Studies in statistical physics:
Stochastic strategies in a minority game and continuum percolation of overlapping discs

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By

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Declaration

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

The work was done under the guidance of Professor Deepak Dhar, at the Tata Institute of Fundamental Research, Mumbai.

(V Sasidevan)

In my capacity as the supervisor of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

(Deepak Dhar)

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Collaborators

The work presented in Chapter 1 is based on the work done in collaboration with Prof. Bikas K Chakrabarti, Saha Institute of Nuclear Physics, Kolkata and it has appeared in print in Physica A, **390** 3477 (2011).

To

My wife Priya
whose love made this possible

Contents

Synopsis

In this thesis, we study two different problems. In the first problem, we study a variant of the Minority Game, which is an application of methods of statistical physics to a model problem in economics. The Minority Game (MG) was introduced by Challet and Zhang [1]. It is based on the the El-Farol Bar problem [2] and is a prototypical model of socio-economic systems such as markets consisting of interacting agents where there is competition between agents for the limited available resources. The model consists of agents who have to repeatedly choose between two alternatives, and at each step those who belong to the minority group are considered as winners. The game has been studied as model of learning, adaptation and co-evolution in scarce resource conditions especially in the context of financial markets [3, 4, 5]. We study performance of stochastic strategies in a variant of the Minority Game. We analyze the problem first using the commonly used solution concept of Nash equilibrium. We show that Nash equilibrium concept is very unsatisfactory for this model giving rise to trapping states. In our first attempt to solve this problem, we made an ad-hoc assumption to avoid the trapping states [6]. Later, we realized that to avoid such arbitrary ad-hoc strategies, we need to introduce a new solution concept to be called Co-action equilibrium which takes care of the problem of trapping state in a natural way. This is discussed next. Using this new solution concept, we characterize the steady state of the system for a finite number of agents. We study how the parameters of the optimal strategy depend on the future time horizon of agents. We find that for large enough future time horizon,the optimal strategy switches from fully random choice to the one in which winners do not change their choice. This general win-stay lose-shift strategy is often used in many other real life situations [7].

In the second problem, we study a model of continuum percolation [8]. A host of problems in nature ranging from gelation to disease spreading in a community involve spatially random structures. Percolation is the simplest model describing such systems, which exhibit a geometrical phase transition. We study continuum percolation problem of overlapping discs with a distribution of radii having a power-law tail. We show that the power-law tail of the distribution affect the critical behavior of the system for low enough powers of the distribution. For high enough powers, the critical behavior of the model is the same as that for the usual percolation problem. We propose an approximate RG scheme to analyze the problem and determine the percolation threshold and correlation length exponent from Monte-Carlo simulations.

Stochastic strategies in a Minority Game

Standard Minority Game

In the standard MG setting, there are N agents, with N odd, and each of them has to choose between two alternatives (say two restaurants A and B) independently on each day. The agents in the restaurant with fewer people get a payoff 1, and others 0. No communication is allowed between the agents. The agents make their decision based on the information of the record of the winning group (A or B) in the last H_b days. The information is thus a binary string of length H_b and since each agent is affected only by the total behavior of all other agents, the interaction is of mean-field type. A strategy is defined as a rule which for each possible history will give a prediction for the possible minority group on the next day. Each agent has a small number S of strategies available with him/her which is drawn randomly in the beginning of the game from the set of all possible strategies which is $2^{2^{H_b}}$ in number. On any day, an agent decides which restaurant to go to, using the strategy with her which performed best in the recent past. If agents are choosing randomly between A and B, the average number of winners per day is $\approx N/2 - K\sqrt{N}$ where is K is some constant. The fact that make this model interesting is that for a range of values of H_b, effective co-operation emerges between the agents, in the sense that the number K is reduced substantially compared to the random choice case. One would say that resource utilization is better in this case since the number of winners per day is better than the case in which agents select randomly between A and B. Thus, though the agents are selfish in their nature, they seemingly self organize into a state of high social efficiency.

The mean-field nature of the MG where each person interacts with everybody else through the common information, makes it a system whose behavior could be understood using the techniques of statistical physics [3]. It was found that $\alpha = 2^{H_b}/N$ is the relevant parameter in the model and the effective co-operation between the agents become maximum at a critical value of α say α_c. Great effort has been put into the understanding of this emergent behavior and exact results are available for the ergodic phase $\alpha > \alpha_c$ in the limit $N, H_b \to \infty$ with α finite. The behavior of the model in the non-ergodic phase $\alpha < \alpha_c$ is relatively less understood analytically (See [9] for a list of results and references). A more detailed description of these results and references will be given in the thesis.

In this thesis, the focus is on the performance of stochastic strategies in a Minority Game. We consider a variation of the Minority Game where agents use probabilistic strategies. Unlike the standard MG, where each agent is endowed with a small subset of the whole strategy space, in our model, the entire strategy set is made available to all the agents including probabilistic ones. Also the agents are assumed to be fully rational. We study how the self organization to a socially efficient state by the agents is affected in such a model.

Definition of the model

In this work, we will consider stochastic strategies in a variant of the MG. The agents in our model are assumed to be selfish and rational. The model consists of an odd number of agents $N = 2M + 1$ where M is a positive integer. They have to select one of the two alternatives A or B at each step simultaneously and independently. An agent will receive a payoff 1 if she is in the minority. i.e., if she is in a restaurant with attendance $\leq M$. Otherwise she receives a payoff 0. The agents can not communicate with each other in any direct way in deciding their choice and make their choice based on the information that how many agents were there in each of the two restaurants in the past H_b days. More precisely, if we denote the number of agents who were in restaurant A on the t -th day by, $M - \Delta(t)$, then the time series $\{\Delta(t')\}$, for $t' = t, t-1, ...t - (H_b - 1)$ is known to all agents at the end of day t. In the standard MG, the information is not the value of $\Delta(t)$, but only the sign of it. Any agent X wants to optimize her weighted expected future payoff,

$$\text{ExpPayoff}(X) = \sum_{\tau=0}^{\infty} [(1-\lambda)\lambda^{\tau}] \langle W_X(\tau+1) \rangle, \tag{1}$$

where $\langle W_X(\tau) \rangle$ is the expected payoff of the agent X on the τ-th day ahead, and λ is a parameter $0 \leq \lambda < 1$, same for all agents. In the problem, we allow agents to have probabilities strategies. So for a given history $\{\Delta(t')\}$, a strategy will specify the probability p with which she should switch her choice. We will restrict ourselves to the simplest case $H_b = 1$ so that the agent's strategy depend only on the attendance on the last day.

If $H_b = 0$, then we have the situation in which agents do not have any information to base their decision. Their optimum strategy then is to select A or B randomly. In such a case, the expectation value of the number of agents who will show up at either restaurant is $N/2$. We can measure the inefficiency of the system by the parameter η usually defined by,

$$\eta = \lim_{N \to \infty} \frac{4}{N} \langle (r - N/2)^2 \rangle, \tag{2}$$

where variable r denote the attendance in restaurant A (or B). $\langle \ \rangle$ denotes averaging over long time and over different initial conditions. The normalization has been chosen, so that the inefficiency parameter η of the system with agents choosing randomly between A and B is 1.

An efficient probabilistic strategy for MG

Consider the simplest case, when agents are optimizing only next days payoff, so that $\lambda = 0$ in Eq. 1. With $H_b = 1$, the information available to the agents is the attendance in the restaurants on the past day. A general strategy applicable in such a situation is the win-stay lose-shift strategy, where if an agent wins on a given day, she sticks to her choice on the following day and if she loses, she will switch her choice with some probability. We first analyze this strategy within the frame work of the Nash solution concept and show that this will give a highly efficient system, where inefficiency can be made of order $(1/N^{1-\epsilon})$, for any $\epsilon > 0$.

The strategy is defined as follows. On the first day $t = 0$, each agent choose one of the restaurants randomly. On all subsequent days, an agent who has found herself on the last day on the minority side will stick to her choice on the next day, but an agent who was in the majority will switch her choice with probability p independent of other agents. The value of p depend only on $\Delta(t)$ and is approximately Δ/M for $\Delta > 0$. The precise dependence of p on $\Delta(t)$

is obtained from the Nash condition on expected payoffs. We can easily show that within a time of order $\log\log N$, the magnitude of Δ will become of $\mathcal{O}(1)$, and then remain of order 1. So the system quickly reaches a highly efficient state and will remain there.

This seems to indicate that the Nash solution concept leads to a socially efficient state in which number of agents in the minority is near maximum possible on each day. However, this solution concept has the problem that it will lead to an absorbing state which we call as trapping state, in which the same set of agents win on each day. To understand this, consider the case $\Delta = 0$. In this case there are exactly M agents in A and $M+1$ agents in B. It is easy to see that, in this case all agents staying put is a Nash solution. There is no optimum non-zero value of p in this case. Under the Nash solution concept, the optimum strategy for each agent in B is to stay put, which will lead to 0 payoff for all of them in all subsequent days. With selfish, rational agents such a solution make little sense. An ad-hoc solution to the problem is that, whenever such a trapping state is reached, all agents irrespective of whether they were in minority or not on day t, switch their choice on next day with a probability $M^{\epsilon-1}$, where ϵ is a real number $0 \leq \epsilon \leq 1$. We shall refer to this step as a major resetting event. The value of ϵ is not determined by our model, but is assumed to have a preset value.

For a given value of ϵ, the value of $|\Delta|$ just after resetting is of order $M^{\epsilon/2}$. Then it takes time of order $\log\log M$ to reach the value $\Delta = 0$. It is easy to see that the inefficiency parameter would vary as $M^{\epsilon-1}/\log\log M$. Then, for more efficiency, the agents should keep ϵ small.

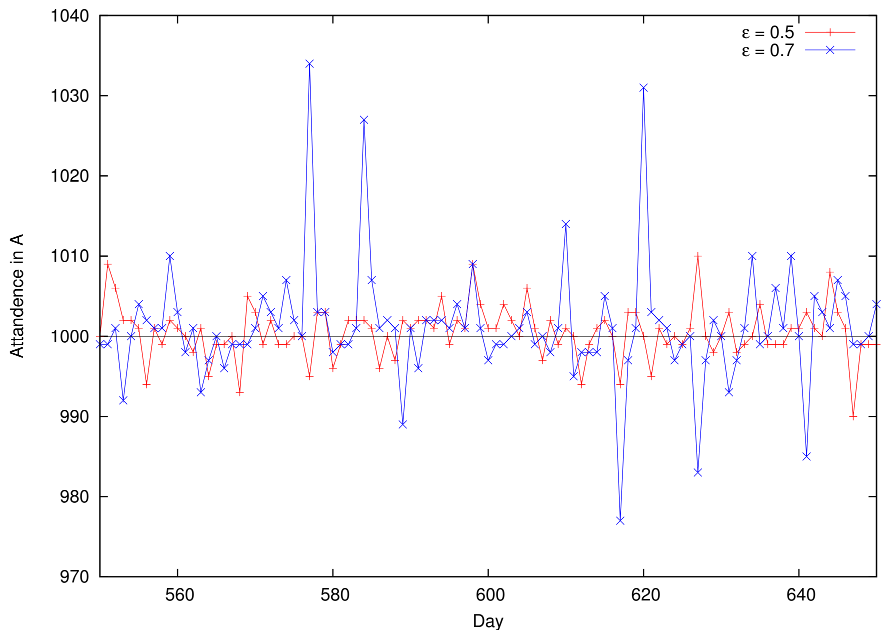

Figure 1: A typical evolution of a system of 2001 agents for two different choices of the parameter $\epsilon = 0.5$ and 0.7. The large deviations correspond to major resetting events (see text).

Simulation results

We study the time evolution of a set of N agents using this strategy using Monte Carlo simulations, with $N = 2001$. If the restaurant with greater attendance has $M + 1 + \Delta$ agents on a given day, with $\Delta > 0$, the next day each of them switches her choice with probability $p(\Delta)$, and the agents in the minority restaurant stick to their choice. If there are exactly $M + 1$ agents in the majority restaurant, all agents switch their restaurant with a probability $1/(2M^{1-\epsilon})$.

The result of a typical evolution is shown in Fig. 1, for two choices of ϵ: 0.5 and 0.7. We see that the majority restaurant changes quite frequently. The large peaks in $|\Delta|$ correspond to re-settings of the system, and clearly, their magnitude decreases if ϵ is decreased. There is very little memory of the location of majority restaurant in the system. To be specific, let $S(t)$ is $+1$ if the minority restaurant is A on the t-th day, and -1 if it is B. Then the auto-correlation function $\langle S(t)S(t+\tau)\rangle$ decays exponentially with τ, approximately as $\exp(-K\tau)$. The value of K depends on ϵ, but is about 2, and the correlation is negligible for $\tau > 3$.

Fig. 2a gives a plot of the inefficiency parameter η as a function of ϵ.

We define $A_i(t)$ equal to $+1$ if the i-th agent was in the restaurant A at time t, and -1 otherwise. We define the auto-correlation function of the A-variables in the steady state as

$$C(\tau) = \frac{1}{N}\sum_i \langle A_i(t)A_i(t+\tau)\rangle. \quad (3)$$

In Fig. 2b, we show the variation of $C(\tau)$ with τ. We see that this function has a large amount of persistence. This is related to the fact that only a small fraction of agents actually switch their choice at any time step. Clearly, the persistence time is larger for smaller ϵ.

Strategy switching and Co-action equilibrium

We saw in the previous section that the Nash solution concept when applied to the problem leads to trapping state. Trapping state arises due to the Nash solution concept of optimizing over strategies of one agent, assuming that all other agent would continue to do as expected; or the Nash condition doesn't take into account the fact that all agents with the same information will reach the same conclusion about their optimum strategy and hence will choose the

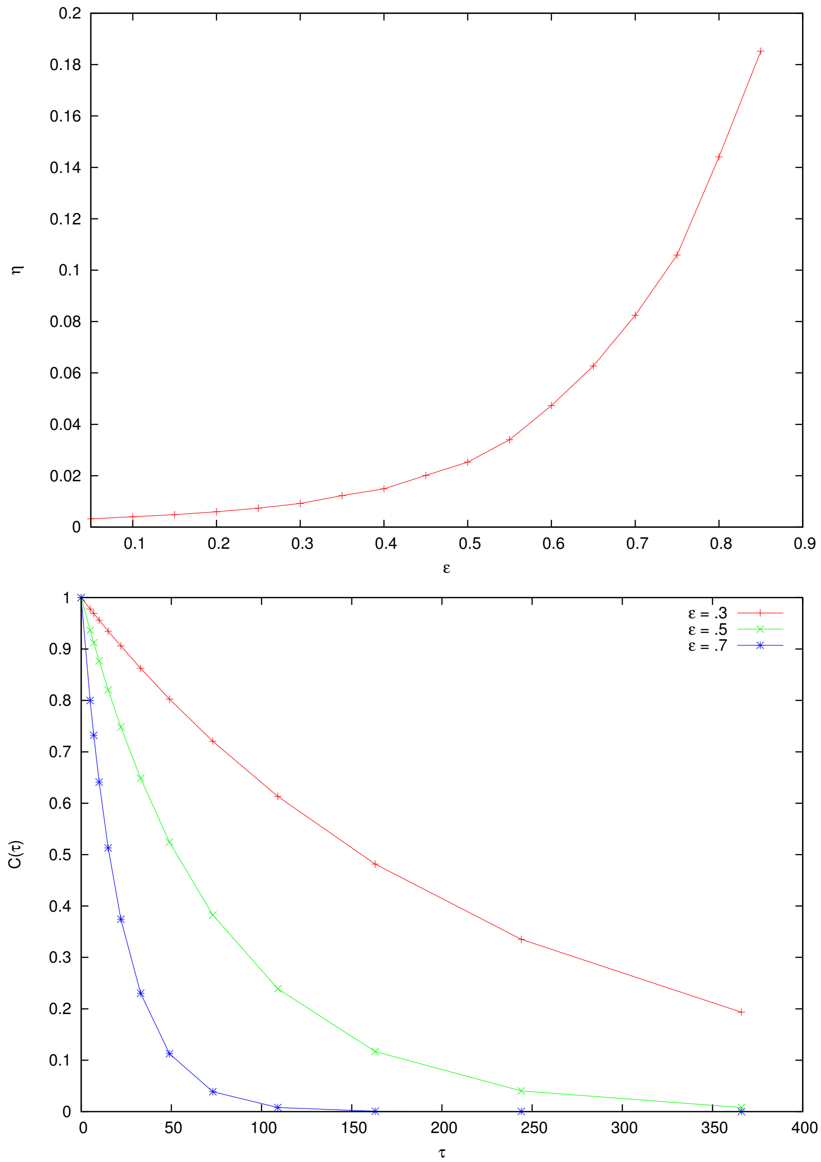

Figure 2: (a) Variation of inefficiency parameter η with ϵ (b) $C(\tau)$ as a function of τ for $\epsilon = .3$, .5 and .7. Each data point is obtained by averaging over 10000 simulation steps.Total number of agents is N = 2001.

same switch probability. In the alternate Co-action equilibrium, agent realizes that whatever switch probability she chooses, others in the same state as her will choose the same probability. Hence she should optimize her switch probability taking this into account. The situation is thus similar to a two-person game where the two 'persons'are the majority and the minority groupings who select the optimal values of their strategy parameters. But these grouping are temporary and change with time.

We will allow the discount parameter λ in Eq. 1 to have non-zero values. An agent is said to be in state C_i when she in a restaurant with total number of agents i. Let p_i be the switch probability chosen by an agent when she is state C_i. For a given N, a strategy $\mathbb{P}$ is defined by the set of N numbers $\mathbb{P} \equiv \{p_1, p_2, ...p_N\}$. If $|Prob(t)\rangle$ is an N-dimensional vector, whose j-th element is $Prob_j(t)$, the probability that a marked agent X finds herself in state C_i on the t-th day, then $|Prob(t)\rangle$ undergoes Markovian evolution described by

$$|Prob(t+1)\rangle = \mathbb{T}|Prob(t)\rangle, \tag{4}$$

where $\mathbb{T}$ is the $N \times N$ Markov transition matrix. Explicit matrix elements are easy to write down. Then total expected payoff of X, given that she is in state C_j at time $t = 0$ is

$$W_j = (1-\lambda)\left\langle L \left| \frac{\mathbb{T}}{1-\lambda\mathbb{T}} \right| j \right\rangle, \tag{5}$$

where $|j\rangle$ is the vector with only the j-th element 1, and rest zero; and $\langle L|$ is the left-vector $\langle 1,1,1,1,..0,0,0..|$, with first $M = (N-1)/2$ elements 1 and rest zero.

Let us denote the equilibrium strategy with N agents by $\{p_1^*, p_2^*, \ldots p_N^*\}$.

One simple choice is that $p_i = 1/2$ for all i. We will denote this strategy by $\mathbb{P}_{rand}$. In this case it is easy to see that average expected payoff W_j is independent of j and is given by

$$W_j = W_{rand} = 1/2 - \binom{N-1}{M} 2^{-N}, \quad \text{for all } j. \tag{6}$$

For a strategy $\mathbb{P}$, it is more convenient to use the inefficiency parameter η defined as follows instead of that in Eq. 2,

$$\eta\left(\mathbb{P}\right) = \left(W_{max} - W_{avg}\left(\mathbb{P}\right)\right) / \left(W_{max} - W_{rand}\right), \tag{7}$$

where $W_{max} = M/N$ is the maximum possible payoff per agent, $W_{avg}\left(\mathbb{P}\right)$ is the average payoff per agent in the steady state for a given λ. By the symmetry of the problem, $p_N^* = 1/2$ for all λ. Now consider the strategy $\{p_i^*\} = \{p_1^*, 1/2, 1/2, 1/2...\}$. If an agent X is in state C_1, and next day all other agents will switch with probability 1/2, it does not matter if X switches or not: payoffs W_1 and W_{N-1} are independent of p_1^*. Hence p_1^* can be chosen to be of any value. It can be shown that the strategy $\mathbb{P}'_{rand}$, in which $p_1^* = 0$, and $p_{N-1}^* < 1/2$, chosen to maximize W_{N-1}, is better for all agents and stable, and hence is always preferred over $\mathbb{P}_{rand}$.

As a simple application, consider first the case $N = 3$. Since $p_1^* = 0$, $p_3^* = 1/2$, the only free parameter is p_2^*. This is determined by maximizing $W_2(\lambda)$ with respect to p_2, and this gives the optimal strategy for any λ. It is found that p_2^* monotonically decreases with λ from its value 1/2 at $\lambda = 0$ as shown in Fig. 3a. The payoff of agents in various states with this optimum strategy is shown in Fig. 3b. The average payoff per agent per day W_{avg} in the steady state for $N = 3$ is a monotonically increasing function of λ, and leads to the best possible solution $W_{max} = 1/3$ as $\lambda \to 1$.

We can similarly determine the optimal strategies for $N = 5$. The optimal strategy is characterized by the five parameters $(p_1^*, p_2^*, p_3^*, p_4^*, p_5^*)$. As discussed before, the strategy $\mathbb{P}'_{rand} = (0, 1/2, 1/2, p_4^*(\lambda), 1/2)$ gives higher payoff than $\mathbb{P}_{rand}$ for all agents, for all λ. The optimum values p_2^* and p_3^* can be determined from the behavior of the payoff functions W_2 and W_3 in the $p_2 - p_3$ plane. It is found that for $\lambda \leq \lambda_{c1} = .195 \pm .001$, $p_2 = p_3 = 1/2$ is the optimum choice and for $\lambda > \lambda_{c1}$ the optimum values are $p_2 = 0$ and $p_3 = p_3^*(\lambda) \leq 1/2$. There is also a continuous transition at $\lambda_{c2} = .737 \pm .001$. The optimum switch probabilities and the corresponding payoffs and inefficiency are shown in fig 4.

One can analyze the problem with higher N as well. For eg. with $N = 7$, we find that there are four transitions with thresholds λ_{ci}, where $i = 1$ to 4. For $\lambda < \lambda_{c1}$, the optimal strategy has the form $(0, 1/2, 1/2, 1/2, 1/2, p_6^*, 1/2)$. For $\lambda_{c1} \leq \lambda \leq \lambda_{c2}$, we get $p_3^* = 0$, and $p_4^* < 1/2$. For still higher values $\lambda_{c2} < \lambda \leq \lambda_{c3}$, agents in state C_2 and C_5 also find it better to switch to win-stay lose-shift strategy, and we get $p_2^* = 0$, $p_5^* < 1/2$. Thus the general structure of the optimum strategy is that as λ is increased, it changes from random switching to a complete win-stay lose-shift strategy in steps.

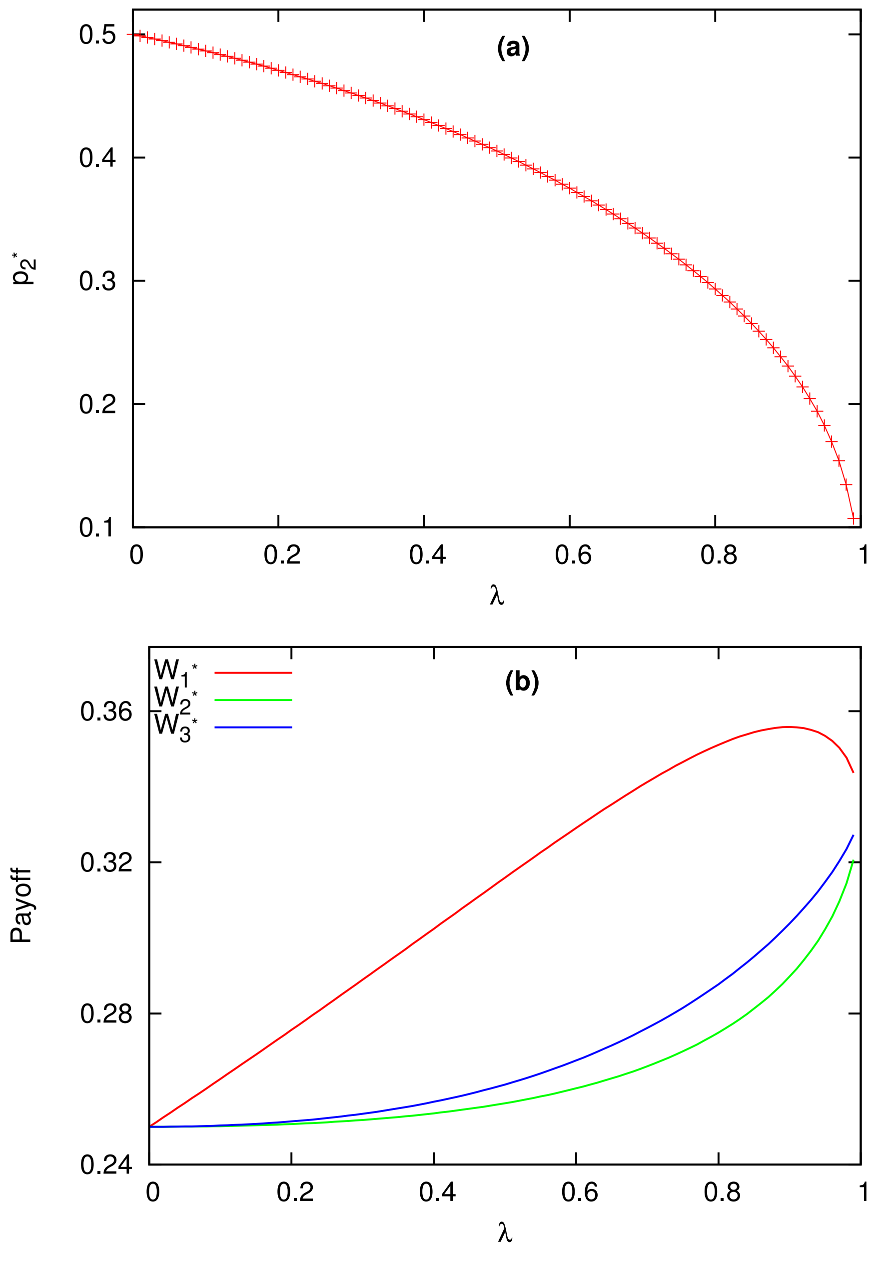

Figure 3: $N = 3$: (a) Variation of p_2^* with λ, and (b) The optimum payoffs W_i, ($i = 1$ to 3), as functions of λ.

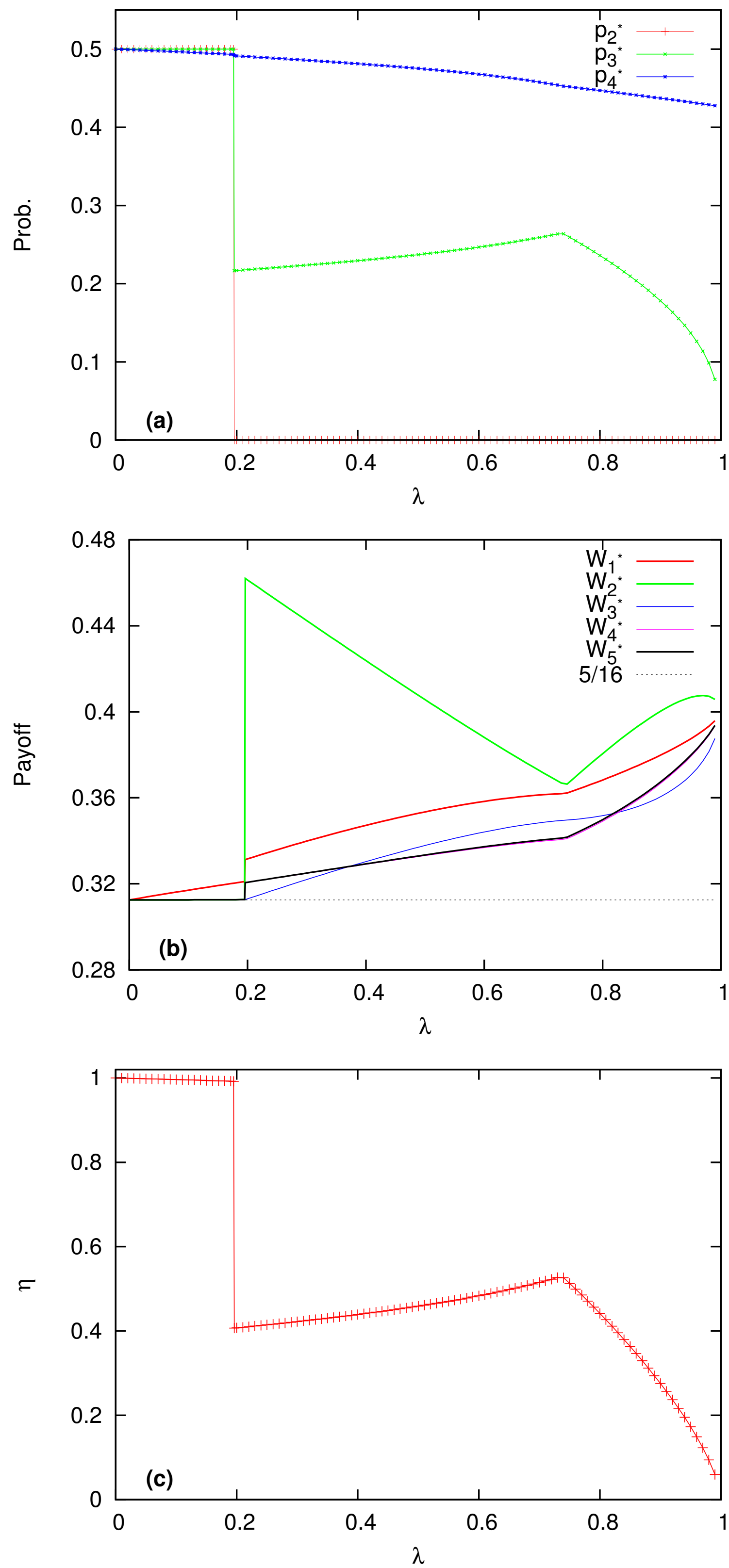

Figure 4: $N = 5$: (a) Variation of p_2^*, p_3^* and p_4^* with λ , (b) Optimum payoffs as functions of λ, (c) Inefficiency η as a function of λ.

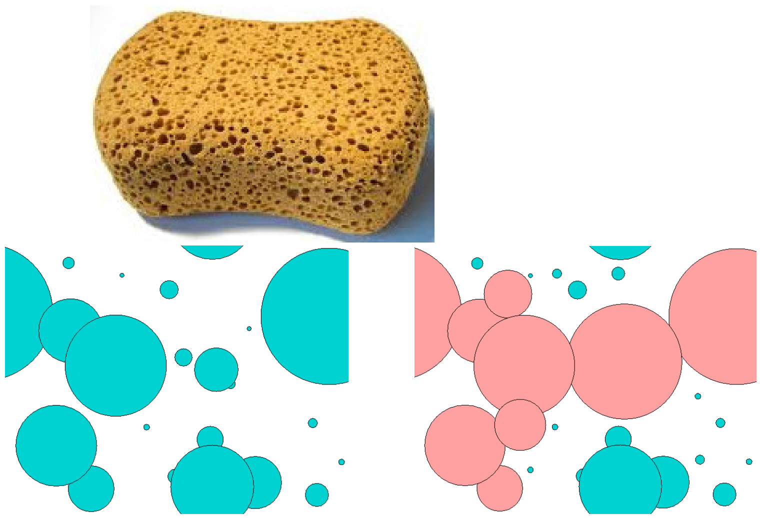

Figure 5: a) Synthetic sponge [12]. b) A 2D model below percolation threshold and c) above it. Spanning cluster is shown in red.

Continuum Percolation problem of overlapping discs with a distribution of radii having a power-law tail

In general, two classes of percolation models are studied in the literature namely lattice and continuum percolation. Disordered systems with discrete geometric structure are modeled by percolation on lattice and those, where underlying space is a continuum by continuum percolation. In problems like effective modeling of disordered systems, the continuum models of percolation are more realistic than their lattice counterparts because often the basic geometric structure of the system is not discrete as in a lattice percolation model (See Fig. 5). In two dimensions, model continuum percolation systems studied in the literature involve discs, squares etc of same or varying size distributed randomly on a plane [13, 14, 15]. The problem of disc percolation where discs have bounded size has been studied a lot, mainly by simulation [14, 16, 17]. If n is the number density of the discs in the system, there is a critical number density n^* such that for $n < n^*$, system is non-percolating and for $n > n^*$ the system is percolating where there exists a spanning cluster in the system (See Fig. 5). For the single sized disc percolation, the threshold is known to a very high degree of accuracy, $n^* \simeq .359081$ [16]. Also simulation studies have shown that the disc percolation in 2D with discs of bounded size falls in the same universality class as that of lattice percolation in 2D [18].

An interesting sub-class of these problems is where the basic percolating units have an unbounded size distribution. This is similar to systems like Lennard-Jones fluid or Ising model with long-range interactions where interaction potential decays as a power-law. It is known that such long-range interaction can affect the critical behavior of the system for slow enough decay of the interaction [19]. Continuum percolation with percolating units having an unbounded size distribution are comparatively less studied though a few formal results are available which establishes the existence of a non-zero percolation threshold [20]. Here, we consider a continuum percolation problem where the basic percolating units are overlapping discs which has a power-law distribution for the radii.

The model is defined as follows. Let n be the number density of discs. The probability that any small area element dA has the center of a disc in it is ndA, independent of all other area elements. For each disc, we assign a radius, independently of other discs, from a probability distribution $Prob(R)$. We consider the case when $Prob(R)$ has a power-law tail; the probability of radius being greater than R varies as R^{-a} for large R. For simplicity, consider the case when radii take only discrete values $R_0\Lambda^j$ where $j = 0, 1, 2, ...,$ with probabilities $(1-p)p^j$ where $p = \Lambda^{-a}$. Here R_0 is the size of smallest disc, and Λ is a constant > 1. The fraction of the entire plane which is covered by at least one disc, called the covered area fraction $f_{covered}$, is given by

$$f_{covered} = 1 - \exp\left(-A\right), \tag{8}$$

where A is the areal density - mean area of the discs per unit area of the plane - which is finite only for $a > 2$. For $a \leq 2$, in the thermodynamic limit, all points of the plane are eventually covered, and $f_{covered} = 1$.

We define two point function $\mathrm{Prob}(1 \rightsquigarrow 2)$ as the probability that points 1 and 2 in the plane at a distance r_{12} from each other are connected by overlapping discs. Let $\mathrm{Prob}^{(1)}(1 \rightsquigarrow 2)$ denote the probability that there is at least one disc that covers both points 1 and 2. Then, clearly,

$$\mathrm{Prob}(1 \rightsquigarrow 2) \geq \mathrm{Prob}^{(1)}(1 \rightsquigarrow 2). \tag{9}$$

We can show that $\mathrm{Prob}^{(1)}(1 \rightsquigarrow 2)$ decays as r_{12}^{2-a}. A comparison with the Ising model with long-range interaction or fluids with long -range potential [19, 21] where similar scenario occurs gives the result that a deviation from the standard critical behavior is expected when $a < 3 - \eta_{sr}$ and the critical exponents will take their short-range values for $a > 3 - \eta_{sr}$ where η_{sr} is the anomalous dimension exponent for the usual percolation problem. Also mean-field behavior is expected when $a \leq 2$. However for this range of a, the entire plane is covered for all non-zero number densities and hence there is no phase transition.

We propose an approximate RG scheme to analyze the behavior of continuum percolation models near the percolation threshold, when the percolating units have a distribution of sizes. We assume that we can replace discs of one size having a number density n with discs of another size and number density n', provided the correlation length remains the same. From this we get the results that the correlation-length exponent for our problem is the same as that for single sized disc problem and the critical number density is given by,

$$n^* = n_c \left(1 - \Lambda^{(2-a-1/\nu)}\right) / \left(1 - \Lambda^{-a}\right), \qquad (10)$$

where n_c is the critical number density for percolation with single sized discs of unit radius.

Simulation results

We determine the exponent ν and the percolation threshold n^* by simulating the continuum percolation system in 2D, with discs having a power law distribution for their radii. We assume a continuous distribution for the radii where, given a disc, the probability that it has a radius between R and $R+dR$ is equal to $aR^{-(a+1)}$ where $a > 2$. We use cyclic boundary conditions and consider the system as percolating whenever it has a path through the discs from the left to the right boundary. We drop discs one at a time on to a region of a plane of size $L \times L$, each time checking whether the system has formed a spanning cluster or not. From this we determine the distribution of the number density of discs to be dropped to achieve spanning $\Pi(n, L)$. The number of realizations sampled for a particular value of L varies from a maximum of 2.75×10^7

for $a = 2.05$ and $L = 90$ to a minimum of 4000 for $a = 9.0$ and $L = 1020$. The total computation time spent is approximately 10000 CPU hours. From the scaling form for the spanning probability,

$$\Pi'(n, L) = \phi((n^* - n)L^{1/\nu}), \tag{11}$$

we can determine the percolation threshold n^* and the correlation length exponent ν [22]. Values of $1/\nu$ obtained for various values of a are shown in Fig. 6a. We can see that the estimates for $1/\nu$ are very much in line with the standard percolation value for $a > 3-\eta_{sr}$ while it varies with a for $a < 3-\eta_{sr}$. Fig. 6b shows the variation of the percolation threshold n^* with a. As expected, with increasing a, the percolation threshold increases and tends to the single sized disc value as $a \to \infty$.

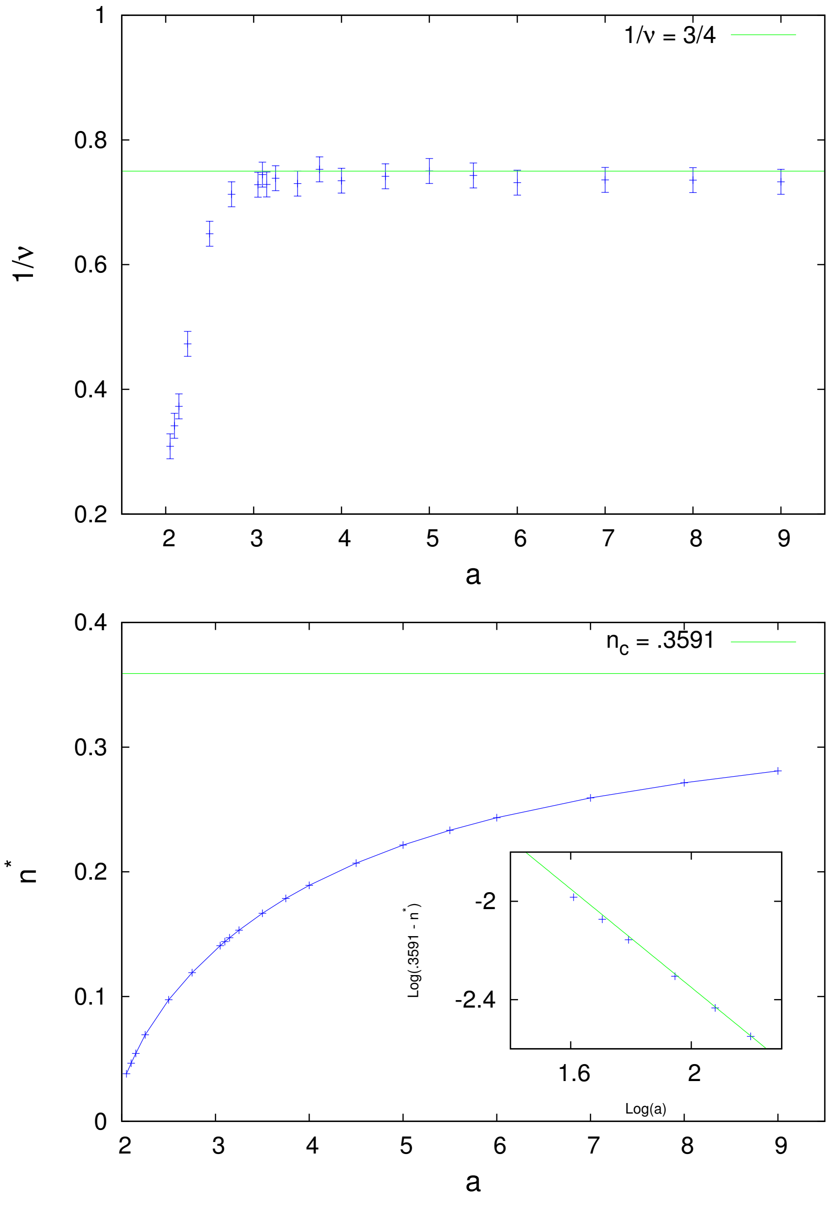

Figure 6: a) Variation of $1/\nu$ with a. The horizontal line corresponds to the standard 2D percolation value $1/\nu = 3/4$. b) Variation of percolation threshold n^* with a. The horizontal line corresponds to the threshold for the single sized discs case. (Inset) Asymptotic approach of n^* to the single sized discs value $n_c = .3591$ along with a straight line of slope -1

Bibliography

[1] D. Challet and Y. C. Zhang, Physica A **246**, 407 (1997).

[2] W. Brian Arthur, Amer. Econ. Rev. (Papers and Proc.) **84**, 406 (1994).

[3] D. Challet, M. Marsili and Y. C. Zhang, *Minority Games: Interacting agents in financial markets* (Oxford University Press, Oxford, 2005).

[4] E. Moro, in *Advances in Condensed Matter and Statistical Mechanics*, Eds. E. Korutcheva and R. Cuerno (Nova Science Publishers, New York, 2004). arXiv:cond-mat/0402651v1 [cond-mat.dis-nn].

[5] W. Kets, Journal of Economic Surveys, Vol. **26**, Issue 5, pp. 865-878 (2012).

[6] D. Dhar, V. Sasidevan and Bikas K. Chakrabarti, Physica A **390** , 3477 (2011).

[7] M. Nowak and K. Sigmund, Nature **364**, 56 (1993).

[8] R. Meester and R. Roy, *Continuum Percolation* (Cambridge University Press, Cambridge, 1996).

[9] http://www3.unifr.ch/econophysics/minority/.

[10] J. B. Satinover and D. Sornette, Eur. Phys. J. B **60**, 369 (2007).

[11] G. Reents, R. Metzler and W. Kinzel, Physica A **299**, 253 (2001).

[12] ©Johan, http://commons.wikimedia.org/wiki/File:Sponge-viscose.jpg, CC-BY-SA-3.0/ GFDL.

[13] E.T. Gawlinski and H.E. stanley, J. Phys. A: Math.Gen. **14**, L291 (1981).

[14] J. Quintanilla, Phys. Rev. **E 63**, 061108 (2001).

[15] R. Ogata, T. Odagaki and K. Okazaki, J. Phys.:Condens. matter **17**, 4531 (2005)

[16] J.A. Quintanilla and R.M. Ziff, Phys. Rev. **E 76**, 051115 (2007).

[17] M. D. Rintoul and S. Torquato, J. Phys. A: Math. Gen. **30**, L585 (1997).

[18] T. Vicsek and J. Kertesz, J. Phys. A: Math. Gen. **14**, L31 (1981).

[19] M. Aizenman and R. Fernandez, Lett. Math. Phys. **16**, 39 (1988).

[20] J.B. Gouere, The Ann. of Probab. **36**, 4 1209 (2008).

[21] R. F. Kayser and H. J. Raveche, Phys. Rev **A 29**, 1013 (1984).

[22] D. Stauffer and A. Aharony, *Introduction to percolation theory* (Taylor and Francis, London, 2003)

Publications

Included in this thesis:

1. V. Sasidevan and Deepak Dhar, *Strategy switches and co-action equilibria in a minority game*, Physica A, 10.1016/j.physa.2014.02.007, (2014).

2. V. Sasidevan, *Continuum percolation of overlapping disks with a distribution of radii having a power-law tail*, Phys. Rev. **E 88**, 022140 (2013).

3. Deepak Dhar, V. Sasidevan and Bikas K. Chakrabarti, *Emergent cooperation amongst competing agents in minority games*, Physica A **390**, 3477 (2011).

Introduction

Statistical physics deals with properties of systems of interacting particles. In such systems, exact solution of dynamical evolution equations is often difficult and perhaps not much of use, but average properties can be determined quite accurately. One very a priori unexpected property of a system of many particles is the existence of sharp phase transitions which is a collective effect. More is different: such properties emerge only when we consider a collection of entities. In this thesis, we study two examples of such transitions: one in non-equilibrium and one in equilibrium system. In the first problem, we study a variant of the Minority Game, which is an application of methods of statistical physics to a model problem in economics. Here, interacting 'particles' are the players of the game and we are interested in their collective behavior resulting from simple individual goals. In the second problem, we study a model of continuum percolation which undergoes a continuous phase transition. We study a model where the percolating units are overlapping discs with a distribution of radii having a power-law tail. In this chapter, we will give an introduction and necessary background to the two problems and will summarize our main results. An outline of each of the chapters may be found in Sec. 0.3.

0.1 The Minority Game

There has been a lot of interest in applying techniques of statistical physics to socio-economic problems in the past two decades. Naturally, a physicists' approach to problems in economics is somewhat different from that of a conventional economist and often this has lead to discussions about the need for an interdisciplinary approach to the problems [1]. While the usefulness and insights the conventional approach provided is unquestionable, the new approaches only add to our understanding of the problems. An example is the understanding of the scaling behaviour of the fat-tailed distribution observed in many kinds of financial data [2]. In particular features like fluctuations about the steady state and universality can be treated and understood better using tools from statistical physics [3, 4, 5].

While these new approaches have drawn their share of criticisms, it is fair to say that these have led to better modelling and analysis of the collective behaviour of interacting agents, as in a market. The Minority Game (MG) introduced by Challet and Zhang [6] is the prototypical model in this subject which drew inspiration from the El Farol Bar problem introduced by Brian Arthur in 1994 [7]. The model consists of agents who have to repeatedly choose between two alternatives say A and B and at each step, those who belong to the minority group are considered as winners. We can find examples of many complex systems which have such a minority rule in play. A few are vehicular traffic on multilane highways where drivers would like to be on the less congested road, markets where the general wisdom is that it is better to be a buyer when most agents are selling and vice versa, ecologies of animals looking for food where each one would like to occupy a region with plentiful food which is less frequented by other animals and so on.

In MG, at any stage of the game, each agent is concerned with following question. What is her optimum choice on the next step ? or in other words which of the two alternatives is going to be the minority side on the next step ?. The agents have to make their choices simultaneously and independently. Hence, the answer to this question in general depends on her expectation of what other agents are going to do collectively.

In MG each agent is affected only by the collective behaviour of all other agents and is like a mean-field situation in statistical physics. It is easy to see that in such a situation any common expectation held by the agents will negate themselves. If most of the agents expect that a particular option is going to be the minority choice on the next day, then all of them will take that option invalidating their belief. We will make these ideas concrete and define the game more precisely as we go along.

MG is thus an interesting model, where each agent has to see what other agents are doing collectively and adapt accordingly. This in turn, leads to other agents modifying their expectations and thus the expectations of agents co-evolve. These features make MG an interesting model of learning, adaptation and co-evolution. These features also make MG a problem of interdisciplinary research which has been studied by researchers from diverse areas ranging from physicists and sociologists to economists [8, 9, 10, 11].

In a standard MG setting [6], there are $N = 2M + 1$ agents where M is a positive integer, who has to choose between the options A and B (say two restaurants) at each step independently. At each step, each agent in the minority group get a payoff 1 and others get payoff 0. There is public information available to the agents which is the choice by the minority in the past H_b days. This information is thus a binary string of A and B of length H_b. Given this information, agents have certain rules called strategies to recommend what should be their future action. A strategy is defined as a rule which will predict the minority group on the next step for all possible inputs. Note that for a given agent, this prediction may turn out to be right or wrong depending upon the net outcome of strategies used by all agents at a step. For e.g., if all agents use the same strategy, all of them will turn out to be wrong, since all of them will turn up in the same restaurant making it a majority. So a strategy is just a rule which will recommend a future course of action (whether to go to A or B) for each possible input information. The whole strategy set is thus finite, having $2^{2^{H_b}}$ strategies in total. An example for a strategy with $H_b = 3$ is shown in table 1. One can think of a strategy as a look up table which contains a unique prediction for each possible input information. For e.g.. the strategy in table 1 predicts that when the minority group is A on three consecutive days (corresponds to first row), restaurant A will have the minority group on the following day.

History	Prediction
A A A	A
A A B	A
A B A	B
A B B	A
B A A	B
B A B	B
B B A	B
B B B	A

Table 1: An example of a strategy when $H_b = 3$.

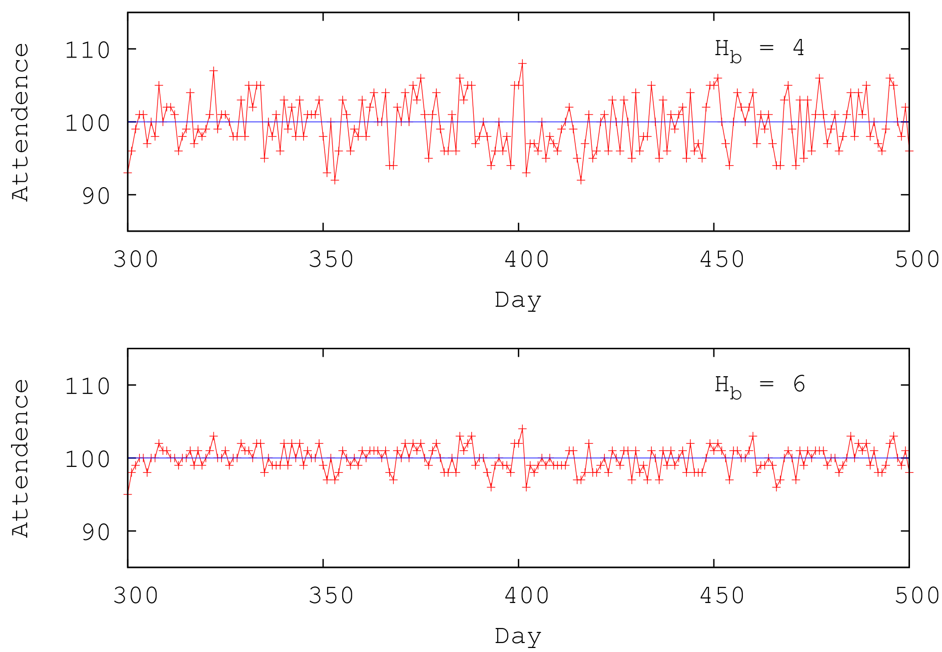

Figure 7: Schematic representation of the attendance in restaurant A for two different values of memory length H_b when total number of agents is $N = 201$. Fluctuation in the attendance is reduced below that of the random choice case as H_b is increased from 4 to 6.

In the model as defined in [6], each agent is given a small number of strategies, say S, randomly picked out of the whole set at the beginning and at any given step, an agent will use the one with the best performance in the recent past. Performance of strategies is measured by an agent by keeping a virtual score for each of the strategies with her. So, even though each agent will be using only one of the several strategies with her, she keeps a tab on the performance of all the strategies with her.

With these evolution rules, simulation studies of the model [12] have shown that the attendance in each restaurant fluctuates around $N/2$ while the fluctuation in the attendance vary with the history length H_b for a given N (see Fig. 7). The observation that the average attendance in each restaurant is

$N/2$ is easy to understand. This is because, the two restaurants are equivalent in every respect and hence none of them can contain the minority group consistently. If each agent is making a random choice between A and B, then the fluctuation in the attendance in the minority restaurant is $\approx N/2 - K\sqrt{N}$ where $K = 1$. It was found that for a range of H_b values, value of K is substantially reduced below 1 implying that agents self organize into a state which is better for all agents. Thus agents effectively cooperate even though individual agents are selfish and care only about their individual payoff. In such a case, we would say that resource utilization is better or social efficiency is higher since the number of winners per day is better than the case in which agents select randomly between A and B.

One should distinguish between the idea of a high social efficiency and high personal gain. Agents striving for their personal gain may or may not attain a high social efficiency. A state of high social efficiency is better for each of the individuals only when there is enough 'social mobility' so that all agents are on an average benefited equally. Understanding this emergent behaviour of a high social efficiency with the strategy set defined above is the subject of much of the study in the MG (See Sec. 1.1 for a review of earlier work).

With a setting such as that of MG, the possibilities to modify it to adapt to different situations are immense. Several variants of the model have been introduced later which share the same basic features of the game mentioned above. An example from an earlier period is the Thermal MG [13] which is a continuous and stochastic extension of the standard MG in which agents select the strategy to use at any step probabilistically from the strategy set allocated to them at the beginning. Here, the probability distribution is assumed to be like a Gibbs distribution, $\propto \exp(U/T)$, where U is the virtual score of a strategy and T is a temperature like parameter. When $T = \infty$, the agents selects completely randomly from their allocated strategy set and when $T = 0$, they behave as in the standard MG where they use the strategy with the best score. The continuum time limit of this model is described by stochastic dynamical equations [13, 14] (See also [15, 16]). The relation to the continuum time limit of the standard MG was made clear in a later work [17].

Another example is the evolutionary MG [18, 19] where strategies of a fraction of worst performing agents are replaced periodically with strategies selected randomly from the common pool. As in the standard MG, the central theme in these variants is that a higher social efficiency is achieved compared to the random choice case for some parameter range of the model. A more recent extension of the model is the market-directed resource allocation game (MDRAG) [20, 21], where agent's strategies have inbuilt heterogeneities so that each agent has a bias towards one of the two restaurants. Minority games with finite score memory [22], Multi-resource MG models [23] and the spherical MG [24] are a few other ones. Of these, the pherical MG model in which each agent uses a linear combination of strategies available with her has the virtue that it can be exactly solved for all parameter ranges of the model. in which each agent uses a linear combination of strategies available with her which has the virtue that it can be exactly solved for all parameter ranges of the model are a few other ones. It is impossible to give a detailed account of all different modifications of MG that has been considered since its inception here. A good place to look up is the recent review [25]. A detailed but not exhaustive list of references may be found in the Minority Game website [26].

One of the motivations to introduce the standard MG was to model the behavior of agents with bounded rationality who resort to inductive reasoning. Agents follow rather simple rules to decide when to switch strategies, based on the performance score of the strategies. Initial allocation of the strategy set to each agent is made randomly. So agents are assumed to have some fixed 'response modes'. Thus one may imagine that these agents are unthinking machines, following some pre-programmed instructions. At any stage of the game, the agents use the strategy which worked well in the recent past, from the set of response modes available. In this respect, MG can be considered as a model of learning with fixed strategy set (response modes) to each agent. For a discussion of this and the interpretation of the behaviour of the agents in standard MG see the review by Kets [27] and references therein.

As a game, MG belong to the general class of congestion games [28] where a player's benefit, who selects one of the several options from a common pool, depends upon the number of players selecting the same option. A congestion game thus models scarcity of a resource which is available to all the players. For a recent review of MG in the context of statistical mechanics of competitive resource allocation see [25].

A game, very similar in spirit to MG, is the much studied market entry game introduced by Selten and Guth [29] where each player has to decide whether to enter a market or stay out. The market is assumed to have a fixed capacity and the payoff of those who enter is a decreasing function of the number of entrants. The payoff of agents who do not enter is assumed to be constant. So unlike MG, here there is no symmetry between the two options.

Though, using a deterministic strategy set as described in the standard

MG results in a relatively higher payoff per agent and thus result in a higher social efficiency compared to the random choice case, it was found that simple probabilistic strategies could perform better in this regard [30]. This is intuitively clear, since in games of imperfect information where agents have to make their choice simultaneously, any pattern in choices made by some agents may be utilized by others to their advantage. So randomizing is considered as a better option. For example it was found in the standard MG setting that, if an agent is randomly selecting between her available strategies rather than using the one with the best performance score, she could do better than others [31].

Also, the basic premise of MG or the El Farol Bar problem was that, agents had to behave inductively rather than deductively because of the fact that there is no single optimum strategy applicable to all agents. However it is easy to see that this is true only when agents make their choice in a deterministic way. We will consider these issues in detail in Chapter 1.

In our work, we explore in detail stochastic strategies in a variation of the Minority game in which the entire strategy set is made available to all the agents. We address the question that what is the optimum strategy of an agent if she were to play the minority game rationally. We will show that contrary to the popular belief, rational deductive agents can perform much better than the inductive agents in the standard MG, and this variation is analytically tractable. We will end this section with a summary of our main results.

0.1.1 Summary of main results

- We show the the most commonly used solution concept of Nash equilibrium is very unsatisfactory for the model leading to trapping state.
- We propose a new solution concept to be called as co-action equilibrium which take care of the problem of trapping state in a natural way.
- We work out the optimum strategy of agents for small N, and find common properties which are expected to hold for larger N.
- The optimum strategy is more efficient than possible under the standard MG.
- The solution shows multiple transitions as a function of the future time horizon of agents even with finite N.

0.2 A continuum percolation problem

Several problems in nature like gelation, disease spreading etc. involve spatially random structures. Percolation is the simplest model describing the geometry of such structures and has contributed greatly in understanding problems in a diverse set of topics in physics, material science, complex networks, epidemiology etc. Apart from modelling such disordered systems, percolation models are important in statistical physics in understanding the general nature of critical phenomena associated with a continuous phase transition [32, 33, 34].

Disordered systems may have either a discrete geometric structure or a continuous one. The former is modelled by percolation on lattice and the latter by continuum percolation models. In the former, each site or bond of a lattice is occupied with some probability say p. A maximal group of connected sites or bonds is called a cluster. Then, above a critical value of p say p_c, a giant cluster spans the system. In contrast, in a continuum percolation model, percolating units which are basic geometric shapes like discs, sticks, spheres etc. are thrown in randomly into a space with a specified number density say n. Then above a critical value of n say n^*, a giant cluster spans the system (see Fig. 8).

The lattice models have been studied extensively over the past 50 years or so. The percolation threshold is known exactly for a few models like bond percolation on a square lattice [35]. Below the percolation threshold, it was shown that the probability that a given site/bond is contained in a cluster of size r decays exponentially in r [36, 35]. For the critical behavior, scaling theory predicts the existence of critical exponents and relations among them [37]. Values of these exponents which are believed to be exact, but not rigorously established are available in two dimensions (2D) [38] with backing from numerical simulation data [32, 34].

Compared to its lattice counter part, continuum models of percolation are less explored though the latter is a more realistic model of the geometry of disordered systems. In two dimensions, model continuum percolation systems studied in the literature involve discs, squares etc. of same or varying size, distributed randomly on a plane [40, 41, 42, 43]. The problem of disc percolation (See Fig. 8) where discs have bounded size has been studied a lot, mainly by simulation [41, 44, 45]. For the single sized disc percolation, the threshold is known to a very high degree of accuracy, $n^* \simeq .359081$ [44]. Also simulation

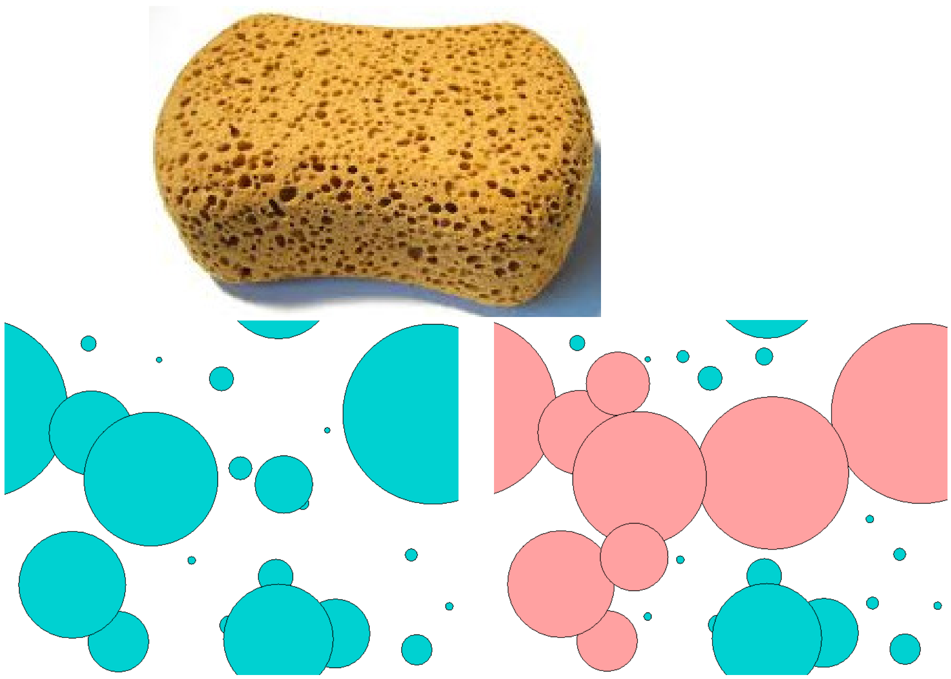

Figure 8: a) Synthetic sponge [39]. b) A 2D model below percolation threshold and c) above it. Spanning cluster is shown in red.

studies have shown that the disc percolation in 2D with discs of bounded size falls in the same universality class as lattice percolation in 2D [46].

An interesting sub-class of these problems is where the basic percolating units have an unbounded size distribution. Continuum percolation with percolating units having an unbounded size distribution are comparatively less studied, though a few formal results are available which show that a non-zero percolation threshold exists if and only if the expectation value of the $D-th$ moment of the size distribution is finite where D is the dimension [47].

In our work, we consider a continuum percolation problem where the basic percolating units are overlapping discs which has a power-law distribution of the radii. Thus given a disc, the probability of its radius being greater than R varies as R^{-a} for large R. We address questions like whether the power-law tail of the distribution affect the critical behaviour of the system and how does the percolation threshold depend on the power of the power-law tail. From an application point of view, a power-law polydispersity for an extended range of object sizes is quite common in nature especially for fractal systems [48]. Disordered systems like carbonate rocks often contain pores of widely varied sizes covering many decades in length scales ranging from few microns to several milli meters [49, 50], whose geometry may be well modelled by a power-law distribution of pore sizes.

0.2.1 Summary of main results

- Power-law tail of the distribution strongly affects the nature of the phase transition for low enough values of a.

  There are two-regimes:

  $a < 3 - \eta_{sr}$: Critical exponents depend on a.

  $a > 3 - \eta_{sr}$: Critical exponents take standard percolation values.

  where η_{sr} is the anomalous dimension exponent for the standard percolation problem.

- The entire low density non-percolating phase has power-law correlations for any value of a in contrast to the exponential decay for the standard percolation.

- We propose an approximate RG scheme which is good for relatively large values of a. We obtain an expression for the percolation threshold which is asymptotically exact.

- We determine the percolation threshold as a function of a using Monte-

Carlo simulations.

0.3 Outline

In Chapter 1, after a review of some earlier work in MG, we define our model precisely and analyze the system using the most commonly used solution concept of Nash equilibrium. We show that this solution concept is very unsatisfactory for this model leading to an absorbing state of the system where the same set of agents are the winners on all subsequent days. We call this as the trapping state. An ad hoc assumption is made to avoid this, which will reset the system after reaching the trapping state. We work out the Nash equilibrium solution for states of the system other than the trapping state and characterize the steady state behavior of the system. We then present simulation results for the problem as a function of the resetting parameter.

In Chapter 2, we propose a solution concept to be called as co-action equilibrium which take care of the problem of trapping state in a natural way. We analyze the system using this solution concept and characterize the optimum strategies of the agents. We work out the solution exactly for small number of agents $N = 3, 5, 7$ indicating the general structure of the solution for larger N. We show that the parameters of the optimal strategy depend on the future time horizon of the agents and show sudden transitions as the future time horizon is increased. The resulting optimal strategy perform better than the deterministic strategy set in the standard MG.

In Chapter 3, we consider the continuum percolation problem of overlapping discs with a distribution of radii having a power-law tail. First we show that in the low-density non percolating phase, the two-point correlation function shows a power-law decay with distance, even at arbitrarily low number densities of the discs, unlike the exponential decay in the usual percolation problem. As in the problem of fluids with long-range interaction, we argue that in our problem, the critical exponents take their short range values for $a > 3 - \eta_{sr}$, whereas they depend on a for $a < 3 - \eta_{sr}$, where η_{sr} is the anomalous dimension exponent for the usual percolation problem. We propose an approximate renormalization scheme to determine the correlation length exponent ν and the percolation threshold. We carry out Monte-Carlo simulations and determine the exponent ν as a function of a. The determined values of ν shows that it is independent of the parameter a for $a > 3 - \eta_{sr}$ and is equal to that for the lattice percolation problem, whereas ν varies with a for $a < 3 - \eta_{sr}$. We also determine the percolation threshold as a function of the parameter a.

Chapter 1

Stochastic strategies in a Minority Game

As we mentioned earlier in the introduction, randomizing is found to be a better option for the players in many games where they have to make their choices at the same time. In this chapter, we consider stochastic strategies in a variant of the MG where agents are assumed to be selfish and rational. So each agent wants to optimize only her payoff and will make use of the available information to do so.

The chapter is organized as follows. In Sec. 1.1, we review some of the earlier work in MG. In Sec. 1.2, we give the motivation for our study and emphasize the difference between our version and the standard MG. In Sec. 1.3 we define our model precisely. In Sec. 1.4 and 1.5, we analyze the game within the frame work of the solution concept of Nash equilibrium and show that this leads to very unsatisfactory trapping states for the problem. Then in Sec. 1.6, we describe our first attempt to solve the problem of trapping states by making an ad-hoc assumption. This may be found in [51]. Later in chapter 2 we propose a new solution concept called co-action equilibrium, which will take care of the problem of trapping state in a natural way. In Sec. 1.7, we characterize the steady state behavior of the system by simulation studies. Sec. 1.8 contains the summary.

1.1 Earlier work in MG

Much of the literature on MG is devoted to understanding the higher social efficiency achieved by the agents for some parameter range of the model. Much of the earlier insights into the behavior of MG came by simulation studies [6, 52, 53]. It was made clear from simulation studies that the relevant parameter in the model is the ratio $2^{H_b}/N$ between the number of possible histories and the number of agents [53, 54]. Later an explanation was given based on the fact that even though the number of possible strategies is very large, many of them are not very different [52]. There are strategies which predict the same outcome for most of the possible inputs. The nature of the game dictates that if a significant number of agents use similar strategies, the fluctuation in the attendance will be higher. A precise quantification of similarity between two strategies can be given in terms of the Hamming distance which gives a quantitative measure of the difference between two strings in terms of the number of positions at which the corresponding bits are different in the two strings. Based on this, it was shown that the number of independent strategies whose normalized Hamming distance from each other is equal to 1/2 is 2^{H_b} rather than $2^{2^{H_b}}$ [52]. If the number of agents N is much smaller than 2^{H_b}, we expect that the behavior of agents will be more or less similar to the case when they are choosing randomly between the restaurants. If $N \gg 2^{H_b}$, we expect a large number of agents to choose the same option at each step since they are using more or less similar strategies. This is called as herding effect and leads to oscillations in the attendance of a restaurant. So there will be large fluctuations in the attendance of a restaurant when $N \gg 2^{H_b}$.

The identification of this key parameter is crucial to a better understanding of the behavior of MG. Thus if σ^2 denotes the variance in the attendance difference between the two restaurants, a plot of σ^2/N against α will look like as shown in Fig. 1.1. When agents select randomly between the two restaurants, it is easy to see that $\sigma^2/N = 1$. We can see from Fig. 1.1 that for a whole range of α, the fluctuation is less than its value for random choice behavior and attains a minimum at some value say α_c. One can say that this observed minima in the fluctuation which signifies apparent cooperation is the most significant factor which aroused much of the interest in MG. It should be noted that for large N, with fixed memory length H_b, the behavior of the system is much worse than when the agents make random choices. Thus for the emergence of this apparent coordination, agents require large memory length of the order of $\log N$.

An analytical understanding of the behavior of MG was made possible later through a series of works (see [55] and references therein). An important step was the realization that replacing the history of length H_b by a random binary string does not affect the qualitative behavior of the model [56]. This means that we can feed the agents a randomly chosen H_b bit information and obtain the same qualitative behavior of the system as in standard MG. So what is

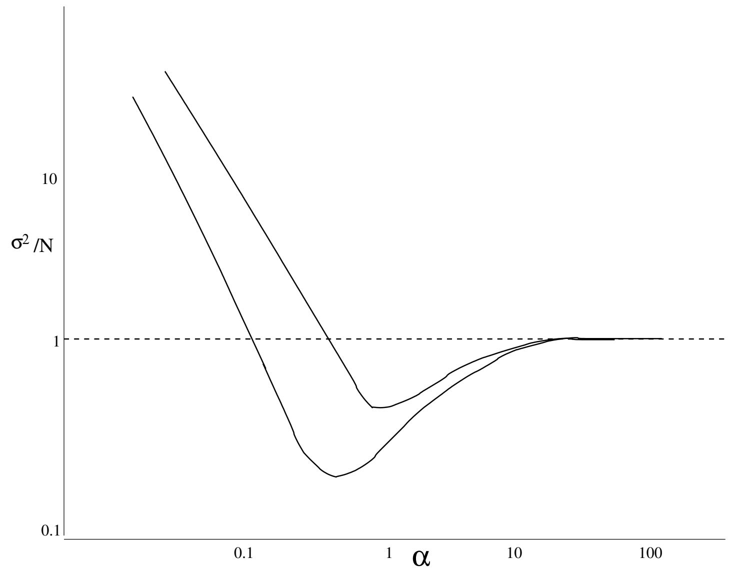

Figure 1.1: Schematic representation of the variation of the normalized global efficiency σ^2/N with the parameter $\alpha = 2^m/N$ for $S = 2$ and $S = 3$. The curve with lower minimum corresponds to $S = 2$ and the other curve corresponds to $S = 3$. The dashed horizontal line shows the value of σ^2/N when agents choose randomly between the two restaurants.

important in the model is that all agents react to the same information rather than to the information generated by their own evolution.

Along with some other simplifications which do not affect the qualitative behavior of the system, this lead to an exact dynamical solution of the model for $\alpha > \alpha_c$ using concepts and formalism developed originally for the spin-glass problem [57, 58, 55]. These modifications include introducing a temperature like parameter which introduces a probability distribution over the strategies of an agent so that agents select probabilistically from the set of strategies assigned to them [13]. This is actually the Thermal MG we mentioned in the introduction. Another modification is to make the payoffs of agents linear in the minority attendance to avoid some mathematical difficulties associated with the discrete payoff structure [59]. These involve the limits $N, H_b \to \infty$

with $\alpha = 2^{H_b}/N$ fixed. With $S = 2$, it was shown that $\alpha_c \approx 0.3374$ [57]. For a discussion of the properties of the two phases $\alpha > \alpha_c$ and $\alpha < \alpha_c$ and relevant references, see [55, 26].

Another approach which is exact for the entire parameter regime was developed based on the generating functional method [60]. The method is exact and can give analytical predictions for the phase $\alpha > \alpha_c$. The phase $\alpha < \alpha_c$ is relatively less understood analytically [61]. With $S = 2$ strategies per agent, it was shown that $\alpha_c \approx 0.3374$ and an expression for the volatility σ^2 valid in the $\alpha > \alpha_c$ phase can be obtained [62, 63]. Later the method was extended to $S > 2$ strategies per agent in [64, 65]. A detailed account of the generating functional formalism as applied to MG may be found in [11]. Though rigorous, the method is mathematically very heavy and estimation of quantities is often difficult. As in the replica analysis, the theory relies on the limits $N, H_b \to \infty$.

1.1.1 Experiments related to MG

There are interesting questions one can ask related to the actual play of a Minority Game by humans (and by other species as in [66] where fish is made to play MG!). For e.g. one can ask how does the volatility vary over time and whether the system perform better than the random choice case. Another important question is how does the available information affect the behavior of the agents. A tougher question is how agents actually decide which option to pick ?.

A handful of studies exist which explores these questions [10, 67, 68, 69, 70, 71, 72]. We can say that a better than random choice efficiency is definitely found to emerge in these studies, even when the number of agents is small, showing that there is an emergent co-operation between the agents. However, how an individual agent decides her choice is obviously a tough question to answer. Studies with respect to MG exist which try to address this question [67, 73, 72]. In these, though not all individual behaviors observed could not be accounted for by a single decision rule, the findings provide some support for modeling the behavior of agents using a low-rationality reinforcement learning model [67]. A recent study concludes that agents do use mixed strategies in MG [72]. Devetag *et. al* [67] gives a nice literature review on earlier experiments on various congestion games. Chmura and Guth [73] also gives a brief summary of various learning rules and useful references to earlier literature. Interested readers are requested to look into these papers for more details.

1.2 Our work

Our considerations differ from the standard MG described above in one key aspect. We do not restrict the agents to have only specified deterministic strategies as in the standard MG. Thus the entire strategy set is available to all the agents including probabilistic ones. Allowing agents to have access to the full strategy set may appear to be against the spirit of standard MG which was introduced as a model of collective behavior of heterogeneous agents who make their decisions by inductive reasoning. However, the reason for introducing such a heterogeneity in the first place was based on the argument that agents have an incentive to act as differently as possible [7]. In Brian Arthur's words, "Expectations will be forced to differ" and "there is no deductively rational solution - no 'correct' expectational model. From the agents' viewpoint, the problem is ill-defined and they are propelled into a world of induction ". However it is easy to see that these are true only when agents use deterministic strategies and not when agents use probabilistic strategies. Also one could argue that even the agents using the same probabilistic strategies are inherently heterogeneous in the sense that different agents in practice will be making different choices at each step. A comparative study of performance of rational deductive agents and agents who resort to inductive reasoning as in the standard MG described above is thus of interest.

Stochastic strategies have been considered in the context of MG before. For e.g. Reents, Metzler and Kinzel proposed a simple probabilistic strategy in [30], which will result in a highly efficient system. Also there are different strategies introduced in the context of MG which will give a highly efficient system [74, 75, 76]. The point of view taken here is different from these in the following respect. The idea is not to come up with a strategy from 'outside the system' (a centralized strategy) which will give a high social efficiency, but to give the freedom of choice to the agents playing the game who want to optimize only their personal gain, and study how does this affect the social efficiency. The agents are assumed to be symmetric so that they do not have any inherent preferences for one option over the other.

As we will see, the absence of heterogeneity or quenched disorder in the form of strategies assigned to agents in the beginning of the game make our variation much more tractable than the standard MG and use of stochastic instead of deterministic strategies make the system much more efficient. We will find that the optimal strategies of agents can be determined by a mean-field theory like self consistency requirement and for N-agents case, we get coupled algebraic equations in N variables. The simplicity of our analysis thus makes it an interesting and instructive prototypical toy model for the minority situations. The model also shows interesting features like non analytic dependence of the non equilibrium steady state on a control parameter even for finite number of agents. The general probabilistic 'win-stay lose-shift' strategy, which is the strategy found to be optimal in our analysis is often found in real-life situations [77]. This has a simple interpretation in terms of the behavior of agents. It says that agents retain their successful previous action while they modify their behavior when their action resulted in a loss. Thus, the variation studied here is as close an idealization of the real-life minority situations as the standard MG. For a discussion of experiments related to MG, see Sec. 1.1.1

1.3 Definition of the model

The model consists of an odd number of agents $N = 2M + 1$ where M is a positive integer. Each agent has to select one of the two alternatives A or B (say two restaurants) at each step simultaneously and independently. An agent will receive a payoff 1 if she is in the minority. i.e., if she is in a restaurant with attendance $\leq M$. Otherwise she receives a payoff 0. Agents can not communicate with each other in any way in deciding their choice, and make their choice based only on the information that how many agents were there in each of the two restaurants in the past H_b days and their own payoff in the past H_b days. More precisely, if we denote the number of agents who were in restaurant A on the t -th day by, $M - \Delta(t)$, then the time series $\{\Delta(t')\}$, for

$t' = t, t-1, ...t-(H_b-1)$ is known to all agents at the end of day t. In the standard MG [6], the information is not the value of $\Delta(t)$, but only the sign of it. We note that in the El Farol bar problem [7] from which MG was motivated has the same information as in our variation. In our model, any agent X has a future time horizon, and wants to optimize not only her next day's payoff but also payoffs she might receive far into the future. More precisely, she wants to optimize her weighted expected future payoff,

$$\text{ExpPayoff}(X) = \sum_{\tau=0}^{\infty}[(1-\lambda)\lambda^{\tau}]\langle W_X(\tau+1)\rangle, \tag{1.1}$$

where $\langle W_X(\tau)\rangle$ is the expected payoff of the agent X on the τ-th day ahead, and λ is a parameter $0 \leq \lambda < 1$, same for all agents. The parameter λ is called as the discount parameter in the literature [78]. $\lambda = 0$ thus models the situation where agents are only optimizing next day's payoff and $\lambda = 1$ corresponds to the case where agents give equal weightage to payoffs of all future days. In other words, agents have a future time horizon of the order of $1/(1-\lambda)$ days and lower values of λ means agents are impatient to receive a payoff.

In our variation of the problem, we allow agents to have probabilistic strategies. So for a given history $\{\Delta(t')\}$, a strategy will specify the probability $p(\{\Delta\})$ with which she should switch her current choice. We will restrict ourselves to the simplest case $H_b = 1$ so that the agent's strategy depend only on the attendance on the last day.

If $H_b = 0$, then we have the situation in which agents do not have any information to base their decision. Their optimum strategy then is to select A

or B randomly. In such a case, the expectation value of the number of agents who will show up at either restaurant is $N/2$. We can measure the global inefficiency of the system by the parameter η defined by,

$$\eta = \frac{4}{N}\langle(r - N/2)^2\rangle, \tag{1.2}$$

where variable r denote the attendance in restaurant A (or B). $\langle\ \rangle$ denotes averaging over long time and over different initial conditions. The normalization has been chosen, so that the inefficiency parameter η of the system with agents choosing randomly between A and B is 1.

1.4 The problem of trapping state

Given the definition of the game, we would like to answer the following question. What is the optimum strategy of an agent at a given stage of the game?. The answer to this question in game theory is provided by a solution concept which is a formal rule for predicting how a game will be played. The prediction is called the solution to the game and describes which strategies will be employed by the agents under given conditions. We will first analyze the system within the frame work of the most commonly used solution concept of Nash equilibrium [78, 79]. Informally it states that, it is a state at which no individual agent has an incentive to change her strategy unilaterally. For a discussion of other alternatives to Nash equilibrium see [80]. Most of these alternatives are for equilibrium refinements which refers to the exclusion of some of the several possible Nash equilibrium for a game. An example is the sub game perfect Nash equilibrium [79] which is used to eliminate the possibility of 'non-credible threats'. Another example is Mertens-stable equilibrium [81], which takes the stability of the solution also into account.

A state of the system in which agent i uses a strategy $\mathcal{S}_i$ is a Nash equilibrium, if for all i, $\mathcal{S}_i$ is the best response of i, assuming that all agents $j \neq i$ use the strategy $\mathcal{S}_j$. In other words, in Nash equilibrium, each player is assumed to know the equilibrium strategies of other players and no player benefit by changing only her own strategy unilaterally.

In our problem, consider, for simplicity the case $\lambda = 0$, where agents optimize only next day's payoff. Then, the state of the system with M agents in one restaurant and $M + 1$ agents in the other, with all agents staying put ($p_i = 0$ for all agents i) is a Nash equilibrium, as no agent can gain by switching, if other agents stay put. However, then, the next day the state remains the same. Thus we get a frozen steady state, that is very unsatisfactory for *the majority of agents* since they are on the losing side for all future days, even though it maximizes the number of happy people per day.

In fact, in the Nash equilibrium concept, an agent in the majority restaurant, with all agents in the minority restaurant staying put, is advised that her best strategy is to stay put. If other agents in the majority restaurant switch with a non zero probability, this is the 'optimal' solution because she has hope of receiving some payoff if at least one agent switches to the other restaurant. *This does not take into account the fact that if all agents follow this advice, their expected future gain is zero*, which is clearly unsatisfactory: No other advice could do worse!. From an outsider's perspective, agents in the majority restaurant could have clearly done better by jumping, since they then have a hope of receiving payoff in the future. The problem with the analysis lies in the Nash assumption of optimizing over strategies of one agent, *assuming that other agents would do as before.* So within the Nash solution concept, there is no escape from such a trapping state.

It is easy to come up with ad-hoc strategies which if ordered by a central authority, and followed by all agents, will avoid the trapping state. In our first work [51], this is precisely what we did, where all agents are assumed to switch their choice after reaching the trapping state (see Sec. 1.6). Another approach as done in [75] is to introduce one (or more) agents who selects randomly between the two restaurants. An optimum and fair performance will be achieved when the number of such agents is two. As mentioned in Sec. 1.2, we would like to see why rational agents get into the trapping state in which majority of the agents are unhappy forever.

Now what about other states where the attendance in the minority is less than M. In the next section we analyze these states using the Nash solution concept. We will show that with $M - \Delta$ agents in A and $M + \Delta + 1$ agents in B for $\Delta > 0$, agents in the minority staying put and agents in majority switching restaurant with a probability $\approx \Delta/M$ is a Nash solution.

1.5 Non-trapping states

As described in Sec. 1.3, consider the state of the system on the t-th day where there are $M - \Delta$ agents in A and $M + \Delta + 1$ agents in B. We may assume that $\Delta > 0$ without loss of any generality ($\Delta = 0$ correspond to the trapping state). Again for simplicity we will only consider the case $\lambda = 0$ so that agents are only optimizing next days payoff. Extending to higher values of λ will not change the qualitative nature of the solution. We will first describe a simple $H_b = 1$ probabilistic strategy for $\Delta > 0$, and then show that it is a Nash solution to the problem and gives a highly efficient system. (Note that as we mentioned earlier, agents are symmetric and they do not have preferences for one restaurant over the other. There are several Nash equilibriums possible for the case with asymmetric agents [74, 9]

The strategy is defined as follows: At $t = 0$, each agent chooses one of the two restaurants with probability 1/2. At any subsequent time $t + 1$, each agent follows the following simple strategy : If at time t, she found herself in the minority, she chooses the same restaurant as at time t. If she found herself in the majority, and the total number of people visiting the same restaurant as her was $M + \Delta(t) + 1$, with $\Delta(t) > 0$, she changes her choice with a small probability p independent of other agents. The value of p depends only on $\Delta(t)$. It is approximately equal to Δ/M. The precise dependence of p on Δ is discussed below.

For large M, the number of agents changing their choice at each step is distributed according to the Poisson distribution, with mean approximately equal to Δ, and width varying as $\sqrt{\Delta(t)}$. Thus we have the approximate recursion $\Delta(t+1) \approx \sqrt{\Delta(t)}$, for $\Delta(t) \gg 1$. This shows that within a time of order $\log \log N$, the magnitude of Δ will become of $\mathcal{O}(1)$, and then remain of order 1. So the system quickly reaches a highly efficient state where the number of people in the minority is near optimum.

Now consider a particular agent Alice, who went to A on the t-th day, and found herself in the happy situation of being in the minority. Alice assumes that all other agents follow the proposed strategy. Then, all other agents who went to A will go there again on day $(t+1)$. There are $M+\Delta+1$ agents who went to B. Each of these agents will change her choice with probability p. Let r be the number of agents that actually change their choice at time $(t+1)$. Then, r is a random variable, with a distribution given by

$$\text{Prob}_p(r) = \binom{M+\Delta+1}{r} p^r (1-p)^{M+\Delta+1-r}. \tag{1.3}$$

For $M \gg 1$, this distribution tends to the Poisson distribution with parameter $\Lambda = p(M+\Delta+1)$, given by

$$\text{Prob}_\Lambda(r) = \Lambda^r e^{-\Lambda}/r!. \tag{1.4}$$

If Alice chooses to go to A on the next day, she will be in the winning position, if $r \leq \Delta$. Hence her expected payoff $EP(Alice|stay)$, if she chooses to stay with her present choice is

$$EP(Alice|stay) = \sum_{r=0}^{\Delta} \text{Prob}_p(r). \tag{1.5}$$

On the other hand, if Alice switches her choice, she would win if $r \geq \Delta+2$. Hence, we have her expected payoff $EP(Alice|switch)$ if she chooses to switch, given by

$$EP(Alice|switch) = \sum_{r=\Delta+2}^{\infty} \text{Prob}_p(r). \tag{1.6}$$

If Alice does not follow the prescribed strategy of staying put, we will call it as 'cheating '. So for Alice to have no incentive to cheat, we must have

$$EP(Alice|stay) \geq EP(Alice|switch). \tag{1.7}$$

This sets the Nash equilibrium condition for agents in restaurant A. Now consider the agent Bob, who went to B on day t. He also assumes that all other people will follow the strategy: those who went to A will stick to their choice, and those who went to B will switch their choice with probability p. There are $M + \Delta$ other persons who went to B. If Bob chooses to cheat, and decide to stay put, without using a random number generator, the number of agents switching would be a random number $\tilde{r}$, with a distribution given by

$$\mathrm{Prob}'_p(\tilde{r}) = \binom{M+\Delta}{\tilde{r}} p^{\tilde{r}} (1-p)^{M+\Delta-\tilde{r}}. \tag{1.8}$$

He would be in the minority, if $\tilde{r} \geq \Delta + 1$. Thus, if he chooses to stay, we have his expected payoff $EP(Bob|stay)$ given by

$$EP(Bob|stay) = \sum_{\tilde{r}=\Delta+1}^{\infty} \mathrm{Prob}'_p(\tilde{r}). \tag{1.9}$$

On the other hand, if Bob decide to switch his choice, he would win if $\tilde{r} \leq \Delta - 1$. In that case, his expected payoff $EP(Bob|switch)$ is given by

$$EP(Bob|switch) = \sum_{\tilde{r}=0}^{\Delta-1} \mathrm{Prob}'_p(\tilde{r}). \tag{1.10}$$

We choose the value of p to make these equal so that any agent unilaterally deviating from this strategy do not have any advantage over those who follow it. In other words, we demand that the proposed strategy is a Nash equilibrium. Thus the equation determining p, for a given Δ and N is

$$EP(Bob|stay) = EP(Bob|switch). \tag{1.11}$$

In the limit of $M \gg \Delta$, Eq. (1.11) simplifies, as the dependence on M drops out, and we get a simple equation for the dependence of the Poisson parameter Λ on Δ. Then, Eq. (1.11) becomes

$$\sum_{r=0}^{\Delta-1} \frac{\Lambda^r}{r!} e^{-\Lambda} = \sum_{r=\Delta+1}^{\infty} \frac{\Lambda^r}{r!} e^{-\Lambda}. \tag{1.12}$$

This equation may be rewritten, avoiding the infinite summation, as

$$2 \sum_{r=0}^{\Delta-1} \frac{\Lambda^r e^{-\Lambda}}{r!} = 1 - \frac{\Lambda^{\Delta} e^{-\Lambda}}{\Delta!}. \tag{1.13}$$

It is easy to see that Eq. (1.13) implies that Eq. (1.7) is also satisfied. *So the strategy described is a Nash solution.*

Thus, for any given value of $\Delta > 0$, the optimum value of Λ is determined by the solution of Eq. (1.13). This equation is easily solved numerically and the resulting values of Λ for different Δ are shown in Table 1.1.

Table 1.1:

Δ	Λ	Δ	Λ
1	1.14619	8	8.16393
2	2.15592	9	9.16423
3	3.15942	10	10.16448
4	4.16121	20	20.16557
5	5.16229	30	30.16594
6	6.16302	40	40.16612
7	7.16354	50	50.16623

1.5.1 Asymptotic behavior

Here we will show that for large Δ, $(\Lambda - \Delta)$ tends to $1/6$. Let us rewrite Eq. 1.12 as

$$\sum_{r=0}^{\Delta-1} f_\Lambda(r) = \sum_{r=\Delta+1}^{\infty} f_\Lambda(r), \tag{1.14}$$

where $f_\Lambda(r) = \Lambda^r \exp(-\Lambda)/\Gamma(r+1)$, for r not necessarily integer. We want to solve for Λ, when Δ is given to be a large positive integer. We want to show in the limit of large Δ, $\Lambda - \Delta$ tends to $1/6$.

For large Λ, the Poisson distribution tends to a Gaussian centered at Λ, of variance Λ. If the distribution for large Λ were fully symmetric about the mean, the solution to the above equation would be $\Lambda = \Delta$. The fact that difference between these remains finite is due to the residual asymmetry in the Poisson distribution, for large Λ.

For large Λ, $f_\Lambda(r)$ is a slowly varying function of its argument. We add $f(\Delta)/2$ to both sides of Eq. (1.12), and approximate the summation by an integration. Then, Eq. (1.12) can be approximated by

$$\int_0^\Delta f_\Lambda(r)dr = \int_\Delta^{+\infty} f_\Lambda(r)dr = 1/2, \tag{1.15}$$

where we have used the trapezoid rule

$$\left[f(r) + f(r+1)\right]/2 \approx \int_r^{r+1} dr' f(r'),$$

It can be shown that the discrepancy between Eqs. (1.14) and (1.15) is at most of order $(1/\Lambda)$.

Then, for large Λ, deviations of $f_\Lambda(r)$ from the limiting Gaussian form can be expanded in inverse half-integer powers of Λ

$$f_\Lambda(r) = \frac{1}{\sqrt{\Lambda}}\phi_0(x) + \frac{1}{\Lambda}\phi_1(x) + \ldots . \tag{1.16}$$

where x is a scaling variable defined by $x = (r - \Lambda)/\sqrt{\Lambda}$. Here $\phi_0(x)$ is the asymptotic Gaussian part of the distribution, as expected from the central limit theorem, and $\phi_1(x)$ describes the first correction term.

The characteristic function for the Poisson distribution $\tilde{\Phi}_\Lambda(k)$ defined by

$$\begin{aligned} \tilde{\Phi}_\Lambda(k) = \langle e^{ikr} \rangle = \sum_{r=0}^{\infty} e^{ikr}\mathrm{Prob}_\Lambda(r) &= \exp\left[\Lambda e^{ik} - \Lambda\right], \\ &= \exp\left[ik\Lambda - k^2\Lambda/2 - ik^3\Lambda/6 + ..\right]. \end{aligned} \tag{1.17}$$

Keeping the terms up to quadratic in k gives the asymptotic Gaussian form of the central limit theorem

$$\phi_0(x) = \frac{1}{\sqrt{2\pi}}\exp(-x^2/2).$$

The first order correction to this asymptotic form of $\tilde{\Phi}_\Lambda(k)$ is given by

$$\tilde{\phi}_1(k) = \frac{-ik^3}{6}\exp(-k^2/2),$$

which gives on taking inverse Fourier transform

$$\phi_1(x) = \frac{1}{6}\frac{d^3}{dx^3}\phi_0(x). \tag{1.18}$$

Substituting the functional forms for $\phi_0(x)$ and $\phi_1(x)$ in Eq. (1.15), we get

$$\int_{-\infty}^{\frac{\Delta-\Lambda}{\sqrt{\Lambda}}} dx \left[\phi_0(x) + \frac{1}{\sqrt{\Lambda}}\phi_1(x)\right] = 1/2.$$

Now, $\phi_1(x)$ is an odd function of x, and is zero for $x = 0$. As $\Delta - \Lambda$ is small, in the coefficient of $1/\sqrt{\Lambda}$, we can replace the upper limit of the integral by zero. Thus we write

$$\int_{-\infty}^{(\Delta-\Lambda)/\sqrt{\Lambda}} \phi_1(x')dx' \approx \int_{-\infty}^{0} \phi_1(x')dx'. \tag{1.19}$$

But using Eq. (1.18), we get

$$\int_{-\infty}^{0} \phi_1(x')dx' = \frac{1}{6}\frac{d^2}{dx^2}\phi_0(x)|_{x=0} = -\phi_0(0)/6.$$

Substituting in Eq. (1.19), we get

$$\int_{-\infty}^{(\Delta-\Lambda)/\sqrt{\Lambda}} \phi_0(x')dx' = 1/2 - \frac{\phi_0(0)}{6\sqrt{\Lambda}} + \mathcal{O}(1/\Lambda),$$

and comparing terms of order $\Lambda^{-1/2}$ we get

$$\Lambda - \Delta = 1/6 + \mathcal{O}(\frac{1}{\sqrt{\Lambda}}). \tag{1.20}$$

1.6 An ad-hoc solution to the trapping state problem

As described in Sec. 1.4, reaching the $\Delta = 0$ state, the system get struck there forever. An ad-hoc way to avoid this is to assume that the agents reset the system after reaching the trapping state by switching with some largish probability. A simple dictatorial solution in this case is to propose that, all agents irrespective of whether they were in minority or not, switch with a probability $M^{\epsilon-1}$ after reaching the trapping state, where ϵ is a real number $0 \leq \epsilon \leq 1$. We shall refer to this step as a major resetting event. The value of ϵ is not determined by the model, but is assumed to have a preset value.

For a given value of ϵ, the value of $|\Delta|$ just after resetting is of order $M^{\epsilon/2}$. Then it lakes time of order $\log\log M$ to reach the value $\Delta = 0$. Then the maximum contribution to the mean efficiency parameter comes from the major resetting events, and it is easy to see that the mean inefficiency parameter would vary as $M^{\epsilon-1}/\log\log M$. Then, for more efficiency, we should keep ϵ small.

1.7 Simulation results

We study the time evolution of a set of N agents using the strategy described above using Monte Carlo simulations, with $N = 2001$. If the restaurant with greater attendance has $M+1+\Delta$ agents on a given day, with $\Delta > 0$, the next day each of them switches her choice with probability $\Lambda(\Delta)/(M+\Delta+1)$, and the agents in the minority restaurant stick to their choice. If there are exactly

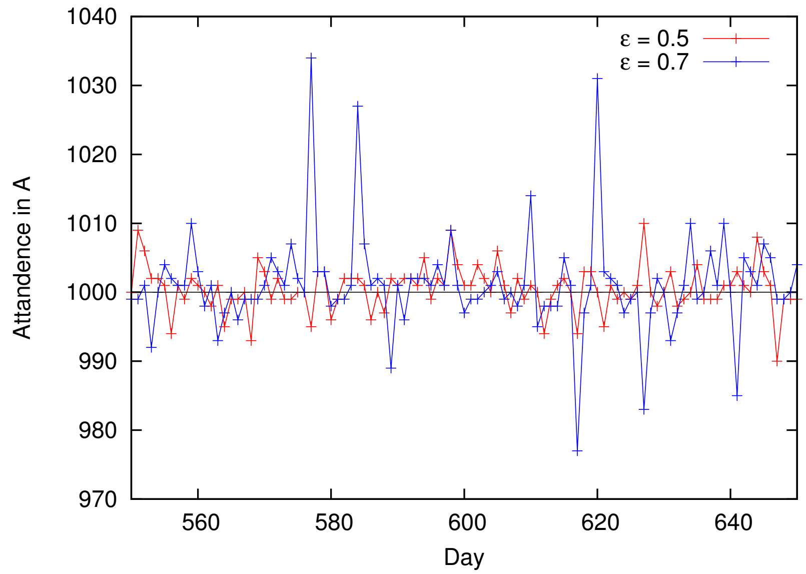

Figure 1.2: A typical evolution of a system of 2001 agents for two different choices of the parameter $\epsilon = 0.5$ and 0.7. The large deviations correspond to major events (see text).

$M + 1$ agents in the majority restaurant, all agents switch their restaurant with a probability $1/(2M^{1-\epsilon})$.

The result of a typical evolution is shown in Fig. 1.2, for two choices of ϵ: 0.5 and 0.7. We see that the minority restaurant changes quite frequently. In fact, the system reaches the steady state fairly quickly, within about 10 steps. The large peaks in $|\Delta|$ correspond to resettings of the system, and clearly, their magnitude decreases if ϵ is decreased. There is very little memory of the location of minority restaurant in the system. To be specific, let $S(t)$ is $+1$ if the minority restaurant is A in the t-th step, and -1 if it is B. Then the autocorrelation function $\langle S(t)S(t+\tau)\rangle$ decays exponentially with τ, approximately as $\exp(-K\tau)$ (See Fig. 1.5). The value of K depends on ϵ, but is about 2, and the correlation is negligible for $\tau > 3$.

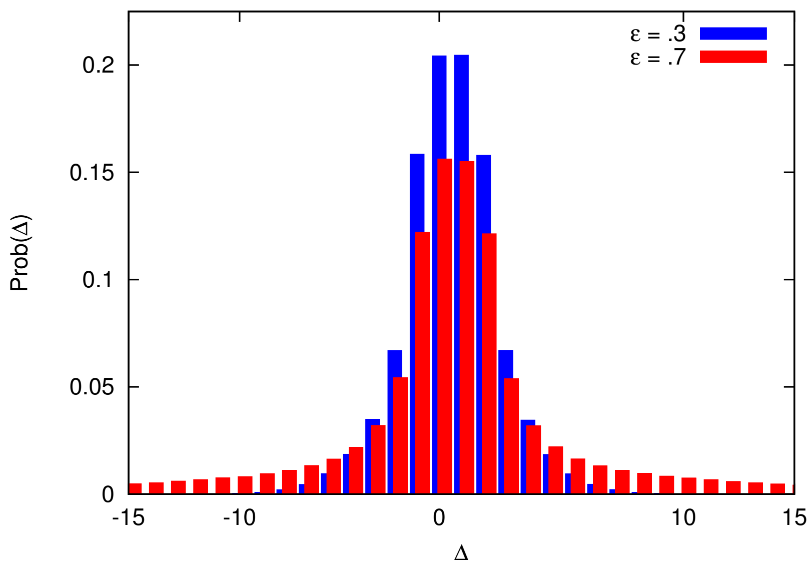

Figure 1.3: Probability distribution of Δ in the steady state for $\epsilon = .3, .7$ obtained by evolving $N = 2001$ agents for 10^6 time steps. The red bars have been shifted a bit to the right for visual clarity.

Fig. 1.3 shows the probability distribution of Δ in the steady state for two different values of ϵ. Fig. 1.4 gives a plot of the inefficiency parameter η as a function of ϵ. In each case, the estimate of η was obtained using a single evolution of the system for 10000 time steps. The fractional error of estimate is less than the size of symbols used.

To see the correlation in the position of agents in the restaurants, we define a variable $A_i(t)$ which is equal to $+1$ if the i-th agent was in the restaurant A at time t, and -1 otherwise. We define the auto-correlation function of the A-variables in the steady state as

$$C(\tau) = \frac{1}{N} \sum_i \langle A_i(t) A_i(t + \tau) \rangle. \tag{1.21}$$

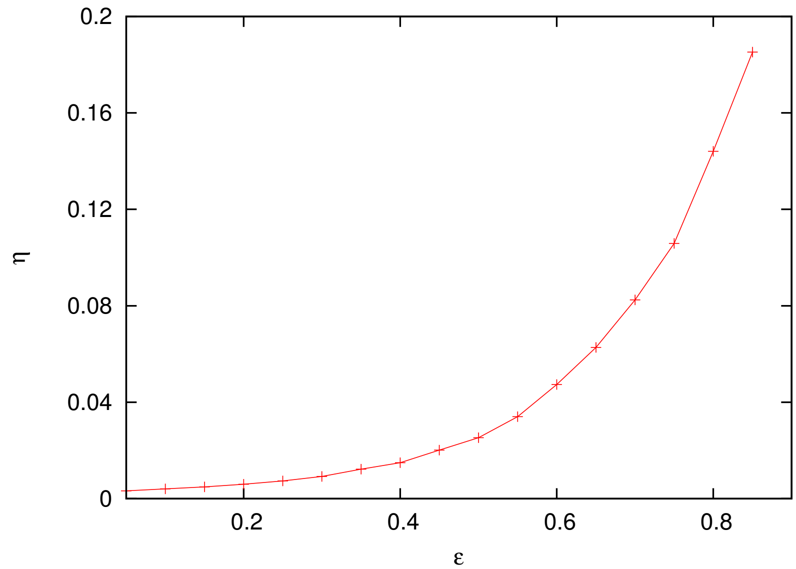

Figure 1.4: Variation of inefficiency parameter η with ϵ, obtained by averaging the evolution of $N = 2001$ agents for 10000 time steps.

In Fig. 1.6, we have shown the variation of $C(\tau)$ with τ. We see that this function has a large amount of persistence. This is related to the fact that only a small fraction of agents actually switch their choice at any time step. Clearly, the persistence is larger for smaller ϵ. The qualitative picture is thus that the agents themselves find in the same restaurant for longer period of time (characterized by) whereas the minority restaurant itself changes quite frequently. Thus if we define a persistence time as the time spend by an agent in the minority, it's distribution has an exponential decay with typical persistence time of the order of 2 or 3 days.

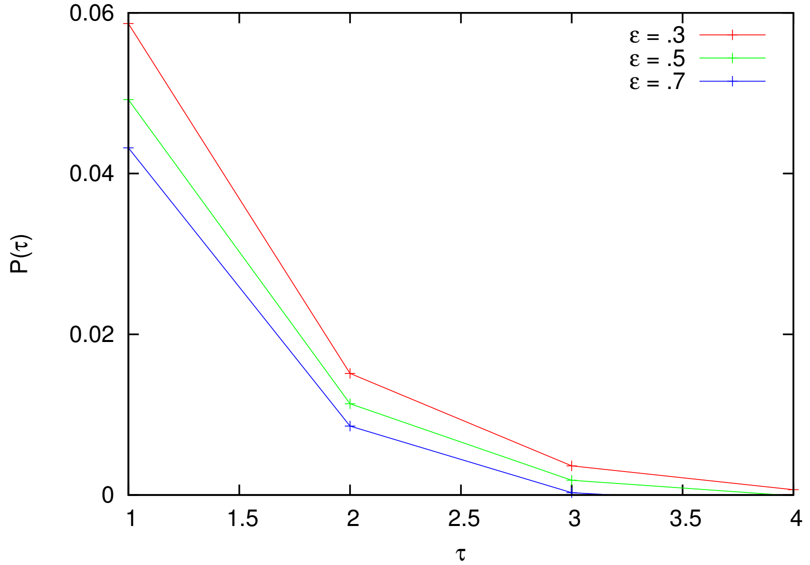

Figure 1.5: $< S(t)S(t+\tau) >$ as a function of τ for $\epsilon = 0.3, 0.5$ and 0.7. Each data point is obtained by averaging over 10000 simulation steps.Total number of agents is $N = 2001$.

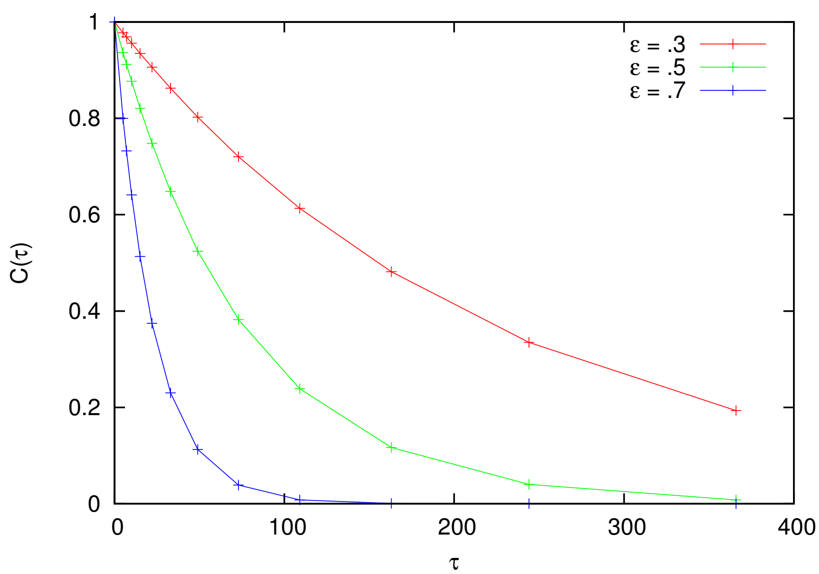

Figure 1.6: $C(\tau)$ as a function of τ for $\epsilon = 0.3, 0.5$ and 0.7. Each data point is obtained by averaging over 10000 simulation steps.Total number of agents is $N = 2001$.

1.8 Summary

In this chapter, we considered the performance of stochastic strategies in MG where N agents select one of the two available choices at each time step and want to be in the minority. We analyzed the game within the frame work of Nash solution concept. We showed that the Nash solution concept is very unsatisfactory here leading to an absorbing state of the system where having reached the state in which the minority restaurant contain exactly M agents, nobody switches their choices. So in this state, the same set of agents benefit on all days. We then proposed an ad-hoc solution to avoid this by resetting the system after reaching the trapping state. We worked out the Nash equilibrium solution for non trapping states. Together with the ad-hoc assumption, we characterized the steady state of the system. We showed that the strategy leads to a much more efficient utilization of resources, and the average deviation from the maximum possible can be made $\mathcal{O}(N^{\epsilon})$, for any $\epsilon > 0$. The time required to reach this level increases with N as only $\log \log N$.

One may attribute the fast learning rate in this game compared to the standard MG to the fact that more information is provided to the agents. Unlike the agents in the standard MG who know only which one is the minority restaurant, agents in our model are assumed to know the exact value of M and Δ. However, it is not very difficult to imagine situations in which agents have knowledge about both the comfort level (M) and how better they are doing compared to the comfort level (Δ). Also, as shown in a subsequent paper to ours by S. Biswas *et al.* [75], it is not really necessary to have the exact information for the agents to have a fast learning rate. Even rough guesses about the values of Δ or appropriate assumption about the time variation of

$\Delta(t)$ (termed as an annealing schedule) by the agents is sufficient [75].

Chapter 2

Co-action equilibrium

In this chapter, we propose a new solution concept to be called as co-action equilibrium which will take care of the problem of trapping states described in chapter 1. As we saw in chapter 1, within the solution concept of Nash equilibrium, an agent in the restaurant with $M+1$ agents in it, is advised that her best strategy is to stay put and this lead to the trapping state where the same set of agents become the winners on all subsequent days. The problem with this analysis lies in the Nash concept of optimizing over strategies of one agent, *assuming that other agents would do as before*. In the alternate co-action equilibrium concept proposed here, an agent i in a restaurant with total M agents in it realizes that she can choose her switching probability p_i, but all the other fully rational $(M-1)$ agents in the same restaurant, with the same information available, would argue similarly, *and choose the same value of* p_i. In determining the optimum value of p_i, agents will take this into account. Determining the optimal value of p_i that maximizes payoff of an agent does not need communication between agents.

One can think of co-action equilibrium on any given day as an equilibrium for a two-person game, where the two persons are the majority and the minority groups, and they select the optimal values of their strategy parameters. *But these groupings are temporary, and change with time.* In our model, the complete symmetry between the agents, and the assumption of their being fully rational, ensures that they will reach co-action equilibrium.

The chapter is organized as follows. In Sec. 2.1, we set up the problem in the language of Markovian evolution. In Sec. 2.2, we discuss the exact solution of the problem with small number of agents N. We also indicate the general structure of the solution for arbitrary N. Sec. 2.4 contains the summary.

2.1 Co-action equilibrium

We say that an agent is in state C_i when she is in a restaurant with total number of people i in it (note that, here index i refers to a group of agents rather than a single agent). Let p_i be the switch probability chosen by an agent when she is in the state C_i. For a given N, a strategy $\mathbb{P}$ is defined by the set of N numbers $\mathbb{P} \equiv \{p_1, p_2, ...p_N\}$. Clearly, as all agents in the restaurant with i agents switch independently with probability p_i, the system undergoes a Markovian evolution, described by a master equation. As each agent can be in one of the two restaurants, the state space of the Markov chain is 2^N dimensional. However, we use the symmetry under permutation of agents to reduce the Markov transition matrix to $N \times N$ dimensional. Let $|Prob(t)\rangle$ be an N-dimensional vector, whose j-th element is $Prob_j(t)$, the probability that a marked agent X finds herself in the state C_j on the t-th day. On the next day, each agent will switch according to the probabilities given by $\mathbb{P}$, and we get

$$|Prob(t+1)\rangle = \mathbb{T}|Prob(t)\rangle, \tag{2.1}$$

where $\mathbb{T}$ is the $N \times N$ Markov transition matrix. Explicit matrix elements are easy to write down. For example, $\mathbb{T}_{11}$ is the conditional probability that the marked agent is in state C_1 on the next day, given that she is in C_1 today. This is the sum of two terms: one corresponding to everybody staying with the current choice [the probability of this is $(1-p_1)(1-p_{N-1})^{N-1}$], and another corresponding to all switching their respective restaurant [the probability is $p_1 p_{N-1}^{N-1}$].

The total expected payoff of X, given that she is in the state C_j at time $t=0$ is easily seen to be

$$W_j = (1-\lambda)\left\langle L \left| \frac{\mathbb{T}}{1-\lambda\mathbb{T}} \right| j \right\rangle, \tag{2.2}$$

where $|j\rangle$ is a column vector with only the j-th element 1, and rest zero; and $\langle L|$ is the left-vector $\langle 1,1,1,1,..0,0,0..|$, with first $M=(N-1)/2$ elements 1 and rest zero. The left vector thus encodes the payoff structure of the game.

As mentioned earlier, one can think of co-action equilibrium on any given day as a solution concept for a two-person game, where the two persons are the majority and the minority groups who select the optimal values of their strategy parameters p_i and p_{N-i}.

We now discuss the equilibrium choice $\{p_1^*, p_2^*, \ldots p_N^*\}$. The co-action equilibrium condition implies N conditions on the N parameters $\{p_i^*\}$. There can be more than one self-consistent solution to the equations, and each solution corresponds to a possible steady state.

One simple choice is that $p_i^* = 1/2$ for all i, which is the random choice strategy, where each agent just picks a restaurant totally randomly each day, independent of history. We will denote this strategy by $\mathbb{P}_{rand}$. In the corresponding steady state, it is easy to see that W_j is independent of j, and given by

$$W_j = W_{rand} = 1/2 - \binom{N-1}{M} 2^{-N}, \quad \text{for all } j. \tag{2.3}$$

For a strategy $\mathbb{P}$, it is more convenient to use the inefficiency parameter η defined as follows instead of that in Eq. (1.2).

$$\eta\left(\mathbb{P}\right) = \left(W_{max} - W_{avg}\left(\mathbb{P}\right)\right) / \left(W_{max} - W_{rand}\right), \tag{2.4}$$

where $W_{max} = M/N$ is the maximum possible payoff per agent, $W_{avg}\left(\mathbb{P}\right)$ is the average payoff per agent in the steady state for a given $\lambda > 0$.

By the symmetry of the problem, it is clear that $p_N^* = 1/2$ for all λ. Now consider the more general possible equilibria $\{p_i^*\} = \{p_1^*, 1/2, 1/2, 1/2...\}$. If X is in the state C_1, and next day all other agents would switch with probability 1/2, it does not matter if X switches or not: payoffs W_1 and W_{N-1} are independent of p_1^*. Hence p_1^* can be chosen to be of any value. It is easy to see that the strategy $\mathbb{P}'_{rand}$, in which $p_1^* = 0$, and $p_{N-1}^* < 1/2$, chosen to maximize W_{N-1}, is better for all agents and is stable. The stability of this solution is easy to see, because none of the agents gain by deviating from their respective strategies. The result that this strategy is better for all agents than random switching can be shown by an analysis similar to the one discussed in section 2.2.2. In short, $\mathbb{P}'_{rand}$ is always preferred over $\mathbb{P}_{rand}$ by all agents.

2.2 Explicit calculation of the steady state for small N

2.2.1 $N = 3$

We consider first the simplest case $N = 3$. Since $p_1^* = 0$, $p_3^* = 1/2$, the only free parameter is p_2. In this case, the transfer matrix is easily seen to be

$$\mathbb{T} = \begin{bmatrix} q_2^2 & p_2 q_2 & 1/4 \\ 2p_2q_2 & q_2 & 1/2 \\ p_2^2 & p_2^2 & 1/4 \end{bmatrix} \tag{2.5}$$

where $q_2 = 1 - p_2$. The payoff W_2 is given by

$$\begin{aligned} W_2 &= (1-\lambda)\,[1\ 0\ 0]\,\frac{\mathbb{T}}{(1-\lambda\mathbb{T})}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \\ &= \frac{4p_2q_2 - \lambda p_2(q_2 - p_2)}{(1-\lambda q_2(q_2-p_2))(4+\lambda(4p_2^2-1))}. \end{aligned} \tag{2.6}$$

The eigenvalues and eigen vectors of the transfer matrix $\mathbb{T}$ are easily determined. Eigenvalues are, $\left(1,\ \frac{1}{4}\left(1-4p_2^2\right),\ q_2\left(q_2-p_2\right)\right)$. The right eigen vectors corresponding to these eigenvalues are $[1,\ \ 2,\ \ 4p_2^2]$, $[1,\ 2,\ \ -3]$, $[1,\ \ -1,\ \ 0]$ and the left eigenvectors are $[1,\ \ 1,\ \ 1]$, $[4p_2^2,\ 4p_2^2,\ \ -3]$, $[2,\ -1,\ 0]$ respectively. Normalizing the eigenvector corresponding to eigenvalue 1 gives the average gain in the steady state W_{avg} as

$$W_{avg} = \frac{1}{3+4(p_2^*)^2}. \tag{2.7}$$

The value of p_2 that maximizes W_2 is easily seen to be root of the following cubic equation in λ.

$$\begin{aligned}&16 - 32p_2^* - (24 - 56p_2^* + 32p_2^{*2})\lambda + (9 - 28p_2^* + 40p_2^{*2}\\&- 96p_2^{*3} + 144p_2^{*4} - 64p_2^{*5})\lambda^2 - (1 - 4p_2^* + 8p_2^{*2} - 24p_2^{*3}\\&+ 48p_2^{*4} - 32p_2^{*5})\lambda^3 = 0.\end{aligned}$$

The variation of p_2^* with λ is shown in Fig. 2.1a. p_2^* monotonically decreases with λ from its value $1/2$ at $\lambda = 0$, and tends to 0 as λ tends to 1. Therefore, W_{avg} is a monotonically increasing function of λ, and leads to the best possible solution $W_{avg} = 1/3$ as $\lambda \to 1$. The payoff of agents in various states with this optimum strategy is shown in Fig. 2.1b. Thus as $\lambda \to 1$, $p_2^* \to 0$ and the expected payoffs in each state become equal. So as λ is increased, the system tends to stay in the state C_1 (or C_2) for more and more time and only rarely flipping to the state C_3. Note that here C_i is used to denote the state of the system rather than that of an agent. Both C_1 and C_2 refers to the same state of the system where there is 1 agent in one of the restaurants and 2 in the other. The variation of inefficiency with λ is shown in Fig. 2.1c.

We can obtain an expression for p_2^* as $\lambda \to 1$ by the following argument. As we saw, $p_2^* \to 0$ as $\lambda \to 1$ or the system stay in the state C_1 (or C_2) for longer and longer. Then the expected payoff of an agent in state C_2 for H_f days into future can be written as

$$W_2(\lambda \to 1) \approx H_f/3 - Const.(1/p_2) - H_f.(p_2^2). \qquad (2.8)$$

Here the first term on R.H.S is the payoff if all agents are equally benefited. However, an agent in state C_2 will be flipped to C_1 only on an average $\sim 1/p_2$ days. So an agent who is starting in state C_1 will have an advantage of $\sim 1/p_2$ over H_f days . This is reflected in the second loss term. The third term indicate the fact that with probability p_2^2, both the agents in state C_2 jump resulting in zero payoff for all the agents. Maximizing W_2 in Eq. (2.8) with respect to p_2, we get

$$p_2^*(\lambda \to 1) \sim H_f^{-1/3}. \tag{2.9}$$

Now in terms of the parameter λ, $H_f \sim 1/(1-\lambda)$. Therefore as $\lambda \to 1$,

$$p_2^*(\lambda \to 1) \sim (1-\lambda)^{1/3},$$

and the average gain in the steady state W_{avg},

$$W_{avg}(\lambda \to 1) \approx 1/3[1 - 4/3(1-\lambda)^{2/3}].$$

So for $\lambda = 1 - \varepsilon$, the average payoff per agent per day in the steady state is given by

$$W_{avg} = 1/3 - K\varepsilon^{2/3} + O(\varepsilon), \tag{2.10}$$

where K is a numerical constant.

2.2.2 $N = 5$

We can similarly determine the optimal strategy for $N = 5$. This is characterized by the five parameters $(p_1^*, p_2^*, p_3^*, p_4^*, p_5^*)$. The simplest strategy is $\mathbb{P}_{rand}$, which corresponds to $p_i^* = 1/2$, for all i. As explained above, the strategy

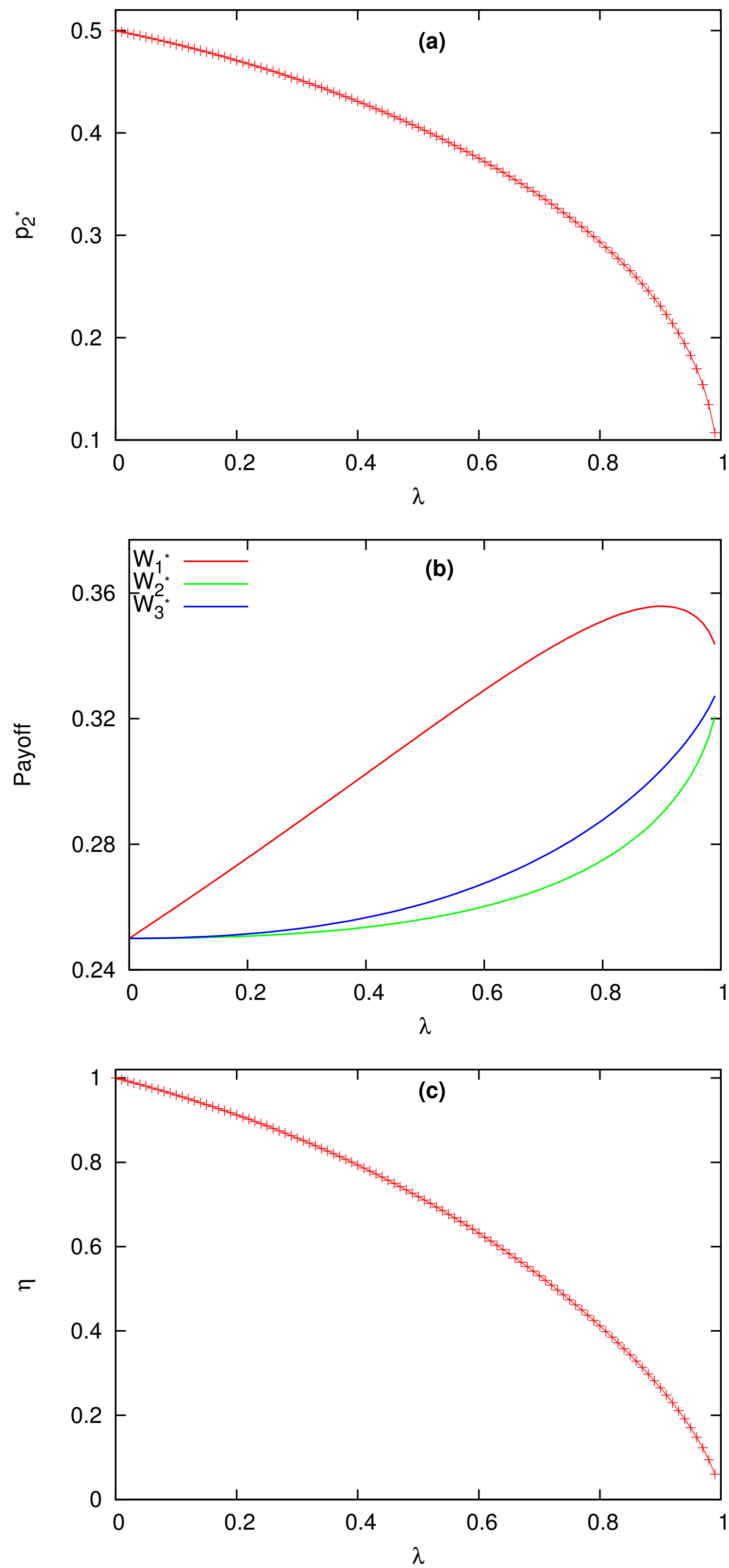

Figure 2.1: $N = 3$: (a) Variation of p_2^* with λ, (b) The optimum payoffs W_i^*, ($i = 1$ to 3), as functions of λ and (c) Variation of inefficiency η with λ.

$\mathbb{P}'_{rand} = (0, 1/2, 1/2, p_4^*(\lambda), 1/2)$, gives higher payoff than $\mathbb{P}_{rand}$ for all agents, for all λ.

Now consider agents in the states C_2 and C_3. What values of p_2 and p_3 they would select, given their expectation or belief about the selected values of p_1, p_4 and p_5 ?. We can determine these by an analysis of the variation of payoffs W_2 and W_3 as functions of p_2 and p_3 for fixed values of p_1, p_4, p_5 and λ, as we discuss below.

Let us denote the best response of agents in state C_2, (that maximizes W_2), if the agents in the opposite restaurant jump with probability p_3 by $r_2^{opt}(p_3)$. Similarly, $r_3^{opt}(p_2)$ denotes the best response of agents in state C_3, when those in the opposite restaurant jump with probability p_2.

In Fig. 2.2, we plot the functions $r_2^{opt}(p_3)$ (OAP) and $r_3^{opt}(p_2)$ (BP), in the (p_3, p_2) plane, for three representative values of λ. For small p_3, $r_2^{opt}(p_3)$ remains zero, and its graph sticks to x-axis initially (segment OA in figure), and then increases monotonically with p_3. The strategy $\mathbb{P}'_{rand}$ is the point $(1/2, 1/2)$, denoted by P. We also show the lines PC corresponding to $W_3 = W'$, and PD, corresponding to $W_2 = W'$, where W' is the expected gain of agents in state C_2 or C_3 under $\mathbb{P}'_{rand}$. For all points in the curvilinear triangle PCD, both W_2 and $W_3 \geq W'$. Clearly, possible equilibrium points are the points lying on the lines $r_2^{opt}(p_3)$ or $r_3^{opt}(p_2)$ that lie within the curvilinear triangle PCD. However, along the blue curve OAP representing $r_2^{opt}(p_3)$, maximum value for W_2 is achieved when $p_2 = 0$. Therefore we can restrict the discussion of possible equilibrium points to the line segment CD in Fig. 2.2.

For small λ (shown in Fig. 2.2a for $\lambda = 0.1$), The point A is to the left of C, and the only possible self-consistent equilibrium point is P. This implies

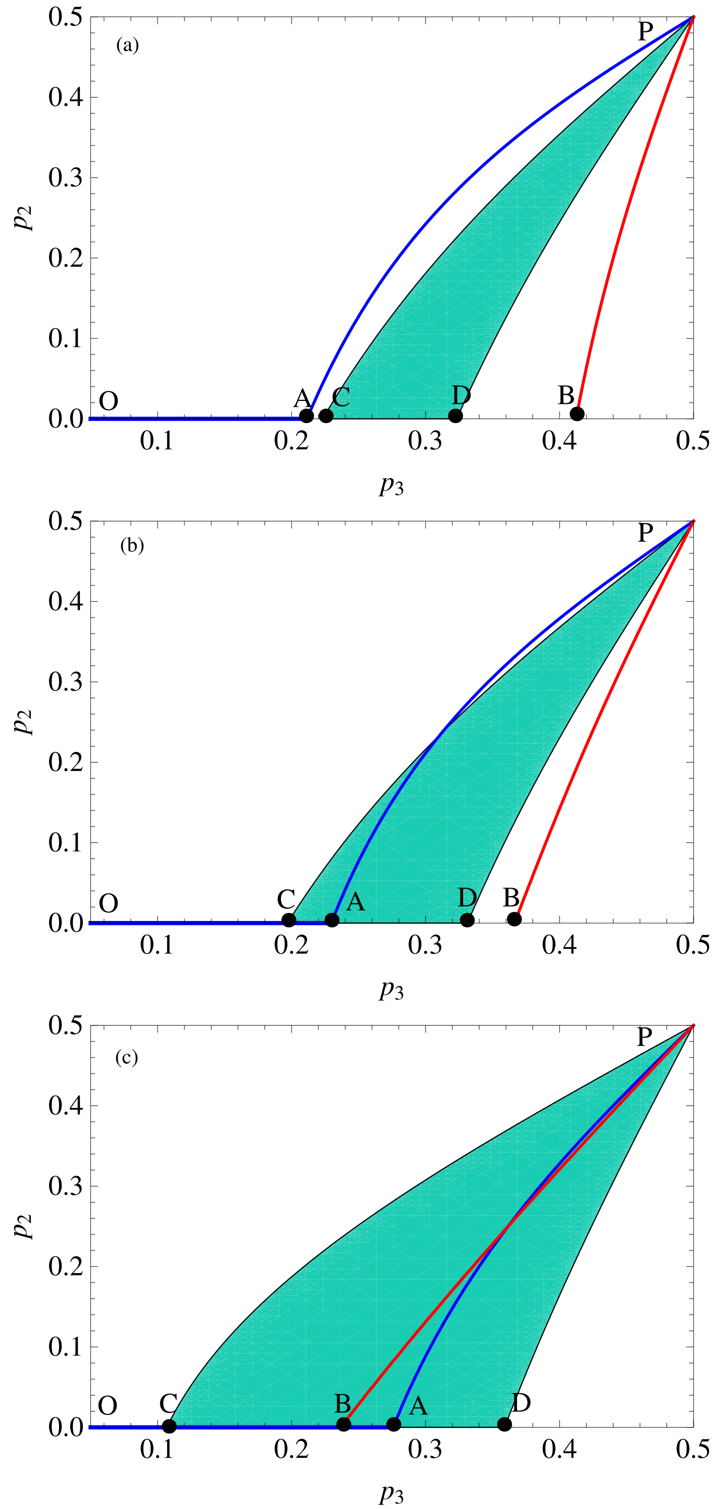

Figure 2.2: Region in the p_2-p_3 plane showing the best responses $r_2^{opt}(p_3)$ (blue) and $r_3^{opt}(p_2)$ (red) for agents in state $|2\rangle$ and $|3\rangle$ respectively, for (a) $\lambda = .1$, (b) $\lambda = .4$ and (c) $\lambda = .8$. The line PC and PD show the curves $w_3 = W'$ and $W_2 = W'$ respectively. In the curvilinear triangle PCD, all agents do at least as well as at P.

that the agents would choose $p_2^* = p_3^* = 1/2$. This situation continues for all $\lambda < \lambda_{c1} = 0.195 \pm .001$.

For $\lambda > \lambda_{c1}$, the point A is to the right of C. This is shown in Fig. 2.2b, for $\lambda = 0.4$. In this case, possible equilibrium points lie on the lie-segment CA, and out of these, A will be chosen by agents in state C_3. At A, both W_2 and W_3 are greater than W', and hence this would be preferred by all. Further optimization of p_4 changes p_3 and p_4 only slightly.

As we increase λ further, for $\lambda > \lambda_{c2}$ [numerically, $\lambda_{c2} = 0.737 \pm .001$], the point B comes to the left of A. Out of possible equilibria lying on the line-segment CA, the point preferred by agents in state C_3 is no longer A, but B. The self-consistent values of p_2^*, p_3^*, and p_4^* satisfying these conditions and the corresponding payoffs are shown in Fig. 2.3a and Fig. 2.3b respectively.

In Fig. 2.3c, we have plotted the inefficiency parameter η as a function of λ. Interestingly, we see that in the range $\lambda_{c1} < \lambda < \lambda_{c2}$, the inefficiency rises as the agents optimize for farther into future. This may appear paradoxical at first, as certainly, the agents could have used strategies corresponding to lower λ. This happens because though the state for larger λ is slightly less efficient overall, in it the majority benefits more, as the difference between W_2^* and W_3^* is decreased substantially (Fig. 2.3b).

We note that the optimal strategies, and hence the (non-equilibrium) steady state of the system shows a non-analytic dependence on λ even for finite N. This is in contrast to the case of systems in thermal equilibrium, where sharp phase transitions can occur only in the limit of infinite number of degrees of freedom. This may be understood by noting that the fully optimizing agents in our model make it more like an equilibrium system at zero-temperature. How-

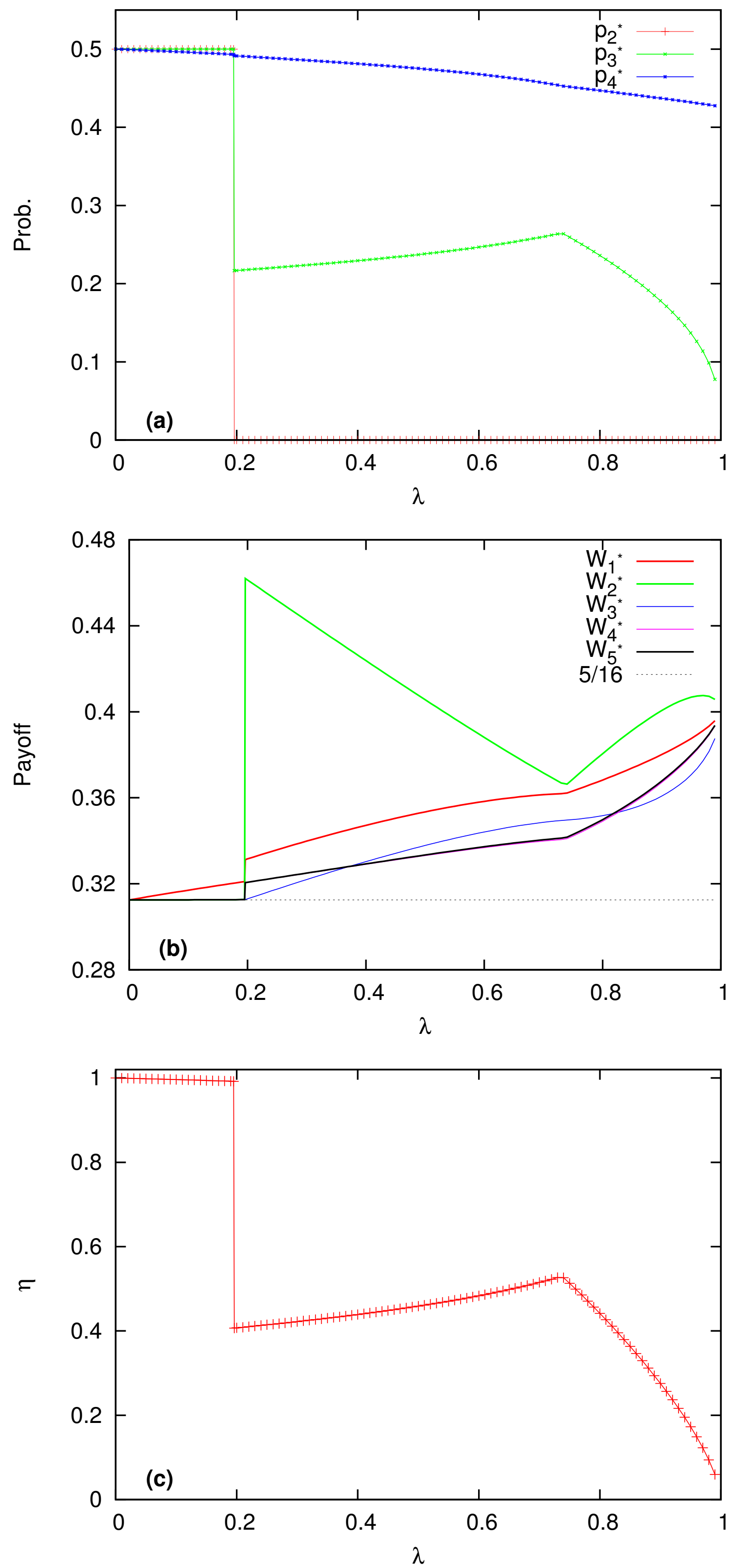

Figure 2.3: $N = 5$: (a) Variation of p_2^*, p_3^* and p_4^* with λ , (b) Optimum payoffs as functions of λ, (c) Inefficiency η as a function of λ.

ever note that unlike the latter, here the system shows a lot of fluctuations in the steady state.

2.2.3 $N = 7$

For higher values of N, the analysis is similar. For the case $N = 7$, we find that there are four thresholds λ_{ci}, with $i = 1$ to 4. For $\lambda < \lambda_{c1}$, the optimal strategy has the form $(0, 1/2, 1/2, 1/2, 1/2, p_6^*, 1/2)$. For $\lambda_{c1} \leq \lambda \leq \lambda_{c2}$, we get $p_3^* = 0$, and $p_4^* < 1/2$. For still higher values $\lambda_{c2} < \lambda \leq \lambda_{c3}$, agents in the states C_2 and C_5 also find it better to switch to a win-stay lose-shift strategy, and we get $p_2^* = 0$, $p_5^* < 1/2$. The transitions at λ_{c3} and λ_{c4} are similar to the second transition for $N = 5$, in the (p_4, p_3) and (p_5, p_2) planes respectively. Numerically, we find $\lambda_{c1} \approx .465, \lambda_{c2} \approx .515, \lambda_{c3} \approx .83$ and $\lambda_{c4} \approx .95$. *The general structure of the optimum strategy is thus clear. As λ is increased, it changes from random switching to a complete win-stay lose-shift strategy in stages.*

We present some graphs for the solution for $N = 7$. Fig. 2.4a shows variation of the optimum switch probabilities in various states and Fig. 2.4b shows the variation of the optimum payoffs. Fig. 2.4c shows the variation of inefficiency with λ.

2.2.4 Higher N

We note that using the symmetry under permutation of agents, we can block diagonalize the transfer matrix $\mathbb{T}$ into two blocks of size M and $M+1$. This is achieved by a change of basis, from vectors $|i\rangle$ and $|N-i\rangle$ to the basis vectors

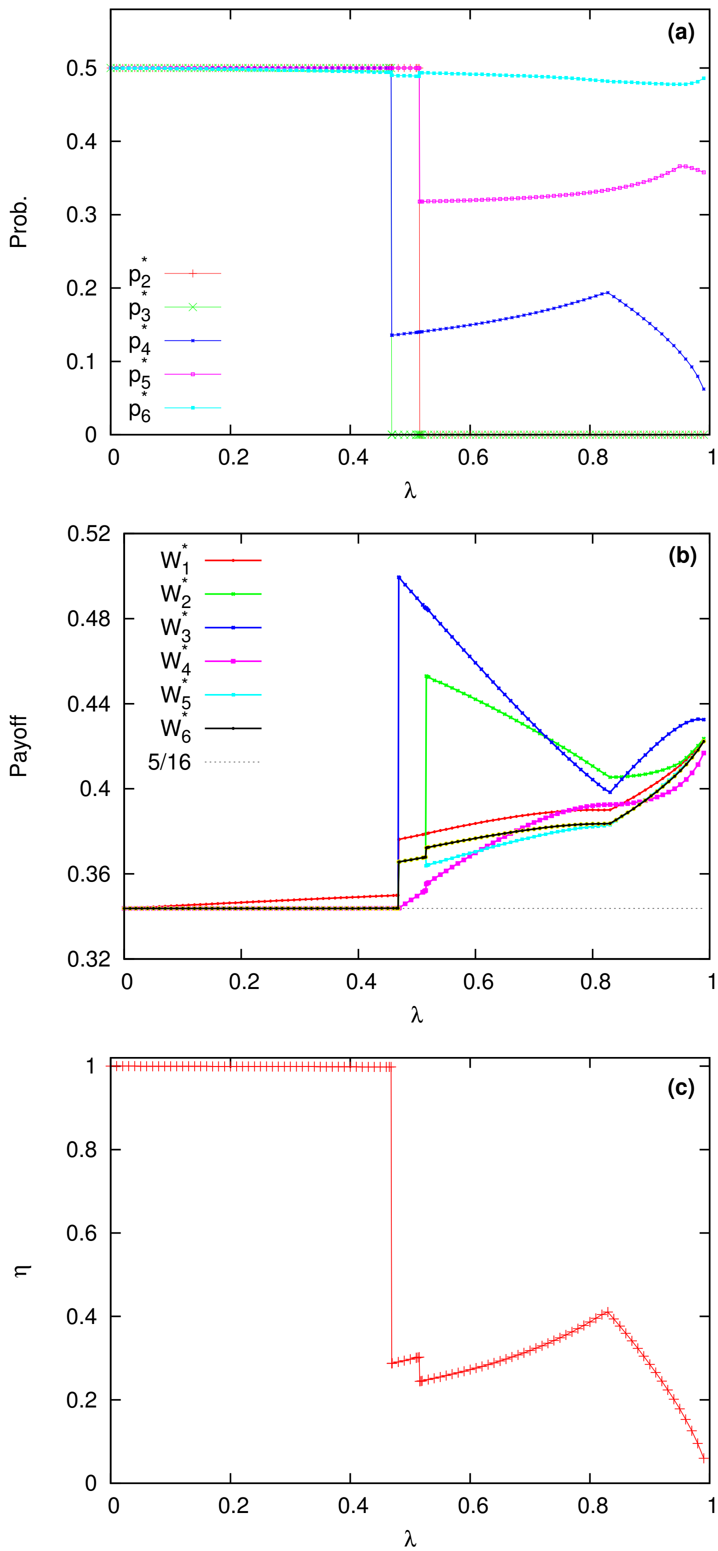

Figure 2.4: $N = 7$: (a) Variation of optimum switch probabilities with λ, (b) Optimum payoffs as functions of λ. Payoff W_7^* is bit less than, but indistinguishable from W_6^* and hence not shown here, (c) Inefficiency η as a function of λ.

$|s_i\rangle$ and $|a_i\rangle$, where

$$|s_i\rangle = |i\rangle + |N-i\rangle,$$
$$|a_i\rangle = (N-i)|i\rangle - i|N-i\rangle. \tag{2.11}$$

This comes from the fact that in the steady state

$$Prob(\text{being in}|i\rangle)/i = Prob(\text{being in}|N-i\rangle)/(N-i),$$

and this property suggests using the basis $|s_i\rangle$ and $|a_i\rangle$ as in Eq. (2.11). An interesting consequence of the symmetry between the two restaurants is the following: If there is a solution $\{p_i^*\}$ of the self-consistent equations, another solution with all payoffs unchanged can be obtained by choosing for any j, a solution $\{p_i^{*\prime}\}$, given by $p_j^{*\prime} = 1 - p_j^*$, and ${p_{N-j}^*}' = 1 - p_{N-j}$, and $p_i' = p_i$, for $i \neq j$ or $(N-j)$. How agents choose between these symmetry related 2^M equilibria can only be decided by local conventions. For example if 'win-stay lose-shift' behavior is 'genetically wired' or culturally accepted, then we have a unique optimum strategy set which is the 'natural' solution to the problem.

2.3 The Large-N limit

In this section, we discuss the transition from the random strategy $\mathbb{P}_{rand}$, with all $p_j = 1/2$, to the strategy $\mathbb{P}_1$, in which with $p_M^* = p_{M+1}^* = 1/2$, and $p_j = 1/2$, for all other j. We will determine the value of $\lambda_{c1}(N)$ where this switch occurs.

The difference between the average payoffs in the strategies $\mathbb{P}_{rand}$ and $\mathbb{P}'_{rand}$ is only of order 2^{-N}, and may be ignored for large N.

In calculating the expected payoffs for strategy $\mathbb{P}_1$, it is convenient to group the states of the system into three groups: $|M\rangle, |M+1\rangle$, and the rest. These will de denoted by $|e_1\rangle, |e_2\rangle$ and $e_3\rangle$ respectively.

The transition matrix $\mathbb{T}$ may be taken as a 3×3 matrix. We consider the case when p_{M+1} is $\mathcal{O}(M^{-5/4})$. Then $\mathbb{T}_{21}$ is $\mathcal{O}(M^{-1/4})$. It is convenient to write $\mathbb{T}_{21} = aM^{-1/4}$, and use a as variational parameter, rather than p_{M+1}. We also write $b = (1-\lambda)M^{3/4}$. We consider the case where a and b are finite, and $\mathcal{O}(1)$. The transition probabilities $\mathbb{T}_{12} = \mathbb{T}_{21} = aM^{-1/4}$, and $\mathbb{T}_{31} = \mathbb{T}_{32} = a^2M^{-1/2}/2$, to leading order in M. Also $\mathbb{T}_{13} = \mathbb{T}_{23}$ is the probability that, when all agents are jumping at random, the marked agent will find himself in the state $|M\rangle$, (equivalently in state $|M+1\rangle$). For large N, this is well-approximated by the Gaussian approximation, and keeping only the leading term, we write $\mathcal{W}_{13} = \mathcal{W}_{23} = cM^{-1/2}$, where $c = 1/\sqrt{\pi}$.

Therefore we can write the transition matrix $\mathbb{T}$, keeping terms only up to $\mathcal{O}(M^{-1/2})$ as,

$$\mathbb{T} = \begin{bmatrix} 1 - aM^{-1/4} - \dfrac{a^2M^{-1/2}}{2} & aM^{-1/4} & cM^{-1/2} \\ aM^{-1/4} & 1 - aM^{-1/4} - \dfrac{a^2M^{-1/2}}{2} & cM^{-1/2} \\ \dfrac{a^2M^{-1/2}}{2} & \dfrac{a^2M^{-1/2}}{2} & 1 - 2cM^{-1/2} \end{bmatrix}. \tag{2.12}$$

Using the symmetry between the states $|e_1\rangle$ and $|e_2\rangle$, it is straight forward to diagonalize $\mathcal{W}$. Let the eigenvalues be μ_i, with $i = 1, 2, 3$, and the corresponding left and right eigenvectors be $\langle L_i|$ and $|R_i\rangle$.

For the steady state eigenvalue $\mu_1 = 1$, we have

$$\langle L_1| = [1, 1, 1]\,; \quad |R_1\rangle = \frac{1}{a^2 + 4c}\begin{bmatrix} 2c \\ 2c \\ a^2 \end{bmatrix}.$$

The second eigenvalue is $\mu_2 = 1 - \frac{a^2+4c}{2}M^{-1/2}$, and we have

$$\langle L_2| = \frac{1}{a^2 + 4c}\left[a^2, a^2, -4c\right]; \quad |R_2\rangle = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \end{bmatrix}.$$

The third eigenvalue is $\mu_3 = 1 - 2aM^{-1/4} - a^2M^{-1/2}/2$, and we have

$$\langle L_3| = [1/2, -1/2, 0]\,; \quad |R_3\rangle = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

It is easily verified that $\langle L_i|R_j\rangle = \delta_{ij}$.

Now, we calculate the expected values of the payoff. We note that if an agent is in the state $|e_3\rangle$, not only her exact state is uncertain, but even her expected payoff depends on whether she reached this state from $|e_3\rangle$ in the previous day, or from $|e_2\rangle$. This is because the expected payoff in this state depends on previous history of agent. However, her expected payoff *next day* depends only on her current state (whether $|e_1\rangle$ or $|e_2\rangle$ or $|e_3\rangle$).

The expected payoff vector for the next day is easily seen to be

$$\left[W_{e_1}^{(0)}, W_{e_2}^{(0)}, W_{e_3}^{(0)}\right] = \left[1 - aM^{-1/4} - a^2M^{-1/2}/2,\ aM^{-1/4} + a^2M^{-1/2}/2,\ 1/2 - dM^{-1/2}\right], \tag{2.13}$$

where $d = 1/(2\sqrt{\pi})$. The expected payoff after n days is given by $\left[W_1^{(0)}, W_2^{(0)}, W_3^{(0)}\right]\mathbb{T}^{n-1}$. Then the discounted expected payoff with parameter λ is given by

$$[W_{e_1}, W_{e_2}, W_{e_3}] = \left[W_{e_1}^{(0)}, W_{e_2}^{(0)}, W_{e_3}^{(0)}\right]\frac{(1-\lambda)}{(1-\lambda\mathbb{T})}. \tag{2.14}$$

We write

$$\mathbb{T} = \sum_{i=1}^{3} |R_i\rangle\mu_i\langle L_i|, \tag{2.15}$$

and hence write

$$[W_{e_1}, W_{e_2}, W_{e_3}] = \sum_{i=1}^{3} U_i\langle L_i|, \tag{2.16}$$

where

$$U_i = \left[W_{e_1}^{(0)}, W_{e_2}^{(0)}, W_{e_3}^{(0)}\right]|R_i\rangle\frac{(1-\lambda)}{(1-\lambda\mu_i)}. \tag{2.17}$$

Now, explicitly evaluate U_i. We see that U_1 is independent of λ, and is the expected payoff in the steady state. The terms involving $M^{-1/4}$ cancel, and we get

$$U_1 = \frac{1}{2} - \frac{da^2}{(a^2+4c)}M^{-1/2}. \tag{2.18}$$

For U_2, we note that $\left[W_{e_1}^{(0)}, W_{e_2}^{(0)}, W_{e_3}^{(0)}\right]|R_2\rangle$ is of order $M^{-1/2}$, and $\frac{(1-\lambda)}{(1-\lambda\mu_2)}$ is of order $M^{-1/4}$, hence this term does not contribute to order $M^{-1/2}$.

The third term is U_3. Here the matrix element $\left[W_{e_1}^{(0)}, W_{e_2}^{(0)}, W_{e_3}^{(0)}\right]|R_3\rangle$ is

$\mathcal{O}(1)$, and $\frac{(1-\lambda)}{(1-\lambda\mu_3)}$ is of $\mathcal{O}(M^{-1/2})$, giving

$$U_3 = (b/2a)M^{-1/2} + \mathcal{O}(M^{-3/4}). \tag{2.19}$$

Putting these together, we get that W_{e_2} is given by

$$W_{e_2} = 1/2 + M^{-1/2}\left[-\frac{b}{4a} - d + \frac{4dc}{a^2+4c}\right] + \mathcal{O}(M^{-3/4}). \tag{2.20}$$

The agents in state $|e_2\rangle$ will choose the value $a = a^*$ to maximize this payoff W_{e_2} with respect to a. Hence we have

$$b = \frac{32a^{*3}dc}{(a^{*2}+4c)^2}. \tag{2.21}$$

For any given b, we can solve this equation for a^*. Then, at this point, the expected payoff W_{e_2} is

$$W_{e_2} = 1/2 - dM^{-1/2}\left[1 - \frac{4c(4c-a^{*2})}{(a^{*2}+4c)^2}\right]. \tag{2.22}$$

This quantity is greater than the expected payoff in the fully random state, so long as $a^{*2} < 4c$, i.e.

$$b < b_{max} = 2\pi^{-3/4}. \tag{2.23}$$

Thus, we see that if $\lambda > 1 - b_{max}M^{-3/4}$, there exists a non-trivial solution $a^*(b)$ satisfying Eq. (2.21), with $(a^*)^2 < 4c$, and then the strategy in which agents in state C_M stay, and C_{M+1} shift with a small probability is beneficial to all. Note that the future time horizon of agents only grows as a sub-linear power of M, while in the large M limit, in standard MG, the time-scales grow (at least) linearly with M.

This large M limit is somewhat subtle, as there are three implicit time scales in the problem: The average time-interval between transitions between the states $|e_1\rangle$ and $|e_2\rangle$ is of $\mathcal{O}(M^{1/4})$ days. Jumps into the state $|e_3\rangle$ occur at time-scales of $\mathcal{O}(M^{1/2})$ days. Once in the state $|e_3\rangle$, the system tends to stay there for a time of $\mathcal{O}(M^{1/2})$ days, before a fluctuation again brings it to the state $|e_1\rangle$ or $|e_2\rangle$. The third time scale of $\mathcal{O}(M^{3/4})$ is the minimum scale of future horizon of agents required if the small per day benefit of a more efficient steady state of $\mathcal{O}(M^{-1/2})$ is to offset the cumulative disadvantage to the agents in state $|e_2\rangle$ of $\mathcal{O}(M^{1/4})$.

Note that the above analysis only determines the critical value of λ above which the strategy $\mathbb{P}_1$ becomes preferred over $\mathbb{P}_{rand}$. This would be the actual critical value of λ if the transition to the win-stay-lose-shift occurs in stages, as is suggested by the small N examples we worked out explicitly. However, we cannot rule out the possibility that for N much larger than 7, the shift does not occur in stages, but in one shot, and such a strategy (similar to the one described in [51]) may be preferred over $\mathbb{P}_{rand}$ at much lower values of λ.

2.4 Summary

In this chapter, we have analyzed a variant of the minority game in which rational agents use stochastic strategies and have a future time horizon. We proposed a new solution concept of co-action equilibrium which take care of the problem of trapping states in a natural way. We determined the optimal choice of probabilities of different actions exactly in terms of simple self-consistent equations. Optimal choice gives higher expected payoff for all agents. The optimal strategy is characterized by N real parameters which depend on the future time-horizon of agents, parametrized by a real variable λ, and are non-analytic functions of λ, even for a finite number of agents. The solution for $N \leq 7$ is worked out explicitly and we indicate the nature of the solution for general N. For large enough future time horizon, the optimal strategy switches from random choice to a win-stay lose-shift strategy, with the shift probability depending on the current state and λ. It shows multiple sharp transitions as a function of the discount parameter λ, even for finite N. The optimal strategy is more efficient than possible under the deterministic MG thus showing that contrary to the popular belief, symmetric rational agents can perform optimally in minority situations.

Generalizations of the model for larger backward time horizon, or when all agents are not identical etc. are easy to define, and appear to be interesting subjects for further study. The technique may be used to study similar games with different payoff functions, e.g. agents win when their restaurant has attendance exactly r.

Chapter 3

Disc percolation with a distribution of radii having a power-law tail

In this chapter, we consider a continuum percolation model of overlapping discs in two dimensions (2D) where distribution of the radii of the discs has a power-law tail. We address questions like whether the power-law tail in the distribution of radii changes the critical behavior of the system, and how does the percolation threshold depend on the power of the power-law tail. The power-law distribution of the radii makes this system similar to the Ising or fluid system with long-range interactions. For the latter case, it is known that the long-range nature of the interaction does affect the critical behavior of the system for slow enough decay of the interaction [82].

The behavior of our model differs from that of the standard continuum percolation model in two aspects. First, the entire low density regime in our model shows a power-law decay of the two-point correlation function in contrast to the exponential decay in the standard continuum percolation. Thus the whole low density regime is 'critical'. However, there is a non-zero percolation threshold below which there is no infinite cluster exist in the system. Second, the critical exponents are functions of the power a of the power-law distribution for low enough a. So while the system belong to the same universality class as the standard continuum percolation for high enough a, the critical behavior is quite different for low values of a.

The chapter is organized as follows: In Sec. 3.1, we review some earlier work for the continuum percolation problem of overlapping discs. In Sec. 3.2, we define our model of disc percolation precisely. In Sec. 3.3, using a rigorous lower bound on the two-point correlation function, we show it decays only as a power-law with distance for arbitrarily low coverage densities. In Sec. 3.4,we discuss the critical behavior of the system. In Sec. 3.5, we propose an approximate renormalization scheme to calculate the correlation length exponent ν and the percolation threshold in such models. In Sec. 3.6, we discuss results from simulation studies and Sec. 3.7 contains summary.

3.1 Some earlier results

3.1.1 Percolation with single sized discs

The simplest continuum percolation model one can think of is the one in which percolating units are overlapping discs of same radius say R [41]. In such a model let p_0 be the probability that a point O in the plane is not covered by any of the discs. This is same as the probability that there are no centers of

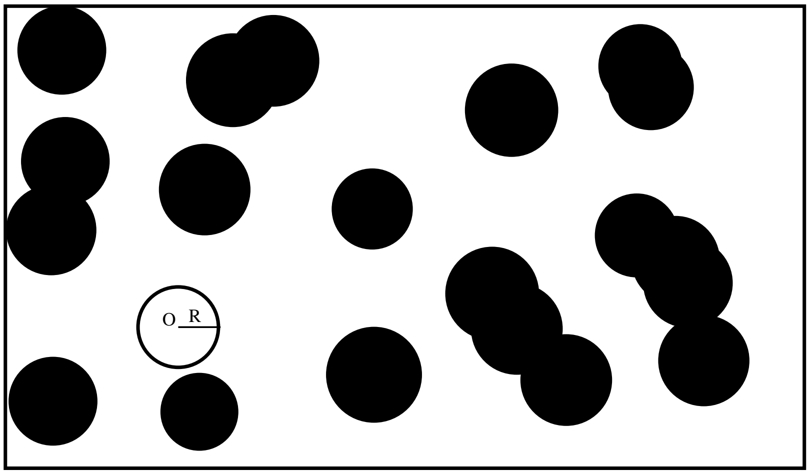

Figure 3.1: A sub critical percolation system where percolating units are discs of same size R. The point O of the plane will not be covered by any of the discs, if a circular area of radius R centered at O do not contain center of any of the discs.

any discs present within a circular area of radius R (See Fig. 3.1). Then, for an infinite system,

$$p_0 = \exp\left(-n\pi R^2\right). \tag{3.1}$$

The quantity $A = n\pi R^2$ is called the areal density which is the average total area of the discs whose centers lie within unit area of the plane. p_0 also gives the fraction of the entire plane not covered by any of the discs. Therefore the fraction of the entire plane which is covered by at least one disc (called as the covered area fraction to be denoted by $f_{covered}$) will be,

$$f_{covered} = 1 - p_0 = 1 - \exp(-A). \tag{3.2}$$

The critical areal density A^* and the critical covered area fraction are defined respectively as $n^*\pi R^2$ and $1 - \exp(-n^*\pi R^2)$ where n^* is the critical density which is the minimum number density of discs for which the system percolates.

Since areal density $A = n\pi R^2$ remains invariant under length rescaling, the critical areal density A^* for percolation problem with discs of single size is independent of the size of the discs. So if R_1 and R_2 are some fixed reals and $R_1 \neq R_2$,

$$n^*(R_1)\pi R_1^2 = n^*(R_2)\pi R_2^2 = A^*. \tag{3.3}$$

Even though an analytical determination of A^* has not yet made possible, simulation studies yield fairly accurate results; the best up to date being $A^* = 1.128085$ [83]. This corresponds to a critical covered area fraction, $f_{covered} = 1 - \exp(-A^*) = 0.6763475(5)$.

We define the two-point correlation function $\text{Prob}(1 \rightsquigarrow 2)$ as the probability that points P_1 and P_2 in the plane which are at a distance r_{12} from each other are connected by overlapping discs. For a sub critical system, $\text{Prob}(1 \rightsquigarrow 2)$ varies with r_{12} as

$$\text{Prob}(1 \rightsquigarrow 2) \sim \exp\left(-r_{12}/\xi\right),$$

where ξ is the correlation length [84]. Since probability should remain invariant under length rescaling, we must have $\xi(A) = R\, g(A)$ where $g(A)$ determines how the correlation length varies with areal density A and is independent of R. Near the critical areal density A^*, it is expected to vary as $g(A) \sim (A^* - A)^{-\nu}$ with $\nu = 4/3$ [38].

3.1.2 Percolation with discs having a distribution of radii

If we have a distribution $\rho(R)$ for the radii variable R, then Eqs. (3.1) and (3.2) generalize into

$$p_0 = \exp\left(-n\pi \left\langle R^2 \right\rangle\right), \tag{3.4}$$

where $\langle R^2 \rangle = \int R^2 \rho(r) dr$ and

$$f_{covered} = 1 - \exp(-A), \tag{3.5}$$

where areal density $A = n\pi \langle R^2 \rangle$.

Note that if $\langle R^2 \rangle$ diverges, then $f_{covered} = 1$ and for every $n > 0$, entire plane will be covered in the thermodynamic limit. So one will always consider the case where $\langle R^2 \rangle$ is finite. The critical areal density A^* and the critical covered area fraction $f_{covered}$ are defined as in the percolation with single sized disc but with $\langle R^2 \rangle$ in place of R^2.

The constancy of A^* for the percolation with single sized discs suggested the conjecture [85] that for all random variables R with bounded support, A^* is a constant independent of $\rho(R)$. For earlier references to this 'constant volume fraction rule' see [86, 87]. However Phani and Dhar in [88] argued that for a percolation problem with variable disc size, this conjecture do not hold. Later these arguments were made rigorous by Meester *et al.* [89] who showed that the critical areal density for a percolation problem with a distribution of radii is greater than or equal to that when the radii takes only single value. So the threshold indeed depends on size distribution of the basic percolating units.

3.2 Definition of the model

We consider a continuum percolation model of overlapping discs in two dimensions. The number density of discs is n, and the probability that any small area element dA has the center of a disc in it is ndA, independent of all other area elements. For each disc, we assign a radius, independently of other discs, from a probability distribution $Prob(R)$. We consider the case when $Prob(R)$ has a power-law tail; the probability of radius being greater than R varies as R^{-a} for large R. For simplicity, we consider the case when radii take only discrete values $R_0\Lambda^j$ where $j = 0, 1, 2, ...$, with probabilities $(1-p)p^j$ where $p = \Lambda^{-a}$. Here R_0 is the size of smallest disc, and Λ is a constant > 1. We refer to the disc of size $R_0\Lambda^j$ as the disc of type j.

It is easy to see that the covered area fraction $f_{covered}$, is finite only for $a > 2$. For $a \leq 2$, in the thermodynamic limit all points of the plane are eventually covered, and $f_{covered} = 1$. If $a > 2$, we have areal density,

$$A = n\pi R_0^2(1-p)/(1-p\Lambda^2). \tag{3.6}$$

We define the percolation probability P_∞ as the probability that a randomly chosen disc belongs to an infinite cluster of overlapping discs. One expects that there is a critical number density n^* such that for $n < n^*$, P_∞ is exactly zero, but $\mathrm{P}_\infty > 0$, for $n > n^*$. We shall call the phase $n < n^*$ the non-percolating phase, and the phase $n > n^*$ as the percolating phase.

It is easy to show that $n^* < \infty$. We note that for percolation of discs where all discs have the same size R_0, there is a finite critical number density n_1^*, such that for $n > n_1^*$, $\mathrm{P}_\infty > 0$. Then, for the polydisperse case, where all discs have

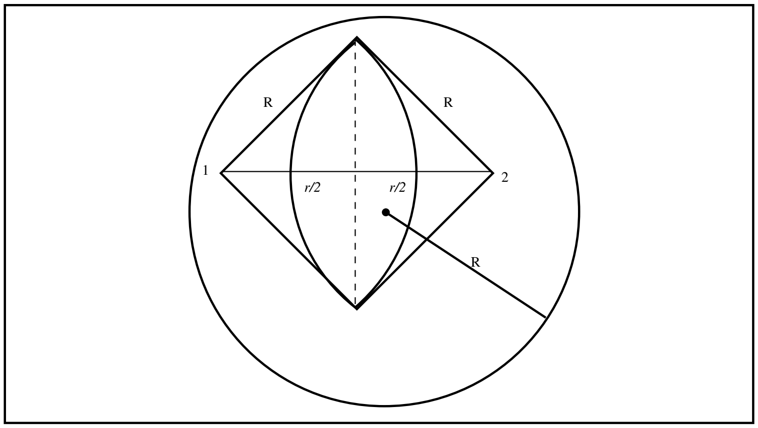

Figure 3.2: Points 1 and 2 in the plane at a distance r from each other will be covered by a single disc of radius R, if the center of such a disc falls in the area of intersection of two circles with radius R and centers at 1 and 2.

radii R_0 or larger, the percolation probability can only increase, and hence $n^* < n_1^*$. Also as noted earlier, whenever we have a bounded distribution of radii of the discs, the critical areal density is greater than that for a system with single sized discs [89]. Our simulation results show that this remains valid for unbounded distribution of radii of the discs.

3.3 Non-percolating phase

By rotational invariance of the problem, $\mathrm{Prob}(1 \rightsquigarrow 2)$ is only a function of the euclidean distance r_{12} between the two points. Let $\mathrm{Prob}^{(1)}(1 \rightsquigarrow 2)$ denote the probability that there is at least one disc that covers both P_1 and P_2. Then, clearly,

$$\mathrm{Prob}(1 \rightsquigarrow 2) \geq \mathrm{Prob}^{(1)}(1 \rightsquigarrow 2). \tag{3.7}$$

It is straightforward to estimate $\mathrm{Prob}^{(1)}(1 \rightsquigarrow 2)$ for our model. Let j be the minimum number such that radius of disc of type j is greater than or equal to

r_{12}, i.e. $R_0\Lambda^j \geq r_{12}$. Let S be the region of plane such that the distance of any point in S from P_1 or P_2 is less than or equal to $R_0\Lambda^j$. This region S is greater than or equal to the region where each point is within a distance r_{12} from both P_1 and P_2. Using elementary geometry, the area of region S is greater than or equal to $(2\pi/3 - \sqrt{3}/4)r_{12}^2$ (See Fig. 3.2). The number density of discs with radius greater than or equal to $R_0\Lambda^j$ is $n\Lambda^{-aj}$. Therefore, the probability that there is at least one such disc in the region S is $1 - \exp\left(-n|S|\Lambda^{-aj}\right)$, where $|S|$ is the area of region S. Thus we get,

$$\mathrm{Prob}^{(1)}(1 \rightsquigarrow 2) \geq 1 - \exp\left[-nK\Lambda^{-aj}r_{12}^2\right], \tag{3.8}$$

where $K = 2\pi/3 - \sqrt{3}/4$.

Now, as assumed, $R_0\Lambda^j < r_{12}\Lambda$. Hence we have $\Lambda^{-aj} > r_{12}^{-a}\Lambda^{-a}/R_0^{-a}$. Putting this in Eq. (3.8), we get

$$\mathrm{Prob}^{(1)}(1 \rightsquigarrow 2) \geq 1 - \exp\left[-nK\Lambda^{-a}r_{12}^{-a+2}\right], \tag{3.9}$$

where some constant factors have been absorbed into K. For large r_{12}, it is easy to see that this varies as r_{12}^{2-a}. Hence the two-point correlation function is bounded from below by a power-law.

We can extend this calculation, and write the two-point correlation function as an expansion

$$\mathrm{Prob}(1 \rightsquigarrow 2) = \sum_{n=1}^{\infty} \mathrm{Prob}^{(n)}(1 \rightsquigarrow 2), \tag{3.10}$$

where $\mathrm{Prob}^{(n)}(1 \rightsquigarrow 2)$ is the probability that the path of overlapping discs connecting points P_1 and P_2 requires at least n discs. The term $n = 2$ corresponds to a more complicated integral over two overlapping discs. But it is easy to see that for large r_{12}, this also decays as r_{12}^{-a+2}. Assuming that similar behavior holds for higher order terms as well, we expect that the two-point correlation function decays as a power law even for arbitrarily low densities of discs.

We note that this is consistent with the result that for continuum percolation in d dimensions, the diameter of the connected component containing the origin say $\langle D \rangle$ is divergent even for arbitrarily small number densities when $\langle R^{d+1} \rangle$ is divergent [47]. Here R denote the radii variable. In our case $\langle D \rangle = \int r_{12} \frac{dProb(r_{12})}{dr_{12}} dr_{12} \sim \int r_{12}^{2-a} dr_{12}$ (where P_1 is the origin) is divergent when $a \leq 3$, consistent with the above.

3.4 Critical behavior

The power-law decay of the two-point correlation function is the result of the fact that for any distance r, we have discs of radii of the order of r. However for large values of r, we can imagine that there would also be a contribution from a large number of overlapping discs of radii much smaller than r connecting the two points separated by the distance r, which as in the usual percolation problem decays exponentially with distance. Therefore it is reasonable to write the two-point correlation function in our problem as a sum of two parts; the first part say $G_{sr}(r)$ due to the 'short range' connections which has an exponential decay with distance for large r and the second one say $G_{lr}(r)$ due to the 'long range' connections which has a power law decay with distance.

Therefore,

$$G(r) = G_{sr}(r) + G_{lr}(r), \tag{3.11}$$

where

$$G_{lr}(r) \sim D(A)/r^{a-2} + higher\ order\ terms, \tag{3.12}$$

where $D(A)$ is assumed to go to a non-zero constant as $A \to A^*$ and its dependence on A is a slowly varying one.

The power-law distribution of the radii, makes this system similar to a long range interaction problem in statistical physics in the sense that given two points in the plane, a direct connection by a single disc overlapping both the points is possible. In fact similar behavior for the two-point correlation function exists whenever we have long range interactions in a system, such as in Ising model with long range potentials or fluid with long range interactions [90, 91]. In such systems, the two-point correlation function shows a power-law decay just as in our problem [92]. The effect of such long range potentials on the critical exponents have been studied earlier [93, 94, 95, 82, 90, 96] with the general conclusion that the long range part of the interaction can influence the critical behavior of the system [97]. More precisely, if we have an attractive pair potential in d dimensions of the form $-\phi(r) \sim \dfrac{1}{r^{d+\ sigma}}$ where $\sigma > 0$, then critical exponents take their short-range values for all $\sigma \geq 2 - \eta_{sr}$ where η_{sr} is the anomalous dimension. For $\sigma < 2 - \eta_{sr}$, two kinds of behavior exist. For $0 < \sigma \leq d/2$, the exponents take their mean-field values and for $d/2 < \sigma < 2 - \eta_{sr}$, the exponents depend on the value of σ (See [82] and references therein). So $\sigma = 2 - \eta_{sr}$ is the dividing line between the region dominated by short range interactions and the region dominated by long-range interactions.

Though there is a well established connection between the lattice percolation problem and the Ising model [98], there is no similar result connecting the continuum percolation problem to any simple Hamiltonian system. However, the following simple argument provide us with a prediction about the values of the parameter a for which the power-law nature of the distribution is irrelevant and the system is similar to a continuum percolation system with a bounded size distribution for the percolating units. Assuming that the strength of the long range interaction from a given point in the Ising/fluid system (which decays like $\sim \dfrac{1}{r^{2+\sigma}}$ in 2D) is like the strength of the connectivity from the center of a given disc which is given by the distribution of the radii; in our problem, we expect the dividing line between the region dominated by short-range connectivity and the region dominated by long-range connectivity to be the same as that for an Ising system with long range potential of the form $-\phi(r) \sim \dfrac{1}{r^{a+1}}$ where $a > 2$. Then the results for the long-range Ising system discussed in the last paragraph should carry over with $\sigma = a - 1$. So a deviation from the standard critical behavior is expected when $a < 3 - \eta_{sr}$ and the critical exponents will take their short-range values for $a > 3 - \eta_{sr}$. For 2D percolation, $\eta_{sr} = 5/24$ [99]. Also mean-field behavior is expected when $a \leq 2$. However for this range of a, the entire plane is covered for all non-zero number densities and hence there is no phase transition.

In the next two sections, we investigate for the dependence of exponents on the power-law tail of the distribution of the radii of the discs. First we develop an approximate RG method. Then we carry out simulation studies which show that the correlation length exponent ν takes its short range value for $a > 3 - \eta_{sr}$, while it depends upon a for $a < 3 - \eta_{sr}$.

3.5 An approximate RG scheme

In this section, we propose an approximate RG method to analyze the behavior of continuum percolation models near the percolation threshold, when the percolating units have a distribution of sizes. We assume that we can replace discs of one size having a number density n with discs of another size and number density n', provided the correlation length remains the same. Application of a similar idea in disc percolation problem with only two sizes of discs may be found in [100].

We will illustrate the method by considering a problem in which the radii of discs take only two possible values, say R_1 and R_2. Let their areal densities be A_1 and A_2 respectively, and assume that both A_1 and A_2 are below A^*, the critical threshold for the percolation problem with only single sized discs present ($A^* \approx 1.128085$ [44]). Also let ξ_1 represent the correlation length when only discs of size R_1 are present in the system and ξ_2 represent that when only discs of size R_2 are present. Invariance of the two-point correlation function under length rescaling requires that the expression for the correlation length ξ is of the form $\xi = Rg(A)$, where the function $g(A)$ determines how the correlation length depends on the areal density A and is independent of the radius R. Let $\tilde{A}_2$ is the areal density of the discs of size R_2 which will give the same correlation length as the discs of size R_1. i.e.,

$$\xi_1 (A_1) = \xi_2 \left(\tilde{A}_2\right), \tag{3.13}$$

or

$$R_1 g\left(A_1\right) = R_2 g\left(\tilde{A}_2\right). \tag{3.14}$$

Given the form of the function $g(A)$, we can invert the above equation to find $\tilde{A}_2$. Formally,

$$\tilde{A}_2 = g^{-1}\left(\frac{R_1}{R_2} g\left(A_1\right)\right). \tag{3.15}$$

So the problem is reduced to one in which only discs of size R_2 are present, whose net areal density is now given by,

$$A_2' = \tilde{A}_2 + A_2. \tag{3.16}$$

System percolates when $A_2' = A^*$. Now, when areal density A is close to A^*, we have

$$g(A) = K\left(A^* - A\right)^{-\nu}. \tag{3.17}$$

where K is some constant independent of A and ν is the correlation-length exponent in the usual percolation problem. Using this in Eq. (3.15), we get

$$\tilde{A}_2 = A^* - \left(A^* - A_1\right)\left(R_2/R_1\right)^{1/\nu}. \tag{3.18}$$

Therefore, for a given value of $A_1 < A^*$, the areal density of discs of radius R_2, so that the system becomes critical is given by,

$$\begin{aligned} A_2 &= A^* - \tilde{A}_2, \\ &= \left(A^* - A_1\right)\left(R_2/R_1\right)^{1/\nu}. \end{aligned} \tag{3.19}$$

So the total areal density at the percolation threshold is,

$$A_1 + A_2 = A_1 + (A^* - A_1)\,(R_2/R_1)^{1/\nu}\,,$$
$$= A_1(1-x) + A^* x.$$

where $x = (R_2/R_1)^{1/\nu}$. Without loss of generality we may assume $R_2 > R_1$. Then $x > 1$ and we can see from the above expression that the percolation threshold $A_1 + A_2 > A^*$, a result well known from both theoretical studies [89] and simulation studies [44].

Now in our problem assume that areal density of discs of type 0 do not exceed A^*. Renormalizing discs up to type m in our problem gives the equation for the effective areal density of the m-th type discs A'_m as

$$A'_m = A^* - \left(A^* - A'_{m-1}\right)\Lambda^{1/\nu} + \rho_m, \tag{3.20}$$

where $m \geq 1$, $A'_0 = \rho_0$ and $\rho_m = n_0 \pi \Lambda^{(2-a)m}$ denote the areal density of discs of radius Λ^m. Here n_0 is the number density of discs of radius R_0 (or of type 0), which for convenience we have set equal to unity. If we denote $A^* - A'_m$ by ε_m which is the distance from the criticality after m-th step of the renormalization, then the above expression becomes

$$\varepsilon_m = \varepsilon_{m-1}\Lambda^{1/\nu} - \rho_m. \tag{3.21}$$

The equation describes the flow near the critical point when we start with a value of ρ_0, the areal density of the first type of discs. Here ε_m gives the

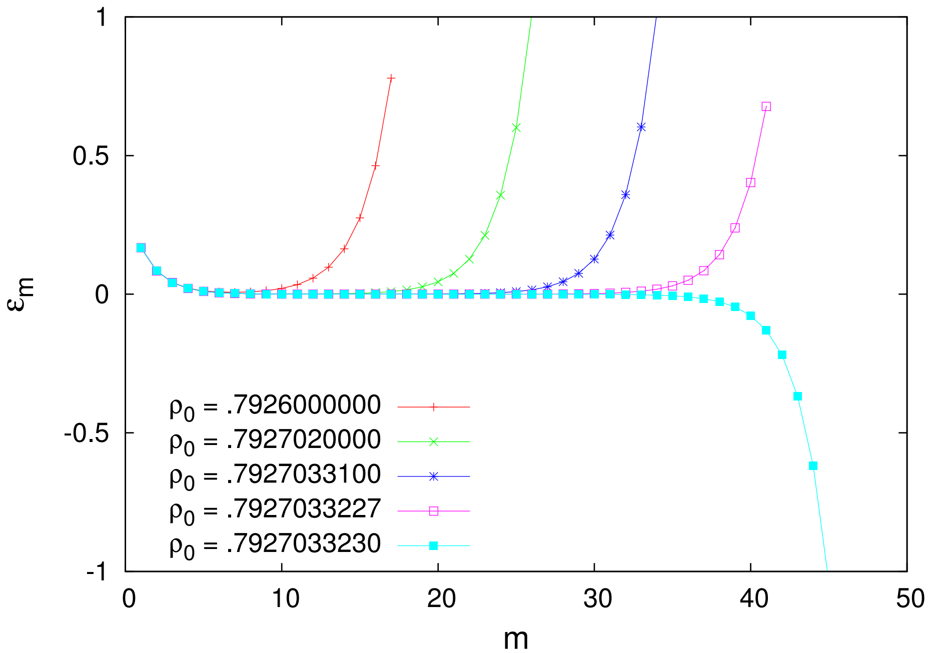

Figure 3.3: Variation of ε_m with m for different values of ρ_0 showing sub critical and supercritical regimes. We have used $a = 3$ and $\Lambda = 2$.

effective distance from criticality of the m-th order discs in the system, in which now only m-th and higher order discs are present. Now for given values of the parameters a and Λ, we can evaluate ε_m in Eq. (3.21) using a computer program and plot ε_m versus m.

Depending upon the value of ρ_0, we get three different behaviors. For value of ρ_0 below the critical value denoted by ρ_0^*, ε_m will go to A^* asymptotically (System is sub critical) and when it is above ρ_0^*, ε_m will go to $-\infty$ asymptotically (System is super critical). As $\rho_0 \to \rho_0^*$, we get the critical behavior characterized by ε_m tending to the RG fixed point 0 asymptotically. Typical result using Eq. (3.21) with $\Lambda = 2$ and $a = 3$ is shown in Fig. 3.3. We can see that as we tune ρ_0, the system approaches criticality, staying closer to the $\varepsilon_m = 0$ line longer and longer. Critical behavior here can be characterized by the value of m at which the curve deviates from the approach to $\varepsilon_m = 0$ line. To understand how the correlation length diverges as we approach criticality, we assume that we can replace the sub critical system with a system where only discs of type m' is present and has a fixed areal density below A^*, where m' is the value of m at which ε_m shows a substantial increase - say ε_m becomes $A^*/2$. For continuum percolation problem with single sized discs, the correlation length $\xi = Rg(A)$, where $g(A)$ is a function with no explicit dependence on radius R. Therefore, correlation length in our problem,

$$\xi \propto \Lambda^{m'}. \tag{3.22}$$

We can write the recurrence relation Eq.(3.21) in terms of the areal density ρ_n as

$$\varepsilon_m = A^* \Lambda^{\frac{m}{\nu}} - \sum_{n=0}^{m} \rho_n \Lambda^{\left[\frac{m-n}{\nu}\right]}. \tag{3.23}$$

But $\rho_n = \rho_0 \Lambda^{n(2-a)}$. Therefore,

$$\varepsilon_m = A^* \Lambda^{\left[\frac{m}{\nu}\right]} - \frac{\rho_0 \Lambda^{\left[\frac{m}{\nu}\right]} \left[1 - \Lambda^{m(2-a-1/\nu)}\right]}{\left[1 - \Lambda^{(2-a-1/\nu)}\right]}. \tag{3.24}$$

For large values of m, the last term in the above equation involving $\Lambda^{m(2-a-1/\nu)}$ can be neglected. Then,

$$\varepsilon_m = \Lambda^{\left[\frac{m}{\nu}\right]} \left[A^* - \frac{\rho_0}{1 - \Lambda^{(2-a-1/\nu)}}\right]. \tag{3.25}$$

Therefore,

$$\Lambda^{\left[\frac{m}{\nu}\right]} = \frac{\varepsilon_m}{\left[A^* - \frac{\rho_0}{1 - \Lambda^{(2-a-1/\nu)}}\right]}. \tag{3.26}$$

For a given value of $\rho_0 \leq A^*$, the order m' at which ε_m is increased substantially, say to a value $A^*/2$ is given by

$$\begin{aligned} m' = [&\log_\Lambda (A^*/2) - \log_\Lambda (\rho_0^* - \rho_0) \\ &+ \log_\Lambda \left(1 - \Lambda^{(2-a-1/\nu)}\right)]\nu \end{aligned} \tag{3.27}$$

So for ρ_0 close to ρ_0^* and large values of a,

$$m' \sim \log_\Lambda (\rho_0^* - \rho_0)^{-\nu} . \tag{3.28}$$

so that

$$\xi \propto (\rho_0^* - \rho_0)^{-\nu} . \tag{3.29}$$

Thus we find that the correlation length exponent ν is independent of the parameters a and Λ of the distribution. From Eq. (3.26), we can also obtain the percolation threshold ρ_0^* as a function of the parameters a and Λ. In Eq. (3.26) left hand side is positive definite. So for values of ρ_0 for which $\frac{\rho_0}{1-\Lambda^{(2-a-1/\nu)}} < A^*$, we will have $\varepsilon_m > 0$ for large values of m. Similarly for values of ρ_0 for which $\frac{\rho_0}{1-\Lambda^{(2-a-1/\nu)}} > A^*$, we will have $\varepsilon_m < 0$ for large values of m. Hence the critical areal density ρ_0^* must be given by

$$\rho_0^* = A^* \left[1 - \Lambda^{(2-a-1/\nu)}\right] . \tag{3.30}$$

Or in terms of the total number density, the percolation threshold n^* is given by,

$$n^* = n_c \left(1 - \Lambda^{(2-a-1/\nu)}\right) / \left(1 - \Lambda^{-a}\right) , \tag{3.31}$$

where $n_c = A^*/\pi$, the critical number density for percolation with single sized discs of unit radius. Note that this approximate result does not give the correct limit, $n^* \to 0$ as $a \to 2$. The RG scheme depends on the approximation that the effect of size R_1 of areal density A_1 is the same as that of discs of radius R_2 of density A_2, as in Eq. (3.13). This is apparently good only for $a > 3-\eta_{sr}$.Fig. 3.4 shows the variation of the critical threshold with a for two different values of Λ using Eq. (3.31) along with simulation results (See section 3.6 for details of simulation studies). We see that a reasonable agreement is obtained between the two. As one would expect, for large values of a, n^* tends to n_c.

From Eq. (3.31), we can obtain the asymptotic behavior of the critical number density n^* as $\Lambda \to 1$. This is useful since it corresponds to the threshold for a continuous distribution of radii with a power-law tail and we no more have to consider the additional discretization parameter Λ. It is easy to see that in the limit $\Lambda \to 1$, Eq. (3.31) becomes

$$n^*_{\Lambda\to 1} = n_c \left(1 - \frac{5}{4a}\right), \tag{3.32}$$

where we have used the value $\nu = 4/3$. Therefore we expect that for large values of a, a log-log plot of $(n_c - n^*_{\Lambda\to 1})$ against a will be a straight line with slope -1 and y-intercept $\ln(5n_c/4) \approx -0.35$ for large values of a. A comparison with the thresholds obtained from simulation studies show that Eq. (3.32) indeed predicts the asymptotic behavior correctly (see Fig. 3.8).

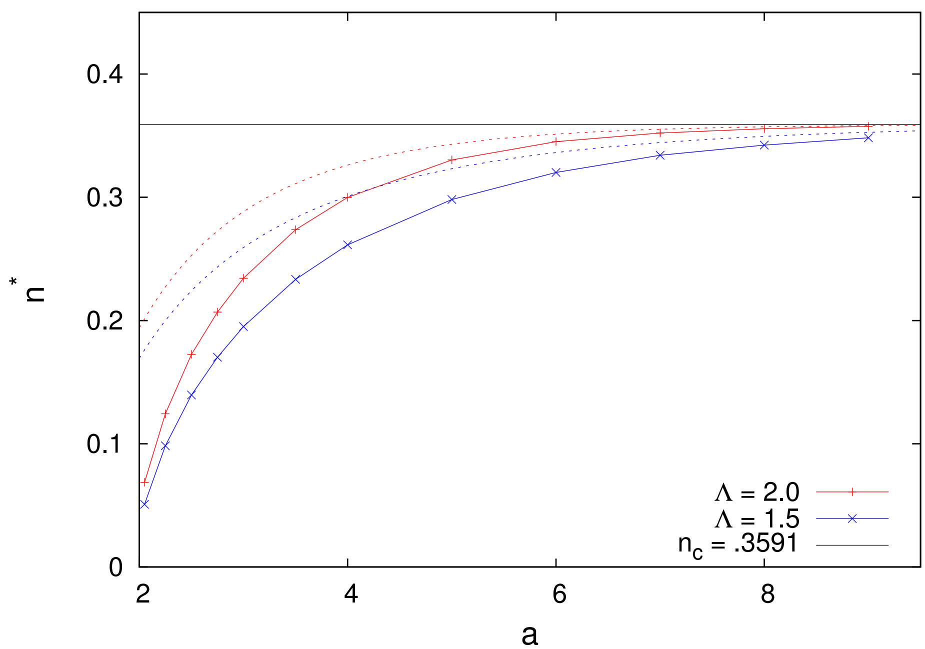

Figure 3.4: Variation of n^* with a for two different values of Λ. Dashed curves correspond to values given by Eq. (3.31) and continuous ones correspond to those from simulation studies. The horizontal line corresponds to the threshold for the single sized discs case.

3.6 Simulation results

We determine the exponent ν and the percolation threshold n^* by simulating the continuum percolation system in 2D, with discs having a power law distribution for their radii. We consider two cases for the distribution of the radii variable. To explicitly compare the prediction of the approximate RG scheme for the percolation threshold given in Sec. 3.5, we use a discrete distribution for the radii variable, with discretization factor Λ as in section 3.2. The results for the thresholds thus obtained is shown in Fig. 3.4. To determine the correlation length exponent ν, we consider the radii distribution in the limiting case $\Lambda \to 1$, so that we do not have to consider the additional parameter Λ. In this case, given a disc, the probability that it has a radius between R and $R+dR$ is equal to $aR^{-(a+1)}$ where $a > 2$. We also obtain the percolation threshold with this continuous distribution for the radii and compare it with the predicted asymptotic behavior in Eq. (3.32). The minimum radius is assumed to be unity.

For $a \leq 2$ the entire plane is covered for arbitrarily low densities of the discs. We use cyclic boundary conditions and consider the system as percolating whenever it has a path through the discs from the left to the right boundary. We drop discs one at a time on to a region of a plane of size $L \times L$, each time checking whether the system has formed a spanning cluster or not. Thus number density is increased in steps of $1/L^2$. So after dropping the $n-th$ disc, the number density is n/L^2. Now associated with each number density we have a counter say f_n which is initialized to 0 in the beginning. If the system is found to span after dropping the n'-th disc, then all counters for $n \geq n'$ is incremented by one. After a spanning cluster is formed, we stop. By this way we can determine the spanning probability $\Pi(n, L) = f_n/N$ where N is the number of realizations sampled. The number of realizations sampled varies from a maximum of 2.75×10^7 for $a = 2.05$ and $L = 90$ to a minimum of 4000 for $a = 10.0$ and $L = 1020$ [For obtaining the results for the threshold in Fig. 3.4, the number of realizations sampled is 20000 for all values of a and Λ]. This method of dropping basic percolating units one by one until the spanning cluster is formed has been used before [101] in the context of stick percolation which was based on the algorithm developed in [102], and allows us to study relatively large system sizes with large number of realizations within reasonable time.

The probability that there is at least a single disc which span the system of size L at number density n is $1-\left(\exp^{-n2^a}\right)/L^{a-2}$. It is easy to see that to leading order in n, this 'long range 'part of the spanning probability $\Pi(n,L)_{lr}$ is $\dfrac{n2^a}{L^{a-2}}$. So one can write a scaling form for the spanning probability,

$$\Pi(n,L)=\Pi(n,L)_{lr}+(1-\Pi(n,L)_{lr})\phi((n^*-n)L^{1/\nu}). \tag{3.33}$$

Therefore we can define the 'short range 'part of the spanning probability $\Pi'(n,L)=(\Pi(n,L)-\Pi(n,L)_{lr})/(1-\Pi(n,L)_{lr})$ where the leading long range part is subtracted out. Therefore, we have

$$\Pi'(n,L)=\phi((n^*-n)L^{1/\nu}), \tag{3.34}$$

and the scaling relations, (See for e.g.. [32])

$$\Delta(L)\propto L^{-1/\nu}, \tag{3.35}$$

$$n^*_{eff}(L)-n^*\propto\Delta, \tag{3.36}$$

where $n^*_{eff}(L)$ is a suitable defined effective percolation threshold for the system of size L, and Δ is the width of the percolation transition obtained from the spanning probability curves $\Pi'(n,L)$. Note that Eqs. (3.35) and (3.36) are applicable with any consistent definition of the effective percolation threshold and width Δ [32]. A good way to obtain n^*_{eff} and Δ is to fit the sigmoidal shaped curves of the spanning probability $\Pi'(n,L)$ with the function $1/2[1+erf[(n-n^*_{eff}(L))/\Delta(L)]]$ (see [45]), which defines the effective percolation threshold n^*_{eff} as the number density at which the spanning probability is 1/2. We determined n^*_{eff} and Δ for each value of a and L and determined $1/\nu$ and n^* for different values of a using Eqs. (3.35) and (3.36) respectively. Typical examples are shown in Fig. 3.5 and Fig. 3.6.

At first, we determined the percolation threshold and the exponent for a system of single sized discs of unit radius. We obtained $n^* = 0.3589(\pm 0.0001)$ (or areal density ≈ 1.12752) and $1/\nu = 0.758(\pm 0.018)$ in very good agreement with the known value for the threshold [44] and the conjectured value of $1/\nu = 3/4$ for the exponent. Values of $1/\nu$ obtained for various values of a are shown in fig.3.7. We scan the low a regime more closely for any variation from the standard answer. We can see that the estimates for $1/\nu$ are very much in line with the standard percolation value for $a > 3 - \eta_{sr}$ while it varies with a for $a < 3 - \eta_{sr}$. Fig. 3.8 shows the variation of the percolation threshold n^* with a. As expected, with increasing a, the percolation threshold increases and tends to the single sized disc value as $a \to \infty$, and as $a \to 2$, the threshold tends to zero. The data also shows that n^* converges to the threshold for the single sized disc value as $1/a$ as predicted by Eq. (3.32). Values of the threshold for some values of a are given in Table 3.1.

Finally as a check, we plot the spanning probability $\Pi'(n, L)$ (see Eq. (3.34)) against $(n - n^*)L^{1/\nu}$ to be sure that a good scaling collapse is obtained. We show two such plots for $a = 2.50$ and $a = 4$ in fig. 3.9 and Fig. 3.10. We can see that a very good collapse is obtained. Similar good collapse is obtained for other values of a as well.

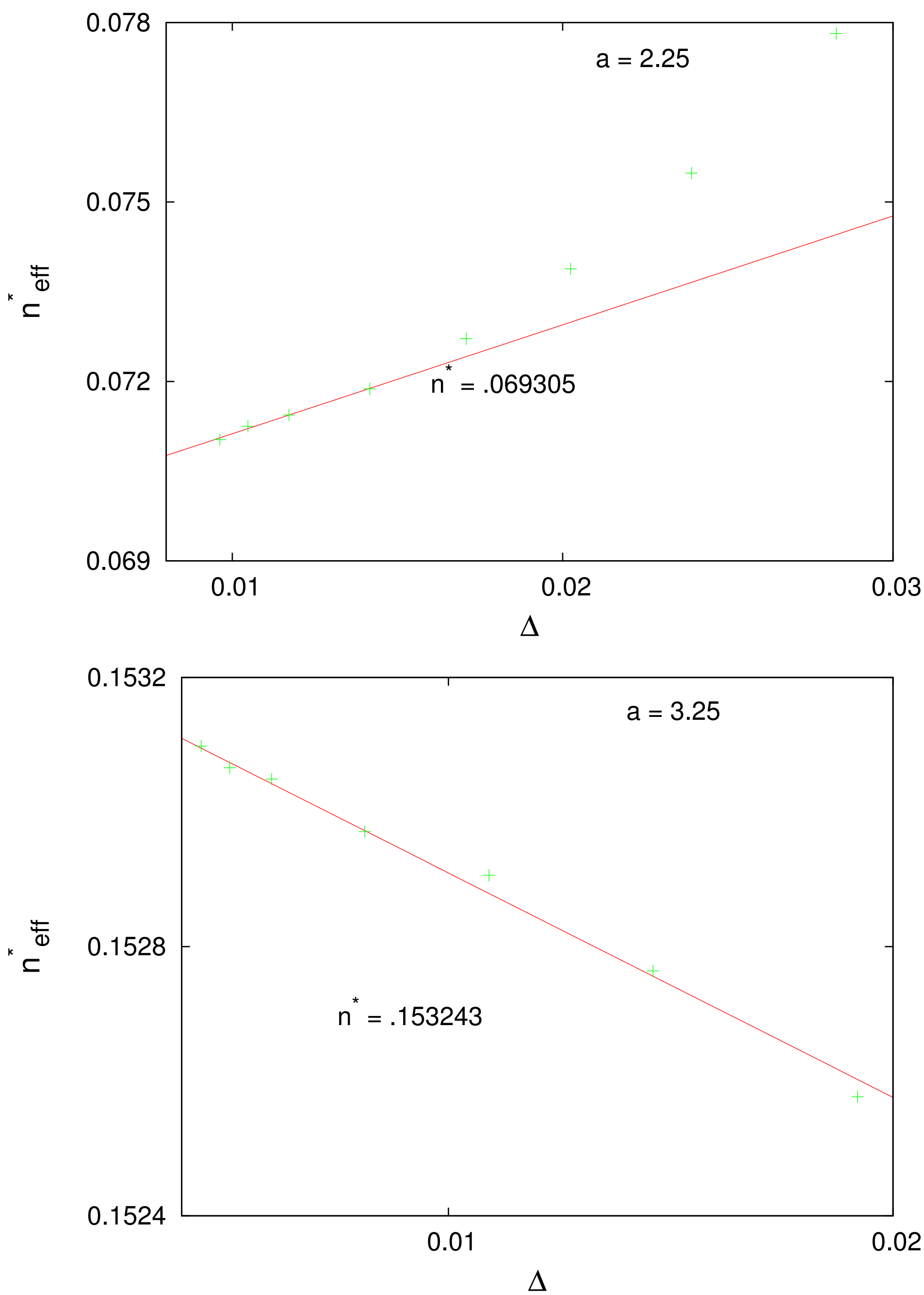

Figure 3.5: Plot of effective percolation threshold n^*_{eff} against Δ for $a = 2.25$ and $a = 3.25$. The best straight line fit is obtained with the last four data points.

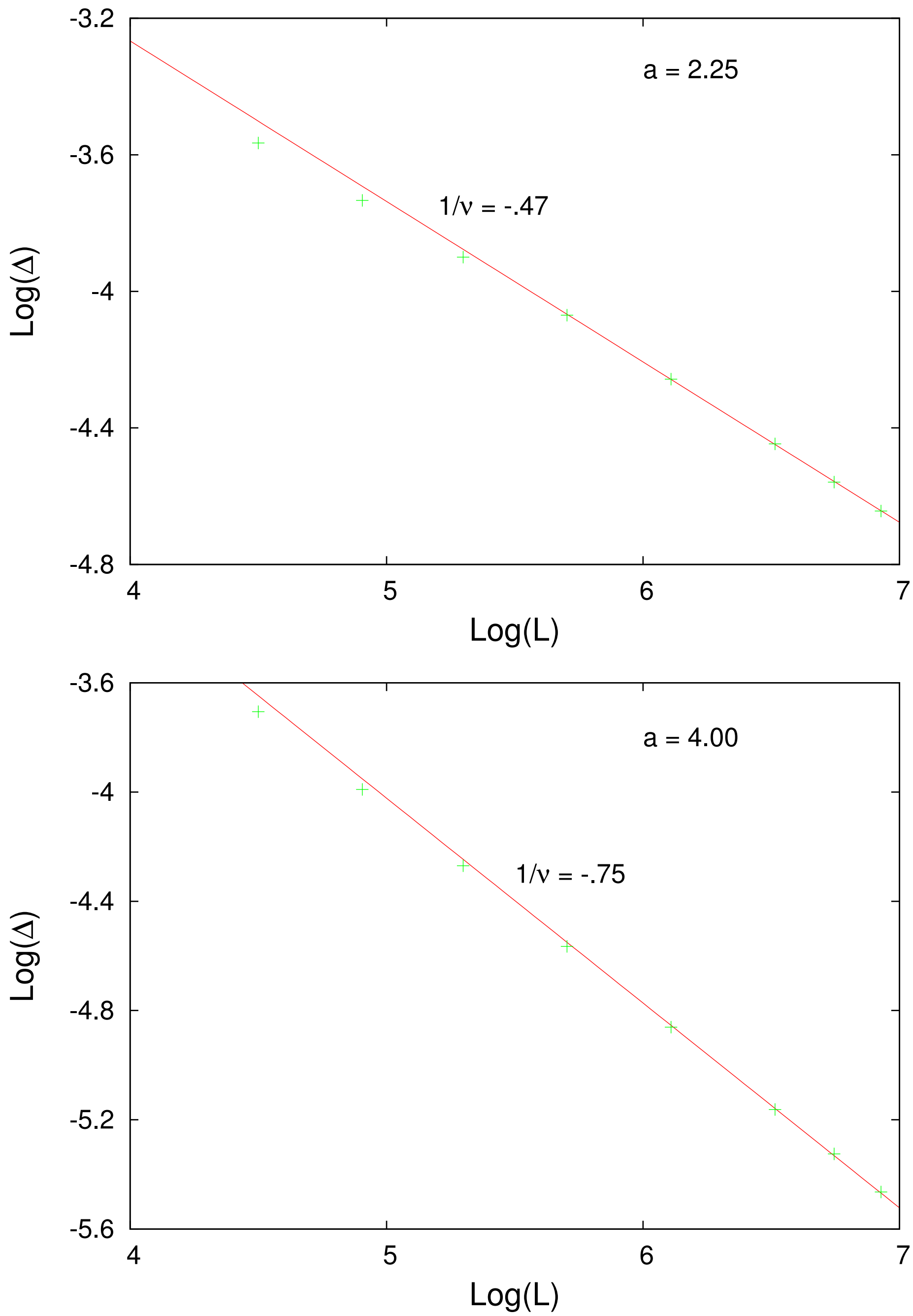

Figure 3.6: Log-Log plot of Δ against L for $a = 2.25$ and $a = 4.0$ along with lines of slope $-.47$ and $-.75$.

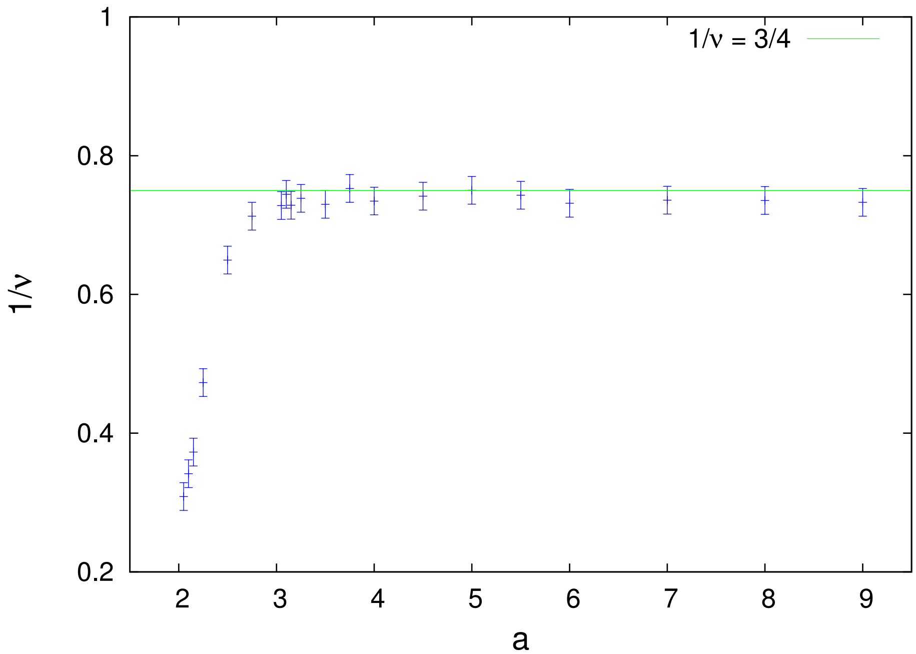

Figure 3.7: Variation of $1/\nu$ with a. The horizontal line corresponds to the standard 2D percolation value $1/\nu = 3/4$.

3.7 Summary

In this chapter, we discussed the effect of a power-law distribution of the radii on the critical behavior of a disc percolation system. If the distribution of radii is bounded, then one would expect the critical exponents to be unchanged. However, if the distribution of radii has a power-law tail, we show that this strongly influence the nature of the phase transition. The whole of the low-density non-percolating phase has power-law decay of correlations. And this occurs for any value of the power a, howsoever large. The critical exponents depend on the value of a for $a < 3 - \eta_{sr}$ and take their short-range values for $a > 3 - \eta_{sr}$. We also proposed an approximate RG scheme to analyze such systems. Using this, we computed the correlation-length exponent and

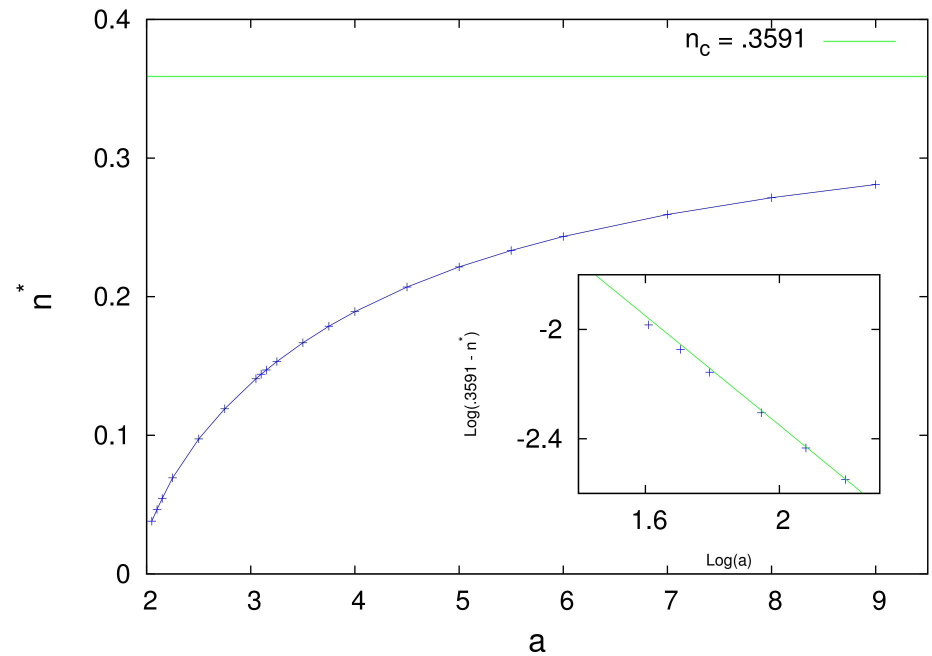

Figure 3.8: Variation of percolation threshold n^* with a. The horizontal line corresponds to the threshold for the single sized discs case. (Inset) Asymptotic approach of n^* to the single sized discs value $n_c = .3591$ along with a straight line of slope -1 and y-intercept -0.35 (See Eq. (3.32)).

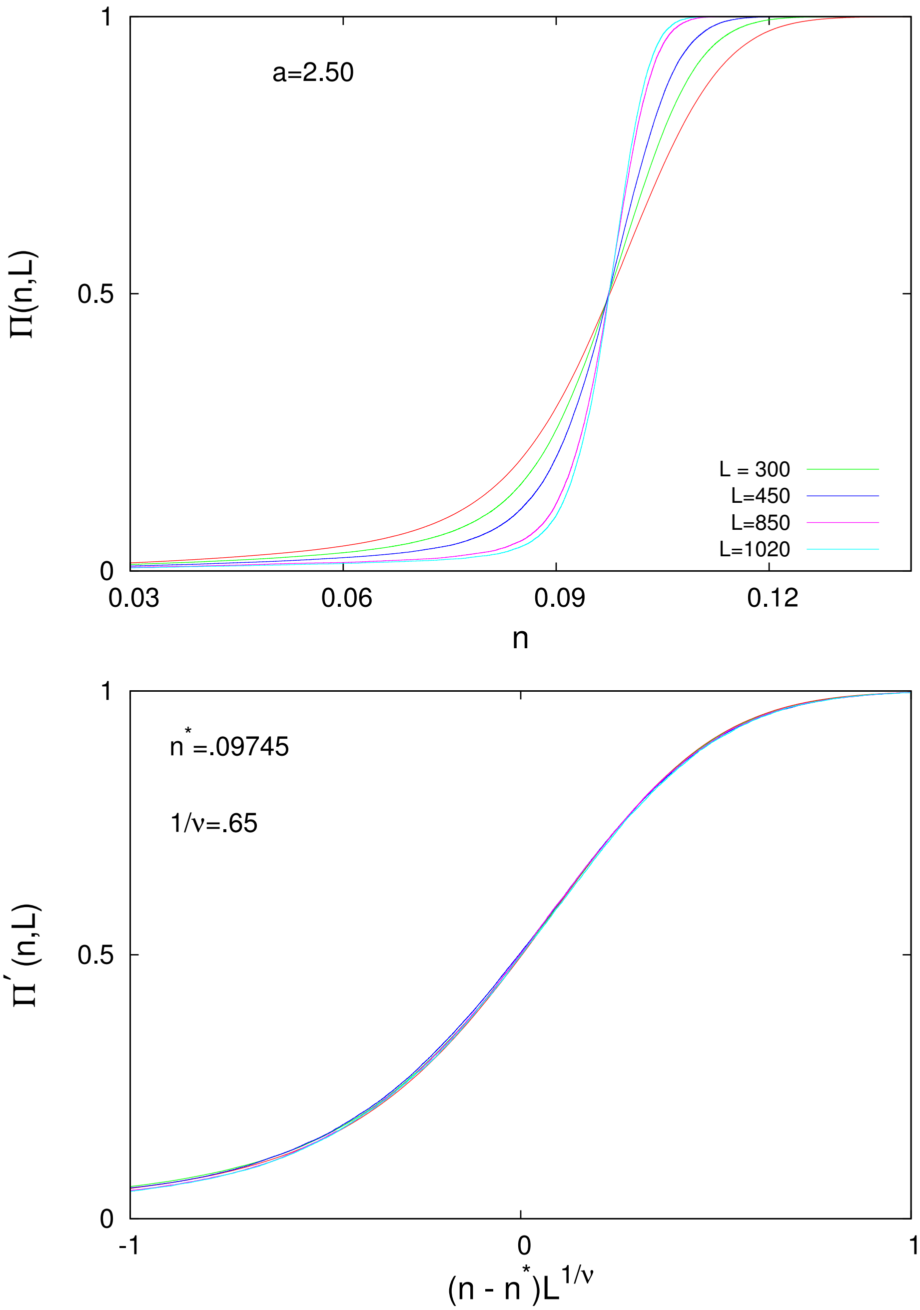

Figure 3.9: Variation of $\Pi(n, L)$ with n (top) and the scaling collapse (bottom) for $a = 2.50$.

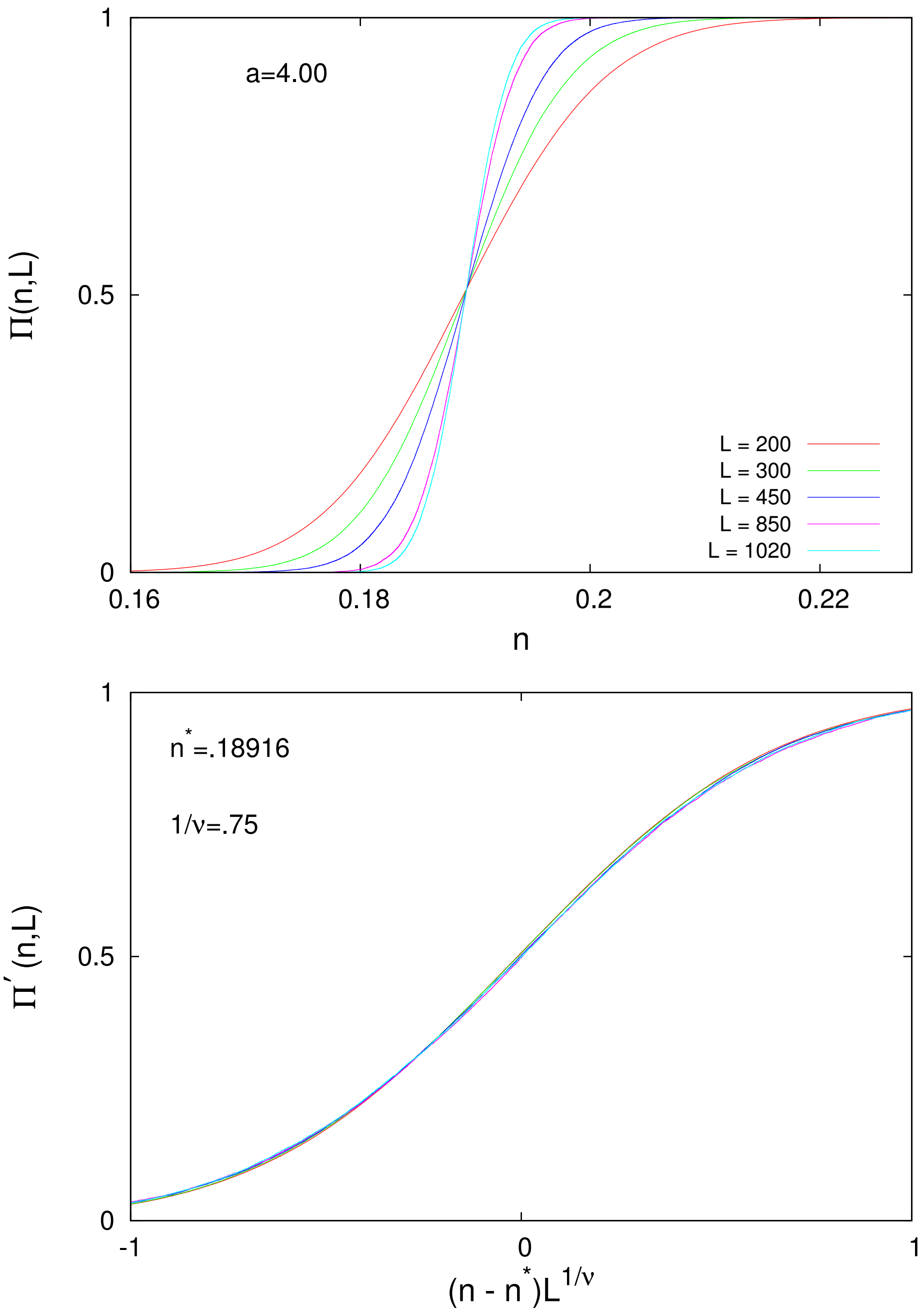

Figure 3.10: Variation of $\Pi(n, L)$ with n (top) and the scaling collapse (bottom) for $a = 4.0$.

a	n^*	$\eta^* = n^*\pi a/(a-2)$	$\phi^* = 1 - \exp^{-\eta^*}$
2.05	0.0380(6)	4.90(7)	0.993(1)
2.25	0.0693(1)	1.959(3)	0.8591(5)
2.50	0.09745(11)	1.5307(17)	0.7836(4)
3.50	0.16679(8)	1.2226(6)	0.70555(17)
4.00	0.18916(3)	1.1885(2)	0.69543(6)
5.00	0.22149(8)	1.1597(4)	0.68643(13)
6.00	0.24340(5)	1.1470(2)	0.68241(8)
7.00	0.2593(2)	1.1406(7)	0.6804(2)
8.00	0.27140(7)	1.1368(3)	0.67917(9)
9.00	0.28098(9)	1.1349(4)	0.67856(12)

Table 3.1: Percolation threshold n^* for a few values of a along with corresponding critical areal density η^* and the critical covered area fraction ϕ^*.

the percolation threshold. The approximate RG scheme found to be good for relatively large values of a. We determined percolation threshold and the correlation-length exponent from Monte-Carlo simulation studies.

We can easily extend the discussion to higher dimensions, or other shapes of objects. It is easy to see that the power law correlations will exist in corresponding problems in higher dimensions as well.

References

[1] D. Helbing and S. Balietti, Science and Culture, Vol **76**, No. 9-10 September-October 2010, Special Issue on 15 Years of Econophysics Research.

[2] R. Mantegna and H. G. Stanley **376**, 46 (1995).

[3] R. Mantegna and H. G. Stanley, *An Introduction to Econophysics: Correlations and complexity in finance* (Cambridge University Press, Cambridge, 2000).

[4] I. Giardina, J.-P. Bouchaud and M. Mezard, Physica A **299**, 28 (2001).

[5] J. -P. Bouchaud, *Power laws in economics and finance, some ideas from physics*, Quantitative finance **1**, 105 (2001).

[6] D. Challet and Y. C. Zhang, Physica A, **246** 407 (1997).

[7] W. Brian Arthur, Amer. Econ. Rev. (Papers and Proc.) **84**, 406 (1994).

[8] M. Marsili and D. Challet, *Adv. Complex Systems* **1**, 1 (2008).

[9] W. Kets and M. Voorneveld, CentER discussion paper, 2007-61. arXiv:0708.3542v1 [physics.soc-ph].

[10] T. Chmura and T. Pitz, Physica A, **363**, 477 (2006).

[11] A. C. C. Coolen *The mathematical theory of minority games: Statistical mechanics of interacting agents* (Oxford Univ. Press, Oxford, 2005).

[12] E. Moro, in *Advances in Condensed Matter and Statistical Mechanics*, Eds. E. Korutcheva and R. Cuerno (Nova Science Publishers, New York, 2004).
arXiv:cond-mat/0402651v1 [cond-mat.dis-nn].

[13] A. Cavagna, J. P. Garrahan, I. Giardina and D. Sherrington, Phys. Rev. Lett. **83**, 4429 (1999).

[14] J. P. Garrahan, E. Moro and D. Sherrington, Phys. Rev. **E 62**, R9 (2000).

[15] D. Challet, M. Marsili and R. Zecchina, Phys. Rev. Lett **85**, 5008 (2000).

[16] A. Cavagna, J. P. Garrahan, I. Giardina and D. Sherrington, Phys. Rev. Lett. **85**, 5009 (2000).

[17] M. Marsili and D. Challet, Phys. Rev. **E 64**, 056138 (2001).

[18] Y. Li, R. Riolo and R. Savit, Physica A **276**, 234 (2000).

[19] Y. Li, R. Riolo and R. Savit, Physica A **276**, 265 (2000).

[20] W. wang, Y. Chen, J. P. Huang *Proc natl Acad Sci USA* 106, 8423 (2009).

[21] L. Zhao, G. Yang, W. Wang, Y. Chen, J. P. Huang, H. Ohashi and H. E. Stanley, *Proc natl Acad Sci USA* 108, 15058 (2011).

[22] D. Challet, A. D. Martino, M. Marsili and I. P. Castillo, J. Stat. Mech. P03004 (2006).

[23] Z. G. Huang, J. Q. Zhang, J. Q Dong, L. Huang and Y. C. Lai, Sci. Rep. **2**, 703 (2012).

[24] T. Galla, A. C. C. Coolen and D. Sherrington, J. Phys. A: Math. Gen. **36**, 11159 (2003).

[25] A. Chakrabarti *et al*, arXiv:1305.2121v1[physics.soc-ph] (2013).

[26] http://www3.unifr.ch/econophysics/minority/.

[27] W. Kets, Journal of Economic Surveys, Vol. **26**, Issue 5, pp. 865-878, 2012.

[28] R. Rosenthal, Int. J. Game Theo. **2**, 65 (1973).

[29] R. Selten and W. Guth, In M. Diestler, E. Furst and G. Schwodiauer (Eds.), *Games, Economic Dynamics, and Time Series Analysis*, Wien-Wurzburg: Physics-Verlag (1982).

[30] G. Reents, R. Metzler and W. Kinzel, Physica A **299**, 253 (2001).

[31] J. B. Satinover and D. Sornette, Eur. Phys. J. B **60**, 369 (2007).

[32] D. Stauffer and A. Aharony, *Introduction to percolation theory*, (Taylor and Francis, London, 2003).

[33] B. Ballobas and O. Riordan, *Percolation*, (Cambridge University Press, Cambridge, 2006).

[34] G. Grimmet, *Percolation*, (Springer-Verlag, Newyork, 1999).

[35] H. Kesten, *Percolation theory for mathematicians*, (Birkhauser, 1982).

[36] M. Menshikov, Soviet Mathematics Doklady **33**, 856 (1986).

[37] D. Stauffer, Physics reports **54**, 1 (1979).

[38] M. P. M. den Nijs, J. Phys. A: Math. Gen. **12**, 1857 (1979).

[39] ©Johan, http://commons.wikimedia.org/wiki/File:Sponge-viscose.jpg, CC-BY-SA-3.0/ GFDL.

[40] E.T. Gawlinski and H.E. stanley, J. Phys. A: Math.Gen. **14**, L291 (1981).

[41] J. Quintanilla, Phys. Rev. **E 63**, 061108 (2001).

[42] R. Ogata, T. Odagaki and K. Okazaki, J. Phys.:Condens. matter **17**, 4531 (2005).

[43] N. Wagner, I. Balberg and D. Klein, Phys. Rev. **E 74**, 011127 (2006).

[44] J.A. Quintanilla and R.M. Ziff, Phys. Rev. **E 76**, 051115 (2007).

[45] M. D. Rintoul and S. Torquato, J. Phys. A: Math. Gen. **30**, L585 (1997).

[46] T. Vicsek and J. Kertesz, J. Phys. A: Math. Gen. **14**, L31 (1981).

[47] J.B. Gouere, The Ann. of Probab. **36**, 4 1209 (2008).

[48] J. E. Martin, J. Appl. Cryst. **19**, 25 (1986).

[49] B. Biswal *et al.*, Phys. Rev. **E 75**, 061303 (2007).

[50] S. Roth et al. AAPG Bulletin **95**, 6 925 (2011).

[51] D. Dhar, V. Sasidevan and Bikas K. Chakrabarti, Physica A **390** , 3477 (2011).

[52] D. Challet and Y. C. Zhang, Physica A **256**, 514 (1998).

[53] R. Manuca, Y. Li, R. Riolo and R. Savit, Physica A **282**, 559 (2000).

[54] R. Savit, R. Manuca and R. Riolo, Phys. Rev. Lett. **82**, 2203 (1999).

[55] D. Challet, M. Marsili and Y. C. Zhang, *Minority Games: Interacting agents in financial markets* (Oxford Univ. Press, Oxford, 2005).

[56] A. Cavagna, Phys. Rev. **E 59**, R3783 (1999).

[57] D. Challet, M. Marsili and R. Zecchina, Phys. Rev. Lett. **84**, 1824 (2000).

[58] A. D. Martino and M. Marsili, J. Phys. A: Math. Gen. **34**, 2525 (2001).

[59] N. F. Johnson *et al*, Physica A **258**, 230 (1998).

[60] C. De Dominicis, Phys. Rev. B **18**, 4913 (1978).

[61] A. de Martino and T. Galla, New Math. Natural Comput. **7** 249 (2011).

[62] J. A. F Heimel and A C C Coolen, Phys. Rev. **E 63**, 056121 (2001).

[63] A. C. C. Coolen and J. A. F. Heimel, J. Phys. A: Math. Gen, **34** 10783 (2001).

[64] N. Shayeghi and A. C. C. Coolen, J. Phys. A: Math. Gen. **39** 13921 (2006).

[65] A. de Martino, I. C. Perez and D. Sherrington J. Stat. Mech: Theory Exp P01006 (2007).

[66] R. T. Liu, F. F. Chung and S. S. Liaw, EPL, **97** 20004 (2012).

[67] G. Devetag and F. Pancotto, T. Brenner, *University of Bologna, working paper*, DSE 770 (2011).

[68] R. Selten, T. Chmura, T. Pitz, S. Kube and M. Schreckenberg, *Games and economic behavior* **58**, 394 (2007).

[69] G. Bottazzi and G. Devetag, J. Evol. Econ, **17**, 241 (2007).

[70] G. Bottazzi and G. Devetag, Physica A **324** 124 (2003).

[71] T. Platkowski and M. Ramsza, Physica A **323**, 726 (2003).

[72] Linde, Jona, J. Sonnemans and J. Tuinstra. *Strategies and Evolution in the Minority Game: A Multi-Round Strategy Experiment*, No. 13-043/I. Tinbergen Institute, 2013.

[73] T. Chmura and W. Guth, *Games* **2**, 333 (2011).

[74] M. Marsili and D. Challet, Adv. Complex Systems **4**, 3 (2001).

[75] S. Biswas *et al*, Phys. Rev. **E 85**, 031104 (2012).

[76] Y.B. Xie *et al*, Eur. Phys. J. B **47**, 587 (2005).

[77] M. Nowak and K. Sigmund, Nature **364**, 56 (1993).

[78] D. Fundenberg and J. Tirole, *Game Theory*, (MIT Press, Cambridge, MA, 1991).

[79] Osborne, J. Martin and A. Rubinstein, *A course in game theory*, (MIT Press, Cambridge, MA, 1994).

[80] J. C. Harsanyi and R. Selten, *A general theory of equlibrium selection in games*, (MIT Press, Cambridge, 1988).

[81] E. Kohlberg and J-F. Mertens, Econometrica **54**, 5 1003 (1986).

[82] M. Aizenman and R. Fernandez, Lett. Math. Phys. **16**, 39 (1988).

[83] J. Quintanilla and R. M. Ziff, Phys. Rev. E **76**, 051115 (2007).

[84] R. Meester and R. Roy, *Continuum Percolation* (Cambridge University Press, Cambridge, 1996).

[85] J. Kertesz and T. Viscek, Z. Phys. B:Condensed Matter **45**, 345 (1982).

[86] H. Scher and R. Zallen, J. Chem. Phys. **53**, 3759 (1970).

[87] G. E. Pike and C. H. Seager, Phys. Rev. B, **10**, 1421 (1974).

[88] M.K. Phani and D. Dhar, J. Phys. A: Math. Gen. **17** L645 (1984).

[89] R. Meester, R. Roy and A. Sarkar, J. Stat. Phys. **75**, 123 (1994).

[90] J. Sak, Phys. Rev. **B 8** 281 (1973).

[91] R. F. Kayser and H. J. Raveche, Phys. Rev **A 29**, 1013 (1984).

[92] D. Iagolnitzer and B. Souillard, Phys, Rev. **A 16**, 1700 (1977).

[93] G. Stell, Phys. Rev. **B 1**, 2265 (1970).

[94] G. Stell, Phys. Rev. **B 8**, 1271 (1973).

[95] M. E. Fisher, S. Ma, and B. G. Nickel, Phys. Rev. Lett. **29**, 917 (1972).

[96] D. Dantchev, Eur. Phys. J. B **23**, 211 (2001).

[97] G. Stell, Phys. Rev., **B 5**, 981 (1972).

[98] C. M. Fortuin and P. W. Kasteleyn, Physica **57**, 536 (1972).

[99] B. Nienhuis, E. K. Riedel and M. Schick, J. Phys. A: Math. Gen. **13**, L189 (1980).

[100] A.C. Balram and D. Dhar, Pramana **74**, 109 (2010).

[101] Jiantong Li and Shi-Li Zhang, Phys. Rev. **E 80**, 040104(R) (2009).

[102] M. E. J. Newman and R. M. Ziff, Phys. Rev. **E 64**, 016706 (2001).

Papers

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Emergent cooperation amongst competing agents in minority games

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ABSTRACT

We study a variation of the minority game. There are N agents. Each has to choose between one of two alternatives every day, and there is a reward to each member of the smaller group. The agents cannot communicate with each other, but try to guess the choice others will make, based only on the past history of the number of people choosing the two alternatives. We describe a simple probabilistic strategy using which the agents, acting independently, and trying to maximize their individual expected payoff, still achieve a very efficient overall utilization of resources, and the average deviation of the number of happy agents per day from the maximum possible can be made $\mathcal{O}(N^{\epsilon})$, for any $\epsilon > 0$. We also show that a single agent does not expect to gain by not following the strategy.

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1. Introduction

The Minority Game (MG) is a particular version of the El Farol Bar problem. The latter was introduced by Brian Arthur as a prototypical model for the complex emergent behavior in a system of many interacting agents having only incomplete information, and bounded rationality [1]. This problem is about N agents, who have to repeatedly make choices between two alternatives, and at each step, the winners are those who selected the alternative chosen by fewer agents. MG has been studied a lot as a mathematical model of learning, adaptation, and co-evolution of agents [2,3]. An overview and bibliography may be found in Ref. [4–6]. Similar models have been discussed earlier under the name of congestion games [7], and market entry games [8]. The interesting feature of the minority game is that the agents seem to be able to coordinate their actions, without any direct communication with each other, and the system can self-organize to a state in which the fluctuations in the steady state are much less than what would be expected if each agent made a random choice. This is called the efficiency of the markets.

In a system of N interacting agents, with N odd, the degree of efficiency of the system may be measured by how close is the average number of happy agents in the steady state to the maximum possible value $(N-1)/2$. Simulations of MG have shown that typically the difference is of order $N^{1/2}$. The coefficient of the $N^{1/2}$ depends on details of the model, like how far back in the past the agents look to decide their action, but it can be much less than the value for agents making random choices. The minimum value of the coefficient attained in several variants of the MG is about 1/6 [5].

In our formulation of the problem, all agents are equally smart, and hence the average expected payoff per day achievable by any agent is the same as for any other. Then, maximizing average payoff for one agent is the same as maximizing the average number of happy agents per day. Hence the strategy that optimizes the expected individual payoff of one particular agent also maximizes the overall utilization of resources.

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A variation of the minority game, focusing on the efficient utilization of resources was studied by Chakrabarti et al. as the Kolkata Paise Restaurant problem [9–11]. In this variation, there are N restaurants, and N agents, and there is rank order amongst the restaurants. Each restaurant can take only one agent per day, and agents prefer to go to a higher ranked restaurant. In spite of this complication, it was found that an egalitarian probabilistic strategy exists in which the agents visit restaurants in a cyclic order. Also, the agents can reach this cyclic state in a short time.

In this paper, we describe a probabilistic strategy, inspired by the strategy suggested in Ref. [11], for the minority games, that is very simple, but is more efficient than those previously studied in the literature. In this strategy, the average deviation of the number of people in the minority from the maximum $(N-1)/2$ can be reduced to be of order N^{ϵ}, for any $\epsilon > 0$, and the time required to reach this level increases with N only as $\log \log N$. In addition, we show that a game where all agents follow this strategy is stable against individual cheaters.

Our strategy is an application of the general win–stay–lose–shift strategy [12], an adaptation of which to MG was discussed earlier by Reents et al. [13]. In the latter, the deviation from best possible can be made of order 1, but the time required grows as $N^{1/2}$. We are able to get a much faster approach to optimum by using a shift probability that depends on the current distance from optimum. Other probabilistic strategies for minority games have also been discussed in the literature [14–16], and it has been noted that in minority games, random choices by agents give better results on average than the deterministic strategies [17]. While the strategy discussed here seems more or less obvious, we could not find such a discussion in the published literature, and it seems worthwhile to study it quantitatively.

We will show that if all the agents follow the proposed common strategy, the social inefficiency of the system is considerably reduced. The emergence of effective cooperation amongst selfish agents in our problem may seem rather paradoxical at first. After all, the main point of MG is that agents gain by differentiating, and not following the same strategy. We note that the differentiation in our case is achieved by the random number generators used by the agents. The main result of this paper is that this is more efficient than using different deterministic strategies.

The plan of the paper is as follows: in Section 2, we define the rules of the game precisely and argue that the strategy defined leads to a very efficient use of resources. In Section 3, we show that individual agents have no incentive to cheat, if every body else follows the same strategy. Section 4 contains the results of our simulations of the model, and Section 5 contains some concluding remarks.

2. Definition of the model

The model we consider is a variation of the El Farol Bar problem. We consider a small city with exactly two restaurants. There are N people in the city, called agents, each of whom goes for dinner every evening to one of the two restaurants. The prices and quality of food are quite similar in both, and the only thing that governs the choice of agents about which restaurant they go to on a particular day is that the quality of service is worse if the restaurant is crowded. We assume that N is odd, and write $N = 2M + 1$. The restaurant is said to be crowded on a particular day if the number of people turning up to eat there that day exceeds M. An agent is happy if he goes to a restaurant that is uncrowded, and will be said to have a payoff of 1. If he turns up at a crowded restaurant, his payoff is 0. Once the choice of which restaurant to go to is made, an agent cannot change it for that day.

The agents cannot communicate with each other in any way directly in deciding which restaurant to go to. However, each of them has available to him/her the entire earlier history of how many people chose to go to the first restaurant (call it A), on any earlier day. Let us denote the number of agents turning up at A on the t-th day by $M - \Delta(t)$. Then the number of agents turning up at Restaurant B is $M + \Delta(t) + 1$. At the end of day t, the value of $\Delta(t)$ is made public, and is known to all the agents. Using the information $\{\Delta(t')\}$, for $t' = 1, 2, \ldots, t$, the agents try to guess the choice that other customers who share the same public knowledge will make, and decide which restaurant to go to on the day $(t + 1)$, and try to optimize their payoff.

In the standard MG, the public information is not the value of $\Delta(t)$, but only whether it is negative or not [2,3]. In contrast, in our model, the agents have better quality of information, and this difference is important. Also, in MG each agent has a finite set of strategies available to him/her, which uses only the history $\{\Delta(t)\}$ for m previous days, where m is a fixed non-negative integer. Each strategy is deterministic: for a given history, it tells which restaurant agent should go to. While the agent has more than one strategy available to him/her, he chooses the strategy that has the best 'performance score' in the recent past. As noted in Ref. [17], this particular method of selecting the 'best' strategy does not seem to be very good. In fact, an agent choosing a strategy with the worst score may do better than others choosing the ones with the best score!. Here the only probabilistic component is in the initial allocation of a subset of strategies to each agent out of the set of all possible strategies. For a given history, the future choices of all agents for all subsequent days are fully determined.

In the problem we study here, we allow agents to have probabilistic strategies. For a given history $\{\Delta(t)\}$, a strategy will specify a probability p with which he should go to restaurant A. Another important difference from the MGs is that we allow the strategy to depend explicitly on the payoffs received in the m previous days. In MG, the strategy does not explicitly involve previous payoffs. The payoff only affect the outcome indirectly, through the performance scores that determine which strategy is used by the agent.

To completely specify the model, in addition to the payoff function, we also need to specify the agents' priorities, and time-horizons. The usual assumption in game theory is that agents are fully selfish, and only try to maximize their personal payoffs. This is, somewhat incorrectly, termed rational behavior. In this model, we assume that the agents are selfish, but try to maximize their expected payoff over the next D days. The usually studied cases are $D = 1$ and D very large, tending to infinity. However, in situations where outcomes of games are correlated in time, optimizing strategies for different D can be quite different.

For simplicity, we will assume that all agents have the same value of D. For most of our discussion, we restrict ourselves to the simple case $D = 1$. But we will show that the purely $D = 1$ optimization leads to an impasse. As already noted in the introduction, the assumed symmetry between agents immediately leads to the conclusion that the best performing strategy for any particular individual in the limit of D large, also maximizes the average resource utilization for the society as a whole.

The simplest case corresponds to $m = 0$ and $D = 1$. In this case, an agent has no information. His probabilistic strategy is to make a random choice of which restaurant to go to, with equal probability. In this case, the probability that r people show up at Restaurant A is clearly

$$\mathrm{Prob}(r) = \binom{N}{r} 2^{-N}. \tag{1}$$

The expectation value of r is $N/2$, and for large N, the distribution is nearly gaussian, with a width proportional to $\sqrt{N}$. We can measure the inefficiency of the system by a parameter η defined as

$$\eta = \lim_{N\to\infty} \frac{4}{N} \langle (r - N/2)^2 \rangle \tag{2}$$

where $\langle\rangle$ denotes averaging over a long time evolution, and over different initial conditions.

The normalization has been chosen, so that the inefficiency parameter η of the system with agents using his /her choice randomly is 1.

We now describe a simple $m = 1$ probabilistic strategy, that gives a highly efficient system, where the inefficiency parameter can be made of order $(1/N^{1-\epsilon})$, for any $\epsilon > 0$.

The strategy is defined as follows: at $t = 0$, each agent chooses one of the two restaurants with probability 1/2 each, independently of others. At any subsequent time $t + 1$, each agent follows the same simple strategy: if at time t, he found himself in the minority, he chooses the same restaurant as at time t. If he found himself in the majority, and the number of people visiting the same restaurant as him was $M + \Delta(t) + 1$, with $\Delta(t) \geq 0$, he changes his choice with a small probability p, and sticks to the earlier choice with probability $1 - p$, independent of other agents. The value of p depends only on $\Delta(t)$. It is approximately equal to Δ/M for $\Delta > 0$. The precise dependence of p on Δ is discussed later in the paper.

For large M, the number of people changing their choice is distributed according to the Poisson distribution, with mean approximately equal to Δ, and width varying as $\sqrt{\Delta(t)}$. Thus we have the approximate recursion $\Delta(t + 1) \approx \sqrt{\Delta(t)}$, for $\Delta(t) \gg 1$. This shows that within a time of order $\log\log N$, the magnitude of Δ will become of $\mathcal{O}(1)$, and then remain of order 1.

3. Stability against individual cheaters

While the strategy given in the previous section leads to a very efficient utilization of resources, selfish agents may not do what is expected of them for the social good, and act differently, if it gives them profit. In this section, we show that if all the other people are following the common strategy outlined above, there is a specially selected value of p, for each $\Delta > 0$, such that if other agents follow the strategy with this value of p, a single individual gains no advantage by cheating.

If rational agents with $D = 1$ strategies know that they cannot improve their immediate individual expected gain by cheating, they might then change their optimization criterion, and try to maximize their individual long-term payoff. This they can do, if they follow the same common strategy. *This cooperative strategy is beneficial for everybody in the long run, and deviating from it has no advantage*. This is the reason for the emergent cooperation between agents in our model.

Let us consider any particular day t. Let the number of people who showed up in Restaurant A be $M - \Delta(t)$. We may assume $\Delta(t) \geq 0$, without loss of generality.

We consider first the case $\Delta > 0$. We consider a particular agent Alice, who went to A on the t-th day, and found herself in the happy situation of being in the minority. Alice assumes that all other agents follow the strategy. Then, all other agents who went to A will go to it again on day $(t + 1)$. There are $M + \Delta + 1$ agents that went to B. Each of these agents will change his/her choice with probability p. Let r be the number of agents that actually change their choice at time $(t + 1)$. Then, r is a random variable, with a distribution given by

$$\mathrm{Prob}_p(r) = \binom{M + \Delta + 1}{r} p^r (1 - p)^{M+\Delta+1-r}. \tag{3}$$

For $M \gg 1$, this distribution tends to the Poisson distribution with parameter $\lambda = p(M + \Delta + 1)$, given by

$$\mathrm{Prob}_\lambda(r) = \lambda^r e^{-\lambda}/r!. \tag{4}$$

If Alice chooses to go to A the next day, she will be in the winning position, if $r \leq \Delta$. Hence her expected payoff $EP(Alice|stay)$, if she chooses to stay with her present choice is

$$EP(Alice|stay) = \sum_{r=0}^{\Delta} \mathrm{Prob}_p(r). \tag{5}$$

If, on the other hand, Alice would switch her choice, she would win if $r \geq \Delta + 2$. Clearly, her expected payoff $EP(Alice|switch)$ if she chooses to switch is given by

$$EP(Alice|switch) = \sum_{r=\Delta+2}^{\infty} \mathrm{Prob}_p(r). \tag{6}$$

For Alice to have no incentive to cheat, we must have

$$EP(Alice|stay) \geq EP(Alice|switch). \tag{7}$$

Now consider the agent Bob, who went to B on day t. He also assumes that all other people will follow the strategy: those who went to A will stick to their choice, and those who went to B switch their choice with probability p. There are $M + \Delta$ other persons who went to B. If Bob chooses to cheat, and decide to stay put, without using a random number generator, the number of agents switching would be a random number $\tilde{r}$, with a distribution given by

$$\mathrm{Prob}'_p(\tilde{r}) = \binom{M+\Delta}{\tilde{r}} p^{\tilde{r}}(1-p)^{M+\Delta-\tilde{r}}. \tag{8}$$

He would be in the minority, if $\tilde{r} \geq \Delta + 1$. Thus, if he chooses to stay, we have his expected payoff $EP(Bob|stay)$ given by

$$EP(Bob|stay) = \sum_{\tilde{r}=\Delta+1}^{\infty} \mathrm{Prob}'_p(\tilde{r}). \tag{9}$$

On the other hand, if Bob decides to switch his choice, he would win if $\tilde{r} \leq \Delta - 1$. In that case, his expected payoff $EP(Bob|switch)$ is given by

$$EP(Bob|switch) = \sum_{\tilde{r}=0}^{\Delta-1} \mathrm{Prob}'_p(\tilde{r}). \tag{10}$$

We choose the value of p to make these equal. Thus the equation determining p, for a given Δ and N is

$$EP(Bob|stay) = EP(Bob|switch). \tag{11}$$

If the above condition is satisfied, Bob can choose to stay, or switch, and his expected payoff is the same. More generally, he can choose to switch with a probability α, and his payoff is independent of α. In that case, what is the optimum value of α for Bob? One has to bring in a different optimization rule to decide this, and it seems reasonable that Bob would choose a value that optimizes his long-time average payoff, (which is the same for any other agent), and hence choose the value p.

In the limit of $M \gg \Delta$, Eq. (11) simplifies, as the dependence on M drops out, and we get a simple equation determining the dependence of the Poisson parameter λ on Δ. Then, Eq. (11) becomes

$$\sum_{r=0}^{\Delta-1} \frac{\lambda^r}{r!} e^{-\lambda} = \sum_{r=\Delta+1}^{\infty} \frac{\lambda^r}{r!} e^{-\lambda}. \tag{12}$$

This equation may be rewritten, avoiding the infinite summation, as

$$2\sum_{r=0}^{\Delta-1} \frac{\lambda^r e^{-\lambda}}{r!} = 1 - \frac{\lambda^{\Delta} e^{-\lambda}}{\Delta!}. \tag{13}$$

It is easy to see that Eq. (13) implies that the inequality (7) is also satisfied. For the sake of simplicity, we will only consider this limit of large M in the following. The extension to finite M presents no special difficulties.

Thus, for any given value of $\Delta > 0$, the optimum value of λ is determined by the solution of Eq. (13). This equation is easily solved. The resulting values of λ for different Δ are shown in Table 1. For large Δ, we show in Appendix that $(\lambda - \Delta)$ tends to $1/6$.

We note that the values of λ do not have to be broadcast to the agents by any central authority. Each individual rational agents will be able to deduce them as optimal, without any need to communicate with others. Fig. 1 shows the variation of the expected payoff for the next day of Alice and Bob with Δ. As expected we can see that for large values of Δ, the expected payoff of an agent in either restaurant tends to the value 1/2. Alice's payoff is a bit bigger than 1/2, but this advantage is short-lived. Also, Bob cannot utilize this predictability of the system, as an attempt to switch by him changes the outcome with finite probability.

Now, we consider the case $\Delta = 0$. In this case, Restaurant A has exactly M, and B has $M + 1$ people. In this case, it is easy to see that the solution of Eq. (11) is $\lambda = 0$, and this also satisfies Eq. (7). Qualitatively, this may be explained as being

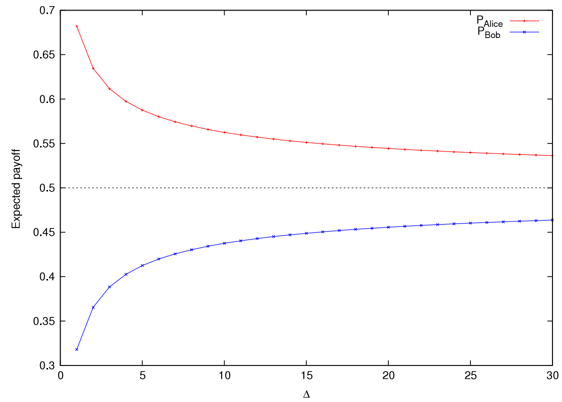

Fig. 1. Variation of expected payoff for the next day of an agent in Restaurant A (P_{Alice}) and Restaurant B (P_{Bob}) with Δ.

Table 1
The optimal values of the shift probability parameter λ, when the number of persons in the minority restaurant is $M - \Delta$.

Δ	λ	Δ	λ
1	1.14619	8	8.16393
2	2.15592	9	9.16423
3	3.15942	10	10.16448
4	4.16121	20	20.16557
5	5.16229	30	30.16594
6	6.16302	40	40.16612
7	7.16354	50	50.16623

due to the fact if there is a nonzero λ, and agents from B switch to A, Bob has an incentive to cheat, because if he goes to A, he would be sure to be in the majority. If he cheats, and stays back, but at least one other agent leaves from B to A (which occurs with nonzero probability for any non-zero λ), he has some chance to be on the winning side. Since all agents in B would reason this way, we get $\lambda = 0$.

Clearly, $\lambda = 0$ is actually a very stupid strategy from Bob's point of view, as then nobody switches, and the state at day $(t + 1)$ is the same as on day t, and the same situation is met again. While this is a solution which minimizes wastage of resources, and is socially as efficient as possible, this is clearly a very *unfair* state of affairs, where a subset of people are privileged, and have payoff 1 every day, and another set has no chance of any payoff. In fact, if Bob is a short-sighted agent, with only a myopic $D = 1$ optimization goal, he can do no better individually on the next day. If Alice knows that Bob has only a myopic $D = 1$ strategy, she would prefer not to switch, which then makes Bob reason that he should not switch, and so on.

The only way out of this impasse is for Bob to realize that he should switch with some probability, even if it does not lead to any better payoff that day, and not stay back hoping that other B's will switch. That increases his expected payoff tomorrow, even if it does not increase it today. The existence of these "trapping states" at $\Delta = 0$ is a problem caused by Bob's restricting himself to immediate-payoff $D = 1$ optimization.

We make a small modification of the basic strategy outlined above to take care of the problem when $\Delta = 0$. We note that in this case, though Bob does not expect to gain anything on the next day by switching, he would still like to do that to upset the status quo, and improve his chance of winning the day after. Of course, as Alice realizes that some people from B are likely to switch, she would like to switch as well. Consider the case when all people who went to A switch with probability λ'/M, and all who went to B switch with probability $\lambda''/(M + 1)$, with both λ' and λ'' non-zero. Let r' and r'' be the random variables denoting the number of people switching sides from A to B, and from B to A respectively. Then, r' and r'' are Poisson-distributed independent random variables with mean λ' and λ'' respectively. Repeating the analysis above, we see that the condition that Alice has no incentive to cheat gives

$$\mathrm{Prob}(r' < r'' - 2) = \mathrm{Prob}(r' \geq r''). \tag{14}$$

Similarly, for the absence of an incentive to cheat for Bob, we should have

$$\mathrm{Prob}(r' < r'' - 1) = \mathrm{Prob}(r' \geq r'' + 1). \tag{15}$$

It is easy to see that Eqs. (14) and (15) are mutually inconsistent, as the LHS of the former is strictly less than the LHS of the latter, and for the RHS it is the opposite. Thus, we cannot find nonzero finite values λ' and λ'', which will give a stable strategy against individuals cheating.

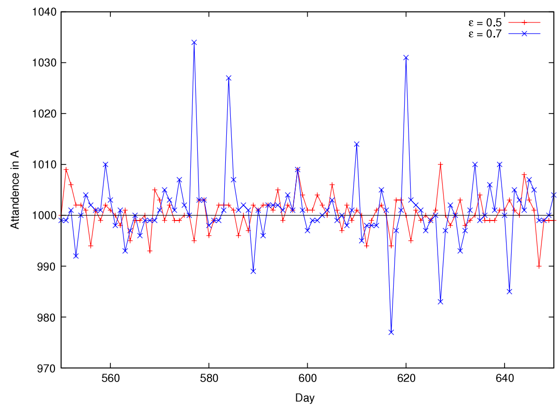

Fig. 2. A typical evolution of a system of 2001 agents for two different choices of the parameter $\epsilon = 0.5$ and 0.7. The large deviations correspond to major events (see text).

While there is no stable strategy solution to our equations for Poisson-distributed random variables r' and r'', with finite means λ' and λ'', there is a stable solution where each agent switches with probability 1/2. It is easy to see that this is also stable against cheating by individual agents.

A slight generalization of this strategy is that in the case $\Delta = 0$, all agents irrespective of whether they were in the minority or not on day t, switch the next day with a probability $M^{\epsilon-1}$, where ϵ is a real number $0 \leq \epsilon \leq 1$. This corresponds to both λ' and λ'' very large, of order M^{ϵ}. We shall refer to this step as a major resetting event.

For a given value of ϵ, the value of $|\Delta|$ just after resetting is of order $M^{\epsilon/2}$. Then it lakes a time of order $\log\log M$ to reach the value $\Delta = 0$. Then the maximum contribution to the mean efficiency parameter comes from the major resetting events, and it is easy to see that the mean inefficiency parameter would vary as $M^{\epsilon-1}/\log\log M$. Then, for more efficiency, we should keep ϵ small.

4. Monte Carlo simulations

We have studied the time evolution of a set of N agents using this strategy using Monte Carlo simulations, with $N = 2001$. If the restaurant with greater attendance has $M + 1 + \Delta$ agents on a given day, with $\Delta > 0$, the next day each of them switches his/her choice with probability $\lambda(\Delta)/(M + \Delta + 1)$, and the agents in the minority restaurant stick to their choice. If there are exactly $M + 1$ agents in the majority restaurant, all agents switch their restaurant with a probability $1/(2M^{1-\epsilon})$.

The result of a typical evolution is shown in Fig. 2, for two choices of ϵ: 0.5 and 0.7. We see that the majority restaurant changes quite frequently. In fact, the system reaches the steady state fairly quickly, within about 10 steps. The large peaks in $|\Delta|$ correspond to resettings of the system, and clearly their magnitude decreases if ϵ is decreased. There is very little memory of the location of the majority restaurant in the system. To be specific, let $S(t)$ is $+1$ if the minority restaurant is A in the t-th step, and -1 if it is B. Then the autocorrelation function $\langle S(t)S(t+\tau)\rangle$ decays exponentially with τ, approximately as $\exp(-K\tau)$. The value of K depends on ϵ, but is about 2, and the correlation is negligible for $\tau > 3$.

Fig. 3 shows the probability distribution of Δ in the steady state for two different values of ϵ. Fig. 4 gives a plot of the inefficiency parameter η as a function of ϵ. In each case, the estimate of η was obtained using a single evolution of the system for 10000 time steps. The fractional error of estimate is less than the size of the symbols used.

We define $A_i(t)$ equal to $+1$ if the i-th agent was in Restaurant A at time t, and -1 otherwise. We define the autocorrelation function of the A-variables in the steady state as

$$C(\tau) = \frac{1}{N}\sum_i \langle A_i(t)A_i(t+\tau)\rangle. \tag{16}$$

In Fig. 5, we have shown the variation of $C(\tau)$ with τ. We see that this function has a large amount of persistence. This is related to the fact that only a small fraction of agents actually switch their choice at any time step. Clearly, the persistence time is larger for smaller ϵ.

5. Discussion

In our analysis of the strategy discussed, we assumed that whenever the system reaches the state $\Delta = 0$ and it is not possible to find a nearby state with only a few agents switching, the system undergoes a major resetting. However, consider a situation where because of shared common history, the agents agree to a convention that if such a state is reached, it continues for T more days without change, as it is socially efficient, and on the $(T + 1)$-th day, the major resetting occurs.

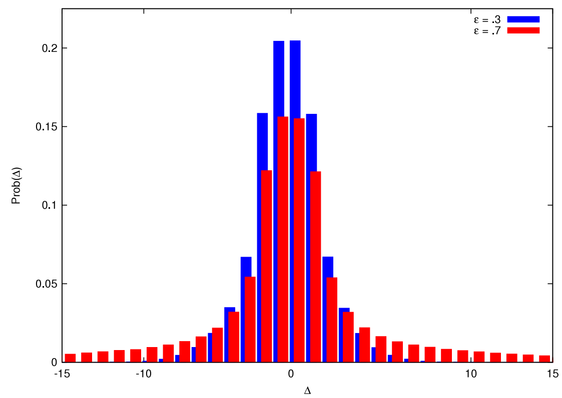

Fig. 3. Probability distribution of Δ in the steady state for $\epsilon = .3, .7$ obtained by evolving $N = 2001$ agents for 10^6 time steps. The red bars have been shifted a bit to the right for visual clarity.

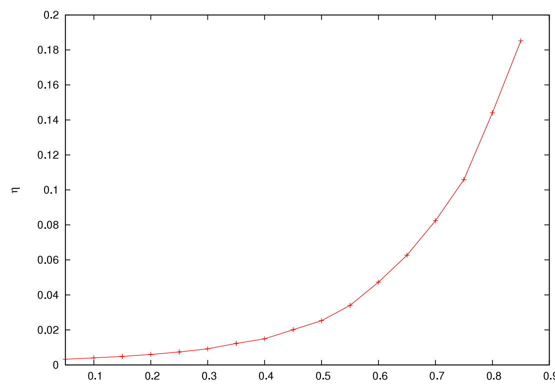

Fig. 4. Variation of inefficiency parameter η with ϵ, obtained by averaging the evolution of $N = 2001$ agents for 10 000 time steps.

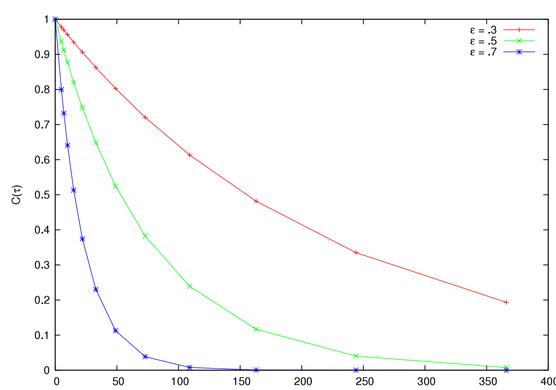

Fig. 5. $C(\tau)$ as a function of τ for $\epsilon = .3, .5$ and .7. Each data point is obtained by averaging over 10 000 simulation steps. Total number of agents is $N = 2001$.

The rationale for such a choice would be that all agents recognize that this state has overall maximum social benefit, and in the long run, any agent would spend an equal amount of time in the privileged class. Clearly, for realistic modeling, T should not be too large. It has to be significantly less than the expected lifetime of an agent.

The number of consecutive days when Δ is nonzero is of order $\log\log N$, and then for T consecutive days Δ remains zero. Then, the inefficiency parameter η in such a strategy is given by

$$\eta \simeq \frac{K_1 N^{\epsilon-1}}{T + K_2 \log\log N} \tag{17}$$

where K_1 and K_2 are some constants.

This conclusion is not very surprising. A society that has a larger value of T has more overall social benefit than one with a shorter value. However, agents have to look for something other than payoff on the next day to realize this, and one needs to go beyond immediate payoff optimization $D = 1$. A detailed analysis of optimum strategies for larger D seems difficult at present, but seems like an interesting open question.

Generalization of the strategy discussed here to the Kolkata Paise Restaurant problem is straightforward. The strategy is as follows: if an agent was fed at a restaurant of rank k at time step t, he goes to a restaurant of rank $k - 1$ at time $t + 1$. If he found no food at time step t, He picks at random one restaurant, out of the restaurants that had no customers at step t. If the picked restaurant has rank k', he goes to the restaurant with rank $k' - 1$. Then, the average time required to reach a cyclic state is of order $\log N$. And in the cyclic state, each agent gets to sample all the restaurants. The strategy can be made robust against cheaters, if we make the additional rule that if more than one customer shows up at the restaurant of rank k, preference is given to the customer who was served at the rank $(k + 1)$ restaurant the previous day.

An interesting question is the effect of heterogeneity in agents, as far as the value of ϵ is concerned. There may be impatient agents who do not want to wait, and switch with probability 1/2 as soon as the value $\Delta = 0$ is reached. If the number of such agents is N^a, with $a < 1$, it is easy to see that the final efficiency parameter cannot be less than N^{a-1}. In order to get a substantial decrease in inefficiency, the number of such agents should be small.

The optimum value of T, or of the parameter ϵ is not decidable within the framework of our model, without bringing in new parameters like the relative weight of desirables like fairness or social equality, social efficiency or immediate gratification in determining the optimum choice. Also there have to be some general shared values amongst the agents to make this possible. Clearly, a discussion of these issues is beyond the scope of our work.

Acknowledgments

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Appendix

In this Appendix, we discuss the solution of Eq. (12)

$$\sum_{r=0}^{\Delta-1} f_\lambda(r) = \sum_{r=\Delta+1}^{\infty} f_\lambda(r) \tag{A.1}$$

where $f_\lambda(r) = \lambda^r \exp(-\lambda)/\Gamma(r + 1)$, for r not necessarily integer. We want to solve for λ, when Δ is given to be a large positive integer. We want to show in the limit of large Δ, $\lambda - \Delta$ tends to 1/6.

For large λ, the Poisson distribution tends to a gaussian centered at λ, of variance λ. If the distribution for large λ were fully symmetric about the mean, the solution to the above equation would be $\lambda = \Delta$. The fact that the difference between these remains finite is due to the residual asymmetry in the Poisson distribution, for large λ.

For large λ, $f_\lambda(r)$ is a slowly varying function of its argument. We add $f(\Delta)/2$ to both sides of Eq. (12), and approximate the summation by an integration. Then, Eq. (12) can be approximated by

$$\int_0^{\Delta} f_\lambda(r)\mathrm{d}r = \int_{\Delta}^{+\infty} f_\lambda(r)\mathrm{d}r = 1/2. \tag{A.2}$$

We have used the trapezoid rule

$$[f(r) + f(r+1)]/2 \approx \int_r^{r+1} \mathrm{d}r' f(r'). \tag{A.3}$$

It can be shown that the discrepancy between Eqs. (12) and (A.2) is at most of order $(1/\lambda)$.

Then, for large λ, deviations of $f_\lambda(r)$ from the limiting gaussian form can be expanded in inverse half-integer powers of λ

$$f_\lambda(r) = \frac{1}{\sqrt{\lambda}}\phi_0(x) + \frac{1}{\lambda}\phi_1(x) + \cdots \tag{A.4}$$

where x is a scaling variable defined by $x = (r - \lambda)/\sqrt{\lambda}$. Here $\phi_0(x)$ is the asymptotic gaussian part of the distribution, as expected from the central limit theorem, and $\phi_1(x)$ describes the first correction term.

The characteristic function for the Poisson distribution $\tilde{\Phi}_\lambda(k)$ is defined by

$$\tilde{\Phi}_\lambda(k) = \langle e^{ikr} \rangle = \sum_{r=0}^{\infty} e^{ikr} \mathrm{Prob}_\lambda(r) = \exp[\lambda e^{ik} - \lambda]$$
$$= \exp[ik\lambda - k^2\lambda/2 - ik^3\lambda/6 + \cdots]. \tag{A.5}$$

Keeping the terms up to quadratic in k gives the asymptotic gaussian form of the central limit theorem

$$\phi_0(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2). \tag{A.6}$$

The first order correction to this asymptotic form of $\tilde{\Phi}_\lambda(k)$ is given by

$$\tilde{\phi}_1(k) = \frac{-ik^3}{6} \exp(-k^2/2) \tag{A.7}$$

which gives on taking inverse Fourier transforms

$$\phi_1(x) = \frac{1}{6} \frac{d^3}{dx^3} \phi_0(x). \tag{A.8}$$

Substituting the functional forms for $\phi_0(x)$ and $\phi_1(x)$ in Eq. (A.2), we get

$$\int_{-\infty}^{\frac{\Delta-\lambda}{\sqrt{\lambda}}} dx \left[\phi_0(x) + \frac{1}{\sqrt{\lambda}} \phi_1(x) \right] = 1/2. \tag{A.9}$$

Now, $\phi_1(x)$ is an odd function of x, and is zero for $x = 0$. As $\Delta - \lambda$ is small, in the coefficient of $1/\sqrt{\lambda}$, we can replace the upper limit of the integral by zero. Thus we write

$$\int_{-\infty}^{(\Delta-\lambda)/\sqrt{\lambda}} \phi_1(x')dx' \approx \int_{-\infty}^{0} \phi_1(x')dx'. \tag{A.10}$$

But using Eq. (A.8), we get

$$\int_{-\infty}^{0} \phi_1(x')dx' = \frac{1}{6} \frac{d^2}{dx^2} \phi_0(x)|_{x=0} = -\phi_0(0)/6. \tag{A.11}$$

Substituting in Eq. (A.10), we get

$$\int_{-\infty}^{(\Delta-\lambda)/\sqrt{\lambda}} \phi_0(x')dx' = 1/2 - \frac{\phi_0(0)}{6\sqrt{\lambda}} + \mathcal{O}(1/\lambda) \tag{A.12}$$

and comparing terms of order $\lambda^{-1/2}$ we get

$$\lambda - \Delta = 1/6 + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right). \tag{A.13}$$

References

[1] W. Brian Arthur, Amer. Econ. Rev. (Papers and Proc.) 84 (1994) 406.
[2] D. Challet, Y.C. Zhang, Physica A 246 (1997) 407.
[3] D. Challet, Y.C. Zhang, Physica A 256 (1998) 514.
[4] D. Challet, M. Marsili, Y.C. Zhang, Minority Games, Oxford Univ. Press, Oxford, 2005.
[5] E. Moro, in: E. Korutcheva, R. Cuerno (Eds.), Advances in Condensed Matter and Statistical Mechanics, Nova Science Publishers, New York, 2004, arXiv:cond-mat/0402651v1 [cond-mat.dis-nn].
[6] W. Kets, preprint: arXiv:0706.4432v1 [q-fin.GN].
[7] R.W. Rosenthal, Int. J. Game Theory 2 (1973) 65–67.
[8] A. Rapoport, Group Decision and Negotiation 4 (1995) 117–133.
[9] A.S. Chakrabarti, B.K. Chakrabarti, A. Chatterjee, M. Mitra, Physica A 388 (2009) 2420.
[10] A. Ghosh, A.S. Chakrabarti, B.K. Charabarti, in: B. Basu, B.K. Charabarti, A. Ghosh, K. Gangopadhyay (Eds.), Econophysics and Economics of Games, Social Choices and Quantitative Techniques, Springer, Milan, 2010.
[11] A. Ghosh, A. Chatterjee, M. Mitra, B.K. Chakrabarti, New J. Phys. 12 (2010) 075033.
[12] M. Nowak, K. Sigmund, Nature 364 (1993) 56.
[13] G. Reents, R. Metzler, W. Kinzel, Physica A 299 (2001) 253.
[14] A. Cavagna, J.P. Garrahan, I. Giardina, D. Sherrington, Phys. Rev. Lett. 83 (1999) 4429.
[15] D. Challet, M. Marsili, R. Zecchina, Phys. Rev. Lett. 85 (2000) 5008;
A. Cavagna, J.P. Garrahan, I. Giardina, D. Sherrington, Phys. Rev. Lett. 85 (2000) 5009.
[16] Y.B. Xie, B.H. Wang, C.K. Hu, T. Zou, Eur. Phys. J. B 47 (2005) 587.
[17] J.B. Satinover, D. Sornette, Eur. Phys. J. B 60 (2007) 369.

Strategy switches and co-action equilibria in a minority game

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We propose an analytically tractable variation of the minority game in which rational agents use probabilistic strategies. In our model, N agents choose between two alternatives repeatedly, and those who are in the minority get a pay-off 1, others zero. The agents optimize the expectation value of their discounted future pay-off, the discount parameter being λ. We propose an alternative to the standard Nash equilibrium, called co-action equilibrium, which gives higher expected pay-off for all agents. The optimal choice of probabilities of different actions are determined exactly in terms of simple self-consistent equations. The optimal strategy is characterized by N real parameters, which are non-analytic functions of λ, even for a finite number of agents. The solution for $N \leq 7$ is worked out explicitly indicating the structure of the solution for larger N. For large enough future time horizon, the optimal strategy switches from random choice to a win-stay lose-shift strategy, with the shift probability depending on the current state and λ.

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I. INTRODUCTION

There has been a lot of interest in applying techniques of statistical physics to economics in the past two decades, in particular for a better understanding of the behaviour of fluctuations in systems with many interacting agents, as in a market. A prototypical model is the El Farol bar problem [1] in which agents optimize their personal pay-offs by guessing what other agents would be doing. A particular realization of this is the Minority Game (MG) introduced in 1997 by Challet and Zhang [2]. It has been described by Arthur as a classic 'model problem that is simple to describe but offers a wealth of lessons' [3]. In this model, an odd number of agents repeatedly make choices between two alternatives, and at each step the agents who belong to the minority group are considered as winners. In MG, the agents cannot communicate with each other, and base their decision on the common-information which is the history of the winning choices in the last few days.

Each agent has a small number of strategies available with her, and at any time uses her best-performing strategy to decide her immediate future action. The agents are adaptive, and if they find that the strategy they are using is not working well, they will change it. This in turn affects the performance of other agents, who may then change their strategies and so on. Thus, this provides a very simple and instructive model of learning, adaptation, self-organization and co-evolution in a group of interacting agents.

Simulation studies of this model showed that the agents self-organize into a rather efficient state where there are more winners per day than would be expected if agents made the choice by a random throw of a coin, for a range of the parameters of the model. This sparked a flurry of interest in the model, and soon after the original paper of Challet and Zhang, a large number of papers appeared, discussing several aspects of the model, or variations. Several good reviews are available in the literature [4, 5], and there are excellent monographs that describe the known results [6, 7]. It is one of the few non-trivial models of interacting agents that is also theoretically tractable.

However, one would like to understand how well a particular strategy for learning and adaptation works, and compare it with alternate strategies. This strategy to select strategies may be called a meta-strategy. Clearly, the meta-strategy that gives better pay-off to its user will be considered better. In this respect, the meta-strategy used in MG does not work so well. While in some range of parameters, the agents are found to self-organize into a globally efficient state, in other regions of the parameter space (for large number of agents), its overall efficiency is worse than if the agents simply chose the restaurants at random. This is related to the fact that in MG, agents use deterministic strategies, and each agent has only a limited number of deterministic strategies available to her. Also, the rule to select the strategy to use, in terms of performance scores of strategies, is known to be not very effective. In fact, simulation studies have shown [8] that, for some range of parameters, if a small fraction of agents always select the

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strategy with the worst performance score, their average performance, is better than of other agents who are using the usual MG rule of choosing the strategy with the best performance score !.

It seems worthwhile, if only to set a point of reference, to determine how well agents in a Minority-like game could do, if they use some other meta-strategy. In this paper, we will study a model where the agents use mixed strategies. In our formulation, each agent, on each day, selects a probability p, and then generates a new random number uniformly between 0 and 1, and switches her choice from the previous day if it is $\leq p$. The choice of p depends on the history of the game, and her own history of pay-offs in the last m days, and constitutes the strategy of the agent [The deterministic strategies are special cases when p is 0 or 1]. We discuss how the optimal value of this parameter depends on the history of the game.

For this purpose, we propose a new solution concept, as an alternative to the usual notion of Nash equilibrium [9]. We show that the Nash equilibrium states are not very satisfactory for our model, giving rise to 'trapping states' (discussed below), and our proposed alternative, to be called co-action equilibrium, avoids this problem. To distinguish it from the original Minority game, we will refer to the new game as the Co-action Minority game (CAMG), and refer to the original MG as Challet-Zhang Minority Game (CZMG).

In CZMG, on any day, each agent selects one strategy from a small set of deterministic strategies given to her at the beginning of the game. We make the basket of strategies given to the agents much bigger, and make all strategies, within a specified infinite class, available to all agents. The use of stochastic, instead of the deterministic, strategies makes the CAMG more efficient than CZMG. Also, the absence of quenched disorder - in the form of assigning strategies to agents in the beginning of the game - in our model makes it much more tractable. One can determine the behaviour of many quantities of interest in more detail, using only elementary algebra. The theoretical analysis of CZMG requires more sophisticated mathematical techniques such as functional integrals, and taking special limits of large number of agents, large backward time horizon, and large times (explained later in the paper).

We find that the optimal strategies of agents can be determined by a mean-field theory like self-consistency requirement. For the N-agents case, we get coupled algebraic equations in N variables [10]. The simplicity of our analysis makes this model an interesting and instructive, analytically tractable model of interacting agents. Interestingly, this also provides us with a non-trivial example of a non-equilibrium steady state which shows a non-analytic dependence on a control parameter *even for finite number of agents.*

The plan of the paper is as follows: In Sec. II, we recapitulate the main features of the CZMG, and what is known about its behaviour. In Sec. III, we introduce the CAMG game. In Sec. IV, we show that the model has Nash equilibrium states that are trapping states, where all agents stay with the same choice next day, and the system gets into a frozen state. In Sec. V, we introduce the solution concept of co-action equilibrium to avoid these trapping states. Sec. VI develops the general theoretical framework of Markov chains to calculate the expected pay-off functions of agents in CAMG, which is used to determine the optimal strategies by agents. In Sec VII, we work out explicitly, the optimal strategies when the number of agents $N = 3, 5$ and 7, and discuss what one may expect for larger N. In Sec. VIII, we discuss the case of large N, and study the first transition from random state to one where some of the agents choose not to jump. Sec. IX contains a summary of our results, and some concluding remarks.

II. THE CHALLET-ZHANG MINORITY GAME

In CZMG [2], each of the N agents, with N odd, has to choose between two alternatives, say two restaurants A and B, on each day and those in the restaurant with fewer people get a pay-off 1, and others 0. The agents cannot communicate with each other, and make their choice based only on the information of which was the minority restaurant for each of the last m days. A strategy gives which one of the two choices (A or B) is preferred, for each of the 2^m possible histories of the game. The total number of possible strategies is 2^{2^m}. Each agent has a small fixed number k of strategies randomly picked out of all possible strategies at the beginning of the game. For each of the strategies assigned to an agent, she keeps a performance score which tells how often in the past the strategy correctly predicted the winning choice. On each day, she decides which restaurant to go to, using the strategy that performed best in the recent past.

We write $N = 2M + 1$. Clearly, on any day, the number of people that are happy (i.e. having a positive pay-off) is $\leq M$. The amount by which the average number of happy people per day differs from the maximum possible value M is a measure of the social inefficiency of the system. For a system of agents in a steady state $\mathbb{S}$, we will characterize the inefficiency of the system in terms of a parameter η, called the inefficiency parameter, defined as,

$$\eta_{\mathbb{S}} = \frac{W_{max} - \langle W \rangle_{\mathbb{S}}}{W_{max} - W_{rand}}, \tag{1}$$

where $W_{max} = M/N$ is the maximum possible pay-off per agent, $\langle W \rangle_{\mathbb{S}}$ is the average pay-off per agent in the steady state $\mathbb{S}$, and W_{rand} is the average pay-off per agent when agents select randomly between A and B.

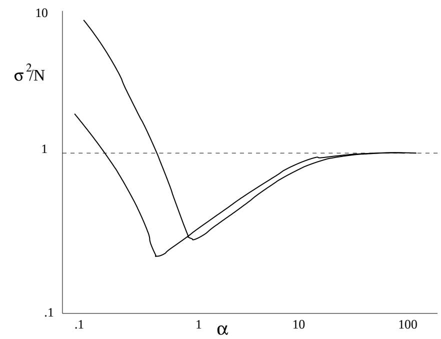

FIG. 1: Schematic representation of the variation of the normalized fluctuation in the attendance difference between the two restaurants σ^2/N with the parameter $\alpha = 2^m/N$ for two different values of k - the number of strategies with each agent. The curve with lower minimum corresponds to $k = 2$ and the other curve corresponds to $k = 3$. The dashed horizontal line shows the value of σ^2/N when agents choose randomly between the two restaurants.

The general qualitative behaviour of MG is quite well understood from simulations. Fig. 1 shows the schematic behaviour of the system as seen in simulations. The theoretical analysis is rather complicated, and involves several limits: large N and m, with $2^m/N = \alpha$ held fixed. Also, one has to rescale time with N, and the exact theoretical results are possible for fixed $\tau = t/N$. The asymptotic behaviour in the steady state can be determined exactly only in these limits, and only for α greater than a critical value α_c, using concepts and formalism developed originally for the spin-glass problem. For a more detailed discussion, see [6, 7].

III. THE CO-ACTION MINORITY GAME

We would like to construct a game which preserves the basic simplicity of the Challet-Zhang minority game, but changes it in several important ways, to make it more tractable. We will keep the allowed actions, and pay-off function the same, but consider different strategies used by agents. We discuss these changes one by one.

A. Stochastic versus deterministic strategies

It is well-known that in repeated games where agents have to make their choices simultaneously, probabilistic strategies are much more effective than deterministic ones. In fact, Arthur, in the forward of [6], recalls that when he introduced the El-Farol Bar problem at a conference, the session chair Paul Krugman had objected that the problem has a simple efficient strategy, where each agent uses a mixed strategy, and decides between the two options by tossing a coin. Of course, in some minority-like games, like managing hedge funds, the agents do not have the option of using probabilistic strategies. In the CAMG, we allow the agents to use mixed strategies. We will show that this results in a different emergent behaviour of the system.

In the CZMG, the agents are assigned a small number of strategies at random, and different agents have different basket of strategies. In CAMG, we allow each agent to choose his shift probability to be any value p, with the only constraint being $0 \leq p \leq 1$. Since the choice of p constitutes the strategy of an agent, each agent is allowed to choose from an infinite set of strategies. Also, the same set of strategies is available to all the agents. Thus we do not have any quenched disorder in the model, and this simplifies the analytical treatment of the model considerably.

Mixed strategies in the context of minority game have been discussed before. An example is the thermal minority game [12]. A somewhat similar model to ours, involving a minority game, with probabilistic strategies was studied earlier by Reents et al [13]. However, there is an important difference between these earlier studies, and ours. In the earlier studies, the probabilities of different actions was thought to be due to a kind of noise in the system, not under the control of the agents. The probability of ‘non-optimal’ choice is externally prescribed in the beginning. In our model, the agent is free to choose the value of p, and chooses it to maximize her pay-off. Also, the agent’s choice can vary from day to day.

B. Rational versus adaptive agents

Another important difference from the CZMG is that we will treat our agents as intelligent and rational who also expect other agents to act rationally. The agents are selfish, and choose their actions only to maximize their personal expected gain. This is an important difference from CZMG, where one of the motivations for introducing the game was to model the behaviour of agents with only bounded rationality who resort to inductive reasoning. In CZMG, the agents follow rather simple rules to decide when to switch their strategies, based on the performance scores of strategies. One may imagine that these agents are unthinking machines, following some pre-programmed instructions.

We will also assume that all agents are equally intelligent. Thus, there is no built-in heterogeneity in the behaviour of agents. When the agents follow deterministic strategies as in CZMG, they are forced to differentiate amongst themselves in the strategies they use, as many agents following the same strategy is clearly not good. When agents use mixed strategies, the strategies need not be different, as the differentiation is naturally provided by the random number generators used by the agents. In fact, this differentiation is rather efficient. We will see that even without any assumed heterogeneity of agents, the system shows non-trivial emergent behaviour, and reaches a more efficient state quicker.

We need not discuss here the question whether full rationality or bounded rationality can describe the behaviour of real-life agents. Clearly, there would be situations where one or the other is a better model. We only note that the general probabilistic 'win-stay lose-shift' strategy has been seen to be used in many real-world learning situations [14], and this is also the strategy that is found to be optimal by rational agents in CAMG.

C. Agents' optimization aim

The next issue is deciding the pay-off function that is optimized by the rational agents. Clearly, maximizing the probability of winning next day is a possible goal. But agents in repeated games need not be concerned only about their immediate payoffs.

We note that if all agents had the same optimization goal of minimizing the system inefficiency, there is no competition between them, and we get a trivial game. There is a simple strategy which will give the best possible result of long-term average of the inefficiency parameter being zero. In this strategy, each day, agents choose some shift probability strictly between 0 and 1, until one finds M people in one restaurant, and $M+1$ in the other. Once this state is reached, each agent goes to the same restaurant on all subsequent days. In this simple strategy, the long-time average gain per agent per day is $M/(2M+1)$, as each agent is equally likely to end up being in the winning set.

Clearly, this optimization goal makes the game trivial. A more reasonable goal for selfish agents would be to try to optimize their personal long-term average pay-off. Selfish agents in the majority restaurant would not be interested in pursuing the strategy outlined above. But the game with selfish agents who aim to maximize personal long-term average pay-off is also easily solved. If we allow agents in the state with $(M, M+1)$ decomposition to shift with a small probability ϵ, then in the long-time steady state, the system jumps between different decompositions of the type $(M, M+1)$. When ϵ is small, in the steady state, the state $(M, M+1)$ still occurs with a large weight, and this weight will tend to 1 as ϵ tends to zero. However, for any non-zero value of ϵ, for times $T \gg 1/\epsilon$, the fraction of time spent in the minority by each agent will be nearly the same. Thus, for any specified history, this strategy still gives expected future long-term pay-off that tends to the highest possible, as ϵ tends to zero.

The problem with the the above game is that, for small ϵ, an agent in the unhappy situation in the $(M, M+1)$ breakup may have to wait very long before her pay-off changes. This suggests that a reasonable model of the agent behaviour would be that he/she does not want to wait for too long for the next winning day. We therefore consider agents who have a finite future time-horizon. Clearly, we can think of agents who try to maximize their net pay-off over the next H days.

It is more convenient to introduce a real parameter λ, lying between 0 and 1, and assume that any agent X only wants to optimize her weighted expected future pay-off,

$$\mathrm{ExpPayoff}(X) = \sum_{\tau=0}^{\infty} [(1-\lambda)\lambda^{\tau}] \langle W_X(\tau+1) \rangle, \tag{2}$$

where $\langle W_X(\tau) \rangle$ is the expected pay-off of the agent X on the τ-th day ahead, and λ is a parameter $0 \leq \lambda < 1$. It is called as the discount parameter in the literature and is easier to deal with than the discrete parameter H. The factor $(1-\lambda)$ has been introduced in the definition of $\mathrm{ExpPayoff}(X)$ so that the maximum possible value of the expected payoff is 1.

The advantage of using the real parameter λ, instead of the discrete parameter H, is that we can study changes in the steady state of the system as we change the parameter λ continuously. We will find that there are strategy switches in the optimal strategies of agents as λ is varied, which leads to discontinuous changes in several properties of the steady state. These are of interest, as they are analogous to dynamical phase transitions in steady states in non-equilibrium statistical mechanics.

Consistent with our assumption of homogeneity of agents, we assume that all agents use the same value of λ.

D. Common information

We assume that all agents know the total number of players N, and they also know that all players use the stochastic strategy of selecting a jump probability based on previous days outcome, and use the same value of discount parameter λ. In the CZMG, on each day, which restaurant was the minority is announced publicly, and this information is available to all the agents. In CAMG, we assume that the common information is more detailed: each agent knows the time-series $\{n(t)\}$ of how many people went to A on different days in the past. We note that in the El Farol bar problem that led to MG, has the same information as in our variation. On any day, the attendance in restaurant A can take $(N+1)$ possible values. Then history of m previous days can take $(N+1)^m$ values, and each strategy is specified by $(N+1)^m$ real numbers.

We restrict our discussion here to the simplest case, where $m = 1$ for all agents. Then, an agent's strategy is solely determined by the number of people who were in the restaurant she went to the previous day. Since this number cannot be zero, the number of possible histories here is N, not $(N+1)$.

IV. THE PROBLEM OF TRAPPING STATES

This model was first defined in [11]. In that paper, we tried to determine the optimal choice of the shift probabilities using the standard ideas of Nash equilibrium. However, we realized that in this problem, there are special states such that none of the agents in that state would prefer to shift to a different restaurant the next day, following a Nash-like analysis. This frozen steady state may be called the trapping state.

The existence of such a trapping state is paradoxical, as rational agents in the majority restaurant have no reason to pursue a strategy that makes them stay in a losing position for ever. The resolution of this paradox requires a new solution concept, that we discuss now.

The most commonly used notion in deciding optimal strategies in N-person games is that of Nash equilibrium: A state of the system in which agent i uses a strategy $\mathcal{S}_i$ is a Nash equilibrium, if for all i, $\mathcal{S}_i$ is the best response of i, assuming that all agents $j \neq i$ use the strategy $\mathcal{S}_j$. There may be more than one Nash equilibria in a given game, and they need not be very efficient. For CAMG also, the Nash equilibrium is not very satisfactory: it gives rise to a trapping state.

Consider, for simplicity, the case $\lambda = 0$, where agents optimize only next day's pay-off. Now, during the evolution of the game, at some time or the other, the system will reach a state with M agents in one restaurant (assume A), and $M+1$ agents in the other restaurant B. What is the best strategy of these agents who want to maximize their expected pay-off for the next day?

We imagine that each agent hires a consultant to advise them. To an agent in A (we will call her Alice), the advise would be to stay put, if the probability that no person switches from the restaurant B is greater than 1/2. If the agents in the restaurant B switch with probability p_{M+1}, the probability that no one switches is $(1-p_{M+1})^{M+1}$. In this case, the expected pay-off of Alice would be $(1-p_{M+1})^{M+1}$. So long as this p_{M+1} is small enough that this pay-off is $> 1/2$, Alice's best strategy would be to choose $p_M = 0$.

If an agent in the restaurant B (let us call him Bob) expects that agents in A would not switch, what is his best response? The consultant argues that if Bob switches, he would be in the majority, and his pay-off would be zero. Hence his best strategy is to set his switching probability p_{M+1} to zero. Then, there is some possibility that he will be in the winning set the next day, if some other agent from B shifts. In fact, with agents in A staying put ($p_M = 0$), the probability that he wins is proportional to his stay-put probability, and is maximized for $p_{M+1} = 0$.

This value $p_{M+1} = p_M = 0$, is then a self-consistent choice corresponding to the the fact that the choice $p_M = 0$ is an individual agent's best response to opposite restaurant's people choosing $p_{M+1} = 0$, and vice versa. It is a Nash equilibrium.

This advice is given to all agents in restaurant B, and then no one shifts, and the situation next day is the same as before. Thus, the system gets trapped into a state where all agents stick to their previous day's choice. In this state, the total number of happy people is the best possible, and the state has the best possible global efficiency. However, this situation is very unsatisfactory for *the majority of agents* (they are on the losing side for all future days). Setting p_{M+1} equal to zero by agents in B, is clearly not an optimal choice.

Let us denote the state of an agent who is in a restaurant with total of i people in it as $\mathcal{C}_i$. In the Nash equilibrium concept, an agent in the state C_{M+1}, who believes that agents in the opposite restaurant would be setting their switch probability $p_M = 0$, is advised that his best response is to set $p_{M+1} = 0$. If other agents are using $p_{M+1} = 0$, one cannot do better by moving. If other agents in the restaurant switch with probability $p_{M+1} \neq 0$, this is the 'optimal' solution. *This does not take into account the fact that if all agents follow this advice, their expected future gain is zero*, which is clearly unsatisfactory: No other advice could do worse!

In our previous paper [11], we realized this problem, but could see no way out within the Nash solution concept. We adopted an *ad hoc* solution, where the agents in state $\mathcal{C}_{M+1}$ were required by external fiat to shift with a non-zero probability ϵ. The alternate solution concept of co-action equilibrium provides a natural and rational solution to this problem. This is explained in the next section. Note that the presence of even a small number of agents who always choose randomly would keep the system away from the trapping state [15].

V. AN ALTERNATE SOLUTION CONCEPT: CO-ACTION EQUILIBRIUM

The problem with the consultant's reasoning lies in the Nash-analysis assumption of optimizing over strategies of one agent, *assuming that other agents would do as before*. Let the marked agent be denoted by X. All agents in the same retaurant, who are not X denoted by X'. Then, the agent X determines his jump probability p_X to optimize his expected payoff $\mathrm{ExpPayoff}(X) = (1 - p(X))(1 - \prod_{X'}(1 - p(X'))$. In this case, by varying with respect to $p(X)$, keeping all $p(X')$ constant, the payoff is clearly maximized at $p(X) = 0$.

In the alternate co-action equilibrium concept proposed here, an agent in state C_i realizes that she can choose her switching probability p_i, but all the other fully rational $(i-1)$ agents in the same restaurant, with the same information available, would argue similarly, *and choose the same value of* p_i. Determining the optimal value of p_i that maximizes the pay-off of agents in state C_i does not need communication between the agents.

If $p_M = 0$, then the expected pay-off W_{M+1} of an agent in restaurant B is clearly given by the probability that he does not shift, but at least one of the other agents in his restaurant does. This is easily seen to be $q_{M+1}(1 - q_{M+1}^M)$, where $q_{M+1} = 1 - p_{M+1}$. This is zero for $q_{M+1} = 0$ or 1, and becomes maximum when $q_{M+1} = (M+1)^{-1/M}$. In particular, q_{M+1} equal to 1 is *no longer the optimal response*.

One may argue that this solution concept is not so different from the usual Nash equilibrium, if one thinks of this as a two-person game each day, where the two persons are the majority and the minority groups, and they select the optimal values of their strategy parameters p_i and p_{N-i}. The important point is that these groupings are temporary, and change with time. For non-zero λ, one cannot think of this game as a series of two-person games.

In our model, the complete symmetry between the agents, and the assumption of their being fully rational, ensures that they will reach the co-action equilibrium.

Note that an agent in B may wants to 'cheat' by deciding not to shift, assuming that other agents would shift with a nonzero probability. But this is equivalent to setting his strategy parameter $p = 0$. Our assumption of rationality then implies that all other agents, in the same situation, would argue in the same way, and do the same.

VI. DETERMINING THE OPTIMAL MIXED STRATEGY

For a given N, a person's full strategy $\mathbb{P}$ is defined by the set of N numbers $\mathbb{P} \equiv \{p_1, p_2, ...p_N\}$. In CAMG, all rational agents would end up selecting the same optimal values of strategy parameters $\{p_1^*, p_2^*, \ldots\}$. It would have been very inefficient for all agents to use the same strategy, if they were using deterministic rules. This is not so in CAMG. We now discuss the equilibrium choice $\{p_1^*, p_2^*, \ldots p_N^*\}$. The co-action equilibrium condition that p_i^* is chosen to maximize the expected pay-off of agent in state C_i, implies N conditions on the N parameters $\{p_i^*\}$. There can be more than one self-consistent solution to the equations, and each solution corresponds to a possible steady state.

Clearly, as all agents in the restaurant with i agents switch independently with probability p_i, the system undergoes a Markovian evolution, described by a master equation. As each agent can be in one of the two states, the state space of the Markov chain is 2^N dimensional. However, we use the symmetry under permutation of agents to reduce the Markov transition matrix to $N \times N$ dimensional. Let $|Prob(t)\rangle$ be an N-dimensional vector, whose j-th element is $Prob_j(t)$, the probability that a marked agent X finds herself in the state C_j on the t-th day. On the next day, each agent will switch according to the probabilities given by $\mathbb{P}$, and we get

$$|Prob(t+1)\rangle = \mathbb{T}|Prob(t)\rangle, \qquad (3)$$

where $\mathbb{T}$ is the $N \times N$ Markov transition matrix.

Explicit matrix elements are easy to write down. For example, $\mathbb{T}_{11}$ is the conditional probability that the marked agent will be in state C_1 on the next day, given that she is in C_1 today. This is the sum of two terms: one corresponding to everybody staying with the current choice [the probability of this is $(1-p_1)(1-p_{N-1})^{N-1}$], and another corresponding to all agents switching their choice [the probability is $p_1 p_{N-1}^{N-1}$].

The total expected pay-off of X, given that she is in the state C_j at time $t=0$ is easily seen to be

$$W_j = (1-\lambda)\left\langle L \left| \frac{\mathbb{T}}{1-\lambda\mathbb{T}} \right| j \right\rangle, \tag{4}$$

where $|j\rangle$ is the column vector with only the j-th element 1, and rest zero; and $\langle L|$ is the left-vector $\langle 1,1,1,1,..0,0,0..|$, with first $M=(N-1)/2$ elements 1 and rest zero.

In fact, we can use the permutation symmetry between the agents to block-diagonalize the matrix $\mathbb{T}$ into a two blocks of size $(M+1)$ and M. This is achieved by a change of basis, from vectors $|i\rangle$ and $|N-i\rangle$ to the basis vectors $|s_i\rangle$ and $|a_i\rangle$, where

$$\begin{aligned} |s_i\rangle &= |i\rangle + |N-i\rangle, \\ |a_i\rangle &= (N-i)|i\rangle - i|N-i\rangle. \end{aligned} \tag{5}$$

This choice is suggested by the fact that in the steady state

$$Prob(C_i)/i = Prob(C_{N-i})/(N-i). \tag{6}$$

It is easily verified that using the basis vectors $|s_i\rangle$ and $|a_i\rangle$, the matrix $\mathbb{T}$ is block-diagonalized.

One simple choice is that $p_i^* = 1/2$ for all i, which is the random choice strategy, where each agent just picks a restaurant totally randomly each day, independent of history. We will denote this strategy by $\mathbb{P}_{rand}$. In the corresponding steady state, it is easy to see that W_j is independent of j, and is given by

$$W_j = W_{rand} = 1/2 - \binom{N-1}{M} 2^{-N}, \quad \text{for all } j. \tag{7}$$

By the symmetry of the problem, it is clear that $p_N^* = 1/2$ for all λ. Now consider the strategy $\{p_i^*\} = \{p_1^*, 1/2, 1/2, 1/2...\}$. If X is in the state C_1, and next day all other agents would switch with probability 1/2, it does not matter if X switches or not: payoffs W_1 and W_{N-1} are independent of p_1^*. Hence p_1^* can be chosen to be of any value. It is easy to see that the strategy $\mathbb{P}'_{rand}$ in which $p_1^* = 0$ and $p_{N-1}^* < 1/2$, chosen to maximize W_{N-1}, is better for all agents and is stable, and hence is always preferred over $\mathbb{P}_{rand}$.

VII. EXACT SOLUTION FOR SMALL N

A. $N = 3$

We consider first the simplest case $N=3$. Since $p_1^* = 0$, $p_3^* = 1/2$, the only free parameter is p_2^*. The value of p_2^* is decided by the agents in state C_2, and they do it by maximizing W_2.

In this case, the transfer matrix is easily seen to be

$$\mathbb{T} = \begin{bmatrix} q_2^2 & p_2 q_2 & 1/4 \\ 2p_2 q_2 & q_2 & 1/2 \\ p_2^2 & p_2^2 & 1/4 \end{bmatrix}, \tag{8}$$

where $q_2 = 1 - p_2$. The pay-off W_2 is given by

$$\begin{aligned} W_2 &= (1-\lambda)\,[1\ 0\ 0]\,\frac{\mathbb{T}}{(1-\lambda\mathbb{T})}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \\ &= \frac{4p_2q_2 - \lambda p_2(q_2-p_2)}{(1-\lambda q_2(q_2-p_2))(4+\lambda(4p_2^2-1))}. \end{aligned} \tag{9}$$

The eigenvalues of the transfer matrix $\mathbb{T}$ are easily seen to be $\left(1,\ \frac{1}{4}\left(1-4p_2^2\right),\ q_2\left(q_2-p_2\right)\right)$. The eigen vectors are easily written down. The average gain in the steady state W_{avg} is seen to be

$$W_{avg} = \frac{1}{3+4p_2^2}. \tag{10}$$

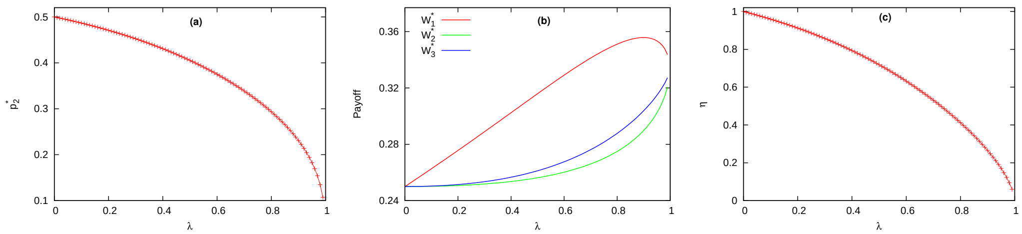

FIG. 2: $N = 3$: (a) Variation of p_2^* with λ, (b) The optimum payoffs W_i^*, ($i = 1$ to 3), as functions of λ and (c) Inefficiency η as a function of λ. .

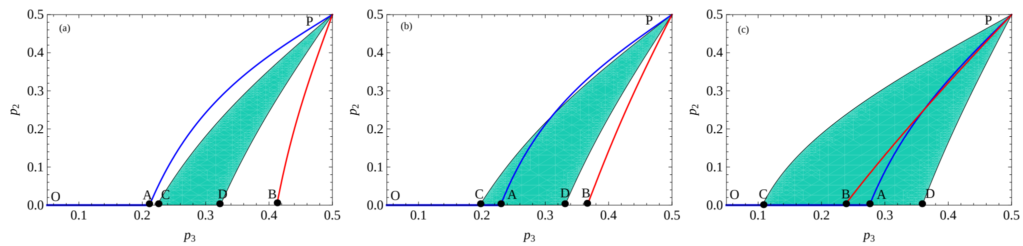

FIG. 3: Region in the p_2-p_3 plane showing the best responses $r_2^{opt}(p_3)$ (blue) and $r_3^{opt}(p_2)$ (red) for agents in state $|2\rangle$ and $|3\rangle$ respectively, for (a) $\lambda = .1$, (b) $\lambda = .4$ and (c) $\lambda = .8$. The line PC and PD show the curves $w_3 = W'$ and $W_2 = W'$ respectively. In the curvilinear triangle PCD, all agents do at least as well as at P.

From Eq. 9, the value of p_2 that maximizes W_2 is easily seen to be root of the following cubic equation in λ.

$$\begin{aligned}&16 - 32p_2^* - (24 - 56p_2^* + 32p_2^{*2})\lambda + (9 - 28p_2^* + 40p_2^{*2}\\&- 96p_2^{*3} + 144p_2^{*4} - 64p_2^{*5})\lambda^2 - (1 - 4p_2^* + 8p_2^{*2} - 24p_2^{*3}\\&+ 48p_2^{*4} - 32p_2^{*5})\lambda^3 = 0.\end{aligned}$$

The variation of p_2^* with λ is shown in Fig 2a. p_2^* monotonically decreases with λ from its value $1/2$ at $\lambda = 0$, and tends to 0 as λ tends to 1. The pay-off of agents in various states with this optimum strategy is shown in Fig. 2b and the variation of the inefficiency parameter η with λ is shown in Fig. 2c.

It is easily seen that W_{avg} is a monotonically increasing function of λ, and tends to the maximum possible value $W_{max} = 1/3$ as $\lambda \to 1$. The variation of the inefficiency parameter η with λ is shown in Fig. 2c. In particular, it is easily seen that η varies as $(1-\lambda)^{2/3}$, as λ tends to 1.

B. $N = 5$

We can similarly determine the optimal strategy for $N = 5$. This is characterized by the five parameters $(p_1^*, p_2^*, p_3^*, p_4^*, p_5^*)$. The simplest strategy is $\mathbb{P}_{rand}$, which corresponds to $p_i^* = 1/2$, for all i. As explained above, the strategy $\mathbb{P}'_{rand} = (0, 1/2, 1/2, p_4^*(\lambda), 1/2)$, gives higher pay-off than $\mathbb{P}_{rand}$ for all agents, for all λ.

Now consider agents in the states C_2 and C_3. What values of p_2 and p_3 they would select, given their expectation/belief about the selected values of p_1, p_4 and p_5 ?. We can determine these by analyzing the variation of payoffs W_2 and W_3 as functions of p_2 and p_3 for fixed values of p_1, p_4, p_5 and λ as discussed below.

Let us denote the best response of agents in state C_2 (that maximizes W_2), if the agents in the opposite restaurant jump with probability p_3 by $r_2^{opt}(p_3)$. Similarly, $r_3^{opt}(p_2)$ denotes the best response of agents in state C_3, when those in the opposite restaurant jump with probability p_2.

In Fig. 3, we plot the functions $r_2^{opt}(p_3)$ (OAP) and $r_3^{opt}(p_2)$ (BP), in the (p_3, p_2) plane, for three representative values of λ. For small p_3, $r_2^{opt}(p_3)$ remains zero, and its graph sticks to x-axis initially, (segment OA in figure), and then increases monotonically with p_3. The strategy $\mathbb{P}'_{rand}$ is the point $(1/2, 1/2)$, denoted by P. We also show the lines PC corresponding to $W_3 = W'$, and PD, corresponding to $W_2 = W'$, where W' is the expected gain of agents in

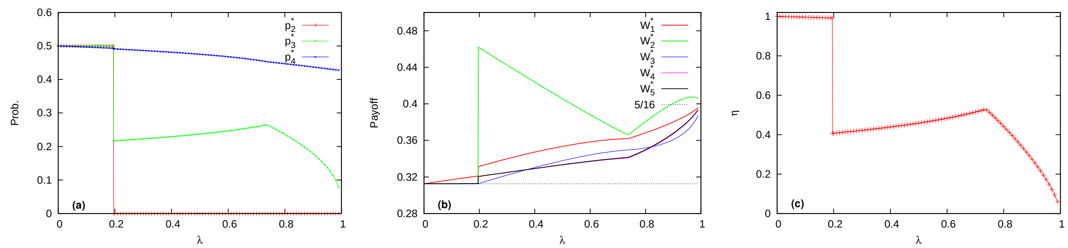

FIG. 4: $N = 5$: (a) Variation of p_2^*, p_3^* and p_4^* with λ , (b) Optimum payoffs as functions of λ, (c) Inefficiency η as a function of λ.

state C_2 or C_3 under $\mathbb{P}'_{rand}$. For all points in the curvilinear triangle PCD, both W_2 and $W_3 \geq W'$. Clearly, possible equilibrium points are the points lying on the lines $r_2^{opt}(p_3)$, or $r_3^{opt}(p_2)$ that lie within the curvilinear triangle PCD. However, along the blue curve OAP representing $r_2^{opt}(p_3)$, maximum value for W_2 is achieved when $p_2 = 0$. Therefore we can restrict the discussion of possible equilibrium points to the line segment CD in Fig. 3.

For small λ (shown in Fig. 3a for $\lambda = 0.1$), The point A is to the left of C, and the only possible self-consistent equilibrium point is P. For example, if agents in the state C_3 (Bob) assumes that agents in the minority restaurant (Alice) is going to set $p_2^* = 0$, Bob can get better pay-off than $\mathbb{P}'_{rand}$, by choosing his probability parameter p_3^* in the range CD in Fig 3a. But a rational Alice would not choose $p_2^* = 0$, if she expects p_3^* to be in the range CD. Similar argument rules out all points in the colored curvilinear triangle PCD as unstable. This implies that the agents would choose $p_2^* = p_3^* = 1/2$. This situation continues for all $\lambda < \lambda_{c1} = 0.195 \pm 0.001$.

For $\lambda > \lambda_{c1}$, the point A is to the right of C. This is shown in Fig. 3b, for $\lambda = 0.4$. In this case, possible equilibrium points lie on the line-segment CA, and out of these, A will be chosen by agents in state C_3. At A, both W_2 and W_3 are greater than W', and hence this would be preferred by all. Further optimization of p_4 changes p_3 and p_4 only slightly.

As we increase λ further, for $\lambda > \lambda_{c2}$ [numerically, $\lambda_{c2} = 0.737 \pm 0.001$], the point B comes to the left of A. Out of possible equilibria lying on the line-segment CA, the point preferred by agents in state C_3 is no longer A, but B. The self-consistent values of p_2^*, p_3^*, and p_4^* satisfying these conditions and the corresponding payoffs are shown in Fig. 4a and Fig. 4b respectively.

In Fig. 4c, we have plotted the inefficiency parameter η as a function of λ. For $\lambda < \lambda_{c1}$, there are possible values of p_2^* and p_3^*, that would increase the expected pay-off for everybody. However, Alice and Bob can not be sure that the other party would not take advantage of them, and hence stick to the default sub-optimal-for-both choice $p_2^* = p_3^* = 1/2$.

Also, in the range $\lambda_{c1} < \lambda < \lambda_{c2}$, the inefficiency rises as the agents optimize for farther into the future. This may appear paradoxical at first, as certainly, the agents could have used strategies corresponding to lower λ. This happens because though the game for larger λ is slightly less efficient overall, in it the majority benefits more, as the difference between the optimum payoffs W_2^* and W_3^* is decreased substantially (Fig. 4b).

We note that the optimal strategies, and hence the (non-equilibrium) steady state of the system shows a non-analytic dependence on λ, even for finite N. This is in contrast to the case of systems in thermal equilibrium, where mathematically sharp phase transitions can occur only in the limit of infinite number of degrees of freedom N. This may be understood by noting that the fully optimizing agents in CAMG make it more like an equilibrium system at zero-temperature. However note that unlike the latter, here the system shows a lot of fluctuations in the steady state.

C. Higher N

For higher values of N, the analysis is similar. For the case $N = 7$, we find that there are four thresholds λ_{ci}, with $i = 1$ to 4. For $\lambda < \lambda_{c1}$, the optimal strategy has the form $(0, 1/2, 1/2, 1/2, 1/2, p_6^*, 1/2)$. For $\lambda_{c1} \leq \lambda \leq \lambda_{c2}$, we get $p_3^* = 0$, and $p_4^* < 1/2$. For still higher values $\lambda_{c2} < \lambda \leq \lambda_{c3}$, agents in the states C_2 and C_5 also find it better to switch to a win-stay lose-shift strategy, and we get $p_2^* = 0$, $p_5^* < 1/2$. The transitions at λ_{c3} and λ_{c4} are similar to the second transition for $N = 5$, in the (p_4, p_3) and (p_5, p_2) planes respectively. Numerically, we find $\lambda_{c1} \approx 0.47, \lambda_{c2} \approx 0.52, \lambda_{c3} \approx 0.83$ and $\lambda_{c4} \approx 0.95$. We present some graphs for the solution for $N = 7$. Fig. 5a shows variation of the optimum switch probabilities in various states and Fig. 5b shows the variation of the optimum payoffs. Fig. 5c shows the variation of inefficiency with λ. The general structure of the optimum strategy is thus

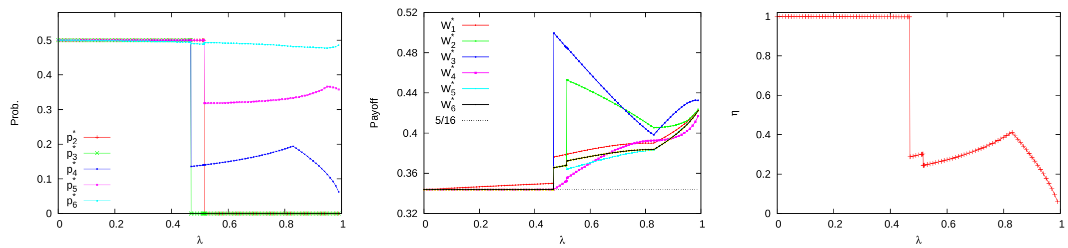

FIG. 5: $N = 7$: (a) Variation of optimum switch probabilities with λ , (b) Optimum payoffs as functions of λ, (c) Inefficiency η as a function of λ.

clear. As λ is increased, it changes from random switching to a complete win-stay lose-shift strategy in stages.

An interesting consequence of the symmetry between the two restaurants is the following: If there is a solution $\{p_i^*\}$ of the self-consistent equations, another solution with all payoffs unchanged can be obtained by choosing for any j, a solution $\{p_i^{*\prime}\}$, given by $p_j^{*\prime} = 1 - p_j^*$, and $p_{N-j}^{*\;\prime} = 1 - p_{N-j}$, and $p_i' = p_i$, for all $i \neq j$ and $i \neq N - j$. How agents choose between these symmetry related 2^M equilibria can only be decided by local conventions. For example, if all agents follow the 'Win-stay lose-shift' convention, this would select a unique equilibrium point.

VIII. THE LARGE-N LIMIT

In this section, we discuss the transition from the random strategy $\mathbb{P}_{rand}$, with all $p_j = 1/2$, to the strategy $\mathbb{P}_1$, in which with $p_M^* = p_{M+1}^* = 1/2$, and $p_j = 1/2$, for all other j. We will determine the value of $\lambda_{c1}(N)$ where this switch occurs.

The difference between the average payoffs in the strategies $\mathbb{P}_{rand}$ and $\mathbb{P}'_{rand}$ is only of order 2^{-N}, and may be ignored for large N.

In calculating the expected payoffs for strategy $\mathbb{P}_1$, it is convenient to group the states of the system into three groups: $|M\rangle, |M+1\rangle$, and the rest. These will de denoted by $|e_1\rangle, |e_2\rangle$ and $e_3\rangle$ respectively.

The transition matrix $\mathbb{T}$ may be taken as a 3×3 matrix. We consider the case when p_{M+1} is $\mathcal{O}(M^{-5/4})$. Then $\mathbb{T}_{21}$ is $\mathcal{O}(M^{-1/4})$. It is convenient to write $\mathbb{T}_{21} = aM^{-1/4}$, and use a as variational parameter, rather than p_{M+1}. We also write $b = (1-\lambda)M^{3/4}$. We consider the case where a and b are finite, and $\mathcal{O}(1)$. The transition probabilities $\mathbb{T}_{12} = \mathbb{T}_{21} = aM^{-1/4}$, and $\mathbb{T}_{31} = \mathbb{T}_{32} = a^2M^{-1/2}/2$, to leading order in M. Also $\mathbb{T}_{13} = \mathbb{T}_{23}$ is the probability that, when all agents are jumping at random, the marked agent will find himself in the state $|M\rangle$, (equivalently in state $|M+1\rangle$). For large N, this is well-approximated by the Gaussian approximation, and keeping only the leading term, we write $\mathcal{W}_{13} = \mathcal{W}_{23} = cM^{-1/2}$, where $c = 1/\sqrt{\pi}$.

Therefore we can write the transition matrix $\mathbb{T}$, keeping terms only up to $\mathcal{O}(M^{-1/2})$ as,

$$\mathbb{T} = \begin{bmatrix} 1 - aM^{-1/4} - \dfrac{a^2M^{-1/2}}{2} & aM^{-1/4} & cM^{-1/2} \\ aM^{-1/4} & 1 - aM^{-1/4} - \dfrac{a^2M^{-1/2}}{2} & cM^{-1/2} \\ \dfrac{a^2M^{-1/2}}{2} & \dfrac{a^2M^{-1/2}}{2} & 1 - 2cM^{-1/2} \end{bmatrix}. \tag{11}$$

Using the symmetry between the states $|e_1\rangle$ and $|e_2\rangle$, it is straight forward to diagonalize $\mathcal{W}$. Let the eigenvalues be μ_i, with $i = 1, 2, 3$, and the corresponding left and right eigenvectors be $\langle L_i|$ and $|R_i\rangle$.

For the steady state eigenvalue $\mu_1 = 1$, we have

$$\langle L_1| = [1, 1, 1]\,; \quad |R_1\rangle = \frac{1}{a^2 + 4c} \begin{bmatrix} 2c \\ 2c \\ a^2 \end{bmatrix}.$$

The second eigenvalue is $\mu_2 = 1 - \frac{a^2+4c}{2} M^{-1/2}$, and we have

$$\langle L_2| = \frac{1}{a^2+4c}\left[a^2, a^2, -4c\right]; \quad |R_2\rangle = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \end{bmatrix}.$$

The third eigenvalue is $\mu_3 = 1 - 2aM^{-1/4} - a^2 M^{-1/2}/2$, and we have

$$\langle L_3| = [1/2, -1/2, 0]; \quad |R_3\rangle = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

It is easily verified that $\langle L_i|R_j\rangle = \delta_{ij}$.

Now, we calculate the expected values of the payoff. We note that if an agent is in the state $|e_3\rangle$, not only her exact state is uncertain, but even her expected payoff depends on whether she reached this state from $|e_3\rangle$ in the previous day, or from $|e_2\rangle$. This is because the expected payoff in this state depends on previous history of agent. However, her expected payoff *next day* depends only on her current state (whether $|e_1\rangle$ or $|e_2\rangle$ or $|e_3\rangle$).

The expected payoff vector for the next day is easily seen to be

$$\left[W_{e_1}^{(0)}, W_{e_2}^{(0)}, W_{e_3}^{(0)}\right] = \left[1 - aM^{-1/4} - a^2M^{-1/2}/2,\ aM^{-1/4} + a^2M^{-1/2}/2,\ 1/2 - dM^{-1/2}\right], \tag{12}$$

where $d = 1/(2\sqrt{\pi})$. The expected payoff after n days is given by $\left[W_1^{(0)}, W_2^{(0)}, W_3^{(0)}\right]\mathbb{T}^{n-1}$. Then the discounted expected payoff with parameter λ is given by

$$[W_{e_1}, W_{e_2}, W_{e_3}] = \left[W_{e_1}^{(0)}, W_{e_2}^{(0)}, W_{e_3}^{(0)}\right]\frac{(1-\lambda)}{(1-\lambda\mathbb{T})}. \tag{13}$$

We write

$$\mathbb{T} = \sum_{i=1}^{3} |R_i\rangle \mu_i \langle L_i|, \tag{14}$$

and hence write

$$[W_{e_1}, W_{e_2}, W_{e_3}] = \sum_{i=1}^{3} U_i \langle L_i|, \tag{15}$$

where

$$U_i = \left[W_{e_1}^{(0)}, W_{e_2}^{(0)}, W_{e_3}^{(0)}\right] |R_i\rangle \frac{(1-\lambda)}{(1-\lambda\mu_i)}. \tag{16}$$

Now, explicitly evaluate U_i. We see that U_1 is independent of λ, and is the expected payoff in the steady state. The terms involving $M^{-1/4}$ cancel, and we get

$$U_1 = \frac{1}{2} - \frac{da^2}{(a^2+4c)} M^{-1/2}. \tag{17}$$

For U_2, we note that $\left[W_{e_1}^{(0)}, W_{e_2}^{(0)}, W_{e_3}^{(0)}\right] |R_2\rangle$ is of order $M^{-1/2}$, and $\frac{(1-\lambda)}{(1-\lambda\mu_2)}$ is of order $M^{-1/4}$, hence this term does not contribute to order $M^{-1/2}$.

The third term is U_3. Here the matrix element $\left[W_{e_1}^{(0)}, W_{e_2}^{(0)}, W_{e_3}^{(0)}\right] |R_3\rangle$ is $\mathcal{O}(1)$, and $\frac{(1-\lambda)}{(1-\lambda\mu_3)}$ is of $\mathcal{O}(M^{-1/2})$, giving

$$U_3 = (b/2a)M^{-1/2} + \mathcal{O}(M^{-3/4}). \tag{18}$$

Putting these together, we get that W_{e_2} is given by

$$W_{e_2} = 1/2 + M^{-1/2}\left[-\frac{b}{4a} - d + \frac{4dc}{a^2+4c}\right] + \mathcal{O}(M^{-3/4}). \tag{19}$$

The agents in state $|e_2\rangle$ will choose the value $a = a^*$ to maximize this payoff W_{e_2} with respect to a. Hence we have

$$b = \frac{32a^{*3}dc}{(a^{*2} + 4c)^2}. \tag{20}$$

For any given b, we can solve this equation for a^*. Then, at this point, the expeced payoff W_{e_2} is

$$W_{e_2} = 1/2 - dM^{-1/2}\left[1 - \frac{4c(4c - a^{*2})}{(a^{*2} + 4c)^2}\right]. \tag{21}$$

This quantity is greater than the expected payoff in the fully random state, so long as $a^{*2} < 4c$, i.e.

$$b < b_{max} = 2\pi^{-3/4}. \tag{22}$$

Thus, we see that if $\lambda > 1 - b_{max}M^{-3/4}$, there exists a nontrivial solution $a^*(b)$ satisfying Eq. (20), with $(a^*)^2 < 4c$, and then the strategy in which agents in state C_M stay, and C_{M+1} shift with a small probability is beneficial to all. Note that the future time horizon of agents only grows as a sub-linear power of M, while in the large M limit, in CZMG, the time-scales grow (at least) linearly with M.

This large M limit is somewhat subtle, as there are three implicit time scales in the problem: The average time-interval between transitions between the states $|e_1\rangle$ and $|e_2\rangle$ is of $\mathcal{O}(M^{1/4})$ days. Jumps into the state $|e_3\rangle$ occur at time-scales of $\mathcal{O}(M^{1/2})$ days. Once in the state $|e_3\rangle$, the system tends to stay there for a time of $\mathcal{O}(M^{1/2})$ days, before a fluctuation again brings it to the state $|e_1\rangle$ or $|e_2\rangle$. The third time scale of $\mathcal{O}(M^{3/4})$ is the minimum scale of future horizon of agents required if the small per day benefit of a more efficient steady state of $\mathcal{O}(M^{-1/2})$ is to offset the cumulative disadvantage to the agents in state $|e_2\rangle$ of $\mathcal{O}(M^{1/4})$.

Note that the above analysis only determines the critical value of λ above which the strategy $\mathbb{P}_1$ becomes preferred over $\mathbb{P}_{rand}$. This would be the actual critical value of λ if the transition to the win-stay-lose-shift occurs in stages, as is suggested by the small N examples we worked out explicitly. However, we cannot rule out the possibility that for N much larger than 7, the shift does not occur in stages, but in one shot, and such a strategy (similar to the one described in [11]) may be preferred over $\mathbb{P}_{rand}$ at much lower values of λ.

IX. SUMMARY AND CONCLUDING REMARKS

In this paper, we have analyzed a stochastic variant of the minority game, where the N agents are equal (no quenched randomness in strategies given to agents). This permits an exact solution in terms of N self-consistently determined parameters. The solution shows multiple sharp transitions as a function of the discount parameter λ, even for finite N. The main reason for the improved efficiency is that random number generators used by agents are much more effective in providing controlled differentiation between them than scoring methods for strategies. Also, the agents actually optimize the value of jump probability, and not use some preassigned noise parameter. In general, the performance using the CAMG is found to be better than in CZMG. Also, there is some numerical evidence, and a qualitative argument that the relaxation time to reach the steady state increases rather slowly, roughly as $\log N$, compared to time of order N days in CZMG [11].

Our treatment of the model here differs from that in [11]: in that paper, the game was discussed only for $\lambda = 0$ (corresponds to agents optimizing only next day's payoff), and in terms of Nash equilibrium. Within the Nash solution concept, it was not clear how to avoid the problem of trapping states, and we had made an ad hoc assumption that whenever the system reaches a trapping state, a major resetting event occurs where all agents switch restaurant with some largish probability. In the co-action equilibrium concept proposed here, the decision to switch to $p = 1/2$ or not is made rationally by the agents themselves depending upon their future time horizon.

Generalizations of the model where agents look back further than last day are easy to define, but even in the case $N = 3$, this already becomes quite complicated, involving a simultaneous optimization over 9 parameters. Introducing inhomogeneity in the agents, say agents with different time horizons, is much more difficult. In such a game, even if agents knew what fraction use what discount parameter, knowing only the record of attendances, would have to guess the fraction in their restaurant, and this makes the problem much harder to analyse. The technique can be used to study other games with different pay-off functions, e.g. agents win only when their restaurant has attendance exactly equal to some specified number r, and these appear to be interesting subjects for further study.

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[1] W. Brian Arthur, Amer. Econ. Rev. (Papers and Proc.) **84** 406, 1994.
[2] D. Challet and Y. C. Zhang, Physica A, **246** 407 (1997).
[3] W. Brian arthur, in the foreword of [6].
[4] E. Moro, in *Advances in Condensed Matter and Statistical Mechanics*, Eds. E. Korutcheva and R. Cuerno (Nova Science Publishers, New York, 2004).
arXiv:cond-mat/0402651v1 [cond-mat.dis-nn].
[5] W. Kets, Journal of Economic Surveys, Vol. 26, Issue 5, pp. 865-878, 2012.
[6] D. Challet, M. Marsili and Y. C. Zhang, *Minority Games: Interacting agents in financial markets* (Oxford Univ. Press, Oxford, 2005).
[7] A. C. C. Coolen *The mathematical theory of minority games: Statistical mechanics of interacting agents* (Oxford Univ. Press, Oxford, 2005).
[8] J. B. Satinover and D. Sornette, Eur. Phys. J. B **60** 369 (2007).
[9] For a discussion of other alternates to Nash Equilibrium, see John. C. Harsanyi and R. Selten, *A general theory of equlibrium selection in games* (MIT Press, Cambridge, 1988).
[10] For other applications of mean-field methods to game theory, see O. Gueant, J.-M. Lastry and P. -L. Lions, Lecture Notes in Mathematics, Vol. 2003, Eds. A. R. Carmona et al., 205 (Springer, Berlin, 2010).
[11] D. Dhar, V. Sasidevan and Bikas K. Chakrabarti, Physica A, **390** 3477 (2011).
[12] A. Cavagna, A. P. Garrahan, I. Giardina and D. Sherrington, Phys. Rev. Lett. **83** 4429 (1999).
[13] G. Reents, R. Metzler and W. Kinzel, Physica A, **299** 253 (2001).
[14] M. Nowak and K. Sigmund, Nature, **364** (1993) 56.
[15] S. Biswas, A. Ghosh, A. Chatterjee, T. Naskar and Bikas K. Chakrabarti, Phys. Rev. **E 85**, 031104 (2012).

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Continuum percolation of overlapping disks with a distribution of radii having a power-law tail

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We study the continuum percolation problem of overlapping disks with a distribution of radii having a power-law tail; the probability that a given disk has a radius between R and $R + dR$ is proportional to $R^{-(a+1)}$, where $a > 2$. We show that in the low-density nonpercolating phase, the two-point function shows a power-law decay with distance, even at arbitrarily low densities of the disks, unlike the exponential decay in the usual percolation problem. As in the problem of fluids with long-range interaction, we argue that in our problem, the critical exponents take their short-range values for $a > 3 - \eta_{sr}$ whereas they depend on a for $a < 3 - \eta_{sr}$ where η_{sr} is the anomalous dimension for the usual percolation problem. The mean-field regime obtained in the fluid problem corresponds to the fully covered regime, $a \leqslant 2$, in the percolation problem. We propose an approximate renormalization scheme to determine the correlation length exponent ν and the percolation threshold. We carry out Monte Carlo simulations and determine the exponent ν as a function of a. The determined values of ν show that it is independent of the parameter a for $a > 3 - \eta_{sr}$ and is equal to that for the lattice percolation problem, whereas ν varies with a for $2 < a < 3 - \eta_{sr}$. We also determine the percolation threshold of the system as a function of the parameter a.

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I. INTRODUCTION

In problems like effective modeling of random media, the continuum models of percolation are more realistic than their lattice counterparts. So, much effort has been put into the study of such systems in the recent past. In two dimensions, the model systems studied involve disks, squares, etc., of the same size or of different sizes [1–6] and in three dimensions spheres, cubes etc., distributed randomly in space [7–11]. An interesting subclass of problems is where the basic percolating units have an unbounded size distribution. These are comparatively less studied, though a few formal results are available [12]. The problem of disk percolation where disks have bounded sizes has been studied a lot, mainly by simulation [2,13,14]. For the single sized disk percolation, the threshold is known to a very high degree of accuracy [13]. Also simulation studies have shown that the disk percolation in two dimensions with disks of bounded size falls in the same universality class as that of lattice percolation in two dimensions [15]. For a review of continuum percolation see [16].

In this paper we consider a continuum percolation model of overlapping disks in two dimensions where distribution of the radii of the disks has a power-law tail. We address questions like whether the power-law tail in the distribution of radii changes the critical behavior of the system, and how does the percolation threshold depend on the power of the power-law tail. From an application point of view, a power-law polydispersity for an extended range of object sizes is quite common in nature, especially for fractal systems [17]. Disordered systems like carbonate rocks often contain pores of widely varied sizes covering many decades in length scales [18,19], whose geometry may be well modeled by a power-law distribution of pore sizes. The power-law distribution of the radii makes our system similar to the Ising or fluid system with long-range interactions. For the latter case, it is known that the long-range nature of the interaction does affect the critical behavior of the system for slow enough decay of the interaction [20]. For similar results in the context of long-range epidemic processes, see [21].

The behavior of our model differs from that of the standard continuum percolation model in two aspects. First, the entire low density regime in our model shows a power-law decay of the two-point function in contrast to the exponential decay in the standard continuum percolation. Thus the whole low density regime is “critical.” However, there is a nonzero percolation threshold below which there is no infinite cluster exist in the system. Second, the critical exponents are functions of the power a of the power-law distribution for low enough a. So while the system belongs to the same universality class as the standard continuum percolation for high enough a, the critical behavior is quite different for low values of a.

The plan of this paper is as follows: In Sec. II, we define the model of disk percolation precisely. In Sec. III, using a rigorous lower bound on the two-point correlation function, we show that it decays only as a power law with distance for arbitrarily low coverage densities. We discuss the two-point function and critical exponents. In Sec. IV, we propose an approximate renormalization scheme to calculate the correlation length exponent ν and the percolation threshold in such models. In Sec. V, we discuss results from simulation, and Sec. VI contains some concluding remarks.

II. DEFINITION OF THE MODEL

We consider a continuum percolation model of overlapping disks in two dimensions. The number density of disks is n, and the probability that any small area element dA has the center of a disk in it is $n dA$, independent of all other area elements. For each disk, we assign a radius, independently of other disks, from a probability distribution Prob(R). We consider the case when Prob(R) has a power-law tail; the probability of the radius being greater than R varies as R^{-a} for large R. For simplicity, we consider the case when the

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radii take only discrete values $R_0\Lambda^j$ where $j = 0,1,2,\ldots$, with probabilities $(1-p)p^j$ where $p = \Lambda^{-a}$. Here R_0 is the size of smallest disk, and Λ is a constant > 1. We refer to the disk of size $R_0\Lambda^j$ as the disk of type j.

The fraction of the entire plane which is covered by at least one disk, called the covered area fraction f_{covered}, is given by

$$f_{\text{covered}} = 1 - \exp(-A), \tag{1}$$

where A is the areal density—mean area of the disks per unit area of the plane—of the disks, which is finite only for $a > 2$. For $a \leqslant 2$, in the thermodynamic limit all points of the plane are eventually covered, and $f_{\text{covered}} = 1$. If $a > 2$, we have areal density,

$$A = n\pi R_0^2(1-p)/(1-p\Lambda^2). \tag{2}$$

We define the percolation probability P_∞ as the probability that a randomly chosen disk belongs to an infinite cluster of overlapping disks. One expects that there is a critical number density n^* such that for $n < n^*$, P_∞ is exactly zero, but $P_\infty > 0$, for $n > n^*$. We shall call the phase $n < n^*$ the nonpercolating phase, and the phase $n > n^*$ the percolating phase.

It is easy to show that $n^* < \infty$. We note that for percolation of disks where all disks have the same size R_0, there is a finite critical number density n_1^*, such that for $n > n_1^*$, $P_\infty > 0$. Then, for the polydisperse case, where all disks have radii R_0 or larger, the percolation probability can only increase, and hence $n^* < n_1^*$. Also it has been proved that whenever we have a bounded distribution of radii of the disks, the critical areal density is greater than that for a system with single sized disks [22]. Our simulation results show that this remains valid for unbounded distribution of radii of the disks.

III. NONPERCOLATING PHASE

We define two point function $\text{Prob}(1 \rightsquigarrow 2)$ as the probability that points P_1 and P_2 in the plane are connected by overlapping disks. Then, by rotational invariance of the problem, $\text{Prob}(1 \rightsquigarrow 2)$ is only a function of the Euclidean distance r_{12} between the two points. Let $\text{Prob}^{(1)}(1 \rightsquigarrow 2)$ denote the probability that there is at least one disk that covers both P_1 and P_2. Then, clearly,

$$\text{Prob}(1 \rightsquigarrow 2) \geqslant \text{Prob}^{(1)}(1 \rightsquigarrow 2). \tag{3}$$

It is straightforward to estimate $\text{Prob}^{(1)}(1 \rightsquigarrow 2)$ for our model. Let j be the minimum number such that the radius of disk of type j is greater than or equal to r_{12}, i.e., $R_0\Lambda^j \geqslant r_{12}$. Let S be the region of plane such that the distance of any point in S from P_1 or P_2 is less than or equal to $R_0\Lambda^j$. This region S is greater than or equal to the region where each point is within a distance r_{12} from both P_1 and P_2. Using elementary geometry, the area of region S is greater than or equal to $(2\pi/3 - \sqrt{3}/4)r_{12}^2$ (see Fig. 1). The number density of disks with radius greater than or equal to $R_0\Lambda^j$ is $n\Lambda^{-aj}$. Therefore, the probability that there is at least one such disk in the region S is $1 - \exp(-n|S|\Lambda^{-aj})$, where $|S|$ is the area of region S. Thus we get

$$\text{Prob}^{(1)}(1 \rightsquigarrow 2) \geqslant 1 - \exp\left[-nK\Lambda^{-aj}r_{12}^2\right], \tag{4}$$

where $K = 2\pi/3 - \sqrt{3}/4$.

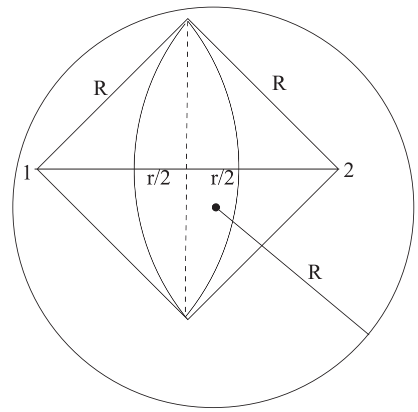

FIG. 1. Points 1 and 2 in the plane at a distance r from each other will be covered by a single disk of radius R, if the center of such a disk falls in the area of intersection of two circles with radius R and centers at 1 and 2.

Now, clearly, $R_0\Lambda^j < r_{12}\Lambda$. Hence we have $\Lambda^{-aj} > r_{12}^{-a}\Lambda^{-a}/R_0^{-a}$. Putting this in Eq. (4), we get

$$\text{Prob}^{(1)}(1 \rightsquigarrow 2) \geqslant 1 - \exp\left[-nK\Lambda^{-a}r_{12}^{-a+2}\right], \tag{5}$$

where some constant factors have been absorbed into K. For large r_{12}, it is easy to see that this varies as r_{12}^{2-a}. Hence the two-point correlation function is bounded from below by a power law.

We can extend this calculation, and write the two-point correlation function as an expansion,

$$\text{Prob}(1 \rightsquigarrow 2) = \sum_{n=1}^{\infty} \text{Prob}^{(n)}(1 \rightsquigarrow 2), \tag{6}$$

where $\text{Prob}^{(n)}(1 \rightsquigarrow 2)$ is the probability that the path of overlapping disks connecting points P_1 and P_2 requires n disks. The term $n = 2$ corresponds to a more complicated integral over two overlapping disks. But it is easy to see that for large r_{12}, this also decays as r_{12}^{-a+2}. Assuming that similar behavior holds for higher order terms as well, we expect that for all nonzero densities n, the two-point correlation function decays as a power law even for arbitrarily low densities of disks.

We note that this is consistent with the result that for continuum percolation in d dimensions, the diameter of the connected component containing the origin, say $\langle D\rangle$, is divergent even for arbitrarily small number densities when $\langle R^{d+1}\rangle$ is divergent [12]. Here R denote the radii variable. In our case $\langle D\rangle = \int r_{12}\frac{d\text{Prob}(r_{12})}{dr_{12}}dr_{12} \sim \int r_{12}^{2-a}dr_{12}$ (where P_1 is the origin) is divergent when $a \leqslant 3$, consistent with the above.

The power-law decay of the two-point function is the result of the fact that for any distance r, we have disks of radii of the order of r. However for large values of r, we can imagine that there would also be a contribution from a large number of overlapping disks of radii much smaller than r connecting the two points separated by the distance r, which as in the usual percolation problem decays exponentially with distance. Therefore it is reasonable to write the two-point function in our problem as a sum of two parts; the first part, say $G_{sr}(r)$, due to the " short-range" connections which has an exponential decay with distance for large r, and the second one, say $G_{lr}(r)$, due to the "long-range" connections which has a power-law decay with distance. Therefore

$$G(r) = G_{sr}(r) + G_{lr}(r), \tag{7}$$

where

$$G_{lr}(r) \sim D(A)/r^{a-2} + \text{higher order terms}, \tag{8}$$

where $D(A)$ is assumed to go to a nonzero constant as A goes to its critical value and its dependence on A is a slowly varying one.

The power-law distribution of the radii makes this system similar to a long-range interaction problem in statistical physics in the sense that given two points in the plane, a direct connection by a single disk overlapping both points is possible. In fact similar behavior for the two-point function exists whenever we have long-range interactions in a system, such as in an Ising model with long-range potentials or fluid with long-range interactions [23,24]. In such systems, the two-point function shows a power-law decay just as in our problem [25]. The effect of such long-range potentials on the critical exponents has been studied earlier [20,23,26–29] with the general conclusion that the long-range part of the interaction does influence the critical behavior of the system [30]. More precisely, if we have an attractive pair potential in d dimensions of the form $-\phi(r) \sim \frac{1}{r^{d+\sigma}}$ where $\sigma > 0$, then critical exponents take their short-range values for all $\sigma \geqslant 2 - \eta_{sr}$ where η_{sr} is the anomalous dimension (for the short-range problem in two dimensions, at criticality, the two-point function decays with distance as $1/r^{\eta_{sr}}$). For $\sigma < 2 - \eta_{sr}$, two kinds of behavior exist. For $0 < \sigma \leqslant d/2$, the exponents take their mean-field values and for $d/2 < \sigma < 2 - \eta_{sr}$, the exponents depend on the value of σ (see [20] and references therein). So $\sigma = 2 - \eta_{sr}$ is the dividing line between the region dominated by short-range interactions and the region dominated by long-range interactions.

Though there is a well established connection between the lattice percolation problem and the Ising model [31], there is no similar result connecting the continuum percolation problem to any simple Hamiltonian system. However, the following simple argument provides us with a prediction about the values of the parameter a for which the power-law nature of the distribution is irrelevant and the system is similar to a continuum percolation system with a bounded size distribution for the percolating units. Assuming that the strength of the long-range interaction from a given point in the Ising or fluid system (which decays like $\sim 1/r^{2+\sigma}$ in two dimensions) is like the strength of the connectivity from the center of a given disk which is given by the distribution of the radii; in our problem, we expect the dividing line between the region dominated by short-range connectivity and the region dominated by long-range connectivity to be the same as that for an Ising system with long-range potential of the form $-\phi(r) \sim 1/r^{a+1}$ where $a > 2$. Then the results for the long-range Ising system discussed in the last paragraph should carry over with $\sigma = a - 1$. So for our problem, a deviation from the standard critical behavior is expected when $a < 3 - \eta_{sr}$ and the critical exponents will take their short-range values for $a > 3 - \eta_{sr}$.

For two-dimensional (2D) percolation, $\eta_{sr} = 5/24$ [32]. Also mean-field behavior is expected when $a \leqslant 2$. However for this range of a, the entire plane is covered for all nonzero number densities and hence there is no phase transition.

In the next two sections, we investigate the dependence of exponents on the power-law tail of the distribution of the radii of the disks. First we develop an approximate renormalization-group (RG) method. Then we carry out simulation studies which show that the correlation length exponent ν takes its short-range value for $a > 3 - \eta_{sr}$, while it depends upon a for $a < 3 - \eta_{sr}$.

IV. CRITICAL BEHAVIOR NEAR THE PERCOLATION THRESHOLD

In this section, we propose an approximate RG method to analyze the behavior of continuum percolation models near the percolation threshold, when the percolating units have a distribution of sizes. We assume that we can replace disks of one size having a number density n with disks of another size and number density n', provided the correlation length remains the same. Application of a similar idea in the disk percolation problem with only two sizes of disks may be found in [5].

We will illustrate the method by considering a problem in which the radii of disks take only two possible values, say R_1 and R_2. Let their areal densities be A_1 and A_2 respectively, and assume that both A_1 and A_2 are below A^*, the critical threshold for the percolation problem with only single sized disks present($A^* \approx 1.128\,085$ [13]). Also let ξ_1 represent the correlation length when only disks of size R_1 are present in the system and ξ_2 represent that when only disks of size R_2 are present. Invariance of the two-point function under length rescaling requires that the expression for the correlation length ξ is of the form $\xi = Rg(A)$, where the function $g(A)$ determines how the correlation length depends on the areal density A and is independent of the radius R. Let $\tilde{A}_2$ be the areal density of the disks of size R_2 which will give the same correlation length as the disks of size R_1, i.e,

$$\xi_1(A_1) = \xi_2(\tilde{A}_2) \tag{9}$$

or

$$R_1 g(A_1) = R_2 g(\tilde{A}_2). \tag{10}$$

Given the form of the function $g(A)$, we can invert the above equation to find $\tilde{A}_2$. Formally,

$$\tilde{A}_2 = g^{-1}\left(\frac{R_1}{R_2} g(A_1)\right). \tag{11}$$

So the problem is reduced to one in which only disks of size R_2 are present, whose net areal density is now given by

$$A'_2 = \tilde{A}_2 + A_2. \tag{12}$$

The system percolates when $A'_2 = A^*$. Now, when areal density A is close to A^*, we have

$$g(A) = C(A^* - A)^{-\nu}, \tag{13}$$

where C is some constant independent of A and ν is the correlation-length exponent in the usual percolation problem. Using this in Eq. (11), we get

$$\tilde{A}_2 = A^* - (A^* - A_1)(R_2/R_1)^{1/\nu}. \tag{14}$$

Therefore, for a given value of $A_1 < A^*$, the areal density of disks of radius R_2, so that the system becomes critical, is given by

$$A_2 = A^* - \tilde{A}_2 = (A^* - A_1)(R_2/R_1)^{1/\nu}. \tag{15}$$

So the total areal density at the percolation threshold is

$$A_1 + A_2 = A_1 + (A^* - A_1)(R_2/R_1)^{1/\nu} = A_1(1-x) + A^* x,$$

where $x = (R_2/R_1)^{1/\nu}$. Without loss of generality we may assume $R_2 > R_1$. Then $x > 1$ and we can see from the above expression that the percolation threshold $A_1 + A_2 > A^*$, a result well known from both theoretical studies [22] and simulation studies [13].

Now in our problem assume that areal density of disks of type 0 do not exceed A^*. Renormalizing disks up to type m in our problem gives the equation for the effective areal density of the mth type disks A'_m as

$$A'_m = A^* - (A^* - A'_{m-1})\Lambda^{1/\nu} + \rho_m, \tag{16}$$

where $m \geqslant 1$, $A'_0 = \rho_0$ and $\rho_m = n_0 \pi \Lambda^{(2-a)m}$ denote the areal density of disks of radius Λ^m. Here n_0 is the number density of disks of radius R_0 (or of type 0), which for convenience we have set equal to unity. If we denote $A^* - A'_m$ by ε_m which is the distance from the criticality after the mth step of the renormalization, then the above expression becomes

$$\varepsilon_m = \varepsilon_{m-1}\Lambda^{1/\nu} - \rho_m. \tag{17}$$

The equation describes the flow near the critical point when we start with a value of ρ_0, the areal density of the first type of disks. Here ε_m gives the effective distance from criticality of the mth order disks in the system, in which now only mth and higher order disks are present. Now for given values of the parameters a and Λ, we can evaluate ε_m in Eq. (17) using a computer program and plot ε_m versus m. Depending upon the value of ρ_0, we get three different behaviors. For values of ρ_0 below the critical value denoted by ρ_0^*, ε_m will go to A^* asymptotically (the system is subcritical) and when it is above ρ_0^*, ε_m will go to $-\infty$ asymptotically (the system is supercritical). As $\rho_0 \to \rho_0^*$, we get the critical behavior characterized by ε_m tending to the RG fixed point 0 asymptotically. A typical result using Eq. (17) with $\Lambda = 2$ and $a = 3$ is shown in Fig. 2. We can see that as we tune ρ_0, the system approaches criticality, staying closer to the $\varepsilon_m = 0$ line longer and longer. Critical behavior here can be characterized by the value of m at which the curve deviates from the approach to the $\varepsilon_m = 0$ line. To understand how the correlation length diverges as we approach criticality, we assume that we can replace the subcritical system with a system where only disks of type m' are present and has a fixed areal density below A^*, where m' is the value of m at which ε_m shows a substantial increase—say ε_m becomes $A^*/2$. For the continuum percolation problem with single sized disks, the correlation length $\xi = Rg(A)$, where $g(A)$ is a function with no explicit dependence on radius R. Therefore, the correlation length in our problem is

$$\xi \propto \Lambda^{m'}. \tag{18}$$

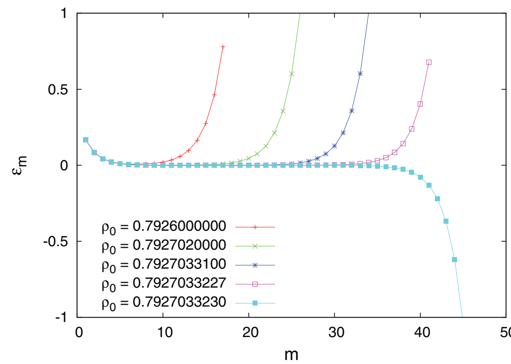

FIG. 2. (Color online) Variation of ε_m with m for different values of ρ_0 showing subcritical and supercritical regimes. We have used $a = 3$ and $\Lambda = 2$.

We can write the recurrence relation Eq. (17) in terms of the areal density ρ_n as

$$\varepsilon_m = A^* \Lambda^{\frac{m}{\nu}} - \sum_{n=0}^{m} \rho_n \Lambda^{[(m-n)/\nu]}. \tag{19}$$

But $\rho_n = \rho_0 \Lambda^{n(2-a)}$. Therefore,

$$\varepsilon_m = A^* \Lambda^{[m/\nu]} - \frac{\rho_0 \Lambda^{[m/\nu]}[1 - \Lambda^{m(2-a-1/\nu)}]}{[1 - \Lambda^{(2-a-1/\nu)}]}. \tag{20}$$

For large values of m, the last term in the above equation involving $\Lambda^{m(2-a-1/\nu)}$ can be neglected. Then,

$$\varepsilon_m = \Lambda^{[m/\nu]} \left[A^* - \frac{\rho_0}{1 - \Lambda^{(2-a-1/\nu)}} \right]. \tag{21}$$

Therefore,

$$\Lambda^{[m/\nu]} = \frac{\varepsilon_m}{[A^* - \frac{\rho_0}{1-\Lambda^{(2-a-1/\nu)}}]}. \tag{22}$$

For a given value of $\rho_0 \leqslant A^*$, the order m' at which ε_m is increased substantially, say to a value $A^*/2$, is given by

$$m' = [\log_\Lambda(A^*/2) - \log_\Lambda(\rho_0^* - \rho_0) + \log_\Lambda(1 - \Lambda^{(2-a-1/\nu)})]\nu. \tag{23}$$

So for ρ_0 close to ρ_0^* and large values of a,

$$m' \sim \log_\Lambda(\rho_0^* - \rho_0)^{-\nu}, \tag{24}$$

so that

$$\xi \propto (\rho_0^* - \rho_0)^{-\nu}. \tag{25}$$

Thus we find that the correlation length exponent ν is independent of the parameters a and Λ of the distribution. From Eq. (22), we can also obtain the percolation threshold ρ_0^* as a function of the parameters a and Λ. In Eq. (22) the left hand side is positive definite. So for values of ρ_0 for which $\frac{\rho_0}{1-\Lambda^{(2-a-1/\nu)}} < A^*$, we will have $\varepsilon_m > 0$ for large values of m. Similarly for values of ρ_0 for which $\frac{\rho_0}{1-\Lambda^{(2-a-1/\nu)}} > A^*$, we will have $\varepsilon_m < 0$ for large values of m. Hence the critical areal density ρ_0^* must be given by

$$\rho_0^* = A^*[1 - \Lambda^{(2-a-1/\nu)}]. \tag{26}$$

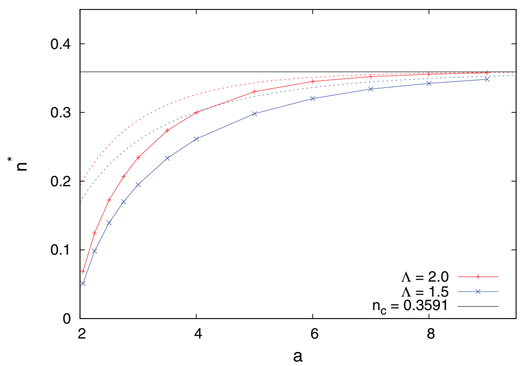

FIG. 3. (Color online) Variation of n^* with a for two different values of Λ. Dashed curves correspond to values given by Eq. (27) and continuous ones correspond to those from simulation studies. The horizontal line corresponds to the threshold for the single sized disks case.

Or in terms of the total number density, the percolation threshold n^* is given by

$$n^* = n_c(1 - \Lambda^{(2-a-1/\nu)})/(1 - \Lambda^{-a}), \qquad (27)$$

where $n_c = A^*/\pi$, the critical number density for percolation with single sized disks of unit radius. Note that this approximate result does not give the correct limit, $n^* \to 0$ as $a \to 2$. The RG scheme depends on the approximation that the effect of size R_1 of areal density A_1 is the same as that of disks of radius R_2 of density A_2, as in Eq. (9). This is apparently good only for $a > 3 - \eta_{sr}$. Figure 3 shows the variation of the critical threshold with a for two different values of Λ using Eq. (27) along with simulation results (see Sec. V for details of simulation studies). We see that a reasonable agreement is obtained between the two for higher values of a. Also, as one would expect, for large values of a, n^* tends to n_c.

From Eq. (27), we can obtain the asymptotic behavior of the critical number density n^* as $\Lambda \to 1$. This is useful since it corresponds to the threshold for a continuous distribution of radii with a power-law tail and we no more have to consider the additional discretization parameter Λ. It is easy to see that in the limit $\Lambda \to 1$, Eq. (27) becomes

$$n^*_{\Lambda\to 1} = n_c\left(1 - \frac{5}{4a}\right), \qquad (28)$$

where we have used the value $\nu = 4/3$. Thus we expect that a log-log plot of $(n_c - n^*_{\Lambda\to 1})$ against a will be a straight line with slope -1 and y-intercept $\ln(5n_c/4) \approx -0.35$ for large values of a. A comparison with the thresholds obtained from simulation studies show that Eq. (28) indeed predicts the asymptotic behavior correctly (see Fig. 7).

V. SIMULATION RESULTS

We determine the exponent ν and the percolation threshold n^* by simulating the continuum percolation system in two dimensions, with disks having a power-law distribution for their radii. We consider two cases for the distribution of the radii variable. To explicitly compare the prediction of the approximate RG scheme for the percolation threshold given in Sec. IV, we use a discrete distribution for the radii variable, with discretization factor Λ as in Sec. II. The results for the thresholds thus obtained are shown in Fig. 3. To determine the correlation length exponent ν, we consider the radii distribution in the limiting case $\Lambda \to 1$, so that we do not have to consider the additional parameter Λ. In this case, given a disk, the probability that it has a radius between R and $R + dR$ is equal to $aR^{-(a+1)}$ where $a > 2$. We also obtain the percolation threshold with this continuous distribution for the radii and compare it with the predicted asymptotic behavior in Eq. (28). The minimum radius is assumed to be unity.

For $a \leqslant 2$ the entire plane is covered for arbitrarily low densities of the disks. We use cyclic boundary conditions and consider the system as percolating whenever it has a path through the disks from the left to the right boundary. We drop disks one at a time onto a region of a plane of size $L \times L$, each time checking whether or not the system has formed a spanning cluster. Thus number density is increased in steps of $1/L^2$. So after dropping the nth disk, the number density is n/L^2. Now associated with each number density we have a counter, say f_n, which is initialized to 0 in the beginning. If the system is found to span after dropping the n'th disk, then all counters for $n \geqslant n'$ are incremented by 1. After a spanning cluster is formed, we stop. In this way we can determine the spanning probability $\Pi(n,L) = f_n/N$ where N is the number of realizations sampled. The number of realizations sampled varies from a maximum of 2.75×10^7 for $a = 2.05$ and $L = 90$ to a minimum of 4000 for $a = 10.0$ and $L = 1020$ (for obtaining the results for the threshold in Fig. 3, the number of realizations sampled is 20 000 for all values of a and Λ). This method of dropping basic percolating units one by one until the spanning cluster is formed has been used before [33] in the context of stick percolation which was based on the algorithm developed in [34], and allows us to study relatively large system sizes with a large number of realizations within a reasonable time.

The probability that there is at least a single disk which spans the system of size L at number density n is

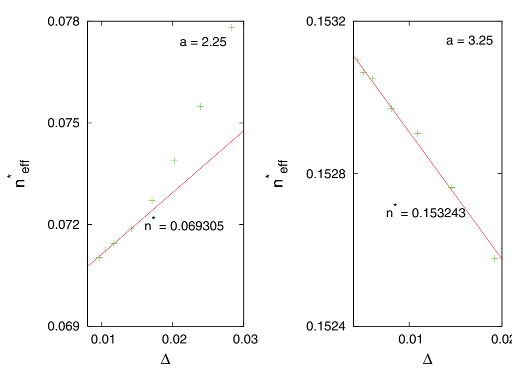

FIG. 4. (Color online) Plot of effective percolation threshold n^*_{eff} against Δ for $a = 2.25$ and $a = 3.25$. The best straight line fit is obtained with the last four data points.

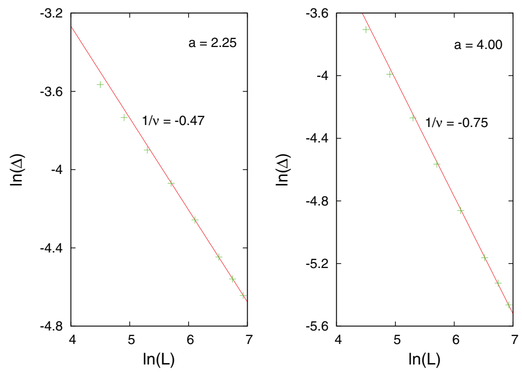

FIG. 5. (Color online) Log-log plot of Δ vs L for $a = 2.25$ and $a = 4.0$ along with lines of slope -0.47 and -0.75.

$1 - \exp^{(-n2^a/L^{a-2})}$. It is easy to see that to leading order in n, this "long-range" part of the spanning probability $\Pi(n,L)_{lr}$ is $n2^a/L^{a-2}$. So one can write a scaling form for the spanning probability,

$$\Pi(n,L) = \Pi(n,L)_{lr} + [1 - \Pi(n,L)_{lr}]\phi[(n^* - n)L^{1/\nu}]. \tag{29}$$

Therefore we can define the "short-range" part of the spanning probability $\Pi'(n,L) = [\Pi(n,L) - \Pi(n,L)_{lr}]/[1 - \Pi(n,L)_{lr}]$, where the leading long-range part is subtracted out. Therefore, we have

$$\Pi'(n,L) = \phi[(n^* - n)L^{1/\nu}] \tag{30}$$

and the scaling relations (see for, e.g., [35])

$$\Delta(L) \propto L^{-1/\nu}, \tag{31}$$

$$n^*_{\text{eff}}(L) - n^* \propto \Delta, \tag{32}$$

where $n^*_{\text{eff}}(L)$ is a suitable defined effective percolation threshold for the system of size L, and Δ is the width of the percolation transition obtained from the spanning probability

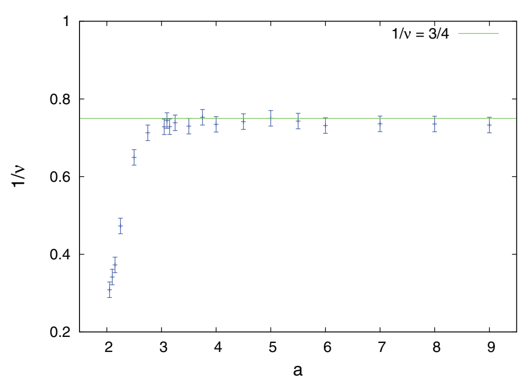

FIG. 6. (Color online) Variation of $1/\nu$ with a. The horizontal line corresponds to the standard 2D percolation value $1/\nu = 3/4$.

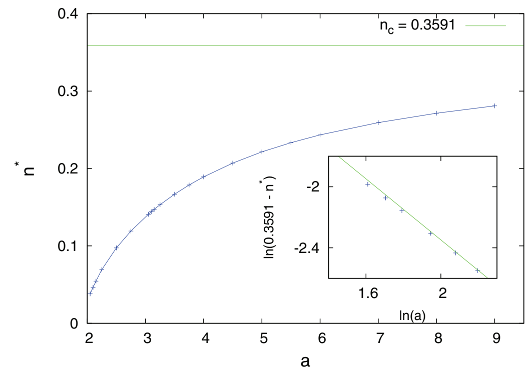

FIG. 7. (Color online) Variation of percolation threshold n^* with a. The horizontal line corresponds to the threshold for the single sized disks case. Inset: Asymptotic approach of n^* to the single sized disks value $n_c = 0.3591$ along with a straight line of slope -1 and y-intercept -0.35 [see Eq. (28)].

curves $\Pi'(n,L)$. Note that Eqs. (31) and (32) are applicable with any consistent definition of the effective percolation threshold and width Δ [35]. A good way to obtain n^*_{eff} and Δ is to fit the sigmoidal shaped curves of the spanning probability $\Pi'(n,L)$ with the function $1/2[1 + \text{erf}([n - n^*_{\text{eff}}(L)]/\Delta(L))]$ (see [14]), which defines the effective percolation threshold n^*_{eff} as the number density at which the spanning probability is $1/2$. We determined n^*_{eff} and Δ for each value of a and L and determined $1/\nu$ and n^* for different values of a using Eqs. (31) and (32) respectively. Typical examples are shown in Figs. 4 and 5.

At first, we determined the percolation threshold and the exponent for a system of single sized disks of unit radius. We obtained $n^* = 0.3589(\pm 0.0001)$ (or areal density $\approx 1.127\,52$) and $1/\nu = 0.758(\pm 0.018)$ in very good agreement with the known value for the threshold [13] and the conjectured value of $1/\nu = 3/4$ for the exponent. Values of $1/\nu$ obtained for various values of a are shown in Fig. 6. We scan the low a regime more closely for any variation from the standard answer. We can see that the estimates for $1/\nu$ are very much in line with the standard percolation value for $a > 3 - \eta_{sr}$ while it varies

TABLE I. Percolation threshold n^* for a few values of a along with corresponding critical areal density η^* and the critical covered area fraction ϕ^*.

a	n^*	$\eta^* = n^*\pi a/(a-2)$	$\phi^* = 1 - \exp^{-\eta^*}$
2.05	0.0380(6)	4.90(7)	0.993(1)
2.25	0.0693(1)	1.959(3)	0.8591(5)
2.50	0.09745(11)	1.5307(17)	0.7836(4)
3.50	0.16679(8)	1.2226(6)	0.70555(17)
4.00	0.18916(3)	1.1885(2)	0.69543(6)
5.00	0.22149(8)	1.1597(4)	0.68643(13)
6.00	0.24340(5)	1.1470(2)	0.68241(8)
7.00	0.2593(2)	1.1406(7)	0.6804(2)
8.00	0.27140(7)	1.1368(3)	0.67917(9)
9.00	0.28098(9)	1.1349(4)	0.67856(12)

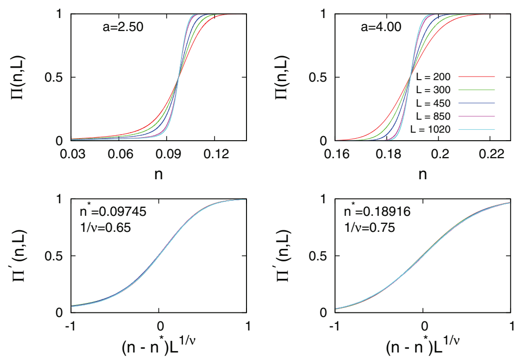

FIG. 8. (Color online) Variation of $\Pi(n,L)$ with n (top row) and the scaling collapse (bottom row) for $a = 2.50$ (on left) and $a = 4.00$ (on right).

with a for $a < 3 - \eta_{sr}$. Figure 7 shows the variation of the percolation threshold n^* with a. As expected, with increasing a, the percolation threshold increases and tends to the single sized disk value as $a \to \infty$, and as $a \to 2$, the threshold tends to zero. The data also show that n^* converges to the threshold for the single sized disk value as $1/a$ as predicted by Eq. (28). Values of the threshold for some values of a are given in Table I.

Finally as a check, we plot the spanning probability $\Pi'(n,L)$ [see Eq. (30)] against $(n - n^*)L^{1/\nu}$ to be sure that a good scaling collapse is obtained. We show two such plots for $a = 2.50$ and $a = 4$ in Fig. 8. We can see that a very good collapse is obtained. Similar good collapse is obtained for other values of a as well.

VI. CONCLUDING REMARKS

In this paper, we discuss the effect of a power-law distribution of the radii on the critical behavior of a disk percolation system. If the distribution of radii is bounded, then one would expect the critical exponents to be unchanged and to be the same as that for standard percolation. However, if the distribution of radii has a power-law tail, we show that this strongly influences the nature of the phase transition. The whole of the low-density nonpercolating phase has power-law decay of correlations in contrast to the exponential decay for the standard percolation and this occurs for any value of the power a, howsoever large. The critical exponents depend on the value of a for $a < 3 - \eta_{sr}$ and take their short-range values for $a > 3 - \eta_{sr}$. We also propose an approximate RG scheme to analyze such systems. Using this, we compute the correlation-length exponent and the percolation threshold. The approximate RG scheme is good only for $a > 3 - \eta_{sr}$. Monte Carlo simulation results for the percolation thresholds and the correlation-length exponent are presented.

We can easily extend the discussion to higher dimensions, or other shapes of objects. It is easy to see that the power-law correlations will exist in corresponding problems in higher dimensions as well.

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[1] E. T. Gawlinski and H. E. Stanley, J. Phys. A: Math. Gen. **14**, L291 (1981).
[2] J. Quintanilla, Phys. Rev. E **63**, 061108 (2001).
[3] R. Roy and H. Tanemura, Adv. Appl. Probab. **34**, 48 (2002).
[4] M. K. Phani and D. Dhar, J. Phys. A: Math. Gen. **17**, L645 (1984).
[5] A. C. Balram and D. Dhar, Pramana **74**, 109 (2010).
[6] R. Ogata, T. Odagaki, and K. Okazaki, J. Phys.: Condens. Matter **17**, 4531 (2005).
[7] S. Akagawa and T. Odagaki, Phys. Rev. E **76**, 051402 (2007).
[8] Y. B. Yi and E. Tawerghi, Phys. Rev. E **79**, 041134 (2009).
[9] R. Consiglio, D. R. Baker, G. Paul, and H. E. Stanley, Physica A **319**, 49 (2003).
[10] M. D. Rintoul, Phys. Rev. E **62**, 68 (2000).
[11] G. Ambrosetti, N. Johner, C. Grimaldi, A. Danani, and P. Ryser, Phys. Rev. E **78**, 061126 (2008).
[12] J. B. Gouere, Ann. Probab. **36**, 1209 (2008).
[13] J. A. Quintanilla and R. M. Ziff, Phys. Rev. E **76**, 051115 (2007).
[14] M. D. Rintoul and S. Torquato, J. Phys. A: Math. Gen. **30**, L585 (1997).
[15] T. Vicsek and J. Kertesz, J. Phys. A: Math. Gen. **14**, L31 (1981).
[16] R. Meester and R. Roy, *Continuum Percolation* (Cambridge University Press, Cambridge, England, 1996).
[17] J. E. Martin, J. Appl. Crystallogr. **19**, 25 (1986).
[18] B. Biswal, P. E. Oren, R. J. Held, S. Bakke, and R. Hilfer, Phys. Rev. E **75**, 061303 (2007).
[19] S. Roth, B. Biswal, G. Afshar, R. J. Held, P.-E. Oren, L. I. Berge, and R. Hilfer, AAPG Bull. **95**, 925 (2011).
[20] M. Aizenman and R. Fernandez, Lett. Math. Phys. **16**, 39 (1988).
[21] F. Linder, J. Tran-Gia, S. R. Dahmen, and H. Hinrichsen, J. Phys. A: Math. Theor. **41**, 185005 (2008).
[22] R. Meester, R. Roy, and A. Sarkar, J. Stat. Phys. **75**, 123 (1994).
[23] J. Sak, Phys. Rev. B **8**, 281 (1973).
[24] R. F. Kayser and H. J. Raveche, Phys. Rev. A **29**, 1013 (1984).
[25] D. Iagolnitzer and B. Souillard, Phys. Rev. A **16**, 1700 (1977).
[26] G. Stell, Phys. Rev. B **1**, 2265 (1970).
[27] G. Stell, Phys. Rev. B **8**, 1271 (1973).
[28] M. E. Fisher, S. Ma, and B. G. Nickel, Phys. Rev. Lett. **29**, 917 (1972).
[29] D. Dantchev, Eur. Phys. J. B **23**, 211 (2001).
[30] G. Stell, Phys. Rev. B **5**, 981 (1972).
[31] C. M. Fortuin and P. W. Kasteleyn, Physica **57**, 536 (1972).
[32] B. Nienhuis, E. K. Riedel, and M. Schick, J. Phys. A: Math. Gen. **13**, L189 (1980).
[33] J. Li and S.-L. Zhang, Phys. Rev. E **80**, 040104(R) (2009).
[34] M. E. J. Newman and R. M. Ziff, Phys. Rev. E **64**, 016706 (2001).
[35] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor and Francis, London, 2003).