the transformation (4.9).

The scalar field Φ becomes in this limit

$$e^{-4\Phi/\sqrt{3}} = \frac{f_p}{f_q}\,, \tag{4.65}$$

where f_p and f_q now refer to the functions f_p and f_q at the horizon:

$$f_p = -M_K^2 \sin^2\theta + M_K\tilde{p} + \frac{\tilde{p}}{\tilde{p}+\tilde{q}}\frac{(\tilde{p}-2M_K)(\tilde{q}-2M_K)}{2} - \frac{\tilde{p}\sqrt{(\tilde{p}^2-4M_K^2)(\tilde{q}^2-4M_K^2)}}{2(\tilde{p}+\tilde{q})}\cos\theta \tag{4.66}$$

$$f_q = -M_K^2 \sin^2\theta + M_K\tilde{q} + \frac{\tilde{q}}{\tilde{p}+\tilde{q}}\frac{(\tilde{p}-2M_K)(\tilde{q}-2M_K)}{2} + \frac{\tilde{q}\sqrt{(\tilde{p}^2-4M_K^2)(\tilde{q}^2-4M_K^2)}}{2(\tilde{p}+\tilde{q})}\cos\theta\,. \tag{4.67}$$

The near horizon gauge field can also be calculated by a tedious but straightforward procedure after taking into account the change in coordinates described above. The final result is of the form given in (4.3) with

$$e - \alpha b(\theta) = \frac{M_K\sqrt{\tilde{p}\tilde{q}}}{4\sqrt{\pi}\,f_q}\left(\frac{1}{2}\frac{\tilde{p}}{\tilde{q}}Q\,\sin^2\theta + P\sqrt{\frac{\tilde{q}}{\tilde{p}}}\cos\theta\right)\,. \tag{4.68}$$

This gives

$$e = \frac{1}{2}\left[(e-\alpha b(\pi)) + (e-\alpha b(0))\right] = -\frac{P^2 Q}{4\sqrt{\pi}\sqrt{J^2-P^2Q^2}}\,, \tag{4.69}$$

$$p = -\frac{2\pi}{\alpha}\left[(e-\alpha b(\pi)) - (e-\alpha b(0))\right] = \sqrt{\pi}\,P\,, \tag{4.70}$$

and

$$q = \frac{8\pi}{\alpha}\left[\frac{e^{2\sqrt{3}\Phi}b'}{\sin\theta}\right]_0^\pi = 4\sqrt{\pi}\,Q\,. \tag{4.71}$$

Finally, the entropy associated with this solution can be easily calculated by computing the area of the horizon, and is given by

$$S_{BH} = 2\pi\sqrt{J^2-P^2Q^2}\,. \tag{4.72}$$

We have explicitly checked that the near horizon ergo-branch field configurations described above satisfy the differential equations (4.4)-(4.7).

The entropy is clearly invariant under the transformation (4.10). However in this case the near horizon background is not invariant under the transformation

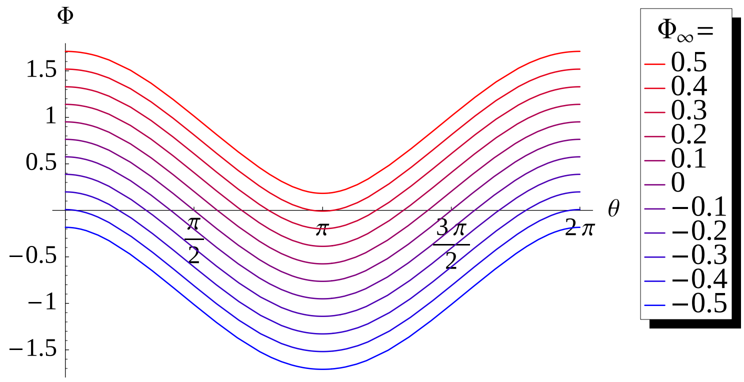


Figure 4.4: Scalar field profile at the horizon for a black hole on the ergo-branch for different asymptotic values of Φ. We take $P = Q = 2\sqrt{\pi}$ and $J = 4\pi\sqrt{2}$ for $\Phi_\infty = 0$, and then change Φ_∞ and P, Q using the transformation (4.9). Clearly the scalar field profile at the horizon depends on its asymptotic value.

(4.9). One way to see this is to note that under the transformation (4.10) the combination $M_K^2 \tilde{p}\tilde{q} = (J^2 - P^2Q^2)/64\pi^2$ remains invariant. This shows that M_K cannot remain invariant under this transformation, since if M_K had been invariant then $\tilde{p}\tilde{q}$ would be invariant, and the invariance of J given in (4.51) would imply that $\tilde{p} + \tilde{q}$ is also invariant. This in turn would mean that M_K, $\tilde{p}$ and $\tilde{q}$ are all invariant under (4.10) and hence P and Q would be invariant which is clearly a contradiction. Given the fact that M_K is not invariant under this transformation we see that the coefficient of the $\sin^2\theta$ term in f_p and f_q are not invariant under (4.10). This in turn shows that ψ, and hence the background metric, is not invariant under the transformation (4.9). This is also seen from figures 4.3 and 4.4 where we have shown respectively the radial evolution of the scalar field and the scalar field profile at the horizon for different asymptotic values of Φ. Nevertheless several components of the near horizon background, *e.g.* $\Omega(\theta)$ and the parameters α and e do remain invariant under this transformation, indicating that at least these components do get attracted towards fixed values as we approach the horizon.

4.4.2 Black Holes in Toroidally Compactified Heterotic String Theory

The theory under consideration is a four dimensional theory of gravity coupled to a complex scalar $S = S_1 + iS_2$, a 4×4 matrix valued scalar field M satisfying the constraint

$$MLM^T = L, \qquad L = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \tag{4.73}$$

and four U(1) gauge fields $A_\mu^{(i)}$ $(1 \le i \le 4)$.[1] Here I_2 denotes 2×2 identity matrix. The bosonic part of the lagrangian density is

$$\begin{aligned} \mathcal{L} \;=\; & R - \frac{1}{2} g^{\mu\nu} S_2^{-2} \partial_\mu \bar{S} \partial_\nu S + \frac{1}{8} g^{\mu\nu} Tr(\partial_\mu ML \partial_\nu ML) \\ & - \frac{1}{4} S_2 g^{\mu\rho} g^{\nu\sigma} F^{(i)}_{\mu\nu} (LML)_{ij} F^{(j)}_{\rho\sigma} + \frac{1}{4} S_1 g^{\mu\rho} g^{\nu\sigma} F^{(i)}_{\mu\nu} L_{ij} \widetilde{F}^{(j)}_{\rho\sigma} , \end{aligned} \tag{4.74}$$

where

$$\widetilde{F}^{(i)\mu\nu} = \frac{1}{2} (\sqrt{-\det g})^{-1} \epsilon^{\mu\nu\rho\sigma} \, \widetilde{F}^{(i)}_{\rho\sigma} . \tag{4.75}$$

General rotating black solution in this theory, carrying electric charge vector $\vec{q}$ and magnetic charge vector $\vec{p}$, has been constructed in [11]. Before we begin analyzing the solution, we would like to note that the lagrangian density (4.74) is invariant under an SO(2,2) rotation:

$$M \to \Omega M \Omega^T, \quad F^{(i)}_{\mu\nu} \to \Omega_{ij} F^{(j)}_{\mu\nu} , \tag{4.76}$$

where Ω is a 4×4 matrix satisfying

$$\Omega L \Omega^T = L . \tag{4.77}$$

Thus given a classical solution, we can generate a class of classical solutions using this transformation. Since the magnetic and electric charges p_i and q_i are proportional to $F^{(i)}_{\theta\phi}$ and $\partial\mathcal{L}/\partial F^{(i)}_{rt}$ respectively, we see that under the transformation (4.76), $p_i \to \Omega_{ij} p_j$, $q_i \to (\Omega^T)^{-1}_{ij} q_j$. Thus if we want the new solution to have the same electric and magnetic charges, we must make compensating transformation

[1] Actual heterotic string theory has 28 gauge fields and a 28×28 matrix valued scalar field, but the truncated theory discussed here contains all the non-trivial information about the theory.

in the parameters labelling the electric and magnetic charges. This shows that we can generate a family of solutions carrying the same electric and magnetic charges by making the transformation:

$$M \to \Omega M \Omega^T, \quad F^{(i)}_{\mu\nu} \to \Omega_{ij} F^{(j)}_{\mu\nu}, \quad Q_i \to \Omega^T_{ij} Q_j, \quad P_i \to \Omega^{-1}_{ij} P_j\,, \tag{4.78}$$

where $\vec{Q}$ and $\vec{P}$ are the parameters which label electric and magnetic charges in the original solution. This transformation changes the asymptotic value of M leaving the charges unchanged. Thus the general argument of section 4.1 will imply that the entropy must remain invariant under such a transformation. Invariance of the entropy under the transformation (4.76), which is a symmetry of the theory, will then imply that the entropy must be invariant under

$$Q_i \to \Omega^T_{ij} Q_j, \quad P_i \to \Omega^{-1}_{ij} P_j\,. \tag{4.79}$$

On the other hand if there is a unique background for a given set of charges then the background itself must be invariant under the transformation (4.78).

The equations of motion derived from the lagrangian density (4.74) is also invariant under the electric magnetic duality transformation:

$$S \to \frac{aS+b}{cS+d}, \qquad F^{(i)}_{\mu\nu} \to (cS_1 + d) F^{(i)}_{\mu\nu} + cS_2 (ML)_{ij} \widetilde{F}^{(j)}_{\mu\nu}\,, \tag{4.80}$$

where a, b, c, d are real numbers satisfying $ad - bc = 1$. We can use this transformation to generate a family of black hole solutions from a given solution. From the definition of electric and magnetic charges it follows that under this transformation the electric and magnetic charge vectors $\vec{q}$, $\vec{p}$ transform as:

$$\vec{q} \to (a\vec{q} - bL\vec{p}), \qquad \vec{p} \to (-cL\vec{q} + d\vec{p})\,. \tag{4.81}$$

Thus if we want the new solution to have the same charges as the old solution we must perform compensating transformation on the electric and magnetic charge parameters $\vec{Q}$ and $\vec{P}$. We can get a family of solutions with the same electric and magnetic charges but different asymptotic values of the scalar field S by the transformation:

$$S \to \frac{aS+b}{cS+d}, \quad F^{(i)}_{\mu\nu} \to (cS_1{+}d) F^{(i)}_{\mu\nu} {+} cS_2 (ML)_{ij} \widetilde{F}^{(j)}_{\mu\nu}, \quad \vec{Q} \to d\vec{Q} {+} bL\vec{P}, \quad \vec{P} \to cL\vec{Q} {+} a\vec{P}\,. \tag{4.82}$$

Arguments similar to the one given for the O(2,2) transformation shows that the entropy must remain invariant under the transformation

$$\vec{Q} \to d\vec{Q} + bL\vec{P}, \quad \vec{P} \to cL\vec{Q} + a\vec{P}\,. \tag{4.83}$$

Furthermore if the entropy function has a unique extremum then the near horizon field configuration must also remain invariant under the transformation (4.82).

4.4.2.1 The black hole solution

Ref.[11] constructed rotating black hole solutions in this theory carrying the following charges:

$$Q = \begin{pmatrix} 0 \\ Q_2 \\ 0 \\ Q_4 \end{pmatrix}, \qquad P = \begin{pmatrix} P_1 \\ 0 \\ P_3 \\ 0 \end{pmatrix}. \tag{4.84}$$

These black holes break all the supersymmetries of the theory. In order to describe the solution we parametrize the matrix valued scalar field M as

$$M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} \tag{4.85}$$

where G and B are 2×2 matrices of the form

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & B_{12} \\ -B_{12} & 0 \end{pmatrix}. \tag{4.86}$$

Physically G and B represent components of the string metric and the anti-symmetric tensor field along an internal two dimensional torus. The solution is given by

$$\begin{aligned}
G_{11} &= \frac{(r+2m\sinh^2\delta_4)(r+2m\sinh^2\delta_2)+l^2\cos^2\theta}{(r+2m\sinh^2\delta_3)(r+2m\sinh^2\delta_2)+l^2\cos^2\theta},\\
G_{12} &= \frac{2ml\cos\theta(\sinh\delta_3\cosh\delta_4\sinh\delta_1\cosh\delta_2-\cosh\delta_3\sinh\delta_4\cosh\delta_1\sinh\delta_2)}{(r+2m\sinh^2\delta_3)(r+2m\sinh^2\delta_2)+l^2\cos^2\theta},\\
G_{22} &= \frac{(r+2m\sinh^2\delta_3)(r+2m\sinh^2\delta_1)+l^2\cos^2\theta}{(r+2m\sinh^2\delta_3)(r+2m\sinh^2\delta_2)+l^2\cos^2\theta},\\
B_{12} &= -\frac{2ml\cos\theta(\sinh\delta_3\cosh\delta_4\cosh\delta_1\sinh\delta_2-\cosh\delta_3\sinh\delta_4\sinh\delta_1\cosh\delta_2)}{(r+2m\sinh^2\delta_3)(r+2m\sinh^2\delta_2)+l^2\cos^2\theta},\\
Im\,S &= \frac{\Delta^{\frac{1}{2}}}{(r+2m\sinh^2\delta_3)(r+2m\sinh^2\delta_4)+l^2\cos^2\theta},
\end{aligned}$$

$$
\begin{aligned}
ds^2 &= \Delta^{\frac{1}{2}}[-\frac{r^2-2mr+l^2\cos^2\theta}{\Delta}dt^2+\frac{dr^2}{r^2-2mr+l^2}+d\theta^2+\frac{\sin^2\theta}{\Delta}\{(r+2m\sinh^2\delta_3) \\
&\times (r+2m\sinh^2\delta_4)(r+2m\sinh^2\delta_1)(r+2m\sinh^2\delta_2)+l^2(1+\cos^2\theta)r^2+W \\
&+ 2ml^2 r\sin^2\theta\}d\phi^2-\frac{4ml}{\Delta}\{(\cosh\delta_3\cosh\delta_4\cosh\delta_1\cosh\delta_2 \\
&- \sinh\delta_3\sinh\delta_4\sinh\delta_1\sinh\delta_2)r+2m\sinh\delta_3\sinh\delta_4\sinh\delta_1\sinh\delta_2\}\sin^2\theta dtd\phi],
\end{aligned} \tag{4.87}
$$

where

$$
\begin{aligned}
\Delta &\equiv (r+2m\sinh^2\delta_3)(r+2m\sinh^2\delta_4)(r+2m\sinh^2\delta_1)(r+2m\sinh^2\delta_2) \\
&+ (2l^2r^2+W)\cos^2\theta, \\
W &\equiv 2ml^2(\sinh^2\delta_3+\sinh^2\delta_4+\sinh^2\delta_1+\sinh^2\delta_2)r \\
&+ 4m^2l^2(2\cosh\delta_3\cosh\delta_4\cosh\delta_1\cosh\delta_2\sinh\delta_3\sinh\delta_4\sinh\delta_1\sinh\delta_2 \\
&- 2\sinh^2\delta_3\sinh^2\delta_4\sinh^2\delta_1\sinh^2\delta_2-\sinh^2\delta_4\sinh^2\delta_1\sinh^2\delta_2 \\
&- \sinh^2\delta_3\sinh^2\delta_1\sinh^2\delta_2-\sinh^2\delta_3\sinh^2\delta_4\sinh^2\delta_2-\sinh^2\delta_3\sinh^2\delta_4\sinh^2\delta_1) \\
&+ l^4\cos^2\theta.
\end{aligned} \tag{4.88}
$$

a, m, δ_1, δ_2, δ_3 and δ_4 are parameters labelling the solution. Ref.[11] did not explicitly present the results for $Re\ S$ and the gauge fields.

The ADM mass M, electric and magnetic charges $\{Q_i, P_i\}$, and the angular momentum J are given by:[1]

$$
\begin{aligned}
M &= 8\pi\, m(\cosh^2\delta_1+\cosh^2\delta_2+\cosh^2\delta_3+\cosh^2\delta_4)-16\pi\, m, \\
Q_2 &= 4\sqrt{2\pi}\, m\cosh\delta_1\sinh\delta_1, \qquad Q_4 = 4\sqrt{2\pi}\, m\cosh\delta_2\sinh\delta_2, \\
P_1 &= 4\sqrt{2\pi}\, m\cosh\delta_3\sinh\delta_3, \qquad P_3 = 4\sqrt{2\pi}\, m\cosh\delta_4\sinh\delta_4, \\
J &= -16\pi\, lm(\cosh\delta_1\cosh\delta_2\cosh\delta_3\cosh\delta_4-\sinh\delta_1\sinh\delta_2\sinh\delta_3\sinh\delta_4).
\end{aligned} \tag{4.89}
$$

The entropy associated with this solution was computed in [11] to be

$$
S_{BH} = 32\pi^2\Big[m^2(\prod_{i=1}^{4}\cosh\delta_i+\prod_{i=1}^{4}\sinh\delta_i)+m\sqrt{m^2-l^2}(\prod_{i=1}^{4}\cosh\delta_i-\prod_{i=1}^{4}\sinh\delta_i)\Big]\,. \tag{4.90}
$$

As in the case of Kaluza-Klein black hole this solution also has two different kinds of extremal limit which we shall denote by ergo-branch and ergo-free branch. The ergo-branch was discussed in [11].

[1] In defining M and J we have taken into account our convention $G_N = 16\pi$, and also the fact that our definition of the angular momentum differs from the standard one by a minus sign. Normalizations of $\vec{Q}$ and $\vec{P}$ are arbitrary at this stage.

4.4.2.2 The ergo-branch

The extremal limit corresponding to the ergo-branch is obtained by taking the limit $l \to m$. In this limit the second term in the expression for the entropy vanishes and the first term gives

$$S_{BH} = 2\pi \sqrt{J^2 + Q_2 Q_4 P_1 P_3}\,. \tag{4.91}$$

Now the most general transformation of the form (4.79) which does not take the charges given in (4.84) outside this family is:

$$\Omega = \begin{pmatrix} e^{\gamma} & 0 & 0 & 0 \\ 0 & e^{\beta} & 0 & 0 \\ 0 & 0 & e^{-\gamma} & 0 \\ 0 & 0 & 0 & e^{-\beta} \end{pmatrix}, \tag{4.92}$$

for real parameters γ, β. This gives

$$P_1 \to e^{-\gamma} P_1, \quad P_3 \to e^{\gamma} P_3, \quad Q_2 \to e^{\beta} Q_2, \quad Q_4 \to e^{-\beta} Q_4\,. \tag{4.93}$$

On the other hand most general transformation of the type (4.83) which keeps the charge vector within the same family is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}. \tag{4.94}$$

This gives

$$P_1 \to a\, P_1, \quad P_3 \to a\, P_3, \quad Q_2 \to a^{-1}\, Q_2, \quad Q_4 \to a^{-1}\, Q_4\,. \tag{4.95}$$

It is easy to see that the entropy given in (4.91) does not change under the transformations (4.93), (4.95).[1]

After some tedious manipulations along the lines described in section 4.4.1.5, the near horizon metric can be brought into the form given in eq.(4.3) with

$$\begin{aligned} \Omega &= \frac{1}{8\pi}\sqrt{J^2 + Q_2 Q_4 P_1 P_3}\,\sin\theta, \qquad e^{-2\psi} = \frac{1}{64\pi^2}\left(J^2 + Q_2 Q_4 P_1 P_3\right)\sin^2\theta\,\Delta^{-1/2}\,, \\ \alpha &= \frac{J}{\sqrt{J^2 + Q_2 Q_4 P_1 P_3}}\,, \end{aligned} \tag{4.96}$$

[1]As in (4.48), (4.49), the parameters $\vec{P}$, $\vec{Q}$ are related to the charges $\vec{p}$, $\vec{q}$ by some overall normalization factors. These factors do not affect the transformation laws of the charges given in (4.93), (4.95).

where Δ has to be evaluated on the horizon $r = m$. We have found that the near horizon metric and the scalar fields are not invariant under the corresponding transformations (4.78) and (4.82) generated by the matrices (4.92) and (4.94) respectively, essentially due to the fact that Δ is not invariant under these transformations. This shows that in this case for a fixed set of charges the entropy function has a family of extrema.

4.4.2.3 The ergo-free branch

The extremal limit in the ergo-free branch is obtained by taking one or three of the δ_i's negative, and then taking the limit $|\delta_i| \to \infty$, $m \to 0$, $l \to 0$ in a way that keeps the Q_i, P_i and J finite. It is easy to see that in this limit the first term in the expression (4.90) for the entropy vanishes and the second term gives[1]

$$S_{BH} = 2\pi \sqrt{-J^2 - Q_2 Q_4 P_1 P_3} \, . \tag{4.97}$$

Again we see that S_{BH} is invariant under the transformations (4.93), (4.95).

On the ergo-free branch the horizon is at $r = 0$. The near horizon background can be computed easily from (4.87) following the approach described in section 4.4.1.3 and has the following form after appropriate rescaling of the time coordinate:

$$\begin{aligned} ds^2 &= \frac{1}{8\pi}\sqrt{-Q_2Q_4P_1P_3 - J^2\cos^2\theta}\left(-r^2dt^2 + \frac{dr^2}{r^2} + d\theta^2\right) \\ &\quad + \frac{1}{8\pi}\,\frac{-Q_2Q_4P_1P_3 - J^2}{\sqrt{-Q_2Q_4P_1P_3 - J^2\cos^2\theta}}\,\sin^2\theta\,(d\phi - \alpha r dt)^2 \, , \end{aligned} \tag{4.98}$$

$$ImS = \sqrt{-\frac{Q_2Q_4}{P_1P_3} - \frac{J^2\cos^2\theta}{(P_1P_3)^2}} \, , \tag{4.99}$$

$$G_{11} = \left|\frac{P_3}{P_1}\right|, \qquad G_{12} = -\frac{J\cos\theta}{P_1Q_2}\left|\frac{Q_2}{Q_4}\right|, \qquad G_{22} = \left|\frac{Q_2}{Q_4}\right|, \qquad B_{12} = \frac{J\cos\theta}{P_1Q_4} \, , \tag{4.100}$$

where

$$\alpha = -J/\sqrt{-Q_2Q_4P_1P_3 - J^2} \, . \tag{4.101}$$

[1]Note that the product $Q_2Q_4P_1P_3$ is negative due to the fact that an odd number of δ_i's are negative.

It is easy to see that the background is invariant under (4.78) and (4.82) for transformation matrices of the form described in (4.92) and (4.94).

4.4.2.4 Duality invariant form of the entropy

In the theory described here a combination of the charges that is invariant under both transformations (4.79) and (4.83) is

$$D \equiv (Q_1Q_3 + Q_2Q_4)(P_1P_3 + P_2P_4) - \frac{1}{4}(Q_1P_1 + Q_2P_2 + Q_3P_3 + Q_4P_4)^2 \,. \quad (4.102)$$

Thus we expect the entropy to depend on the charges through this combination. Now for the charge vectors given in (4.84) we have

$$D = Q_2Q_4P_1P_3 \,. \quad (4.103)$$

Using this result we can express the entropy formula (4.91) in the ergo-branch in the duality invariant form[11]:

$$S_{BH} = 2\pi \sqrt{J^2 + D} \,. \quad (4.104)$$

On the other hand the formula (4.97) on the ergo-free branch may be expressed as

$$S_{BH} = 2\pi \sqrt{-J^2 - D} \,. \quad (4.105)$$

We now note that the Kaluza-Klein black hole described in section (4.4.1) also falls into the general class of black holes discussed in this section with charges:

$$Q = \sqrt{2} \begin{pmatrix} Q \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad P = \sqrt{2} \begin{pmatrix} P \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (4.106)$$

Thus in this case

$$D = -P^2Q^2 \,. \quad (4.107)$$

We can now recognize the entropy formulæ (4.45) and (4.72) as special cases of (4.105) and (4.104) respectively.

Finally we can try to write down the near horizon metric on the ergo-free branch in a form that holds for the black hole solutions analyzed in this as well as in the previous subsection and which makes manifest the invariance of the background under arbitrary transformations of the form described in (4.78), (4.82). This is of the form:

$$\begin{aligned} ds^2 &= \frac{1}{8\pi}\sqrt{-D-J^2\cos^2\theta}\left(-r^2dt^2+\frac{dr^2}{r^2}+d\theta^2\right) \\ &\quad +\frac{1}{8\pi}\frac{-D-J^2}{\sqrt{-D-J^2\cos^2\theta}}\sin^2\theta\,(d\phi-\alpha r dt)^2\,, \end{aligned} \tag{4.108}$$

where

$$\alpha = -\frac{J}{\sqrt{-D-J^2}}\,. \tag{4.109}$$

(4.38) and (4.98) are special cases of this equation.

Chapter 5

Extension to black rings

In this chapter, we study the entropy of extremal four dimensional black holes and five dimensional black holes and black rings is a unified framework using Sen's entropy function and dimensional reduction. The five dimensional black holes and black rings we consider project down to either static or stationary black holes in four dimensions. The analysis is done in the context of two derivative gravity coupled to abelian gauge fields and neutral scalar fields. We apply this formalism to various examples including $U(1)^3$ minimal supergravity.

5.1 Black thing entropy function and dimensional reduction

We wish to apply the entropy function formalism [6, 49], and its generalisation to rotating black holes [51], to the five dimensional black rings and black holes — black things. These objects are characterised by the topology of their horizons. Black ring horizons have $S^2 \times S^1$ topology while black holes have S^3 topology.

We consider a five dimensional Lagrangian with gravity, abelian gauge fields, $\bar{F}^I$, neutral massless scalars, X^S, and a Chern-Simons term:

$$S = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\Big(R - h_{ST}(\vec{X})\partial_\mu X^S\partial^\mu X^T - f_{IJ}(\vec{X})\bar{F}^I_{\mu\nu}\bar{F}^{J\,\mu\nu} - c_{IJK}\epsilon^{\mu\nu\alpha\beta\gamma}\bar{F}^I_{\mu\nu}\bar{F}^J_{\alpha\beta}\bar{A}^K_\gamma\Big), \tag{5.1}$$

where $\epsilon^{\mu\nu\alpha\beta\gamma}$ is the completely antisymmetric *tensor* with $\epsilon^{tr\psi\theta\phi} = 1/\sqrt{-g}$. The gauge couplings, f_{IJ}, and the sigma model metric, h_{ST}, are functions of the scalars, X^S, while the Chern-Simons coupling, c_{IJK}, a completely symmetric tensor, is taken to be independent of the scalars. The gauge field strengths are related to the gauge potentials in the usual way: $\bar{F}^I = d\bar{A}^I$.

Since the Lagrangian density is not gauge invariant, we need to be slightly careful about applying the entropy function formalism. Following [52] (who consider a gravitational Chern-Simons term in three dimensions) we dimensionally reduce to a four dimensional action which is gauge invariant. This allows us to find a reduced Lagrangian and in turn the entropy function. As a bonus we will also obtain a relationship between the entropy of four dimensional and five dimensional extremal solutions – this is the $4D$-$5D$ lift of [53, 54] in a more general context.

Assuming all the fields are independent of a compact direction ψ, we take the ansatz[1]

$$\begin{aligned} ds^2 &= w^{-1} g_{\mu\nu} dx^\mu dx^\nu + w^2 (d\psi + A^0_\mu dx^\mu)^2, && (5.2) \\ \bar{A}^I &= A^I_\mu dx^\mu + a^I(x^\mu) \left(d\psi + A^0_\mu dx^\mu \right), && (5.3) \\ \Phi^S &= \Phi^S(x^\mu). && (5.4) \end{aligned}$$

Whether space-time indices above run over 4 or 5 dimensions should be clear from the context. Performing dimensional reduction on ψ, the action becomes

$$S = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g_4} \Big(R - h_{st}(\vec{\Phi}) \partial\Phi^s \partial\Phi^t - f_{ij}(\vec{\Phi}) F^i_{\mu\nu} F^{j\,\mu\nu} - \tilde{f}_{ij}(\vec{\Phi}) \epsilon^{\mu\nu\alpha\beta} F^i_{\mu\nu} F^j_{\alpha\beta} \Big) \quad (5.5)$$

where $(\int d\psi) G_4 = G_5$, $F^i = (F^0, F^I)$, $F^0 = dA^0$, $\Phi^s = (w, a^I, X^S)$ and

$$f_{ij} = \begin{array}{c c} & \begin{array}{cc} 0 & J \end{array} \\ \begin{array}{c} 0 \\ I \end{array} & \begin{pmatrix} \frac{1}{4} w^3 + w f_{LM} a^L a^M & w f_{JL} a^L \\ w f_{IL} a^L & w f_{IJ} \end{pmatrix} \end{array} \quad (5.6)$$

$$\tilde{f}_{ij} = \begin{array}{c c} & \begin{array}{cc} 0 & J \end{array} \\ \begin{array}{c} 0 \\ I \end{array} & \begin{pmatrix} 4 c_{KLM} a^K a^L a^M & 4 c_{JKL} a^K a^L \\ 6 c_{IKL} a^K a^L & 12 c_{IJK} a^K \end{pmatrix} \end{array} \quad (5.7)$$

$$h_{rs} = \mathrm{diag}\left(\tfrac{9}{2} w^{-2}, 2 w f_{IJ}, h_{RS} \right) \quad (5.8)$$

[1] For simplicity, we will work in units in which the Taub-Nut modulus is set to 1.

The index, i, labelling the four dimensional gauge fields, runs over 0 and $\{I\}$. The additional gauge field, A^0 comes from the off-diagonal part of the five dimensional metric while the remaining ones descend from the original five dimensional gauge fields. The index, s, labelling the four dimensional scalars, runs over 0, $\{I\}$ and $\{S\}$. The first additional scalar w, comes from the size of the Kaluza-Klein circle. Then next set, which we label a^I, come from the ψ-components of the five-dimensional gauge fields and become axions in four dimensions. Lastly, the original five dimensional scalars, X^S, descend trivially. Finally, notice that the coupling, $\tilde{f}_{ij}(\vec{\Phi})$, is built up out of the five-dimensional Chern-Simons coupling and the axions.

In the next two sections we shall consider what happens when the near-horizon symmetries are $AdS_2 \times S_2 \times U(1)$ or $AdS_2 \times U(1)^2$, where the $U(1)$'s may be non-trivially fibred. Firstly, we will look at black things with a higher degree of symmetry, namely $AdS_2 \times S_2 \times U(1)$. Upon dimensional reduction we obtain a static, spherically symmetric, extremal black hole near-horizon geometry — $AdS_2 \times S_2$ — for which the analysis is much simpler. The entropy function formalism only involves algebraic equations. After that we will look at black things whose near horizon symmetries are $AdS_2 \times U(1)^2$ in five dimensions. After dimensional reduction, we get an extremal, rotating, near horizon geometry — $AdS_2 \times U(1)$ — for which the entropy function analysis was performed in [51]. For this case, the formalism involves differential equations in general.

5.2 Algebraic entropy function analysis

In this section, we will construct and analyse the entropy function for five dimensional black things sitting in Taub-NUT space with $AdS_2 \times S_2 \times U(1)$ near horizon symmetries (with the $U(1)$ non-trivially fibred). Once we have an appropriate ansatz, it is straight forward to calculate the entropy function. We will apply the analysis to static black holes with $AdS_2 \times S^3$ horizons and black rings with $AdS_3 \times S^2$ horizons. We will see that these black rings are in some sense dual to the black holes. We will then consider Lagrangians with real special geometry and study to the case of $U(1)^3$ super-gravity in some detail.

5.2.1 Set up

Before proceeding to the analysis we need to establish some notation and consider some geometry.

We either use $\tilde{p}^0$, to denote the Taub-NUT charge of the space a black ring is sitting in, or p^0, to denote the charge of a black hole sitting at the center of the space. In each case the $U(1)$ will be modded out by either $\tilde{p}^0$ or p^0. Unlike the black hole, the black ring does not carry Taub-NUT charge. Since we are only looking at the near horizon geometry, the only influence of the charge on the ring will be modding the $U(1)$. We can impose this by hand. To encode asymptotically flat space we simply set the Taub-NUT charge to 1 in both cases. To present things in a unified way, we include p^0 and $\tilde{p}^0$ in the formulae below. Given this notation, when we consider black rings, we must remember to set $p^0 = 0$ and mod out the $U(1)$ by $\tilde{p}^0$. When considering black holes, p^0 is non-zero and, since we do need to mod out by hand, we set $\tilde{p}^0 = 1$.

For black holes, we can fibre the $U(1)$ over the S^2 to get $S^3/\mathbb{Z}_{p^0}$ while for the rings it will turn out that we can fibre the $U(1)$ over the AdS_2 to get $AdS_3/\mathbb{Z}_{\tilde{p}^0}$. These fibrations will only work for specific values of the radius of the Kaluza-Klein circle, w, depending on the radii of the base spaces, S^2 or AdS^2, and the parameters, p^0 or e^0 respectively.[1] Even though we start out treating w as an arbitrary parameter, we will see below that the "correct" value for w will pop out of the entropy function analysis. The fibration which gives us S^3 is the standard Hopf fibration and the one for AdS, which is very similar, is discussed towards the end of section 5.4. The black ring and black hole geometries are schematically illustrated in figure 5.1 and 5.2.

Now, to study the near horizon geometry of black things in Taub-NUT space, with the required symmetries, we specialise our Kaluza-Klein ansatz, (5.2-5.4), to

$$\begin{aligned} ds^2 &= w^{-1}\left[v_1\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + v_2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \\ &\quad + w^2\left(d\psi + e^0\, rdt + p^0\cos\theta d\phi\right)^2, \qquad (5.1) \\ A^I &= e^I\, rdt + p^I\cos\theta d\phi + a^I\left(d\psi + e^0\, rdt + p^0\cos\theta d\phi\right), \qquad (5.2) \end{aligned}$$

[1] e^0 is conjugate to the angular momentum of the ring

$$\Phi^S = u^S, \tag{5.3}$$

where the coordinates, θ and ϕ have periodicity π and 2π respectively. The coordinate ψ has periodicity 4π for black holes and $4\pi/\tilde{p}^0$ for black rings. This ansatz, (5.1-5.3), is consistent with the near horizon geometries of the solutions of [55, 56, 57, 58] as discussed in section 5.4.

Now that we have an appropriate five dimensional ansatz, we can construct the entropy function from the dimensionally reduced four dimensional Lagrangian. From the four dimensional action, we can evaluate the reduced Lagrangian, f, evaluated at the horizon subject to our ansatz. The entropy function is then given by the Legendre transformation of f with respect to the electric fields and their conjugate charges.

The reduced four dimensional action, f, evaluated at the horizon is given by

$$f = \frac{1}{16\pi G_4}\int_H d\theta d\phi\sqrt{-g_4}\mathcal{L}_4 = \frac{1}{16\pi}\left(\frac{4\pi}{\tilde{p}^0 G_5}\right)\int_H d\theta d\phi\sqrt{-g_4}\mathcal{L}_4. \tag{5.4}$$

The equations of motion are equivalent to

$$f_{,v_1} = f_{,v_2} = f_{,w} = f_{,\vec{a}} = f_{,\vec{\Phi}} = 0, \tag{5.5}$$

$$f_{,e^i} = Nq_i, \tag{5.6}$$

where $e^i = (e^0, e^I)$ and q_i are its (conveniently normalised) conjugate charges. We choose the the normalisation $N = 4\pi/\tilde{p}^0 G_5 = 1/G_4$. Using the ansatz, (5.1), we find

$$\begin{aligned} f &= \left(\frac{2\pi}{\tilde{p}^0 G_5}\right)\Big\{v_1 - v_2 - \frac{v_1}{v_2}\left(\tfrac{1}{4}w^3(p^0)^2 + wf_{IJ}(p^I+p^0a^I)(p^J+p^0a^J)\right) \\ &\qquad + \frac{v_2}{v_1}\left(\tfrac{1}{4}w^3(e^0)^2 + wf_{IJ}(e^I+e^0\,a^I)(e^J+e^0\,a^J)\right)\Big\} \\ &\quad + \left(\frac{48\pi}{\tilde{p}^0 G_5}\right)c_{IJK}\left\{p^Ie^Ja^K + \tfrac{1}{2}(p^0e^I+e^0p^I)a^Ja^K + \tfrac{1}{3}p^0e^0a^Ia^Ja^K\right\}, \end{aligned} \tag{5.7}$$

while (5.6) gives the following relationship between the electric fields e^i and their conjugate charges q_i:

$$q_I - \tilde{f}_{Ij}p^j = \left(\frac{v_2}{v_1}\right)wf_{IJ}(e^J + e^0a^J), \tag{5.8}$$

$$q_0 - \tilde{f}_{0j}p^j - a^I\hat{q}_I = \left(\frac{v_2}{v_1}\right)(\tfrac{1}{4}w^3e^0), \tag{5.9}$$

where, $p^i = (p^0, p^I)$ and $\tilde{f}_{ij}$ is given by (5.7). The entropy function is the Legendre transform of f with respect to the charges q_i:

$$\mathcal{E} \;=\; 2\pi(Nq_i e^i - f). \tag{5.10}$$

In terms of $\mathcal{E}$ the equations of motion become

$$\mathcal{E}_{,v_1} = \mathcal{E}_{,v_2} = \mathcal{E}_{,w} = \mathcal{E}_{,\vec{a}} = \mathcal{E}_{,\vec{\Phi}} = \mathcal{E}_{,\vec{e}} = 0. \tag{5.11}$$

Evaluating the entropy function gives

$$\mathcal{E} = 2\pi(Nq^i e_i - f) = \frac{4\pi^2}{\tilde{p}^0 G_5}\left\{v_2 - v_1 + \frac{v_1}{v_2}V_{eff}\right\}, \tag{5.12}$$

where we have defined the effective potential

$$V_{eff} \;=\; f^{ij}\hat{q}_i\hat{q}_j + f_{ij}p^i p^j \tag{5.13}$$

where f_{ij} is given by (5.6), the shifted charges, $\hat{q}_i$, are given by

$$\hat{q}_i \;=\; q_i - \tilde{f}_{ij}p^j, \tag{5.14}$$

and f^{ij}, the inverse of f_{ij}, is given by

$$f^{ij} \;=\; \begin{array}{cc} & \begin{array}{cc} 0 & J \end{array} \\ \begin{array}{c} 0 \\ I \end{array} & \begin{pmatrix} 4w^{-3} & -4w^{-3}a^J \\ -4w^{-3}a^I & w^{-1}f^{IJ} + 4w^{-3}a^I a^J \end{pmatrix} \end{array}, \tag{5.15}$$

where f^{IJ} is the inverse of f_{IJ}. More explicitly, the effective potential is given by

$$\begin{aligned} V_{eff} \;=\; & \tfrac{1}{4}w^3(p^0)^2 + 4w^{-3}(q_0 - \tilde{f}_{0j}(\vec{a})p^j - a^I(q_I - \tilde{f}_{Ij}(\vec{a})p^j))^2 \\ & + w f_{IJ}(\vec{X})(p^I + a^I p^0)(p^J + a^J p^0) \\ & + w^{-1} f^{IJ}(\vec{X})(q_I - \tilde{f}_{Ik}(\vec{a})p^k)(q_J - \tilde{f}_{Jl}(\vec{a})p^l), \end{aligned} \tag{5.16}$$

5.2.2 Preliminary analysis

While the effective potential V_{eff} is in general quite complicated, the dependence of the entropy function, (5.12), on the S^2 and AdS^2 radii is quite simple. Extremising the entropy function with respect to v_1 and v_2, one finds that, at the extremum,

$$\mathcal{E} = \frac{4\pi^2}{\tilde{p}^0 G_5} V_{eff}|_{\partial V = 0}, \tag{5.17}$$

with

$$v_1 = v_2 = V_{eff}|_{\partial V=0}, \tag{5.18}$$

where the effective potential is evaluated at its extremum:

$$\partial_{\{w,\vec{a},\vec{X}\}} V_{eff} = 0. \tag{5.19}$$

As a check, we note that, the result, (5.17), agrees with the Hawking-Bekenstein entropy since,

$$S = \frac{A_H}{4G_5} = \frac{\left(\frac{16\pi^2}{\tilde{p}^0} v_2\right)}{4G_5} = \mathcal{E}. \tag{5.20}$$

Also, the result, (5.18), tells us that the radii of the S^2 and AdS^2 are equal with the scale set by size of the charges.

Finding extrema of the general effective potential, V_{eff}, given by (5.16) may in principle be possible but in practice not simple. In the following sections we consider simpler cases with only a subset of charges turned on.

5.2.3 Black rings

We are now really to specialise to the case of black rings. As discussed at the beginning of the section, for black rings, we take $p^0 = 0$ so that our $AdS_2 \times U(1) \times S^2$ ansatz[1] becomes

$$ds^2 = w^{-1}\left[v_1\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + v_2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] + w^2\left(d\psi + e^0\, rdt\right)^2, \tag{5.21}$$

$$A^I = e^I\, rdt + p^I\cos\theta d\phi + a^I\left(d\psi + e^0\, rdt\right), \tag{5.22}$$

$$\Phi^S = u^S. \tag{5.23}$$

In this case the gauge field (or in 4-D language the axion) equations simplify considerably and it is convenient to analyse them first. Varying f with respect to $\vec{a}$ we find

$$d_{IJ}(e^J + e^0 a^J) = 0 \tag{5.24}$$

[1]It will turn out that once we solve the equations of motion, the value of w is such that the geometry is $AdS_3 \times S^2$.In section A, we have discussed the near horizon geometry of supersymmetric black ring solution.

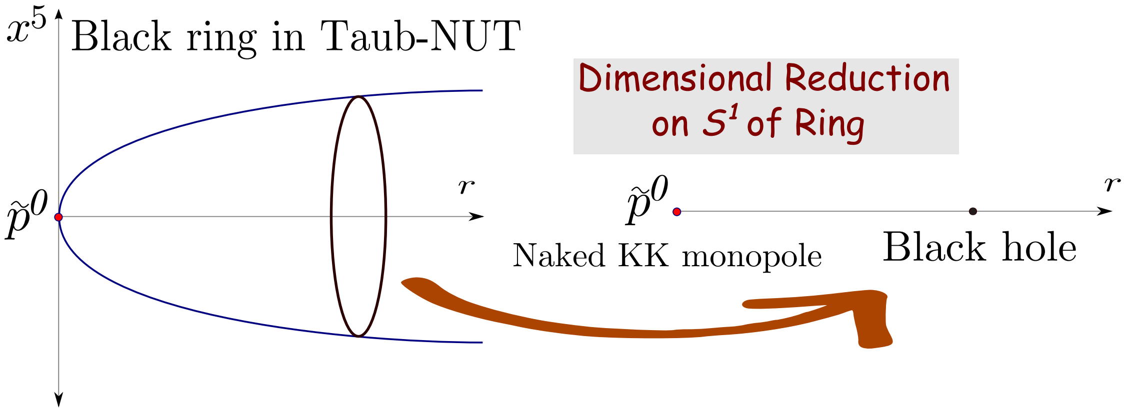


Figure 5.1: A Black ring away from the center of Taub-NUT is projected down to a black hole and naked KK magnetic monopole in four dimensions. The angular momentum carried in the compact dimension will translate to electric charge in four dimensions. An $AdS^2 \times S^2 \times U(1)$ near horizon geometry will project down to $AdS^2 \times S^2$. On the other hand, an $AdS^2 \times U(1)^2$ will go to $AdS^2 \times U(1)$.

where $d_{IJ} = wf_{IJ} + 12c_{IJK}p^K$. Assuming d_{IJ} has no zero eigenvalues, (5.24) implies that the electric field F^I_{tr} is zero. Using (5.8,5.14) this in turn implies $\hat{q}_j = 0$,

$$\hat{q}_0 = q_0 - \tfrac{1}{2}c^{IJ}q_I q_J \tag{5.25}$$

$$a^K = c^{KJ}q_J, \tag{5.26}$$

where c^{KJ} is the inverse of $12c_{IJK}p^K$. Notice that c^{IJ} is equal to the inverse of $\tilde{f}^{IJ}(a^K)$ with a^K replaced by p^K. Now the effective potential becomes

$$V_{eff} = wf_{IJ}p^Ip^J + (4w^{-3})(\hat{q}_0)^2. \tag{5.27}$$

Using $\partial_w V_{eff} = 0$ we find

$$w^4 = \frac{12\hat{q}_0^2}{V_M} \tag{5.28}$$

where we have defined the magnetic potential, $V_M = f_{IJ}p^Ip^J$. So

$$V_{eff} = \frac{4}{3}wV_M = 16w^{-3}(\hat{q}_0)^2 \tag{5.29}$$

Eliminating w from $\mathcal{E}$ we find

$$\mathcal{E} = \frac{8\pi^2}{\tilde{p}^0 G_5}\sqrt{\hat{q}_0(\tfrac{4}{3}V_M)^{\frac{3}{2}}}. \tag{5.30}$$

Finally we note that

$$e_0^2 w^2 = v_1 w^{-1} \tag{5.31}$$

which, we see by comparison with (5.36), means that we have a $S^2 \times AdS_3/\mathbb{Z}_{\tilde{p}^0}$ near horizon geometry.

5.2.3.1 Magnetic potential

Now we study an example in which the magnetic potential, V_M, has a simple form. We consider the 11-dimensional supergravity action dimensionally reduced on T^6 to give 5-dimensional $U(1)^3$ supergravity. In this case the action is of the form (5.1) with $2f_{IJ} = h_{IJ} = \frac{1}{2}\mathrm{diag}\left((X^1)^{-2}, (X^2)^{-2}, (X^3)^{-2}\right)$, and $c_{IJK} = |\epsilon_{IJK}|/24$. We assume the X^I are subject to the constraint

$$X^1 X^2 X^3 = 1 \tag{5.32}$$

so that the T^6 has constant volume.

Evaluating the effective magnetic potential and imposing the constraint (5.32) gives

$$V_M = f_{ij} p^i p^j = \frac{1}{4}\left(\frac{(p^1)^2}{(X^1)^2} + \frac{(p^2)^2}{(X^2)^2} + (p^3)^2 (X^1)^2 (X^2)^2\right) \tag{5.33}$$

which has the extrema

$$(X^1)^3 = \frac{(p^1)^2}{p_2 p_3}, \qquad (X^2)^3 = \frac{(p^2)^2}{p^3 p^1}, \tag{5.34}$$

so the value of V_M at the extremum is

$$V_M = \frac{3}{4}(p^1 p^2 p^2)^{\frac{2}{3}}. \tag{5.35}$$

Solving for the axions we find

$$a^1 = c^{1J} q_J = \frac{-p^1 q_1 + p^2 q_2 + p^3 q_3}{p^2 p^3} \tag{5.36}$$

with a^2 and a^3 given by cyclic permutations which in turn gives

$$\hat{q}_0 = q_0 - \frac{\frac{1}{2}(p^I q_I)^2 - (p^I)^2 (q_I)^2}{p^1 p^2 p^3} \tag{5.37}$$

So finally

$$\mathcal{E} = \frac{8\pi^2}{\tilde{p}_0 G_5}\sqrt{\hat{q}_0(\tfrac{4}{3}V_M)^{3/2}}. \tag{5.38}$$

$$= \frac{8\pi^2}{\tilde{p}_0 G_5}\sqrt{q_0\tilde{p}^0 p^1 p^2 p^3 + (p^I)^2(q_I\tilde{p}^0)^2 - \tfrac{1}{2}(p^I q_I \tilde{p}^0)^2} \tag{5.39}$$

where we have reintroduced a factor of $\tilde{p}^0$ by redefining the normalisation of $\vec{q}$ to be $N = 4\pi/G_5$.

5.2.4 Static 5-d black holes

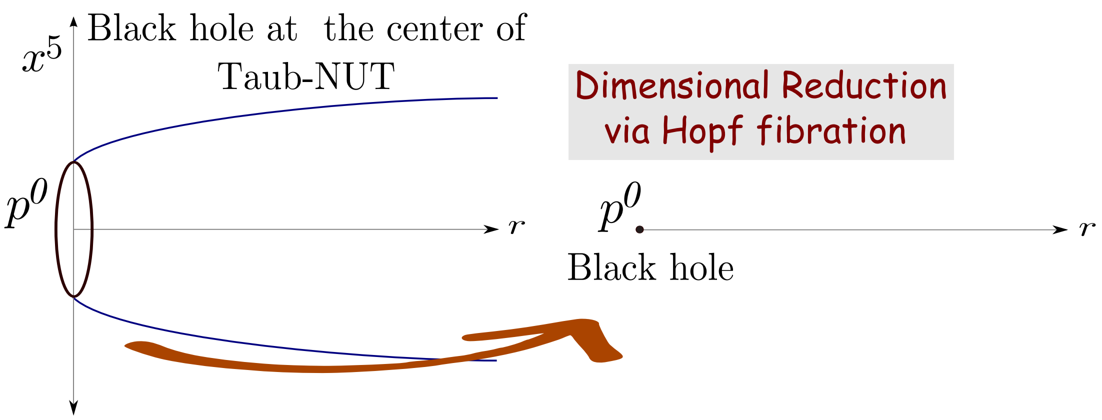


Figure 5.2: A black hole at the center of Taub-NUT caries NUT charge. Using the Hopf fibration it can be projected down to black hole carrying magnetic charge. A spherically symmetric black hole with near horizon geometry of $AdS^2 \times S^3$ will project down to an $AdS^2 \times S^2$. On the other hand, a rotating black hole with a $AdS^2 \times U(1)^2$ geometry will go to $AdS^2 \times U(1)$.

We now consider five dimensional static spherically symmetric black holes. Since they are not rotating we take $e^0 = 0$. This is in some sense "dual" to taking $p^0 = 0$ for black rings. To examine this analogy further, we will relax the natural assumption of an $AdS_2 \times S^3$ geometry to $AdS_2 \times S^2 \times U(1)$. We will see that the analysis for the black holes is very similar to the analysis of the black rings with the magnetic potential replaced by an electric potential. Once we solve the equations of motion we recover an $AdS_2 \times S^3$ geometry via the Hopf fibration.

This is analogous to the black ring where we got $AdS_3 \times S^2$ with the $U(1)$ fibred over the AdS_2 rather than the S^2.

With $e^0 = 0$, our ansatz becomes

$$\begin{aligned} ds^2 &= w^{-1}\left[v_1\left(-r^2dt^2+\frac{dr^2}{r^2}\right)+v_2\left(d\theta^2+\sin^2\theta d\phi^2\right)\right]+w^2\left(d\psi+p^0\cos\theta d\phi\right)^2, && (5.40)\\ A^I &= e^I rdt + p^I\cos\theta d\phi + a^I\left(d\psi+p^0\cos\theta d\phi\right), && (5.41)\\ \Phi^S &= u^S. && (5.42)\end{aligned}$$

In this case the gauge field equation becomes

$$\tilde{d}_{IJ}(p^J+p^0a^J)=0 \tag{5.43}$$

where $\tilde{d}_{IJ} = wf_{IJ} - 12c_{IJK}e^K$. Assuming $\tilde{d}_{IJ}$ has no zero eigenvalues, (5.9,5.14,5.43) now imply,

$$\begin{aligned} \hat{q}_0 - a^I\hat{q}_i &= 0 && (5.44)\\ a^K &= -p^K/p^0, && (5.45)\\ \hat{q}_I &= q_I + 6c_{IJK}p^Jp^K/p^0 && (5.46)\end{aligned}$$

and the effective potential becomes

$$V_{eff} = (\tfrac{1}{4}w^3)(p_0)^2 + w^{-1}f^{IJ}\hat{q}_I\hat{q}_J. \tag{5.47}$$

Using $\partial_w V_{eff} = 0$ we find

$$w^4 = \frac{4V_E}{3p_0^2} \tag{5.48}$$

where we have defined the electric potential $V_E = f^{ij}\hat{q}_i\hat{q}_j$. So

$$V_{eff} = \frac{4}{3}w^{-1}V_E = w^3(p^0)^2 \tag{5.49}$$

Eliminating w from $\mathcal{E}$ we find

$$\mathcal{E} = \frac{4\pi^2}{G_5}\sqrt{p_0(\tfrac{4}{3}V_E)^{\frac{3}{2}}}. \tag{5.50}$$

We note that, analogous to the ring case where we had $e_0^2w^2 = v_1w^{-1}$,

$$p_0^2w^2 = v_2w^{-1} \tag{5.51}$$

which, via the Hopf fibration, gives us an $AdS_2 \times S^3/\mathbb{Z}_{p^0}$ near horizon geometry.

5.2.4.1 Electric potential

As we did for the black ring, we taking the 11-dimensional supergravity action dimensionally reduced on T^6 to give 5-dimensional $U(1)^3$ supergravity. Extremising V_E we find

$$\begin{aligned} \mathcal{E} &= \frac{8\pi^2}{G_5}\sqrt{p_0 \hat{q}_1 \hat{q}_2 \hat{q}_3} && (5.52)\\ &= 2\pi\sqrt{p_0\left(q_1+\tfrac{1}{4}p^2p^3/p^0\right)\left(q_2+\tfrac{1}{4}p^1p^3/p^0\right)\left(q_3+\tfrac{1}{4}p^2p^3/p^0\right)} && (5.53)\end{aligned}$$

5.2.5 Very Special Geometry

We now consider $\mathcal{N}=2$ supergravity in five dimensions corresponding to M-theory on a Calabi-Yau threefold – this gives what has been called real or very special geometry [59, 60, 61, 62, 63, 64, 65] The case of T^6, considered above, is a simple example of very special geometry. Some properties of very special geometry which we use are recorded in section 5.5.

5.2.5.1 Black rings and very special geometry

From the attractor equations, for a black ring governed by V_M, we have the equation

$$\partial_i V_M = \partial_i(H_{IJ}p^I p^J) special : lower : 2\tfrac{1}{2}\partial_i(p_I p^I) = 0 \qquad (5.54)$$

These equations have a solution

$$\lambda X_I = p_I \qquad (5.55)$$

This condition follows from one of the BPS conditions found in [13]. From (5.54) we now have

$$\begin{aligned} \partial_i(p_I p^I) \quad &= \quad \partial_i(p_I)p^I && (5.56)\\ BPS : anz \quad & \lambda\partial_i(X_I)p^I && (5.57)\\ special : covariant \quad & -2\lambda H_{IJ}p^I\partial_i(X^J) && (5.58)\\ special : lower \quad & -\lambda p_J\partial_i(X^J) && (5.59)\\ BPS : anz \quad & -\lambda^2 X_J\partial_i(X^J)X : norm : 20 && (5.60)\end{aligned}$$

We can fix the constant λ using (5.38) which gives

$$X_I = \frac{p_I}{(\frac{1}{6}C_{IJK}p^Ip^Jp^K)^{\frac{1}{3}}} \tag{5.61}$$

so finally we get for X^I,

$$X^I = \frac{p^I}{(\frac{1}{6}C_{IJK}p^Ip^Jp^K)^{\frac{1}{3}}} \tag{5.62}$$

and

$$V_M|_{\partial V=0} = H_{IJ}p^Ip^J = \lambda^2 H_{IJ}X^IX^J = \frac{3}{2}\lambda^2 \tag{5.63}$$

$$= \frac{3}{2}\left(\frac{1}{6}C_{IJK}p^Ip^Jp^K\right)^{\frac{2}{3}} \tag{5.64}$$

This is the supersymmetric solution of [13] derived from the BPS attractor equations.

Notice that (5.57) can be rewritten as extremising the magnetic central charge, $Z_M = X_Ip^I$:

$$\partial_i(X_I)p^I = \partial_i Z_M = 0 \tag{5.65}$$

So we see that $\partial_i V_M = 0$ together with the BPS condition (5.55) implies Z_M extremised. The converse is not necessarily true suggesting there are non-BPS black ring extrema of V_M — this is discussed below.

5.2.5.2 Static black holes and very special geometry

The analysis for these black holes is analogous to the black rings. From the attractor equations for a static black hole, governed by V_E, we will get the equation:

$$\partial_i V_E = \partial_i(H^{IJ}\hat{q}_I\hat{q}_J) = \tfrac{1}{2}\partial_i(\hat{q}^I\hat{q}_I) = 0 \tag{5.66}$$

This will have similar solutions

$$X^I = \frac{\hat{q}^I}{(\frac{1}{6}C_{IJK}\hat{q}^I\hat{q}^J\hat{q}^K)^{\frac{1}{3}}} \tag{5.67}$$

Similarly, extremising the electric central charge Z_e of [13] together with the BPS condition implies V_E is extremised. The converse is not necessarily true suggesting there are non-BPS black hole extrema of V_E as noted in [66].

5.2.5.3 Non-supersymmetric solutions of very special geometry

In 4 dimensional $\mathcal{N} = 2$ special geometry we can write V_{eff} or the "blackhole potential function" as [67]

$$V_{BH} = |Z|^2 + |DZ|^2. \tag{5.68}$$

As noted in [67] and [5] (in slightly different notation), for BPS solutions, each term of the potential is separately extremised while for non-BPS solutions V_{BH} is extremised but $DZ \neq 0$. It is perhaps not surprising that a similar thing happens in very special geometry. In five dimensions, the potential functions are real functions roughly of the form

$$V = Z^2 + (DZ)^2. \tag{5.69}$$

where the inner product is with respect to H_{IJ}.

In fact, this generalisation of the non-BPS attractor equations to five dimensional static black holes has already be shown in [66] using a reduced Lagrangian approach.

The electric potential V_E can be written

$$V_E = H^{IJ}\hat{q}_J\hat{q}_J = H^{IJ}(D_I\hat{Z}_E)(D_J\hat{Z}_E) + \tfrac{2}{3}(\hat{Z}_E)^2. \tag{5.70}$$

Solving $D_I V_E = 0$ we find a BPS solution, $D_I\hat{Z}_E = 0$, and another solution

$$\tfrac{2}{3}H_{IJ}\hat{Z}_E + D_I D_J \hat{Z}_E = 0. \tag{5.71}$$

Similarly the magnetic potential V_M can be written

$$V_M = H_{IJ}p^I p^J = \tfrac{1}{3}Z_M^2 + H^{IJ}D_I Z_M D_J Z_M \tag{5.72}$$

and solving $D_I V_M = 0$ we find a BPS solution, $D_I Z_M = 0$, and another solution

$$\tfrac{1}{3}H_{IJ}Z_M + D_I D_J Z_M = 0. \tag{5.73}$$

Presumably, on can obtain some five dimensional non-SUSY solutions by lifting non-SUSY5s5d solutions in four dimensions which have $AdS_2 \times S^2$ near horizon geometries using the $4D$-$5D$ lift. Furthermore the analysis of [5] should go through so that for such solutions to exist we require that extremum of V_{eff} is a minimum – in other words the matrix

$$\left.\frac{\partial^2 V_{eff}}{\partial\Phi^S \partial\Phi^T}\right|_{\partial V=0} > 0 \tag{5.74}$$

should have non-zero eigenvalues.

5.2.5.4 Rotating black holes

We now consider the case of black holes which break the $AdS_2 \times S^3$ symmetries of the static black hole only partially down to $AdS_2 \times S^2 \times U(1)$. Following section C, the ansatz for the near horizon geometry is given by,

$$\begin{aligned} ds^2 &= w^{-1}\left[v_1\left(-r^2dt^2+\frac{dr^2}{r^2}\right)+v_2\left(d\theta^2+\sin^2\theta d\phi^2\right)\right]+w^2\left(d\psi+e^0\,rdt+p^0\cos\theta d\phi\right)^2 & (5.75)\\ A &= e^1\,rdt + a^1\left(d\psi+e^0\,rdt+p^0\cos\theta d\phi\right), & (5.76)\\ \Phi^1 &= u^1, & (5.77)\end{aligned}$$

and let $f = f_{11}(\Phi^1)$ and $c = 12c_{111}$. This gives us the effective potential

$$V_{eff} = \tfrac{1}{4}w^3(p^0)^2 + 4w^{-3}(q_0 - \tfrac{1}{3}c(a^1)^3 - (a^1)q_1)^2 + wf(a^1)^2(p^0)^2 + (wf)^{-1}q_1^2. \tag{5.78}$$

Extremising V_{eff} we find that we require $q_1 < 0$ and

$$\begin{aligned} a^1 &= \pm\sqrt{\frac{2|q_1|}{cp^0}} & (5.79)\\ w &= (\tfrac{4}{3})^{\frac{1}{3}}(|aq_1| \pm 3q^0)^{1/3} & (5.80)\\ f &= \frac{|a^1|c}{2w} & (5.81)\end{aligned}$$

with

$$V_{eff}|_{\partial V=0} = V_\pm = \frac{8}{3}\sqrt{\frac{2|q_1|^3 p^0}{c}} \pm 2p^0q^0 \tag{5.82}$$

Interestingly, this relatively simple effective potential has two extrema with different values of the entropy at each extremum. For both extrema to exist we require $|a^1 q| > 3|q^0|$.

We note that, [68], which recently appeared, discuss rotating black holes in very special geometry.

5.3 General Entropy function

We now relax our symmetry assumptions to $AdS_2 \times U(1)^2$, taking the following ansatz

$$
\begin{aligned}
ds^2 &= w^{-1}(\theta)\Omega^2(\theta)e^{2\Psi(\theta)}\left(-r^2dt^2+\frac{dr^2}{r^2}+\beta^2 d\theta^2\right)+w^{-1}(\theta)e^{-2\Psi(\theta)}(d\phi+e_\phi r dt)^2 && (5.1)\\
&\quad +w^2(\theta)(d\psi+e_0 r dt+b_0(\theta)d\phi)^2 && (5.2)\\
A^I &= e^I r dt + b^I(\theta)(d\phi+e_\phi r dt)+a^I(\theta)(d\psi+e_0 r dt+b_0(\theta)d\phi) && (5.3)\\
\phi^S &= u^S(\theta). && (5.4)
\end{aligned}
$$

Now, using (5.5) and then following [51], the entropy function is

$$
\begin{aligned}
\mathcal{E} &\equiv 2\pi(J_\phi e_\phi + \vec{q}\cdot\vec{e} - \int d\theta d\phi\sqrt{-\det g}\,\mathcal{L}_4)\\
&= 2\pi J_\phi e_\phi + 2\pi\vec{q}\cdot\vec{e} - \frac{\pi^2}{\tilde{p}_0 G_5}\int d\theta\Big[2\Omega(\theta)^{-1}\beta^{-1}(\Omega'(\theta))^2 - 2\Omega(\theta)\beta - 2\Omega(\theta)\beta^{-1}(\Psi'(\theta))^2\\
&\quad +\frac{1}{2}\alpha^2\Omega(\theta)^{-1}\beta e^{-4\Psi(\theta)} - \beta^{-1}\Omega(\theta)h_{rs}(\vec{u}(\theta))u_r'(\theta)u_s'(\theta) + 4\tilde{f}_{ij}(\vec{u}(\theta))(e_i-\alpha b_i(\theta))b_j'(\theta)\\
&\quad +2f_{ij}(\vec{u}(\theta))\left\{\beta\Omega(\theta)^{-1}e^{-2\Psi(\theta)}(e_i-\alpha b_i(\theta))(e_j-\alpha b_j(\theta)) - \beta^{-1}\Omega(\theta)e^{2\Psi(\theta)}b_i'(\theta)b_j'(\theta)\right\}\Big]\\
&\quad +\frac{2\pi^2}{\tilde{p}_0 G_5}\left[\Omega(\theta)^2 e^{2\Psi(\theta)}\sin\theta(\Psi'(\theta)+2\Omega'(\theta)/\Omega(\theta))\right]_{\theta=0}^{\theta=\pi}. \qquad (5.5)
\end{aligned}
$$

where f_{ij}, $\tilde{f}_{ij}$,h_{rs} and u^s related to five dimensional quantities as discussed in section 5.1. Now extremising the entropy function gives us differential equations.

Using the near horizon geometry of the non-SUSY black ring of [69], which we evaluate in section 5.7, we find that the entropy function gives the correct entropy.

5.4 Supersymmetric black ring near horizon geometry

Here, we will consider the black ring solution of [56], and find the near horizon limit of the metric and the gauge fields. This will enable us to compare with the charges defined in Section 3.1.

The metric is

$$ds^2 = -f^2(dt+\omega)^2 + f^{-1}ds^2(M_4)\,, \tag{5.6}$$

where

$$f^{-1} = 1 + \frac{Q-q^2}{2R^2}(x-y) - \frac{q^2}{4R^2}(x^2-y^2) \tag{5.7}$$

$$ds^2(\mathbb{R}^4) = \frac{R^2}{(x-y)^2}\left\{\frac{dy^2}{y^2-1} + (y^2-1)d\psi^2 + \frac{dx^2}{1-x^2} + (1-x^2)d\phi^2\right\} \tag{5.8}$$

and $\omega = \omega_\psi(x,y)d\psi + \omega_\phi(x,y)d\phi$ with

$$\begin{aligned} \omega_\phi &= -\frac{q}{8R^2}(1-x^2)\left[3Q - q^2(3+x+y)\right]\,, \\ \omega_\psi &= \frac{3}{2}q(1+y) + \frac{q}{8R^2}(1-y^2)\left[3Q - q^2(3+x+y)\right]\,. \end{aligned} \tag{5.9}$$

The gauge field is expressed as,

$$A = \frac{\sqrt{3}}{2}\left[f\,(dt+\omega) - \frac{q}{2}((1+x)\,d\phi + (1+y)\,d\psi)\right]\,. \tag{5.10}$$

The ADM charges are given by

$$\begin{aligned} M &= \frac{3\pi}{4G}Q\,, \qquad J_\phi = \frac{\pi}{8G}\,q\,(3Q-q^2)\,, \\ J_\psi &= \frac{\pi}{8G}\,q\,(6R^2+3Q-q^2)\,. \end{aligned} \tag{5.11}$$

Near Horizon Geometry

In these coordinates, the horizon lies at $y \to -\infty$. To examine the near horizon geometry, it is convenient to define a new coordinate $r = -R/y$ (so the horizon is at $r = 0$). Then consider a coordinate transformation of the form

$$\begin{aligned} dt &= dv - B(r)dr, \\ d\phi &= d\phi' - C(r)dr, \qquad (5.12) \\ d\psi &= d\psi' - C(r)dr, \qquad (5.13) \end{aligned}$$

where

$$B(r) = \frac{B_2}{r^2} + \frac{B_1}{r} + B_0, \qquad C(r) = \frac{C_1}{r} + C_0. \tag{5.14}$$

where $B_2 = q^2 L/(4R)$ and $C_1 = -q/(2L)$, with

$$L \equiv \sqrt{3\left[\frac{(Q-q^2)^2}{4q^2} - R^2\right]}, \tag{5.15}$$

and

$$B_1 = (Q+2q^2)/(4L) + L(Q-q^2)/(3R^2)$$
$$C_0 = -(Q-q^2)^3/(8q^3RL^3)$$
$$B_0 = q^2L/(8R^3) + 2L/(3R) - R/(2L) + 3R^3/(2L^3) + 3(Q-q^2)^3/(16q^2RL^3)$$

The metric (5.8) becomes

$$\begin{aligned} ds^2 &= -\frac{16r^4}{q^4}dv^2 + \frac{2R}{L}dvdr + \frac{4r^3\sin^2\theta}{Rq}dvd\phi' \\ &+ \frac{4Rr}{q}dvd\psi' + \frac{3qr\sin^2\theta}{L}drd\phi' \\ &+ 2\Big[\frac{qL}{2R}\cos\theta + \frac{3qR}{2L} + \frac{(Q-q^2)(3R^2-2L^2)}{3qRL}\Big]drd\psi' \\ &+ L^2 d\psi'^2 + \frac{q^2}{4}\left[d\theta^2 + \sin^2\theta\,(d\phi'-d\psi')^2\right] + \ldots \end{aligned} \tag{5.16}$$

and the gauge field (5.10) becomes:

$$A = \frac{\sqrt{3}}{2}\left[f\,(dv+\omega') - \frac{q}{2}((1+x)\,d\phi' + (1+y)\,d\psi')\right. \tag{5.17}$$

$$\left. -\left(fB + C\left\{f\omega_\phi + f\omega_\psi - \frac{q}{2}(1+x) - \frac{q}{2}(1-\frac{R}{r})\right\}\right)dr\right] \tag{5.18}$$

with $\omega' = \omega_\psi d\psi' + \omega_\phi d\phi'$ In the limit of small r

$$f = \frac{1}{1 + x(f_1 - f_2 x) + f_1 r^{-1} + f_2 r^{-2}} \tag{5.19}$$

$$= \frac{r^2}{f_2}\left(1 - f_1 f_2^{-1} r + \left(f_1^2 f_2^{-1} - 1 + x(x - f_1 f_2^{-1})\right) f_2^{-1} r^2 + \mathcal{O}\left(r^3\right)\right) \tag{5.20}$$

where $f_1 = (Q-q^2)/2R^2$ and $f_2 = q^2/4R^2$. Expanding ω in the limit of small r, we have,

$$\omega_\phi = \left\{-\frac{q^3(1-x^2)}{8R}\right\}\frac{1}{r} + \left\{\frac{q\,(xq^2+3q^2-3Q)\,(1-x^2)}{8R^2}\right\} \tag{5.21}$$

$$\omega_\psi = \left\{-\frac{q^3R}{8}\right\}\frac{1}{r^3} + \left\{\frac{xq^3}{8} + \frac{3q^3}{8} - \frac{3Qq}{8}\right\}\frac{1}{r^2} + \left\{\frac{q^3}{8R} - \frac{3qR}{2}\right\}\frac{1}{r} \tag{5.22}$$

$$+ \left\{-\frac{xq^3}{8R^2} - \frac{3q^3}{8R^2} + \frac{3Qq}{8R^2} + \frac{3q}{2}\right\} \tag{5.23}$$

Expanding out the gauge field (neglecting some terms which can be gauged away) we obtain:

$$\begin{aligned} A &= \frac{\sqrt{3}}{2}\left\{-\left[\frac{q}{2}+\frac{Q}{2q}+\mathcal{O}(r)\right]d\psi' + \left[-\frac{q}{2}(x+1)+\mathcal{O}(r)\right]d\chi \right. \\ &\quad \left. + \left[\frac{4}{q^2}r^2+\mathcal{O}(r^3)\right]dv + \left[c_r x\, r+\mathcal{O}(r^2)\right]dr\right\} \end{aligned}$$

where $\chi = \phi - \psi$ and

$$c_r = \frac{L\left(R^2\left(2R^4-3\right)q^4 + \left(-4QR^6+6QR^2+2\right)q^2 + Q^2R^2\left(2R^4-3\right)\right)}{2q^2}$$

Finally taking the near-horizon limit $r = \epsilon L\tilde{r}/R$, $v = \tilde{v}/\epsilon$, $\epsilon \to 0$ we obtain

$$A = -\frac{\sqrt{3}}{4}\left[q+\frac{Q}{q}\right]d\psi' - \frac{\sqrt{3}q}{4}(x+1)\,d\chi \tag{5.24}$$

So comparing (5.24) with (5.2) we get

$$\frac{p^1}{\pi} = \sqrt{3}q \qquad a = -\frac{e^1}{e^0} = -\frac{\sqrt{3}}{4}\left[q+\frac{Q}{q}\right] \tag{5.25}$$

Taking the same near horizon limit for the metric we obtain

$$ds^2 = 2d\tilde{v}d\tilde{r} + \frac{4L}{q}\tilde{r}d\tilde{v}d\psi' + L^2 d\psi'^2 + \frac{q^2}{4}\left[d\theta^2 + \sin^2\theta d\chi^2\right] \tag{5.26}$$

Let us for the moment consider the metric for constant θ and χ. If we perform the coordinate transformation

$$d\psi' = d\psi - \frac{q}{2L}\frac{d\tilde{r}}{\tilde{r}} \tag{5.27}$$

$$d\tilde{v} = dt + \frac{q^2}{4}\frac{d\tilde{r}}{\tilde{r}^2} \tag{5.28}$$

we get

$$ds^2 = \frac{4L}{q}\tilde{r}dtd\psi + L^2 d\psi^2 + \frac{q^2}{4}\frac{d\tilde{r}}{\tilde{r}^2} \tag{5.29}$$

Letting

$$dt = dt' + \frac{q}{2}d\psi \tag{5.30}$$

we obtain the more familiar form of BTZ

$$ds^2 = \frac{4L}{q}\tilde{r}dtd\psi + (L^2 + 2L\tilde{r})d\psi^2 + \frac{q^2}{4}\frac{d\tilde{r}}{\tilde{r}^2} \tag{5.31}$$

Now we define

$$l = \quad q \qquad r_+ = L \tag{5.32}$$

$$\tilde{r} = \quad \tfrac{r^2 - r_+^2}{2r_+} \qquad \tilde{\phi} = \psi + t'/l \tag{5.33}$$

we get the standard form of the BTZ metric

$$ds^2 = -\frac{(r^2 - r_+^2)}{l^2 r^2} dt'^2 + \frac{l^2 r^2}{(r^2 - r_+^2)} dr^2 + r^2 \left(d\tilde{\phi} - \frac{r^2 - r_+^2}{l r^2} dt' \right)^2 \tag{5.34}$$

Returning to (5.29) and letting

$$t = l^2 \tau / 4 \text{ and } e^0 = l/2L \tag{5.35}$$

we obtain

$$ds^2 = \frac{1}{4} l^2 \left(-\tilde{r}^2 d\tau^2 + \frac{d\tilde{r}^2}{\tilde{r}^2} \right) + \frac{l^2}{4(e^0)^2} \left(d\psi + e^0 \tilde{r} d\tau \right)^2 \tag{5.36}$$

5.5 Notes on Very Special Geometry

Here we collect some useful relations and define some notation from very special geometry along the lines of [13, 66], which are used in section 3.3.

- We take our CY_3 to have Hodge numbers $h^{1,1}$ with the index $I \in 1, 2, \ldots, h^{1,1}$.
- The Kähler moduli, X^I which are *real*, correspond to the volumes of the 2-cycles.
- C_{IJK} are the triple intersection numbers.
- The volumes of the 4-cycles Ω_I are given by

$$X_I = \frac{1}{2} C_{IJK} X^J X^K \ . \tag{5.37}$$

- The prepotential is given by

$$\mathcal{V} = \frac{1}{6} C_{IJK} X^I X^J X^K \ = 1 \tag{5.38}$$

- The volume constraint (5.38) implies there are $n_v = h^{1,1} - 1$ independent vector-multiplets.

- denote the independent vector-multiplet scalars as ϕ^i, and the corresponding derivatives $\partial_i = \frac{\partial}{\partial \phi^i}$.

- The kinetic terms for the gauge fields are governed by the metric

$$H_{IJ} = -\tfrac{1}{2}\, \partial_I \partial_J \ln \mathcal{V}|_{\mathcal{V}=1} = -\tfrac{1}{2}(C_{IJK}X^K - X_I X_J)\ , \tag{5.39}$$

  where we use the notation for derivatives: $\partial_I = \frac{\partial}{\partial X^I}$.

- The electric central charge is given by

$$Z_E = X^I q_I . \tag{5.40}$$

  We generalise this to

$$\hat{Z}_E = X^I \hat{q}_I . \tag{5.41}$$

- The magnetic central charge is given by

$$Z_M = X_I p^I . \tag{5.42}$$

- From (5.37) it follows that

$$X_I X^I = 3\ , \tag{5.43}$$

  so

$$X^I \partial_i X_I = \partial_i X^I X_I = 0\ . \tag{5.44}$$

  which in turn together with (5.39) gives

$$X_I \;=\; 2H_{IJ}X^J\ , \tag{5.45}$$

$$\partial_i X_I \;=\; -2H_{IJ}\partial_i X^J\ . \tag{5.46}$$

- As suggested by, (5.45) , we will use $2H_{IJ}$ to lower indices, so for example,

$$p_I = 2H_{IJ}p^J, \tag{5.47}$$

  which in turn implies we should raise indices with $\frac{1}{2}H^{IJ}$,

$$q^I = \tfrac{1}{2}H^{IJ}q_J \tag{5.48}$$

  where H^{IJ} is the inverse of H_{IJ}.

- In order to take the volume constraint (5.38) into account, it is convenient to define a covariant derivative D_I,

$$D_I f = (\partial_I - \tfrac{1}{3}(\partial_I \ln \mathcal{V})|_{\mathcal{V}=1})f. \tag{5.49}$$

Rather than extremise things with respect to the real degrees of freedom using ∂_i, we can take covariant derivatives.

5.6 Spinning black hole near horizon geometry

In this section, we will obtain the near horizon geometry of the rotating black holes mentioned in Section 3.3.4 , which break the $AdS_2 \times S^3$ symmetries of the static black holes, partially down to $AdS_2 \times S^2 \times U(1)$. We consider supersymmetric black holes in Taub-NUT [70] whose near horizon geometry can be written [54] :

$$\begin{aligned} ds^2 &= -\tilde{Q}^{-2}(rdt + \tilde{J}(d\psi + p^0 \cos\theta d\phi))^2 + p^0\tilde{Q}\frac{dr^2}{r^2} + p^0\tilde{Q}(d\theta^2 + \sin^2\theta d\phi^2) && (5.50) \\ &\quad + \frac{\tilde{Q}}{p^0}(d\psi + p^0\cos\theta d\phi)^2 && (5.51) \\ A &= \frac{\sqrt{3}}{2\tilde{Q}}(rdt + \tilde{J}(d\psi + p^0\cos\theta d\phi)) && (5.52) \end{aligned}$$

letting $dt \to dt\sqrt{p^0\tilde{Q}^3 - (Jp^0)^2}$, one can rewrite the metric as

$$ds^2 = (V_{eff})w^{-1}\left(-r^2dt^2 + \frac{dr^2}{r^2} + d\theta^2 + \sin^2\theta d\phi^2\right) + w^2(d\psi + p^0\cos\theta d\phi + e^0 rdt)^2 \tag{5.53}$$

which agrees with 5.75. Here,

$$\begin{aligned} V_{eff} &= \sqrt{p^0\tilde{Q}^3 - (\tilde{J}p^0)^2} && (5.54) \\ w &= \frac{V_{eff}}{p^0\tilde{Q}} && (5.55) \\ e^0 &= -\frac{(\tilde{J}p^0)^2}{V_{eff}} && (5.56) \end{aligned}$$

and the gauge field as

$$A = erdt + a(d\psi + e^0 rdt + p^0\cos\theta d\phi) \tag{5.57}$$

where

$$e = \frac{\sqrt{3}p^0\tilde{Q}^2}{2V_{eff}} \tag{5.58}$$

$$a = \frac{\sqrt{3}p^0\tilde{J}}{2\tilde{Q}} \tag{5.59}$$

5.7 Non-supersymmetric ring near horizon geometry

In section 4, we construct the general entropy function for solutions with near horizon geometries $AdS_2 \times U(1)^2$. Here, we begin with non-supersymmetric black ring solution of [69], and show that it falls into the general class of solutions mentioned in Section 4. Then we also evaluate the entropy of the black ring by extremizing the entropy function. We consider the action

$$I = \frac{1}{16\pi G_5}\int \sqrt{-g}\Big(R - \frac{1}{4}F^2 - \frac{1}{6\sqrt{3}}\,\epsilon^{\mu\alpha\beta\gamma\delta}A_\mu F_{\alpha\beta}F_{\gamma\delta}\Big)\,, \tag{5.1}$$

The metric for the non-SUSY solution is [69]

$$\begin{aligned} ds^2 &= -\frac{1}{h^2(x,y)}\frac{H(x)}{H(y)}\frac{F(y)}{F(x)}\Big(dt + A^0\Big)^2 \\ &\quad + h(x,y)F(x)H(x)H(y)^2 \\ &\quad \times \frac{R^2}{(x-y)^2}\Big[-\frac{G(y)}{F(y)H(y)^3}\,d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)H(x)^3}\,d\phi^2\Big]\,, \end{aligned} \tag{5.2}$$

The functions appearing above are defined as

$$F(\xi) = 1 + \lambda\xi, \qquad G(\xi) = (1-\xi^2)(1+\nu\xi), \qquad H(\xi) = 1 - \mu\xi, \tag{5.3}$$

and

$$h(x,y) = 1 + \frac{s^2}{F(x)H(y)}(x-y)(\lambda+\mu) \tag{5.4}$$

with

$$s = \sinh\alpha \qquad c = \cosh\alpha \tag{5.5}$$

The components of the gauge field are

$$
\begin{aligned}
A^1_t &= \sqrt{3}c/hs\,, & (5.6)\\
A^1_\psi &= \sqrt{3}\frac{R(1+y)s}{h}\left[\frac{C_\lambda(c^2-h)}{s^2F(y)}c^2 - C_\mu\frac{3c^2-h}{H(y)}\right], & (5.7)\\
A^1_\phi &= -\sqrt{3}\frac{R(1+x)c}{h}\left[\frac{C_\lambda}{F(x)}s^2 - C_\mu\frac{3c^2-2h}{H(x)}\right], & (5.8)
\end{aligned}
$$

$$
C_\lambda = \epsilon\sqrt{\lambda(\lambda+\nu)\frac{1+\lambda}{1-\lambda}}\,, \qquad C_\mu = \epsilon\sqrt{\mu(\mu+\nu)\frac{1-\mu}{1+\mu}} \tag{5.9}
$$

A choice of sign $\epsilon = \pm 1$ has been included explicitly. The components of the one-form $A^0 = A^0_\psi\, d\psi + A^0_\phi\, d\phi$ are

$$
A^0_\psi(y) \;=\; R(1+y)c\Big[\frac{C_\lambda}{F(y)}c^2 - \frac{3C_\mu}{H(y)}s^2\Big] \tag{5.10}
$$

$$
A^0_\phi(x) \;=\; = -R\,\frac{1-x^2}{F(x)H(x)}\,\frac{\lambda+\mu}{1+\lambda}\,C_\lambda s^3\,, \tag{5.11}
$$

The coordinates x and y take values in the ranges

$$
-1 \le x \le 1\,, \qquad -\infty < y \le -1\,, \qquad \mu^{-1} < y < \infty\,. \tag{5.12}
$$

The solution has three Killing vectors, ∂_t, ∂_ψ, and ∂_ϕ, and is characterized by four dimensionless parameters, λ, μ, α and ν, and the scale parameter R, which has dimension of length.

Without loss of generality we can take $R > 0$. The parameters λ, μ are restricted as

$$
0 \le \lambda < 1\,, \qquad 0 \le \mu < 1\,. \tag{5.13}
$$

The parameters are not all independent — they are related by

$$
\frac{C_\lambda}{1+\lambda}s^2 = \frac{3C_\mu}{1-\mu}c^2 \tag{5.14}
$$

$$
\frac{1+\lambda}{1-\lambda} = \left(\frac{1+\nu}{1-\nu}\right)^2\left(\frac{1+\mu}{1-\mu}\right)^3. \tag{5.15}
$$

which, in the extremal limit, $\nu \to 0$, implies

$$
\lambda = \frac{\mu(3+\mu^2)}{1+3\mu^2} \tag{5.16}
$$

and

$$s^2 = \frac{3}{4}(1/\mu^2 - 1) \tag{5.17}$$

To avoid conical defects, the periodicites of ψ and ϕ are

$$\Delta\psi = \Delta\phi = 2\pi\sqrt{1-\lambda}(1+\mu)^{\frac{3}{2}}\,. \tag{5.18}$$

5.7.1 Near horizon geometry

In the metric given by (5.2),there is a coordinate singularity at $y = -1/\nu$ which is the location of the horizon. It can be removed by the coordinate transformation [69]:

$$dt = dv + A^0_\psi(y)\,\frac{\sqrt{-F(y)H(y)^3}}{G(y)}dy\,, \qquad d\psi = d\psi' - \frac{\sqrt{-F(y)H(y)^3}}{G(y)}dy\,. \tag{5.19}$$

Letting, $\nu \to 0$, making the coordinate change

$$x = \cos\theta, \qquad y = -\frac{R}{(\sqrt{\lambda\mu})\tilde{r}}, \tag{5.20}$$

and expanding in powers of r, the metric becomes

$$\begin{aligned} ds^2 &= \left[\frac{\lambda F(x)H(x)}{\mu K(x)^2}\right](dv + c_\psi(Rd\psi') + A^0_\phi(x)d\phi)^2 && (5.21)\\ &+[H(x)K(x)]\left(-\tilde{r}^2 d\psi'^2 + 2\mu R d\psi' d\tilde{r} + \mu^2R^2d\theta^2\right) && (5.22)\\ &+\left[\frac{\mu^2K(x)}{H(x)^2F(x)}\right]R^2\sin^2\theta d\phi^2 + \ldots && (5.23)\end{aligned}$$

where

$$c_\psi = c\left(\frac{C_\lambda c^2}{\lambda} + \frac{3s^2C_\mu}{\mu}\right) \tag{5.24}$$

and

$$K(x) = F(x) + s^2(1+\lambda/\mu) \tag{5.25}$$

and we have neglected higher order terms in r which will disappear when we take the near horizon limit below. Letting

$$\begin{aligned} \tilde{\psi} &= \psi' + v/Rc_\psi && (5.26)\\ u &= v/Rc_\psi\epsilon && (5.27)\\ r &= \epsilon\tilde{r} && (5.28)\end{aligned}$$

and taking the near horizon limit, $\epsilon \to 0$, the metric becomes

$$
\begin{aligned}
ds^2 &= [\ldots]\left(-r^2du^2 + 2\mu R du dr + \mu^2 R^2 d\theta^2\right) && (5.29)\\
&\quad +[\ldots]R^2 \sin^2\theta d\phi^2 && (5.30)\\
&\quad [\ldots](c_\psi(Rd\tilde{\psi}) + A^0_\phi(x)d\phi)^2 && (5.31)
\end{aligned}
$$

Now we let

$$
\begin{aligned}
du &= du' + \frac{\mu R dr}{r^2} && (5.32)\\
t &= \frac{u'}{\mu R} && (5.33)
\end{aligned}
$$

Now we use the periodicities of ϕ and $\tilde{\psi}$ to redefine our coordinates,

$$d\phi \to L d\phi \tag{5.34}$$

$$d\tilde{\psi} \to \frac{L d\tilde{\psi}}{2} \tag{5.35}$$

where $L = \sqrt{1-\lambda}(1+\mu)^{3/2}$ Now the metric reduces to (5.1) ,

$$
\begin{aligned}
ds^2 &= w^{-1}(\theta)\Omega^2(\theta)e^{2\Psi(\theta)}\left(-r^2dt^2 + \frac{dr^2}{r^2} + \beta^2 d\theta^2\right) + w^{-1}(\theta)e^{-2\Psi(\theta)}(d\phi + e_\phi r dt)^2 && (5.36)\\
&\quad + w^2(\theta)(d\psi + e_0 r dt + b_0(\theta)d\phi)^2 && (5.37)\\
A^1 &= e^1 r dt + b^1(\theta)(d\phi + e_\phi r dt) + a^1(\theta)(d\psi + e_0 r dt + b_0(\theta)d\phi) && (5.38)
\end{aligned}
$$

where,

$$\Omega = \mu^{3/2}\lambda^{1/2}\frac{L^2}{2}c_\psi R^3 \sin\theta \tag{5.39}$$

$$e^{-2\psi} = \frac{L^3\mu^{3/2}\lambda^{1/2}c_\psi R^3 \sin^2\theta}{2H(x)^{3/2}F(x)^{1/2}} \tag{5.40}$$

$$e_\phi = 0$$
$$e^0 = 0$$

$$w = \sqrt{\frac{L\lambda F(x)H(x)}{2\mu}}\frac{c_\psi R}{K(x)} \tag{5.41}$$

$$b^0(\theta) = \frac{2A^0_\phi(\cos\theta)}{Lc_\psi R} \tag{5.42}$$

The expression for the gauge fields reduce to,

$$A^1_t = 0 \tag{5.43}$$

$$A^1_\psi = a^1(\theta) = \frac{\sqrt{3}Rs}{h}\left(\frac{C_\lambda(c^2-h)c^2}{\lambda s^2} - C_\mu\frac{(3c^2-h)}{\mu}\right) \tag{5.44}$$

$$A^1_\phi = b^1(\theta) + a^1(\theta)b^0(\theta) = -\frac{\sqrt{3}Rc(1+\cos\theta)}{h}\left(\frac{C_\lambda s^2}{1+\lambda\cos\theta} - C_\mu\frac{(3c^2-2h)}{1-\mu\cos\theta}\right) \tag{5.45}$$

and the expression for b^1 is,

$$\begin{aligned} b^1(\theta) = &-\frac{\sqrt{3}Rc(1+\cos\theta)}{h}\left(\frac{C_\lambda s^2}{1+\lambda\cos\theta} - C_\mu\frac{(3c^2-2h)}{1-\mu\cos\theta}\right) \\ &-\frac{\sqrt{3}Rs}{h}\left(\frac{C_\lambda(c^2-h)c^2}{s^2} - C_\mu(3c^2-h)\right)\frac{A^0_\phi(\cos\theta)}{c_\psi R} \end{aligned} \tag{5.46}$$

where

$$h = 1 + s^2\frac{(\lambda+\mu)\cos\theta}{1+\lambda\cos\theta} \tag{5.47}$$

Then using the entropy function(4.5), the entropy of the non-supersymmetric black ring can be expressed as,

$$\mathcal{E} = 8\pi^2\mu^{3/2}\lambda^{1/2}L^2c_\psi R^3 \tag{5.48}$$

Chapter 6

Conclusions

In this thesis, we have demonstrated that the attractor mechanism is independent of supersymmetry. We have shown that, it works for non-supersymmetric black holes both non-rotating and rotating and also works for black rings. For the spherically symmetric, static black holes, we have also demonstrated the stability of attractor in terms of an effective potential. In other cases, we assume such a stable attractor exists.

1) By studying the equations of motion directly we have shown that the attractor mechanism can work for non-supersymmetric extremal black holes [5]. Two conditions sufficient for this are conveniently stated in terms of the effective potential involving the scalars and the charges carried by the black hole. We also obtained similar results for extremal black holes in asymptotically Anti-de Sitter space and in higher dimensions.

2) We have presented a c-function for spherically symmetric, static and asymptotically flat solutions in theories of four-dimensional gravity coupled to gauge fields and moduli [71]. The c-function is valid for both extremal and non-extremal black holes. It monotonically decreases from infinity and in the static region acquires its minimum value at the horizon, where it equals the entropy of the black hole. We also generalized this result to higher dimensional cases, involving p-form gauge fields.

3)We have proved that, in a general higher derivative theory of gravity coupled to abelian gauge fields and neutral scalar fields, the entropy and the near horizon background of a rotating extremal black hole is obtained by extremizing an entropy function which depends only on the parameters labeling the near horizon background and the electric and magnetic charges and angular momentum carried by the black hole [51]. We have illustrated these results in the context of two derivative theories of gravity in several examples. These include Kerr black hole, Kerr-Newman black hole, black holes in Kaluza-Klein theory, and black holes in toroidally compactified heterotic string theory.

4) We have constructed the entropy function for $5D$ extremal black holes and black rings [14]. We considered five dimensional extremal black holes and black rings which project down to either static or stationary black holes. This is done in the context of two derivative gravity coupled to abelian gauge fields and neutral scalar fields.

We have demonstrated the existence of attractor mechanism for extremal, non-supersymmetric configurations. These results should help us in a microscopic understanding of the entropy of non supersymmetric extremal black holes.

References

[1] S. W. Hawking, *Particle creation by black holes*, *Commun. Math. Phys.* **43** (1975) 199–220. 3

[2] S. Ferrara and R. Kallosh, *Supersymmetry and attractors*, *Phys. Rev.* **D54** (1996) 1514–1524, [hep-th/9602136]. 4

[3] S. Ferrara and R. Kallosh, *Universality of supersymmetric attractors*, *Phys. Rev.* **D54** (1996) 1525–1534, [hep-th/9603090]. 4

[4] S. Ferrara, R. Kallosh, and A. Strominger, *N=2 extremal black holes*, *Phys. Rev.* **D52** (1995) 5412–5416, [hep-th/9508072]. 4

[5] K. Goldstein, N. Iizuka, R. P. Jena, and S. P. Trivedi, *Non-supersymmetric attractors*, *Phys. Rev.* **D72** (2005) 124021, [hep-th/0507096]. 5, 66, 73, 80, 81, 102, 103, 139, 153

[6] A. Sen, *Black hole entropy function and the attractor mechanism in higher derivative gravity*, *JHEP* **09** (2005) 038, [hep-th/0506177]. 6, 9, 90, 92, 126

[7] D. Z. Freedman, S. S. Gubser, K. Pilch, and N. P. Warner, *Renormalization group flows from holography supersymmetry and a c-theorem*, *Adv. Theor. Math. Phys.* **3** (1999) 363–417, [hep-th/9904017]. 8, 70, 73, 75, 76

[8] J. M. Bardeen and G. T. Horowitz, *The extreme kerr throat geometry: A vacuum analog of ads(2) x s(2)*, *Phys. Rev.* **D60** (1999) 104030, [hep-th/9905099]. 9, 86, 104, 105

[9] F. Larsen, *Rotating kaluza-klein black holes*, *Nucl. Phys.* **B575** (2000) 211–230, [hep-th/9909102]. 10, 106, 107

[10] D. Rasheed, *The rotating dyonic black holes of kaluza-klein theory*, *Nucl. Phys.* **B454** (1995) 379–401, [hep-th/9505038]. 10, 106, 107, 109, 113

[11] M. Cvetic and D. Youm, *Entropy of non-extreme charged rotating black holes in string theory*, *Phys. Rev.* **D54** (1996) 2612–2620, [hep-th/9603147]. 10, 118, 120, 121, 124

[12] R. Emparan and H. S. Reall, *A rotating black ring in five dimensions*, *Phys. Rev. Lett.* **88** (2002) 101101, [hep-th/0110260]. 10

[13] P. Kraus and F. Larsen, *Attractors and black rings*, *Phys. Rev.* **D72** (2005) 024010, [hep-th/0503219]. 10, 137, 138, 145

[14] K. Goldstein and R. P. Jena, *One entropy function to rule them all*, hep-th/0701221. 10, 154

[15] F. Denef, *Supergravity flows and d-brane stability*, *JHEP* **08** (2000) 050, [hep-th/0005049]. 18, 30

[16] F. Denef, B. R. Greene, and M. Raugas, *Split attractor flows and the spectrum of bps d-branes on the quintic*, *JHEP* **05** (2001) 012, [hep-th/0101135]. 30

[17] G. W. Gibbons and K.-i. Maeda, *Black holes and membranes in higher dimensional theories with dilaton fields*, *Nucl. Phys.* **B298** (1988) 741. 30, 31, 43, 53, 84

[18] R. Kallosh, A. D. Linde, T. Ortin, A. W. Peet, and A. Van Proeyen, *Supersymmetry as a cosmic censor*, *Phys. Rev.* **D46** (1992) 5278–5302, [hep-th/9205027]. 34, 49

[19] G. W. Gibbons and R. E. Kallosh, *Topology, entropy and witten index of dilaton black holes*, *Phys. Rev.* **D51** (1995) 2839–2862, [hep-th/9407118]. 34

[20] P. Dobiasch and D. Maison, *Stationary, spherically symmetric solutions of jordan's unified theory of gravity and electromagnetism*, *Gen. Rel. Grav.* **14** (1982) 231–242. 34, 57

[21] G. W. Gibbons, D. Kastor, L. A. J. London, P. K. Townsend, and J. H. Traschen, *Supersymmetric selfgravitating solitons*, *Nucl. Phys.* **B416** (1994) 850–880, [hep-th/9310118]. 34

[22] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, *Charged ads black holes and catastrophic holography*, *Phys. Rev.* **D60** (1999) 064018, [hep-th/9902170]. 39

[23] G. W. Gibbons and D. L. Wiltshire, *Black holes in kaluza-klein theory*, *Ann. Phys.* **167** (1986) 201. 43, 57

[24] A. Masood-ul Alam, *Uniqueness of a static charged dilaton black holenon-supersymmetric attractors*, *Class. Quant. Grav.* **10** (1993) 2649. 43

[25] M. Mars and W. Simon, *On uniqueness of static einstein-maxwell-dilaton black holes*, *Adv. Theor. Math. Phys.* **6** (2003) 279, [gr-qc/0105023]. 43

[26] G. Gibbons, D. Ida, and T. Shiromizu, *Uniqueness of (dilatonic) charged black holes and black p-branes in higher dimensions*, *Phys. Rev. D* **66** (2002) 044010, [hep-th/0206136]. 43

[27] G. Gibbons, D. Ida, and T. Shiromizu, *Uniqueness and non-uniqueness of static black holes in higher dimensions*, *Phys. Rev. Lett.* **89** (2002) 0441101, [hep-th/0206049]. 43

[28] G. W. Gibbons, R. Kallosh, and B. Kol, *Moduli, scalar charges, and the first law of black hole thermodynamics*, *Phys. Rev. Lett.* **77** (1996) 4992–4995, [hep-th/9607108]. 43

[29] J. Podolosky, *The structure of the extreme schwarzschild-desitter space-timeuniqueness and non-uniqueness of static black holes in higher dimensions*, *Gen. Rel. Grav.* **31** (1999) 1703, [gr-qc/9910029]. 44

[30] H. Nariai, *Nariai de sitter*, *Sci. Rep. Res. Inst. Tohuku Univ. A* **35** (1951) 62. 45

[31] M. Toda, *Theory of nonlinear lattices (2nd ed.)*, *Springer-Verlang* (1988). 53

[32] H. Lu and C. N. Pope, *Sl(n+1,r) toda solitons in supergravities*, *Int. J. Mod. Phys.* **A12** (1997) 2061–2074, [hep-th/9607027]. 53, 57, 60

[33] P. K. Tripathy and S. P. Trivedi, *Non-supersymmetric attractors in string theory*, *JHEP* **03** (2006) 022, [hep-th/0511117]. 65

[34] S. Ferrara, G. W. Gibbons, and R. Kallosh, *Black holes and critical points in moduli space*, *Nucl. Phys.* **B500** (1997) 75–93, [hep-th/9702103]. 66

[35] V. Sahakian, *Holography, a covariant c-function and the geometry of the renormalization group*, *Phys. Rev.* **D62** (2000) 126011, [hep-th/9910099]. 70

[36] E. Alvarez and C. Gomez, *Holography and the c-theorem*, hep-th/0009203. 70

[37] M. Henningson and K. Skenderis, *The holographic weyl anomaly*, *JHEP* **07** (1998) 023, [hep-th/9806087]. 75

[38] S. Nojiri, S. D. Odintsov, and S. Ogushi, *Finite action in d5 gauged supergravity and dilatonic conformal anomaly for dual quantum field theory*, *Phys. Rev.* **D62** (2000) 124002, [hep-th/0001122]. 75

[39] L. Girardello, M. Petrini, M. Porrati, and A. Zaffaroni, *Novel local cft and exact results on perturbations of $n = 4$ super yang-mills from ads dynamics*, *JHEP* **12** (1998) 022, [hep-th/9810126]. 75

[40] T. Banks and E. J. Martinec, *The renormalization group and string field theory*, *Nucl. Phys.* **B294** (1987) 733. 76

[41] S. R. Das, G. Mandal, and S. R. Wadia, *Stochastic quantization on two-dimensional theory space and morse theory*, *Mod. Phys. Lett.* **A4** (1989) 745. 76

[42] C. Vafa, *c theorem and the topology of 2-d qfts*, *Phys. Lett.* **B212** (1988) 28. 76

[43] A. B. Zamolodchikov, *Irreversibility of the flux of the renormalization group in a 2d field theory*, *JETP Lett.* **43** (1986) 730–732. 76

[44] G. T. Horowitz and A. Strominger, *Black strings and p-branes*, *Nucl. Phys.* **B360** (1991) 197–209. 84

[45] V. Iyer and R. M. Wald, *Some properties of noether charge and a proposal for dynamical black hole entropy*, *Phys. Rev.* **D50** (1994) 846–864, [gr-qc/9403028]. 90

[46] T. Jacobson, G. Kang, and R. C. Myers, *Black hole entropy in higher curvature gravity*, gr-qc/9502009. 90

[47] T. Jacobson, G. Kang, and R. C. Myers, *On black hole entropy*, *Phys. Rev.* **D49** (1994) 6587–6598, [gr-qc/9312023]. 90

[48] R. M. Wald, *Black hole entropy in the noether charge*, *Phys. Rev.* **D48** (1993) 3427–3431, [gr-qc/9307038]. 90

[49] A. Sen, *Entropy function for heterotic black holes*, *JHEP* **03** (2006) 008, [hep-th/0508042]. 92, 126

[50] T. Matos and C. Mora, *Stationary dilatons with arbitrary electromagnetic field*, *Class. Quant. Grav.* **14** (1997) 2331–2340, [hep-th/9610013]. 106, 107

[51] D. Astefanesei, K. Goldstein, R. P. Jena, A. Sen, and S. P. Trivedi, *Rotating attractors*, *JHEP* **10** (2006) 058, [hep-th/0606244]. 126, 128, 141, 154

[52] B. Sahoo and A. Sen, *Higher derivative corrections to non-supersymmetric extremal black holes in n = 2 supergravity*, *JHEP* **09** (2006) 029, [hep-th/0603149]. 127

[53] D. Gaiotto, A. Strominger, and X. Yin, *5d black rings and 4d black holes*, *JHEP* **02** (2006) 023, [hep-th/0504126]. 127

[54] D. Gaiotto, A. Strominger, and X. Yin, *New connections between 4d and 5d black holes*, *JHEP* **02** (2006) 024, [hep-th/0503217]. 127, 147

[55] I. Bena and N. P. Warner, *One ring to rule them all ... and in the darkness bind them?*, *Adv. Theor. Math. Phys.* **9** (2005) 667–701, [hep-th/0408106]. 130

[56] H. Elvang, R. Emparan, D. Mateos, and H. S. Reall, *A supersymmetric black ring*, *Phys. Rev. Lett.* **93** (2004) 211302, [hep-th/0407065]. 130, 141

[57] H. Elvang, R. Emparan, D. Mateos, and H. S. Reall, *Supersymmetric black rings and three-charge supertubes*, *Phys. Rev.* **D71** (2005) 024033, [hep-th/0408120]. 130

[58] J. P. Gauntlett and J. B. Gutowski, *General concentric black rings*, *Phys. Rev.* **D71** (2005) 045002, [hep-th/0408122]. 130

[59] I. Antoniadis, S. Ferrara, and T. R. Taylor, *N=2 heterotic superstring and its dual theory in five dimensions*, *Nucl. Phys.* **B460** (1996) 489–505, [hep-th/9511108]. 137

[60] A. C. Cadavid, A. Ceresole, R. D'Auria, and S. Ferrara, *Eleven-dimensional supergravity compactified on calabi-yau threefolds*, *Phys. Lett.* **B357** (1995) 76–80, [hep-th/9506144]. 137

[61] A. H. Chamseddine, S. Ferrara, G. W. Gibbons, and R. Kallosh, *Enhancement of supersymmetry near 5d black hole horizon*, *Phys. Rev.* **D55** (1997) 3647–3653, [hep-th/9610155]. 137

[62] B. de Wit and A. Van Proeyen, *Broken sigma model isometries in very special geometry*, *Phys. Lett.* **B293** (1992) 94–99, [hep-th/9207091]. 137

[63] M. Gunaydin, G. Sierra, and P. K. Townsend, *The geometry of n=2 maxwell-einstein supergravity and jordan algebras*, *Nucl. Phys.* **B242** (1984) 244. 137

[64] M. Gunaydin, G. Sierra, and P. K. Townsend, *Gauging the d = 5 maxwell-einstein supergravity theories: More on jordan algebras*, *Nucl. Phys.* **B253** (1985) 573. 137

[65] G. Papadopoulos and P. K. Townsend, *Compactification of d = 11 supergravity on spaces of exceptional holonomy*, *Phys. Lett.* **B357** (1995) 300–306, [hep-th/9506150]. 137

[66] F. Larsen, *The attractor mechanism in five dimensions*, hep-th/0608191. 138, 139, 145

[67] R. Kallosh, *New attractors*, *JHEP* **12** (2005) 022, [hep-th/0510024]. 139

[68] G. L. Cardoso, J. M. Oberreuter, and J. Perz, *Entropy function for rotating extremal black holes in very special geometry*, *JHEP* **05** (2007) 025, [hep-th/0701176]. 140

[69] H. Elvang, R. Emparan, and P. Figueras, *Non-supersymmetric black rings as thermally excited supertubes*, *JHEP* **02** (2005) 031, [hep-th/0412130]. 141, 148, 150

[70] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis, and H. S. Reall, *All supersymmetric solutions of minimal supergravity in five dimensions*, *Class. Quant. Grav.* **20** (2003) 4587–4634, [hep-th/0209114]. 147

[71] K. Goldstein, R. P. Jena, G. Mandal, and S. P. Trivedi, *A c-function for non-supersymmetric attractors*, *JHEP* **02** (2006) 053, [hep-th/0512138]. 153