CONSTRAINTS ON FLUID DYNAMICS FROM EQUILIBRIUM PARTITION FUNCTION

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1 Synopsis

1.1 Introduction

Relativistic Hydrodynamics arises as the universal long wavelength effective description of Lorentz invariant quantum field theories in translationally invariant phases. Its a minimalistic description based on symmetries where the entire dynamics of the system is governed by local conservation laws following from these symmetries. Specifically, the homogeneity gives rise to stress tensor conservation and other global symmetry, if present, leads to corresponding conservation law. To make it into a closed dynamical system hydrodynamics is supplemented with so called 'constitutive relations' which express the conserved currents in terms of local fluid variables namely velocity, temperature and chemical potential fields. These local variables are assumed to be varying only over length scales large compared to the 'mean free path' whose scale is set by the local temperature of the fluid. That is, for the validity of fluid dynamical description the derivatives of the fluid variables have to be small. This lead to the simplification that the constitutive relation can be specified order by order in a long wavelength(derivatives) expansion of the fluid variables.

As in any other effective theory, hydrodynamics is characterized by parameters(e.g. the transport coefficients in constitutive relations) which parametrized our ignorance. Although these parameters are in principal determined from the underlying microscopic field theory by an averaging out procedure, in practice it is extremly difficult to implement for any interesting enough system. A natural question to ask in such a case is: Are there any constraints on the possible values of these coefficients ? or are they completely arbitrary and any possible value of these would result from some underlying field theory.

In the context of hydrodynamics this question becomes even more pressing for the following reason. Hydrodynamics in its current formulation is only at the level of equations of motion and does not have an action formulation. So it is quite possible that some the properties which are built into action formulation are missing here and hydrodynamics have to be supplimented with them as external constraints.

The answer to what are the most general constitutive relations which could be realized in nature is greatly constrained, over and above symmetry considerations, by the macroscopic laws of thermodynamics. Landau-Lifshitz [1], and several subsequent authors, have emphasized consistency with a local form of the second law of thermodynamics as a source of constraints on the equation of hydrodynamics. As is well known, this requirement imposes inequalities on several parameters (like viscosities and conductivities) that appear in the equations of hydrodynamics. It is perhaps less well appreciated that the requirement of local entropy increase also yields equalities relating otherwise distinct fluid dynamical parameters, and so reduces the number of free parameters that appear in the equations of fluid dynamics (see e.g. [1, 2], for more recent work inspired by the AdS/CFT correspondence see e.g. [3–11]).

The second law of thermodynamics is a macroscopic law and its microscopic origin is not well understood. This makes the constraints obtained using it somewhat mysterious. In this synopsis we report on some progress we have made, in hydrodynamical context, towards demystifying a subset of these constraints using very general physical requirements on the equations of hydrodynamics. We explore the structural constraints imposed on the equations of relativistic hydrodynamics by two related physical requirements. First that these dynamical equations admit a stationary solution on an arbitrarily weakly curved stationary background spacetime. Second that the conserved currents (e.g. the stress tensor) on the corresponding solution follow from an equilibrium partition function[1]. In various examples that we have studied so far we demonstrate that the equalities obtained from the comparison with equilibrium (described in the previous paragraph) agree precisely with the equalities between coefficients obtained from the local second law of thermodynamics. These results lead us to conjecture that the constraints obtained from these two naively distinct physical requirements infact always coincide.

In this synopsis we nowhere utilize the AdS/CFT correspondence. However our work is motivated by the potential utility of our results in an investigation of the constraints imposed by the second law of thermodynamics on higher derivative corrections to Einstein's equations [12], via the Fluid - Gravity map of AdS/CFT ([13], see [14, 15] for reviews).

The rest of this synopsis is organized as follows. In section 1.2 we discuss the equilibrium partition function on weakly curved manifolds. In section 1.3 we briefly review standard formulation of hydrodynamics and then describe our general procedure to constrain non-dissipative transport coefficients using equilibrium partition function. In section 1.4 we elaborate the method using the example of parity violating charged fluids in 3+1d at first order in derivative expansion. In section 1.5 we present a brief give summary of results for many other cases that we have worked out using the set of ideas described in sections 1.2 and 1.4.

1.2 Equilibrium partition function and constraints on hydrodynamics

Consider a relativistically invariant quantum field theory with a global $U(1)$ symmetry on a manifold with a timelike killing vector and a background gauge field turned on. By a suitable choice of coordinates, the metric and the gauge field on any such manifold can be put in the form

$$\begin{aligned} ds^2 &= -e^{2\sigma(\vec{x})}\left(dt + a_i(\vec{x})dx^i\right)^2 + g_{ij}(\vec{x})dx^i dx^j \\ \mathcal{A} &= \mathcal{A}_0(\vec{x})dx^0 + \mathcal{A}_i(\vec{x})dx^i \end{aligned} \tag{1.1}$$

where $i = 1 \ldots p$. ∂_t is the killing vector on this manifold, while the coordinates $\vec{x}$ parametrize spatial slices. Here $\sigma, a_i, g_{ij}, \mathcal{A}_0$ and $\mathcal{A}_i$ are smooth functions of coordinates $\vec{x}$.

Let H denote the Hamiltonian that generates translations of the time coordinate t and Q be the charge that generates the global $U(1)$ transformations. Let us address the following

[1]Although the existence of a generating function for equilibrium conserved currents do not follow from the existence of equilibrium solution, as there is a non trivial integrability condition here. We find that in all cases we study the two infact give the same results

question. What can we say, on general symmetry grounds, about the dependence of the the partition function of the system

$$Z = \mathrm{Tr} e^{-\frac{H-\mu_0 Q}{T_0}} \tag{1.2}$$

on σ, g_{ij}, a_i, $\mathcal{A}_0$ and $\mathcal{A}_i$? Here we focus on the long wavelength limit, i.e. on manifolds whose curvature length scales and the scale of gauge field variations are much larger than the 'mean free path' of the thermal fluid. In this limit the question formulated above may addressed using the techniques of effective field theory. In the long wavelength limit the background manifold may be thought of as a union of approximately flat patches, in each of which the system is in a local flat space thermal equilibrium at the locally red shifted temperature

$$T(x) = e^{-\sigma} T_0 + \dots \tag{1.3}$$

(where T_0 is the equilibrium temperature of the system and the ... represent derivative corrections). Consequently the partition function of the system is given by

$$\ln Z = \int d^p x \sqrt{g_p} \frac{1}{T(x)} P(T(x), \mu(x)) + \dots \tag{1.4}$$

where $P(T, \mu)$ is the thermodynamical function that computes the pressure as a function of temperature in flat space. Substituting (1.3) into (1.4) we find

$$\ln Z = \int d^p x \sqrt{g_p} \frac{e^{\sigma}}{T_0} P(T_0 e^{-\sigma}, A_0 e^{-\sigma}) + \dots \tag{1.5}$$

The ... in (1.5) denote corrections to $\ln Z$ in an expansion in derivatives of the background metric. At any given order in the derivative expansion these correction are determined, by the requirement of the left over diffeomorphism invariance and gauge invariance(upto anomalies) in terms of a finite number of unspecified functions of σ and $\mathcal{A}_0$. For the case of parity violating charged fluids in 3+1 dimensions, which we will describe in some detail in this synopsis, the requirements of three dimensional diffeomorphism invariance, Kaluza Klein gauge invariance, and $U(1)$ gauge invariance upto an anomaly[2] force the partition function to take the form[3]

$$\begin{aligned}
\ln Z &= W^0 + W^1_{inv} + W^1_{anom} \\
W^0 &= \int \sqrt{g_3} \frac{e^{\sigma}}{T_0} P\left(T_0 e^{-\sigma}, e^{-\sigma} A_0\right) \\
W^1_{inv} &= \frac{C_0}{2} \int A dA + \frac{T_0^2 C_1}{2} \int a da + \frac{T_0 C_2}{2} \int A da \\
W^1_{anom} &= \frac{C}{2} \left(\int \frac{A_0}{3T_0} A dA + \frac{A_0^2}{6T_0} A da \right)
\end{aligned} \tag{1.6}$$

[2]Here we only consider the effect of $U(1)^3$ anomalies ignoring the effects of, for instance, mixed gauge-gravitational anomalies. A systematic study of the effect of these anomalies in fluid dynamics would require analysis at second order in derivative expansion.

[3]Our convention is

$$\frac{1}{2} \int X dY = \int d^3 x \sqrt{g_3} \epsilon^{ijk} X_i \partial_j Y_k \ , \quad \frac{1}{2} \int dY = \int d^2 x \sqrt{g_2} \epsilon^{ij} \partial_i Y_j \ .$$

where A_i

$$\begin{aligned} A_0 &= \mathcal{A}_0 + \mu_0 \\ A_i &= \mathcal{A}_i - A_0 a_i \end{aligned} \tag{1.7}$$

(1.6) is written in terms of A_i because A_i, unlike $\mathcal{A}_i$, is Kaluza Klein gauge invariant[4].

W^0 in (1.6) is zero derivative contribution to the partition function, and is the patchwise approximation to equilibrium, in the spirit of (1.5). W^1_{inv} is the most general diffeomorphism and gauge invariant one derivative correction to W^0. Note that W^1 is the sum of a Chern Simons term for the connection A, a Chern Simons term for the connection a and a mixed Chern Simons term in A and a. As usual, the Chern Simons terms are gauge invariant only upto boundary terms, and thus local gauge invariance forces the coefficients C_0, C_1 and C_2 of these Chern Simons terms to be constants.

(1.6) is the most general form of the partition function of our system that satisfies the requirements of 3 dimensional diffeomorphism invariance and gauge invariance. If we, in addition, impose the requirement of CPT invariance of the underlying four dimensional field theory then it turns out that $C_0 = C_1 = 0$. In other words, the requirement of CPT invariance allows only the mixed Chern Simons term, setting the 'pure' Chern Simons terms to zero.

W^1_{anom} is the part of the effective action that is not gauge invariant under $U(1)$ gauge transformations. [5] Its gauge variation under $A_\mu \to A_\mu + \partial_\mu \phi(\vec{x})$ is given by

$$\delta W^1_{anom} = \frac{C}{24T_0} \int d^3x \sqrt{-g_4} * (\mathcal{F} \wedge \mathcal{F})\ \phi(x) \tag{1.8}$$

This is exactly the variation of the effective action predicted by the anomalous conservation equation

$$\nabla_\mu \tilde{J}^\mu = -\frac{C}{8} * (\mathcal{F} \wedge \mathcal{F}) \tag{1.9}$$

where $\tilde{J}$ is the gauge invariant $U(1)$ charge current, and $*$ denotes the Hodge dual.

1.3 Constraints using stationary equilibrium

1.3.1 Relativistic Hydrodynamics

In this subsection we present a lightening review of the structure of the equations of charged relativistic hydrodynamics. The equations of hydrodynamics are simply the equations of conservation of the stress tensor and the charge current

$$\nabla_\mu T^\mu_\nu = \mathcal{F}_{\nu\mu} \tilde{J}^\mu, \quad \nabla_\mu \tilde{J}^\mu = -\frac{C}{8} * (\mathcal{F} \wedge \mathcal{F}), \tag{1.10}$$

[4] The background data can be taken as gauge field $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_i)$ with constant chemical potential μ_0 and temperature T_0. Equivalently we can think of the system to have background gauge field $B = (\mathcal{A}_0 + \mu_0, \mathcal{A}_i)$ with no chemical potential. These two are equivalent physical statements as μ_0 can be absorbed in the constant part of $\mathcal{A}_0$.

[5] It is striking that the effect of the anomaly can be captured by a local term in the 3 dimensional effective action. Note that W^1 cannot be written as the dimensional reduction of a local contribution to the 4 dimensional action, in agreement with general expectations.

where $\mathcal{F}$ is the field strength of the gauge field $\mathcal{A}$. These equations constitute a closed dynamical system when supplemented with constitutive relations that express $T_{\mu\nu}$ and J_μ as a function of the fluid temperature, chemical potential and velocity. These constitutive relations are presented in an expansion in derivatives and take the form

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + Pg^{\mu\nu} + \pi^{\mu\nu}, \quad J^\mu = qu^\mu + J^\mu_{diss}, \tag{1.11}$$

The pressure P, proper energy density ϵ and proper charge density q are those functions of T and μ predicted by flat space equilibrium thermodynamics. $\pi_{\mu\nu}$ refers to the sum of all corrections to the stress tensor that are of first or higher order in the derivative expansion (the derivatives in question could act either on the T, μ, u^μ, or the background metric and gauge field $g_{\mu\nu}$ and $\mathcal{A}_\mu$). Similarly J^μ_{diss} refers to corrections to the perfect fluid charge current that depend on atleast one spacetime derivative. Field redefinitions of the T μ and u^μ may be used to impose $p+2$ constraints on $\pi_{\mu\nu}$ and J^μ_{diss}, referred to as frame choice. In this synopsis we will work in the so called Landau Frame which is defined by the conditions

$$u^\mu \pi_{\mu\nu} = 0, \quad u^\mu J^{diss}_\mu = 0 \tag{1.12}$$

Terms in $\pi_{\mu\nu}$ and J^μ_{diss} are both graded according to the number of spacetime derivatives they contain, i.e.

$$\begin{aligned} \pi^{\mu\nu} &= \pi^{\mu\nu}_{(1)} + \pi^{\mu\nu}_{(2)} + \pi^{\mu\nu}_{(3)} + \dots \\ J^\mu_{diss} &= J^\mu_{diss,(1)} + J^\mu_{diss,(2)} + J^\mu_{diss,(3)} + \dots \end{aligned} \tag{1.13}$$

where the subscript counts the number of derivatives.

Symmetry considerations immediately constrain the possible expansions for $\pi_{\mu\nu}$ and J^μ_{diss} as follows. At any given point in spacetime, the fluid velocity u^μ is a particular timelike vector. The value of the velocity breaks the local $SO(p,1)$ Lorentz symmetry of the theory down to the rotational subgroup $SO(p)$. In the Landau frame (1.10) $\pi_{\mu\nu}$ may be decomposed into an $SO(p)$ tensor and $SO(p)$ scalar. J^μ_{diss} is an $SO(p)$ vector.

In order to parameterize freedom in the equations of hydrodynamics, it is useful to define some terminology. Let t^n_f, v^n_f and s^n_f respectively denote the number of onshell inequivalent tensor, vector and scalar expressions made up of a total of n derivatives acting on T, u^μ, μ, $g_{\mu\nu}$ and A_μ. It follows immediately that the most general symmetry allowed expression for $\pi^{\mu\nu}_{(n)}$ is given in terms of $t^n_f + s^n_f$ unknown functions of the two variables T and μ. In a similar manner the most general expression for the $J^\mu_{diss(n)}$, permitted by symmetries, is given in terms of v^n_f unknown functions of the same two variables.

It turns out that the $(t^n_f + s^n_f + v^n_f)$ n^{th} order transport coefficients are not all independent. The requirement that the hydrodynamical equations be consistent with the existence of an entropy current that is of positive divergence in every conceivable fluid flow imposes several relationships between these coefficients cutting down the number of parameters in these equations(see e.g. [1, 3, 5, 10]). We now turn to a description of a simpler physical principal that appears predict the same relations between these coefficients. These relations may all be constructively determined by comparison of the equations of hydrodynamics with a partition function.

1.3.2 Constraints from stationary equilibrium

As we have explained in the previous subsection, it follows from symmetry considerations that the equations of charged hydrodynamics, at n^{th} order in the derivative expansion, are parameterized by $t^n_f + v^n_f + s^n_f$ unknown functions of two variables(σ and A_0) appearing in the constitutive relation at nth order. We will now argue that these functions are not all independent, but instead are determined in terms of a smaller number of functions.

It is easy to verify that the equations of perfect fluid hydrodynamics (hydrodynamics at lowest order in the derivative expansion) admit a stationary 'equilibrium' solution in the backgrounds (1.1) given by

$$u^\mu_{(0)}(\vec{x}) = e^{-\sigma}(1, 0, \dots, 0), \quad T_{(0)}(\vec{x}) = T_o e^{-\sigma}, \quad \mu_{(0)}(\vec{x}) = e^{-\sigma} A_0 \tag{1.14}$$

As explained above, this is also the equilibrium solution one expects of the fluid on intuitive ground. At higher order in the derivative expansion this solution is corrected; the corrected solution can itself be expanded in derivatives

$$\begin{aligned} u^\mu &= u^\mu_{(0)} + u^\mu_{(1)} + u^\mu_{(2)} + \dots \\ T &= T_{(0)} + T_{(1)} + T_{(2)} + \dots \\ \mu &= \mu_{(0)} + \mu_{(1)} + \mu_{(2)} + \dots \end{aligned} \tag{1.15}$$

where $u^\mu_{(n)}$, $T_{(n)}$ and $\mu_{(n)}$ are expressions of n^{th} order in derivatives acting on σ, A_0, a_i, A_i and g_{ij}. What can we say about the form of the corrections $u^\mu_{(n)}$, $T_{(n)}$ and $\mu_{(n)}$? Adopting the notation defined in the last paragraph of the previous subsection, symmetries determine the expression for $u^\mu_{(n)}$ in terms of v^n_e as yet unknown functions of σ and A_0, while T and μ are each determined in terms of s^n_e as yet unknown equations of A_0 and σ.

The stress tensor and charge current in equilibrium are given by plugging (1.15) into (1.13). The result is an expression for $\pi^{\mu\nu}$ and J^μ_{diss} written entirely in terms of σ, A_0, a_i, A_i, g_{ij} and their derivatives.

This expressions for the stress tensor and charge current so obtained depend only on a subset of the transport coefficients that appear in the expansion of $\pi^{\mu\nu}$ and J^μ_{diss}. For instance, the expansion of the n^{th} order tensor part of $\pi^{\mu\nu}$ has t^n_f terms in general. When evaluated on (1.14), however, this expression reduces to a sum over $t^n_e \leq t^n_f$ terms. The coefficients of these terms define t^n_e subspace of the t^n_f dimensional set of n^{th} order transport coefficients. We refer to this subspace as the subspace of *non dissipative* transport coefficients.

We demand that the expressions for the equilibrium stress tensor and charge current, obtained as described in the previous paragraph, agree with the corresponding expressions obtained from the equilibrium partition function by varying with respect to the metric and gauge field respectively. This requirement yields a set of $t_e^n + 2v_e^n + 3s_e^n$ equations[6] that completely determine both the n^{th} order corrections to the equilibrium solutions T_n μ_n and u_n^μ ($v_e^n + 2s_e^n$ coefficients in all) as well as the $t_e^n + v_e^n + s_e^n$ non dissipative hydrodynamical transport coefficients. Note that the number of variables precisely equals the number of equations. Dissipative hydrodynamical transport coefficients are completely unconstrained by this procedure.

We emphasize that the shifted equilibrium velocities, temperatures and chemical potentials obtained from the procedure just described automatically obey the equations of hydrodynamics. By construction, the shifted fluid variables, together with the constitutive relations determined above yield the stress tensor that follows from the functional variation of an equilibrium partition function, and the stress tensor obtained from the variation of *any* diffeomorphically invariant functional is automatically conserved. Very similar remarks apply to the charge current.

The procedure described above may also be used to derive constraints on the form of the fluid entropy current. The entropy current must obey two constraints. First its divergence must vanish on all the equilibrium configurations derived above. Second, the integral over the entropy density (obtained from the entropy current) must equal the thermodynamical entropy that follows from the partition function. These requirements impose constraints on the form of the (non dissipative) part of the most general symmetry allowed hydrodynamical entropy current.

In the case of parity violating fluid dynamics in 3+1 dimensions at first order in derivative expansion, which we will discuss in some detail below, the results for transport coefficients computed from (1.6) match perfectly with those of Son and Surowka [3] (generalized in [16],[4]) once we impose the additional requirement of CPT invariance. Before imposing the requirement of CPT invariance, we have an additional one parameter freedom that is not captured by the the generalized Son-Surowka analysis. The reason for this is that Son and Surowka (and subsequent authors) assumed that the entropy current was necessarily gauge invariant. This does not seem to be physically necessary. It seems to us that an entropy current whose divergence is gauge invariant - and whose integral over a compact manifold in equilibrium is gauge invariant - is perfectly acceptable. As we explain below, it is easy to find a one parameter generalization of the Son-Surowka solution that meets these conditions, and that gives rise to the additional term C_0 in the partition function (1.6). However it turns out that the requirement of CPT invariance sets C_0 (along with C_1) to zero in (1.6), so this possible ambiguity is never realized in the hydrodynamical description of a quantum field theory. Later we will present the results for anomalous charged fluids in arbitrary even dimensions and see that this turns out to be true there as well.

In the next section we present our study of the particularly interesting case of parity vio-

[6]The counting goes as follows. The stress tensor decomposes into one $SO(p)$, tensor, one vector and two scalars. The charge current decomposes into a vector and a scalar. Equating the hydrodynamical equilibrium stress tensor and charge current to the expressions obtained by varying the equilibrium yields $3s_e^n + 2v_e^n + t_e^n$ equations.

lating charged fluids in 3+1d at first order in derivative expansions to illustrate this powerful method.

1.4 3+1d parity violating charged fluids at first order

Throughout rest of synopsis we will be working with the background metric and gauge field configuration (1.1). Before proceeding we first define some useful notation. Let u_K^μ be the unit normalized vector in the Killing direction. In components

$$u_K^\mu = e^{-\sigma}(1, 0, \ldots, 0) \tag{1.16}$$

Let ${\mathcal{P}_\mathcal{K}}^{\mu\nu}$ denote the projector orthogonal to u_K^μ

$${\mathcal{P}_\mathcal{K}}^{\mu\nu} = g^{\mu\nu} + u_K^\mu u_K^\nu \tag{1.17}$$

Explicitly in matrix form

$$(\mathcal{P}_K)_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & g_{ij} \end{pmatrix}$$

Let us also define the shear tensor, vorticity and expansion and acceleration of this Killing 'velocity' field by

$$\begin{aligned} &\Theta_K = \nabla.u_K = \text{Expansion}, \quad \mathfrak{a}_K^\mu = (u_K.\nabla)u_K^\mu = \text{Acceleration} \\ &\sigma_K^{\mu\nu} = P^{\mu\alpha}P^{\nu\beta}\left(\frac{\nabla_\alpha(u_K)_\beta + \nabla_\beta(u_K)_\alpha}{2} - \frac{\Theta_K}{3}g_{\alpha_\beta}\right) = \text{Shear tensor} \\ &\omega_K^{\mu\nu} = P^{\mu\alpha}P^{\nu\beta}\left(\frac{\nabla_\alpha(u_K)_\beta - \nabla_\beta(u_K)_\alpha}{2}\right) = \text{Vorticity} \end{aligned} \tag{1.18}$$

A straightforward computation yields

$$\begin{aligned} &\Theta_K = 0, \quad (\mathfrak{a}_K)_\mu = (\mathcal{P}_\mathcal{K})_{\mu i}\nabla^i\sigma \\ &\sigma_K^{\mu\nu} = 0 \\ &(\omega_K)_{\mu\nu} = \frac{e^\sigma}{2}(\mathcal{P}_\mathcal{K})_{\mu i}(\mathcal{P}_\mathcal{K})_{\nu j}f^{ij} \end{aligned} \tag{1.19}$$

1.4.1 Equilibrium from hydrodynamics

In this subsection we evaluate the most general hydrodynamical stress tensor and charge current, at first order in derivatives, and evaluate it in as yet undetermined equilibrium configuration.

In table 1 we list the onshell equivalent first order fluid data. From this table this follows that the most general symmetry allowed correction the constitutive relations at first order in derivatives is given by

$$\begin{aligned} &\pi^{\mu\nu} = -\zeta\theta\mathcal{P}_{\mu\nu} - \eta\sigma_{\mu\nu} \\ &J_{diss}^\mu = \sigma\left(E_\mu - T\mathcal{P}_\mu^\alpha\partial_\alpha\nu\right) + \alpha_1 E^\mu + \alpha_2\mathcal{P}^{\mu\alpha}\partial_\alpha T + \xi_\omega\omega^\mu + \xi_B B^\mu \end{aligned} \tag{1.20}$$

Type	Data	Evaluated at equilibrium $T = T_0 e^{-\sigma},\ \mu = e^{-\sigma} A_0,\ u^\mu = u^\mu_K$
Scalars	$\nabla.u$	0
Vectors	$E_\mu = F_{\mu\nu} u^\nu$, $\mathcal{P}^{\mu\alpha}\partial_\alpha T$, $(E^\mu - T\mathcal{P}^{\mu\alpha}\partial_\alpha \nu)$	$e^{-\sigma}\partial_i A_0$ -$T_0 e^{-\sigma}\partial^i \sigma$ 0
Pseudo-Vectors	$\epsilon_{\rho\lambda\alpha\beta} u^\lambda \nabla^\alpha u^\beta$ $B_\mu = \frac{1}{2}\epsilon_{\rho\lambda\alpha\beta} u^\lambda F^{\alpha\beta}$	$\frac{e^\sigma}{2}\epsilon_{ijk} f^{jk}$ $B_i = \frac{1}{2} g_{ij}\epsilon^{jkl}(F_{kl} + A_0 f_{kl})$
Tensors	$\mathcal{P}_{\mu\alpha}\mathcal{P}_{\nu\beta}\left(\frac{\nabla^\alpha u^\beta + \nabla^\beta u^\alpha}{2} - \frac{\nabla.u}{3} g^{\alpha\beta}\right)$	0

Table 1. One derivative fluid data

Scalars	None
Vectors	$\partial^i A_0$, $\partial^i \sigma$
Pseudo-Vectors	$\epsilon^{ijk}\partial_j A_k$, $\epsilon^{ijk}\partial_j a_k$
Tensors	None

Table 2. One derivative background data

where the shear viscosity η, bulk viscosity ζ, conductivity σ and the remaining possible transport coefficients α_1, α_2, ξ_ω and ξ_B are arbitrary functions of σ and A_0.

Solutions in equilibrium are determined entirely by the background fields σ, A_0, a_i, A_i and g^{ij}. In Table(2,1) we have listed all coordinate and gauge invariant one derivative scalars, vectors and tensors constructed out of this background data. As Table (2,1) lists no one derivative scalars, it follows immediately that the equilibrium temperature field $T(x) = e^{-\sigma} T_0$ and chemical potential field $\mu(x) = e^{-\sigma} A_0$ receive no corrections at first order in the derivative expansion. The velocity field in equilibrium can, however, be corrected. The most general correction to first order is proportional to the vectors and pseudo vectors listed in Table (2,1) and is given by

$$\delta u^i = -\frac{e^{-\sigma} b_1}{4}\epsilon^{ijk} f_{jk} + b_2 B^i_K + b_3 \partial^i \sigma + b_4 \partial^i A_0 \tag{1.21}$$

where

$$\begin{aligned} &f_{jk} = \partial_j a_k - \partial_k a_j, \quad F_{jk} = \partial_j A_k - \partial_k A_j, \quad A_j = \mathcal{A}_j - a_j A_0, \\ &B^i_K = \frac{1}{2}\epsilon^{ijk}(F_{jk} + A_0 f_{jk}), \quad \epsilon^{123} = \frac{1}{\sqrt{g_3}}. \end{aligned} \tag{1.22}$$

The fluid stress tensor evaluated on this equilibrium configuration has two source of first derivative corrections. The first set of corrections arises from the corrections (1.20) evaluated on the zero order equilibrium fluid configuration (1.14). [7] The second source of corrections arises from inserting the velocity correction (1.21) into the zero order (perfect fluid)

[7]When $u^\mu \propto (1, 0 \ldots, 0)$ the Landau frame condition sets $\pi_{00} = \pi_{0i} = J^{diss}_0 = 0$. Consequently T_{00}, T_{0i} and J_0 receive no one derivative corrections of this sort.

constitutive relations. The net change in the constitutive relations is the sum of above two contributions and is given by

$$
\begin{aligned}
&\delta T_{00} = \delta J_0 = \delta T^{ij} = 0, \\
&\delta T^i_0 = -e^{\sigma}(\epsilon + P)\Big[\frac{1}{2}(b_2 A_0 - \frac{1}{2}b_1 e^{\sigma})\epsilon^{ijk} f_{jk} + \frac{1}{2}b_2\epsilon^{ijk}F_{jk} - b_3 T_0 e^{-\sigma}\partial^i\sigma + b_4\partial^i A_0\Big] \\
&\delta \tilde{J}^i = \Big[\frac{1}{2}\Big((\xi_B + qb_2)A_0 - \frac{1}{2}(\xi_\omega + qb_1)e^{\sigma}\Big)\epsilon^{ijk} f_{jk} + \frac{1}{2}(\xi_B + qb_2)\epsilon^{ijk}F_{jk} \\
&\qquad - (qb_3 + \alpha_2)T_0 e^{-\sigma}\partial^i\sigma + (qb_4 + \alpha_1)\partial^i A_0\Big].
\end{aligned}
\tag{1.23}
$$

1.4.2 Equilibrium from the Partition Function

In this subsection we compute the first order corrections to the equilibrium stress tensor and charge current from the equilibrium partition function. From the fact that Table (2) lists no gauge invariant scalars, one might be tempted to conclude that the equilibrium partition function can have no gauge invariant one derivative corrections. But as we have already explained earlier the three(constant) parameter set of Chern Simons terms listed in the third line of (1.6) yield perfectly local and gauge invariant contributions to the partition function, even though they cannot be written as integrals of local gauge invariant expressions. In addition to these gauge invariant pieces we need a term in the action that results in its anomalous gauge transformation property. This requirement is precisely met by the term in the last line of (1.6). With the action (1.6) in hand it is straightforward to compute the stress tensor and charge current in equilibrium using

$$
\begin{aligned}
T_{00} &= -\frac{T_0 e^{2\sigma}}{\sqrt{-g_{(p+1)}}}\frac{\delta W}{\delta\sigma}, \quad T^i_0 = \frac{T_0}{\sqrt{-g_{(p+1)}}}\left(\frac{\delta W}{\delta a_i} - A_0\frac{\delta W}{\delta A_i}\right), \\
T^{ij} &= -\frac{2T_0}{\sqrt{-g_{(p+1)}}}g^{il}g^{jm}\frac{\delta W}{\delta g^{lm}}, \quad J_0 = -\frac{e^{2\sigma}T_0}{\sqrt{-g_{(p+1)}}}\frac{\delta W}{\delta A_0}, \quad J^i = \frac{T_0}{\sqrt{-g_{(p+1)}}}\frac{\delta W}{\delta A_i}.
\end{aligned}
\tag{1.24}
$$

where, for instance, the derivative w.r.t A_0 is taken at constant σ, a_i, A_i, g^{ij}, T_0 and μ_0. Using these we find

$$
\begin{aligned}
T_{00} &= 0, \quad T^{ij} = 0, \\
T^i_0 &= e^{-\sigma}\epsilon^{ijk}\left[(-\frac{1}{2}CA_0^2 + 2C_0 A_0 + C_2)\nabla_j A_k + (2C_1 - \frac{C}{6}A_0^3 - C_2 A_0)\nabla_j a_k\right] \\
J_0 &= -e^{\sigma}\epsilon^{ijk}\left[\frac{C}{3}A_i\nabla_j A_k + \frac{C}{3}A_0 A_i\nabla_j a_k\right] \\
J^i &= e^{-\sigma}\epsilon^{ijk}\left[2\left(\frac{C}{3}A_0 + C_0\right)\nabla_j A_k + \left(\frac{C}{6}A_0^2 + C_2\right)\nabla_j a_k + \frac{C}{3}A_k\nabla_j A_0\right],
\end{aligned}
\tag{1.25}
$$

The current obtained above is gauge non invariant. The more familiar gauge invariant 'covariant' current is related to this 'consistent' current by [17]

$$
\tilde{J}^{\mu} = J^{\mu} - \frac{C}{6}\epsilon^{\mu\nu\gamma\delta}\mathcal{A}_\nu\mathcal{F}_{\gamma\delta}
\tag{1.26}
$$

Using (1.26) it follows that

$$
\begin{aligned}
&\tilde{J}_0 = 0, \\
&\tilde{J}^i = e^{-\sigma}\epsilon^{ijk}\big[(CA_0 + 2C_0)\nabla_j A_k + (\frac{1}{2}CA_0^2 + C_2)\nabla_j a_k\big],
\end{aligned}
\tag{1.27}
$$

1.4.3 Constraints on Hydrodynamics

In this subsection we compare the equilibrium stress tensor and charge current obtained in two different ways in 1.4.1 and 1.4.2 to obtain constraints Equating the coefficients of independent terms in the two expressions for T_0^i (1.23),(1.25) determines the one derivative corrections of the velocity field in equilibrium. We find.

$$
\begin{aligned}
b_1 &= \frac{T^3}{\epsilon + P}\big(\frac{2}{3}\nu^3 C + 4\nu^2 C_0 - 4\nu C_2 + 4C_1\big), \\
b_2 &= \frac{T^2}{\epsilon + P}\big(\frac{1}{2}\nu^2 C + 2\nu C_0 - C_2\big), \\
b_3 &= b_4 = 0.
\end{aligned}
\tag{1.28}
$$

where $\nu = \frac{\mu}{T} = \frac{A_0}{T_0}$.

Equating coefficients of independent terms in J^i in equations 1.23 and 1.27 and using (1.28) gives

$$
\begin{aligned}
\xi_\omega &= C\nu^2 T^2\big(1 - \frac{2q}{3(\epsilon+P)}\nu T\big) + T^2\big[(4\nu C_0 - 2C_2) - \frac{qT}{\epsilon + P}(4\nu^2 C_0 - 4\nu C_2 + 4C_1)\big], \\
\xi_B &= C\nu T\big(1 - \frac{q}{2(\epsilon+P)}\nu T\big) + T\big(2C_0 - \frac{qT}{\epsilon+P}(2\nu C_0 - C_2)\big), \\
\alpha_1 &= \alpha_2 = 0
\end{aligned}
\tag{1.29}
$$

Let us summarize. We have found that the hydrodynamical charge current and stress tensor are given by

$$
\begin{aligned}
\pi^{\mu\nu} &= -\zeta\theta\mathcal{P}_{\mu\nu} - \eta\sigma_{\mu\nu} \\
J^\mu_{diss} &= \sigma\left(E_\mu - T\mathcal{P}^\alpha_\mu\partial_\alpha\nu\right) + \xi_\omega\omega^\mu + \xi_B B^\mu
\end{aligned}
\tag{1.30}
$$

In (1.30) the viscosities ζ and η together with the conductivity σ are all dissipative parameters. These parameters multiply expressions that vanish in equilibrium and are completely unconstrained by the analysis of this subsection. On the other hand ζ_ω and ζ_B - together with α_1 and α_2 in (1.20) - are non dissipative parameters. They multiply expressions that do not vanish in equilibrium. The analysis of this section has demonstrated that α_1 and α_2 vanish and that ζ_ω and ζ_B are given by (1.30). The expressions (1.30) agree exactly with the results of Son and Surowka - based on the requirement of positivity of the entropy current - upon setting $C_0 = C_1 = C_2 = 0$. Upon setting $C_0 = 0$ they agree with the generalized results of [16] (see also [4],[8]).

1.4.4 The Entropy Current

The entropy of our fluid system can easily be computed from the equilibrium partition function as

$$\begin{aligned} S &= \frac{\partial}{\partial T_0}(T_0 \log Z) \\ &= \int d^3x \sqrt{g_3}\epsilon^{ijk}\left[C_0 A_i \nabla_j A_k + 3C_1 T_0^2 a_i \nabla_j a_k + 2C_2 T_0 A_i \nabla_j a_k\right]. \end{aligned} \tag{1.31}$$

Although we can cannot determined the full entropy current(J_S^μ) from equilibrium partition function, but the requirement that total entropy

$$S = \int d^3x \sqrt{-g_4} J_S^0 \tag{1.32}$$

should match with the (1.31), can be used to constrain the equilibrium entropy current. As mentioned in the introduction and as is apparent by looking at the (1.31) that we have to allow for gauge non invariant terms in the entropy current, but keeping associated physical quantities, namely the divergence of entropy current and total entropy, gauge invariant. The most general physically allowed form for the entropy current, at one derivative order, may then be read off from Table 2

$$\begin{aligned} J_S^\mu = {} & su^\mu - \nu J^\mu_{diss} + D_\theta \Theta u^\mu + D_c\left(E^\mu - T\mathcal{P}^{\mu\alpha}\partial_\alpha \nu\right) + D_E E^\mu + D_a \mathfrak{a}^\mu \\ & + D_\omega \omega^\mu + D_B B^\mu + h\epsilon^{\mu\nu\lambda\sigma}\mathcal{A}_\nu \partial_\lambda \mathcal{A}_\sigma \\ & \text{where } h \text{ is a constant} \end{aligned} \tag{1.33}$$

From this the first order correction to the total entropy in equilibrium is easily computed to be

$$\begin{aligned} & \int d^3x \sqrt{-g_4} J_S^0|_{correction} \\ = & \int d^3x \sqrt{g_3}\epsilon^{ijk}\Bigg[T_0^2\left(3C_1 + h\nu^2 + \frac{d_\omega}{2} - \nu d_B\right) a_i \partial_j a_k \\ & \qquad + T_0(2C_2 + 2h\nu - d_B) a_i \partial_j A_k + h A_i \partial_j A_k\Bigg] \end{aligned} \tag{1.34}$$

where

$$\nu = \frac{\mu}{T} = \frac{A_0}{T_0}, \quad d_B = \frac{D_B}{T} - \left(\frac{C\nu^2}{2} - C_2\right), \quad d_\omega = \frac{D_\omega}{T^2} - \left(\frac{C\nu^3}{3} - 2C_2\nu + 2C_1\right) \tag{1.35}$$

Comparing this expression with (1.31) we find

$$h = C_0, \quad d_B = 2C_0\nu, \quad d_\omega = 2C_0\nu^2 \tag{1.36}$$

This result agrees precisely with that of Son and Surowka as generalized in [5].

To end this section let us finally look at the the CPT properties of our partition function (1.6). From table 3 terms appearing with C_0 and C_1 and thus imposing the requirement of CPT invariance on the partition function sets these coefficients to zero.

Field	C	P	T	CPT
σ	$+$	$+$	$+$	$+$
a_i	$+$	$-$	$-$	$+$
g_{ij}	$+$	$+$	$+$	$+$
A_0	$-$	$+$	$+$	$-$
A_i	$-$	$-$	$-$	$-$

Table 3. Action of CPT

1.5 A summary of results

In previous sections we have discussed the generalities of our equilibrium partition function technique and illustrated it some detail with the case of first order parity violating fluids in 3+1d. Using these set of ideas we have worked many other cases. In these section we briefly summarize these results one by one.

1.5.1 3+1d uncharged fluids at 2nd order

For uncharged fluids, the first correction to the equilibrium partition function appears at 2nd order in derivative expansion. Without assuming anything about the parity properties, the most general 2nd order correction to the equilibrium partition function for 3+1d uncharged fluids is given by

$$\begin{split} W = \log Z = -\frac{1}{2}\int d^3x\ \sqrt{g_3}\left[\tilde{P}_1(T_0e^{-\sigma})R + T_0^2\tilde{P}_2(T_0e^{-\sigma})f_{ij}f^{ij} + \tilde{P}_3(T_0e^{-\sigma})(\partial\sigma)^2\right] \\ \text{where}\ \ \tilde{P}_i(T_0e^{-\sigma}) = P_i(\sigma)\ \ \text{and}\ \ P_i' \equiv \frac{dP_i(\sigma)}{d\sigma}\quad (i=1,2,3) \end{split} \tag{1.37}$$

where P_1, P_2, P_3 are three arbitrary function of σ and from now on we will remove the explicit dependence. In partition function, the fourth scalar $\nabla^2\sigma$ and the pseudo-scalar $\epsilon_{ijk}\partial^i\sigma f^{jk}$ do not appear as they are total derivatives.

The most general constitutive relations for this case are parametrized by

$$\begin{split} \Pi_{\mu\nu} = &-\eta\sigma_{\mu\nu} - \zeta P_{\mu\nu}\Theta \\ &+ T\Big[\tau\ (u.\nabla)\sigma_{\langle\mu\nu\rangle} + \kappa_1\tilde{R}_{\langle\mu\nu\rangle} + \kappa_2 K_{\langle\mu\nu\rangle} + \lambda_0\ \Theta\sigma_{\mu\nu} \\ &+ \lambda_1\ \sigma_{\langle\mu}{}^{a}\sigma_{a\nu\rangle} + \lambda_2\ \sigma_{\langle\mu}{}^{a}\omega_{a\nu\rangle} + \lambda_3\ \omega_{\langle\mu}{}^{a}\omega_{a\nu\rangle} + \lambda_4\ \mathfrak{a}_{\langle\mu}\mathfrak{a}_{\nu\rangle}\Big] \\ &+ TP_{\mu\nu}\Big[\zeta_1(u.\nabla)\Theta + \zeta_2\tilde{R} + \zeta_3\tilde{R}_{00} + \xi_1\Theta^2 + \xi_2\sigma^2 + \xi_3\omega^2 + \xi_4\mathfrak{a}^2\Big] \\ &+ T\Big[\sum_{i=1}^{4}\delta_i t_{\mu\nu}^{(i)} + \delta_5 P_{\mu\nu}\mathfrak{a}_\alpha l^\alpha\Big] \end{split} \tag{1.38}$$

where

$$
\begin{aligned}
&u^\mu = \text{The normalized four velocity of the fluid}\\
&P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu = \text{Projector perpendicular to } u^\mu\\
&\Theta = \nabla . u = \text{Expansion}, \quad \mathfrak{a}_\mu = (u.\nabla) u_\mu = \text{Acceleration}\\
&\sigma^{\mu\nu} = P^{\mu\alpha} P^{\nu\beta} \left(\frac{\nabla_\alpha u_\beta + \nabla_\beta u_\alpha}{2} - \frac{\Theta}{3} g_{\alpha_\beta} \right) = \text{Shear tensor}\\
&\omega^{\mu\nu} = P^{\mu\alpha} P^{\nu\beta} \left(\frac{\nabla_\alpha u_\beta - \nabla_\beta u_\alpha}{2} \right) = \text{Vorticity}\\
&K^{\mu\nu} = \tilde{R}^{\mu a \nu b} u_a u_b, \quad \tilde{R}^{\mu\nu} = \tilde{R}^{a\mu b\nu} g_{ab} \quad (\tilde{R}^{abcd} = \text{Riemann tensor})\\
&\sigma^2 = \sigma_{\mu\nu}\sigma^{\mu\nu}, \qquad \omega^2 = \omega_{\mu\nu}\omega^{\nu\mu}
\end{aligned}
\tag{1.39}
$$

and

$$
A_{\langle\mu\nu\rangle} \equiv P_\mu^\alpha P_\nu^\beta \left(\frac{A_{\alpha\beta} + A_{\beta\alpha}}{2} - \left[\frac{A_{ab} P^{ab}}{3} \right] g_{\alpha\beta} \right) \quad \text{For any tensor } A_{\mu\nu}
$$

The expansion (1.38) is given in terms of 15 parity even and 5 parity odd arbitrary transport coefficients, each of which is, as yet, an arbitrary function of temperature). Seven of these fifteen parity even terms and two of the five parity odd terms vanish in equilibrium. Using the techniques used in previous sections we can determine the equilibrium solution as well as the non dissipative transport coefficients in terms of the coefficient functions P_1, P_2, P_3 of the equilibrium partition function. Eliminating P_1, P_2, P_3, we find relation among the non dissipative transport coefficients

$$
\begin{aligned}
\kappa_2 &= \kappa_1 + T \frac{d\kappa_1}{dT}\\
\zeta_2 &= \frac{1}{2} \left[s \frac{d\kappa_1}{ds} - \frac{\kappa_1}{3} \right]\\
\zeta_3 &= \left(s \frac{d\kappa_1}{ds} + \frac{\kappa_1}{3} \right) + \left(s \frac{d\kappa_2}{ds} - \frac{2\kappa_2}{3} \right) + \frac{s}{T} \left(\frac{dT}{ds} \right) \lambda_4\\
\xi_3 &= \frac{3}{4} \left(\frac{s}{T} \right) \left(\frac{dT}{ds} \right) \left(T \frac{d\kappa_2}{dT} + 2\kappa_2 \right) - \frac{3\kappa_2}{4} + \left(\frac{s}{T} \right) \left(\frac{dT}{ds} \right) \lambda_4\\
&\quad + \frac{1}{4} \left[s \frac{d\lambda_3}{ds} + \frac{\lambda_3}{3} - 2 \left(\frac{s}{T} \right) \left(\frac{dT}{ds} \right) \lambda_3 \right]\\
\xi_4 &= -\frac{\lambda_4}{6} - \frac{s}{T} \left(\frac{dT}{ds} \right) \left(\lambda_4 + \frac{T}{2} \frac{d\lambda_4}{dT} \right) - T \left(\frac{d\kappa_2}{dT} \right) \left(\frac{3s}{2T} \frac{dT}{ds} - \frac{1}{2} \right)\\
&\quad - \frac{Ts}{2} \left(\frac{dT}{ds} \right) \left(\frac{d^2\kappa_2}{dT^2} \right)
\end{aligned}
\tag{1.40}
$$

This is in perfect agreement with the relations obtained in [10] using the second law of thermodynamics.

1.5.2 Anomalous charged fluids in arbitrary dimensions

The story described in 1.4 can be generalized to anomalous charged fluid dynamics in arbitrary even dimensions. Although the parity even contributions to the partition functions could appear at all orders in derivative expansion, the parity odd part receive first non trivial contribution at $n-1$ order in derivatives. The anomalous part of the equilibrium partition function, constrained by $2n-1$ dimensional diffeomorphisms invariance, kaluza-klein gauge invariance and the $U(1)$ gauge invariance upto anomaly, is given by

$$\begin{aligned} W_{anom} = \frac{1}{T_0}\int d^{2n-1}x\sqrt{g_{2n-1}}\Bigg\{ \sum_{m=1}^{n} \alpha_{m-1}(A_0,T_0)\ [\epsilon A(da)^{m-1}(dA)^{n-m}] \\ + \alpha_n(T_0)\ [\epsilon a(da)^{n-1}]\Bigg\}. \end{aligned} \tag{1.41}$$

where, $\epsilon^{ijk\dots}$ is the $(2n-1)$ dimensional tensor density defined via

$$\epsilon^{i_1 i_2 \dots i_{d-1}} = e^{-\sigma}\varepsilon^{0 i_1 i_2 \dots i_{d-1}}$$

The indices (i,j) run over $(2n-1)$ values. We have used the following notation for the sake of brevity

$$\begin{aligned} &[\epsilon A(da)^{m-1}(dA)^{n-m}] \\ &\quad \equiv \epsilon^{ij_1k_1\dots j_{m-1}k_{m-1}p_1q_1\dots p_{n-m}q_{n-m}} A_i \partial_{j_1} a_{k_1} \dots \partial_{j_{m-1}} a_{k_{m-1}} \partial_{p_1} A_{q_1} \dots \partial_{p_{n-m}} A_{q_{n-m}} \\ &[\epsilon (da)^{m-1}(dA)^{n-m}]^i \\ &\quad \equiv \epsilon^{ij_1k_1\dots j_{m-1}k_{m-1}p_1q_1\dots p_{n-m}q_{n-m}} \partial_{j_1} a_{k_1} \dots \partial_{j_{m-1}} a_{k_{m-1}} \partial_{p_1} A_{q_1} \dots \partial_{p_{n-m}} A_{q_{n-m}} \end{aligned} \tag{1.42}$$

The coefficient function α_m's are all determined by gauge invariance upto the anomaly equation to be

$$\begin{aligned} \alpha_m &= -\mathcal{C}_{anom}\binom{n}{m+1} A_0^{m+1} + \tilde{C}_m T_0^{m+1}, \quad m = 0,\dots,n-1 \\ \alpha_n &= \tilde{C}_n T_0^{n+1} \end{aligned} \tag{1.43}$$

Here, $\tilde{C}_m$ are arbitrary constants. The requirement of CPT invariance forces all $\tilde{C}_{2k} = 0$.

Using the analysis similar to the that used in section 1.4 we obtain the anomalous part of the constitutive relation to be

$$\begin{aligned} &\delta T^{\mu\nu}_{odd} = 0 \\ &\delta J^{\mu}_{odd} = \sum_{m=1}^{n} \xi_m \varepsilon^{\mu\nu\ \gamma_1\delta_1\dots\gamma_{m-1}\delta_{m-1}\ \alpha_1\beta_1\dots\alpha_{n-m}\beta_{n-m}} u_\nu (\partial_\gamma u_\delta)^{m-1} (\partial_\alpha \mathcal{A}_\beta)^{n-m} + \dots . \end{aligned} \tag{1.44}$$

where the coefficients ξ_m's are determined to be

$$\begin{aligned} \xi_m = &\left[m\frac{q\mu}{\epsilon+p} - (m+1)\right]\mathcal{C}_{anom}\binom{n+1}{m+1}\mu^m \\ &+ \sum_{k=0}^{m}\left[m\frac{q\mu}{\epsilon+p} - (m-k)\right](-1)^{k-1}\tilde{C}_k\binom{n-k}{m-k} T^{k+1}\mu^{m-k-1} \end{aligned} \tag{1.45}$$

This then is the prediction of this transport coefficient via partition function methods. This exactly matches with the expression from [8](see section (2) of [18] for conversion of results of [8] to our conventions).

1.5.3 Parity violating charged fluids in 2+1d at first order

For this case, here, we will only discuss the parity odd part of the partition function and the constraints obtained from it on the the constitutive relations. The parity even sector works as in 3+1d.

The most general partition function in this case is

$$\mathcal{W} = \frac{1}{2}\int \Big(\alpha(\sigma, A_0) dA + T_0 \beta(\sigma, A_0) da\Big), \tag{1.46}$$

where α and β are two arbitrary functions.

The most general first order constitutive relations are given by

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + \left(P - \zeta\nabla_\alpha u^\alpha - \tilde{\chi}_B B - \tilde{\chi}_\Omega \Omega\right) P^{\mu\nu} - \eta\sigma^{\mu\nu} - \tilde{\eta}\tilde{\sigma}^{\mu\nu}\,, \tag{1.47a}$$

$$J^\mu = \rho u^\mu + \sigma V^\mu + \tilde{\sigma}\tilde{V}^\mu + \tilde{\chi}_E \tilde{E}^\mu + \tilde{\chi}_T \tilde{T}^\mu\,. \tag{1.47b}$$

The various quantities appearing in the constitutive relations (1.47) are defined as

$$\Omega = -\epsilon^{\mu\nu\rho} u_\mu \nabla_\nu u_\rho, \qquad B = -\frac{1}{2}\epsilon^{\mu\nu\rho} u_\mu F_{\nu\rho}, \tag{1.48a}$$

$$E^\mu = F^{\mu\nu} u_\nu, \qquad V^\mu = E^\mu - T P^{\mu\nu}\nabla_\nu \frac{\mu}{T}, \tag{1.48b}$$

$$P^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu}, \qquad \sigma^{\mu\nu} = P^{\mu\alpha} P^{\nu\beta}\left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - g_{\alpha\beta}\nabla_\lambda u^\lambda\right)\,, \tag{1.48c}$$

and

$$\tilde{E}^\mu = \epsilon^{\mu\nu\rho} u_\nu E_\rho\,, \qquad \tilde{V}^\mu = \epsilon^{\mu\nu\rho} u_\nu V_\rho\,, \tag{1.48d}$$

$$\tilde{\sigma}^{\mu\nu} = \frac{1}{2}\left(\epsilon^{\mu\alpha\rho} u_\alpha {\sigma_\rho}^\nu + \epsilon^{\nu\alpha\rho} u_\alpha {\sigma_\rho}^\mu\right)\,, \qquad \tilde{T}^\mu = \epsilon^{\mu\nu\rho} u_\nu \nabla_\rho T. \tag{1.48e}$$

Out the 6 parity odd transport coefficients, namely $\{\tilde{\chi}_B, \tilde{\chi}_\omega, \tilde{\eta}, \tilde{\sigma}, \tilde{\chi}_E,\ \tilde{\chi}_T\}$ appearing in the constitutive relations (1.47), only 4 $\{\tilde{\chi}_B, \tilde{\chi}_\omega, \tilde{\chi}_E,\ \tilde{\chi}_T\}$ survive in equilibrium. These are determined in terms of the partition function coefficients α and β to be

$$\begin{aligned}
\tilde{\chi}_B &= \frac{\partial P}{\partial \epsilon}\Big(-T_0 e^{-\sigma}\frac{\partial \alpha}{\partial \sigma}\Big) + \frac{\partial P}{\partial \rho}\Big(T_0 \frac{\partial \alpha}{\partial A_0}\Big),\\
\tilde{\chi}_\Omega &= \frac{\partial P}{\partial \epsilon}\Big(T_0 e^{-2\sigma}\left(T_0\frac{\partial \beta}{\partial \sigma} - A_0\frac{\partial \alpha}{\partial \sigma}\right)\Big) + \frac{\partial P}{\partial \rho}\Big(-T_0 e^{-\sigma}\left(T_0\frac{\partial \beta}{\partial A_0} - A_0\frac{\partial \alpha}{\partial A_0}\right)\Big),\\
\tilde{\chi}_E &= \left(T_0\frac{\partial \alpha}{\partial A_0}\right) - \frac{\rho}{\epsilon + P}\Big(-T_0 e^{-\sigma}\left(T_0\frac{\partial \beta}{\partial A_0} - A_0\frac{\partial \alpha}{\partial A_0}\right)\Big)\\
T\tilde{\chi}_T &= \left(-T_0 e^{-\sigma}\frac{\partial \alpha}{\partial \sigma}\right) - \frac{\rho}{\epsilon + P}\Big(T_0 e^{-2\sigma}\left(T_0\frac{\partial \beta}{\partial \sigma} - A_0\frac{\partial \alpha}{\partial \sigma}\right)\Big)
\end{aligned} \tag{1.49}$$

Eliminating the α and β we find the following relation between the four nondissipative transport coefficients.

$$\tilde{\chi}_B - \frac{\rho}{\epsilon + P}\ \tilde{\chi}_\Omega = \frac{\partial P}{\partial \rho}\tilde{\chi}_E + \frac{\partial P}{\partial \epsilon}\ T\ \tilde{\chi}_T. \tag{1.50}$$

which matches precisely with the relation obtained in [19].

1.5.4 Superfluids in 3+1d

In field theories where a global $U(1)$ symmetries is spontaneously broken, e.g. by condensation of charged scalar field, the long wavelength hydrodynamical description has, over and above the usual hydrodynamical modes, a massless goldstone mode corresponding to the spontaneously broken symmetry. The presence of this massless mode renders the equilibrium partition function nonlocal and its is hard then to make sense of the derivative expansions in such a case. To deal with this problem we shall follow the following strategy. Rather then working with the non local partition function we will work with the euclidean effective action for the goldstone mode. This effective action, unlike the partition function, is local and can usefully be studies in a derivative expansion.

We limit ourselves here to the case where the underlying field theories in an appropriate large N limit in which the effective action for the goldstone boson comes with suitable positive power of N, so that effective dynamics of the the goldstone mode becomes classical in the large N limit. [8] In this classical approximation the partition function of our system is just the Goldstone effective action evaluated on shell.

The most general goldstone effective action for the superfluids upto first order in derivative expansions is given by

$$\begin{aligned}
&S = S^{(0)} + S^{(1)}_{even} + S^{(1)}_{odd} + S_{anom},\\
\text{where}\quad & S^{(0)} = \int d^3x\sqrt{g}\frac{1}{\hat{T}}P(\hat{T},\hat{\mu},\chi),\\
&S^{(1)}_{even} = \int d^3y\sqrt{g}\left[\frac{f_1}{\hat{T}}(\zeta.\partial)\hat{T} + \frac{f_2}{\hat{T}}(\zeta.\partial)\hat{\nu} - f_3\nabla_i\left(\frac{f}{\hat{T}}\zeta^i\right)\right]\\
&S^{(1)}_{odd} = \int \sqrt{g}d^3x\left(g_1\,\epsilon^{ijk}\zeta_i\partial_j A_k + T_0 g_2\epsilon^{ijk}\zeta_i\partial_j a_k\right) + \frac{C_1}{2}\int ada\\
&S_{anom} = \frac{C}{2T_0}\left(\int \frac{A_0}{3}AdA + \frac{A_0^2}{6}Ada\right)
\end{aligned} \tag{1.51}$$

where

$$\begin{aligned}
&\hat{T} = T_0 e^{-\sigma}, \quad \hat{\mu} = A_0 e^{-\sigma}, \quad \hat{u}^\mu = (1,0,0,0)e^{-\sigma}, \quad \hat{\nu} = \frac{\hat{\mu}}{\hat{T}} = \frac{A_0}{T_0},\\
&\xi_i = -\partial_i\phi + \mathcal{A}_i, \quad \zeta_i = \xi_i - a_i A_0 = -\partial_i\phi + A_i, \quad \zeta_0 = \xi_0 = A_0,\\
&\chi = \xi^2 = -\xi^\mu\xi_\mu, \quad \psi = \frac{\xi^2}{\hat{T}^2}.
\end{aligned} \tag{1.52}$$

[8] Outside such a large N limit the quantum corrections to the classical answers, which are suppressed by appropriate powers of N(e.g. by $\frac{1}{N^2}$ in adjoint theories, like $\mathcal{N}=4$ SYM, in t'Hooft limit) may have interesting structure, see e.g. [20–23] for related work, but we would restrict here to the strict large N limit.

The parity even coefficients $f_i \equiv f_i(\hat{T}, \hat{\nu}, \zeta^2)$ for i=1,2,3 while the coefficients $g_i \equiv g_i(\hat{T}, \hat{\nu}, \psi)$. The function $P(\hat{T}, \hat{\nu}, \chi)$ is the thermodynamical pressure and

$$f(\hat{T}, \hat{\nu}, \zeta^2) = -2\frac{\partial P}{\partial \zeta^2}$$

.

A few comments are in order.

- Under gauge transformations we have

$$\mathcal{A}_i \to \mathcal{A}_i + \partial_i \alpha, \quad \phi \to \phi + \alpha.$$

  Gauge invariance thus forces that the effective action can only depend on the ξ_i which is gauge invariant. ζ_i is useful since it is not only $U(1)$ gauge invariant but also Kaluza Klein gauge invariant.

- The coefficient f_3 multiplies the zeroth order equation of motion for the the field ϕ and is thus shifted by field redefinition of ϕ and its effect on physical quantities would be rather trivial.

- While the fields σ, μ and χ are even under the action of time reversal, the fields ξ_i and ζ_i are odd under this operation. Thus the simultaneous requirement of parity and time reversal invariance simply sets $S^1_{even} = 0$.

The most general constitutive relations at first order in this case and the relations obtained between the non dissipative transport coefficients from our method are rather cumbersome to state here, so for sake of brevity we shall just give the counting of the transport coefficients and relations this case. Working in the frame invariant formalism of [4] we find

- In the parity even sector there are 22 non dissipative transport coefficients which are determined in term of 2 free field redefinition invariant functions f_1 and f_2 in the partition function.

- In the parity odd sector there are 18 non dissipative transport coefficients are determined in terms of two free functions g_1 and g_2 appearing in the partition function.

These explicit results obtained, [24], agree precisely with those obtained using the local entropy increase principle [4] and slightly generalized in [24].

1.6 Conclusion

In this synopsis we reported on the progress we have made in better understanding the constraints on the non dissipative transport coefficients from a very physical point of view of demanding that there must exist time independent solutions when a fluid is put on a time independent background and that the equilibrium conserved currents should come from an equilibrium partition function. We showed that the constraints thus obtained match precisely with those obtained by the use of a local form of second law of thermodynamics in number of nontrivial examples. The extensive matching of the relations, in cases like superfluids, where the number of relation are numerous and rather intricate, makes provides extensive support for our conjecture that such a relation is true to all orders in long wavelength expansion. It would be certainly be interesting to find a proof or a counterexample against this conjecture.

Although in this synopsis we have nowhere used the AdS/CFT correspondence, it is one of the main motivation behind the work presented in this synopsis. The local entropy increase of the boundary fluid dynamics maps, under the fluid-gravity map, to Hawking's black hole area(/entropy) increase theorem in two derivative Einstein-Hilbert gravity. Although a generalization of the Bekenstein-Hawking entropy by Wald has been long proposed, a corresponding Wald entropy increase theorem has not yet been proven. The second law of thermodynamics for fluids dual to higher derivative gravity theories would map to a Wald entropy increase theorem. This leads to a exciting possibility of either proving the Wald entropy increase theorem for higher derivative theories of gravity or constraining the possible higher derivative corrections to Einstein-Hilbert gravity by the requirement of an entropy increase principle.

Recently, using the ideas of the equilibrium partition function other authors have also made progress in understanding the effects of anomalies in other global symmetries like weyl symmetry, diffeomorphisms and other non-abelian global symmetries(see e.g. [25–28]). An interesting related queston is, whether anomalies also affect the dissipative transport coefficients.

We have proposed a very simple and powerful technique to analyse the equality type constraints in hydrodynamics. It is natural to wonder if there is a simillar simpler understanding of the inequalities for the dissipative transport coefficients e.g. by the consideration of stability of equilibrium solution. If true, this would be some progress towards a better understanding of second law of thermodynamics at least in the hydrodynamical context.

2 List of publications

2.1 Publications by the author summarized in this synopsis/thesis

- Nabamita Banerjee, Suvankar Dutta, Sachin Jain, R. Loganayagam, Tarun Sharma,
  "Constraints on Anomalous Fluid in Arbitrary Dimensions "
  JHEP 1303 (2013) 048, arXiv:1206.6499 [hep-th].

- Sayantani Bhattacharyya, Sachin Jain, Shiraz Minwalla, Tarun Sharma,
  "Constraints on Superfluid Hydrodynamics from Equilibrium Partition Functions "
  JHEP 1301 (2013) 040, arXiv:1206.6106 [hep-th].

- Sachin Jain, Tarun Sharma,
  "Anomalous charged fluids in 1+1d from equilibrium partition function"
  JHEP 1301 (2013) 039, arXiv:1203.5308 [hep-th].

- Nabamita Banerjee, Jyotirmoy Bhattacharya, Sayantani Bhattacharyya, Sachin Jain, Shiraz Minwalla, Tarun Sharma,
  "Constraints on Fluid Dynamics from Equilibrium Partition Functions"
  JHEP 1209 (2012) 046, arXiv:1203.3544 [hep-th].

2.2 All publications by author during the course of graduate studies at TIFR

- Amin A. Nizami, Tarun Sharma, V. Umesh,
  "Superspace formulation of correlation functions of 3d superconformal field theories"
  arxiv:1308.4778 [hep-th].

- Sachin Jain, Shiraz Minwalla, Tarun Sharma, Tomohisa Takimi, Spenta R. Wadia, Shuichi Yokoyama,
  "Phases of large N vector Chern-Simons theories on $S^2 \times S^1$"
  arXiv:1301.6169 [hep-th].

- Chi-Ming Chang, Shiraz Minwalla, Tarun Sharma, Xi Yin,
  "ABJ Triality: from Higher Spin Fields to Strings"
  J.Phys. A46 (2013) 214009, arXiv:1207.4485 [hep-th].

- Nabamita Banerjee, Suvankar Dutta, Sachin Jain, R. Loganayagam, Tarun Sharma,
  "Constraints on Anomalous Fluid in Arbitrary Dimensions "
  JHEP 1303 (2013) 048, arXiv:1206.6499 [hep-th].

- Sayantani Bhattacharyya, Sachin Jain, Shiraz Minwalla, Tarun Sharma,
  "Constraints on Superfluid Hydrodynamics from Equilibrium Partition Functions "
  JHEP 1301 (2013) 040, arXiv:1206.6106 [hep-th].

- Sachin Jain, Tarun Sharma,
  "Anomalous charged fluids in 1+1d from equilibrium partition function"
  JHEP 1301 (2013) 039, arXiv:1203.5308 [hep-th].

- Nabamita Banerjee, Jyotirmoy Bhattacharya, Sayantani Bhattacharyya, Sachin Jain, Shiraz Minwalla, Tarun Sharma,
  "Constraints on Fluid Dynamics from Equilibrium Partition Functions"
  JHEP 1209 (2012) 046, arXiv:1203.3544 [hep-th].

- Shiraz Minwalla, Prithvi Narayan, Tarun Sharma, V. Umesh, Xi Yin,
  "Supersymmetric States in Large N Chern-Simons-Matter Theories "
  JHEP 1202 (2012) 022, arXiv:1104.0680 [hep-th].

3 Constraints on Fluid Dynamics From Equilibrium Partition Function

3.1 Introduction and Summary

In this chapter we explore, in great detail, the structural constraints imposed on the equations of relativistic hydrodynamics by two related physical requirements. First that these equations admit a stationary solution on an arbitrarily weakly curved stationary background spacetime. Second that the conserved currents (e.g. the stress tensor) on the corresponding solution follow from an equilibrium partition function.

Landau-Lifshitz [1], and several subsequent authors, have emphasized another source of constraints on the equations of hydrodynamics, namely that the equations are consistent with a local form of the second law of thermodynamics. As is well known, this requirement imposes inequalities on several parameters (like viscosities and conductivities) that appear in the equations of hydrodynamics. It is perhaps less well appreciated that the requirement of local entropy increase also yields equalities relating otherwise distinct fluid dynamical parameters, and so reduces the number of free parameters that appear in the equations of fluid dynamics (see e.g. [1, 2], for more recent work inspired by the AdS/CFT correspondence see e.g. [3–11]). In three specific examples we demonstrate below that the equalities obtained from the comparison with equilibrium (described in the previous paragraph) agree exactly with the equalities between coefficients obtained from the local second law of thermodynamics. These results lead us to conjecture that the constraints obtained from these two a naively distinct physical requirements infact always coincide.

In the rest of this section we summarize our procedure and results in detail. In subsection 3.1.1 below we describe the structure of equilibrium partition functions for field theories on stationary spacetimes in an expansion in derivatives of the background spacetime metric (and gauge fields). In subsection 3.1.2 we then describe the constraints on the equations of relativistic hydrodynamics imposed by the structure of the partition functions described in subsection 3.1.1. In three examples we compare these constraints to those obtained from the requirement of entropy increase and find perfect agreement in each case.

3.1.1 Equilibrium partition functions on weakly curved manifolds

Consider a relativistically invariant quantum field theory on a manifold with a timelike killing vector. By a suitable choice of coordinates, any such manifold may be put in the form

$$ds^2 = -e^{2\sigma(\vec{x})}\left(dt + a_i(\vec{x})dx^i\right)^2 + g_{ij}(\vec{x})dx^i dx^j \tag{3.1}$$

where $i = 1 \dots p$. ∂_t is the killing vector on this manifold, while the coordinates $\vec{x}$ parametrize spatial slices. Here σ, a_i, g_{ij} are smooth functions of coordinates $\vec{x}$.

Let H denote the Hamiltonian that generates translations of the time coordinate t. In this subsection we address the following question. What can we say, on general symmetry grounds, about the dependence of the the partition function of the system

$$Z = \text{Tr} e^{-\frac{H}{T_0}}, \tag{3.2}$$

on σ, g_{ij} and a_i? We will focus on the long wavelength limit, i.e. on manifolds whose curvature length scales are much larger than the 'mean free path' of the thermal fluid [9]. In this limit the question formulated above may addressed using the techniques of effective field theory. In the long wavelength limit the background manifold may be thought of as a union of approximately flat patches, in each of which the system is in a local flat space thermal equilibrium at the locally red shifted temperature

$$T(x) = e^{-\sigma} T_0 + \dots \tag{3.3}$$

(where T_0 is the equilibrium temperature of the system and the $\dots$ represent derivative corrections, see below). Consequently the partition function of the system is given by

$$\ln Z = \int d^p x \sqrt{g_p} \frac{1}{T(x)} P(T(x)) + \dots \tag{3.4}$$

where $P(T)$ is the thermodynamical function that computes the pressure as a function of temperature in flat space. Substituting (3.3) into (3.4) we find

$$\ln Z = \int d^p x \sqrt{g_p} \frac{e^{\sigma}}{T_0} P(T_0 e^{-\sigma}) + \dots \tag{3.5}$$

The $\dots$ in (3.5) denote corrections to $\ln Z$ in an expansion in derivatives of the background metric. At any given order in the derivative expansion these correction are determined, by the requirement of diffeomorphism invariance, in terms of a finite number of unspecified functions of σ. For example, to second order in the derivative expansion, p dimensional diffeomorphism invariance and $U(1)$ gauge invariance of the Kaluza Klein field a constrain the action to take the form

$$\begin{aligned} \log Z = W = -\frac{1}{2}\Bigg(&\int d^p x \sqrt{g_p} \frac{e^{\sigma}}{T_0} P(T_0 e^{-\sigma}) \\ &+ \int d^p x \sqrt{g_p} \big(P_1(\sigma) R + T_0^2 P_2(\sigma) (\partial_i a_j - \partial_j a_i)^2 + P_3(\sigma) (\nabla \sigma)^2 \big) \Bigg) \end{aligned} \tag{3.6}$$

where $P_1(\sigma)$, $P_2(\sigma)$ and $P_3(\sigma)$ are arbitrary functions. It is possible to demonstrate on general grounds that the temperature dependence of these functions is given by

$$P_i(\sigma) = \tilde{P}_i(T_0 e^{-\sigma}) \tag{3.7}$$

so that

$$\begin{aligned} \log Z = W = -\frac{1}{2}\Bigg(&\int d^p x \sqrt{g_p} \frac{e^{\sigma}}{T_0} P(T_0 e^{-\sigma}) \\ &+ \int d^p x \sqrt{g_p} \big(\tilde{P}_1(T_0 e^{-\sigma}) R + T_0^2 \tilde{P}_2(T_0 e^{-\sigma}) (\partial_i a_j - \partial_j a_i)^2 + \tilde{P}_3(T_0 e^{-\sigma}) (\nabla \sigma)^2 \big) \Bigg) \end{aligned} \tag{3.8}$$

[9] Equilibrium Partition functions special curved manifolds or with particular background gauge fields have been studied before in [19, 29, 30]

The discussion above is easily generalized to the study of a relativistic fluid which possesses a conserved current J_μ corresponding to a global $U(1)$ charge. We work on the manifold (3.1) in the presence of a time independent background $U(1)$ gauge connection

$$\mathcal{A} = \mathcal{A}_0(\vec{x})dx^0 + \mathcal{A}_i(\vec{x})dx^i \tag{3.9}$$

and study the partition function

$$Z = \text{Tr} e^{-\frac{H-\mu_0 Q}{T_0}} \tag{3.10}$$

Later in the section we present a detailed study of the special case of charged fluid dynamics in $p = 3$ and $p = 2$ spatial dimensions, at first order in the derivative expansion, without imposing the requirement of parity invariance. Let us first consider the case $p = 3$. The requirements of three dimensional diffeomorphism invariance, Kaluza Klein gauge invariance, and $U(1)$ gauge invariance upto an anomaly[10] (see below) force the partition function to take the form[11]

$$\begin{aligned}
\ln Z &= W^0 + W^1_{inv} + W^1_{anom} \\
W^0 &= \int \sqrt{g_3}\frac{e^\sigma}{T_0} P\left(T_0 e^{-\sigma}, e^{-\sigma} A_0\right) \\
W^1_{inv} &= \frac{C_0}{2}\int AdA + \frac{T_0^2 C_1}{2}\int ada + \frac{T_0 C_2}{2}\int Ada \\
W^1_{anom} &= \frac{C}{2}\left(\int \frac{A_0}{3T_0} AdA + \frac{A_0^2}{6T_0} Ada\right)
\end{aligned} \tag{3.11}$$

where A_i

$$\begin{aligned}
A_0 &= \mathcal{A}_0 + \mu_0 \\
A_i &= \mathcal{A}_i - A_0 a_i
\end{aligned} \tag{3.12}$$

(3.11) is written in terms of A_i because A_i, unlike $\mathcal{A}_i$, is Kaluza Klein gauge invariant[12].

W^0 in (3.11) is zero derivative contribution to the partition function, and is the patchwise approximation to equilibrium, in the spirit of (3.5). W^1_{inv} is the most general diffeomorphism

[10] In this thesis we only consider the effect of $U(1)^3$ anomalies ignoring the effects of . for instance, mixed gravity-gauge anomalies. A systematic study of the effect of these anomalies in fluid dynamics would require us to extend our analysis of charged fluid dynamics to 2nd order, a task we leave for the future (see however section 3.6). It is possible that C_2 above will turn out to be determined in terms of such an anomaly coefficient. We thank R. Loganayagam for pointing this out to us.

[11] Our convention is

$$\frac{1}{2}\int XdY = \int d^3x\sqrt{g_3}\epsilon^{ijk}X_i\partial_j Y_k\ , \quad \frac{1}{2}\int dY = \int d^2x\sqrt{g_2}\epsilon^{ij}\partial_i Y_j\ .$$

[12] The background data can be taken as gauge field $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_i)$ with constant chemical potential μ_0 and temperature T_0. Equivalently we can think of the system to have background gauge field $B = (\mathcal{A}_0 + \mu_0, \mathcal{A}_i)$ with no chemical potential. These two are equivalent physical statements as μ_0 can be absorbed in the constant part of $\mathcal{A}_0$.

and gauge invariant one derivative correction to W^0. Note that W^1 is the sum of a Chern Simons term for the connection A, a Chern Simons term for the connection a and a mixed Chern Simons term in A and a. As usual, gauge invariance forces the coefficients C_0, C_1 and C_2 of these Chern Simons terms to be constants.

(3.11) is the most general form of the partition function of our system that satisfies the requirements of 3 dimensional diffeomorphism invariance and gauge invariance. If we, in addition, impose the requirement of CPT invariance of the underlying four dimensional field theory then it turns out that $C_0 = C_1 = 0$ (see subsection 3.3.6). In other words, the requirement of CPT invariance allows only the mixed Chern Simons term, setting the 'pure' Chern Simons terms to zero.

W^1_{anom} is the part of the effective action that is not gauge invariant under $U(1)$ gauge transformations. [13] Its gauge variation under $A_\mu \to A_\mu + \partial_\mu \phi(\vec{x})$ is given by

$$\delta W^1_{anom} = \frac{C}{24T_0} \int d^3x \sqrt{-g_4} * (\mathcal{F} \wedge \mathcal{F})\ \phi(x) \tag{3.13}$$

As we explain in much more detail below, this is exactly the variation of the effective action predicted by the anomalous conservation equation

$$\nabla_\mu \tilde{J}^\mu = -\frac{C}{8} * (\mathcal{F} \wedge \mathcal{F}) \tag{3.14}$$

where $\tilde{J}$ is the gauge invariant $U(1)$ charge current, and $*$ denotes the Hodge dual.

Let us now turn to parity violating charged fluid dynamics in $p = 2$ spatial dimensions. In this case there is no anomaly in the system and the parity odd sector is qualitatively much different from its $p = 3$ spatial dimension counterpart. For this system we primarily focus on the parity odd sector upto the first order in derivative expansion and the manifestly gauge invariant partition function in this case takes the form

$$\ln Z = \mathcal{W}^0 + \mathcal{W}, \tag{3.15}$$

where

$$\begin{aligned} \mathcal{W}^0 &= \int \sqrt{g_2} \frac{e^\sigma}{T_0} P\left(T_0 e^{-\sigma}, e^{-\sigma} A_0\right) \\ \mathcal{W} &= \frac{1}{2} \int \left(\alpha(\sigma, A_0)\ dA + T_0\ \beta(\sigma, A_0)\ da\right). \end{aligned} \tag{3.16}$$

Where A_0 and A_i are defined in (3.12) and α and β are arbitrary functions.

It is straightforward, if tedious, to generalize the form of the partition function presented in special examples above to higher orders in the derivative expansion. To any given order in the derivative expansion, the dependence of $\ln Z$, on g_{ij}, a_i, σ, A_0 and A_i is fixed by the

[13] It is striking that the effect of the anomaly can be captured by a local term in the 3 dimensional effective action. Note that W^1 cannot be written as the dimensional reduction of a local contribution to the 4 dimensional action, in agreement with general expectations.

requirements of p dimensional diffeomorphism invariance and gauge invariance in terms of a finite number of unspecified functions of two variables, σ and A_0.

We will now define some terminology that will prove useful in the sequel. Let s_e^n denote the number of independent gauge invariant scalar expressions that one can construct out of σ, a_i (and A_0 and A_i in the case that the fluid is charged) at n^{th} order in the derivative expansion. In a similar manner, v_e^n and t_e^n will denote the number of n^{th} order independent gauge invariant vectors and (traceless symmetric two index) tensors formed out of the same quantities. Finally let st_e^n denote the total number of n^{th} order scalar expressions that happen to be total derivatives (including the contribution of a coefficient function) and so integrate to zero [14] It is clear that at n^{th} order in the derivative expansion, the equilibrium action $\ln Z$ depends on $s_e^n - st_e^n$ unknown functions of two variables.

3.1.2 Constraints on Fluid Dynamics from stationary equilibrium

Relativistic Hydrodynamics In this subsubsection we present a lightening review of the structure of the equations of charged relativistic hydrodynamics. The equations of hydrodynamics are simply the equations of conservation of the stress tensor and the charge current

$$\nabla_\mu T^\mu_\nu = \mathcal{F}_{\nu\mu} \tilde{J}^\mu, \quad \nabla_\mu \tilde{J}^\mu = -\frac{C}{8} * (\mathcal{F} \wedge \mathcal{F}), \tag{3.17}$$

where $\mathcal{F}$ is the field strength of the gauge field $\mathcal{A}$ in (3.9). These equations constitute a closed dynamical system when supplemented with constitutive relations that express $T_{\mu\nu}$ and J_μ as a function of the fluid temperature, chemical potential and velocity. These constitutive relations are presented in an expansion in derivatives and take the form

$$T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu + P g^{\mu\nu} + \pi^{\mu\nu}, \quad J^\mu = q u^\mu + J^\mu_{diss}, \tag{3.18}$$

The pressure P, proper energy density ϵ and proper charge density q are those functions of T and μ predicted by flat space equilibrium thermodynamics. $\pi_{\mu\nu}$ refers to the sum of all corrections to the stress tensor that are of first or higher order in the derivative expansion (the derivatives in question could act either on the T, μ, u^μ, or the background metric and gauge field $g_{\mu\nu}$ and $\mathcal{A}_\mu$). Similarly J^μ_{diss} refers to corrections to the perfect fluid charge current that depend on atleast one spacetime derivative. Field redefinitions of the T μ and u^μ may be used to impose $p+2$ constraints on $\pi_{\mu\nu}$ and J^μ_{diss}; throughout this section we will work in the so called Landau Frame in which

$$u^\mu \pi_{\mu\nu} = 0, \quad u^\mu J^{diss}_\mu = 0 \tag{3.19}$$

Terms in $\pi_{\mu\nu}$ and J^μ_{diss} are both graded according to the number of spacetime derivatives they contain, i.e.

$$\begin{aligned} \pi^{\mu\nu} &= \pi^{\mu\nu}_{(1)} + \pi^{\mu\nu}_{(2)} + \pi^{\mu\nu}_{(3)} + \ldots \\ J^\mu_{diss} &= J^\mu_{diss,(1)} + J^\mu_{diss,(2)} + J^\mu_{diss,(3)} + \ldots \end{aligned} \tag{3.20}$$

[14] For example, the two derivative scalar $\nabla_\mu h(\sigma) \nabla^\mu \sigma$ is a total derivative for arbitrary $h(\sigma)$.

where the subscript counts the number of derivatives.

Symmetry considerations immediately constrain the possible expansions for $\pi_{\mu\nu}$ and J^μ_{diss} as follows. At any given point in spacetime, the fluid velocity u^μ is a particular timelike vector. The value of the velocity breaks the local $SO(p,1)$ Lorentz symmetry of the theory down to the rotational subgroup $SO(p)$. In the Landau frame (3.17) $\pi_{\mu\nu}$ may be decomposed into an $SO(p)$ tensor and $SO(p)$ scalar. J^μ_{diss} is an $SO(p)$ vector.

In order to parameterize freedom in the equations of hydrodynamics, it is useful to define some terminology. Let t^n_f, v^n_f and s^n_f respectively denote the number of onshell inequivalent tensor, vector and scalar expressions that can be formed out expressions made up of a total of n derivatives acting on T, u^μ, μ, $g_{\mu\nu}$ and A_μ. It follows immediately that the most general symmetry allowed expression for $\pi^{\mu\nu}_{(n)}$ is given in terms of $t^n_f + s^n_f$ unknown functions of the two variables T and μ. In a similar manner the most general expression for the $J^\mu_{diss(n)}$, permitted by symmetries, is given in terms of v^n_f unknown functions of the same two variables.

It turns out that the $(t^n_f + s^n_f + v^n_f)$ n^{th} order transport coefficients are not all independent. The requirement that the hydrodynamical equations are consistent with the existence of an entropy current that is of positive divergence in every conceivable fluid flow imposes several relationships between these coefficients cutting down the number of parameters in these equations; we refer the reader to [1, 3, 5, 10], for example, for a fuller discussion. We now turn to a description of a simpler physical principal that appears predict the same relations between these coefficients.These relations may all be constructively determined by comparison of the equations of hydrodynamics with a partition function.

Constraints from stationary equilibrium As we have explained in the previous subsubsection, it follows from symmetry considerations that the equations of charged hydrodynamics, at n^{th} order in the derivative expansion, are parameterized by $t^n_f + v^n_f + s^n_f$ unknown functions of two variables (or $t^n_f + s^n_f$ functions of one variable for uncharged hydrodynamics). We will now argue that these functions are not all independent, but instead are determined in terms of a smaller number of functions.

It is easy to verify that the equations of perfect fluid hydrodynamics (hydrodynamics at lowest order in the derivative expansion) admit a stationary 'equilibrium' solution in the backgrounds (3.1) and (3.9) given by

$$u^\mu_{(0)}(\vec{x}) = e^{-\sigma}(1, 0, \dots, 0), \quad T_{(0)}(\vec{x}) = T_o e^{-\sigma}, \quad \mu_{(0)}(\vec{x}) = e^{-\sigma} A_0 \tag{3.21}$$

As explained above, this is also the equilibrium solution one expects of the fluid on intuitive ground. At higher order in the derivative expansion this solution is corrected; the corrected solution may be expanded in derivatives

$$\begin{aligned} u^\mu &= u^\mu_{(0)} + u^\mu_{(1)} + u^\mu_{(2)} + \dots \\ T &= T_{(0)} + T_{(1)} + T_{(2)} + \dots \\ \mu &= \mu_{(0)} + \mu_{(1)} + \mu_{(2)} + \dots \end{aligned} \tag{3.22}$$

where $u^\mu_{(n)}$, $T_{(n)}$ and $\mu_{(n)}$ are expressions of n^{th} order in derivatives acting on σ, A_0, a_i, A_i and g_{ij}. What can we say about the form of the corrections $u^\mu_{(n)}$, $T_{(n)}$ and $\mu_{(n)}$? Adopting the notation defined in the last paragraph of the previous subsection, symmetries determine the expression for $u^\mu_{(n)}$ in terms of v^n_e as yet unknown functions of σ and A_0, while T and μ are each determined in terms of s^n_e as yet unknown equations of A_0 and σ.

The stress tensor and charge current in equilibrium are given by plugging (3.22) into (3.20). The result is an expression for $\pi^{\mu\nu}$ and J^μ_{diss} written entirely in terms of σ, A_0, a_i, A_i, g_{ij} and their derivatives.

This expressions for the stress tensor and charge current so obtained depend only on a subset of the transport coefficients that appear in the expansion of $\pi^{\mu\nu}$ and J^μ_{diss}. For instance, the expansion of the n^{th} order tensor part of $\pi^{\mu\nu}$ has t^n_f terms in general. When evaluated on (3.21), however, this expression reduces to a sum over $t^n_e \leq t^n_f$ terms. The coefficients of these terms define t^n_e subspace of the t^n_f dimensional set of n^{th} order transport coefficients. We refer to this subspace as the subspace of *non dissipative* transport coefficients.

In this thesis we demand that the expressions for the equilibrium stress tensor and charge current, obtained as described in the previous paragraph, agree with the corresponding expressions obtained by differentiating the equilibrium partition function of subsection 3.1.1 with respect to the background gauge field and metric. This requirement yields a set of $t^n_e + 2v^n_e + 3s^n_e$ equations[15] that completely determine both the n^{th} order corrections to the equilibrium solutions T_n μ_n and u^μ_n ($v^n_e + 2s^n_e$ coefficients in all) as well as the $t^n_e + v^n_e + s^n_e$ non dissipative hydrodynamical transport coefficients. Note that the number of variables precisely equals the number of equations. Dissipative hydrodynamical transport coefficients are completely unconstrained by this procedure.

We emphasize that the shifted equilibrium velocities, temperatures and chemical potentials obtained from the procedure just described automatically obey the equations of hydrodynamics. By construction, the shifted fluid variables, together with the constitutive relations determined above yield the stress tensor that follows from the functional variation of an equilibrium partition function, and the stress tensor obtained from the variation of *any* diffeomorphically invariant functional is automatically conserved. Very similar remarks apply to the charge current.

Let us summarize. In general $\pi^{\mu\nu}$ and J^μ_{diss} are expanded in terms of $t^n_f + s^n_f$ and v^n_f transport coefficients, each of which is a function of temperature and chemical potential. However $t^n_f - t^n_e + s^n_f - s^n_e$ of these coefficients in $\pi^{\mu\nu}$ and $v^n_f - v^n_e$ of these coefficients in J^μ_{diss} evaluate to zero on the 'equilibrium' configuration (3.21). The remaining $t^n_e + v^n_e + s^n_e$ non dissipative transport coefficients multiply expressions that do not vanish on (3.21). Comparison with the equilibrium partition function algebraically determines all non dissipative transport coefficients in terms of the $s^n_e - st^n_e$ functions (and derivatives thereof) that appear as coefficients

[15] The counting goes as follows. The stress tensor decomposes into one $SO(p)$, tensor, one vector and two scalars. The charge current decomposes into a vector and a scalar. Equating the hydrodynamical equilibrium stress tensor and charge current to the expressions obtained by varying the equilibrium yields $3s^n_e + 2v^n_e + t^n_e$ equations.

in the derivative expansion of the partition function. In other words the $t_e^n + v_e^n + s_e^n$ non dissipative transport coefficients are not all independent; there exist $t_e^n + v_e^n + st_e^n$ relations between these coefficients.

The procedure described above may also be used to derive constraints on the form of the fluid entropy current. The entropy current must obey two constraints. First its divergence must vanish on all the equilibrium configurations derived above. Second, the integral over the entropy density (obtained from the entropy current) must equal the thermodynamical entropy that follows from the partition function (3.10). These requirements impose constraints on the form of the (non dissipative) part of the most general symmetry allowed hydrodynamical entropy current.

We have implemented the procedure described above in detail in three separate examples which we describe in more detail immediately below. In each case we have obtained detailed expressions for all non dissipative hydrodynamical coefficients in terms of the parameters that appear in the action. In each case, the relations obtained between non dissipative transport coefficients, after eliminating the action parameters, agree exactly with the relations obtained between the same quantities by previous investigations based on the study study of the second law of thermodynamics.

In the case of parity violating first order fluid dynamics in 3+1 dimensions, the results for transport coefficients computed from (3.11) match perfectly with those of Son and Surowka [3] (generalized in [16],[4]) once we impose the additional requirement of CPT invariance. [16]

In the case of parity preserving fluid dynamics in 3+1 dimensions, the results obtained from the partition function (3.6) agree perfectly with those of Bhattacharyya [10]. Finally, in the case of parity non preserving charged fluid dynamics in 2+1 dimensions, the results from section 3.4 agree perfectly with those of [19].

In ending this introduction let us note the following. As we have described at the beginning of the introduction, the physical principles that yield constraints on the transport relations of fluid dynamics are twofold. First, that these equations are consistent with the existence of a stationary solution in every background of the form (3.1), (3.9). Second, that the stress tensor and charge current evaluated on this equilibrium configuration obeys the integrability constraints that follow if these expressions can be obtained by differentiating a partition function. In the presentation described above we have mixed these two conditions together (as the partition function is the starting point of our discussion). However it is also possible to separate these two conditions. For each of the three examples discussed above, in

[16]Before imposing the requirement of CPT invariance, we have an additional one parameter freedom that is not captured by the the generalized Son-Surowka analysis. The reason for this is that Son and Surowka (and subsequent authors) assumed that the entropy current was necessarily gauge invariant. This does not seem to us to be physically necessary. It seems to us that an entropy current whose divergence is gauge invariant - and whose integral over a compact manifold in equilibrium is gauge invariant - is perfectly acceptable. As we explain below, it is easy to find a one parameter generalization of the Son-Surowka solution that meets these conditions, and that gives rise to the additional term C_0 in the partition function (3.11). However it turns out that the requirement of CPT invariance sets C_0 (along with C_1) to zero in (3.11), so this possible ambiguity is never realized in the hydrodynamical description of a quantum field theory.

Appendix 3.8.1, 3.8.2 and 3.8.3 we present a detailed study of the constraints on the equations of fluid dynamics obtained merely from the existence of stationary solutions in arbitrary backgrounds of the form (3.1), (3.9). In each case we find that all of the relations between transport coefficients, derived in this thesis, are implied already by this weaker condition. In these three examples, once equilibrium exists, the requirement that it follows from a partition function turns out to be automatic. We do not expect this always to be the case. In more complicated cases we expect the existence of a partition function to imply further constraints than those implied merely by the existence of equilibrium. However we leave the study of such effects to future work.

3.2 Preparatory Material

In this subsection we present background material that we will need in the main part of this section. In subsubsection 3.2.1 we present some Kaluza Klein reduction formulae for metrics of the form (3.1). In subsubsection 3.2.2 we describe the transformation properties of various quantities of interest under Kaluza Klein gauge transformations. In subsubsection 3.2.3 we discuss how the stress tensor and charge current of our system is related to the partition function. We also discuss the thermodynamical energy, entropy and entropy of our system, and compare these quantities to those obtained from integrals over local currents. In subsubsection 3.2.6 we discuss the relation between consistent currents (those obtained from the variation of an action) and gauge invariant currents in systems with a $U(1)$ anomaly. In subsubsection3.2.7 we describe how the equations of perfect fluid hydrodynamics may be 'derived' starting from a zero derivative equilibrium partition function.

3.2.1 Kaluza Klein Reduction Formulae

As explained in the introduction, in this thesis we study theories on metric and gauge fields in the Kaluza Klein form

$$\begin{aligned} ds^2 &= -e^{2\sigma(\vec{x})}\left(dt + a_i(\vec{x})dx^i\right)^2 + g_{ij}(\vec{x})dx^i dx^j \\ \mathcal{A}^\mu &= (A^0(\vec{x}), \mathcal{A}^i(\vec{x})) \end{aligned} \tag{3.23}$$

The inverse of this metric is given by

$$g^{\mu\nu} = \begin{pmatrix} (-e^{-2\sigma} + a^2) & -a^i \\ -a^i & g^{ij} \end{pmatrix}$$

where the first row and column refer to time and g^{ij} is the inverse of g_{ij}. Christoffel symbols, $\tilde{\Gamma}$, of the $p+1$ dimensional metric are given in terms of those of the p dimensional Christoffel symbols Γ by

$$
\begin{aligned}
\tilde{\Gamma}^0_{00} &= -e^{2\sigma}(a.\partial)\sigma \\
\tilde{\Gamma}^i_{00} &= e^{2\sigma} g^{im}\partial_m\sigma \\
\tilde{\Gamma}^0_{i0} &= \partial_i\sigma - e^{2\sigma}(a.\partial)\sigma a_i + \frac{e^{2\sigma} f_{im} a^m}{2} \\
\tilde{\Gamma}^i_{j0} &= e^{2\sigma} g^{ik}(-\frac{1}{2} f_{jk} + \partial_k\sigma a_j) \\
\tilde{\Gamma}^0_{ij} &= -a_n\Gamma^n_{ij} + \frac{e^{2\sigma}}{2}\Big[a_j a^m \partial_i a_m + a_i a^m \partial_j a_m\Big] \\
&\quad - \frac{1}{2} a.\partial(e^{2\sigma} a_i a_j) + \frac{e^{-2\sigma}}{2}\Big[\partial_i(e^{2\sigma} a_j) + \partial_j(e^{2\sigma} a_i)\Big] \\
\tilde{\Gamma}^k_{ij} &= \Gamma^k_{ij} - \frac{e^{2\sigma}}{2} g^{km}\Big[a_j\partial_i a_m + a_i \partial_j a_m\Big] \\
&\quad + \frac{1}{2} g^{km}\partial_m(e^{2\sigma} a_i a_j)
\end{aligned}
\tag{3.24}
$$

Curvature symbols of the $p+1$ dimensional metric (e.g. the Ricci scalar $\tilde{R}$) are given in terms of p dimensional curvature data (e.g. the p dimensional Ricci Scalar R) by[17]

$$
\begin{aligned}
\tilde{R} &= R + \frac{1}{4} e^{2\sigma} f^2 - 2(\nabla\sigma)^2 - 2\nabla^2\sigma \\
\tilde{R}^{ij} &= R^{ij} - \nabla^i\sigma\nabla^j\sigma - \nabla^i\nabla^j\sigma + \frac{1}{2} e^{2\sigma} f^{im} f^j{}_m \\
K^{ij} &\equiv \tilde{R}_0{}^i{}_0{}^j (u^0)^2 = \nabla^i\sigma\nabla^j\sigma + \nabla^i\nabla^j\sigma + \frac{1}{4} e^{2\sigma} f^{im} f^j{}_m,
\end{aligned}
\tag{3.26}
$$

where $f_{ij} = \partial_i a_j - \partial_j a_i$

Let us define u^μ_K to be the unit normalized vector in the Killing direction. In components

$$
u^\mu_K = e^{-\sigma}(1, 0, \ldots, 0) \tag{3.27}
$$

Let $\mathcal{P_K}^{\mu\nu}$ denote the projector orthogonal to u^μ_K

$$
\mathcal{P_K}^{\mu\nu} = g^{\mu\nu} + u^\mu_K u^\nu_K \tag{3.28}
$$

Explicitly in matrix form

$$
(\mathcal{P}_K)_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & g_{ij} \end{pmatrix}
$$

[17] The definitions we adopt in this thesis are

$$
\begin{aligned}
R_{\mu\nu\rho}{}^\sigma &= \partial_\nu\Gamma^\sigma_{\mu\rho} - \partial_\mu\Gamma^\sigma_{\nu\rho} + \Gamma^\alpha_{\mu\rho}\Gamma^\sigma_{\alpha\nu} - \Gamma^\alpha_{\nu\rho}\Gamma^\sigma_{\alpha\mu}, \\
R_{\mu\nu} &= R_{\mu\sigma\nu}{}^\sigma.
\end{aligned}
\tag{3.25}
$$

We always use the mostly positive signature.

Let us also define the shear tensor, vorticity and expansion and acceleration of this Killing 'velocity' field by

$$\begin{aligned}
&\Theta_K = \nabla . u_K = \text{Expansion}, \quad \mathfrak{a}^\mu_K = (u_K.\nabla) u^\mu_K = \text{Acceleration} \\
&\sigma^{\mu\nu}_K = P^{\mu\alpha} P^{\nu\beta} \left(\frac{\nabla_\alpha (u_K)_\beta + \nabla_\beta (u_K)_\alpha}{2} - \frac{\Theta_K}{3} g_{\alpha_\beta} \right) = \text{Shear tensor} \\
&\omega^{\mu\nu}_K = P^{\mu\alpha} P^{\nu\beta} \left(\frac{\nabla_\alpha (u_K)_\beta - \nabla_\beta (u_K)_\alpha}{2} \right) = \text{Vorticity}
\end{aligned} \tag{3.29}$$

A straightforward computation yields

$$\begin{aligned}
&\Theta_K = 0, \quad (\mathfrak{a}_K)_\mu = (\mathcal{P}_\mathcal{K})_{\mu i} \nabla^i \sigma \\
&\sigma^{\mu\nu}_K = 0 \\
&(\omega_K)_{\mu\nu} = \frac{e^\sigma}{2} (\mathcal{P}_\mathcal{K})_{\mu i} (\mathcal{P}_\mathcal{K})_{\nu j} f^{ij}
\end{aligned} \tag{3.30}$$

3.2.2 Kaluza Klein gauge transformations

The form of the metric and gauge fields in (3.23) is preserved by p dimensional spatial diffeomorphisms together with redefinitions of time of the form

$$t' = t + \phi(\vec{x}), \quad x' = x. \tag{3.31}$$

Under coordinate changes of the form (3.31) the Kaluza Klein gauge field a_i transforms like a connection:

$$a'_i = a_i - \partial_i \phi.$$

Let us now examine the transformation of $p+1$ dimensional tensors under the coordinate transformations (3.31). It is not difficult to verify that *upper spatial* indices and *lower temporal* indices are gauge invariant. So, for instance, if $A_{\mu\nu}$ is any $p+1$ dimensional two tensor, the p dimensional scalar A_{00}, the p dimensional vector A^i_0 and the p dimensional tensor A^{ij} are all Kaluza Klein gauge invariant. On the other hand lower spatial indices and upper temporal indices transform under the Kaluza Klein gauge transformation (3.31) according to

$$V'_i = V_i - \partial_i \phi V_0, \quad (V')^0 = V^0 + \partial_i \phi V^i. \tag{3.32}$$

Note that the p dimensional oneforms

$$g_{ij} V^j = V_i - a_i V_0$$

are gauge invariant. In the sequel we will make heave use of the p dimensional oneforms

$$A_i = \mathcal{A}_i - a_i A_0 \tag{3.33}$$

This oneform is Kaluza Klein gauge invariant and transform as connections under $U(1)$ gauge transformations. This is the reason that the partition function (3.11) was written in terms of A_i rather than $\mathcal{A}_i$.

3.2.3 Stress Tensor and $U(1)$ current

The $p+1$ dimensional tensors that will be of most interest to us in this section are the stress tensor, the charge current and the entropy current. The stress tensor and charge current are defined in terms of variation of the action with respect to the higher dimensional metric and gauge field according to the formulas

$$\delta S = \int dx^{p+1}\sqrt{-g_{p+1}}\left(-\frac{1}{2}T_{\mu\nu}\delta g^{\mu\nu} + J^{\mu}\delta\mathcal{A}^{\mu}\right) \tag{3.34}$$

As we have described in the introduction, in this thesis we will be interested in the partition function $\ln Z$ of our system on the background (3.23). This partition function may be thought of as the Euclidean action of our system on the metric (3.1) with coordinate time t compactified on a circle of length $\frac{1}{T_0}$. The change of $\ln Z$ under time independent variations of the metric and gauge field is thus given by

$$\begin{aligned}\delta \ln Z &= \int dx^{p+1}\sqrt{-g_{p+1}}\left(-\frac{1}{2}T_{\mu\nu}\delta g^{\mu\nu} + J^{\mu}\delta\mathcal{A}_{\mu}\right)\\ &= \frac{1}{T_0}\int dx^{p}\sqrt{-g_{p+1}}\left(-\frac{1}{2}T_{\mu\nu}\delta g^{\mu\nu} + J^{\mu}\delta\mathcal{A}_{\mu}\right)\end{aligned} \tag{3.35}$$

It follows that

$$\begin{aligned}T_{\mu\nu} &= -2T_0\frac{\delta \ln Z}{\delta g^{\mu\nu}}\\ J^{\mu} &= T_0\frac{\delta \ln Z}{\delta \mathcal{A}_{\mu}}\end{aligned} \tag{3.36}$$

The formulae (3.36) are not written in the most useful form for the purposes of this section. As we have described in the introduction, we find it useful to regard our partition function as a functional of

$$\ln Z = W(e^{\sigma}, A_0, a_i, A_i, g^{ij}, T_0, \mu_0). \tag{3.37}$$

By application of the chain rule to the formulas (3.35) we find

$$\begin{aligned}T_{00} &= -\frac{T_0 e^{2\sigma}}{\sqrt{-g_{(p+1)}}}\frac{\delta W}{\delta\sigma}, \quad T_0^i = \frac{T_0}{\sqrt{-g_{(p+1)}}}\left(\frac{\delta W}{\delta a_i} - A_0\frac{\delta W}{\delta A_i}\right),\\ T^{ij} &= -\frac{2T_0}{\sqrt{-g_{(p+1)}}}g^{il}g^{jm}\frac{\delta W}{\delta g^{lm}}, \quad J_0 = -\frac{e^{2\sigma}T_0}{\sqrt{-g_{(p+1)}}}\frac{\delta W}{\delta A_0}, \quad J^i = \frac{T_0}{\sqrt{-g_{(p+1)}}}\frac{\delta W}{\delta A_i}.\end{aligned} \tag{3.38}$$

where, for instance, the derivative w.r.t A_0 is taken at constant σ, a_i, A_i, g^{ij}, T_0 and μ_0.

3.2.4 Dependence of the partition function on T_0 and μ_0

From the viewpoint of a Euclidean path integral, the parameter T_0 in the partition function (3.10) is the coordinate length of the time circle. Moreover, every quantum field of charge q is twisted by the phase $q\frac{\mu_0}{T_0}$ as it winds the temporal circle in Euclidean space. As usual, such a twist is gauge equivalent to a shift in the ' temporal gauge field $\mathcal{A}_0 \to \mathcal{A}_0 + \mu_0 = A_0$ [18] holding $\mathcal{A}_i$ fixed. It follows that $\ln Z$ is a function of $\mathcal{A}_0$, $\mathcal{A}_i$ and μ_0 only in the combination A_0 and A_i . The dependence of $\ln Z$ on T_0 may be deduced in a similar fashion. The Euclidean time coordinate $t' = tT_0$ has unit periodicity. When rewritten in terms of t', the metric and gauge field retain the form (3.23) with

$$e^{\sigma'} = \frac{e^{\sigma}}{T_0}, \quad a'_i = a_i T_0, \quad A'_0 = \frac{A_0}{T_0}$$

It follows from all these considerations that

$$W(e^{\sigma}, \mathcal{A}_0, a_i, \mathcal{A}_i, g^{ij}, T_0, \mu_0) = \mathcal{W}(\frac{e^{\sigma}}{T_0}, \frac{A_0}{T_0}, T_0 a_i, A_i, g^{ij}). \tag{3.39}$$

We will never use the function $\mathcal{W}$ below; all our explicit formulae will be written in terms of the function W. Nonetheless (3.39) will allow us to relate thermodynamical derivatives w.r.t. T_0 and μ_0 to functional derivatives of the partition function w.r.t. background fields.

3.2.5 Conserved charges and entropy

In this subsubsection we will compute the $U(1)$ charge and energy of our system from integrals over the appropriate charge currents, and compare the expressions so obtained with thermodynamical formulas.

The $U(1)$ charge of our system in equilibrium is given by

$$Q = \int d^p x \sqrt{-g_{p+1}} J^0 \tag{3.40}$$

where the integral is taken over the p dimensional spatial manifold. Let us now define the (conserved) energy of our system. Whenever the divergence of the stress tensor vanishes, the current $-v^{\lambda} T^{\mu}_{\lambda}$ is conserved provided v^{λ} is a killing vector field. We cannot directly apply this result to the killing vector field $v^{\lambda} = (1, \ldots, 0)$, as the stress tensor in this section is not divergence free in general (see (3.17)). However it is easily verified that the shifted current

$$J^{\mu}_E = -T^{\mu}_0 - \mathcal{A}_0 J^{\mu} \tag{3.41}$$

is conserved in equilibrium. As a consequence we define

$$E = \int d^p x \sqrt{-g_{p+1}} J^0_E = \int d^p x \sqrt{-g_{p+1}} \left(-T^0_0 - \mathcal{A}_0 J^0\right) \tag{3.42}$$

Q and E defined in (3.40) and (3.42) may be shown to be Kaluza Klein gauge invariant. For instance, the Kaluza Klein gauge variation of the RHS of (3.40) is given by

$$\begin{aligned} &\int d^p x \sqrt{-g_{p+1}} J^i \partial_i \phi \\ &= \int d^p x \sqrt{-g_{p+1}} J^{\mu} \partial_{\mu} \phi \\ &= -\int d^p x \sqrt{-g_{p+1}} \; \phi \nabla_{\mu} J^{\mu} = 0 \end{aligned} \tag{3.43}$$

[18] In this formula $\mathcal{A}_0$ is refers to the gauge field in Lorentzian space. Note that μ_0 is gauge equivalent to an imaginary shift of $\mathcal{A}_0$ in Euclidean space.

(where we have used the fact that the gauge parameter ϕ is independent of t, integrated by parts, and used the fact that J^μ is a conserved current). The gauge invariance of E follows from an almost identical argument.

We will now demonstrate that the expressions (3.40) and (3.42) agree exactly with the thermodynamical definitions of the charge and energy that follow from the partition function. In great generality, the charge of any thermodynamical system may be obtained from its partition function (3.10) via the thermodynamical formula

$$Q = T_0 \frac{\partial W}{\partial \mu_0}$$

where the partial derivative is taken at constant $T_0, \mathcal{A}_0, \mathcal{A}_i, g^{ij}, a_i, \sigma$. In the current context

$$\begin{aligned} T_0 \frac{\partial W}{\partial \mu_0} &= T_0 \int d^p x \left(\frac{\delta \mathcal{W}}{\delta A_0(x)} - a_i \frac{\delta \mathcal{W}}{\delta A_i(x)} \right) \\ &= \int d^p x \sqrt{-g_{p+1}} J^0 = Q \end{aligned} \tag{3.44}$$

where we have used (3.39), $J^0 = -e^{-2\sigma} J_0 - a_i J^i$ (this follows from the fact that $J_0 = g_{00} J^0 + g_{0i} J^i$) and explicit expressions for J_0 and J^i listed in (3.38). Let us note that, in the presence of anomaly 3.17 current J^μ is neither gauge invariant nor conserved[19].

The thermodynamical energy

$$T_0^2 \frac{\partial W}{\partial T_0} + \mu_0 Q$$

(where the partial derivative is taken at constant $\mu_0, \mathcal{A}_0, \mathcal{A}_i, g^{ij}, a_i, \sigma$). may be processed, in the current context, as

$$\begin{aligned} & T_0^2 \frac{\partial W}{\partial T_0} + \mu_0 Q = \\ & = T_0 \int \left[-\frac{\delta \mathcal{W}}{\delta \sigma} + a_i \frac{\delta \mathcal{W}}{\delta a_i} - \frac{\delta \mathcal{W}}{\delta A_0} A_0 + \frac{\delta \mathcal{W}}{\delta A_0} \mu_0 - \mu_0 a_i \frac{\delta \mathcal{W}}{\delta A_i} \right] \\ & = \int \sqrt{-g_{p+1}} \left[\left(e^{-2\sigma} T_{00} + a_i T_0^i \right) - \mathcal{A}_0 J^0 \right] \\ & = \int \sqrt{-g_{p+1}} \left[-T_0^0 - \mathcal{A}_0 J^0 \right] \\ & = E \end{aligned} \tag{3.45}$$

[19]One can construct a conserved current which is given by

$$\hat{J}^\mu = J^\mu + \frac{C}{12} \epsilon^{\mu\nu\rho\sigma} \mathcal{A}_\nu F_{\rho\sigma}.$$

where we have used (3.39), the fact that $-T^0_0 = e^{-2\sigma}T_{00} + a_i T^i_0$ and the explicit expressions for T_{00} and T^i_0 in (3.38)). In summary

$$\begin{aligned} E &= T_0^2 \frac{\partial W}{\partial T_0} + \mu_0 Q \\ Q &= T_0 \frac{\partial W}{\partial \mu_0} \end{aligned} \tag{3.46}$$

Even in the presence of anomaly one can show that the current J^μ_E in 3.41 remains conserved, where J^μ is defined as in 3.36. Thus, the thermodynamic formula 3.45 holds for anomalous system as well.

We conclude that the conserved charge and energy in our system are given, in terms of the partition function, by the usual thermodynamical formulae. It follows that the entropy of our system should also be given by the standard statistical formula

$$S = \frac{\partial (T_0 W)}{\partial T_0} \tag{3.47}$$

Later in this thesis we obtain constraints on the entropy current of our system by equating (3.47) with $\int d^p x \sqrt{-g_{p+1}} J^0_S$.

3.2.6 Consistent and Covariant Anomalies

[20] In this subsubsection we discuss the relationship between the consistent charge current (the current obtained by differentiating the partition function w.r.t. the background gauge field) and the gauge invariant charge current in arbitrary $3+1$ dimensional $U(1)$ gauge theories with a $U(1)^3$ anomaly. Readers who are familiar with the issue of consistent and covariant anomalies in quantum field theories can skip this section. The equations which will be used later are (3.53),(3.48),(3.59).

In this thesis we will have occasion to study field theories in 4 spacetime dimensions whose $U(1)$ current obeys the anomalous conservation

$$\nabla_\mu J^\mu = -\frac{C}{24} * (\mathcal{F} \wedge \mathcal{F}) \tag{3.48}$$

J^μ in (3.48) is the so called 'consistent' current defined by $J^\mu = \frac{\delta W}{\delta \mathcal{A}_\mu}$. As all gauge fields in this thesis are always time independent

$$* (\mathcal{F} \wedge \mathcal{F}) = -8 e^{-\sigma} \epsilon^{ijk} \partial_i A_0 \partial_j \mathcal{A}_k \tag{3.49}$$

(here $\epsilon^{123} = \frac{1}{\sqrt{g_3}}$) so that the anomaly equation may be rewritten as

$$\nabla_\mu J^\mu = \frac{C}{3} e^{-\sigma} \epsilon^{ijk} \partial_i A_0 \partial_j \mathcal{A}_k \tag{3.50}$$

[20] We would like to thank S. Trivedi and S. Wadia for discussions and on this topic and S. Wadia for referring us to [17].

[21] It follows that the variation of the action under a gauge transformation is given by

$$\delta S = \int \sqrt{-g_4}\frac{\delta S}{\delta \mathcal{A}_\mu}\partial_\mu \phi = \frac{C}{24}\int d^4x\sqrt{-g_4}\phi * (\mathcal{F}\wedge\mathcal{F}) = -\frac{C}{3}\int d^4x\sqrt{g_3}\phi\epsilon^{ijk}\partial_i A_0 \partial_j \mathcal{A}_k \quad (3.51)$$

We now follow the discussion of Bardeen and Zumino [17] to determine the gauge transformation property of J^μ. The principle that determines this transformation law is simply that the result of first performing an arbitrary variation of the gauge field $\mathcal{A}_\mu \to \delta\mathcal{A}_\mu$ and then a gauge transformation generated by $\delta\phi$ must be the same as that obtained upon reversing the order of these operations. The variation of the action under the first order of operations, to quadratic order in variations, is given by

$$\int \sqrt{-g_4}\delta J^\mu(\delta \mathcal{A}_\mu)$$

(where δJ^μ denotes the variation of the consistent current J^μ under the gauge transformation $\delta\phi$). The reverse order gives

$$\frac{C}{24}\int \delta\phi \frac{\delta(\mathcal{F}\wedge\mathcal{F})}{\delta\mathcal{A}_\mu}\delta\mathcal{A}_\mu = \frac{C}{24}\int \delta\phi \frac{\delta(\mathcal{F}\wedge\mathcal{F})}{\delta\mathcal{A}_\mu}\delta\mathcal{A}_\mu = \frac{C}{6}\int \sqrt{-g_4}\delta\mathcal{A}_\alpha\epsilon^{\alpha\beta\gamma\delta}\partial_\beta\phi\mathcal{F}_{\gamma\delta}$$

Comparing the two expressions it follows that under a gauge transformation

$$\delta J^\alpha = \frac{C}{6}\epsilon^{\alpha\beta\gamma\delta}\partial_\beta\phi\mathcal{F}_{\gamma\delta} \quad (3.52)$$

It follows that the shifted current

$$\tilde{J}^\mu = J^\mu - \frac{C}{6}\epsilon^{\mu\nu\gamma\delta}\mathcal{A}_\nu\mathcal{F}_{\gamma\delta} \quad (3.53)$$

is gauge invariant. $\tilde{J}^\mu$ is the current that is most familiar to most field theorists; for instance it is the current whose divergence is computed by the usual triangle diagram in standard text books. It follows from (3.53) that the divergence of $\tilde{J}^\mu$ is given by

$$\nabla_\mu \tilde{J}^\mu = -\frac{C}{8} * (\mathcal{F}\wedge\mathcal{F}) \quad (3.54)$$

Using (3.49), the anomaly equations may be rewritten as

$$\begin{aligned}\nabla_\mu J^\mu &= \frac{C}{3}e^{-\sigma}\epsilon^{ijk}\partial_i A_0\partial_j\mathcal{A}_k \\ \nabla_\mu \tilde{J}^\mu &= Ce^{-\sigma}\epsilon^{ijk}\partial_i A_0\partial_j\mathcal{A}_k\end{aligned} \quad (3.55)$$

Let us summarize. $\tilde{J}^\mu$ is the gauge invariant current that we will use in the fluid dynamical analysis in this section. It obeys the anomalous conservation equation (3.54). On the other

[21] In order to forestall all possible confusion we list our conventions. $\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu$, $*(\mathcal{F}\wedge\mathcal{F}) = \epsilon^{\mu\nu\alpha\beta}\mathcal{F}_{\mu\nu}\mathcal{F}_{\alpha\beta}$ where $\epsilon^{0123} = \frac{1}{\sqrt{-g_4}}$. The variation of the gauge field $\mathcal{A}_\mu$ under a gauge transformation is given by $\delta\mathcal{A}_\mu = \partial_\mu\phi$.

hand the non gauge invariant current J^μ is simply related to the action W (it is the functional derivative of W w.r.t. $\mathcal{A}_\mu$). These two currents are related by (3.53).

To end this subsection we will now derive the stress tensor conservation equation (3.17) in the presence of a potential anomalous background gauge field. We start by noting that the variation of W under an arbitrary variation of $g^{\mu\nu}$ and $\mathcal{A}_\mu$ is given by

$$\delta W = \int \sqrt{-g_4}\left(-\frac{1}{2}\delta g^{\mu\nu}T_{\mu\nu} + J^\mu \delta\mathcal{A}_\mu\right) \tag{3.56}$$

Let us now choose the variations of the metric and gauge fields to be of the form generated by an infinitesimal coordinate transformation, i.e.

$$\delta g^{\mu\nu} = \nabla^\mu \epsilon^\nu + \nabla^\nu \epsilon^\mu, \quad \delta\mathcal{A}_\mu = -\left(\nabla_\mu \epsilon^\nu \mathcal{A}_\nu + \epsilon^\nu \nabla_\nu \mathcal{A}_\mu\right)$$

General coordinate invariance (which we assume to be non anomalous) demands that $\delta W = 0$ in this special case. Plugging these variations into (3.56), setting the LHS to zero and integrating by parts yields

$$\int d^4x\sqrt{-g_4}\epsilon^\nu\left(\nabla^\mu T_{\mu\nu} - J^\mu\left(\nabla_\nu \mathcal{A}_\mu - \nabla_\mu \mathcal{A}_\nu\right) + \nabla_\mu J^\mu \mathcal{A}_\nu\right) \tag{3.57}$$

Using (3.48) together with the identity

$$\mathcal{A}_\sigma \epsilon^{\mu\nu\alpha\beta}\mathcal{F}_{\mu\nu}\mathcal{F}_{\alpha\beta} = -4\epsilon^{\mu\nu\alpha\beta}\mathcal{A}_\nu \mathcal{F}_{\alpha\beta}\mathcal{F}_{\sigma\mu} \tag{3.58}$$

we conclude that

$$\nabla_\mu T^\mu_\nu = \mathcal{F}_{\nu\mu}(J^\mu - \frac{C}{6}\epsilon^{\mu\sigma\alpha\beta}\mathcal{A}_\sigma \mathcal{F}_{\alpha\beta}) = \mathcal{F}_{\nu\mu}\tilde{J}^\mu \tag{3.59}$$

Thus the two equation of motion of charged fluids are given by (3.48),(3.59).

3.2.7 Perfect fluid hydrodynamics from the zero derivative partition function

It is well known (and obvious on physical grounds) that the equations of perfect fluid dynamics are completely determined by the equation of state of the fluid (i.e, for instance, the pressure as a function of temperature and velocity).

In this subsubsection we will 'rederive' the fact that the equations of hydrodynamics, at zero derivative order, are determined in terms of a single function of two variables, by comparison with the equilibrium partition function on a general background of the form (3.1). The the results we obtain in this subsection are obvious on physical grounds. However this subsubsection illustrates the basic idea behind the work out in subsequent subsections.

At zero order in the derivative expansion, the most general symmetry allowed constitutive relations of fluid dynamics take the form

$$T^{\mu\nu} = (\epsilon + \mathcal{P})u^\mu u^\nu + \mathcal{P}g^{\mu\nu}, \quad J^\mu = qu^\mu, \tag{3.60}$$

At this stage ϵ, $\mathcal{P}$ and q are arbitrary functions of any two thermodynamical fluid variables. ϵ, $\mathcal{P}$ and q (which will, of course, eventually turn out to be the fluid energy density, pressure and charge density) are as yet independent and arbitrary functions of the temperature and velocity.

We will now show that ϵ, $\mathcal{P}$ and q cannot be independent functions, but are all determined in terms of a single 'master' function of two variables. In order to do that we note that the most general p dimensional gauge and diffeomorphism invariant partition function for our system on (3.1) must take the form

$$W = \ln Z = \int d^3x \sqrt{g_3}\ \frac{e^{\sigma}}{T_0} P\left(T_0 e^{-\sigma}, e^{-\sigma} A_0\right) \tag{3.61}$$

for some function of two variables P (it is convenient to regard P as a function of $e^{-\sigma}$ and $e^{-\sigma}A_0$ rather than simply σ and A_0 as we will see below). The stress tensor and charge current that follows from the partition function (3.61) are easily evaluated using (3.38). The results are most simply written once we introduce some notation. Let

$$a = e^{-\sigma} T_0, \quad b = e^{-\sigma} A_0$$

Let P_a denote the partial derivative of P w.r.t its first argument, and P_b the partial derivative of P w.r.t. its second argument. Below, unless otherwise specified, the functions P, P_a and P_b will always evaluated at (a, b), and we will notationally omit the dependence of these functions on their arguments. In terms of this notation

$$T^{ij} = P g^{ij}, \quad T_{00} = e^{2\sigma}\left(P - aP_a - bP_b\right), \qquad J^0 = e^{-\sigma} P_b \tag{3.62}$$

$$T^i_0 = 0, \quad J^i = 0, \tag{3.63}$$

Comparing the expression for J^i in (3.60) with the same quantity in (3.62) we conclude that

$$u^\mu = e^{-\sigma}(1, 0, \dots, 0)$$

Comparing the other quantities it follows that

$$\mathcal{P} = P, \quad \epsilon = -P + aP_a + bP_b, \quad q = P_b \tag{3.64}$$

In the special case of flat space the variables a and b reduce to the temperature and chemical potential. It is clear on physical grounds that $\mathcal{P}$, ϵ and q are functions only of local values of thermodynamical variables. Consistency requires us to identify the local value of the temperature with a and the local chemical potential with b. The function P that appears in the partition is simply the pressure as a function of T and μ. Standard thermodynamical identities then allow us to identify ϵ with the energy density of the fluid and q with the the charge density of the fluid.

Let us summarize the net upshot of this analysis. Symmetries determine the form of the perfect fluid constitutive relations upto three undetermined functions ϵ, $\mathcal{P}$ and q, of the temperature and chemical potential. On the other hand the equilibrium partition function is given by a single unknown function, P, of two variables. Comparison of the partition function with the fluid hydrodynamics allow us to determine $\mathcal{P}$, ϵ and q in terms of P; as a bonus we also find expressions for the temperature and chemical potential in equilibrium on an arbitrary background of the form (3.1), (3.9).

As the results of this subsection are obvious, and very well known. However a similar procedure leads non obvious constraints for higher derivative corrections of the fluid constitutive relations, as we now explain.

3.3 3 + 1 dimensional Charged fluid dynamics at first order in the derivative expansion

In this subsection we will derive the constraints imposed on the equations of charged fluid dynamics, at first order in the derivative expansion, by comparison with the most general equilibrium partition function.

The final results of this subsection agree with the slight generalization of Son and Surowka [3] presented in [16],[4] as we now explain.

Recall that [3] argued that the hydrodynamic charge currents in field theories with a $U(1)^3$ anomaly must contain a term proportional to the vorticity and another term proportional to the background magnetic field. [3] used the principle of entropy increase to find a set of differential equations that constrain these coefficients, and determined one solution to these differential equations. It was later demonstrated that the most general solution to these differential equations is a two parameter generalization of the Son Surowka result [16],[4]. The further requirement of CPT invariance disallows one of these two additional coefficients.

As we describe in detail below, our method for determining the hydrodynamical expansion starts with the action (3.131), and then proceeds to determine the coefficients terms in the charge current proportional to vorticity and the magnetic field in a purely algebraic manner. Nowhere in this procedure do we solve a differential equation, so our procedure generates no integration constants. However the starting point of our procedure, the partition function (3.131) itself, depends on the three constants C_0, C_1 and C_2. As we demonstrate below, C_1 and C_2 map to the integration constants obtained from the differential equations of [3]. The third constant C_0 is new, and does not arise from the analysis of [3]. As we explain below, this coefficient corresponds to the freedom of adding a $U(1)$ gauge non invariant term to the entropy current, subject to the physical requirement that the contribution to entropy production from this term is gauge invariant. It turns out, however, that the requirement of CPT invariance forces C_0 to vanish. As a consequence this new term cannot arise in the hydrodynamical expansion of any system that obeys the CPT theorem.

3.3.1 Equilibrium from Hydrodynamics

In Table (4) we have listed all scalar, vector and tensor expressions that one can form out of fluid fields and background metric and gauge fields (not necessarily in equilibrium) at first order in the derivative expansion. It follows from the listing of this table that the most general

Type	Data	Evaluated at equilibrium $T=T_0e^{-\sigma},\ \mu=e^{-\sigma}A_0,\ u^\mu=u^\mu_K$
Scalars	$\nabla.u$	0
Vectors	$E_\mu=F_{\mu\nu}u^\nu$, $\mathcal{P}^{\mu\alpha}\partial_\alpha T$, $(E^\mu-T\mathcal{P}^{\mu\alpha}\partial_\alpha\nu)$	$e^{-\sigma}\partial_i A_0$ -$T_0e^{-\sigma}\partial^i\sigma$ 0
Pseudo-Vectors	$\epsilon_{\rho\lambda\alpha\beta}u^\lambda\nabla^\alpha u^\beta$ $B_\mu=\frac{1}{2}\epsilon_{\rho\lambda\alpha\beta}u^\lambda F^{\alpha\beta}$	$\frac{e^\sigma}{2}\epsilon_{ijk}f^{jk}$ $B_i=\frac{1}{2}g_{ij}\epsilon^{jkl}(F_{kl}+A_0f_{kl})$
Tensors	$\mathcal{P}_{\mu\alpha}\mathcal{P}_{\nu\beta}\left(\frac{\nabla^\alpha u^\beta+\nabla^\beta u^\alpha}{2}-\frac{\nabla.u}{3}g^{\alpha\beta}\right)$	0

Table 4. One derivative fluid data

Scalars	None
Vectors	$\partial^i A_0$, $\partial^i\sigma$
Pseudo-Vectors	$\epsilon^{ijk}\partial_j A_k$, $\epsilon^{ijk}\partial_j a_k$
Tensors	None

Table 5. One derivative background data

symmetry allowed one derivative expansion of the constitutive relations is given by

$$\begin{aligned}\pi^{\mu\nu}&=-\zeta\theta\mathcal{P}_{\mu\nu}-\eta\sigma_{\mu\nu}\\ J^\mu_{diss}&=\sigma\left(E_\mu-T\mathcal{P}^\alpha_\mu\partial_\alpha\nu\right)+\alpha_1E^\mu+\alpha_2\mathcal{P}^{\mu\alpha}\partial_\alpha T+\xi_\omega\omega^\mu+\xi_BB^\mu\end{aligned}\tag{3.65}$$

where the shear viscosity η, bulk viscosity ζ, conductivity σ and the remaining possible transport coefficients b_1, b_2, b_3 and b_4 are arbitrary functions of σ and A_0.

We are interested in the stationary equilibrium solutions of these equations. In general, every fluid variable can receive derivative corrections in terms of derivatives of the back ground data. The equilibrium temperature, chemical potential and velocity of our system to first order is given by,

$$\begin{gathered}T=T_{(0)}+\delta T=T_0e^{-\sigma}+\delta T,\qquad \mu=\mu_{(0)}+\delta\mu=e^{-\sigma}A_0+\delta\mu,\\ u^\mu=u^\mu_{(0)}+\delta u^\mu\ =e^{-\sigma}(1,0,0,0)+\delta u^\mu,\end{gathered}$$

δu^0 is determined in terms of δu^i (which we would specify in a moment) as follows. Since both u^μ and $u^\mu_{(0)}$ is normalized to (-1), we have

$$u_{(0)}{}_\mu\delta u^\mu=0\ \ \Rightarrow\ \ \delta u^0=-a_i\delta u^i.\tag{3.66}$$

Thus, the nontrivial part of velocity correction δu^μ is encoded in δu^i.

Solutions in equilibrium are determined entirely by the background fields σ, A_0, a_i, A_i and g^{ij}. In Table(5,4) we have listed all coordinate and gauge invariant one derivative scalars, vectors and tensors constructed out of this background data. As Table (5,4) lists no one derivative scalars, it follows immediately that the equilibrium temperature field $T(x) = e^{-\sigma}T_0$ and chemical potential field $\mu(x) = e^{-\sigma}A_0$ receive no corrections at first order in the derivative expansion. The velocity field in equilibrium can, however, be corrected. The most general correction to first order is proportional to the vectors and pseudo vectors listed in Table (5,4) and is given by

$$\delta u^i = -\frac{e^{-\sigma}b_1}{4}\epsilon^{ijk}f_{jk} + b_2 B^i_K + b_3\partial^i\sigma + b_4\partial^i A_0 \tag{3.67}$$

where

$$\begin{aligned} f_{jk} &= \partial_j a_k - \partial_k a_j \\ F_{jk} &= \partial_j A_k - \partial_k A_j \\ A_j &= \mathcal{A}_j - a_j A_0 \\ B^i_K &= \frac{1}{2}\epsilon^{ijk}(F_{jk} + A_0 f_{jk}) \\ \epsilon^{123} &= \frac{1}{\sqrt{g_3}} \end{aligned} \tag{3.68}$$

The fluid stress tensor evaluated on this equilibrium configuration evaluates to (3.62) corrected by an expression of first order in the derivative expansion. The one derivative corrections have two sources.

The first set of corrections arises from the corrections (3.65) evaluated on the zero order equilibrium fluid configuration (3.21). [22] Using Table(5), we then conclude that the change in the stress tensors and charge current due to the modified constitutive relations is given by

$$\begin{aligned} &\delta T_{00} = \delta T^i_0 = \delta J_0 = \delta T^{ij} = 0 \\ &\delta \tilde{J}^i = \alpha_1 e^{-\sigma}\partial^i A_0 - \alpha_2 T_0 e^{-\sigma}\partial^i\sigma + \frac{1}{2}(\xi_B A_0 - \frac{1}{2}\xi_\omega e^{\sigma})\epsilon^{ijk}f_{jk} + \frac{1}{2}\xi_B\epsilon^{ijk}F_{jk} \end{aligned} \tag{3.69}$$

The second source of corrections arises from inserting the velocity correction (3.67) into the zero order (perfect fluid) constitutive relations. At the order at which we work these velocity corrections do not modify T_{00}, J_0 or T^{ij}. A short calculation shows that the modification of the stress tensor and charge corrections due to these corrections takes the form

$$\begin{aligned} \delta T_{00} &= \delta J_0 = \delta T^{ij} = 0 \\ \delta T^i_0 &= -e^{\sigma}(\epsilon + P)\Big[\frac{1}{2}(b_2 A_0 - \frac{1}{2}b_1 e^{\sigma})\epsilon^{ijk}f_{jk} + \frac{1}{2}b_2\epsilon^{ijk}F_{jk} - b_3 T_0 e^{-\sigma}\partial^i\sigma + b_4\partial^i A_0\Big] \\ \delta \tilde{J}^i &= \Big[\frac{1}{2}\Big(qb_2 A_0 - \frac{1}{2}qb_1 e^{\sigma}\Big)\epsilon^{ijk}f_{jk} + \frac{1}{2}qb_2\epsilon^{ijk}F_{jk} \\ &\quad - qb_3 T_0 e^{-\sigma}\partial^i\sigma + qb_4\partial^i A_0\Big] \end{aligned} \tag{3.70}$$

[22] When $u^\mu \propto (1, 0 \ldots, 0)$ the Landau frame condition employed in this section sets $\pi_{00} = \pi_{0i} = J_0^{diss} = 0$. Consequently T_{00}, T_{0i} and J_0 receive no one derivative corrections of this sort.

The net change in T_0^i and J^i is given by summing (3.70) and (3.69) and is given by

$$\delta T_0^i = -e^{\sigma}(\epsilon+P)\Big[\frac{1}{2}(b_2 A_0 - \frac{1}{2}b_1 e^{\sigma})\epsilon^{ijk}f_{jk} + \frac{1}{2}b_2\epsilon^{ijk}F_{jk} - b_3 T_0 e^{-\sigma}\partial^i\sigma + b_4\partial^i A_0\Big]$$
$$\delta\tilde{J}^i = \Big[\frac{1}{2}\Big((\xi_B + qb_2)A_0 - \frac{1}{2}(\xi_\omega + qb_1)e^{\sigma}\Big)\epsilon^{ijk}f_{jk} + \frac{1}{2}(\xi_B + qb_2)\epsilon^{ijk}F_{jk}$$
$$- (qb_3 + \alpha_2)T_0 e^{-\sigma}\partial^i\sigma + (qb_4 + \alpha_1)\partial^i A_0\Big]. \tag{3.71}$$

3.3.2 Equilibrium from the Partition Function

We now turn to the study of the first correction to the perfect fluid equilibrium partition function (3.61) at first order in the derivative expansion. From the fact that Table (5,4) lists no gauge invariant scalars, one might be tempted to conclude that the equilibrium partition function can have no gauge invariant one derivative corrections. We have already explained in the introduction that this is not the case; the three (constant) parameter set of Chern Simons terms listed in the third line of (3.11) yield perfectly local and gauge invariant contributions to the partition function, even though they cannot be written as integrals of local gauge invariant expressions. In addition to these gauge invariant pieces we need a term in the action that results in its anomalous gauge transformation property (3.51). This requirement is precisely met by the term in the last line of (3.11). [23]

With the action (3.11) in hand it is straightforward to use (3.38) to obtain the stress tensor and current corresponding to this equilibrium solution. We find

$$\begin{aligned}
T_{00} &= 0, \quad T^{ij} = 0,\\
T_0^i &= e^{-\sigma}\epsilon^{ijk}\Big[(-\frac{1}{2}CA_0^2 + 2C_0A_0 + C_2)\nabla_j A_k + (2C_1 - \frac{C}{6}A_0^3 - C_2A_0)\nabla_j a_k\Big]\\
J_0 &= -e^{\sigma}\epsilon^{ijk}\left[\frac{C}{3}A_i\nabla_j A_k + \frac{C}{3}A_0A_i\nabla_j a_k\right]\\
J^i &= e^{-\sigma}\epsilon^{ijk}\left[2\left(\frac{C}{3}A_0 + C_0\right)\nabla_j A_k + \left(\frac{C}{6}A_0^2 + C_2\right)\nabla_j a_k + \frac{C}{3}A_k\nabla_j A_0\right],
\end{aligned} \tag{3.73}$$

Using (3.53) it follows that

$$\begin{aligned}
\tilde{J}_0 &= 0,\\
\tilde{J}^i &= e^{-\sigma}\epsilon^{ijk}\big[(CA_0 + 2C_0)\nabla_j A_k + (\frac{1}{2}CA_0^2 + C_2)\nabla_j a_k\big],
\end{aligned} \tag{3.74}$$

[23] In order to see this we first note that the last line of (3.11) may be rewritten as

$$\frac{C}{3}\int d^3x\sqrt{g_3}A_0\epsilon^{ijk}\mathcal{A}_i\partial_j\mathcal{A}_k$$

The variation of this term under a gauge transformation is given by

$$-\frac{C}{3}\int d^3x\sqrt{g_3}\epsilon^{ijk}\partial_i A_0\mathcal{A}_i\partial_j A_k \tag{3.72}$$

in perfect agreement with (3.51).

3.3.3 Constraints on Hydrodynamics

Equating the coefficients of independent terms in the two expressions for T^i_0 (3.71),(3.73) determines the one derivative corrections of the velocity field in equilibrium. We find.

$$\begin{aligned} b_1 &= \frac{T^3}{\epsilon+P}\big(\frac{2}{3}\nu^3 C + 4\nu^2 C_0 - 4\nu C_2 + 4C_1\big), \\ b_2 &= \frac{T^2}{\epsilon+P}\big(\frac{1}{2}\nu^2 C + 2\nu C_0 - C_2\big), \\ b_3 &= b_4 = 0. \end{aligned} \tag{3.75}$$

where $\nu = \frac{\mu}{T} = \frac{A_0}{T_0}$.

Equating coefficients of independent terms in J^i in equations 3.71 and 3.74 and using (3.75) gives

$$\begin{aligned} \xi_\omega &= C\nu^2 T^2\big(1 - \frac{2q}{3(\epsilon+P)}\nu T\big) + T^2\big[(4\nu C_0 - 2C_2) - \frac{qT}{\epsilon+P}(4\nu^2 C_0 - 4\nu C_2 + 4C_1)\big], \\ \xi_B &= C\nu T\big(1 - \frac{q}{2(\epsilon+P)}\nu T\big) + T\big(2C_0 - \frac{qT}{\epsilon+P}(2\nu C_0 - C_2)\big), \\ \alpha_1 &= \alpha_2 = 0 \end{aligned} \tag{3.76}$$

Let us summarize. We have found that the hydrodynamical charge current and stress tensor are given by

$$\begin{aligned} \pi^{\mu\nu} &= -\zeta\theta\mathcal{P}_{\mu\nu} - \eta\sigma_{\mu\nu} \\ J^\mu_{diss} &= \sigma\left(E_\mu - T\mathcal{P}^\alpha_\mu\partial_\alpha\nu\right) + \xi_\omega\omega^\mu + \xi_B B^\mu \end{aligned} \tag{3.77}$$

In (3.77) the viscosities ζ and η together with the conductivity σ are all dissipative parameters. These parameters multiply expressions that vanish in equilibrium and are completely unconstrained by the analysis of this subsection. On the other hand ζ_ω and ζ_B - together with α_1 and α_2 in (3.65) - are non dissipative parameters. They multiply expressions that do not vanish in equilibrium. The analysis of this subsection has demonstrated that α_1 and α_2 vanish and that ζ_ω and ζ_B are given by (3.77). The expressions (3.77) agree exactly with the results of Son and Surowka - based on the requirement of positivity of the entropy current - upon setting $C_0 = C_1 = C_2 = 0$. Upon setting $C_0 = 0$ they agree with the generalized results of [16] (see also [4],[8]). We will return to the role of the additional parameter C_0 later in this section.

3.3.4 The Entropy Current

The entropy of our system is given by

$$\begin{aligned} S &= \frac{\partial}{\partial T_0}(T_0 \log Z) \\ &= \int d^3x\sqrt{g_3}\epsilon^{ijk}\big[C_0 A_i\nabla_j A_k + 3C_1 T_0^2 a_i\nabla_j a_k + 2C_2 T_0 A_i\nabla_j a_k\big]. \end{aligned} \tag{3.78}$$

In this subsubsection we determine the constraints on the entropy current J_S^μ of our system from the requirement that (3.78) agree with the local integral

$$S = \int d^3x \sqrt{-g_4} J_S^0 \tag{3.79}$$

Notice that the first term in (3.78) (the term proportional to C_0) cannot be written as the integral of a $U(1)$ gauge invariant entropy density. It follows immediately that (3.79) and (3.78) cannot agree unless J_S^μ has a non gauge invariant term proportional to C_0. Is it permissible for the entropy current of a system to be non gauge invariant (and therefore ambiguous)? Entropy in equilibrium is physical and should be well defined. Moreover, if we start a system in equilibrium, kick the system (by turning on time dependent background metric and gauge fields) and let it settle back into equilibrium, then the difference between the entropy of the initial and final state, is also unambiguous. It follows that the entropy production (i.e. divergence of the entropy current) as well as (3.79) are necessarily gauge invariant. However these requirements leaves room for the entropy current itself to be gauge dependent.

Over the next few paragraphs we find it useful to dualize the entropy current to a 3 form. The addition of an exact form to the entropy three form contributes neither to entropy production nor to the total integrated value of the entropy in equilibrium. For this reason we regard any two entropy 3-forms that differ by an exact three form as equivalent. With this understanding, the unique non gauge invariant entropy 3 form whose exterior derivative (the Hodge dual of entropy production) is gauge invariant is given by

$$\mathcal{A} \wedge d\mathcal{A}$$

The requirement that the exterior derivative of this 3 form to be gauge invariant forces its coefficient to be constant.[24]

The most general physically allowed form for the entropy current, at one derivative order, may then be read off from Table 5

$$\begin{aligned} J_S^\mu = {} & s u^\mu - \nu J^\mu_{diss} + D_\theta \Theta u^\mu + D_c \left(E^\mu - T\mathcal{P}^{\mu\alpha}\partial_\alpha \nu\right) + D_E E^\mu + D_a \mathfrak{a}^\mu \\ & + D_\omega \omega^\mu + D_B B^\mu + h\epsilon^{\mu\nu\lambda\sigma}\mathcal{A}_\nu \partial_\lambda \mathcal{A}_\sigma \end{aligned} \tag{3.81}$$

where h is a constant

[24] Naively, another candidate for a non gauge invariant contribution to the entropy three form is given by

$$\mathcal{A} \wedge d\left(h(T,\mu)U\right) \tag{3.80}$$

where $U = u_\mu dx^\mu$ and h is an arbitrary function of temperature and chemical potential. But this term can be rewritten as follows.

$$\mathcal{A} \wedge d\left(h(T,\mu)\; U\right) = d\left(h(T,\mu)\; U \wedge \mathcal{A}\right) - h(T,\mu)\; U \wedge d\mathcal{A}$$

It follows that this addition is actually equivalent to a gauge invariant addition to the entropy 3 form.

How is the entropy current (3.81) constrained by the requirement that its integral agrees with (3.78)? The one derivative entropy, as computed from the formula $\int d^3x\sqrt{g_4}J_S^0$ has two sources. First, the perfect fluid entropy current su^0 has a first derivative piece that comes from the one derivative correction of the equilibrium fluid velocity (see above). Second, from the one derivative correction to the entropy current (evaluated on the leading order equilibrium fluid configuration). The terms with coefficients D_θ and D_c vanish on the leading order equilibrium fluid configuration. Therefore these two coefficients can not be determined by comparing with the total entropy as derived from action. All the other correction terms computed from this procedure are parity odd, except those multiplying D_a and D_E. It is possible to verify that the integrals of the terms multiplying D_a and D_E are nonvanishing and linearly independent. As all first derivative entropy corrections in (3.78) are parity odd, it follows immediately that

$$D_a = D_E = 0.$$

Therefore the zero component of the entropy current at first derivative order is given by the following expression.

$$J_S^0|_{correction} = s\delta u^0 + \left(-\nu\xi_B + D_B\right)B^0 + \left(-\nu\xi_\omega + D_\omega\right)\omega^0 + h\epsilon^{0\nu\lambda\sigma}\mathcal{A}_\nu\partial_\lambda\mathcal{A}_\sigma \tag{3.82}$$

Using

$$\begin{aligned}
&\nu = \frac{A_0}{T_0}\\
&B^0 = -\epsilon^{ijk}a_i\partial_j\left(A_k + T_0\nu a_k\right)\\
&\omega^0 = \frac{e^\sigma}{2}\epsilon^{ijk}a_i\partial_j a_k\\
&\epsilon^{0\nu\lambda\sigma}\mathcal{A}_\nu\partial_\lambda\mathcal{A}_\sigma = e^{-\sigma}\epsilon^{ijk}\left[A_i\partial_j A_k + 2T_0\nu a_i\partial_j A_k + T_0^2\nu^2 a_i\partial_j a_k + \partial_i\left(T_0\nu a_j A_k\right)\right]\\
&\delta u^0 = -a_i\delta u^i = b_1\left[\frac{e^\sigma}{2}\epsilon^{ijk}a_i\partial_j a_k\right] - b_2\left[\epsilon^{ijk}a_i\partial_j\left(A_k + T_0\nu a_k\right)\right]
\end{aligned} \tag{3.83}$$

and the expressions for ξ_B, ξ_ω, b_1 and b_2 as computed in the previous subsection (see (3.75) and (3.76)), we find

$$\begin{aligned}
&\int d^3x\sqrt{-g_4}J_s^0|_{correction}\\
= &\int d^3x\sqrt{g_3}\epsilon^{ijk}\Bigg[T_0^2\left(3C_1 + h\nu^2 + \frac{d_\omega}{2} - \nu d_B\right)a_i\partial_j a_k\\
&\qquad + T_0(2C_2 + 2h\nu - d_B)a_i\partial_j A_k + hA_i\partial_j A_k\Bigg]
\end{aligned} \tag{3.84}$$

where

$$d_B = \frac{D_B}{T} - \left(\frac{C\nu^2}{2} - C_2\right), \quad d_\omega = \frac{D_\omega}{T^2} - \left(\frac{C\nu^3}{3} - 2C_2\nu + 2C_1\right)$$

Comparing this expression with (3.78) we find

$$h = C_0, \quad d_B = 2C_0\nu, \quad d_\omega = 2C_0\nu^2 \tag{3.85}$$

This result agrees precisely with that of Son and Surowka as generalized in [5]

3.3.5 Entropy current with non-negative divergence

In the previous subsubsection we have determined the entropy current by comparing with the total entropy derived from the equilibrium partition function and we have allowed for terms which are not gauge invariant provided their divergence is gauge-invariant.

Now we shall try to constrain the most general entropy current (as given in (3.81)) by demanding that its divergence is always non-negative for every possible fluid flow, consistent with the equations of motion. The analysis will be a small modification of [3] because of the new gauge non-invariant term with constant coefficient C_0 added. The steps are as follows.

- First we have to compute the divergence of the current given in (3.81). The new term in the entropy current contributes to the divergence in the following way.

$$\nabla_\mu \left[C_0 \epsilon^{\mu\nu\alpha\beta} \mathcal{A}_\nu \nabla_\alpha \mathcal{A}_\beta \right] = \frac{C_0}{4} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} = -2C_0 E_\mu B^\mu$$

  The full divergence of the entropy current is given by

$$\begin{aligned}
\nabla_\mu J_S^\mu = {} & \eta\sigma_{\mu\nu}\sigma^{\mu\nu} + \zeta\Theta^2 + \sigma T Q_\mu Q^\mu - \xi_E(Q_\mu E^\mu) - \xi_a(Q_\mu \mathfrak{a}^\mu) \\
& + \Theta(u.\nabla)D_\theta + (\mathfrak{a}.\nabla)D_a + (Q.\nabla)D_c + (E.\nabla)D_E \\
& + D_\theta(u.\nabla)\Theta + D_a(\nabla.\mathfrak{a}) + D_c(\nabla.Q) + D_E(\nabla.E) \\
& + \left[\frac{\partial D_B}{\partial T} - \frac{D_B}{T} \right] (B^\mu \partial_\mu T) + \left[\frac{\partial D_\omega}{\partial T} - 2\frac{D_\omega}{T} \right] (\omega^\mu \partial_\mu T) \\
& + \left[\frac{\partial D_B}{\partial \nu} - CT\nu - 2C_0 T \right] (B^\mu \partial_\mu \nu) + \left[\frac{\partial D_\omega}{\partial \nu} - 2D_B \right] (\omega^\mu \partial_\mu \nu) \\
& + \left[-\xi_\omega - \frac{2qT}{\epsilon + P} D_\omega + 2D_B T \right] (\omega_\mu Q^\mu) \\
& + \left[-\xi_B - \frac{qT}{\epsilon + P} D_B + CT\nu + 2C_0 T \right] (B_\mu Q^\mu)
\end{aligned} \tag{3.86}$$

  where

$$Q_\mu = \partial_\mu \nu - \frac{E_\mu}{T}$$

  and

$$J^\mu_{diss} = -\sigma T Q^\mu + \xi_E E^\mu + \xi_a \mathfrak{a}^\mu + \xi_\omega \omega^\mu + \xi_B B^\mu, \quad \text{and} \quad \pi^{\mu\nu} = -\eta\sigma^{\mu\nu} - \zeta\Theta P^{\mu\nu}$$

- As explained in [4], the divergence computed in (3.86) can be non-negative if D_θ, D_E, D_c, ξ_E and ξ_a are set to zero in the parity even sector.

Field	C	P	T	CPT
σ	$+$	$+$	$+$	$+$
a_i	$+$	$-$	$-$	$+$
g_{ij}	$+$	$+$	$+$	$+$
A_0	$-$	$+$	$+$	$-$
A_i	$-$	$-$	$-$	$-$

Table 6. Action of CPT

- Since there is no B^2 or ω^2 term present in (3.86), for positivity, in the parity odd sector we need all the terms that are linear in B_μ and ω_μ to vanish. This condition imposes the following 6 constraints.

$$\begin{aligned}
&\frac{\partial D_B}{\partial T} - \frac{D_B}{T} = 0, \qquad \frac{\partial D_\omega}{\partial T} - \frac{2D_\omega}{T} = 0 \\
&\frac{\partial D_B}{\partial \nu} - CT\nu - 2C_0 T = 0, \qquad \frac{\partial D_\omega}{\partial \nu} - 2D_B = 0 \\
&-\xi_\omega - \frac{2qT}{\epsilon + P} D_\omega + 2D_B T = 0 \\
&-\xi_B - \frac{qT}{\epsilon + P} D_B + CT\nu + 2C_0 T = 0
\end{aligned} \tag{3.87}$$

- We can determine ξ_ω, ξ_B, D_B and D_ω by solving these equations. The solution is identical to the solution determined from the partition function (as given in (3.76) and (3.85)).

3.3.6 CPT Invariance

In this subsubsection we explore the constraints imposed on the partition function (3.11) by the requirement of 4 dimensional CPT invariance. In Table 3.3.6 we list the action of CPT on various fields appearing in the partition function. Using this table one can easily see that the terms with coefficient C_1 and C_0 change sign under CPT transformation while the terms with coefficient C_2 and C remains invariant. Thus the requirement of CPT invariance of the partition function forces $C_1 = 0$ and $C_0 = 0$. Further it also tell us that the function P appearing in the perfect fluid partition function, W^0, must be an even function of A_0 (i.e. that equilibrium does not distinguish between positive and negative charges).

3.4 Parity odd first order charged fluid dynamics in 2+1 dimensions

In this subsection we will derive the constraints imposed on the equations of 2+1 dimensional charged fluid dynamics, at first order in the derivative expansion, by comparison with the most general equilibrium partition function. The parity even constraints are identical to the ones found in 3+1 dimensions (which has been extensively discussed in §3.3). Therefore in this subsection we shall primarily focus on the parity odd constraints which are qualitatively much different from their 3+1 dimensional counterpart. These constraints have been obtained using a local form of the second law of thermodynamics in [19], which we shall reproduce starting from the most general equilibrium partition function.

3.4.1 Equilibrium from Hydrodynamics

Partially borrowing some notations from equation (1.2) in [19], the the most general symmetry allowed one derivative expansion of the constitutive relations is given by [25]

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + \left(P - \zeta \nabla_\alpha u^\alpha - \tilde{\chi}_B B - \tilde{\chi}_\Omega \Omega\right) P^{\mu\nu} - \eta \sigma^{\mu\nu} - \tilde{\eta}\tilde{\sigma}^{\mu\nu}\,, \tag{3.88a}$$

$$J^\mu = \rho u^\mu + \sigma V^\mu + \tilde{\sigma}\tilde{V}^\mu + \tilde{\chi}_E \tilde{E}^\mu + \tilde{\chi}_T \tilde{T}^\mu\,. \tag{3.88b}$$

The various quantities appearing in the constitutive relations (3.88) are defined as

$$\Omega = -\epsilon^{\mu\nu\rho} u_\mu \nabla_\nu u_\rho, \qquad B = -\frac{1}{2}\epsilon^{\mu\nu\rho} u_\mu F_{\nu\rho}, \tag{3.89a}$$

$$E^\mu = F^{\mu\nu} u_\nu, \qquad V^\mu = E^\mu - T P^{\mu\nu} \nabla_\nu \frac{\mu}{T}, \tag{3.89b}$$

$$P^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu}, \qquad \sigma^{\mu\nu} = P^{\mu\alpha} P^{\nu\beta}\left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - g_{\alpha\beta} \nabla_\lambda u^\lambda\right)\,, \tag{3.89c}$$

and

$$\tilde{E}^\mu = \epsilon^{\mu\nu\rho} u_\nu E_\rho\,, \qquad \tilde{V}^\mu = \epsilon^{\mu\nu\rho} u_\nu V_\rho\,, \tag{3.89d}$$

$$\tilde{\sigma}^{\mu\nu} = \frac{1}{2}\left(\epsilon^{\mu\alpha\rho} u_\alpha {\sigma_\rho}^\nu + \epsilon^{\nu\alpha\rho} u_\alpha {\sigma_\rho}^\mu\right)\,, \qquad \tilde{T}^\mu = \epsilon^{\mu\nu\rho} u_\nu \nabla_\rho T. \tag{3.89e}$$

The thermodynamic quantities P, ϵ and ρ are the values of the pressure, energy density and charge density respectively in equilibrium. The transport coefficients $\tilde{\chi}_B$, $\tilde{\chi}_\Omega$, $\tilde{\chi}_E$ and $\tilde{\chi}_T$ are arbitrary functions of σ and A_0. The only non-zero quantities in equilibrium are B, $\omega, \tilde{E}^\mu$ and $\tilde{T}^\mu$. The rest of the first order quantities appearing on the RHS of (3.88) vanish on our equilibrium configuration. In Table 7 we list all the parity odd diffeomorphism invariant background field data. In Table 8 we list the first order quantities occurring in the constitutive relations that are non-zero in equilibrium and express them in terms of the background metric and gauge fields[26].

We are interested in the stationary equilibrium solutions of the fluid equations arising from constitutive relations (3.88). Solutions in equilibrium are determined entirely by the background fields σ, A_0, a_i, A_i and g^{ij}. Just like in 3+1 dimensions the zeroth order solution of the fluid fields are given by

$$u_0^\mu = \{e^{-\sigma}, 0, 0\}; \quad T_{(0)} = T_0 e^{-\sigma}; \quad \mu_{(0)} = e^{-\sigma} A_0. \tag{3.90}$$

[25] Note that in this constitutive relation the parity even constraint, namely the Einstein relation, have already been taken into account.

[26] In the following, we shall use $\epsilon^{12} = \frac{1}{\sqrt{g_2}}$

Pseudo-scalars	$\epsilon^{ij}\partial_i A_j$, $\epsilon^{ij}\partial_i a_j$
Pseudo-vectors	$\epsilon^{ij}\partial_i A_0$, $\epsilon^{ij}\partial^i \sigma$
Pseudo-tensors	None

Table 7. One derivative parity odd diffeomorphism and gauge invariant background data. Here ϵ^{ij} is defined so that $\epsilon^{12} = \frac{1}{\sqrt{g_2}}$.

Type	Data	Evaluated at equilibrium
Pseudo-Scalars	B	$\epsilon^{ij}\partial_i A_j + A_0 \epsilon^{ij}\partial_i a_j$
	Ω	$-e^{\sigma}\epsilon^{ij}\partial_i a_j$
Pseudo-Vectors	$\tilde{E}^{\mu}$ $\tilde{T}^{\mu}$	$\tilde{E}_0 = 0,\ \tilde{E}^i = e^{-\sigma}\epsilon^{ij}\partial_j A_0$ $\tilde{T}_0 = 0,\ \tilde{T}^i = -e^{-\sigma}\epsilon^{ij}\partial_j \sigma$
Pseudo-Tensors	none	

Table 8. One derivative fluid data which are non-zero in equilibrium.

The unit normalized vector in the killing direction is,

$$u_K^{\mu} = e^{-\sigma}(1, 0, 0)$$

In Table(7,8) we have listed all coordinate and gauge invariant one derivative parity odd scalars, vectors and tensors constructed out of this background data. Since there are 2 pseudo-scalars and 2 pseudo-vectors we can have the following most general parity odd corrections to the fluid fields at first order

$$\begin{aligned} u_{\mu} &= u^{\mu}_{(0)} + \xi_E\ \tilde{E}^{\mu}_K + \xi_T\ \tilde{T}^{\mu}_K, \\ T &= T_{(0)} + \tau_B\ B_K + \tau_{\Omega}\ \Omega_K, \\ \mu &= \mu_{(0)} + m_B\ B_K + m_{\Omega}\ \Omega_K, \end{aligned} \tag{3.91}$$

where $\xi_E, \xi_T, \tau_B, \tau_\Omega, m_B$ and m_Ω are taken to be arbitrary functions of σ and A_0 to be determined by matching with the equilibrium partitions function in §3.4.3. $E^{\mu}_K, T^{\mu}_K, B_K, \omega_K$ are the vectors and scalars, defined in equations3.89a and 3.89c, velocity u replaced by u_K.

Just like in 3+1 dimensions the fluid stress tensor evaluated on this equilibrium configuration evaluates to (3.62) corrected by an expression of first order in the derivative expansion. The one derivative corrections again have two sources.

The first set of corrections arises from the corrections (3.88) evaluated on the zero order equilibrium fluid configuration (3.90). [27] The second source of corrections arises from inserting the fluid field corrections in (3.91) into the zero order (perfect fluid) constitutive relations. The net change in the stress tensor and the charge current at first order is obtained

[27] When $u^{\mu} \propto (1, 0 \ldots, 0)$ the Landau frame condition employed in this section sets $\pi_{00} = \pi_{0i} = J_0^{diss} = 0$. Consequently T_{00}, T_{0i} and J_0 receive no one derivative corrections of this sort.

by summing these two contributions and is given by

$$
\begin{aligned}
\delta T^{\mu\nu} &= \left(\frac{\partial P}{\partial T}\tau_B + \frac{\partial P}{\partial \mu}m_B - \tilde{\chi}_B\right) B_K P^{\mu\nu}_{(0)} + \left(\frac{\partial P}{\partial T}\tau_\Omega + \frac{\partial P}{\partial \mu}m_\Omega - \tilde{\chi}_\Omega\right) \Omega_K P^{\mu\nu}_{(0)} \\
&+ \left(\frac{\partial \epsilon}{\partial T}\tau_B + \frac{\partial \epsilon}{\partial \mu}m_B\right) B_K u^\mu_{(0)} u^\nu_{(0)} + \left(\frac{\partial \epsilon}{\partial T}\tau_\Omega + \frac{\partial \epsilon}{\partial \mu}m_\Omega\right) \Omega_K u^\mu_{(0)} u^\nu_{(0)} \\
&+ (\epsilon + P)\xi_E (u^\mu_{(0)} \tilde{E}^\nu_K + u^\nu_{(0)} \tilde{E}^\mu_K) + (\epsilon + P)\xi_T (u^\mu_{(0)} \tilde{T}^\nu_K + u^\nu_{(0)} \tilde{T}^\mu_K). \\
\delta J^\mu &= \left(\frac{\partial \rho}{\partial T}\tau_B + \frac{\partial \rho}{\partial \mu}m_B\right) B_K u^\mu_{(0)} + \left(\frac{\partial \rho}{\partial T}\tau_\Omega + \frac{\partial \rho}{\partial \mu}m_\Omega\right) \Omega_K u^\mu_{(0)} \\
&+ (\tilde{\chi}_E + \rho\xi_E)\tilde{E}^\mu_K + (\tilde{\chi}_T + \rho\xi_T)\tilde{T}^\mu_K
\end{aligned}
\tag{3.92}
$$

For future reference it will be convenient to to write down some of the components of the stress tensor and current in (3.92) purely in terms of the background fields using the expressions listed in the third column of Table 8.

$$
\begin{aligned}
\delta T^{ij} &= \left(\frac{\partial P}{\partial T}\tau_B + \frac{\partial P}{\partial \mu}m_B - \tilde{\chi}_B\right) B_K g^{ij} + \left(\frac{\partial P}{\partial T}\tau_\Omega + \frac{\partial P}{\partial \mu}m_\Omega - \tilde{\chi}_\Omega\right) \Omega_K g^{ij} \\
\delta T_{00} &= e^{2\sigma} \left(\left(\frac{\partial \epsilon}{\partial T}\tau_B + \frac{\partial \epsilon}{\partial \mu}m_B\right) (\epsilon^{ij}\partial_i A_j + A_0 \epsilon^{ij}\partial_i a_j) - e^\sigma \left(\frac{\partial \epsilon}{\partial T}\tau_\Omega + \frac{\partial \epsilon}{\partial \mu}m_\Omega\right) \epsilon^{ij}\partial_i a_j \right), \\
\delta T^i_0 &= \left(-(\epsilon + P)\xi_E \epsilon^{ij}\partial_j A_0 + (\epsilon + P)\xi_T \epsilon^{ij}\partial_j \sigma\right) \\
\delta J_0 &= e^\sigma \left(- \left(\frac{\partial \rho}{\partial T}\tau_B + \frac{\partial \rho}{\partial \mu}m_B\right) (\epsilon^{ij}\partial_i A_j + A_0 \epsilon^{ij}\partial_i a_j) + e^\sigma \left(\frac{\partial \rho}{\partial T}\tau_\Omega + \frac{\partial \rho}{\partial \mu}m_\Omega\right) \epsilon^{ij}\partial_i a_j \right), \\
\delta J^i &= e^{-\sigma} \left((\tilde{\chi}_E + \rho\xi_E)\, \epsilon^{ij}\partial_j A_0 - (\tilde{\chi}_T + \rho\xi_T)\epsilon^{ij}\partial_j \sigma \right).
\end{aligned}
\tag{3.93}
$$

3.4.2 Equilibrium from the Partition Function

We now turn to the study of the first correction to the perfect fluid equilibrium partition function (3.61) at first order in the derivative expansion. From the fact that Table (7,8) lists two gauge invariant Hence the most general parity odd equilibrium partition function is given by

$$
\mathcal{W} = \frac{1}{2}\int \left(\alpha(\sigma, A_0)dA + T_0 \beta(\sigma, A_0)da\right), \tag{3.94}
$$

where α and β are two arbitrary functions in terms of which all the 4 transport coefficients and the 6 first order corrections to the velocity, temperature and chemical potential are to be determined.

With the action (3.94) in hand it is straightforward to use (3.38) to obtain the stress tensor and current corresponding to this equilibrium solution. We find

$$
\begin{aligned}
&T^{ij} = 0, \\
&T_{00} = -T_0 e^{\sigma} \left(\frac{\partial \alpha}{\partial \sigma} \epsilon^{ij} \partial_i A_j + T_0 \frac{\partial \beta}{\partial \sigma} \epsilon^{ij} \partial_i a_j \right), \\
&T_0^i = T_0 e^{-\sigma} \left(\left(T_0 \frac{\partial \beta}{\partial \sigma} - A_0 \frac{\partial \alpha}{\partial \sigma} \right) \epsilon^{ij} \partial_j \sigma + \left(T_0 \frac{\partial \beta}{\partial A_0} - A_0 \frac{\partial \alpha}{\partial A_0} \right) \epsilon^{ij} \partial_j A_0 \right), \\
&J_0 = -T_0 e^{\sigma} \left(\frac{\partial \alpha}{\partial A_0} \epsilon^{ij} \partial_i A_j + T_0 \frac{\partial \beta}{\partial A_0} \epsilon^{ij} \partial_i a_j \right), \\
&J^i = T_0 e^{-\sigma} \left(\frac{\partial \alpha}{\partial \sigma} \epsilon^{ij} \partial_j \sigma + \frac{\partial \alpha}{\partial A_0} \epsilon^{ij} \partial_j A_0 \right).
\end{aligned}
\tag{3.95}
$$

3.4.3 Constraints on Hydrodynamics

In this subsubsection we shall equate the coefficients of independent terms in (3.92) (or (3.93)) with those in (3.95), to determine the first order transport coefficients and fluid corrections in terms of the two arbitrary functions in the action (3.94).

The fact that T^{ij} as evaluated from the action (3.94) vanishes immediately implies from (3.93)

$$
\begin{aligned}
\tilde{\chi}_B &= \frac{\partial P}{\partial T} \tau_B + \frac{\partial P}{\partial \mu} m_B, \\
\tilde{\chi}_\Omega &= \frac{\partial P}{\partial T} \tau_\Omega + \frac{\partial P}{\partial \mu} m_\Omega,
\end{aligned}
\tag{3.96}
$$

Comparing T_{00} from (3.93) and (3.95) we have

$$
\begin{aligned}
&\frac{\partial \epsilon}{\partial T} \tau_B + \frac{\partial \epsilon}{\partial \mu} m_B = -T_0 e^{-\sigma} \frac{\partial \alpha}{\partial \sigma}, \\
&\frac{\partial \epsilon}{\partial T} \tau_\Omega + \frac{\partial \epsilon}{\partial \mu} m_\Omega = T_0 e^{-2\sigma} \left(T_0 \frac{\partial \beta}{\partial \sigma} - A_0 \frac{\partial \alpha}{\partial \sigma} \right),
\end{aligned}
\tag{3.97}
$$

Comparing T_0^i from (3.93) and (3.95) we have

$$
\begin{aligned}
\xi_E &= -\frac{T_0 e^{-\sigma}}{(\epsilon + P)} \left(T_0 \frac{\partial \beta}{\partial A_0} - A_0 \frac{\partial \alpha}{\partial A_0} \right), \\
\xi_T &= \frac{T_0 e^{-\sigma}}{(\epsilon + P)} \left(T_0 \frac{\partial \beta}{\partial \sigma} - A_0 \frac{\partial \alpha}{\partial \sigma} \right).
\end{aligned}
\tag{3.98}
$$

Comparing J_0 from (3.93) and (3.95) we have

$$
\begin{aligned}
&\frac{\partial \rho}{\partial T} \tau_B + \frac{\partial \rho}{\partial \mu} m_B = T_0 \frac{\partial \alpha}{\partial A_0}, \\
&\frac{\partial \rho}{\partial T} \tau_\Omega + \frac{\partial \rho}{\partial \mu} m_\Omega = -T_0 e^{-\sigma} \left(T_0 \frac{\partial \beta}{\partial A_0} - A_0 \frac{\partial \alpha}{\partial A_0} \right),
\end{aligned}
\tag{3.99}
$$

Finally, comparing J^i from (3.93) and (3.95) we have

$$\begin{aligned}\tilde{\chi}_E + \rho\xi_E &= T_0 \frac{\partial \alpha}{\partial A_0},\\ \tilde{\chi}_T + \rho\xi_T &= -\frac{\partial \alpha}{\partial \sigma}.\end{aligned} \tag{3.100}$$

In order to compare the constraints obtained in this section with that in [19] we find the following thermodynamical identities useful

$$\begin{aligned}\frac{\partial P}{\partial \epsilon} &= \left(\frac{\partial P}{\partial T}\frac{\partial \rho}{\partial \mu} - \frac{\partial P}{\partial \mu}\frac{\partial \rho}{\partial T}\right)\Big/\left(\frac{\partial \rho}{\partial \mu}\frac{\partial \epsilon}{\partial T} - \frac{\partial \rho}{\partial T}\frac{\partial \epsilon}{\partial \mu}\right),\\ \frac{\partial P}{\partial \rho} &= \left(-\frac{\partial P}{\partial T}\frac{\partial \epsilon}{\partial \mu} + \frac{\partial P}{\partial \mu}\frac{\partial \epsilon}{\partial T}\right)\Big/\left(\frac{\partial \rho}{\partial \mu}\frac{\partial \epsilon}{\partial T} - \frac{\partial \rho}{\partial T}\frac{\partial \epsilon}{\partial \mu}\right),\end{aligned} \tag{3.101}$$

Now solving for τ_B, τ_Ω, m_B and m_Ω from (3.97) and (3.99), plugging the answer in to (3.96) and using the thermodynamical identities (3.101), we have

$$\begin{aligned}\tilde{\chi}_B &= \frac{\partial P}{\partial \epsilon}\left(-T_0 e^{-\sigma}\frac{\partial \alpha}{\partial \sigma}\right) + \frac{\partial P}{\partial \rho}\left(T_0\frac{\partial \alpha}{\partial A_0}\right),\\ \tilde{\chi}_\Omega &= \frac{\partial P}{\partial \epsilon}\left(T_0 e^{-2\sigma}\left(T_0\frac{\partial \beta}{\partial \sigma} - A_0\frac{\partial \alpha}{\partial \sigma}\right)\right) + \frac{\partial P}{\partial \rho}\left(-T_0 e^{-\sigma}\left(T_0\frac{\partial \beta}{\partial A_0} - A_0\frac{\partial \alpha}{\partial A_0}\right)\right),\end{aligned} \tag{3.102}$$

Finally plugging in the values of ξ_E and ξ_T from (3.98) into (3.100) we have

$$\begin{aligned}\tilde{\chi}_E &= \left(T_0\frac{\partial \alpha}{\partial A_0}\right) - \frac{\rho}{\epsilon + P}\left(-T_0 e^{-\sigma}\left(T_0\frac{\partial \beta}{\partial A_0} - A_0\frac{\partial \alpha}{\partial A_0}\right)\right)\\ T\tilde{\chi}_T &= \left(-T_0 e^{-\sigma}\frac{\partial \alpha}{\partial \sigma}\right) - \frac{\rho}{\epsilon + P}\left(T_0 e^{-2\sigma}\left(T_0\frac{\partial \beta}{\partial \sigma} - A_0\frac{\partial \alpha}{\partial \sigma}\right)\right)\end{aligned} \tag{3.103}$$

Thus through (3.102) and (3.103) we are able to express the 4 transport coefficients in terms two arbitrary functions in the action. The two dimensional manifold of allowed transport coefficients is identical to that in equation (1.8) in [19][28]. In particular it easy to eliminate α and β from (3.102) and (3.103) so as to obtain the following relation between the transport coefficients

$$\tilde{\chi}_B - \frac{\rho}{\epsilon + P}\ \tilde{\chi}_\Omega = \frac{\partial P}{\partial \rho}\tilde{\chi}_E + \frac{\partial P}{\partial \epsilon}\ T\ \tilde{\chi}_T. \tag{3.104}$$

Note that this relation is identical to equation (4.29) in [19].

3.4.4 The Entropy Current

In this system $\ln Z$ is simply given by the action

$$\ln Z = \frac{1}{2}\int \left(\alpha(\sigma, A_0)dA + T_0\beta(\sigma, A_0)da\right) \tag{3.105}$$

[28]Note that the additional function $f_\Omega(T)$ may be reabsorbed into a redefinition of $M_\Omega(\mu, T)$ in equation (1.8) in [19].

The entropy that follows from this partition function is

$$\begin{aligned} S &= \frac{\partial}{\partial T_0}(T_0 \ln Z) \\ &= \frac{1}{2}\int \left(\alpha - \frac{\partial \alpha}{\partial \sigma} - A_0 \frac{\partial \alpha}{\partial A_0}\right) dA + T_0 \left(2\beta - \frac{\partial \beta}{\partial \sigma} - A_0 \frac{\partial \beta}{\partial A_0}\right) da \end{aligned} \tag{3.106}$$

We will now utilize (3.106) to constrain the hydrodynamical entropy current of the system. The entropy current must take the 'canonical' form $su^\mu - \nu J^\mu_{diss}$ corrected by first derivative terms. As in the rest of this section we keep track only of parity odd terms. It follows from Table 8 that the most general one derivative entropy current is given by [29]

$$J^\mu_{(s)} = su^\mu - \frac{\mu}{T}\left(\tilde{\chi}_E \tilde{E}^\mu + \tilde{\chi}_T \tilde{T}^\mu\right) + \left(n_E \tilde{E}^\mu + n_T \tilde{T}^\mu\right) + \left(n_B B + n_\Omega \Omega\right) u^\mu, \tag{3.108}$$

n_E, n_T, n_B and n_Ω are functions of temperature and the chemical potential. On substituting the equilibrium values of temperature and chemical potential they turn into functions of σ and A_0. We find it convenient to define the quantities

$$\begin{aligned} \tilde{n}_E &= n_E - \frac{\mu}{T}\tilde{\chi}_E + s\xi_E, \\ \tilde{n}_T &= n_T - \frac{\mu}{T}\tilde{\chi}_T + s\xi_T. \end{aligned} \tag{3.109}$$

in terms of which the first order part of the entropy current is given by

$$\begin{aligned} \delta J^\mu_{(s)} = &\left(\left(\frac{\partial s}{\partial T}\tau_B + \frac{\partial s}{\partial \mu} m_B + n_B\right) B + \left(\frac{\partial s}{\partial T}\tau_\Omega + \frac{\partial s}{\partial \mu} m_\Omega + n_\Omega\right)\Omega\right) u^\mu_{(0)} \\ &+ \tilde{n}_E \tilde{E}^\mu + \tilde{n}_T \tilde{T}^\mu \end{aligned} \tag{3.110}$$

As we have explained above, the entropy current is necessarily divergence free in equilibrium. This condition yields one condition

$$\frac{\partial \tilde{n}_E}{\partial \sigma} = -T_0 \frac{\partial \tilde{n}_T}{\partial A_0}. \tag{3.111}$$

(3.111) is solved by the ansatz

$$\tilde{n}_E = T_0 \frac{\partial n}{\partial A_0}; \quad \tilde{n}_T = -\frac{\partial n}{\partial \sigma}, \tag{3.112}$$

where n is a arbitrary function of σ and A_0. Plugging in this solution, we now have a 3 parameter set of entropy currents parameterized by n_B, n_ω and n. The entropy (3.106) is an integral over the two parity odd scalars of the system. Equating (3.106) with $\int d^3x \sqrt{-g_3} J^0_S$,

[29] The map between the corrections to the entropy current in this section to that in [19], considering first order terms which are non-zero in the equilibrium, is given by

$$n_T = \tilde{\nu}_1 + \frac{\tilde{\nu}_5}{T}; \quad n_E = \tilde{\nu}_2 + \tilde{\nu}_4 + \frac{\tilde{\nu}_3}{T}; \quad n_B = \tilde{\nu}_4; \quad n_\Omega = \tilde{\nu}_5. \tag{3.107}$$

and equating the coefficients of these two scalars, yields two equations for n_B, n_ω and n. We now explain how this works in more detail

Using the fact

$$\tilde{E}^0 = -e^{-\sigma}\epsilon^{ij}a_i\partial_j A_0; \quad \tilde{T}^0 = e^{-\sigma}\epsilon^{ij}a_i\partial_j\sigma, \tag{3.113}$$

and the expressions of B and Ω in terms of the background field (from Table 8), the entropy can be evaluated from the entropy current in a manifestly Kaluza-Klein gauge invariant way

$$\begin{aligned} S &= \int d^2x\sqrt{-g_3}J^0_{(s)} \\ &= \frac{1}{2}\int \Bigg(\left(\frac{\partial s}{\partial T}\tau_B + \frac{\partial s}{\partial \mu}m_B + n_B\right)(dA + A_0 da) \\ &- \left(\frac{\partial s}{\partial T}\tau_\Omega + \frac{\partial s}{\partial \mu}m_\Omega + n_\Omega\right)e^\sigma da - T_0 n da\Bigg). \end{aligned} \tag{3.114}$$

Comparing (3.114) with (3.106) and using the thermodynamic identities

$$\frac{\partial s}{\partial \epsilon} = \frac{1}{T}, \quad \frac{\partial s}{\partial \rho} = -\frac{\mu}{T}, \tag{3.115}$$

we get the following simple expressions

$$\begin{aligned} n_B &= \alpha \\ n_\Omega &= T_0 e^{-\sigma}\left(A_0\alpha - 2\beta - n\right) \end{aligned} \tag{3.116}$$

In other words, we have managed to evaluate n_B, and one linear combination of n_Ω and n in terms of the functions, α and β, that appear in the partition function of our system. Note that we have not been able to completely determine the non dissipative part of the entropy current using our method (the method based on positivity of the entropy current achieves this determination). However, it straightforward to verify that the constraints (equations 3.11, 3.17, 3.18, and 3.20) in [19] on the corrections to the entropy current from the second law of thermodynamics, are consistent with the relations (3.116) and (3.112).

3.4.5 Comparison with Jensen et.al.

In this subsubsection we shall give a precise connection between partition function coefficients α, β in equation (3.94) and M_Ω, M that appears in [19]. Comparing equations (3.102),(3.103) with equation 1.8 of [19], we get the following differential equations

$$\begin{aligned} \frac{\partial M}{\partial \mu} &= T_0\frac{\partial \alpha}{\partial A_0}, \\ T\frac{\partial M}{\partial T} + \mu\frac{\partial M}{\partial \mu} - M &= -T\frac{\partial \alpha}{\partial \sigma}, \\ \frac{\partial M_\Omega}{\partial \mu} - M &= -T\left(T_0\frac{\partial \beta}{\partial A_0} - A_0\frac{\partial \alpha}{\partial A_0}\right), \\ T\frac{\partial M_\Omega}{\partial T} + \mu\frac{\partial M_\Omega}{\partial \mu} + f_\Omega - 2M_\Omega &= Te^{-\sigma}\left(T_0\frac{\partial \beta}{\partial \sigma} - A_0\frac{\partial \alpha}{\partial \sigma}\right). \end{aligned} \tag{3.117}$$

Field	C	P	T	CPT
σ	+	+	+	+
a_1	+	−	−	+
a_2	+	+	−	−
A_0	−	+	+	−
A_1	−	−	−	−
A_2	−	+	−	+

Table 9. Action of CPT

By solving first two equations in 3.117, we get

$$\alpha = \frac{M}{T_0 e^{-\sigma}} + c \tag{3.118}$$

where c is some constant. Infact the entropy current presented in this section matches that of [19] only if we set $c = 0$ (see equation (3.22) of [19]). Solving last two equations in 3.117, we get

$$\beta = -\frac{e^{2\sigma}}{T_0^2} M_\Omega + \frac{1}{T_0^2} \int f_\Omega (T_0 e^{-\sigma}) e^{2\sigma} d\sigma + \frac{A_0}{T_0} \left(\frac{M}{T_0 e^{-\sigma}} \right) + c_1, \tag{3.119}$$

where c_1 is some other constant.

Also comparing (3.102) and (3.103) with equations (3.17) and (3.18) in [19] one can express the entropy current corrections in terms of α and β in the following way

$$\begin{aligned} T^2 \frac{\partial \tilde{\nu}_4}{\partial T} &= -T_0 e^{-\sigma} \frac{\partial \alpha}{\partial \sigma}, \\ \frac{\partial \tilde{\nu}_4}{\partial (\frac{\mu}{T})} &= T_0 \frac{\partial \alpha}{\partial A_0}, \\ T^2 \left(\frac{\partial \tilde{\nu}_5}{\partial T} + \tilde{\nu}_1 \right) &= T_0 e^{-2\sigma} \left(T_0 \frac{\partial \beta}{\partial \sigma} - A_0 \frac{\partial \alpha}{\partial \sigma} \right), \\ \frac{\partial \tilde{\nu}_5}{\partial (\frac{\mu}{T})} + \tilde{\nu}_3 &= -T_0 e^{-\sigma} \left(T_0 \frac{\partial \beta}{\partial A_0} - A_0 \frac{\partial \alpha}{\partial A_0} \right). \end{aligned} \tag{3.120}$$

Note that with this identification, the equation (3.20) in [19] automatically follows.

3.4.6 Constraints from CPT invariance

Imposing CPT invariance of the partition function 3.94 constrains the form of the otherwise arbitrary functions α, β. (and hence all transport coefficients determined in terms of α and β). Note that we define parity in 2+1 dimensions as $x_1 \to -x_1$ and $x_2 \to x_2$. In Table 9 we list the action of CPT on various fields appearing in the partition function 3.94. Based on the Table 9 we see, in 3.94 "dA" changes sign where as "da" does not, which implies α is odd under CPT and β is even under CPT.

3.5 3+1 dimensional uncharged fluid dynamics at second order in the derivative expansion

In this subsection we will derive the constraints imposed on the equations of uncharged fluid dynamics, at second order in the derivative expansion, by comparison with the most general equilibrium partition function. We do not assume that our system enjoys invariance under parity transformations.

Before getting into the details let us summarize our results. Symmetry considerations determine the expansion of the hydrodynamical stress tensor upto 15 parity even and 5 parity odd transport coefficients. It turns out the 7 of the parity even and 2 of the odd terms vanish in equilibrium. In other words, on symmetry grounds our system has 7 parity odd and 2 parity even dissipative coefficients. In addition we have 8 parity even and 3 parity odd non dissipative coefficients. The most general fluid dynamical partition function, on the other hand, is given in terms of three functions of σ. It turns out that this partition function is automatically even under parity transformations. As a consequence, implementing the procedure spelt out in the introduction, we are able to show that the three nondissipative parity odd coefficients all vanish. In addition the 8 nondissipative parity even coefficients are all determined in term of three functions. In other words we are able to derive 5 relations between these 8 parity even coefficients.

The problem of constraining fluid dynamics at second order in the derivative expansion, using the principle of entropy increase, was studied by one of the coauthors of this work in [10]. In that work the fluid was assumed to enjoy invariance under parity transformations. It was demonstrated that the principle of entropy increase indeed implies 5 relations between the 8 non dissipative transport coefficients. It turns out that the five relations determined in this section agree exactly with those of [10].

Even from a practical point of view the method used in this subsection appears to have some advantages over the more traditional entropy method utilized in [10]. To start with the algebra required for the analysis in this subsection is considerably less formidable than that employed in [10]. As a consequence we are able, rather effortlessly, to generalize our results to allow for the possibility of parity violation. Such a generalization would involve considerable extra effort using the method of [10], and has not yet been done.

3.5.1 Equilibrium from Hydrodynamics

In Tables 1, 2, 3, 7 of [10], all scalar, vector and tensor expressions that one can form out of fluid fields and background metric (not necessarily in equilibrium) at second order in the derivative expansion are listed. It follows from the listing of these tables that the most general symmetry allowed two derivative expansion of the constitutive relations is given by

$$\begin{aligned}
\Pi_{\mu\nu} = \; & -\eta\sigma_{\mu\nu} - \zeta P_{\mu\nu}\Theta \\
& + T\Bigg[\tau\ (u.\nabla)\sigma_{\langle\mu\nu\rangle} + \kappa_1\tilde{R}_{\langle\mu\nu\rangle} + \kappa_2 K_{\langle\mu\nu\rangle} + \lambda_0\ \Theta\sigma_{\mu\nu} \\
& + \lambda_1\ \sigma_{\langle\mu}{}^{a}\sigma_{a\nu\rangle} + \lambda_2\ \sigma_{\langle\mu}{}^{a}\omega_{a\nu\rangle} + \lambda_3\ \omega_{\langle\mu}{}^{a}\omega_{a\nu\rangle} + \lambda_4\ \mathfrak{a}_{\langle\mu}\mathfrak{a}_{\nu\rangle}\Bigg] \\
& + TP_{\mu\nu}\Bigg[\zeta_1(u.\nabla)\Theta + \zeta_2\tilde{R} + \zeta_3\tilde{R}_{00} + \xi_1\Theta^2 + \xi_2\sigma^2 + \xi_3\omega^2 + \xi_4\mathfrak{a}^2\Bigg] \\
& + T\Bigg[\sum_{i=1}^{4}\delta_i t^{(i)}_{\mu\nu} + \delta_5 P_{\mu\nu}\mathfrak{a}_\alpha l^\alpha\Bigg]
\end{aligned} \tag{3.121}$$

where

$$\begin{aligned}
& u^\mu = \text{The normalised four velocity of the fluid} \\
& P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu = \text{Projector perpendicular to } u^\mu \\
& \Theta = \nabla.u = \text{Expansion}, \quad \mathfrak{a}_\mu = (u.\nabla)u_\mu = \text{Acceleration} \\
& \sigma^{\mu\nu} = P^{\mu\alpha}P^{\nu\beta}\left(\frac{\nabla_\alpha u_\beta + \nabla_\beta u_\alpha}{2} - \frac{\Theta}{3}g_{\alpha_\beta}\right) = \text{Shear tensor} \\
& \omega^{\mu\nu} = P^{\mu\alpha}P^{\nu\beta}\left(\frac{\nabla_\alpha u_\beta - \nabla_\beta u_\alpha}{2}\right) = \text{Vorticity} \\
& K^{\mu\nu} = \tilde{R}^{\mu a\nu b}u_a u_b, \quad \tilde{R}^{\mu\nu} = \tilde{R}^{a\mu b\nu}g_{ab} \quad (\tilde{R}^{abcd} = \text{Riemann tensor}) \\
& \sigma^2 = \sigma_{\mu\nu}\sigma^{\mu\nu}, \qquad \omega^2 = \omega_{\mu\nu}\omega^{\nu\mu}
\end{aligned} \tag{3.122}$$

and

$$A_{\langle\mu\nu\rangle} \equiv P^\alpha_\mu P^\beta_\nu\left(\frac{A_{\alpha\beta} + A_{\beta\alpha}}{2} - \left[\frac{A_{ab}P^{ab}}{3}\right]g_{\alpha\beta}\right) \quad \text{For any tensor } A_{\mu\nu}$$

The parity odd terms in the last bracket in (3.121) are defined in Table 10.

The expansion (3.121) is given in terms of 15 undetermined parity even and five undetermined parity odd transport coefficients, each of which is, as yet, an arbitrary function of temperature).

We are interested in the stationary equilibrium solutions of these equations. Solutions in equilibrium are determined entirely by the background fields σ, a_i and g^{ij}. In Table(5,4) we have seen that the Θ and $\sigma_{\mu\nu}$ evaluates to zero in equilibrium. This sets seven of the fifteen parity even terms in equation 3.121 to zero. Two of the five parity odd terms two terms ($t^{(2,3)}_{\mu\nu}$ in table 10) evaluate to zero in equilibrium. The remaining 8 parity even and 3 parity odd coefficients are non dissipative; the non dissipative part of $\Pi_{\mu\nu}$ is given by

$$\begin{aligned}
\frac{\Pi_{\mu\nu}}{T} = \; & \kappa_1\tilde{R}_{\langle\mu\nu\rangle} + \kappa_2 K_{\langle\mu\nu\rangle} + \lambda_3\omega_{\langle\mu}{}^{\alpha}\omega_{\alpha\nu\rangle} + \lambda_4\mathfrak{a}_{\langle\mu}\mathfrak{a}_{\nu\rangle} \\
& + P_{\mu\nu}(\zeta_2\tilde{R} + \zeta_3\tilde{R}_{00}(u^0)^2 + \xi_3\omega^2 + \xi_4\mathfrak{a}^2) \\
& + \delta_1 t^1_{\mu\nu} + \delta_4 t^4_{\mu\nu} + \delta_5 P_{\mu\nu}\mathfrak{a}_\alpha l^\alpha
\end{aligned} \tag{3.123}$$

Type	Data	Evaluated on equilibrium
Pseudo-Scalars	$l^\mu \mathfrak{a}_\mu$	$\frac{1}{2}e^\sigma \epsilon^{ijk}\partial_i\sigma f_{jk}$
Pseudo-Vectors	$(\nabla.u)l_\mu$, $\sigma_{\mu\nu}l^\nu$, $u.\nabla l_\mu$	0 0 $\frac{1}{2}e^{2\sigma}\ (\epsilon^{ijk}\partial_i\sigma f_{jk}\ ,\ \ f_{ij}\epsilon^{jkl}f_{kl})$
Pseudo-Tensors	$t^{(1)}_{\mu\nu} = l_{<\mu}\mathfrak{a}_{\nu>}$, $t^{(2)}_{\mu\nu} = \epsilon^{\lambda\rho\alpha\beta}u_\lambda \mathfrak{a}_\rho \sigma_{\alpha<\mu}g_{\nu>\beta}$, $t^{(3)}_{\mu\nu} = \epsilon^{\lambda\rho\alpha\beta}u_\lambda \nabla_\rho \sigma_{\alpha<\mu}g_{\nu>\beta}$, $t^{(4)}_{\mu\nu} = u_b \tilde{R}^{bcd}_{<\mu}\epsilon_{\nu>cdq}u^q$	$\frac{1}{2}e^\sigma \partial_{\langle i}\sigma\epsilon_{j\rangle kl}f^{kl}$ 0 0 $\frac{1}{2}d_1 e^\sigma \partial_{\langle i}\sigma\epsilon_{j\rangle kl}f^{kl} + \frac{1}{2}d_2 e^\sigma \nabla_{\langle i}\epsilon_{j\rangle kl}f^{kl}$

Table 10. Two derivative parity violating fluid data(Here $d_{1,2}$ are function of σ determined by evaluating $t^4_{\mu\nu}$ on equilibrium, but we will not need there explicit expression.)

In order to proceed further, we list all coordinate invariant two derivative scalars, vectors and tensors constructed out of background data are listed in table (11). The temperature and velocity in equilibrium receives correction at second order. The most general symmetry allowed form of corrected temperature and velocity is

$$
\begin{aligned}
u^\mu &= b_0 u_0^\mu + \left(\sum_{m=1}^{2} v_m V^i_{(m)}\right) + \tilde{v}\tilde{V}^i, \\
T &= T_0 e^{-\sigma} + \left(\sum_{m=1}^{4} t_m S_m\right) + \tilde{t}\tilde{S}
\end{aligned}
$$

where, $V_m(\tilde{V})$ and $S_i(\tilde{S})$are Vectors(pseudo) and scalars(pseudo) respectively that are listed in table 11. Also b_0 can be fixed following equation 3.66 as,

$$
b_0 = 1 - e^\sigma a. \left(\sum_{m=1}^{2} v_m V_{(m)} + \tilde{v}\tilde{V}\right) \tag{3.124}
$$

As in previous sections, the stress tensor in equilibrium received corrections at second order in the derivative expansion. The two derivative corrections have two sources. The first set of corrections arises from the corrections (3.121) evaluated on the zero order equilibrium fluid configuration. Using

$$
\begin{aligned}
\tilde{R}_{00}(u^0)^2 &= \tilde{R}_{\mu\nu}u^\mu u^\nu = \frac{1}{4}e^{2\sigma}f^2 + (\nabla\sigma)^2 + \nabla^2\sigma \\
\omega^{ij} &= -\frac{e^\sigma}{2}f^{ij}, \quad \mathfrak{a}^i = g^{im}\partial_m\sigma,
\end{aligned} \tag{3.125}
$$

Scalars	$S_1 = R,\ S_2 = \nabla^2\sigma,\ S_3 = (\nabla\sigma)^2,\ S_4 = f^2 e^{2\sigma}$
Pseudo-Scalars	$\tilde{S} = \epsilon^{ijk}\partial_i\sigma f_{jk}$
Vectors	$V_1 = e^{\sigma}\nabla_i\sigma f^{ij}, \quad V_2 = e^{\sigma}\nabla_i f^{ij},$
Pseudo-Vectors	$\tilde{V_i} = f_{ij} f_{kl}\epsilon^{jkl}$
Tensors	$R_{ij}, f_i{}^k f_{kj}, \nabla_i\nabla_j\sigma, \nabla_i\sigma\nabla_j\sigma$
Pseudo-Tensors	$\partial_{\langle i}\sigma\epsilon_{j\rangle kl} f^{kl},\ \nabla_{\langle i}\epsilon_{j\rangle kl} f^{kl}$

Table 11. Two derivative background data

$$\begin{aligned}
\tilde{R}_{<ij>} &= R_{ij} - \nabla_i\sigma\nabla_j\sigma - \nabla_i\nabla_j\sigma + \frac{1}{2} f_i{}^k f_{jk} e^{2\sigma} \\
&\quad - \frac{1}{3}\left(R - (\nabla\sigma)^2 - \nabla^2\sigma + \frac{1}{2} f^2 e^{2\sigma}\right) g_{ij} \\
K_{<ij>} &= \nabla_i\sigma\nabla_j\sigma + \nabla_i\nabla_j\sigma + \frac{1}{4} f_i{}^k f_{jk} e^{2\sigma} \\
&\quad - \frac{1}{3}\left((\nabla\sigma)^2 + \nabla^2\sigma + \frac{1}{4} f^2 e^{2\sigma}\right) g_{ij} \\
\omega^a_{<i}\omega_{aj>} &= e^{2\sigma}\left(f_i{}^a f_{ja} - \frac{1}{3} f^2 g_{ij}\right) \\
\mathfrak{a}_{<i}\mathfrak{a}_{j>} &= \nabla_i\sigma\nabla_j\sigma - \frac{1}{3}(\nabla\sigma)^2 g_{ij}
\end{aligned} \tag{3.126}$$

we find that these corrections are given by

$$\begin{aligned}
\Pi^{eq}_{ij} &= a_1\left(R_{ij} - \frac{R}{2} g_{ij}\right) + a_2\left(\nabla_i\nabla_j\sigma - \nabla^2\sigma g_{ij}\right) + a_3\left(\nabla_i\sigma\nabla_j\sigma - \frac{(\nabla\sigma)^2}{2} g_{ij}\right) \\
&+ a_4\left(f_i^k f_{kj} + \frac{f^2}{4} g_{ij}\right) e^{2\sigma} + g_{ij}\left(b_1 R + b_2\nabla^2\sigma + b_3(\nabla\sigma)^2 + b_4 f^2 e^{2\sigma}\right) \\
&+ \frac{1}{2} T(\delta_1 + \delta_4 d_1) e^{\sigma}\partial_{\langle i}\sigma\epsilon_{j\rangle kl} f^{kl} + \frac{1}{2} T d_2\delta_4 e^{\sigma}\epsilon_{kl\langle i}\nabla_{j\rangle} f^{kl} + \frac{1}{2} T\delta_5 e^{\sigma} g_{ij}\epsilon_{mlk}\partial^m\sigma f^{lk} \quad \text{where,}
\end{aligned} \tag{3.127}$$

$$\begin{aligned}
\frac{b_1}{T} &= \zeta_2 - \frac{1}{6}\kappa_1, \quad \frac{b_2}{T} = \frac{2}{3}(\kappa_2 - \kappa_1) + \frac{1}{3}\lambda_4 + 2\zeta_2 + \zeta_3 + \xi_4 \\
\frac{b_3}{T} &= -\frac{1}{3}(\kappa_2 - 2\kappa_2) - \frac{1}{3}\lambda_3 + \frac{1}{2}\lambda_4 + \frac{1}{4}(\zeta_2 + \zeta_3) + \xi_3 \\
\frac{b_4}{T} &= \frac{1}{24}(11\kappa_1 - 5\kappa_2) - 2\zeta_2 + \zeta_3 + \frac{1}{4}\lambda_4, \quad \frac{a_1}{T} = \kappa_1 \\
\frac{a_2}{T} &= \kappa_2 - \kappa_1, \quad \frac{a_3}{T} = \kappa_2 - \kappa_1 + \lambda_4, \quad \frac{a_4}{T} = -\frac{1}{4}(2\kappa_1 + \kappa_2) + \lambda_4.
\end{aligned} \tag{3.128}$$

Here, the indicies are contracted with the lower dimensional metric g_{ij} and its inverse. The coefficients are determined by evaluating the $t^{(1,4)}_{\mu\nu}$ in equilibrium, but we will not need the explicit expressions.

The second source of corrections arises from inserting the velocity correction (3.67) into the zero order (perfect fluid) constitutive relations. We find that the modification of the stress tensor due to these corrections is given by

$$\begin{aligned} T^{ij} &= P_T g^{ij} \left(\sum t_m S_m + \tilde{t}\tilde{S}\right) \\ T_{00} &= T_0^2 \frac{P_{TT}}{T} \left(\sum t_m S_m + \tilde{t}\tilde{S}\right) \\ T_0^i &= -(\epsilon + P) e^{\sigma} \left(\sum v_m V_m^i + \tilde{v}\tilde{V}^i\right) \end{aligned} \tag{3.129}$$

The net change in T_0^i and J^i is given by summing(3.127) and (3.129) and is given by

$$\begin{aligned} T^{ij} &= P_T g^{ij} \left(\sum t_m S_m + \tilde{t}\tilde{S}\right) + \Pi^{ij}_{eq} \\ T_{00} &= T_0^2 \frac{P_{TT}}{T} \left(\sum t_m S_m + \tilde{t}\tilde{S}\right) \\ T_0^i &= -(\epsilon + P) e^{\sigma} \left(\sum v_m V_m^i + \tilde{v}\tilde{V}^i\right) \end{aligned} \tag{3.130}$$

where Π^{ij}_{eq} was listed in (3.127).

3.5.2 Equilibrium from the Partition Function

We now turn to the study of the first correction to the perfect fluid equilibrium partition function (3.61) at second order in the derivative expansion. We observe that the Table (11) lists four scalars and one pseudo-scalar. The most generic partition function for this system at two derivative order is,

$$\begin{aligned} W = \log Z = -\frac{1}{2}\int d^3x \sqrt{g_3}\left[\tilde{P}_1(T_0 e^{-\sigma}) R + T_0^2 \tilde{P}_2(T_0 e^{-\sigma}) f_{ij} f^{ij} + \tilde{P}_3(T_0 e^{-\sigma})(\partial\sigma)^2\right] \\ \text{where} \;\; \tilde{P}_i(T_0 e^{-\sigma}) = P_i(\sigma) \;\; \text{and} \;\; P_i' \equiv \frac{dP_i(\sigma)}{d\sigma} \quad (i = 1, 2, 3) \end{aligned} \tag{3.131}$$

where P_1, P_2, P_3 are three arbitrary function of σ and from now on we will remove the explicit dependence. In partition function, the fourth scalar $\nabla^2\sigma$ and the pseudo-scalar $\epsilon_{ijk}\partial^i\sigma f^{jk}$ do not appear as they are total derivatives.

With the action (3.131) in hand it is straightforward to use analog of (3.38) for uncharged case[30] to obtain the equilibrium stress tensor. We find

$$\begin{aligned}
T^{ij} &= TP_1\big(R^{ij} - \frac{1}{2}Rg^{ij}\big) + 2T_0^2TP_2\big(f^{ik}f_{jk} - \frac{1}{4}f^2g^{ij}\big) + T(P_3 - P_1'')\big(\nabla^i\sigma\nabla^j\sigma \\
&\quad - \frac{1}{2}(\nabla\sigma)^2g^{ij}\big) - TP_1'\big(\nabla^i\nabla^j\sigma - g^{ij}\nabla^2\sigma\big) + \frac{1}{2}TP_1''(\nabla\sigma)^2g^{ij} \\
T_{00} &= \frac{T_0^2}{2T}\big(P_1'R + T_0^2P_2'f^2 - P_3'(\nabla\sigma)^2 - 2P_3\nabla^2\sigma)\big) \\
T_0^i &= 2T_0^2T\big(P_2'\nabla_j\sigma f^{ji} + P_2\nabla_j f^{ji}\big),
\end{aligned} \tag{3.133}$$

where $'$ denotes derivative with respect to σ.

3.5.3 Constraints on Hydrodynamics

Comparing non trivial components of the stress tensor T_0^i, T_{00} in equations 3.130,3.133 and equating coefficients of independent sources one obtains the velocity and temperature corrections in terms of the coefficients P appearing in 3.131. We find

$$\begin{aligned}
&v_1 = -\frac{2T^2}{P_T}P_2', \quad v_2 = -\frac{2T^2}{P_T}P_2, \quad \tilde{v} = 0, \\
&t_1 = \frac{1}{2P_{TT}}P_1', \quad t_2 = -\frac{1}{P_{TT}}P_3, \quad t_3 = -\frac{1}{2P_{TT}}P_3', \quad t_4 = \frac{T^2}{2P_{TT}}P_2', \quad \tilde{t} = 0.
\end{aligned} \tag{3.134}$$

Now comparing T_{ij} in equations 3.130,3.133, and using expressions for temperature corrections, one can express the transport coefficients in terms of the three coefficients P appearing in 3.131. We find

$$\begin{aligned}
&a_1 = TP_1, \quad a_2 = -TP_1' \quad a_4 = -2T^3P_2, \quad a_3 = T(P_3 - P_1''), \\
&b_1 = -\frac{P_T}{2P_{TT}}P_1', \quad b_2 = \frac{P_T}{P_{TT}}P_3, \quad b_4 = -\frac{P_TT^2}{2P_{TT}}P_2', \\
&b_3 = \frac{1}{2}TP_1'' + \frac{P_T}{2P_{TT}}P_3', \quad \delta_1 = \delta_4 = \delta_5 = 0.
\end{aligned} \tag{3.135}$$

One can eliminate the coefficients $P's$ from above set of relations which gives five relations among transport coefficients,

$$\begin{aligned}
&a_1 + a_2 - T\partial_T a_1 = 0, \quad \frac{TP_{TT}}{P_T}b_1 + \frac{1}{2}(a_1 - T\partial_T a_1) = 0 \\
&\frac{TP_{TT}}{P_T}b_2 + (a_2 - T\partial_T a_2) - a_3 = 0, \quad 4\frac{TP_{TT}}{P_T}b_4 - (3a_4 - T\partial_T a_4) = 0, \\
&2\frac{TP_{TT}}{P_T}b_3 + \big(\frac{TP_{TT}}{P_T} + 1\big)(a_2 - T\partial_T a_2) - T\partial_T(a_2 - T\partial_T a_2) - (a_3 - T\partial_T a_3) = 0.
\end{aligned} \tag{3.136}$$

[30] The stress tensor can be evaluated as

$$\begin{aligned}
T_{00} &= -\frac{T_0e^{2\sigma}}{\sqrt{-g_{(p+1)}}}\frac{\delta W}{\delta\sigma}, \quad T_0^i = \frac{T_0}{\sqrt{-g_{(p+1)}}}\frac{\delta W}{\delta a_i}, \\
T^{ij} &= -\frac{2T_0}{\sqrt{-g_{(p+1)}}}g^{il}g^{jm}\frac{\delta W}{\delta g^{lm}}.
\end{aligned} \tag{3.132}$$

Note that parity odd contributions, both to the equilibrium value of the temperature and velocity, as well as to the constitutive relations, are forced to vanish. The simple reason for this is that the most general two derivative correction to the partition function 3.131 is parity even. Note also that the eight parity even non dissipative transport coefficients are all determined in terms of the three functions that parameterize the two derivative partition function. This leaves us five relations among the transport coefficients; these relations may be obtained by substituting the definitions of the a and b coefficients in (3.127) into (3.136); we find

$$
\begin{aligned}
\kappa_2 &= \kappa_1 + T\frac{d\kappa_1}{dT} \\
\zeta_2 &= \frac{1}{2}\left[s\frac{d\kappa_1}{ds} - \frac{\kappa_1}{3}\right] \\
\zeta_3 &= \left(s\frac{d\kappa_1}{ds} + \frac{\kappa_1}{3}\right) + \left(s\frac{d\kappa_2}{ds} - \frac{2\kappa_2}{3}\right) + \frac{s}{T}\left(\frac{dT}{ds}\right)\lambda_4 \\
\\
\xi_3 &= \frac{3}{4}\left(\frac{s}{T}\right)\left(\frac{dT}{ds}\right)\left(T\frac{d\kappa_2}{dT} + 2\kappa_2\right) - \frac{3\kappa_2}{4} + \left(\frac{s}{T}\right)\left(\frac{dT}{ds}\right)\lambda_4 \\
&\quad + \frac{1}{4}\left[s\frac{d\lambda_3}{ds} + \frac{\lambda_3}{3} - 2\left(\frac{s}{T}\right)\left(\frac{dT}{ds}\right)\lambda_3\right] \\
\xi_4 &= -\frac{\lambda_4}{6} - \frac{s}{T}\left(\frac{dT}{ds}\right)\left(\lambda_4 + \frac{T}{2}\frac{d\lambda_4}{dT}\right) - T\left(\frac{d\kappa_2}{dT}\right)\left(\frac{3s}{2T}\frac{dT}{ds} - \frac{1}{2}\right) \\
&\quad - \frac{Ts}{2}\left(\frac{dT}{ds}\right)\left(\frac{d^2\kappa_2}{dT^2}\right)
\end{aligned}
\tag{3.137}
$$

This is in perfect agreement with the relations obtained in [10] using the second law of thermodynamics.

3.5.4 The Entropy Current

The entropy of our system is given by

$$
S = \frac{\partial}{\partial T_0}(T_0 \log Z)
\tag{3.138}
$$

The partition function of our system is given by

$$
\log Z = -\frac{1}{2}\int d^3x\,\sqrt{g_3}\left[\tilde{P}_1(T_0 e^{-\sigma})R + T_0^2\tilde{P}_2(T_0 e^{-\sigma})f_{ij}f^{ij} + \tilde{P}_3(T_0 e^{-\sigma})(\partial\sigma)^2\right]
\tag{3.139}
$$

(we are careful to explicitly keep track of the temperature dependence in the partition function, see the equation (3.131) for a definition of the functions $\tilde{P}$). The total entropy as evaluated from this partition function is

$$
\begin{aligned}
S &= \frac{\partial}{\partial T_0}(T_0 \log Z) \\
&= \frac{1}{2}\int \sqrt{g}\left[(P_1' - P_1)R + T_0^2(P_2' - 3P_2)f_{ij}f^{ij} + (P_3' - P_3)(\partial\sigma)^2\right]
\end{aligned}
\tag{3.140}
$$

To second order in the derivative expansion, the most general symmetry allowed entropy current is given by [10]

$$
\begin{aligned}
J_S^\mu =& su^\mu + \tilde{J}_S^\mu \\
&\text{where} \\
\tilde{J}_S^\mu =& \nabla_\nu\left[A_1(u^\mu\nabla^\nu T - u^\nu\nabla^\mu T)\right] + \nabla_\nu\left(A_2 T\omega^{\mu\nu}\right) \\
&+ A_3\left(\tilde{R}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\tilde{R}\right)u_\nu + \left[A_4(u.\nabla)\Theta + A_5\tilde{R} + A_6(\tilde{R}_{\alpha\beta}u^\alpha u^\beta)\right]u^\mu \\
&+ (B_1\omega^2 + B_2\Theta^2 + B_3\sigma^2)u^\mu + B_4\left[(\nabla s)^2 u^\mu + 2s\Theta\nabla^\mu s\right] \\
&+ \left[\Theta\nabla^\mu B_5 - P^{ab}(\nabla_b u^\mu)(\nabla_a B_5)\right] + B_6\Theta\mathfrak{a}^\mu + B_7\mathfrak{a}_\nu\sigma^{\mu\nu}
\end{aligned}
\tag{3.141}
$$

The terms above with A_1 and A_2 as coefficients are total derivative and do contribute to the total entropy. It follows that A_1 and A_2 are unconstrained by comparison with equilibrium (even though these terms do not pointwise vanish in equilibrium). Terms with coefficients $A_4,\ B_2,\ B_3,\ B_6$ and B_7 vanish on the equilibrium solution. Consequently these coefficients are also unconstrained by the considerations of this section. The entropy current coefficients that can be are constrained by comparison with (3.140) are $A_3,\ A_5,\ A_6,\ B_1,\ B_4$ and B_5

As above, there are two sources for the second order correction to the entropy of our system. The su^μ part in J_S^μ contributes to the total entropy at second order in derivative expansion because of the second order corrections δu^μ to the equilibrium velocity u^μ and δT to the equilibrium temperature.More precisely, if the equilibrium temperature and velocity of our system to second order is given by

$$
T = T_{(0)} + \delta T = T_0 e^{-\sigma} + \delta T \quad \text{and} \quad u^\mu = u^\mu_{(0)} + \delta u^\mu = e^{-\sigma}(1,0,0,0) + \delta u^\mu
$$

then clearly

$$
su^0|_{\text{2nd order}} = e^{-\sigma}\left(\frac{ds}{dT}\right)\delta T + s\delta u^0
$$

.

Using (3.134) and (3.124) we find

$$
\begin{aligned}
\left(\frac{ds}{dT}\right)\delta T &= \frac{1}{2}\left[P_1'\ R + P_2'\ T_0^2 f^2 - P_3'\ (\partial\sigma)^2 - 2P_3\ \nabla^2\sigma\right] \\
&= \frac{1}{2}\left[P_1'\ R + P_2'\ T_0^2 f^2 + P_3'\ (\partial\sigma)^2 - 2\nabla_i\left(P_3\nabla^i\sigma\right)\right]
\end{aligned}
\tag{3.142}
$$

$$
s\delta u^i = -2e^{-\sigma}T_0^2\left[P_2'\ \nabla_j\sigma f^{ji} + P_2\ \nabla_j f^{ji}\right] = -2T_0^2 e^{-\sigma}\nabla_j\left(P_2\ f^{ji}\right)
$$

Therefore using 3.66, the second order correction to J_S^0, from the perfect fluid piece su^0, evaluates to

$$su^0|_{\text{2nd order}} = \frac{e^{-\sigma}}{2}\left[P_1'\ R + (P_2' - 2P_2)\ T_0^2 f^2 + P_3'\ (\partial\sigma)^2\right] + e^{-\sigma}\nabla_j\left[2T_0^2 P_2\ f^{ji}a_i - P_3\nabla^j\sigma\right] \tag{3.143}$$

The second source of two derivative corrections to the entropy current come from the explicit two derivative corrections to the entropy current (3.141) evaluated on the perfect fluid equilibrium configurations. Using

$$\begin{aligned}
&f^2 \equiv f_{ij}f^{ij}\\
&P^{\mu a}\nabla_a u^\nu = \sigma^{\mu\nu} + \omega^{\mu\nu} + P^{\mu\nu}\frac{\Theta}{3}\\
&\tilde{R} = R - 2(\partial\sigma)^2 - 2\nabla^2\sigma + \frac{e^{2\sigma}}{4}f^2\\
&\tilde{R}_{\alpha\beta}u^\alpha u^\beta = (\partial\sigma)^2 + \nabla^2\sigma + \frac{e^{2\sigma}}{4}f^2\\
&\tilde{R}_0^i = \frac{e^{2\sigma}}{2}\left[\nabla_j f^{ji} + 3(\nabla_j\sigma)f^{ji}\right]\\
&\tilde{R}_0^0 = -\left(e^{-2\sigma}\tilde{R}_{00} + a_i\tilde{R}_0^i\right)\\
&e^\sigma\omega^{0i}\partial_i T = -\frac{Te^{2\sigma}}{2}(\partial_i\sigma)f^{ji}a_j
\end{aligned} \tag{3.144}$$

we find that the zero component of $\tilde{J}_S^\mu$ evaluates on equilibrium to

$$\begin{aligned}
\tilde{J}_S^0 =& e^{-\sigma}\left[A_3\left(\tilde{R}_0^0 - \frac{\tilde{R}}{2}\right) + A_5\tilde{R} + A_6(\tilde{R}_{00}e^{-2\sigma}) + B_1\omega^2 + B_4(\partial s)^2 + e^\sigma\left(\frac{dB_5}{dT}\right)\omega^{0i}(\partial_i T)\right]\\
=& e^{-\sigma}\Bigg[\left(A_5 - \frac{A_3}{2}\right)R + \left(\frac{2A_5 + 2A_6 - 3A_3 - 2B_1}{8}\right)e^{2\sigma}f^2 + T^2\left(\frac{ds}{dT}\right)^2 B_4(\partial\sigma)^2\\
&+ (A_6 - 2A_5)\left[\nabla^2\sigma + (\partial\sigma)^2\right] - \frac{A_3 e^{2\sigma}}{2}a_i\nabla_j f^{ji} - \frac{\left(3A_3 - T\frac{dB_5}{dT}\right)e^{2\sigma}}{2}a_i f^{ji}\partial_j\sigma\Bigg]\\
=& e^{-\sigma}\Bigg[\left(A_5 - \frac{A_3}{2}\right)R + \left(\frac{2A_5 + 2A_6 - A_3 - 2B_1}{8}\right)e^{2\sigma}f^2\\
&+ \left[\left(T\frac{ds}{dT}\right)^2 B_4 + T\frac{d}{dT}(A_6 - 2A_5)\right](\partial\sigma)^2 + \frac{T}{2}\left(\frac{dB_5}{dT} - \frac{A_3}{T} - \frac{dA_3}{dT}\right)a_i f^{ji}\partial_j\sigma\\
&- \frac{1}{2}\nabla_j\left(A_3 e^{2\sigma}a_i f^{ji}\right) + \nabla_i\left[(A_6 - 2A_5)\nabla^i\sigma\right]\Bigg]
\end{aligned} \tag{3.145}$$

Summing (3.143) and (3.145) and ignoring total derivatives, we find our final result for the two derivative correction to the total entropy.

$$
\begin{aligned}
&\text{Total Entropy}\\
&=\int d^3x\sqrt{g_3}\Bigg[\left(A_5-\frac{A_3}{2}+\frac{P_1'}{2}\right)R+\left[\frac{2A_5+2A_6-A_3-2B_1+T_0^2(4P_2'-8P_2)e^{-2\sigma}}{8}\right]e^{2\sigma}f^2\\
&+\left[\left(T\frac{ds}{dT}\right)^2B_4+T\frac{d}{dT}(A_6-2A_5)+\frac{P_3'}{2}\right](\partial\sigma)^2+\frac{T}{2}\left(\frac{dB_5}{dT}-\frac{A_3}{T}-\frac{dA_3}{dT}\right)a_if^{ji}\partial_j\sigma\Bigg]
\end{aligned}
\tag{3.146}
$$

While the first three terms in (3.146) are Kaluza Klein gauge invariant, the last term is not. Let us pause, for a moment to explain this. In subsubsection 3.2.5 we have demonstrated that the integral $\int\sqrt{-g_4}J_S^0$ is Kaluza Klein gauge invariant *provided* that $\partial_\mu J_S^\mu=0$. Now it must certainly be true that the correct entropy current is divergence free in equilibrium. However the most general entropy current (3.141) is not divergence free in equilibrium. The non gauge invariant term ion (3.146) results from such terms. The coefficients of these terms must immediately be set to zero (even without comparison with a particular form of the entropy). The coefficients of the remaining three terms in (3.146) must be equated with the coefficients of the corresponding terms in (3.140). In net we have four equations which allow us to solve for four of the entropy current coefficients, B_5, A_3, B_1 and B_4 in terms of the other two (A_5 and A_6) and P_i (the coefficients that appear in the partition function ie.the P_i).

$$
\begin{aligned}
&\frac{dB_5}{dT}=\frac{A_3}{T}+\frac{dA_3}{dT}\\
&A_3=P_1+A_5\\
&B_1=-\frac{P_1}{2}+2T_0^2e^{-2\sigma}P_2+A_5+A_6\\
&\left(T\frac{ds}{dT}\right)^2B_4=-\frac{P_3}{2}-T\frac{d}{dT}(A_6-2A_5)
\end{aligned}
\tag{3.147}
$$

3.5.5 Entropy current with non-negative divergence

Above we have discussed the constraints on the entropy current from comparison with the total entropy of our system. In this subsubsection we will discuss the relationship between these constraints and those obtained by imposing the requirement of positivity of the entropy current.

In the study of the positivity of the divergence of the entropy current, it turns out that some coefficients in the entropy current are determined in terms of transport coefficients, while others are left free (more precisely these coefficients are constrained by inequalities involving transport coefficients). The determined coefficients turn out to be precisely those that multiply terms that are nonvanishing in equilibrium, namely A_3, A_5, A_6, B_1, B_4 and

B_5. The six equations that determine these six parameters are

$$\begin{aligned} A_5 &= 0 \\ A_6 &= 0 \\ \frac{dB_5}{dT} &= \frac{A_3}{T} + \frac{dA_3}{dT} \\ A_3 &= \kappa_1 \\ B_1 &= \frac{1}{4}\left[-\lambda_3 + T\frac{d\kappa_1}{dT} + \kappa_1\right] \\ \left(T\frac{ds}{dT}\right)^2 B_4 &= -\frac{1}{2}\left[\lambda_4 + 2T\frac{d\kappa_1}{dT} + T^2\frac{d^2\kappa_1}{dT^2}\right] \end{aligned} \tag{3.148}$$

The results (3.148) satisfy the constraints (3.147). In order to verify this one plugs in explicit results

$$\begin{aligned} \xi_3 &= -\frac{P_1}{2} + \frac{2}{3}(P_1' - T_0^2 e^{-2\sigma} P_2) + \left(\frac{s}{T}\frac{dT}{ds}\right)\left(2T_0^2 e^{-2\sigma} P_2' + P_3 - \frac{3}{2}P_1'\right) \\ \xi_4 &= \frac{2}{3}\left(P_1'' - P_1' - \frac{P_3}{4}\right) + \left(\frac{s}{T}\frac{dT}{ds}\right)\left(\frac{P_3'}{2} - P_3\right) \\ \zeta_2 &= -\frac{P_1}{6} - \left(\frac{s}{T}\frac{dT}{ds}\right)\frac{P_1'}{2} \\ \zeta_3 &= \frac{2P_1' - P_1}{3} + \left(\frac{s}{T}\frac{dT}{ds}\right)(P_3 - P_1') \\ \lambda_3 &= 3P_1 - 8T_0^2 e^{-2\sigma} P_2 - P_1' \\ \lambda_4 &= P_3 + P_1' - P_1'' \\ \kappa_2 &= P_1 - P_1' \\ \kappa_1 &= P_1 \end{aligned} \tag{3.149}$$

for the transport coefficients in terms of action parameters into (3.148) and checks that the results are consistent with (3.147)

Our results (3.147) are compatible with but weaker than (3.148). (3.148) is equivalent to (3.147) together with $A_5 = A_6 = 0$. As A_5 and A_6 multiply terms that are nonvanishing in equilibrium, we find it surprising that ' our equilibrium study has not been powerful enough to demonstrate that A_5 and A_5 must actually vanish. It is possible that we have overlooked a simple principle that forces these coefficients to vanish without invoking the principle of entropy increase.

3.5.6 The conformal limit

[31]Let us consider Weyl transformation of the full four dimensional metric

$$\bar{g}_{\mu\nu} = g_{\mu\nu} e^{2\phi(x)}.$$

[31]This subsubsection has been worked out in collaboration with R. Loganayagam.

In this subsection first we would like to write an partition function which is invariant under this transformation. In order to have conformal invariance this partition function will have fewer coefficients than the partition function given in (3.131). Then we shall analyze how it will constrain the stress tensor for a conformal fluid.

Under this transformation several three dimensional quantities transform as follows.

$$\begin{aligned} \bar{\sigma} &= \sigma + \phi, \quad \bar{a}_i = a_i, \quad \bar{g}_{ij} = e^{2\phi} g_{ij} \\ (\nabla\bar{\sigma})^2 &= e^{-2\phi}\left[(\nabla\sigma)^2 + 2(\nabla\sigma).(\nabla\phi) + (\nabla\phi)^2\right] \\ \bar{R} &= e^{-2\phi}\left[R - 4\nabla^2\phi - 2(\nabla\phi)^2\right] \\ \bar{f}_{ij}\bar{f}^{ij} &= e^{-4\phi} f_{ij}f^{ij} \\ \sqrt{\bar{g}_3} &= e^{3\phi}\sqrt{g_3} \end{aligned} \tag{3.150}$$

Using (3.150) we can see that under this transformation the partition function (as given in(3.131)) will be invariant (assuming that the total derivative terms will integrate to zero) only if the coefficients P_i's satisfy the following constraints.

$$P_1(\sigma) = e_1 T_0 e^{-\sigma}, \quad P_2(\sigma) = \frac{e_2}{T_0 e^{-\sigma}} \;\text{ and }\; P_3(\sigma) = 2P_1(\sigma) \tag{3.151}$$

where e_1 and e_2 are two dimensionless constants.

Substituting (3.151) in (3.149) we find

$$\begin{aligned} &\xi_3 = \xi_4 = \zeta_2 = \zeta_3 = \lambda_4 = 0 \\ &\kappa_2 = 2\kappa_1 = 2e_1 T_0 e^{-\sigma} \\ &\lambda_3 = 4T_0 e^{-\sigma}(e_1 - 2e_2) \end{aligned} \tag{3.152}$$

These relations precisely match with our expectation for the independent transport coefficients of a conformally covariant stress tensor. Since for a conformally covariant stress tensor only two terms ($\omega_{\langle\mu a}\omega^a{}_{\nu\rangle}$ with coefficient λ_3 and $\left[R_{\langle\mu\nu\rangle} + K_{\langle\mu\nu\rangle}\right]$ with coefficient κ_1) can be non zero in equilibrium and a conformally invariant action also has only two free parameters, it follows that the existence of a partition function does not constrain the stress tensor of a conformal fluid.

3.6 Counting for second order charged fluids in 3+1 dimensions

In this subsection we will use the methods developed in previous subsections to answer the following question: how many transport coefficients are needed to specify the fluid dynamics of a relativistic charged fluid that may not preserve parity, at second order in the derivative expansion? We do not attempt to derive the detailed form of the equations so obtained; our presentation is merely at the level of counting. If the conjecture at the heart of this work is correct, then an analysis of entropy positivity would yield the same number of transport coefficients; however that analysis is much more difficult to perform (even at the level of counting), and we do not attempt it here.

Type	fluid+background data	In equilibrium
scalars	16	9
vectors	17	6
tensors	18	9

Table 12. parity even data for charged fluids at second order

3.6.1 Parity Invariant case

Let us first consider the parity invariant case. Table 3.6.1 list the number of all the the parity preserving fluid plus background onshell independent data at second order. From this table this we see that the total number of symmetry allowed transport coefficients in stress-energy tensor and charge current in landau frame is

$$\text{tensors}(16) + \text{scalars}(18) + \text{vectors}(17) = 51. \tag{3.153}$$

Now let us consider the equilibrium of this system. The third column of table 3.6.1 also list the number of scalars, vectors and tensors that can be constructed out of σ, A_0, a_i, A_i and g^{ij}. The coefficient of these terms that are survive in equilibrium we refer to as 'non dissipative' coefficients while the remaining we refer to as 'dissipative' coefficients. In this case we have a total of 24 non dissipative coefficients. Now there are 9 scalars than can be constructed in equilibrium. We list them below

$$R^i_i, \;\; \nabla^i\sigma\nabla_i\sigma, \;\; f_{ij}f^{ij}, \;\; F_{ij}F^{ij}, \;\; F_{ij}f^{ij}, \;\; \nabla^i\sigma\nabla_iA_0, \;\; \nabla^iA_0\nabla_iA_0, \;\; \nabla^i\nabla_i\sigma, \;\; \nabla^i\nabla_iA_0 \tag{3.154}$$

The last two scalars are total derivatives and hence do not appear in the partition function. This tell us that the 24 non dissipative coefficients are determined in term of 7 independent coefficients that appear in the partition function which means that there will be 17 relation among the 24 non dissipative coefficients.

In summary the methods developed in this section predict that parity invariant charged fluid dynamics is characterized by 7 non dissipative transport coefficients, together with 28 dissipative coefficients (7 scalars, 12 vectors and 9 tensors). Each of these 35 transport coefficients is an unspecified function of T and μ.

3.6.2 Parity Violating case

Let us now consider the parity non invariant charged fluids at second order. Table 3.6.2 lists all the parity odd data at second order. From this table we see that number of transport coefficients in the parity odd sector is

$$\text{pseudo tensors}(12) + \text{pseudo scalars}(6) + \text{pseudo vectors}(9) = 27. \tag{3.155}$$

The third column of table 3.6.2 that out of the 28 parity odd transport coefficients 12 are non dissipative. Now we have 4 new scalars(pseudo) can be added to the partition function. These are listed below

$$\epsilon^{ijk}\partial_i\sigma f_{jk} \;, \;\; \epsilon^{ijk}\partial_iA_0 f_{jk} \;, \;\; \epsilon^{ijk}\partial_i\sigma F_{jk} \;, \;\; \epsilon^{ijk}\partial_iA_0F_{jk} \tag{3.156}$$

Type	fluid+background	In equilibrium
pseudo scalars	6	4
pseudo vectors	9	2
pseudo tensors	12	6

Table 13. parity odd data for charged fluid at second order

As such all of these 3.156 are total derivatives by themselves but they can not be written as total derivatives in the partition function since the coefficients that they will appear with are arbitrary functions of σ and A_0. Thus we see that the 12 parity odd non dissipative coefficients are determined in terms of 4 parity odd coefficients in the partition function which means that their would be 8 relation in parity odd sector.

In summary we predict that, at second order, we have 4 parity odd nondissipative transport coefficients, together with 2 pseudo scalar, 7 pseudo vector and 6 pseudo tensor dissipative coefficients, and total of 20 new coefficients.

3.7 Discussion

The main result of our section is that two apparently different physical requirements, namely the requirement of existence of equilibrium in appropriate circumstances and the requirement of the existence of a point wise positive divergence entropy current, give the same constraints [32] on the equations of hydrodynamics in three specific contexts. Two questions immediately suggest themselves. Do the results of our paper extend to arbitrary order in the derivative expansion, as we have conjectured in this section? If so, why is this the case? Definitive answers to these questions would be very interesting. A proof that the existence of equilibrium plus certain inequalities imply the existence of a positive divergence entropy current could demystify arguments based on the existence of an entropy current, and lead towards a fuller understanding of the second law of thermodynamics.

In the main text of this section we have derived constraints on the constitutive relations of hydrodynamics starting from the assumption of the existence of a partition function. In the appendices to this section we have, however, demonstrated that all the constraints derived in this paper may also be derived from the weaker assumption that fluid admit stationary equilibrium configurations in stationary backgrounds. The integrability conditions from the demand that the currents and stress tensors in equilibrium follow from an action turned out to be automatic in the three examples studied in this paper. Is this always the case (we find this unlikely). In appropriate situations, do the Onsager relations follow from the demand that equilibrium is generated from a partition function?

In another direction, the analysis of this section has led to the consideration of partition functions dual to equilibrium hydrodynamics as a function of background metrics and gauge fields. Given a partition function as a function of sources, it is standard in quantum field

[32] We ignore the inequalities that follow from the principle of entropy increase in this statement.

theory to Legendre transform this object in order to obtain an offshell 1PI effective action for the theory. It may be possible implement this procedure on our partition function to obtain an offshell action for fluid dynamics (albeit one applies only to equilibrium configurations) (see [31] for related work). If so, what is the interpretation of this action in the context of the fluid gravity map of the AdS/CFT correspondence?

It would be very interesting to generalize the work presented in this paper away from equilibrium. Time dependent partition functions are not in general local functionals of their sources. These partition functions are, however usually generated by coupling local field theory dynamics to sources. Can time dependent correlators (perhaps in a Schwinger - Keldysh set up) be generated by minimally coupling a local 'action' for hydrodynamics to the background metric or gauge field? How does this tie in with the fluid gravity map of the AdS/CFT correspondence?

Apart from the traditional requirement of positivity of the entropy current, and the requirement of the existence of equilibrium, emphasized in this paper, one may also attempt to constrain the equations of fluid dynamics by demanding that correlation functions computed from these equations obey all the symmetry properties that follow from the existence of an underlying action (see e.g. [19]). Any system that posseses a well defined partition function, as studied in this paper, automatically obeys all these symmetry properties for time independent correlators. Do the constraints on hydrodynamics that follow from the existence of an equilibrium partition function automatically also guarantee that the symmetry requirements on time dependent correlators are also met?

Finally, it would be very interesting to investigate the interplay of the principal constraint described in this paper (namely the existence of equilibrium for an arbitrary static metric) with the AdS/CFT correspondence. Is this constraint merely from the structure of AdS/CFT, for an arbitrary bulk Lagrangian, or does it impose constraints on possible α' corrections to the equations of Einstein gravity? Within gravity can one prove directly that the existence of equilibrium implies the existence of a Wald entropy increase theorem (and so the existence [32] of a positive divergence entropy current)(see [33] for related discussion)?

3.8 Appendices to chapter 2

3.8.1 First order charged fluid dynamics from equilibrium in 3+1 dimensions

In this appendix we shall rederive the results obtained in section 3.3 making fewer assumptions than in that section. In this Appendix we make no reference to the equilibrium partition function, and nowhere assume its existence. The only demand that we make on our system is that it admit an equilibrium solution in an arbitrary background of the form (3.1), (3.9). We also assume that the zeroth order equilibrium configuration is given by equation 3.21.

As discussed in section 3.3 for the parity violating first order charged fluid, one can not construct any scalar or pseudo scalar at equilibrium and hence temperature or chemical potential does not get corrected to this order. To the first order, the velocity corrections can be written as

$$\delta u^i = -\frac{e^{-\sigma} b_1}{4}\epsilon^{ijk} f_{jk} + b_2 B_K^i + b_3 \partial^i \sigma + b_4 \partial^i A_0 \tag{3.157}$$

The dissipative part of the stress tensor and the current are written in equation 3.65. Since, $\partial_\alpha \frac{\mu}{T} - \frac{E_\alpha}{T}$, θ, $\sigma_{\mu\nu}$ evaluate to zero on equilibrium, we are left with

$$\begin{aligned} \pi^{\mu\nu} &= 0 \\ J^\mu_{diss} &= \alpha_1 E^\mu + \alpha_2 \mathcal{P}^{\mu\alpha}\partial_\alpha T + \xi_\omega \omega^\mu + \xi_B B^\mu \end{aligned} \tag{3.158}$$

We shall now impose that the equations 3.158,3.157 obeys the conservation laws

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= \mathcal{F}^{\nu\lambda}\tilde{J}_\lambda \\ \nabla_\mu \tilde{J}^\mu &= CE.B \end{aligned} \tag{3.159}$$

where

$$\begin{aligned} & E^\mu = \mathcal{F}^{\mu\nu}u_\nu, \quad B^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu \mathcal{F}_{\rho\sigma} \\ & \omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu \nabla_\rho u_\sigma. \end{aligned} \tag{3.160}$$

For computational simplicity, we shall take thermodynamic variables temperature T and $\nu = \frac{\mu}{T}$ as the independent ones. Some useful formulas that are used in computation are

$$\begin{aligned} & \nabla_\mu \nu = \frac{E_\mu}{T}, \quad \nabla_\mu P = qE_\mu + \frac{\epsilon + P}{T}\nabla_\mu T \\ & \left.\frac{\partial P}{\partial T}\right|_\nu = \frac{\epsilon+P}{T}, \quad \left.\frac{\partial P}{\partial \nu}\right|_T = qT \\ & \nabla_\mu \omega^\mu = -\frac{2}{T}\omega_\mu \nabla^\mu T, \quad \nabla_\mu B^\mu = -2\omega_\mu E^\mu - \frac{1}{T}B_\mu \nabla^\mu T. \end{aligned} \tag{3.161}$$

Now using formulas in equation 3.161, it is straight forward to evaluate the scalar equations namely

$$\begin{aligned} u_\nu \nabla_\mu T^{\mu\nu} &= u_\nu \mathcal{F}^{\nu\lambda}\tilde{J}_\lambda \\ \nabla_\mu \tilde{J}^\mu &= CE.B. \end{aligned} \tag{3.162}$$

On setting coefficients of independent data in 3.162, one obtains

$$\begin{aligned} & T\partial_T(\xi_\omega + qb_1) = 2(\xi_\omega + qb_1), \quad T\partial_T(\xi_B + qb_2) = (\xi_B + qb_2) \\ & \partial_\nu(\xi_\omega + qb_1) = 2T(\xi_B + qb_2), \quad \partial_\nu(\xi_B + qb_2) = CT \\ & \partial_\nu[(\epsilon+P)b_2] = T(\xi_B + qb_2), \quad \partial_T[(\epsilon+P)b_2] = \frac{2}{T}(\epsilon+P)b_2 \\ & \partial_\nu[(\epsilon+P)b_1] = T(\xi_\omega + qb_1) + 2T(\epsilon+P)b_2, \quad \partial_T[(\epsilon+P)b_1] = \frac{3}{T}(\epsilon+P)b_1 \\ & \alpha_1 = \alpha_2 = b_3 = b_4 = 0. \end{aligned} \tag{3.163}$$

The vector equation $\mathcal{P}_{\mu\sigma}\nabla_\mu T^{\mu\nu} = \mathcal{P}_{\mu\sigma}\mathcal{F}^{\nu\lambda}\tilde{J}_\lambda$ gives only one new constraint, which is given by

$$2b_2 = \frac{(\xi_\omega + qb_1)}{\epsilon + P}. \tag{3.164}$$

On solving 3.163, one obtains solution for $b's$ and $\xi's$ but with four arbitrary constants. Now using 3.164 one can eliminate one of the constants in terms of other one. Finally we obtain

$$\begin{aligned} b_1 &= \frac{T^3}{\epsilon + P}\big(\frac{2}{3}\nu^3 C + 2\nu^2 z_0 + 4\nu z_2 + z_1\big), \\ b_2 &= \frac{T^2}{\epsilon + P}\big(\frac{1}{2}\nu^2 C + \nu z_0 + z_2\big) \end{aligned} \tag{3.165}$$

and

$$\begin{aligned} a_1 &= C\nu^2 T^2\big(1 - \frac{2q}{3(\epsilon + P)}\nu T\big) + T^2\big[(2\nu z_0 + 2z_2) - \frac{qT}{\epsilon + P}(2\nu^2 z_0 + 4\nu z_2 + z_1)\big], \\ a_2 &= C\nu T\big(1 - \frac{q}{2(\epsilon + P)}\nu T\big) + T\big(z_0 - \frac{qT}{\epsilon + P}(\nu z_0 + z_2)\big). \end{aligned} \tag{3.166}$$

Now identifying $z_0 = 2C_0$, $z_2 = C_2$ and $z_1 = 4C_1$ we see that equations 3.165, 3.166 are exactly same as equations 3.75, 3.76.

3.8.2 First order parity odd charged fluid dynamics from equilibrium in 2+1 dimension

In this appendix we shall derive the constraints on parity odd charged fluid dynamics in 2+1 dimension at first order using just the assumption that there exists a equilibrium solution. As discussed in §3.4, in this case there are 4 transport coefficients and there are 6 corrections to the fluid fields. In §3.4, we were able to express all these 10 functions in terms of 2 arbitrary functions in the action (3.94). This implies that among these 10 functions only 2 are independent which in turn implies there should exist 8 relations among these 10 functions. In this appendix we shall present these 8 relations which follows just by demanding that there exists a equilibrium solution.

We consider the corrections to the fluid fields as in (3.91) and write down the first order corrections to the stress tensor and the charge current. The equation of motion of fluid dynamics are given by

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= \mathcal{F}^{\nu\lambda} J_\lambda \\ \nabla_\mu J^\mu &= 0 \end{aligned} \tag{3.167}$$

Note in particular that the charge current is conserved even in the presence of a background gauge field due to the absence of any anomaly in 2+1 dimension.

Now the scalar equations $\nabla_\mu J^\mu = 0$ and $u^{(0)}_\nu \left(\nabla_\mu T^{\mu\nu} - \mathcal{F}^{\nu\lambda} J_\lambda\right) = 0$ yields the following constraints respectively

$$\begin{aligned}
&T\frac{\partial}{\partial T}\left(\tilde{\chi}_E + \rho\xi_E\right) - T\frac{\partial}{\partial \mu}\left(\tilde{\chi}_T + \rho\xi_T\right) + \mu\frac{\partial}{\partial \mu}\left(\tilde{\chi}_E + \rho\xi_E\right) = 0, \\
&T\frac{\partial}{\partial T}\left((\epsilon + P)\xi_E\right) - T\frac{\partial}{\partial \mu}\left((\epsilon + P)\xi_T\right) + \mu\frac{\partial}{\partial \mu}\left((\epsilon + P)\xi_E\right) + T(\tilde{\chi}_T + \rho\xi_T) = 0.
\end{aligned} \tag{3.168}$$

The vector fluid equations $P^{(0)}_{\rho\nu} \left(\nabla_\mu T^{\mu\nu} - \mathcal{F}^{\nu\lambda} J_\lambda\right) = 0$, yields the rest of the 6 constraints

$$\begin{aligned}
\tilde{\chi}_B &= \frac{\partial P}{\partial T}\tau_B + \frac{\partial P}{\partial \mu} m_B, \\
\tilde{\chi}_\Omega &= \frac{\partial P}{\partial T}\tau_\Omega + \frac{\partial P}{\partial \mu} m_\Omega, \\
(\epsilon + P)\ \xi_E &= \frac{\partial \rho}{\partial T}\tau_\Omega + \frac{\partial \rho}{\partial \mu} m_\Omega, \\
(\epsilon + P)\ T\ \xi_T &= \frac{\partial \epsilon}{\partial T}\tau_\Omega + \frac{\partial \epsilon}{\partial \mu} m_\Omega, \\
\tilde{\chi}_E + \rho\xi_E &= \frac{\partial \rho}{\partial T}\tau_B + \frac{\partial \rho}{\partial \mu} m_B, \\
T\ \left(\tilde{\chi}_T + \rho\xi_T\right) &= \frac{\partial \epsilon}{\partial T}\tau_B + \frac{\partial \epsilon}{\partial \mu} m_B.
\end{aligned} \tag{3.169}$$

Note that the first two constraints in (3.169) are identical to the constraints (3.96) obtained from comparison with most general equilibrium action in §3.4.3. It is straightforward to show that the rest of the constraints in (3.168) and (3.169) are solved by the (3.97), (3.98), (3.99) and (3.100).

3.8.3 Second order uncharged fluid dynamics from equilibrium in 3+1 dimensions

In this appendix we will do a similar computation as done in last two appendices for 3+1 dimensional uncharged fluids at second order. The non-trivial second order (stress tensor conservation equation orthogonal to fluid velocity) equation is,

$$\nabla_i(T_2 e^\sigma) + e^\sigma \left(\sum_{n=1}^{2} v_n V_n^j + \tilde{v}\tilde{V}\right) f_{ij} + \frac{e^\sigma}{P_T}\nabla_\mu \Pi_i^\mu = 0 \tag{3.170}$$

Since, temperature correction is a scalar, we can assume the most generalized temperature correction to be of the following form,

$$T_2 e^\sigma = \sum_{m=1}^{4} t_m S_m + \tilde{t}\tilde{S} \tag{3.171}$$

Four dimensional divergence can be expressed as,

$$\begin{aligned}\nabla_\mu \Pi^\mu_\nu &= \frac{1}{\sqrt{-g_4}}\partial_\mu\left(\sqrt{-g_4}\tilde{g}^{\mu\alpha}\Pi_{\alpha\nu}\right) - \frac{1}{2}\partial_\nu(g_{\alpha\beta})\Pi^{\alpha\beta} \\ &= \nabla^i\Pi_{ik} + \nabla^i\sigma\Pi_{ik} \\ &= (\alpha_A - T\partial_T\alpha_A)\nabla^m\sigma\Pi^A_{mi} + \alpha_A\nabla^m\Pi^A_{mi},\end{aligned} \tag{3.172}$$

where, we have expressed the two derivative correction to equilibrium stress tensor Π of 3.127 in a compact form as, ($\Pi_{ij} = \alpha_A\Pi^A_{ij}, \quad A = 1, 11$). Using following simple derivative formulae

$$\begin{aligned}&\nabla^i\big(R_{ik} - \frac{R}{2}g_{ik}\big) = 0, \quad \nabla^i(\nabla_i\nabla_k - g_{ik}\nabla^2)\sigma = \nabla^i R_{ik}, \\ &\nabla^i(\nabla_i\sigma\nabla_k\sigma - g_{ik}(\nabla\sigma)^2) = \nabla^2\sigma\nabla_k\sigma, \\ &\nabla^i\big((f_i{}^j f_{jk} + \frac{f^2}{4}g_{ik})e^{2\sigma}\big) = e^{2\sigma}\big((\nabla^i f_{ij})f^j{}_k + 2\nabla^i\sigma(f_i{}^j f_{jk} + \frac{f^2}{4}g_{ik})\big),\end{aligned} \tag{3.173}$$

we solve for complete equilibrium solution. In the equation 3.170, we get following three different kinds of terms in parity even sector

$$\nabla^i(\text{Tensor})_{ik}, \quad \nabla_k(\text{Scalar}), \quad \nabla_k\sigma(\text{Scalar}), \tag{3.174}$$

and following four kinds of terms in the parity odd sector,

$$\begin{aligned}&\epsilon^{mkl}f_{kl}f_{ji}f^j_m, \ \epsilon_{imn}f^{mn}\nabla^2\sigma, \ \epsilon_{mni}\nabla^2 f^{mn} \\ &\epsilon^{mnl}\nabla_i(\nabla_m\sigma f_{nl}), \epsilon^{mnl}\nabla_i\sigma\nabla_m\sigma f_{nl}\end{aligned} \tag{3.175}$$

Setting the coefficients of ∇_k(Scalar) to zero, we get the temperature correction as [33],

$$t_1 = -\frac{b_1}{P_T}, \ t_2 = -\frac{b_2}{P_T}, \ t_4 = -\frac{b_4}{P_T}, \ t_3 = -\frac{b_3 + \frac{1}{2}(a_2 - T\partial_T a_2)}{P_T}. \tag{3.176}$$

Setting the coefficients of the other terms to zero and using 3.176, we get, the velocity corrections as

$$v_1 = \frac{3a_4 - T\partial_T a_4}{T^2 P_T}, \quad v_2 = \frac{a_4}{T^2 P_T} \tag{3.177}$$

and the relations among the transport coefficients as given in 3.136. Similarly, setting the coefficients of the independent terms in the parity odd sector to zero, we get all parity odd coefficients zero, that is

$$\tilde{t} = \tilde{v} = \delta_1 = \delta_4 = \delta_5 = 0.$$

[33] we have used $\nabla^i\sigma\nabla_i\nabla_k\sigma = \frac{1}{2}\nabla_k(\nabla\sigma)^2$.

4 Anomalous charged fluids in 1+1d from equilibrium partition function

4.1 Introduction

In this section we study the anomalous charged fluid dynamics in 1+1 dimensions using the equilibrium partition function method discused in detail in the previous section. This system has earlier been studied in [34] using the second law of thermodynamics as well as from an action point of view. In this section we write down the equilibrium partition function for this system at zero derivative order which reproduces the anomalous charge conservation and on comparison with the most general constitutive relations in fluid dynamics, gives the results obtained in [34].

4.2 1+1d parity violating charged fluid dynamics

Consider the parity violating charged fluids in 1+1 dimensions with background metric and gauge field

$$\begin{aligned} ds^2 &= -e^{2\sigma}(dt + a_1 dx)^2 + g_{11}dx^2 \\ \mathcal{A} &= \mathcal{A}_0 dt + \mathcal{A}_1 dx^1 \quad . \end{aligned} \tag{4.1}$$

The equations of motion are the following anomalous conservation laws

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= \mathcal{F}^{\nu\lambda}\tilde{J}_\lambda \\ \nabla_\mu \tilde{J}^\mu &= c\epsilon^{\mu\nu}\mathcal{F}_{\mu\nu} \\ \nabla_\mu J^\mu &= \frac{c}{2}\epsilon^{\mu\nu}\mathcal{F}_{\mu\nu} \end{aligned} \tag{4.2}$$

here $\tilde{J}$, J are covariant and consistent currents respectively ([35], see also [36]).

The most general partition function consistent with Kaluza-Klein gauge invariance[34], diffeomorphism along the spatial direction and $U(1)$ gauge invariance upto anomaly is

$$\begin{aligned} \mathcal{W} &= \mathcal{W}_{inv} + \mathcal{W}_{anom} \\ \mathcal{W}_{inv} &= C_1 T_0 \int A_1 dx - C_2 T_0 \int a_1 dx \\ \mathcal{W}_{anom} &= -\frac{C}{T_0}\int A_0 A_1 dx \end{aligned} \tag{4.4}$$

where C, C_1 and C_2 are constants independent of σ and A_0 and

$$A_0 = \mathcal{A}_0 + \mu_0, \quad A_i = \mathcal{A}_i - A_0 a_i. \tag{4.5}$$

Equation 4.4 is written in terms of A_i which unlike $\mathcal{A}_i$, are Kaluza-Klein gauge invariant.

[34]
$$V_i' = V_i - \partial_i\phi V_0, \quad (V')^0 = V^0 + \partial_i\phi V^i. \tag{4.3}$$

Field	C	P	T	CPT
σ	+	+	+	+
a_1	+	−	−	+
g_{11}	+	+	+	+
A_0	−	+	+	−
A_1	−	−	−	−

Table 14. Action of CPT

Under $U(1)$ gauge transformation $A_0 \to A_0,\ A_1 \to A_1 + \partial_1\phi$, we obtain[35]

$$\begin{aligned}\delta\mathcal{W}_{inv} &= 0\\ \delta\mathcal{W}_{anom} &= \frac{C}{T_0}\int \phi\ \partial_1 A_0 dx = -\frac{C}{2}\int d^2x\sqrt{-g_2}\ \phi\ \epsilon^{\mu\nu}\mathcal{F}_{\mu\nu}.\end{aligned} \tag{4.6}$$

Table (14) lists the action of 2 dimensional C, P and T on various fields. Requiring CPT invariance sets C_1 to zero since the term with coefficient C_1 is odd under CPT.

Now let us look at the most general constitutive relations allowed by symmetry in the parity violating case at zero derivative order. At this order, there are no gauge invariant parity odd scalar or tensor. But one can construct a gauge invariant vector [36]

$$\tilde{u}^\mu = \epsilon^{\mu\nu}u_\nu. \tag{4.8}$$

The most general allowed constitutive relations allowed by symmetry in Landau frame thus take the form

$$\begin{aligned}T^{\mu\nu} &= (\epsilon + p)u^\mu u^\nu + pg^{\mu\nu}\\ \tilde{J}^\mu &= qu^\mu + \xi_j\tilde{u}^\mu.\end{aligned} \tag{4.9}$$

4.2.1 Equilibrium from Partition Function

In this subsubsection we will use the equilibrium partition function (4.4) to obtain the stress tensor and charge current at zero derivative order. Setting C_1 to zero in (4.4) we have

$$\mathcal{W} = -\frac{C}{T_0}\int A_0 A_1 dx - C_2 T_0\int a_1 dx \tag{4.10}$$

[35]Since we are interested in time independent background fields, we consider only time independent gauge transformations.

[36]In components the parity odd vector is

$$\tilde{u}_0 = 0,\quad \tilde{u}^1 = \epsilon^{10}u_0 = \epsilon^1 \tag{4.7}$$

where $\epsilon^1 = e^\sigma\epsilon^{01} = \frac{1}{\sqrt{g_{11}}}$.

Type	Data	Evaluated at equilibrium $T = T_0 e^{-\sigma},\ \mu = e^{-\sigma} A_0,\ u^\mu = u^\mu_K$
Scalars	None	None
Vectors	u^μ	$\delta^\mu_0 e^{-\sigma}$
Pseudo-Vectors	$\epsilon_{\mu\nu} u^\nu$	ϵ_1
Tensors	None	None

Table 15. Zero derivative fluid data

Scalars	None
Vectors	none , none
Pseudo-Vectors	$\epsilon^1 f(\sigma, A_0)$
Tensors	None

Table 16. Zero derivative background data

Using the partition function (4.10) it is straightforward to compute the stress tensor and charge current[37] in equilibrium to be

$$\begin{aligned} &T_{00} = 0, \quad T^{11} = 0, \quad T^1_0 = e^{-\sigma} \epsilon^1 \left(-T_0^2 C_2 + C A_0^2 \right), \\ &J_0 = C \epsilon^1 A_1 e^{\sigma}, \quad J^1 = -C \epsilon^1 e^{-\sigma} A_0. \end{aligned} \tag{4.12}$$

The covariant current ($\tilde{J}^\mu$) can be obtained from the consistent current (J^μ) by an appropriate shift as follows

$$\tilde{J}^\mu = J^\mu + J^\mu_{sh}, \quad J^\mu_{sh} = C \epsilon^{\mu\nu} A_\nu. \tag{4.13}$$

In components the covariant current is then

$$\tilde{J}_0 = 0, \quad \tilde{J}^1 = -2 C e^{-\sigma} \epsilon^1 A_0. \tag{4.14}$$

4.2.2 Equilibrium from Hydrodynamics

We are interested in the stationary equilibrium solutions to conservation equations corresponding to the constitutive relations (4.9). The equilibrium solution in the parity even sector in background (4.1) at zero derivative order is

$$u^\mu = u^\mu_{(0)} = e^{-\sigma}(1, 0), \quad T = T_0 e^{-\sigma}, \quad \mu = A_0 e^{-\sigma}. \tag{4.15}$$

[37]

$$\begin{aligned} T_{00} &= -\frac{T_0 e^{2\sigma}}{\sqrt{-g_{(p+1)}}} \frac{\delta W}{\delta \sigma}, \quad T^i_0 = \frac{T_0}{\sqrt{-g_{(p+1)}}} \left(\frac{\delta W}{\delta a_i} - A_0 \frac{\delta W}{\delta A_i} \right), \\ T^{ij} &= -\frac{2 T_0}{\sqrt{-g_{(p+1)}}} g^{il} g^{jm} \frac{\delta W}{\delta g^{lm}}, \quad J_0 = -\frac{e^{2\sigma} T_0}{\sqrt{-g_{(p+1)}}} \frac{\delta W}{\delta A_0}, \quad J^i = \frac{T_0}{\sqrt{-g_{(p+1)}}} \frac{\delta W}{\delta A_i}. \end{aligned} \tag{4.11}$$

where, for instance, the derivative w.r.t A_0 is taken at constant σ, a_i, A_i, g^{ij}, T_0 and μ_0. See [36] for details.

Since there are no gauge invariant parity odd scalars in table 16, temperature and chemical potential do not receive any correction. However, the fluid velocity in equilibrium receives correction as

$$u^\mu = u^\mu_{(0)} + b\epsilon^{\mu\nu}u^{(0)}_\nu. \tag{4.16}$$

From (4.9), (4.15) and (4.16) we get the parity odd correction to the equilibrium stress tensor and charge current, which receive contribution from correction to the constitutive relations as well as from correction to the equilibrium fluid velocity, to be

$$\begin{aligned}\delta T_{00} &= \delta J_0 = \delta T^{ij} = 0,\\ \delta T^1_0 &= -e^\sigma(\epsilon + P)b\epsilon^1,\\ \delta\tilde{J}^1 &= (qb + \xi_j)\epsilon^1.\end{aligned} \tag{4.17}$$

4.2.3 Constraints on Hydrodynamics

Comparing the non trivial components of the equilibrium stress tensor and charge current of (4.12) and (4.17) we find that the coefficient of velocity correction (4.16) is

$$b = -\frac{T^2}{\epsilon + p}\left(-C_2 + C\nu^2\right) \tag{4.18}$$

and the coefficient in correction to charge current (4.9) is

$$\xi_j = C\left(\frac{q\mu^2}{\epsilon + p} - 2\mu\right) - C_2\frac{qT^2}{\epsilon + p}. \tag{4.19}$$

where $\nu = \frac{\mu}{T} = \frac{A_0}{T_0}$.

The expressions (4.19) agree exactly with the results of [34] based on the requirement of positivity of the entropy current and effective action.

4.2.4 The Entropy Current

The equilibrium entropy can be obtained from the partition function using

$$\begin{aligned}S &= \frac{\partial}{\partial T_0}(T_0 \log Z)\\ &= -2C_2T_0\int \sqrt{g_{11}}\epsilon^1 a_1 dx\ .\end{aligned} \tag{4.20}$$

In this subsection we determine the constraints on the hydrodynamical entropy current J^μ_S from the requirement that (4.20) agree with the local integral

$$S = \int dx\sqrt{-g_2}J^0_S\ \ . \tag{4.21}$$

The most general form of the entropy current allowed by symmetry [38], at zero derivative order is

$$J^\mu_S = su^\mu + \xi_s\tilde{u}^\mu + h\epsilon^{\mu\nu}\mathcal{A}_\nu\ , \tag{4.22}$$

[38] Let us note that the entropy current need not be gauge invariant, see [36] for more details.

where h is a constant.

The parity odd correction to the entropy current in equilibrium, which receives contributions both from correction to the hydrodynamical entropy current and equilibrium velocity, is given by

$$J^0_S|_{correction} = s\delta u^0 + \xi_s \tilde{u}^0 + h\epsilon^{01}\mathcal{A}_1. \tag{4.23}$$

Now using

$$\nu = \frac{A_0}{T_0}, \quad \delta u^0 = -a_1 \delta u^1 = -b\epsilon^1 a_1, \quad \tilde{u}^0 = -\epsilon^1 a_1$$

the correction to the hydrodynamical entropy in equilibrium is given by

$$\int dx \sqrt{-g_2} J^0_s|_{correction} = \int dx \ e^{\sigma} \left((-sb - \xi_s)\epsilon^1 a_1 + h\epsilon^1 (A_1 + A_0 a_1) \right). \tag{4.24}$$

Comparing this expression with (4.20) and using (4.19) we find

$$\xi_s = C \frac{s\mu^2}{\epsilon + p} + C_2 T \left(1 + \frac{\rho\mu}{\epsilon + p} \right), \qquad h = 0. \tag{4.25}$$

This result is in precise agreement with those of [34].

5 Constraints on anomalous fluids in arbitrary even dimensions

5.1 Introduction

Anomalies are a fascinating set of phenomena exhibited by field theories and string theories. For the sake of clarity let us begin by distinguishing between three quite different phenomena bearing that name.

The first phenomenon is when a symmetry of a classical action fails to be a symmetry at the quantum level. One very common example of an anomaly of this kind is the breakdown of classical scale invariance of a system when we consider the full quantum theory. This breakdown results in *renormalization group flow*, i.e., a scale-dependence of physical quantities even in a classically scale-invariant theory. Often this classical symmetry cannot be restored without seriously modifying the content of the theory. Anomalies of this kind are often serve as a cautionary tale to remind us that the symmetries of a classical action like scale invariance will often not survive quantisation.

The second set of phenomena are what are termed as gauge anomalies. A system is said to exhibit a gauge anomaly if a particular classical gauge redundancy of the system is no more a redundancy at a quantum level. Since such redundancies are often crucial in eliminating unphysical states in a theory, a gauge anomaly often signifies a serious mathematical inconsistency in the theory. Hence this second kind of anomalies serve as a consistency criteria whereby we discard any theory exhibiting gauge anomaly as most probably inconsistent.

The third set of phenomena which we would be mainly interested in this work is when a genuine symmetry of a quantum theory is no more a symmetry when the theory is placed in a non-trivial background where we turn on sources for various operators in the theory. This lack of symmetry is reflected in the fact that the path integral with these sources turned on is no more invariant under the original symmetry transformations. If the sources are non-trivial gauge/gravitational backgrounds (corresponding to the charge/energy-momentum operators in the theory) the path integral is no more gauge-invariant. In fact as is well known the gauge transformation of the path-integral is highly constrained and the possible transformations are classified by the Wess-Zumino descent relations[39].

Note that unlike the previous two phenomena here we make no reference to any specific classical description or the process of quantisation and hence this kind of anomalies are well-defined even in theories with multiple classical descriptions (or theories with no known classical description). Unlike the first kind of anomalies the symmetry is simply recovered at the quantum level by turning off the sources. Unlike the gauge anomalies the third kind of anomalies do not lead to any inconsistency. In what follows when we speak of anomaly we will always have in mind this last kind of anomalies unless specified otherwise.

Anomalies have been studied in detail in the least few decades and their mathematical structure and phenomenological consequence for zero temperature/chemical potential situations are reasonably well-understood. However the anomaly related phenomena in finite

[39]The Wess-Zumino descent relations are dealt with in detail in various textbooks[37–39] and lecture notes [40, 41].

temperature setups let alone in non-equilibrium states are still relatively poorly understood despite their obvious relevance to fields ranging from solid state physics to cosmology. It is becoming increasingly evident that there are universal transport processes which are linked to anomalies present in a system and that study of anomalies provide a non-perturbative way of classifying these transport processes say in solid-state physics[**?**].

While the presence of transport processes linked to anomalies had been noticed before in a diversity of systems ranging from free fermions[40] to holographic fluids[41] a main advance was made in [3]. In that work it was shown using very general entropy arguments that the $U(1)^3$ anomaly coefficient in an arbitrary $3+1d$ relativistic field theory is linked to a specific transport process in the corresponding hydrodynamics. This argument has since then been generalised to finite temperature corrections [8, 16] and $U(1)^{n+1}$ anomalies in $d = 2n$ space time dimensions [8, 46]. In particular the author of [8] identified a rich structure to the anomaly-induced transport processes by writing down an underlying Gibbs-current which captured these processes in a succinct way. Later in a microscopic context in ideal Weyl gases, the authors of [30] identified this structure as emerging from an adiabatic flow of chiral states convected in a specific way in a given fluid flow.

While these entropy arguments are reasonably straightforward they appear somewhat non-intuitive from a microscopic field theory viewpoint. It is especially important to have a more microscopic understanding of these transport processes if one wants to extend the study of anomalies far away from equilibrium where one cannot resort to such thermodynamic arguments. So it is crucial to first rephrase these arguments in a more field theory friendly terms so that one may have a better insight on how to move far away from equilibrium.

Precisely such a field-theory friendly reformulation in $3+1d$ and $1+1d$ was found recently in the references [47] and [48] respectively. Our main aim in this section is to generalise their results to arbitrary even space time dimensions. So let us begin by repeating the basic physical idea behind this reformulation in the next few paragraphs.

Given a particular field theory exhibiting certain anomalies, one begins by placing that field theory in a time-independent gauge/gravitational background at finite temperature/chemical potential. We take the gauge/gravitational background to be spatially slowly varying compared to all other scales in the theory. Using this one can imagine integrating out all the heavy modes[42] in the theory to generate an effective Euler-Heisenberg type effective action for the gauge/gravitational background fields at finite temperature/chemical potential.

In the next step one expands this effective action in a spatial derivative expansion and

[40]It would be an impossible task to list all the references in the last few decades which have discovered (and rediscovered) such effects in free/weakly coupled theories in various disguises using a diversity of methods . See for example [42] for what is probably the earliest study in $3 + 1d$. See [30] for a recent generalisation to arbitrary dimensions.

[41]See for example [43–45] for some of the initial holographic results.

[42]Time-independence at finite temperature and chemical potential essentially means we are doing a Euclidean field theory. Unlike the Lorentzian field theory (which often has light-hydrodynamic modes) the Euclidean field theory has very few light modes except probably the Goldstone modes arising out of spontaneous symmetry breaking. We thank Shiraz Minwalla for emphasising this point.

then imposes the constraint that its gauge transformation be that fixed by the anomaly. This constrains the terms that can appear in the derivative expansion of the Euler-Heisenberg type effective action. As is clear from the discussion above, this effective action and the corresponding partition function have a clear microscopic interpretation in terms of a field-theory path integral and hence is an appropriate object in terms of which one might try to reformulate the anomalous transport coefficients.

The third step is to link various terms that appear in the partition function to the transport coefficients in the hydrodynamic equations. The crucial idea in this link is the realisation that the path integral we described above is essentially dominated by a time-independent hydrodynamic state (or more precisely a hydrostatic state). This means in particular that the expectation value of energy/momentum/charge/entropy calculated via the partition function should match with the distribution of these quantities in the corresponding hydrostatic state.

These distributions in turn depend on a subset of transport coefficients in the hydrodynamic constitutive relations which determine the hydrostatic state. In this way various terms that appear in the equilibrium partition function are linked to/constrain the transport coefficients crucial to hydrostatics. Focusing on just the terms in the path-integral which leads to the failure of gauge invariance we can then identify the universal transport coefficients which are linked to the anomalies. This gives a re derivation of various entropy argument results in a path-integral language thus opening the possibility that an argument in a similar spirit with Schwinger-Keldysh path integral will give us insight into non-equilibrium anomaly-induced phenomena.

Our main aim in this section is twofold - first is to carry through in arbitrary dimensions this program of equilibrium partition function thus generalising the results of [47, 48] and re deriving in a path-integral friendly language the results of [8, 46].

Our second aim is to clarify the relation between the Gibbs current studied in [8, 30] and the partition function of [47, 48]. Relating them requires some care on carefully distinguishing the consistent from covariant charge , the final result however is intuitive : the negative logarithm of the equilibrium partition function (times temperature) is simply obtained by integrating the equilibrium Gibbs free energy density (viz. the zeroeth component of the Gibbs free current) over a spatial hyper surface. This provides a direct and an intuitive link between the local description in terms of a Gibbs current vs. the global description in terms of the partition function.

This section is organised as follows. We will begin by mainly reviewing known results in subection §5.2. First we review the formalism/results of [8] in subsubsection§§5.2.1 where entropy arguments were used to constrain the anomaly-induced transport processes a Gibbs-current was written down which captured those processes in a succinct way. This is followed by subsubsection§§5.2.4 where we briefly review the relevant details of the equilibrium partition function formalism for fluids as developed in [47]. A recap of the relevant results in (3+1) and (1+1) dimensions[47, 48] and a comparison with results in this section are relegated to appendix 5.9.1.

Subsection §5.3 is devoted to the derivation of transport coefficients for $2n$ dimensional anomalous fluid using the partition function method. The next section§5.4 contains construction of entropy current for the fluid and the constraints on it coming from partition function. This mirrors similar discussions in [47, 48]. We then compare these results to the results of [8] presented before in subsubsection§§5.2.1 and find a perfect agreement.

Prodded by this agreement, we proceed in next subsection§5.5 to a deeper analysis of the relation between the two formalisms. We prove an intuitive relation whereby the partition function could be directly derived from the Gibbs current of [8] by a simple integration (after one carefully shifts from the covariant to the consistent charge).

This is followed by subsection §5.6 where we generalise all our results for multiple $U(1)$ charges. We perform a CPT invariance analysis of the fluid in subsection §5.7 and this imposes constraints on the fluid partition function.

Various technical details have been pushed to the appendices for the convenience of the reader. After the appendix 5.9.1 on comparison with previous partition function results in (3+1) and (1+1) dimensions, we have placed an appendix 5.9.4 detailing various specifics about the hydrostatic configuration considered in [47]. We then have an appendix 5.9.5 where we present the variational formulae to obtain currents from the partition function in the language of differential forms. This is followed by an appendix 5.9.6 on notations and conventions (especially the conventions of wedge product etc.).

5.2 Preliminaries

In this subsection we begin by reviewing and generalising various results from [8] where constraints on anomaly-induced transport in arbitrary dimensions were derived using adiabaticity (i.e., the statement that there is no entropy production associated with these transport processes). Many of the zero temperature results here were also independently derived by the authors of [46].

We will then review the construction of equilibrium partition function (free energy) for fluid in the rest of the subsection. The technique has been well explained in [47] and familiar readers can skip this part.

5.2.1 Adiabaticity and Anomaly induced transport

Hydrodynamics is a low energy (or long wavelength) description of a quantum field theory around its thermodynamic equilibrium. Since the fluctuations are of low energy, we can express physical data in terms of derivative expansions of fluid variables (fluid velocity $u(x)$, temperature $T(x)$ and chemical potential $\mu(x)$) around their equilibrium value.

The dynamics of the fluid is described by some conservation equations. For example, the conservation equations of the fluid stress-tensor or the fluid charge current. These are known as constitutive equations. The stress tensor and charged current of fluid can be expressed in terms of fluid variables and their derivatives. At any derivative order, a generic form of the stress tensor and charged current can be written demanding symmetry and thermodynamics of the underlying field theory. These generic expressions are known as constitutive relations.

As it turns out, validity of 2nd law of thermodynamics further constraints the form of these constitutive relations.

The author of [8] assumed the following form for the constitutive relations describing energy, charge and entropy transport in a fluid

$$\begin{aligned} T^{\mu\nu} &\equiv \varepsilon u^\mu u^\nu + pP^{\mu\nu} + q^\mu_{anom} u^\nu + u^\mu q^\nu_{anom} + T^{\mu\nu}_{diss} \\ J^\mu &\equiv q u^\mu + J^\mu_{anom} + J^\mu_{diss} \\ J^\mu_S &\equiv s u^\mu + J^\mu_{S,anom} + J^\mu_{S,diss} \end{aligned} \tag{5.1}$$

where u^μ is the velocity of the fluid under consideration which obeys $u^\mu u_\mu = -1$ when contracted using the space time metric $g_{\mu\nu}$. Further, $P^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$, pressure of the fluid is p and $\{\epsilon, q, s\}$ are the energy,charge and the entropy densities respectively. We have denoted by $\{q^\mu_{anom}, J^\mu_{anom}, J^\mu_{S,anom}\}$ the anomalous heat/charge/entropy currents and by $\{T^{\mu\nu}_{diss}, J^\mu_{diss}, J^\mu_{S,diss}\}$ the dissipative currents.

5.2.2 Equation for adiabaticity

A convenient way to describe adiabatic transport process is via a **covariant** anomalous Gibbs current $\left(\mathcal{G}^{Cov}_{anom}\right)^\mu$.

The adjective **covariant** refers to the fact that the Gibbs free energy and the corresponding partition function are computed by turning on chemical potential for the **covariant** charge. This is to be contrasted with the **consistent** partition function and the corresponding **consistent** anomalous Gibbs current $\left(\mathcal{G}^{Consistent}_{anom}\right)^\mu$.

Since this distinction is crucial let us elaborate this in the next few paragraphs - it is a fundamental result due to Noether that the continuous symmetries of a theory are closely linked to the conserved currents in that theory. Hence when the path integral fails to have a symmetry in the presence of background sources, there are two main consequences - first of all it directly leads to a modification of the corresponding charge conservation and a failure of Noether theorem. The second consequence is that various correlators obtained by varying the path integral are not gauge-covariant and a more general modifications of Ward identities occur.

A simple example is the expectation value of the current obtained by varying the path integral with respect to a gauge field (often termed the **consistent** current) as,

$$J^\mu_{Consistent} \equiv \frac{\partial S}{\partial \mathcal{A}_\mu}.$$

The consistent current is not covariant under gauge transformation.

As has been explained in great detail in [17] thus there exists another current in anomalous theories: the covariant current. The covariant current J^μ_{Cov} is a current shifted with respect to the consistent current by an amount J^μ_c. The shift is such that its gauge transformation is anomalous and it exactly cancels the gauge non invariant part of the consistent current. Thus, the covariant current is covariant under the gauge transformation, as suggested by its name.

The covariant Gibbs current describes the transport of Gibbs free energy when a chemical potential is turned on for the covariant charge. We will take a Hodge-dual of this covariant Gibbs current to get a $d-1$ form in d-space time dimensions. Let us denote this Hodge-dual by $\bar{\mathcal{G}}^{Cov}_{anom}$. The anomalous parts of charge/entropy/energy currents can be derived from this Gibbs current via thermodynamics

$$\begin{aligned}\bar{J}^{Cov}_{anom} &= -\frac{\partial \bar{\mathcal{G}}_{anom}}{\partial \mu}\\ \bar{J}^{Cov}_{S,anom} &= -\frac{\partial \bar{\mathcal{G}}_{anom}}{\partial T}\\ \bar{q}^{Cov}_{anom} &= \bar{\mathcal{G}}_{anom} + T\bar{J}_{S,anom} + \mu \bar{J}_{anom}\end{aligned} \tag{5.2}$$

Then according to [8] the condition for adiabaticity is

$$d\bar{q}^{Cov}_{anom} + \mathfrak{a}\wedge \bar{q}^{Cov}_{anom} - \mathcal{E}\wedge \bar{J}^{Cov}_{anom} = Td\bar{J}^{Cov}_{S,anom} + \mu d\bar{J}^{Cov}_{anom} - \mu\bar{\mathfrak{A}}^{Cov} \tag{5.3}$$

where $\mathfrak{a}, \mathcal{E}$ are the acceleration 1-form and the rest-frame electric field 1-form respectively defined via

$$\mathfrak{a} \equiv (u.\nabla)u_\mu \ dx^\mu \ , \quad \mathcal{E} \equiv u^\nu \mathcal{F}_{\mu\nu} dx^\mu$$

Further the rest frame magnetic field/vorticity 2-forms are defined by subtracting out the electric part from the gauge field strength and the acceleration part from the exterior derivative of velocity, viz.,

$$\mathcal{B} \equiv \mathcal{F} - u\wedge\mathcal{E} \ , \quad 2\omega \equiv du + u\wedge \mathfrak{a}$$

The symbol $\bar{\mathfrak{A}}^{Cov}$ is the d-form which is the Hodge dual of the rate at which the **covariant** charge is created due to anomaly,i.e.,

$$d\bar{J}^{Cov} = \bar{\mathfrak{A}}^{Cov}$$

where $\bar{J}^{Cov}$ is the entire covariant charge current including both the anomalous and the non-anomalous pieces. For simplicity we have restricted our attention to a single U(1) global symmetry which becomes anomalous on a non-trivial background.

In terms of the Gibbs current , we can write the adiabiticity condition (5.3) as,

$$d\bar{\mathcal{G}}^{Cov}_{anom} + \mathfrak{a}\wedge\bar{\mathcal{G}}^{Cov}_{anom} + \mu\bar{\mathfrak{A}}^{Cov} = (dT + \mathfrak{a}T)\wedge \frac{\partial \bar{\mathcal{G}}^{Cov}_{anom}}{\partial T} + (d\mu + \mathfrak{a}\mu - \mathcal{E})\wedge\frac{\partial \bar{\mathcal{G}}^{Cov}_{anom}}{\partial \mu} \tag{5.4}$$

5.2.3 Construction of the polynomial $\mathfrak{F}^\omega_{anom}$

The main insight of [8] is that in d-space time dimensions the solutions of this equation are most conveniently phrased in terms of a single homogeneous polynomial of degree $n+1$ in temperature T and chemical potential μ.

Following the notation employed in [30] we will denote this polynomial as $\mathfrak{F}^\omega_{anom}[T,\mu]$. As was realised in [30], this polynomial is often closely related to the anomaly polynomial of the system[43] . More precisely, for a variety of systems we have a remarkable relation between $\mathfrak{F}^{\omega}_{anom}[T,\mu]$ and the anomaly polynomial $\mathcal{P}_{anom}\left[\mathcal{F},\mathfrak{R}\right]$

$$\mathfrak{F}^{\omega}_{anom}[T,\mu] = \mathcal{P}_{anom}\left[\mathcal{F}\mapsto\mu, p_1(\mathfrak{R})\mapsto -T^2, p_{k>1}(\mathfrak{R})\mapsto 0\right] \tag{5.5}$$

Let us be more specific : on a $(2n-1)+1$ dimensional space time consider a theory with

$$\mathfrak{F}^{\omega}_{anom}[T,\mu] = \mathcal{C}_{anom}\mu^{n+1} + \sum_{m=0}^{n} C_m T^{m+1}\mu^{n-m} \tag{5.6}$$

Assuming that the theory obeys the replacement rule (5.5) such a $\mathfrak{F}^{\omega}_{anom}[T,\mu]$ can be obtained from an anomaly polynomial[44]

$$\mathcal{P}_{anom} = \mathcal{C}_{anom}\mathcal{F}^{n+1} + \sum_{m=0}^{n} C_m\left[-p_1(\mathfrak{R})\right]^{\frac{m+1}{2}}\mathcal{F}^{n-m} + \dots \tag{5.7}$$

where we have presented the terms which do not involve the higher Pontryagin forms. Restricting our attention only to the $U(1)^{n+1}$ anomaly (and ignoring the mixed/pure gravitational anomalies) we can write

$$\begin{aligned} d\bar{J}_{Consistent} &= \mathcal{C}_{anom}\mathcal{F}^n \\ d\bar{J}_{Cov} &= (n+1)\mathcal{C}_{anom}\mathcal{F}^n \end{aligned} \tag{5.8}$$

and their difference is given by

$$\bar{J}_{Cov} = \bar{J}_{Consistent} + n\mathcal{C}_{anom}\hat{\mathcal{A}}\wedge\mathcal{F}^{n-1} \tag{5.9}$$

The solution of (5.4) corresponding to the homogeneous polynomial (5.6) is given by

$$\begin{aligned} \bar{\mathcal{G}}^{Cov}_{anom} = C_0 T\hat{\mathcal{A}}\wedge\mathcal{F}^{n-1} + \sum_{m=1}^{n}\Bigg[&\mathcal{C}_{anom}\binom{n+1}{m+1}\mu^{m+1} \\ &+\sum_{k=0}^{m} C_k\binom{n-k}{m-k}T^{k+1}\mu^{m-k}\Bigg](2\omega)^{m-1}\mathcal{B}^{n-m}\wedge u \end{aligned} \tag{5.10}$$

[43] We remind the reader that the anomalies of a theory living in $d=2n$ spacetime dimensions is succinctly captured by a $2n+2$ form living in *two dimensions higher*. This $2n+2$ form called the anomaly polynomial (since it is a polynomial in external/background field strengths $\mathcal{F}$ and $\mathfrak{R}$) is related to the variation of the effective action δW via *the descent relations*

$$\mathcal{P}_{anom} = d\Gamma_{CS}\ , \qquad \delta\Gamma_{CS} = d\delta W$$

We will refer the reader to various textbooks[37–39] and lecture notes [40, 41] for a more detailed exposition.

[44] Since all relativistic theories only have integer powers of Pontryagin forms the constants C_m should vanish whenever m is even. As we shall see later that another way to arrive at the same conclusion is to impose CPT invariance.

Here $\hat{\mathcal{A}}$ is the $U(1)$ gauge-potential 1-form in some gauge with $\mathcal{F} \equiv d\hat{\mathcal{A}}$ being its field-strength 2-form. Further, $\mathcal{B}, \omega$ are the rest frame magnetic field/vorticity 2-forms and T, μ are the local temperature and chemical potential respectively. They obey

$$(d\mathcal{B}) \wedge u = -(2\omega) \wedge \mathcal{E} \wedge u \ , \quad d(2\omega) \wedge u = (2\omega) \wedge \mathfrak{a} \wedge u \tag{5.11}$$

Using these equations it is a straightforward exercise to check that (5.10) furnishes a solution to (5.4).

We will make a few remarks before we proceed to derive charge/entropy/energy currents from this Gibbs current. Note that if one insists that the Gibbs current be gauge-invariant then we are forced to put $C_0 = 0$ - in the solution presented in [8] this condition was implicitly assumed and the C_0 term was absent. The authors of [47] later relaxed this assumption insisting gauge-invariance only for the covariant charge/energy currents. Since we would be interested in comparison with the results derived in [47] it is useful to retain the C_0 term.

Now we use thermodynamics to obtain the charge current as

$$\begin{aligned} &\bar{J}^{Cov}_{anom} \\ &= -\sum_{m=1}^{n} \left[(m+1)\mathcal{C}_{anom} \binom{n+1}{m+1} \mu^m \right. \\ &\quad \left. + \sum_{k=0}^{m} (m-k) C_k \binom{n-k}{m-k} T^{k+1} \mu^{m-k-1} \right] (2\omega)^{m-1} \mathcal{B}^{n-m} \wedge u \end{aligned} \tag{5.12}$$

and the entropy current is given by

$$\begin{aligned} \bar{J}^{Cov}_{S,anom} &= -C_0 \hat{\mathcal{A}} \wedge \mathcal{F}^{n-1} \\ &\quad - \sum_{m=1}^{n} \sum_{k=0}^{m} (k+1) C_k \binom{n-k}{m-k} T^k \mu^{m-k} (2\omega)^{m-1} \mathcal{B}^{n-m} \wedge u \end{aligned} \tag{5.13}$$

The energy current is given by

$$\begin{aligned} &\bar{q}^{Cov}_{anom} \\ &= -\sum_{m=1}^{n} m \left[\mathcal{C}_{anom} \binom{n+1}{m+1} \mu^{m+1} \right. \\ &\quad \left. + \sum_{k=1}^{m} C_k \binom{n-k}{m-k} T^{k+1} \mu^{m-k} \right] (2\omega)^{m-1} \mathcal{B}^{n-m} \wedge u \end{aligned} \tag{5.14}$$

These currents satisfy an interesting Reciprocity type relationship noticed in [8]

$$\frac{\delta \bar{q}^{Cov}_{anom}}{\delta \mathcal{B}} = \frac{\delta \bar{J}^{Cov}_{anom}}{\delta (2\omega)} \tag{5.15}$$

While this is a solution in a generic frame one can specialise to the Landau frame (where the velocity is defined via the energy current) by a frame transformation

$$
\begin{aligned}
u^\mu &\mapsto u^\mu - \frac{q^\mu_{anom}}{\epsilon + p}, \\
J^\mu_{anom} &\mapsto J^\mu_{anom} - q\frac{q^\mu_{anom}}{\epsilon + p}, \\
J^\mu_{S,anom} &\mapsto J^\mu_{S,anom} - s\frac{q^\mu_{anom}}{\epsilon + p}, \\
q^\mu_{anom} &\mapsto 0
\end{aligned}
\tag{5.16}
$$

to get

$$
\begin{aligned}
\bar{J}^{Cov,Landau}_{anom} &= \sum_{m=1}^{n} \xi_m (2\omega)^{m-1} \mathcal{B}^{n-m} \wedge u \\
\bar{J}^{Cov,Landau}_{S,anom} &= \sum_{m=1}^{n} \xi^{(s)}_m (2\omega)^{m-1} \mathcal{B}^{n-m} \wedge u + \zeta\ \hat{\mathcal{A}} \wedge \mathcal{F}^{n-1}
\end{aligned}
\tag{5.17}
$$

where

$$
\begin{aligned}
\xi_m &\equiv \left[m\frac{q\mu}{\epsilon + p} - (m+1) \right] \mathcal{C}_{anom} \binom{n+1}{m+1} \mu^m \\
&\quad + \sum_{k=0}^{m} \left[m\frac{q\mu}{\epsilon + p} - (m-k) \right] C_k \binom{n-k}{m-k} T^{k+1} \mu^{m-k-1} \\
\xi^{(s)}_m &\equiv \left[m\frac{sT}{\epsilon + p} \right] \mathcal{C}_{anom} \binom{n+1}{m+1} T^{-1} \mu^{m+1} \\
&\quad + \sum_{k=0}^{m} \left[m\frac{sT}{\epsilon + p} - (k+1) \right] C_k \binom{n-k}{m-k} T^k \mu^{m-k} \\
\zeta &= -C_0
\end{aligned}
\tag{5.18}
$$

Often in the literature the entropy current is quoted in the form

$$
\bar{J}^{Cov,Landau}_{S,anom} = -\frac{\mu}{T} \bar{J}^{Cov,Landau}_{anom} + \sum_{m=1}^{n} \chi_m (2\omega)^{m-1} \mathcal{B}^{n-m} \wedge u + \zeta\ \hat{\mathcal{A}} \wedge \mathcal{F}^{n-1}
\tag{5.19}
$$

where

$$
\begin{aligned}
\zeta &= -C_0 \\
\chi_m &\equiv \xi^{(s)}_m + \frac{\mu}{T} \xi_m \\
&= -\mathcal{C}_{anom} \binom{n+1}{m+1} T^{-1} \mu^{m+1} - \sum_{k=0}^{m} C_k \binom{n-k}{m-k} T^k \mu^{m-k}
\end{aligned}
\tag{5.20}
$$

where we have used the thermodynamic relation $sT + q\mu = \epsilon + p$. By looking at (5.10) we recognise these to be the coefficients occurring in the anomalous Gibbs current :

$$\bar{\mathcal{G}}^{Cov}_{anom} = -T\left[\sum_{m=1}^{n} \chi_m (2\omega)^{m-1}\mathcal{B}^{n-m}\wedge u + \zeta\, \hat{\mathcal{A}}\wedge \mathcal{F}^{n-1}\right] \tag{5.21}$$

In fact this is to be expected from basic thermodynamic considerations : the above equation is a direct consequence of the relation $G = -T(S + \frac{\mu}{T}Q - \frac{U}{T})$ and the fact that energy current receives no anomalous contributions in the Landau frame.

This ends our review of the main results of [8] adopted to our purposes. Our aim in the rest of the section would be to derive all these results purely from a partition function analysis.

5.2.4 Equilibrium Partition Function

In this subsubsection we review (and extension) an alternative approach to constrain the constitutive relations, namely by demanding the existence of an equilibrium partition function (or free energy) for the fluid as described in [47, 48] [45].

Let us keep the fluid in a special background such that the background metric has a time like killing vector and the background gauge field is time independent. Any such metric can be put into the following Kaluza-Klein form

$$\begin{aligned} ds^2 &= -e^{2\sigma}(dt + a_i dx^i)^2 + g_{ij}dx^i dx^j, \\ \hat{\mathcal{A}} &= \mathcal{A}_0 dt + \mathcal{A}_i dx^i \end{aligned} \tag{5.22}$$

here $i, j\ \epsilon\ \ (1, 2 \ldots 2n-1)$ are the spatial indices. We will often use the notation $\gamma \equiv e^{-\sigma}$ for brevity. This background has a time-like killing vector ∂_t and let $u^\mu_k = (e^{-\sigma}, 0, 0, \ldots)$ be the unit normalized vector in the killing direction so that

$$u^\mu_k \partial_\mu = \gamma\partial_t \quad \text{and} \quad u_k = -\gamma^{-1}(dt + a)$$

In the corresponding Euclidean field theory description of equilibrium, the imaginary time direction would be compactified into a thermal circle with the size of circle being the inverse temperature of the underlying field theory. In the 2n-1 dimensional compactified geometry, the original 2n background field breaks as follow

- metric($g_{\mu\nu}$) : scalar(σ), KK gauge field(a_i), lower dimensional metric(g_{ij}).
- gauge field($\hat{\mathcal{A}}_\mu$) : scalar($\mathcal{A}_0$), gauge field($\mathcal{A}_i$)

Under this KK type reduction the 2n dimensional diffeomorphisms breaks up into 2n-1 dimensional diffeomorphisms and KK gauge transformations. The components of 2n dimensional tensors which are KK-gauge invariant in 2n-1 dimensions are those with lower

[45] For similar discussions, see for example [49, 50].

time(killing direction) and upper space indices. Given a 1-form J we will split it in terms of KK-invariant components as

$$J = J_0(dt + a_i dx^i) + g_{ij} J^i dx^j$$

Other KK non-invariant components of J are given by

$$\begin{aligned} J^0 &= -\left[\gamma^2 J_0 + a_i J^i\right] \\ J_i &= g_{ij} J^j + a_i J_0 \end{aligned} \tag{5.23}$$

To take care of KK gauge invariance we will identify the lower dimensional U(1) gauge field (denoted by non script letters) as follows

$$\begin{aligned} &A_0 = \mathcal{A}_0 + \mu_0, \quad A^i = \mathcal{A}^i \\ \Rightarrow\ &A_i = \mathcal{A}_i - \mathcal{A}_0 a_i \quad \text{and} \\ &F_{ij} = \partial_i A_j - \partial_j A_i = \mathcal{F}_{ij} - A_0 f_{ij} - (\partial_i A_0\ a_j - \partial_j A_0\ a_i). \end{aligned} \tag{5.24}$$

where $f_{ij} \equiv \partial_i a_j - \partial_j a_i$ and μ_0 is a convenient constant shift in $\mathcal{A}_0$ which we will define shortly. We can hence write

$$\hat{\mathcal{A}} = \mathcal{A}_0 dt + \mathcal{A} = A_0(dt + a_i dx^i) + A_i dx^i - \mu_0 dt$$

We are now working in a general gauge - often it is useful to work in a specific class of gauges : one class of gauges we will work on is obtained from this generic gauge by performing a gauge transformation to remove the $\mu_0 dt$ piece. We will call these class of gauges as the 'zero μ_0' gauges. In these gauges the new gauge field is given in terms of the old gauge field via

$$\hat{\mathcal{A}}_{\mu_0=0} \equiv \hat{\mathcal{A}} + \mu_0 dt$$

We will quote all our consistent currents in this gauge. The field strength 2-form can then be written as

$$\mathcal{F} \equiv d\hat{\mathcal{A}} = dA + A_0 da + dA_0 \wedge (dt + a)$$

We will now focus our attention on the **consistent** equilibrium partition function which is the Euclidean path-integral computed on space adjoined with a thermal circle of length $1/T_0$. We will further turn on a chemical potential μ - since there are various different notions of charge in anomalous theories placed in gauge backgrounds we need to carefully define which of these notions we use to define the partition function[46]. While in the previous subsection we used the chemical potential for a **covariant** charge and the corresponding **covariant** Gibbs free-energy following [8] , in this subsection we will follow [47] in using a chemical potential for the consistent charge to define the partition function. This distinction has to be kept in mind while making a comparison between the two formalisms as we will elaborate later in section§5.5.

[46] See, for example, section§3 of [**?**] for a discussion of some of the subtleties.

The consistent partition function $Z_{Consistent}$ that we write down will be the most general one consistent with 2n-1 dimensional diffeomorphisms, KK gauge invariance and the U(1) gauge invariance up to anomaly. It is a scalar S constructed out of various background quantities and their derivatives. The most generic form of the partition function is

$$W = \ln Z_{Consistent} = \int d^{2n-1}x\sqrt{g_{2n-1}}S(\sigma, A_0, a_i, A_i, g_{ij}). \tag{5.25}$$

Given this partition function, we compute various components of the stress tensor and charged current from it. The KK gauge invariant components of the stress tensor $T_{\mu\nu}$ and charge current J_μ can then be obtained from the partition function as follows [47],

$$\begin{aligned} T_{00} &= -\frac{T_0 e^{2\sigma}}{\sqrt{-g_{2n}}}\frac{\delta W}{\delta\sigma}, \quad J_0^{Consistent} = -\frac{e^{2\sigma}T_0}{\sqrt{-g_{2n}}}\frac{\delta W}{\delta A_0}, \\ T_0^i &= \frac{T_0}{\sqrt{-g_{2n}}}\left(\frac{\delta W}{\delta a_i} - A_0\frac{\delta W}{\delta A_i}\right), \quad J^i_{Consistent} = \frac{T_0}{\sqrt{-g_{2n}}}\frac{\delta W}{\delta A_i}, \\ T^{ij} &= -\frac{2T_0}{\sqrt{-g_{2n}}}g^{il}g^{jm}\frac{\delta W}{\delta g^{lm}}. \end{aligned} \tag{5.26}$$

here $\{\sigma, a_i, g_{ij}, A_0, A_i\}$ are chosen independent sources, so the partial derivative w.r.t any of them in the above equations means that others are kept constant. We will sometimes use the above equation written in terms of differential forms - we will refer the reader to appendix 5.9.5 for the differential-form version of the above equations.

Next we parameterize the most generic equilibrium solution and constitutive relations for the fluid as,

$$\begin{aligned} &u(x) = u_0(x) + u_1(x), \quad T(x) = T_0(x) + T_1(x), \quad \mu(x) = \mu_0(x) + \mu_1(x), \\ &T_{\mu\nu} = (\epsilon + p)u_\mu u_\nu + p g_{\mu\nu} + \pi_{\mu\nu}, \quad J^\mu = q u^\mu + j^\mu_{diss}, \end{aligned} \tag{5.27}$$

where, $u_1, T_1, \mu_1, \pi_{\mu\nu}, j^\mu_{diss}$ are various derivatives of the background quantities. Note that we will work in Landau frame throughout.

These corrections are found by comparing the fluid stress tensor $T_{\mu\nu}$ and current J_μ in Eqn.(5.27) with $T_{\mu\nu}$ and J_μ in Eqn.(5.26) as obtained from the partition function. This exercise then constrains various non-dissipative coefficients that appear in the constitutive relations in Eqn.(5.27).

This then ends our short review of the formalism developed in [47]. In the next section we will apply this formalism to a theory with $U(1)^{n+1}$ anomaly in $d = 2n$ space time dimensions.

5.3 Anomalous partition function in arbitrary dimensions

Let us consider then a fluid in a $2n$ dimensional space time. The fluid is charged under a single $U(1)$ abelian gauge field $\mathcal{A}_\mu$. We will generalise to multiple abelian gauge fields later in section §5.6 and leave the non-abelian case for future study. We will continue to use the notation in the subsection §§5.2.1.

The consistent/covariant anomaly are then given by Eqn.(5.8) which can be written in components as

$$\begin{aligned}\nabla_\mu J^\mu_{Consistent} &= \mathcal{C}_{anom}\varepsilon^{\mu_1\nu_1\ldots\mu_n\nu_n}\partial_{\mu_1}\hat{\mathcal{A}}_{\nu_1}\ldots\partial_{\mu_n}\hat{\mathcal{A}}_{\nu_n}\\ &= \frac{\mathcal{C}_{anom}}{2^n}\varepsilon^{\mu_1\nu_1\ldots\mu_n\nu_n}\mathcal{F}_{\mu_1\nu_1}\ldots\mathcal{F}_{\mu_n\nu_n}.\\ \nabla_\mu J^\mu_{Cov} &= (n+1)\mathcal{C}_{anom}\varepsilon^{\mu_1\nu_1\ldots\mu_n\nu_n}\partial_{\mu_1}\hat{\mathcal{A}}_{\nu_1}\ldots\partial_{\mu_n}\hat{\mathcal{A}}_{\nu_n}\\ &= (n+1)\frac{\mathcal{C}_{anom}}{2^n}\varepsilon^{\mu_1\nu_1\ldots\mu_n\nu_n}\mathcal{F}_{\mu_1\nu_1}\ldots\mathcal{F}_{\mu_n\nu_n}.\end{aligned} \tag{5.28}$$

and Eqn.(5.9) becomes

$$J^\mu_{Cov} = J^\mu_{Consistent} + J^\mu_{(c)}. \tag{5.29}$$

where

$$\begin{aligned}J^\lambda_{(c)} &= n\mathcal{C}_{anom}\varepsilon^{\lambda\alpha\mu_1\nu_1\ldots\mu_{n-1}\nu_{n-1}}\hat{\mathcal{A}}_\alpha\partial_{\mu_1}\hat{\mathcal{A}}_{\nu_1}\ldots\partial_{\mu_{n-1}}\hat{\mathcal{A}}_{\nu_{n-1}}\\ &= n\frac{\mathcal{C}_{anom}}{2^{n-1}}\varepsilon^{\lambda\alpha\mu_1\nu_1\ldots\mu_{n-1}\nu_{n-1}}\hat{\mathcal{A}}_\alpha\mathcal{F}_{\mu_1\nu_1}\ldots\mathcal{F}_{\mu_{n-1}\nu_{n-1}}.\end{aligned} \tag{5.30}$$

The energy-momentum equation becomes

$$\nabla_\mu T^\mu_\nu = F_{\nu\mu}J^\mu_{Cov}, \tag{5.31}$$

where J^μ_{Cov} is the covariant current. This has been explicitly shown in [47] [47].

5.3.1 Constraining the partition function

We want to write the equilibrium free energy functional for the fluid. For this purpose, let us keep the in the following $2n$-dimensional time independent background,

$$ds^2 = -e^{2\sigma}(dt + a_i dx^i)^2 + g_{ij}dx^i dx^j, \quad \mathcal{A} = (A_0, \mathcal{A}_i). \tag{5.32}$$

Now, we write the $(2n-1)$ dimensional equilibrium free energy that reproduces the same anomaly as given in (5.76). The most generic form for the anomalous part of the partition function is ,

$$\begin{aligned}W_{anom} = \frac{1}{T_0}\int d^{2n-1}x\sqrt{g_{2n-1}}\Bigg\{&\sum_{m=1}^{n}\alpha_{m-1}(A_0,T_0)\ \left[\epsilon A(da)^{m-1}(dA)^{n-m}\right]\\ &+\alpha_n(T_0)\ \left[\epsilon a(da)^{n-1}\right]\Bigg\}.\end{aligned} \tag{5.33}$$

where, $\epsilon^{ijk\ldots}$ is the $(2n-1)$ dimensional tensor density defined via

$$\epsilon^{i_1 i_2\ldots i_{d-1}} = e^{-\sigma}\varepsilon^{0 i_1 i_2\ldots i_{d-1}}$$

[47] One required identity is,

$$\hat{\mathcal{A}}_\alpha\varepsilon^{\mu_1\nu_1\ldots\mu_n\nu_n}\mathcal{F}_{\mu_1\nu_1}\ldots\mathcal{F}_{\mu_n\nu_n} = 2n\ \hat{\mathcal{A}}_{\mu_1}\varepsilon^{\mu_1\nu_1\mu_2\nu_2\ldots\mu_n\nu_n}\mathcal{F}_{\alpha\nu_1}\mathcal{F}_{\mu_2\nu_2}\ldots\mathcal{F}_{\mu_n\nu_n}$$

for arbitrary $2n-$dimensions

The indices (i, j) run over $(2n-1)$ values. We have used the following notation for the sake of brevity

$$\begin{aligned}&\left[\epsilon A(da)^{m-1}(dA)^{n-m}\right]\\&\quad\equiv \epsilon^{ij_1k_1\ldots j_{m-1}k_{m-1}p_1q_1\ldots p_{n-m}q_{n-m}}A_i\partial_{j_1}a_{k_1}\ldots\partial_{j_{m-1}}a_{k_{m-1}}\partial_{p_1}A_{q_1}\ldots\partial_{p_{n-m}}A_{q_{n-m}}\\&\left[\epsilon(da)^{m-1}(dA)^{n-m}\right]^i\\&\quad\equiv \epsilon^{ij_1k_1\ldots j_{m-1}k_{m-1}p_1q_1\ldots p_{n-m}q_{n-m}}\partial_{j_1}a_{k_1}\ldots\partial_{j_{m-1}}a_{k_{m-1}}\partial_{p_1}A_{q_1}\ldots\partial_{p_{n-m}}A_{q_{n-m}}\end{aligned}\tag{5.34}$$

The invariance under diffeomorphism implies that α_n is a constant in space .For $m < n$ however α_m can have A_0 dependence, as the gauge symmetry is anomalous, but they are independent of σ, due to diffiomorphism invariance.

The consistent current computed from this partition function is,

$$\begin{aligned}(J_{anom})_0^{Consistent} &= -e^{\sigma}\sum_{m=1}^{n}\frac{\partial\alpha_{m-1}}{\partial A_0}\left[\epsilon A(da)^{m-1}(dA)^{n-m}\right]\\(J_{anom})^i_{Consistent} &= e^{-\sigma}\Bigg\{\sum_{m=1}^{n}(n-m+1)\alpha_{m-1}\left[\epsilon(da)^{m-1}(dA)^{n-m}\right]^i\\&\quad-\sum_{m=1}^{n-1}(n-m)\frac{\partial\alpha_{m-1}}{\partial A_0}\left[\epsilon A dA_0(da)^{m-1}(dA)^{n-m-1}\right]^i\Bigg\}\end{aligned}\tag{5.35}$$

Next, we compute the covariant currents, following (5.29). The correction piece for the 0-component of the current is,

$$(J_{(c)})_0 = -n\mathcal{C}_{anom}e^{\sigma}\sum_{m=1}^{n}A_0^m\binom{n-1}{m-1}\left[\epsilon A(da)^{m-1}(dA)^{n-m}\right]\tag{5.36}$$

where, we have used the following identification for $2n$ dimensional gauge field $\mathcal{A}_\mu$ and $(2n-1)$ dimensional gauge fields A_i, a_i and scalar A_0,

$$\begin{aligned}\mathcal{A}_i &= A_i + a_iA_0\\\mathcal{A}_0 &= A_0.\end{aligned}\tag{5.37}$$

where we are working in a'zero μ_0' gauge.

Thus, the 0-component of the covariant current is,

$$(J_{anom})_0^{Cov} = -e^{\sigma}\epsilon^{ijkl\ldots}\sum_{m=1}^{n}\left[\frac{\partial\alpha_{m-1}}{\partial A_0}+n\binom{n-1}{m-1}A_0^{m-1}\mathcal{C}_{anom}\right]\left[\epsilon A(da)^{m-1}(dA)^{n-m}\right].\tag{5.38}$$

Every term in the above sum is gauge non-invariant. So the covariance of the covariant current demands that we chose the arbitrary functions α_m appearing in the partition function (5.33) such that the current vanishes. Thus, we get,

$$\frac{\partial\alpha_{m-1}}{\partial A_0}+n\binom{n-1}{m-1}A_0^{m-1}\mathcal{C}_{anom} = 0.\tag{5.39}$$

The solution for the above equation is,

$$\alpha_m = -\mathcal{C}_{anom}\binom{n}{m+1}A_0^{m+1} + \tilde{C}_m T_0^{m+1}, \quad m = 0, \ldots, n-1$$
$$\alpha_n = \tilde{C}_n T_0^{n+1} \tag{5.40}$$

Here, $\tilde{C}_m$ are constants that can appear in the partition function.

Thus, at this point, a total of $n+1$ coefficients can appear in the partition function. A further study of CPT invariance of the partition function will reduce this number. We will present that analysis later in details and here we just state the result. CPT forces all $\tilde{C}_{2k} = 0$. For even n, the number of constants are $\frac{n}{2}$ where as for odd n, the number is $(\frac{n+1}{2})$.

5.3.2 Currents from the partition function

With these functions the $i-$component of the covariant current is,

$$\begin{aligned}(J_{anom})^i_{Cov} &= e^{-\sigma}\sum_{m=1}^{n}\left[A_0\frac{\partial\alpha_{m-1}}{\partial A_0} + (n-m+1)\alpha_{m-1}\right]\left[\epsilon(da)^{m-1}(dA)^{n-m}\right]^i \\ &= e^{-\sigma}\sum_{m=1}^{n}\Bigg[-(n+1)\mathcal{C}_{anom}\binom{n}{m}T_0A_0^m \\ &\qquad + (n-m+1)T_0^m\tilde{C}_{m-1}\Bigg]\left[\epsilon(da)^{m-1}(dA)^{n-m}\right]^i,\end{aligned} \tag{5.41}$$

As expected, this current is $U(1)$ gauge invariant. The different components of stress-tensor computed from the partition function are,

$$\begin{aligned}T_{00}^{anom} &= 0, \qquad T^{ij}_{anom} = 0 \\ \left(T_0^i\right)_{anom} &= e^{-\sigma}\sum_{m=1}^{n}\left(m\alpha_m - (n-m+1)A_0\alpha_{m-1}\right)\left[\epsilon(da)^{m-1}(dA)^{n-m}\right]^i \\ &= e^{-\sigma}\sum_{m=1}^{n}\Bigg[m\tilde{C}_mT_0^{m+1} - (n+1-m)\tilde{C}_{m-1}T_0^mA_0 \\ &\qquad + \binom{n+1}{m+1}\mathcal{C}_{anom}A_0^{m+1}\Bigg]\left[\epsilon(da)^{m-1}(dA)^{n-m}\right]^i\end{aligned} \tag{5.42}$$

5.3.3 Comparison with Hydrodynamics

Next, we find the equilibrium solution for the fluid variables. As usual, we keep the fluid in the time independent background (5.32). The equilibrium solutions for perfect charged fluid (with out any dissipation) are,

$$u^\mu\partial_\mu = e^{-\sigma}\partial_t, \quad T = T_0e^{-\sigma}, \quad \mu = A_0e^{-\sigma}. \tag{5.43}$$

The most generic constituitive relations for the fluid can be written as,

$$
\begin{aligned}
T_{\mu\nu} &= (\epsilon+p)u_\mu u_\nu + p g_{\mu\nu} + \eta\sigma_{\mu\nu} + \zeta\Theta\mathcal{P}_{\mu\nu} \\
J^\mu_{Cov} &= q u^\mu + J^\mu_{even} + J^\mu_{odd}, \\
J^\mu_{even} &= \sigma(E^\mu - T\mathcal{P}^{\mu\alpha}\partial_\alpha\nu) + \alpha_1 E^\mu + \alpha_2 T\mathcal{P}^{\mu\alpha}\partial_\alpha\nu + \text{higher derivative terms} \\
J^\mu_{odd} &= \sum_{m=1}^{n} \xi_m \varepsilon^{\mu\nu\ \gamma_1\delta_1\ldots\gamma_{m-1}\delta_{m-1}\ \alpha_1\beta_1\ldots\alpha_{n-m}\beta_{n-m}} u_\nu (\partial_\gamma u_\delta)^{m-1}(\partial_\alpha\mathcal{A}_\beta)^{n-m} + \ldots.
\end{aligned} \tag{5.44}
$$

Here, J^μ_{even} is parity even part of the charge current and J^μ_{odd} is parity odd charge current. $\varepsilon^{\mu\nu\alpha\beta\gamma\delta\ldots}$ is a $2n$ dimensional tensor density whose $(n-m)$ indices are contracted with $\partial_\alpha\mathcal{A}_\beta$ and $(m-1)$ indices are contracted with $\partial_\gamma u_\delta$.

We notice that the higher derivative part of the current gets contribution from both parity even and odd vectors. Parity even vectors can be at any derivative order but parity odd vectors always appear at $(n-1)$ derivative order. Thus, for a generic value of n (other than $n=2$) , the parity even and odd parts corrections to the current will always appear at different derivative orders. From now on, we will only concentrate on the parity odd sector. It is also straight forward to check that $J_0^{odd}=0$.

Next, we look for the equilibrium solution for this fluid. Since, there exist no gauge invariant parity odd scalar, the temperature and chemical potential do not get any correction. Also, in $2n$ dimensional theory, the parity odd vectors that we can write are always $(n-1)$ derivative terms. No other parity odd vector at any lower derivative order exists. Since the fluid velocity is always normalized to unity, we have,

$$
\delta T = 0, \quad \delta\mu = 0, \quad \delta u_0 = -a_i \delta u^i. \tag{5.45}
$$

where, the most generic correction to the fluid velocity is,

$$
\delta u^i = \sum_{m=1}^{n} U_m(\sigma, A_0)\left[\epsilon (da)^{m-1}(dA)^{n-m}\right]^i. \tag{5.46}
$$

Here, $U_m(\sigma, A_0)$ are arbitrary coefficients and factors of e^σ is introduced for later convenience. Similarly, we can parameterize the $i-$component of the parity-odd current as,

$$
J^i_{odd} = \sum_{m=1}^{n} J_m(\sigma, A_0)\left[\epsilon (da)^{m-1}(dA)^{n-m}\right]^i. \tag{5.47}
$$

The coefficients $J_m(\sigma, A_0)$ are related to the transport coefficients ξ_m via

$$
J_m = \sum_{k=1}^{m} \binom{n-k}{m-k} \xi_k \left(-e^\sigma\right)^{k-1} A_0^{m-k}. \tag{5.48}
$$

With all these data, we can finally compute the corrections to the stress tensor and charged currents and they take the following form,

$$\begin{aligned}
\delta T_{00} &= 0, \quad \delta T^{ij} = 0, \quad \delta \tilde{J}_0 = 0 \\
\delta T^i_0 &= -e^{\sigma}(\epsilon+p)\epsilon^{ijk\ldots}\sum_{m=1}^{n} U_m(\sigma, A_0)(da)^{m-1}(dA)^{n-m} \\
\delta J^i_{Cov} &= \epsilon^{ijk\ldots}\sum_{m=1}^{n}(J_m(\sigma, A_0) + qU_m(\sigma, A_0))(da)^{m-1}(dA)^{n-m}
\end{aligned} \tag{5.49}$$

Comparing the expressions for various components of stress tensor and covariant current of the fluid obtained from equilibrium partition function (5.42), (5.41) and fluid constitutive relations (5.49), we get,

$$\begin{aligned}
U_m &= -\frac{e^{-2\sigma}}{\epsilon+p}\left[m\alpha_m - (n-m+1)A_0\alpha_{m-1}\right] \\
&= -\frac{e^{-2\sigma}}{\epsilon+p}\Bigg[m\tilde{C}_m T_0^{m+1} - (n+1-m)\tilde{C}_{m-1}A_0 T_0^m \\
&\qquad + \binom{n+1}{m+1}\mathcal{C}_{anom}A_0^{m+1}\Bigg]
\end{aligned} \tag{5.50}$$

Similarly, we can evaluate $J_m(\sigma, A_0)$ as follows,

$$\begin{aligned}
J_m = e^{-\sigma}&\left[-(m+1)\mathcal{C}_{anom}A_0^m\binom{n+1}{m+1} + (n-m+1)\tilde{C}_{m-1}T_0^m\right] \\
&+ \frac{qe^{-2\sigma}}{\epsilon+p}\Bigg[m\tilde{C}_m T_0^{m+1} - (n+1-m)\tilde{C}_{m-1}A_0 T_0^m \\
&\qquad + \binom{n+1}{m+1}\mathcal{C}_{anom}A_0^{m+1}\Bigg]
\end{aligned} \tag{5.51}$$

We want to now use this to obtain the transport coefficients ξ_m in the last relation of (5.44). For this we have to invert the relations (5.48) for ξ_m. We finally get

$$\begin{aligned}
\xi_m = &\left[m\frac{q\mu}{\epsilon+p} - (m+1)\right]\mathcal{C}_{anom}\binom{n+1}{m+1}\mu^m \\
&+ \sum_{k=0}^{m}\left[m\frac{q\mu}{\epsilon+p} - (m-k)\right](-1)^{k-1}\tilde{C}_k\binom{n-k}{m-k}T^{k+1}\mu^{m-k-1}
\end{aligned} \tag{5.52}$$

This then is the prediction of this transport coefficient via partition function methods. This exactly matches with the expression from [8] in (5.18) provided we make the following identification among the constants $\tilde{C}_m = (-1)^{m-1}C_m$.

5.4 Comments on Most Generic Entropy Current

Another physical requirement which has long been used as a source of constraints on fluid dynamical transport coefficients is the local form of second law of thermodynamics. As we reviewed in the subsection§§5.2.1 this principle had been used in [8] to obtain anomaly induced transports coefficients in arbitrary even dimensions.

In this section we will determine the entropy current in equilibrium by comparing the total entropy with that obtained from the equilibrium partition function. In the examples studied in [47, 48] it was seen that in general the comparison with equilibrium entropy (obtained from partition function) did not fix all the non dissipative coefficients in fluid dynamical entropy current. However it did determine the anomalous contribution exactly. Here we will see that this holds true in general even dimensions.

Let us begin by computing the entropy from the equilibrium partition function. We begin with the anomalous part of the partition function

$$\begin{aligned} W_{anom} = \frac{1}{T_0}\int d^{2n-1}x\sqrt{g_{2n-1}}\bigg\{ \sum_{m=1}^{n} \alpha_{m-1}\left[\epsilon A(da)^{m-1}(dA)^{n-m}\right] \\ + \alpha_n \left[\epsilon a (da)^{n-1}\right]\bigg\} \end{aligned} \tag{5.53}$$

where the functions α_m are given in (5.40).

The anomalous part of the total entropy is easily computed to be

$$\begin{aligned} S_{anom} &= \frac{\partial}{\partial T_0}\left(T_0 W_{anom}\right) \\ &= \int d^{2n-1}x\sqrt{g_{2n-1}}\bigg\{ \sum_{m=1}^{n} m\ T_0^{m-1}\ \tilde{C}_{m-1}\ \left[\epsilon A(da)^{m-1}(dA)^{n-m}\right] \\ &\quad +\ (n+1)\tilde{C}_n\ T_0^n \left[\epsilon a(da)^{n-1}\right]\bigg\} \\ &= \int d^{2n-1}x\sqrt{g_{2n-1}}\bigg\{ \sum_{m=1}^{n} (m+1)\ T_0^m\ \tilde{C}_m\ \left[\epsilon a(da)^{m-1}(dA)^{n-m}\right] \\ &\quad + \tilde{C}_0\left[\epsilon A(dA)^{n-1}\right]\bigg\} \end{aligned} \tag{5.54}$$

Now we will determine the most general form of entropy current in equilibrium by comparison with (5.54). In [47] it was argued that the entropy current by itself is not a physical object, but entropy production and total entropy are. This gave a window for gauge non invariant contribution to entropy current but the contribution was removed by CPT invariance. Here also we will allow for such gauge non invariant terms in the entropy current. The most general form of entropy current, allowing for gauge non invariant pieces, is then

$$\begin{aligned} J_S^\mu = su^\mu - \frac{\mu}{T}J^\mu_{odd} + \sum_{m=1}^{n}\chi_m\varepsilon^{\mu\nu\cdots}u_\nu(\partial u)^{m-1}(\partial\hat{\mathcal{A}})^{n-m} \\ + \zeta\varepsilon^{\mu\nu\cdots}\hat{\mathcal{A}}_\nu(\partial\hat{\mathcal{A}})^{n-1} \end{aligned} \tag{5.55}$$

where χ_m is a function of T and μ whereas ζ is a constant . The correction to the local entropy density (i.e., the time component of the entropy current) can be written after an integration by parts as

$$\delta J^0_S = \varepsilon^{0ij\dots}\left[\zeta A(dA)^{n-1} + \sum_{k=1}^{n} \tilde{f}_k\ a\ (da)^{k-1}\ (dA)^{n-k}\right]_{ij\dots} + \text{total derivatives} \tag{5.56}$$

where

$$\tilde{f}_m \equiv -sU_m + \frac{\mu}{T}J_m + \zeta A_0^m\binom{n}{m} + \sum_{k=1}^{m}\binom{n-k}{m-k}\chi_k\left(-e^{\sigma}\right)^k A_0^{m-k} \tag{5.57}$$

The correction to the entropy is then,

$$\begin{aligned}\delta S &= \int d^{2n-1}x\sqrt{g_{2n}}\ J^0_S \\ &= \int d^{2n-1}x\sqrt{g_{2n-1}}\left[\zeta\left[\epsilon A(dA)^{n-1}\right] + \sum_{m=1}^{n}\tilde{f}_m\ \left[\epsilon a\ (da)^{m-1}\ (dA)^{m-k}\right]\right]\end{aligned} \tag{5.58}$$

Comparing the two expressions of total equilibrium entropy (5.54) and (5.58) we find the following expressions of the various coefficients in the entropy current (5.56),

$$\zeta = \tilde{C}_0 \quad \text{and} \quad \tilde{f}_k = (k+1)\ T_0^k\ \tilde{C}_k \ \ \text{for} \ \ 0 \le k \le n \tag{5.59}$$

This in turn implies that

$$\begin{aligned}&T_0\sum_{k=1}^{m}\binom{n-k}{m-k}\chi_k\left(-e^{\sigma}\right)^k A_0^{m-k} \\ &\qquad = \tilde{C}_m T_0^{m+1} + m\binom{n}{m}\mathcal{C}_{anom}A_0^{m+1} - \tilde{C}_0 T_0 A_0^m\binom{n}{m}\end{aligned} \tag{5.60}$$

which can be inverted to give

$$\begin{aligned}\chi_m &= -\mathcal{C}_{anom}\binom{n+1}{m+1}T^{-1}\mu^{m+1} - \sum_{k=0}^{m}\tilde{C}_k(-1)^{k-1}\binom{n-k}{m-k}T^k\mu^{m-k} \\ \zeta &= \tilde{C}_0\end{aligned} \tag{5.61}$$

which matches with the prediction from [8] in equation (5.20) again with the identification $C_m(-1)^{m-1} = \tilde{C}_m$. We see that in the entropy current we have a total of $n+1$ constants as in the equilibrium partition function.

This completes our partition function analysis and our re derivation of the results of [8] via partition function techniques. We see that the transport coefficients match exactly with the results obtained via entropy current (provided the analysis of [8] is extended by allowing gauge-non-invariant pieces in the entropy current). This detailed match of transport coefficients warrants the question whether the form of the equilibrium partition function itself can be directly derived from the expressions of [8] quoted in 5.2.1. We turn to this question in the next section.

5.5 Gibbs current and Partition function

We begin by repeating the expression for the Gibbs current in (5.10) which was central to the results of [8].

$$\begin{split}\bar{\mathcal{G}}^{Cov}_{anom} = C_0 T \hat{\mathcal{A}} \wedge \mathcal{F}^{n-1} + \sum_{m=1}^{n} \Bigg[\mathcal{C}_{anom} \binom{n+1}{m+1} \mu^{m+1} \\ + \sum_{k=0}^{m} C_k \binom{n-k}{m-k} T^{k+1} \mu^{m-k} \Bigg] (2\omega)^{m-1} \mathcal{B}^{n-m} \wedge u\end{split} \tag{5.62}$$

The subscript 'anom' denotes that we are considering only a part of the entropy current relevant to anomalies. The superscript 'Cov' refers to the fact that this is the Gibbs free energy computed by turning on a chemical potential for the **covariant** charge.

Let us ask how this expression would be modified if the Gibbs free energy was computed by turning on a chemical potential for the **consistent** charge instead. The change from covariant charge to consistent charge/current is simply given by a shift as given by the equation(5.9). This shift does not depend on the state of the theory but is purely a functional of the background gauge fields. Thinking of Gibbs free energy as minus temperature times the logarithm of the Eucidean path integral, a conversion from covariant charge to a consistent charge induces a shift

$$\bar{\mathcal{G}}^{Cov}_{anom} = \bar{\mathcal{G}}^{Consistent}_{anom} - \mu\ n\ \mathcal{C}_{anom} \hat{\mathcal{A}} \wedge \mathcal{F}^{n-1}$$

which gives

$$\begin{split}&\bar{\mathcal{G}}^{Consistent}_{anom} \\ &= \sum_{m=1}^{n} \Bigg[\mathcal{C}_{anom} \binom{n+1}{m+1} \mu^{m+1} + \sum_{k=0}^{m} C_k \binom{n-k}{m-k} T^{k+1} \mu^{m-k} \Bigg] (2\omega)^{m-1} \mathcal{B}^{n-m} \wedge u \\ &\quad + \left[C_0 T + n \mathcal{C}_{anom} \mu\right] \hat{\mathcal{A}} \wedge \mathcal{F}^{n-1}\end{split} \tag{5.63}$$

This now a Gibbs current whose μ derivative gives the consistent current rather than a covariant current. It is easy to check that this solves an adiabaticity equation very similar to the one quoted in equation(5.4)

$$\begin{split}d\bar{\mathcal{G}}^{Consistent}_{anom} + \mathfrak{a} \wedge \bar{\mathcal{G}}^{Consistent}_{anom} + n\mathcal{C}_{anom} \left(\hat{\mathcal{A}} + \mu u\right) \wedge \mathcal{E} \wedge \mathcal{B}^{n-1} \\ = (dT + \mathfrak{a}T) \wedge \frac{\partial \bar{\mathcal{G}}^{Consistent}_{anom}}{\partial T} + (d\mu + \mathfrak{a}\mu - \mathcal{E}) \wedge \frac{\partial \bar{\mathcal{G}}^{Consistent}_{anom}}{\partial \mu}\end{split} \tag{5.64}$$

The question we wanted to address is how this Gibbs current is related to the partition function in equation (5.33).

The answer turns out to be quite intuitive - we would like to argue in this section that

$$W_{anom} = \ln\ Z^{anom}_{Consistent} = -\int_{space} \frac{1}{T} \bar{\mathcal{G}}^{Consistent}_{anom} \tag{5.65}$$

This equation instructs us to pull back the $2n-1$ form in equation (5.63) (divided by local temperature) and integrate it on an arbitrary spatial hyperslice to obtain the anomalous contribution to negative logarithm of the equilibrium path integral. Note that pulling back the Hodge dual of Gibbs current on a spatial hyperslice is essentially equivalent to integrating its zero component (i.e., the Gibbs density) on the slice. Seen this way the above relation is the familiar statement relating Gibbs free energy to the grand-canonical partition function.

5.5.1 Reproducing the Gauge variation

Before giving an explicit proof of the relation(5.65) we will check in this subsection that the relation(5.65) essentially gives the correct gauge variation to the path-integral at equilibrium. This will provide us with a clearer insight on how the program of [47] to write a local expression in the partition function to reproduce the anomaly works.

The gauge variation of(5.65) under $\delta\hat{\mathcal{A}} = d\delta\lambda$ is

$$\begin{aligned}
\delta W_{anom} = \delta \ln\ Z^{anom}_{Consistent} &= -\int_{space} \frac{1}{T}\delta\bar{\mathcal{G}}^{Consistent}_{anom} \\
&= -\int_{space}\left[C_0 + n\mathcal{C}_{anom}\frac{\mu}{T}\right]\delta\hat{\mathcal{A}}\wedge\mathcal{F}^{n-1} \\
&= -\int_{space}\left[C_0 + n\mathcal{C}_{anom}\frac{\mu}{T}\right]d\delta\lambda\wedge\mathcal{F}^{n-1} \\
&= -\int_{surface}\delta\lambda\left[C_0 + n\mathcal{C}_{anom}\frac{\mu}{T}\right]\wedge\mathcal{F}^{n-1} + n\mathcal{C}_{anom}\int_{space}\delta\lambda d\left(\frac{\mu}{T}\right)\wedge\mathcal{F}^{n-1}
\end{aligned} \tag{5.66}$$

We will now ignore the surface contribution and use the fact that chemical equilibrium demands that

$$Td\left(\frac{\mu}{T}\right) = \mathcal{E}$$

where $\mathcal{E} \equiv u^\nu \mathcal{F}_{\mu\nu}dx^\nu$ is the rest frame electric-field. This is essentially a statement (familiar from say semiconductor physics) that in equilibrium the diffusion current due to concentration gradients should cancel the drift ohmic current due to the electric field. Putting this in along with the electric-magnetic decomposition $\mathcal{F} = \mathcal{B} + u\wedge\mathcal{E}$, we get

$$\delta W_{anom} = \delta \ln\ Z^{anom}_{Consistent} = \mathcal{C}_{anom}\int_{space}\frac{\delta\lambda}{T}n\mathcal{E}\wedge\mathcal{B}^{n-1} \tag{5.67}$$

which is the correct anomalous variation required of the equilibrium path-integral ! In $d = 2n = 4$ dimensions for example we get the correct $E.B$ variation along with the $1/T$ factor coming from the integration over euclidean time-circle. The factor of n comes from converting to electric and magnetic fields

$$\mathcal{F}^n = n\ u\wedge\mathcal{E}\wedge\mathcal{B}^{n-1}$$

Thus the shift piece along with the chemical equilibrium conspires to reproduce the correct gauge variation. The reader might wonder why this trick cannot be made to work by just keeping the shift term alone in the Gibbs current - the answer is of course that other terms are required if one insists on adiabaticity in the sense that we want to solve (5.64).

5.5.2 Integration by parts

In this subsubsection we will prove (5.65) explicitly. We will begin by evaluating the consistent Gibbs current in the equilibrium configuration. We will as before work in the 'zero μ_0' gauge.

Using the relations in the appendix 5.9.4 we get the consistent Gibbs current as

$$
\begin{aligned}
&-\frac{1}{T}\bar{\mathcal{G}}^{Consistent}_{anom}\\
&\quad= \frac{1}{T_0}\sum_{m=1}^{n}\Bigg[C_m(-1)^{m-1}T_0^{m+1} - C_0(-1)^{0-1}\binom{n}{m}T_0A_0^m\\
&\qquad\quad -\binom{n}{m+1}\mathcal{C}_{anom}A_0^{m+1}\Bigg](da)^{m-1}(dA)^{n-m}\wedge(dt+a)\\
&\qquad -\frac{1}{T_0}\left[n\mathcal{C}_{anom}A_0 + C_0T_0\right]A\wedge(dA+A_0da)^{n-1}\\
&\qquad -\frac{(n-1)}{T_0}\left[n\mathcal{C}_{anom}A_0 + C_0T_0\right]A\wedge dA_0\wedge(dt+a)\wedge(dA+A_0da)^{n-2}
\end{aligned}
\tag{5.68}
$$

After somewhat long set of manipulations one arrives at the following form for the consistent Gibbs current

$$
\begin{aligned}
&-\frac{1}{T}\bar{\mathcal{G}}^{Consistent}_{anom}\\
&\quad= d\Bigg\{\frac{A}{T_0}\sum_{m=1}^{n-1}\Bigg[C_m(-1)^{m-1}T_0^{m+1} - C_0(-1)^{0-1}\binom{n-1}{m}T_0A_0^m\\
&\qquad\quad +m\binom{n}{m+1}\mathcal{C}_{anom}A_0^{m+1}\Bigg](da)^{m-1}(dA)^{n-1-m}\wedge(dt+a)\Bigg\}\\
&\qquad +\frac{A}{T_0}\sum_{m=1}^{n}\left[C_{m-1}(-1)^{m-2}T_0^m - \binom{n}{m}\mathcal{C}_{anom}A_0^m\right](da)^{m-1}(dA)^{n-m}\\
&\qquad + C_n(-1)^{n-1}T_0^n(da)^{n-1}\wedge(dt+a)
\end{aligned}
\tag{5.69}
$$

Here we have taken out a surface contribution which we will suppress from now on since it does not contribute to the partition function. This final form is easily checked term by term.

Suppressing the surface contribution we can write

$$
\begin{aligned}
&-\frac{1}{T}\bar{\mathcal{G}}^{Consistent}_{anom}\\
&\quad= d\,[\ldots] + \frac{A}{T_0}\sum_{m=1}^{n}\left[C_{m-1}(-1)^{m-2}T_0^m - \binom{n}{m}\mathcal{C}_{anom}A_0^m\right](da)^{m-1}(dA)^{n-m}\\
&\qquad + C_n(-1)^{n-1}T_0^n(da)^{n-1}\wedge(dt+a)\\
&\quad= d\,[\ldots] + \frac{A}{T_0}\wedge\sum_{m=1}^{n}\alpha_{m-1}(da)^{m-1}(dA)^{n-m} + \frac{dt+a}{T_0}\wedge\alpha_n(da)^{n-1}
\end{aligned}
\tag{5.70}
$$

where we have defined

$$\begin{aligned}
\alpha_m &= C_m(-1)^{m-1}T_0^{m+1} - \binom{n}{m+1}\mathcal{C}_{anom}A_0^{m+1} \quad \text{for } m < n \\
\alpha_n &= C_n(-1)^{n-1}T_0^{n+1}
\end{aligned} \tag{5.71}$$

To get the contribution to the equilibrium partition function, we integrate the above equation over the spatial slice (putting $dt = 0$). We will neglect surface contributions to get

$$\begin{aligned}
&(\ln \mathcal{Z})_{anom}^{Consistent} \\
&= \int_{\text{space}} \frac{A}{T_0} \wedge \sum_{m=1}^{n} \left[C_{m-1}(-1)^{m-2}T_0^m - \binom{n}{m}\mathcal{C}_{anom}A_0^m \right] (da)^{m-1}(dA)^{n-m} \\
&\quad + \int_{\text{space}} C_n(-1)^{n-1}T_0^n a \wedge (da)^{n-1} \\
&= \int_{\text{space}} \frac{A}{T_0} \wedge \sum_{m=1}^{n} \alpha_{m-1}(da)^{m-1}(dA)^{n-m} + \int_{\text{space}} \frac{a}{T_0} \wedge \alpha_n (da)^{n-1}
\end{aligned} \tag{5.72}$$

with α_ms given by (5.71). We are essentially done - we have got the form in (5.33) and comparing the equations (5.71) and (5.40) we find a perfect agreement with the usual relation $C_m(-1)^{m-1} = \tilde{C}_m$. Now by varying this partition function we can obtain currents as before (the variation can be directly done in form language using the equations we provide in appendix 5.9.5). With this we have completed a whole circle showing that the two formalisms for anomalous transport developed in [8] and [47] are completely equivalent.

Before we conclude, let us rewrite the partition function in terms of the polynomial $\mathfrak{F}_{anom}^{\omega}[T, \mu]$ as

$$\begin{aligned}
&(\ln \mathcal{Z})_{anom}^{Consistent} \\
&= \int_{\text{space}} \frac{A}{T_0 da} \wedge \left[\frac{\mathfrak{F}_{anom}^{\omega}[-T_0 da, dA] - \mathfrak{F}_{anom}^{\omega}[-T_0 da, 0]}{dA} - \frac{\mathfrak{F}_{anom}^{\omega}[0, dA + A_0 da]}{dA + A_0 da} \right] \\
&\quad + \int_{\text{space}} \frac{\mathfrak{F}_{anom}^{\omega}[-T_0 da, 0]}{(T_0 da)^2} \wedge T_0 a
\end{aligned} \tag{5.73}$$

We will consider an example. Using adiabaticity arguments, the authors of [30] derived the following expression for a theory of free Weyl fermions in $d = 2n$ spacetime dimensions

$$(\mathfrak{F}_{anom}^{\omega})_{d=2n}^{free\ Weyl} = -2\pi \sum_{species} \chi_{d=2n} \left[\frac{\frac{\tau}{2}T}{\sin \frac{\tau}{2}T} e^{\frac{\tau}{2\pi}q\mu} \right]_{\tau^{n+1}} \tag{5.74}$$

where $\chi_{d=2n}$ is the chirality and the subscript τ^{n+1} denotes that one needs to Taylor-expand in τ and retain the coefficient of τ^{n+1}. Substituting this into the above expression gives the anomalous part of the partition function of free Weyl fermions.

5.6 Fluids charged under multiple $U(1)$ fields

In this subsection, we will generalize our results to cases where we have multiple abelian $U(1)$ gauge fields in arbitrary $2n-$dimensions.

We can take

$$\mathfrak{F}^{\omega}_{anom}[T,\mu] = \mathcal{C}^{A_1\ldots A_{n+1}}_{anom}\mu_{A_1}\ldots\mu_{A_{n+1}} + \sum_{m=0}^{n} C_m^{A_1\ldots A_{n-m}} T^{m+1}\mu_{A_1\ldots A_{n-m}}. \tag{5.75}$$

In this case, the anomaly equation takes the following form,

$$\nabla_\mu J^{\mu,A_{n+1}}_{Cov} = \frac{n+1}{2^n}\mathcal{C}^{A_1A_2\ldots A_{n+1}}_{anom}\varepsilon^{\mu_1\nu_1\mu_2\nu_2\ldots\mu_n\nu_n}\left(\mathcal{F}_{\mu_1\nu_1}\right)_{A_1}\ldots\left(\mathcal{F}_{\mu_n\nu_n}\right)_{A_n}. \tag{5.76}$$

Where, in $2n$ dimensions $\mathcal{C}_{anom}$ has $n+1$ indices denoted by $(A_1, A_2\cdot A_{n+1})$ and it is symmetric in all its indices. It is straightforward to carry on the above computation for the case of multiple $U(1)$ charges and most of the computations remains the same. Now, for the multiple $U(1)$ case, in partition function 5.33 the functions α_m and the constants $\tilde{C}_m$ (and the constants C_m appearing in $\mathfrak{F}^{\omega}_{anom}$) have $n-m$ number of indices which are contracted with $n-1-m$ number of dA and one A. The constant ζ appearing in the entropy current has n indices.

The constant $\tilde{C}_n$ (and α_n) has no index. All these constants are symmetric in their indices. Considering the above index structure into account, we can understand that the functions U_m appearing in velocity correction and χ_m appearing in entropy corrections has $n-m$ indices and the function J_m appearing in the charge current has $n-m+1$ indices. Now, we can write the generic form of these functions as follows:

$$\begin{aligned} U_m^{A_1A_2\ldots A_{n-m}} = &-\frac{e^{-2\sigma}}{\epsilon+p}\Big[m\tilde{C}_m^{A_1A_2\ldots A_{n-m}}T_0^{m+1} \\ &-(n+1-m)\tilde{C}_{m-1}^{A_1A_2\ldots A_{n-m}B_1}(A_0)_{B_1}T_0^m \\ &+\binom{n+1}{m+1}\mathcal{C}_{anom}^{A_1\ldots A_{n-m}B_1\ldots B_{m+1}}(A_0)_{B_1}\ldots(A_0)_{B_{m+1}}\Big] \end{aligned} \tag{5.77}$$

where $(A_0)_{B_1}$ comes from the B_1th gauge field.

Similarly, we can write the coefficients appearing in A'th charge current (J^A) as,

$$\begin{aligned} \left(J^A\right)_m^{A_1A_2\ldots A_{n-m}} = e^{-\sigma}&\Big[-(m+1)\mathcal{C}_{anom}^{AA_1\ldots A_{n-m}B_1\ldots B_m}(A_0)_{B_1}\ldots(A_0)_{B_m}\binom{n+1}{m+1} \\ &+(n-m+1)\tilde{C}_{m-1}^{AA_1\ldots A_{n-m}}T_0^m\Big] \\ &+\frac{q^Ae^{-2\sigma}}{\epsilon+p}\Big[m\tilde{C}_m^{A_1A_2\ldots A_{n-m}}T_0^{m+1} \\ &-(n+1-m)\tilde{C}_{m-1}^{A_1A_2\ldots A_{n-m}B_1}(A_0)_{B_1}T_0^m \\ &+\binom{n+1}{m+1}\mathcal{C}_{anom}^{A_1\ldots A_{n-m}B_1\ldots B_{m+1}}(A_0)_{B_1}\ldots(A_0)_{B_{m+1}}\Big] \end{aligned} \tag{5.78}$$

Name	Symbol	CPT
Temperature	T	+
Chemical Potential	μ	-
Velocity 1-form	u	+
Gauge field 1-form	$\hat{\mathcal{A}}$	-
Exterior derivative	d	-
Field strength 2-form	$\mathcal{F} = d\hat{\mathcal{A}}$	+
Magnetic field 2-form	$\mathcal{B}$	+
Vorticity 2-form	ω	-

Table 17. Action of CPT on various forms

We can also express the transport coefficients for fluids charged under multiple $U(1)$ charges, generalising equation (5.52) as,

$$
\begin{aligned}
&\left(\xi^A\right)_m^{A_1 A_2 \ldots A_{n-m}} \\
&= \left[m\frac{q^A \mu_B}{\epsilon + p} - (m+1)\delta_B^A\right] \mathcal{C}_{anom}^{BA_1\ldots A_{n-m}B_1\ldots B_m} \binom{n+1}{m+1} \mu_{B_1}\ldots\mu_{B_m} \\
&+ \sum_{k=0}^{m-1} \left[m\frac{q^A \mu_B}{\epsilon + p} - (m-k)\delta_B^A\right] \\
&\quad \times (-1)^{k-1} \tilde{C}_k^{BA_1\ldots A_{n-m}B_1\ldots B_{m-k-1}} \binom{n-k}{m-k} T^{k+1} \mu_{B_1}\ldots\mu_{B_{m-k-1}} \\
&+ \left[m\frac{q^A}{\epsilon + p}\right] (-1)^{m-1} \tilde{C}_m^{A_1\ldots A_{n-m}} T^{m+1}
\end{aligned}
\tag{5.79}
$$

Similarly the coefficieints χ_m appearing entropy current become

$$
\begin{aligned}
\chi_m^{A_1\ldots A_{n-m}} &= -\mathcal{C}_{anom}^{A_1\ldots A_{n-m}B_1\ldots B_{m+1}} \binom{n+1}{m+1} T^{-1} \mu_{B_1}\ldots\mu_{B_{m+1}} \\
&- \sum_{k=0}^{m} (-1)^{k-1} \binom{n-k}{m-k} T^k \tilde{C}_k^{A_1\ldots A_{n-m}B_1\ldots B_{m-k}} \mu_{B_1}\ldots\mu_{B_{m-k}}
\end{aligned}
\tag{5.80}
$$

This finishes the analysis of anomalous fluid charged under multiple abelian $U(1)$ gauge fields.

5.7 CPT Analysis

In this subsection we analyze the constraints of 2n dimensional CPT invariance on the analysis of our previous sections.

Let us first examine the CPT transformation of the Gibbs current proposed in [8]. Using the Table§17 we see that the Gibbs current in Eqn.(5.10) is CPT-even provided the coefficients $\{\mathcal{C}_{anom}, C_{2k+1}\}$ are CPT-even and the coefficients C_{2k} are CPT-odd. Since in a CPT-invariant

fields	C	P	T	CPT
σ	+	+	+	+
a_i	+	-	-	+
g_{ij}	+	+	+	+
A_0	-	+	+	-
A_i	-	-	-	-

Table 18. Action of CPT on various field

theory all CPT-odd coefficients should vanish, we conclude that $C_m = 0$ for even m. This conclusion can be phrased as

$$CPT \quad : \quad C_m(-1)^{m-1} = C_m \tag{5.81}$$

Note that this is the same conclusion as reached by assuming the relation to the anomaly polynomial.

Next we analyze the constraints of 2n dimensional CPT invariance on the partition function (5.33). Our starting point is a partition function of the fluid and we expect it to be invariant under $2n$dimensional CPT transformation of the fields. Table§18 lists the effect of 2n dimensional C, P and T transformation on various field appearing in the partition function (5.33). Since a_i is even while A_i and ∂_j are odd under CPT, the term with coefficient C_m picks up a factor of $(-1)^{(m+1)}$. Thus CPT invariance tells us that C_m must be

- even function of A_0 for odd m.
- odd function of A_0 for even m.

Now the coefficients C_m are fixed upto constants $\tilde{C}_m$ by the requirement that the partition function reproduces the correct anomaly. Note that the A_0(odd under CPT) dependence of the coefficients C_m thus determined are consistent with the requirement CPT invariance. Further, CPT invariance forces $\tilde{C}_m = 0$ for even m. The last term in the partition function (5.33) is odd under parity and thus its coefficient is set to zero by CPT for even n whereas for odd n it is left unconstrained.

Thus finally we see that CPT invariance allows for a total of

- $\frac{n}{2}$ constant ($\tilde{C}_m$ with m odd) for even n.
- $\frac{n+1}{2}$ constants ($\tilde{C}_m$ with m even and $\tilde{C}_n$) for odd n.

In particular the coefficient $\tilde{C}_0$ always vanishes and thus, for a CPT invariant theory, we never get the gauge-non invariant contribution to th elocal entropy current.

5.8 Discussion

In this section we have shown that the results of [8, 46] based on entropy arguments can be re derived within a more field-theory friendly partition function technique [47–50]. This has led us to a deeper understanding linking the local description of anomalous transport in terms of a Gibbs current [8, 30] to the global description in terms of partition functions.

An especially satisfying result is that the polynomial structure of anomalous transport coefficients discovered in [8] is reproduced at the level of partition functions. There it was shown that the whole set of anomalous transport coefficients are essentially governed by a single homogeneous polynomial $\mathfrak{F}^{\omega}_{anom}[T,\mu]$ of temperature and chemical potentials. The authors of [30] noticed that in a free theory of chiral fermions this polynomial structure is directly linked to the corresponding anomaly polynomial of chiral fermions via a replacement rule

$$\mathfrak{F}^{\omega}_{anom}[T,\mu] = \mathcal{P}_{anom}\left[\mathcal{F}\mapsto\mu, p_1(\mathfrak{R})\mapsto -T^2, p_{k>1}(\mathfrak{R})\mapsto 0\right] \tag{5.82}$$

This result could be generalised for an arbitrary free theory with chiral fermions and chiral p-form fields using sphere partition function techniques which link this polynomial to a specific thermal observable [51, 52].

We have derived in this section a particular contribution to the equilibrium partition function that is linked to the underlying anomalies of the theory. A direct test of this result would be to do a direct holographic computation of the same quantity in AdS/CFT to obtain these contributions. Since the CFT anomalies are linked to the Chern-Simons terms in the bulk the holographic test would be a computation of a generalised Wald entropy for a black hole solution of a gravity theory with Chern-Simons terms. The usual Wald entropy gets modified in the presence of such Chern-Simons terms[53, 54] which are usually a part of higher derivative corrections to gravity. We hope that reproducing the results of this paper would give us a test of generalised Wald formalism for such higher derivative corrections.

We have directly linked the description in terms of a Gibbs current[8, 30] satisfying a kind of adiabticity equation to the global description in terms of partition functions. Further we have noticed in (5.21) that at least in the case of anomalous transport this Gibbs current is closely linked to what has been called 'the non-canonical part of the entropy current ' in various entropy arguments[10]. It would be interesting to see whether this construction can be generalised beyond the anomalous transport coefficients to other partition function computations which appear in [47, 49]. This would give us a more local interpretation of the various terms appearing in the partition function linking them to a specific Gibbs free energy transport process. Hence with such a result one could directly identify the coefficients appearing in the partition function as the transport coefficients of the Gibbs current.

Another interesting observation of [8] apart from the polynomial structure is that the anomalous transport satisfies an interesting reciprocity type relation (5.15)- the susceptibility describing the change in the anomalous charge current with a small change in vorticity is equal to the susceptibility describing the change in the anomalous energy current with a small change in magnetic field. While we see that the results of this section are consistent with this observation made in [8], we have not succeeded in deriving this relation directly from the partition function. It would be interesting to derive such a relation from the partition function hence clarifying how such a relation arises in a microscopic description.

Finally as we have emphasised in the introductions one would hope that the results of this section serve as a starting point for generalising the analysis of anomalies to non-equilibrium phenomena. Can one write down a Schwinger-Keldysh functional which transforms appropriately - does this provide new constraints on the dissipative transport coefficients ? We leave such questions to future work.

5.9 Appendices to chapter 4

5.9.1 Results of $(3+1)-$ dimensional and $(1+1)-$ dimensional fluid

In this appendix we want to specialise our results to $1+1$ and $3+1$ dimensional anomalous fluids.By considering local entropy production of the system, the results for $(3+1)-$dimensional anomalous fluid were obtained in [3], [4, 16] and for $(1+1)-$dimensional fluid were obtained in [11]. The same results have also been obtained in [47] and [48] for $(3+1)-$dimensional and $(1+1)-$dimensional anomalous fluid respectively, by writing the equilibrium partition function, the technique that we have followed in this section. Our goal in this section is to check that the arbitrary dimension results reduce correctly to these special cases.

5.9.2 $(3+1)-$ dimensional anomalous fluids

Let us consider fluid living in $(3+1)-$dimension and is charged under a $U(1)$ current. Take

$$\mathfrak{F}^{\omega}_{anom}[T,\mu] = \mathcal{C}^{d=4}_{anom}\mu^3 + C^{d=4}_0 T\mu^2 + C^{d=4}_1 T^2\mu + C^{d=4}_2 T^2\mu \tag{5.83}$$

the constants $\{C^{d=4}_0, C^{d=4}_2\}$ if non-zero violate CPT since their subscript indices are even.

By the replacement rule of [30] this corresponds to a theory with the anomaly polynomial

$$\mathcal{P}_{anom} = \mathcal{C}^{d=4}_{anom}\mathcal{F}^3 - C^{d=4}_1\ p_1(\mathfrak{R}) \wedge \mathcal{F} \tag{5.84}$$

where $p_1(\mathfrak{R})$ is the first-pontryagin 4-form of curvature.

We have

$$d\bar{J}_{Consistent} = \mathcal{C}^{d=4}_{anom}\mathcal{F}^2$$
$$d\bar{J}_{Cov} = 3\mathcal{C}^{d=4}_{anom}\mathcal{F}^2$$

and their difference is given by

$$\bar{J}_{Cov} = \bar{J}_{Consistent} + 2\mathcal{C}^{d=4}_{anom}\hat{\mathcal{A}}\wedge\mathcal{F}$$

In components we have

$$\begin{aligned}
\nabla_\mu J^\mu_{Consistent} &= \mathcal{C}^{d=4}_{anom}\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma},\\
\nabla_\mu J^\mu_{Cov} &= 3\mathcal{C}^{d=4}_{anom}\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma},\\
J^\mu_{Cov} &= J^\mu_{Consistent} + 2\mathcal{C}^{d=4}_{anom}\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\hat{\mathcal{A}}_\nu\mathcal{F}_{\rho\sigma}
\end{aligned} \tag{5.85}$$

The anomaly-induced transport coefficients (in Landau frame) in this case are given by

$$\begin{aligned}
J^{\mu,anom}_{Cov} &= \xi^{d=4}_1\varepsilon^{\mu\nu\rho\sigma}u_\nu\partial_\rho\hat{\mathcal{A}}_\sigma + \xi^{d=4}_2\varepsilon^{\mu\nu\rho\sigma}u_\nu\partial_\rho u_\sigma\\
\xi^{d=4}_1 &= 3\mathcal{C}^{d=4}_{anom}\mu\left[\frac{q\mu}{\epsilon+p}-2\right] + 2C^{d=4}_0 T\left[\frac{q\mu}{\epsilon+p}-1\right] + C^{d=4}_1 T^2\mu^{-1}\left[\frac{q\mu}{\epsilon+p}\right]\\
\xi^{d=4}_2 &= \mathcal{C}^{d=4}_{anom}\mu^2\left[2\frac{q\mu}{\epsilon+p}-3\right] + C^{d=4}_0 T\mu\left[2\frac{q\mu}{\epsilon+p}-2\right]\\
&\quad + C^{d=4}_1 T^2\mu\left[2\frac{q\mu}{\epsilon+p}-1\right] + C^{d=4}_2 T^3\mu^{-1}\left[2\frac{q\mu}{\epsilon+p}\right]
\end{aligned} \tag{5.86}$$

and

$$
\begin{aligned}
J_S^{\mu,anom} &= -\frac{\mu}{T} J_{Cov}^{\mu,anom} + \chi_1^{d=4} \varepsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho \hat{\mathcal{A}}_\sigma + \chi_2^{d=4} \varepsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho u_\sigma + \zeta^{d=4} \varepsilon^{\mu\nu\rho\sigma} \hat{\mathcal{A}}_\nu \partial_\rho \hat{\mathcal{A}}_\sigma \\
\mathcal{G}_{Cov}^{\mu,anom} &= -T\chi_1^{d=4} \varepsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho \hat{\mathcal{A}}_\sigma - T\chi_2^{d=4} \varepsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho u_\sigma - T\zeta^{d=4} \varepsilon^{\mu\nu\rho\sigma} \hat{\mathcal{A}}_\nu \partial_\rho \hat{\mathcal{A}}_\sigma \\
-\zeta^{d=4} &= C_0^{d=4} \\
-\chi_1^{d=4} &= 3\mathcal{C}_{anom}^{d=4} T^{-1} \mu^2 + 2C_0^{d=4} \mu + C_1^{d=4} T \\
-\chi_2^{d=4} &= \mathcal{C}_{anom}^{d=4} T^{-1} \mu^3 + C_0^{d=4} \mu^2 + C_1^{d=4} T\mu + C_2^{d=4} T^2
\end{aligned}
\tag{5.87}
$$

The anomalous part of the consistent partition function is given by

$$
\begin{aligned}
&(\ln \mathcal{Z})_{anom}^{Consistent} \\
&= \int_{\text{space}} \frac{A}{T_0} \wedge \left\{ \left[C_0^{d=4}(-1)T_0 - 2\mathcal{C}_{anom}^{d=4} A_0 \right] (dA) + \left[C_1^{d=4} T_0^2 - \mathcal{C}_{anom}^{d=4} A_0^2 \right] (da) \right\} \\
&\quad + \int_{\text{space}} C_2^{d=4} (-1) T_0^2 a \wedge (da) \\
&= -\frac{\mathcal{C}_{anom}^{d=4}}{T_0} \int d^3x \sqrt{g_3} \epsilon^{ijk} \left[2A_0 A_i \partial_j A_k + A_0^2 A_i \partial_j a_k \right] \\
&\quad - C_0^{d=4} \int d^3x \sqrt{g_3} \epsilon^{ijk} A_i \partial_j A_k + C_1^{d=4} T_0 \int d^3x \sqrt{g_3} \epsilon^{ijk} A_i \partial_j a_k \\
&\quad - C_2^{d=4} T_0^2 \int d^3x \sqrt{g_3} \epsilon^{ijk} a_i \partial_j a_k
\end{aligned}
\tag{5.88}
$$

The results for the equilibrium partition function and the transport coefficients of the fluid have been obtained in [47] in great detail. We will now compare the results above against the results there. We begin by first fixing the relation between the notation here and the notation employed in [47]. Comparing our partition function in (5.88) against Eqn(1.11) of [47] we get a perfect match with the following relabeling of constants[48]

$$
\mathcal{C}_{anom}^{d=4} = \frac{C}{6}\ , \quad C_0^{d=4} = -C_0\ , \quad C_1^{d=4} = C_2\ , \quad C_2^{d=4} = -C_1 \tag{5.89}
$$

The first of these relations also follows independently from comparing our eqn(5.85) against the corresponding equations in [47] for covariant/consistent anomaly and the Bardeen current. We then proceed to compare the transport coefficients in Eqn(3.12) and Eqn.(3.21) of [47] against our results in (5.86) and (5.87).

We get a match provided one uses (in addition to (5.89)) the following relations arising from comparing definitions here against [47]

$$
\xi_B = \xi_1^{d=4}\ , \quad \xi_\omega = 2\xi_2^{d=4}\ , \quad D_B = \chi_1^{d=4}\ , \quad D_\omega = 2\chi_2^{d=4}\ , \quad h = \zeta^{d=4} \tag{5.90}
$$

[48] We warn the reader that the wedge notation in [47] differs from the one we use by numerical factors. So the comparisons are to be made *after* converting to explicit components to avoid confusion.

5.9.3 $(1+1)-$ dimensional anomalous fluids

Let us consider fluid living in $(1+1)-$dimension and is charged under a $U(1)$ current. Take

$$\mathfrak{F}^{\omega}_{anom}[T,\mu] = \mathcal{C}^{d=2}_{anom}\mu^2 + C^{d=2}_0 T\mu + C^{d=2}_1 T^2 \tag{5.91}$$

the constant $C^{d=2}_0$ if non-zero violates CPT since its subscript index is even.

By the replacement rule of [30] this corresponds to a theory with the anomaly polynomial

$$\mathcal{P}_{anom} = \mathcal{C}^{d=2}_{anom}\mathcal{F}^2 - C^{d=2}_1\ p_1(\mathfrak{R}) \tag{5.92}$$

where $p_1(\mathfrak{R})$ is the first-pontryagin 4-form of curvature.

We have

$$d\bar{J}_{Consistent} = \mathcal{C}^{d=2}_{anom}\mathcal{F}$$
$$d\bar{J}_{Cov} = 2\mathcal{C}^{d=2}_{anom}\mathcal{F}$$

and their difference is given by

$$\bar{J}_{Cov} = \bar{J}_{Consistent} + \mathcal{C}^{d=2}_{anom}\hat{\mathcal{A}}$$

In components we have

$$\begin{aligned}
\nabla_\mu J^\mu_{Consistent} &= \mathcal{C}^{d=2}_{anom}\frac{1}{2}\varepsilon^{\mu\nu}\mathcal{F}_{\mu\nu},\\
\nabla_\mu J^\mu_{Cov} &= 2\mathcal{C}^{d=2}_{anom}\frac{1}{2}\varepsilon^{\mu\nu}\mathcal{F}_{\mu\nu},\\
J^\mu_{Cov} &= J^\mu_{Consistent} + \mathcal{C}^{d=2}_{anom}\varepsilon^{\mu\nu}\hat{\mathcal{A}}_\nu
\end{aligned} \tag{5.93}$$

The anomaly-induced transport coefficients (in Landau frame) in this case are given by

$$\begin{aligned}
J^{\mu,anom}_{Cov} &= \xi^{d=2}_1\varepsilon^{\mu\nu}u_\nu\\
\xi^{d=2}_1 &= \mathcal{C}^{d=2}_{anom}\mu\left[\frac{q\mu}{\epsilon+p}-2\right] + C^{d=2}_0 T\left[\frac{q\mu}{\epsilon+p}-1\right] + C^{d=2}_1 T^2\mu^{-1}\left[\frac{q\mu}{\epsilon+p}\right]
\end{aligned} \tag{5.94}$$

and

$$\begin{aligned}
J^{\mu,anom}_S &= -\frac{\mu}{T}J^{\mu,anom}_{Cov} + \chi^{d=2}_1\varepsilon^{\mu\nu}u_\nu + \zeta^{d=2}\varepsilon^{\mu\nu}\hat{\mathcal{A}}_\nu\\
\mathcal{G}^{\mu,anom}_{Cov} &= -T\chi^{d=2}_1\varepsilon^{\mu\nu}u_\nu - T\zeta^{d=2}\varepsilon^{\mu\nu}\hat{\mathcal{A}}_\nu\\
-\zeta^{d=2} &= C^{d=2}_0\\
-\chi^{d=2}_1 &= \mathcal{C}^{d=2}_{anom}T^{-1}\mu^2 + C^{d=2}_0\mu + C^{d=2}_1 T
\end{aligned} \tag{5.95}$$

The anomalous part of the consistent partition function is given by

$$\begin{aligned}
&(\ln\mathcal{Z})^{Consistent}_{anom}\\
&= \int_{\text{space}}\frac{A}{T_0}\wedge\left[C^{d=2}_0(-1)T_0 - \mathcal{C}^{d=2}_{anom}A_0\right] + \int_{\text{space}} C^{d=2}_1 T_0 a\\
&= -\frac{\mathcal{C}^{d=2}_{anom}}{T_0}\int dx\sqrt{g_1}\epsilon^i A_0 A_i - C^{d=2}_0\int dx\sqrt{g_1}\epsilon^i A_i + C^{d=2}_1 T_0\int dx\sqrt{g_1}\epsilon^i a_i
\end{aligned} \tag{5.96}$$

Now we are all set to compare our results with the results of [48]. The comparison proceeds here the same way as the comparison in $3+1$d before. By comparing Eqn(2.4) of [48] against our (5.96) we get[49]

$$C^{d=2}_{anom} = C \ , \quad C^{d=2}_{0} = -C_1 \ , \quad C^{d=2}_{1} = -C_2 \ , \tag{5.97}$$

and we get a match of transport coefficients using the definitions

$$\xi_j = \xi^{d=2}_1 \ , \quad \xi_s + \frac{\mu}{T}\xi_j = \chi^{d=2}_1 \ , \quad D_\omega = 2\chi^{d=4}_2 \ , \quad h = \zeta^{d=2} \tag{5.98}$$

5.9.4 Hydrostatics and Anomalous transport

In this appendix we will follow [47, 48] in describing a hydrostatic configuration,i.e., a time-independent hydrodynamic configuration in a gauge/gravitational background. We will then proceed to evaluate the anomalous currents derived in previous appendix in this background. This is followed by a computation of consistent partition function by integrating the consistent Gibbs current over a spatial slice. For convenience we will phrase our entire discussion in the language of forms (as in the previous appendix) and refer the reader to the appendix5.9.6 for our form conventions.

Let us consider the special case where we consider a stationary (time-independent) spacetime with a metric given by

$$g_{spacetime} = -\gamma^{-2}(dt + a)^2 + g_{space}$$

where in the notation of [47]we can write $\gamma \equiv e^{-\sigma}$. Following the discussion there, consider a time-independent fluid configuration with local temperature and chemical potential T, μ and

placed in a time-independent gauge-field background

$$\hat{\mathcal{A}} = \mathcal{A}_0 dt + \mathcal{A}$$

We first compute

$$\begin{aligned} \mathcal{E} &\equiv \mathcal{F}_{\mu\nu}dx^\mu u^\nu = \gamma \mathcal{F}_{i0}dx^i = \gamma d\mathcal{A}_0 \\ \mathfrak{a} &\equiv u^\mu \nabla_\mu u_\nu dx^\nu = -\gamma^{-1}d\gamma = \gamma d\gamma^{-1} \\ dT + \mathfrak{a}T &= \gamma d\left(\gamma^{-1}T\right) \\ d\mu + \mathfrak{a}\mu - \mathcal{E} &= \gamma d\left(\gamma^{-1}\mu - \mathcal{A}_0\right) \end{aligned} \tag{5.99}$$

If we insist that

$$\begin{aligned} dT + \mathfrak{a}T &= 0 \\ d\mu + \mathfrak{a}\mu - \mathcal{E} &= 0 \end{aligned} \tag{5.100}$$

[49]Note that authors of [48] set the CPT-violating coefficient $C^{d=2}_0 = -C_1 = 0$ in most of their analysis. This fact has to be accounted for during the comparison.

then it follows that the quantities

$$T_0 \equiv \gamma^{-1} T \quad \text{and} \quad \mu_0 \equiv \gamma^{-1}\mu - \mathcal{A}_0$$

are constant across space. We can invert this to write

$$T = \gamma T_0 \quad \text{and} \quad \mu = \gamma\left(\mathcal{A}_0 + \mu_0\right) \equiv \gamma A_0$$

where we have defined $A_0 \equiv \mathcal{A}_0 + \mu_0$.Following [47]we will split the gauge field as

$$\hat{\mathcal{A}} = \mathcal{A}_0 dt + \mathcal{A} = A_0(dt + a) + A - \mu_0 dt$$

where $A \equiv \mathcal{A} - A_0\ a$. We are now working in a general gauge - often it is useful to work in a specific gauge : one gauge we will work on is obtained from this generic gauge by performing a gauge transformation to remove the $\mu_0 dt$ piece. We will call this gauge as the 'zero μ_0' gauge. In this gauge the new gauge field is given in terms of the old gauge field via

$$\hat{\mathcal{A}}_{\mu_0=0} \equiv \hat{\mathcal{A}} + \mu_0 dt$$

We will quote all our consistent currents in this gauge.

We are now ready to calculate various hydrostatic quantities

$$\begin{aligned}
\mathcal{E} &= \gamma d\mathcal{A}_0 = \gamma dA_0 \\
\mathfrak{a} &= -\gamma^{-1} d\gamma = \gamma d\gamma^{-1} \\
\mathcal{B} &\equiv \mathcal{F} - u \wedge \mathcal{E} = d\left[A_0(dt+a) + A - \mu_0 dt\right] + (dt+a) \wedge dA_0 \\
&= dA + A_0 da \\
2\omega &= du + u \wedge \mathfrak{a} = -\gamma^{-1} da \\
2\omega T &= -T_0 da \\
2\omega\mu &= -A_0 da \\
\hat{A} + \mu u &= A - \mu_0 dt \\
\mathcal{B} + 2\omega\mu &= dA
\end{aligned} \tag{5.101}$$

Now let us compute the various anomalous currents in terms of the hydrostatic fields. Using (5.101) we get the Gibbs current as

$$\begin{aligned}
&-\bar{\mathcal{G}}^{Cov}_{anom} \\
&= \gamma \sum_{m=1}^{n} \left[C_m (-1)^{m-1} T_0^{m+1} - C_0 (-1)^{0-1} \binom{n}{m} T_0 A_0^m \right. \\
&\quad \left. + m \binom{n+1}{m+1} \mathcal{C}_{anom} A_0^{m+1} \right] (da)^{m-1} (dA)^{n-m} \wedge (dt + a) \\
&\quad - \gamma C_0 T_0 \hat{\mathcal{A}}_{\mu_0=0} \wedge \mathcal{F}^{n-1}
\end{aligned} \tag{5.102}$$

In the following we will always write the minus signs in the form $C_m(-1)^{m-1}$ so that once we impose CPT all the minus signs could be dropped.

We can now calculate the charge/entropy/energy currents

$$\begin{aligned}\bar{J}^{Cov}_{anom} = \sum_{m=1}^{n} \Big[& -(n+1-m)C_{m-1}(-1)^{m-2}T_0^m \\ & +(n+1)\binom{n}{m}\mathcal{C}_{anom}A_0^m\Big](da)^{m-1}\wedge(dA)^{n-m}\wedge(dt+a)\end{aligned} \tag{5.103}$$

$$\begin{aligned}\bar{J}^{Cov}_{S,anom} = \sum_{m=1}^{n} \Big[& (m+1)C_m(-1)^{m-1}T_0^m \\ & -C_0(-1)^{0-1}\binom{n}{m}A_0^m\Big](da)^{m-1}(dA)^{n-m}\wedge(dt+a) \\ & - C_0\hat{\mathcal{A}}_{\mu_0=0}\wedge\mathcal{F}^{n-1}\end{aligned} \tag{5.104}$$

and

$$\begin{aligned}& \bar{q}^{Cov}_{anom} \\ & = \gamma\sum_{m=1}^{n}\Big[mC_m(-1)^{m-1}T_0^{m+1} - (n+1-m)C_{m-1}(-1)^{m-2}T_0^m A_0 \\ & \quad +\binom{n+1}{m+1}\mathcal{C}_{anom}A_0^{m+1}\Big](da)^{m-1}(dA)^{n-m}\wedge(dt+a)\end{aligned} \tag{5.105}$$

We can go to the Landau frame as before

$$\begin{aligned} u^\mu &\mapsto u^\mu - \frac{q^\mu_{anom}}{\epsilon+p} \\ J^\mu_{anom} &\mapsto J^\mu_{anom} - q\frac{q^\mu_{anom}}{\epsilon+p} \\ J^\mu_{S,anom} &\mapsto J^\mu_{S,anom} - s\frac{q^\mu_{anom}}{\epsilon+p} \\ q^\mu_{anom} &\mapsto 0 \end{aligned} \tag{5.106}$$

In the Landau frame we can write the corrections to various quatities as

$$\begin{aligned} \delta\bar{u} &\equiv -\gamma^{-1}\sum_{m=1}^{n} U_m(da)^{m-1}\wedge(dA)^{n-m}\wedge(dt+a) \\ \delta\bar{J}^{Cov}_{anom} &\equiv -\gamma^{-1}\sum_{m=1}^{n}(J_m + q\ U_m)(da)^{m-1}\wedge(dA)^{n-m}\wedge(dt+a) \\ \delta\bar{J}^{Cov}_{S,anom} &\equiv -\gamma^{-1}\sum_{m=1}^{n}(S_m + s\ U_m)(da)^{m-1}\wedge(dA)^{n-m}\wedge(dt+a) \end{aligned} \tag{5.107}$$

where

$$
\begin{aligned}
U_m &= -\frac{\gamma^2}{\epsilon+p}\left[mC_m(-1)^{m-1}T_0^{m+1} - (n+1-m)C_{m-1}(-1)^{m-2}T_0^m A_0\right.\\
&\qquad\left. + \binom{n+1}{m+1}\mathcal{C}_{anom}A_0^{m+1}\right]\\
J_m + q\ U_m &= \gamma\left[(n+1-m)C_{m-1}(-1)^{m-2}T_0^m - (n+1)\binom{n}{m}\mathcal{C}_{anom}A_0^m\right]\\
S_m + s\ U_m &= \gamma\left[-(m+1)C_m(-1)^{m-1}T_0^m + C_0(-1)^{0-1}\binom{n}{m}A_0^m\right]
\end{aligned}
\tag{5.108}
$$

which matches with expressions from the partition function.

The corresponding consistent currents can be obtained via the relations

$$
\begin{aligned}
\bar{\mathcal{G}}^{Cov}_{anom} &= \bar{\mathcal{G}}^{Consistent}_{anom} - \mu\ n\ \mathcal{C}_{anom}\hat{\mathcal{A}}\wedge\mathcal{F}^{n-1}\\
\bar{J}^{Cov}_{anom} &= \bar{J}^{Consistent}_{anom} + n\ \mathcal{C}_{anom}\hat{\mathcal{A}}\wedge\mathcal{F}^{n-1}\\
\bar{J}^{Cov}_{S,anom} &= \bar{J}^{Consistent}_{S,anom}\\
\bar{q}^{Cov}_{anom} &= \bar{q}^{Consistent}_{anom}
\end{aligned}
\tag{5.109}
$$

In particular we have

$$
\begin{aligned}
-&\frac{1}{T}\bar{\mathcal{G}}^{Consistent}_{anom}\\
&= \frac{1}{T_0}\sum_{m=1}^{n}\left[C_m(-1)^{m-1}T_0^{m+1} - C_0(-1)^{0-1}\binom{n}{m}T_0A_0^m\right.\\
&\qquad\left.-\binom{n}{m+1}\mathcal{C}_{anom}A_0^{m+1}\right](da)^{m-1}(dA)^{n-m}\wedge(dt+a)\\
&\quad-\frac{1}{T_0}\left[n\mathcal{C}_{anom}A_0 + C_0T_0\right]A\wedge(dA+A_0da)^{n-1}\\
&\quad-\frac{(n-1)}{T_0}\left[n\mathcal{C}_{anom}A_0 + C_0T_0\right]A\wedge dA_0\wedge(dt+a)\wedge(dA+A_0da)^{n-2}
\end{aligned}
\tag{5.110}
$$

5.9.5 Variational formulae in forms

The energy current is defined via the relation

$$
\begin{aligned}
q_\mu dx^\mu &\equiv -T_{\mu\nu}u^\mu dx^\nu\\
&= -\gamma T_{00}(dt+a) - \gamma g_{ij}T_0^i dx^j
\end{aligned}
\tag{5.111}
$$

Hence its Hodge dual is (See 5.9.6 for the definition of Hodge dual)

$$
\bar{q} = \gamma^3 T_{00}d\forall_{d-1} + \gamma T_0^i(dt+a)\wedge(d\Sigma_{d-2})_i
\tag{5.112}
$$

We take the following relations[50] from Eqn(2.16) of [47]

$$
\begin{aligned}
\gamma T_{00} d\forall_{d-1} &= \frac{\delta}{\delta\gamma}\left(T_0 \ln\ \mathcal{Z}\right) \\
T_0^i d\forall_{d-1} &= dx^i \wedge T_0^j \left(d\Sigma_{d-2}\right)_j = \left[\frac{\delta}{\delta a_i} - A_0 \frac{\delta}{\delta A_i}\right]\left(T_0 \ln\ \mathcal{Z}\right)
\end{aligned}
\tag{5.113}
$$

where the independent variables are $\{\gamma, a, g^{ij}, A_0, A, T_0, \mu_0\}$. Converting into forms

$$
\begin{aligned}
\bar{q} &= \left[\gamma^2 \frac{\delta}{\delta\gamma} + \gamma(dt+a)\wedge\frac{\delta}{\delta a} - \gamma A_0 (dt+a)\wedge\frac{\delta}{\delta A}\right]\left(T_0 \ln \mathcal{Z}\right) \\
&= \left[\gamma^2 \frac{\delta}{\delta\gamma} + \gamma(dt+a)\wedge\frac{\delta}{\delta a} - \mu (dt+a)\wedge\frac{\delta}{\delta A}\right]\left(T_0 \ln \mathcal{Z}\right)
\end{aligned}
\tag{5.114}
$$

Similarly for the charge current

$$
\begin{aligned}
-\gamma^2 J_0 d\forall_{d-1} &= \frac{\delta}{\delta A_0}\left(T_0 \ln\ \mathcal{Z}\right) \\
J^i d\forall_{d-1} &= dx^i \wedge J^j \left(d\Sigma_{d-2}\right)_j = \frac{\delta}{\delta A_i}\left(T_0 \ln\ \mathcal{Z}\right)
\end{aligned}
\tag{5.115}
$$

which implies

$$
\begin{aligned}
\bar{J} &\equiv -\gamma^2 J_0 d\forall_{d-1} - J^i (dt+a)\wedge\left(d\Sigma_{d-2}\right)_i \\
&= \left[\frac{\delta}{\delta A_0} - (dt+a)\wedge\frac{\delta}{\delta A}\right]\left(T_0 \ln \mathcal{Z}\right)
\end{aligned}
\tag{5.116}
$$

Putting $T_0 \ln \mathcal{Z} = -\gamma^{-1}\bar{\mathcal{G}}$ we can write

$$
\begin{aligned}
\bar{J} &\equiv -\frac{\partial\bar{\mathcal{G}}}{\partial\mu} = -\gamma^{-1}\left[\frac{\delta}{\delta A_0} - (dt+a)\wedge\frac{\delta}{\delta A}\right]\bar{\mathcal{G}} \\
\bar{J}_S &\equiv -\frac{\partial\bar{\mathcal{G}}}{\partial T} = -\gamma^{-1}\frac{1}{T_0}\left[\gamma\frac{\delta}{\delta\gamma} + (dt+a)\wedge\frac{\delta}{\delta a} - A_0\frac{\delta}{\delta A_0}\right]\bar{\mathcal{G}} \\
\bar{q} &= \bar{\mathcal{G}} + T\bar{J}_S + \mu\bar{J}
\end{aligned}
\tag{5.117}
$$

5.9.6 Convention for Forms

The inner product between two 1-forms $J \equiv J_0(dt+a) + g_{ij}J^i dx^j$ and $J' \equiv J'_0(dt+a) + g_{ij}(J')^i dx^j$ is given in terms of the KK-invariant components as

$$
\langle J, J'\rangle \equiv -\gamma^2 J_0 J'_0 + g_{ij} J^i (J')^j \tag{5.118}
$$

In general, the exterior derivative of a p-form

$$
A_p \equiv \frac{1}{p!} A_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}
$$

[50]we remind the reader that $\gamma \equiv e^{-\sigma}$ and $d\forall_{d-1} = d^{d-1}x\sqrt{-\det\ g_d}$

is given by

$$
\begin{aligned}
(dA)_{p+1} &\equiv \frac{1}{p!}\partial_\lambda A_{\mu_1\ldots\mu_p} dx^\lambda \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \\
&= \frac{1}{(p+1)!}\left[\partial_{\mu_1} A_{\mu_2\ldots\mu_{p+1}} + \text{cyclic}\right] dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{p+1}}
\end{aligned}
\tag{5.119}
$$

The Levi-Civita tensor $\varepsilon^{\mu_1\cdots\mu_d}$ is defined as the completely antisymmetric tensor with

$$
\varepsilon^{012\ldots(d-1)} = \frac{1}{\sqrt{-\det\ g_d}} = \frac{1}{\gamma^{-1}\sqrt{\det\ g_{d-1}}}
$$

We will also often define the spatial Levi-Civita tensor $\epsilon^{i_1 i_2\ldots i_{d-1}}$ such that

$$
\epsilon^{12\ldots(d-1)} = \frac{1}{\sqrt{\det\ g_{d-1}}}
$$

which is related to its spacetime counterpart via

$$
\epsilon^{i_1 i_2\ldots i_{d-1}} = \gamma^{-1}\varepsilon^{0 i_1 i_2\ldots i_{d-1}}
$$

Let us define the spatial volume $(d-1)$-form as

$$
\begin{aligned}
d\forall_{d-1} &\equiv \gamma^{-1}\epsilon_{i_1\ldots i_{d-1}} dx^{i_1} \otimes \ldots \otimes dx^{i_{d-1}} \\
&= \frac{1}{(d-1)!}\gamma^{-1}\epsilon_{i_1\ldots i_{d-1}} dx^{i_1} \wedge \ldots \wedge dx^{i_{d-1}} \\
&= d^{d-1}x\ \gamma^{-1}\sqrt{\det\ g_{d-1}} \\
&= d^{d-1}x\ \sqrt{-\det\ g_d}
\end{aligned}
\tag{5.120}
$$

where $\epsilon_{i_1\ldots i_{d-1}}$ is the spatial Levi-Civita symbol. The form $d\forall_{d-1}$ transforms like a vector with a lower time-index and hence is KK-invariant.

Define the spatial area $(d-2)$-form as

$$
\begin{aligned}
(d\Sigma_{d-2})_j &\equiv \gamma^{-1}\epsilon_{j i_1\ldots i_{d-2}} dx^{i_1} \otimes \ldots \otimes dx^{i_{d-2}} \\
&= \frac{1}{(d-2)!}\gamma^{-1}\epsilon_{j i_1\ldots i_{d-2}} dx^{i_1} \wedge \ldots \wedge dx^{i_{d-2}}
\end{aligned}
\tag{5.121}
$$

This transforms like a vector with a lower time-index and a lower spatial index but is anti-symmetric in these two indices and is hence KK-invariant. The area $(d-2)$-form satisfies

$$
dx^i \wedge (d\Sigma_{d-2})_j = d\forall_{d-1}\ \delta^i_j
$$

The Hodge-dual of a 1-form $J \equiv J_0(dt+a) + g_{ij}J^i dx^j$ is defined as

$$
\bar{J} = -\gamma^2 J_0 d\forall_{d-1} - J^i(dt+a) \wedge (d\Sigma_{d-2})_i \tag{5.122}
$$

This is defined such that

$$J' \wedge \bar{J} = \langle J', J\rangle (dt+a) \wedge d\forall_{d-1} = \langle J', J\rangle dt \wedge d\forall_{d-1} \tag{5.123}$$

In particular

$$d\bar{J} = (\nabla_\mu J^\mu)\, dt \wedge d\forall_{d-1} \tag{5.124}$$

One often useful formula is this

$$\begin{gathered} \bar{J} = \hat{\mathcal{A}} \wedge (d\hat{\mathcal{A}})^{n-1} \\ \text{is equivalent to} \\ J^\mu = \left[\varepsilon \hat{\mathcal{A}}\ (\partial \hat{\mathcal{A}})^{n-1}\right]^\mu \end{gathered} \tag{5.125}$$

Let us take another example which will recur throughout this section - say we are given that the Hodge-dual of a 1-form $J \equiv J_0(dt+a) + g_{ij}J^i dx^j$ is

$$-\bar{J} = A \wedge (da)^{m-1}(dA)^{n-m} + A_0(dt+a) \wedge (da)^{m-1}(dA)^{n-m}$$

where $a = a_i dx^i$ and $A = A_i dx^i$ are two arbitrary 1-forms with only spatial components.

Then we can invert the Hodge-dual using the following statement

$$\begin{aligned} \bar{J} &= -A \wedge (da)^{m-1}(dA)^{n-m} - A_0(dt+a) \wedge (da)^{m-1}(dA)^{n-m} \\ &\text{is equivalent to} \\ J_0 &= \gamma^{-1}\left[\epsilon A (da)^{m-1}(dA)^{n-m}\right] \\ J^i &= \gamma A_0 \left[\epsilon (da)^{m-1}(dA)^{n-m}\right]^i \end{aligned} \tag{5.126}$$

6 Constraints on superfluids from equilibrium partition function

6.1 Introduction

In this section we discuss the application of the equilibrium partition function method to case of superfluids. The equations of charged hydrodynamics are modified when the charge symmetry of the system is spontaneously broken by the condensation of a charged operator in thermal equilibrium. The effective description of such systems has new hydrodynamical degrees of freedom whose origin lies in the Goldstone mode of the charge condensate. The resultant hydrodynamical equations are referred to as the equations of superfluid hydrodynamics, and are the subject of the current section.

More particularly in this section we study 's' wave superfluid hydrodynamics, i.e. the hydrodynamics of a system whose charge condensate is a complex scalar operator. We study the constraints on the equations of first order 's' wave superfluid hydrodynamics imposed by the requirement that these equations admit equilibrium under appropriate situations, and that the charge currents in equilibrium agree with those from an appropriate partition function. We do not assume that the superfluids we study necessarily preserve either parity or time reversal invariance.

As we explain in section 6.2 below, the general analysis presented in this section closely follows that of [47] (for the case of ordinary, i.e. not 'super' fluids) with one important difference. The Euclidean partition function for a superfluid in an arbitrary background [51] is determined by an effective field theory that includes a massless mode: the Goldstone boson of the theory. This effective field theory is local, and may usefully be studied in the derivative expansion. However the partition function that follows after integrating out the Goldstone boson is neither local nor simple. As we explain below, the study of the local effective action of the Goldstone boson (rather than the partition function itself) allows us to usefully constrain the constitutive relations of superfluid hydrodynamics. In this section we present a careful derivation of the relations between otherwise independent transport functions that follow from such a study.

Constraints on the constitutive relations of first order superfluid hydrodynamics have previously been obtained using the local form of the second law in [2, 4, 7, 55] for the case of time reversal invariant superfluids. In this section we generalize the derivation of [4] to include the study of superfluids that do not preserve time reversal invariance. We then compare the results obtained from the two different methods; i.e. the constraints that follow from the requirement of existence of equilibrium and those that follow from the local second law. As in the case of ordinary (i.e. non super) fluids we find perfect agreement between the equality type constraints obtained from these two apparently distinct methods. Our results supply further evidence for the conjecture that the equality type constraints from these two methods agree in a wide range of hydrodynamical contexts and to all orders in the derivative expansion.

[51] See [31, 49] for a discussion of this partition function at the perfect fluid level.

A proof of this conjecture would go some way towards proving the local form of the second law, and would permit the demystification of this law in a hydrodynamical context.

While the work reported in this section is purely hydrodynamical and nowhere uses AdS/CFT, much of the motivation for this work lies within the fluid gravity map of AdS/CFT. The status of the second law of thermodynamics for theories of gravity that include higher derivative corrections to the Einstein Lagrangian remains unclear. In particular it has never been proved that the Hawking area increase theorem generalizes to a Wald entropy increase theorem for arbitrary higher derivative corrections to Einstein's gravity. If the interplay between the existence of equilibrium in appropriate circumstances and entropy increase can be proved on general grounds in a hydrodynamical context, then it seems likely that the lessons learnt can be taken over to the study of entropy increase in higher derivative gravity (at least for asymptotically AdS space) via the fluid gravity map. This could lead to a proof of a Wald entropy increase theorem under appropriate conditions on the higher derivative corrections of the gravitational system.

6.2 Equilibrium effective action for the Goldstone mode

6.2.1 The question addressed

In this subsubsection we study an s wave superfluid propagating on the stationary background metric

$$ds^2 = G_{\mu\nu}dx^\mu dx^\nu = -e^{2\sigma(\vec{x})}\left(dt + a_i(\vec{x})dx^i\right)^2 + g_{ij}(\vec{x})dx^i dx^j \tag{6.1}$$

and background gauge field

$$A = \mathcal{A}_0(\vec{x})dx^0 + \mathcal{A}_i(\vec{x})dx^i \tag{6.2}$$

Below we will often work in terms of the modified gauge fields

$$\begin{aligned} A_i &= \mathcal{A}_i - A_0 a_i \\ A_0 &= \mathcal{A}_0 + \mu_0 \end{aligned} \tag{6.3}$$

All background fields above are assumed to vary slowly; we work in an expansion in derivatives of these fields. We address the following question: what is the most general allowed form of the partition function

$$Z = \text{Tr} e^{-\frac{H-\mu_0 Q}{T_0}} \tag{6.4}$$

as a function of the background fields σ, a_i, g_{ij}, A_0 and A_i in a systematic derivative expansion?

6.2.2 The partition function for charged (non super) fluids

The analogous question was studied for the case of an ordinary (non super) charged fluid in [47]. It was demonstrated that to first order in the derivative expansion the most general allowed form of the partition function for an ordinary charged fluid on the background (6.1), (6.2) is given by

$$\begin{aligned} W &= \ln Z = W^0 + W^1_{inv} + W^1_{anom} \\ W^0 &= \int \sqrt{g}\frac{e^\sigma}{T_0} P\left(T_0 e^{-\sigma}, e^{-\sigma}A_0\right) \\ W^1_{inv} &= \frac{C_0}{2}\int AdA + \frac{C_1}{2}\int ada + \frac{C_2}{2}\int Ada \\ W^1_{anom} &= \frac{C}{2}\left(\int \frac{A_0}{3}AdA + \frac{A_0^2}{6}Ada\right) \end{aligned} \tag{6.5}$$

where $P(T, \mu)$ is the thermodynamical pressure of the system as a function of its temperature and chemical potential and C_0, C_1 C_2 and C are all constants. The constant C specifies the covariant $U(1)^3$ anomaly via the equation

$$\partial_\mu \tilde{J}^\mu = -\frac{C}{8} * (F \wedge F) \tag{6.6}$$

The constants C_0, C_1 and C_2 do not (yet) have similar interpretations. It was demonstrated that $C_0 = C_1 = 0$ in any system that respects CPT invariance.

Notice that the result (6.5) for the partition function of an ordinary (non super) fluid is a *local* function of the background sources g_{ij}, a_i, σ, A_0 and A_i. Locality is a direct consequence of the fact that the path integral that computes the partition function (6.4) has a unique hydrodynamical saddle point (as opposed to a moduli space of saddle points). As a consequence the partition function is generically [52] computed by a path integral over an action with no massless fields. It follows that the result is local on length scales large compared to the inverse mass gap in the action (this mass gap is sometimes referred to as a static screening length of the 4 d thermal system)[53].

6.2.3 Euclidean action for the Goldstone mode for superfluids

Unlike an ordinary charged fluid, the equilibrium configuration of a superfluid in the background (6.1) is not unique. As superfluids break the global $U(1)$ symmetry, every background admits at least a one parameter set of equilibrium configurations that differ by a constant shift in the phase of the expectation value of the condensed scalar. It follows that the path integral that computes (6.4) has a zero mode (the phase of the scalar condensate). Consequently, the partition function (6.4), is *not* a local function of the background source fields. Instead this partition function is generated by a local three dimensional field theory of the *dynamical* phase field ϕ.

The dynamics of the Goldstone boson in general, governed by a 3d massless quantum field theory. In this subsubsection, however, we focus on field theories in an appropriate large

[52] Non hydrodynamical massless modes occur when the system is tuned to a second order phase transition. We assume in what follows that our system has not been tuned to such a phase transition. We leave the study of this interesting special case [56] to future work.

[53] We thank K. Jensen for discussions on this topic

N limit (such as theories with matrix degrees of freedom in the t' Hooft limit). In such a limit the effective action for the Goldstone boson is multiplied by a suitable positive power of N (the factor is N^2 in the t'Hooft limit mentioned above). As a consequence Goldstone dynamics is effectively classical in the large N limit. Quantum corrections to this classical answer, which are suppressed by appropriate powers of N (this power is $\frac{1}{N^2}$ in the t'Hooft limit), may have very interesting structure, see e.g. [20–23] for related work. We leave their study to future work. [54]

In principle, the partition function (6.4) for the superfluid may be obtained from the corresponding local effective action by integrating out the Goldstone boson (i.e. solving its equation of motion and plugging the solution back into the action). [55] In practice the implementation of this procedure requires the solution of a nonlinear partial differential equation. Moreover, even if one could solve this equation the resultant partition function would be highly nonlocal. A direct analysis of the partition function itself seems neither easy nor particularly useful. In order to obtain constraints on the equations of superfluid hydrodynamics below we will work directly with the local effective action for the Goldstone mode rather than the final result for the partition function.

The requirements of gauge invariance significantly constrain the form of Goldstone effective action. Let ϕ denote the phase of the scalar condensate. Under a gauge transformation $\mathcal{A}_i \to \mathcal{A}_i + \partial_i \alpha$, ϕ transforms as $\phi + \alpha$. It follows that the effective action can only depend on the combination

$$\xi_i = -\partial_i \phi + \mathcal{A}_i$$

Note that ξ_μ like $\mathcal{A}_\mu$, is a field of zero order in the derivative expansion [56].

The local field theory for the Goldstone boson must also enjoy invariance under Kaluza Klein gauge transformations ($a_i \to a_i - \partial_i \gamma$, see subsection 2.2 of [47] for details). For this reason we work with the Kaluza Klein invariant fields

$$\zeta_i = \xi_i - a_i A_0 = -\partial_i \phi + A_i. \tag{6.7}$$

We also define

$$\xi_0 = A_0$$

and define

$$\chi = \xi^2 = -\xi_\mu \xi^\mu = \xi_0^2 e^{-2\sigma} - g^{ij} \zeta_i \zeta_j. \tag{6.8}$$

[54] We thank K. Jensen for discussions on this topic.

[55] If the Euclidean 3 dimensional manifold we work on is compact and we demand single valuedness of the field ϕ then it is plausible that the solution to the ϕ equation of motion is (at least generically) unique, see below.

[56] This means that the phase field ϕ is of -1 order in derivatives; this observation does not invalidate the derivative expansion as gauge invariant physical quantities are functions only of ξ^μ and not independently of ϕ.

6.2.4 The Goldstone action for perfect superfluid hydrodynamics

As we have explained above, the euclidean partition function for our system is generated by an effective action S for the Goldstone field ϕ. This Goldstone action may be expanded in a power series in derivatives.

$$S = S_0 + S_1 + S_2 \ldots \tag{6.9}$$

At lowest (zero) order in the derivative expansion symmetries constrain the Goldstone boson effective action to take the form[57]

$$\begin{aligned} S_0 &= \int d^3x \sqrt{g} \frac{1}{\hat{T}} P(\hat{T}, \hat{\mu}, \chi). \\ \hat{T} &= T_0 e^{-\sigma} \\ \hat{\mu} &= A_0 e^{-\sigma} \\ \hat{u}^\mu &= (1,0,0,0) e^{-\sigma} \end{aligned} \tag{6.10}$$

where P is an arbitrary function whose thermodynamical significance we will soon discover, and χ was defined in (6.8). The fields $\hat{T}$, $\hat{\mu}$ and $\hat{u}^\mu$ are the values of the hydrodynamical temperature, chemical potential and velocity fields in equilibrium at zeroth order in the derivative expansion (see [47]).

In the classical (or large N) limit adopted throughout this section, the partition function Z of our system is obtained by evaluating the Goldstone action on shell. Let the solution to the equation of motion be denoted by

$$\zeta_i(x) = \zeta_i^{eq}(x).$$

Then the partition function is given by

$$\ln Z = S(\zeta_i^{eq}(x)) \tag{6.11}$$

At lowest order in the derivative expansion, the action (6.10) depends only on first derivatives of the massless field ϕ. Varying this action w.r.t. ϕ

$$\begin{aligned} \delta S_0 &= \int d^3x \sqrt{g} \frac{e^\sigma}{T_0} \frac{\partial P}{\partial \chi} 2 g^{ij} \zeta_i \partial_j \delta\phi \\ &= -\int d^3x \frac{1}{T_0} \partial_j (\sqrt{-G} f \zeta^j) \delta\phi \end{aligned} \tag{6.12}$$

yields

$$\partial_j (\sqrt{-G} f \zeta^j) = \nabla_\mu^{(4)} (f \xi^\mu) = \nabla_i \left(\frac{f}{T} \zeta^i \right) = 0. \tag{6.13}$$

where

$$f = 2 \frac{\partial P}{\partial \chi}.$$

[57] The action (6.10) was already presented in [49]. The presentation of this subsection differs from [49] only in the emphasis that ϕ be regarded as a dynamical field in (6.10), rather than a background like $\hat{T}$. For related discussions on effective action for superfluid, see for example [31?].

Note this equation of motion is of second order in derivatives of the field ϕ. [58] Plugging the solution to (6.13) back into the (6.12) in principle yields an explicit though complicated and nonlocal expression for the partition function of the system as a function of source fields.

The stress tensor and charge current that follow from the action (6.10) may be computed in a straightforward manner using the formulas listed in eqs.(2.16) of [47]; they are given by

$$\begin{aligned}
J_0 &= -\frac{T_0 e^{\sigma}}{\sqrt{g}}\frac{\delta S_0}{\delta A_0} = -e^{2\sigma}\left[e^{-\sigma}\frac{\partial P}{\partial \mu} + \frac{\partial P}{\partial \chi}\frac{\partial \chi}{\partial A_0}\right] = -q e^{\sigma} - \xi_0 f \\
J^i &= \frac{T_0 e^{-\sigma}}{\sqrt{g}}\frac{\delta S_0}{\delta A_i} = \frac{\partial P}{\partial \chi}\frac{\partial \chi}{\partial A_i} = -f\xi^i \\
T_{00} &= -\frac{T_0 e^{\sigma}}{\sqrt{g}}\frac{\delta S_0}{\delta \sigma} = -e^{2\sigma}\left[P + \frac{\partial P}{\partial T_0 e^{-\sigma}}\frac{\partial T_0 e^{-\sigma}}{\partial \sigma} + \frac{\partial P}{\partial \mu}\frac{\partial \mu}{\partial \sigma} + \frac{\partial P}{\partial \chi}\frac{\partial \chi}{\partial \sigma}\right] \\
&= -e^{2\sigma}\left[P - sT - q\mu - f\xi_0^2 e^{-2\sigma}\right] = e^{2\sigma}\epsilon + f\xi_0^2 \\
T_0^i &= \frac{T_0}{e^{\sigma}\sqrt{g}}\left[\frac{\delta S_0}{\delta a_i} - A_0\frac{\delta S_0}{\delta A_i}\right] = \frac{\partial P}{\partial a_i} - A_0\frac{\partial P}{\partial A_i} = -A_0\frac{\partial P}{\partial \chi}\frac{\partial \chi}{\partial A_i} = f A_0 \xi^i \\
T^{ij} &= \frac{-2T_0}{e^{\sigma}\sqrt{g}} g^{ik}g^{jl}\frac{\delta S_0}{\delta g^{kl}} = -2g^{ik}g^{jl}\left[-\frac{1}{2}g_{kl}P + \frac{\partial P}{\partial \chi}\frac{\partial \chi}{\partial g^{kl}}\right] = P g^{ij} + f\xi^i\xi^j
\end{aligned} \tag{6.14}$$

The gauge and diffeomorphism invariance of the action (6.10) ensure the stress tensor and charge current described above are automatically conserved onshell (i.e. upon imposing the equation of motion (6.13)).

The complicated looking expressions for the conserved currents (6.14) may actually be summarized in a remarkably simple form as

$$\begin{aligned}
T^{\mu\nu} &= (\epsilon + P)\hat{u}^{\mu}\hat{u}^{\nu} + P g^{\mu\nu} + f\xi^{\mu}\xi^{\nu} \\
J^{\mu} &= q\hat{u}^{\mu} - f\xi^{\mu},
\end{aligned} \tag{6.15}$$

where $\hat{u}$ was defined in (6.10) and all terms on the RHS of (6.15) are evaluated on the zero order equilibrium solutions $T(x) = \hat{T}$ and $\mu(x) = \hat{\mu}$, defined in (6.10) and the functions ϵ, s and q are defined in terms of the pressure p by the equations

$$\begin{aligned}
\epsilon + P &= sT + q\mu \\
dP &= sdT + qd\mu + \frac{1}{2}f d\chi
\end{aligned} \tag{6.16}$$

(6.15) and (6.33) are precisely the Landau-Tisza constitutive relations of superfluid hydrodynamics.

6.3 The Goldstone Action at first order in derivatives

One derivative corrections to the Goldstone action (6.10) may be divided into parity even and parity odd terms. We consider these in turn.

[58] The formal similarity of (6.13) to the equation $\nabla^2\phi = 0$ (where the Laplacian is taken in an appropriately rescaled metric) suggests that (6.13) has a unique solution on a compact manifold (up to constant shift in ϕ) provided that ϕ is required to be single valued and smooth on this manifold. However we do not have a proof of this statement.

6.3.1 Parity even one derivative corrections

The most general parity preserving one derivative correction to (6.10) is given by

$$
\begin{aligned}
S &= S_0 + S_1^{even} \\
S_1^{even} &= \int d^3y\sqrt{g}\left[\frac{f_1}{\hat{T}}(\zeta.\partial)\hat{T} + \frac{f_2}{\hat{T}}(\zeta.\partial)\hat{\nu} - f_3\nabla_i\left(\frac{f}{\hat{T}}\zeta^i\right)\right]
\end{aligned}
\tag{6.17}
$$

where $\hat{T}$ was defined in (6.10),

$$
\hat{\nu} = \frac{\hat{\mu}}{\hat{T}} = \frac{A_0}{T_0}
$$

and

$$
f_i = f_i(\hat{T}, \hat{\nu}, \zeta^2) \quad (i = 1\ldots3)
$$

are arbitrary functions while f was defined in the previous subsection

$$
f(\hat{T}, \hat{\nu}, \zeta^2) = -2\frac{\partial P}{\partial \zeta^2}
$$

Two remarks are in order

- 1. In (6.17) the unspecified function f_3 multiplies the zero order equation of motion of the phase field ϕ. As a consequence, under the field redefinition

  $$
  \begin{aligned}
  \phi &= \tilde{\phi} + \delta\phi(\hat{T}, \hat{\nu}, \zeta) \\
  \Rightarrow \xi_\mu &= \tilde{\xi}_\mu - \partial_\mu\left(\delta\phi\right)
  \end{aligned}
  \tag{6.18}
  $$

  we find

  $$
  S_0[\phi] = S_0[\tilde{\phi}] - \int d^3x\sqrt{g}\nabla_j\left(\frac{f}{\hat{T}}\zeta^j\right)\delta\phi \tag{6.19}
  $$

  In other words we are free to use the variable $\tilde{\phi}$ instead of ϕ; however the first derivative correction with this choice of variable, $\tilde{S}_1^{even}$, differs from S_1^{even} by

  $$
  \tilde{S}_1^{even} = S_1^{even} - \int d^3x\sqrt{g}\nabla_j\left(\frac{f}{\hat{T}}\zeta^j\right)\delta\phi \tag{6.20}
  $$

  In other words the field redefinition (6.18) induces the shifts

  $$
  \tilde{f}_1 - f_1 = 0, \quad \tilde{f}_2 - f_2 = 0, \quad \tilde{f}_3 - f_3 = \delta\phi \tag{6.21}
  $$

  (where $\tilde{f}_1$, $\tilde{f}_2$ and $\tilde{f}_3$ are the functions that appear in the expansion of $\tilde{S}_1^{even}$, see (6.17)) For this reason, the dependence of all physical quantities - like the fluid constitutive relations - on f_3 is rather trivial, and easy to deduce on general grounds, as we will see below.

- 2. While the fields σ, μ and χ are even under the action of time reversal, the fields ξ_i and ζ_i are odd under this operation. It follows that each of the three terms in (6.18) is odd under the action of time reversal. In other words the simultaneous requirement of parity and time reversal invariance simply sets $W_1 = 0$. It follows that time reversal invariant superfluids have no non dissipative transport coefficients at first order.

The corrections from (6.10) to the charge current and stress tensor (6.14) in equilibrium are given by

$$\begin{aligned}
\delta J_0 &= -\frac{\hat{T}e^{2\sigma}}{\sqrt{g}}\left[\frac{\delta S_1^{even}}{\delta A_0}\right]_{\zeta=\zeta^{eq}} = -\frac{\hat{T}e^{2\sigma}}{\sqrt{g}}\left(\frac{\delta W_1^{even}}{\delta A_0}\right) = -\frac{e^{\sigma}}{\sqrt{g}}\left(\frac{\delta W_1^{even}}{\delta \hat{\nu}}\right)\\
&= -e^{\sigma}\left[\frac{\partial}{\partial\hat{\nu}}\left(\frac{f_1}{\hat{T}}\right)(\zeta^{eq}.\partial)\hat{T} + \frac{\partial}{\partial\hat{\nu}}\left(\frac{f_2}{\hat{T}}\right)(\zeta^{eq}.\partial)\hat{\nu} + \frac{\partial}{\partial\hat{\nu}}\left(\frac{f}{\hat{T}}\right)(\zeta^{eq}.\partial)f_3 - \frac{f}{\hat{T}}(\zeta^{eq}.\partial)\left(\frac{f_2}{f}\right)\right]
\end{aligned}$$

$$\begin{aligned}
\delta J^i &= \frac{\hat{T}}{\sqrt{g}}\left(\frac{\delta S_1^{even}}{\delta A_i}\right)_{\zeta=\zeta^{eq}} = \frac{\hat{T}}{\sqrt{g}}\left(\frac{\delta W_1^{even}}{\delta A_i}\right)\\
&= 2(\zeta^{eq})^i\left[\frac{\partial f_1}{\partial(\zeta^{eq})^2}(\zeta^{eq}.\partial)\hat{T} + \frac{\partial f_2}{\partial(\zeta^{eq})^2}(\zeta^{eq}.\partial)\hat{\nu} + \frac{\partial f}{\partial(\zeta^{eq})^2}(\zeta^{eq}.\partial)f_3\right]\\
&\quad + g^{ij}\left(f_1\partial_j\hat{T} + f_2\partial_j\hat{\nu} + f\partial_j f_3\right)
\end{aligned} \tag{6.22}$$

$$\begin{aligned}
\delta T_{00} &= -\frac{\hat{T}e^{2\sigma}}{\sqrt{g}}\left[\frac{\delta S_1^{even}}{\delta\sigma}\right]_{\zeta=\zeta^{eq}} = -\frac{\hat{T}e^{2\sigma}}{\sqrt{g}}\left(\frac{\delta W_1^{even}}{\delta\sigma}\right) = \frac{\hat{T}^2e^{2\sigma}}{\sqrt{g}}\left(\frac{\delta W_1^{even}}{\delta\hat{T}}\right)\\
&= \hat{T}^2e^{2\sigma}\Bigg[\frac{\partial}{\partial\hat{T}}\left(\frac{f_1}{\hat{T}}\right)(\zeta^{eq}.\partial)\hat{T} + \frac{\partial}{\partial\hat{T}}\left(\frac{f_2}{\hat{T}}\right)(\zeta^{eq}.\partial)\hat{\nu}\\
&\qquad + \frac{\partial}{\partial\hat{T}}\left(\frac{f}{\hat{T}}\right)(\zeta^{eq}.\partial)f_3 - \frac{f}{\hat{T}}(\zeta^{eq}.\partial)\left(\frac{f_1}{f}\right)\Bigg]
\end{aligned} \tag{6.23}$$

$$\delta T_0^i = \frac{\hat{T}}{\sqrt{g}}\left(\frac{\delta S_1^{even}}{\delta a_i}\right)|_{\zeta=\zeta^{eq}} = \frac{\hat{T}}{\sqrt{g}}\left(\frac{\delta W_1^{even}}{\delta a_i}\right)|_{A_i=Constant} = -A_0\ \delta J^i \tag{6.24}$$

$$\begin{aligned}
\delta T^{ij} &= -\frac{\hat{T}}{\sqrt{g}}g^{il}g^{jm}\left[\frac{\delta S_1^{even}}{\delta g^{ij}}\right]_{\zeta=\zeta^{eq}} = -\frac{\hat{T}}{\sqrt{g}}g^{il}g^{jm}\left(\frac{\delta W_1^{even}}{\delta g^{ij}}\right)\\
&= -\left[(\zeta^{eq})^i\delta J^j + (\zeta^{eq})^j\delta J^i\right] + g^{ij}\left[f_1(\zeta^{eq}.\partial)\hat{T} + f_2(\zeta^{eq}.\partial)\hat{\nu} + f(\zeta^{eq}.\partial)f_3\right]\\
&\quad +2(\zeta^{eq})^i(\zeta^{eq})^j\left[\frac{\partial f_1}{\partial(\zeta^{eq})^2}(\zeta^{eq}.\partial)\hat{T} + \frac{\partial f_2}{\partial(\zeta^{eq})^2}(\zeta^{eq}.\partial)\hat{\nu} + \frac{\partial f}{\partial(\zeta^{eq})^2}(\zeta^{eq}.\partial)f_3\right]
\end{aligned} \tag{6.25}$$

In equations (6.22) and (6.23) all the scalar functions f_1, f_2, f_3 and f have been treated as functions $\hat{T}, \hat{\nu}$ and $(\zeta^{eq})^2$ respectively. In obtaining (6.22) we have used the zeroth order equation of motion for ϕ.

$$\nabla_i \left(\frac{f}{\hat{T}} (\zeta^{eq})^i \right) = 0$$

to simplify the expressions presented above .

6.3.2 Parity violating terms

The most general parity odd contributions to the action are given by[59]

$$\begin{aligned} S^{odd} &= S_1^{odd} + S_{anom} \\ S_1^{odd} &= \int \sqrt{g} d^3x \left(g_1 \epsilon^{ijk} \zeta_i \partial_j A_k + T_0 g_2 \epsilon^{ijk} \zeta_i \partial_j a_k \right) + \frac{C_1}{2} \int ada \\ S_{anom} &= \frac{C}{2} \left(\int \frac{A_0}{3} AdA + \frac{A_0^2}{6} Ada \right) \end{aligned} \tag{6.26}$$

[60] where

$$g_1 = g_1(\hat{T}, \hat{\nu}, \psi), \quad g_2 = g_2(\hat{T}, \hat{\nu}, \psi),$$

C_1 is a constant and

$$\hat{\nu} = \frac{\hat{\mu}}{\hat{T}}, \quad \psi = \frac{\zeta^2}{\hat{T}^2}.$$

(We emphasize that we have slightly changed notation compared to the previous subsection. The independent variables for all functions in this subsection are $\hat{T}$, $\hat{\nu}$ and ψ. The corresponding variables in the previous subsection were $\hat{T}$, $\hat{\nu}$ and ζ^2.)

Note that (6.26) is automatically even under time reversal. The corrections induced by (6.26) to the stress tensor and *consistent* charge current ([17], see section 2.3 equation 2.16

[59] Our convention is $\frac{1}{2} \int XdY = \int d^3x \sqrt{g_3} \epsilon^{ijk} X_i \partial_j Y_k$.

[60] The action for parity odd (non super) fluids (6.5) also contains the terms

$$W = \frac{C_0}{2} \int AdA + \frac{C_2}{2} \int Ada.$$

But using the fact that $\zeta_i = A_i + \partial_i \phi$ and

$$\int \sqrt{g} \epsilon^{ijk} \partial_i \phi \partial_j A_k = 0,$$

we can absorb C_0 in g_1 and C_2 in g_2.

of [47]) in equilibrium are given by

$$\begin{aligned}
T^{ij} &= -\frac{2}{\hat{T}}(\zeta^{eq})^i(\zeta^{eq})^j \left(g_{1,\psi_{eq}}S_1 + T_0 g_{2,\psi_{eq}}S_2\right)\\
T_{00} &= -T_0 e^{\sigma}\left((-\hat{T}g_{1,\hat{T}} + 2\psi_{eq}g_{1,\psi_{eq}})S_1 + T_0(-\hat{T}g_{2,\sigma} + 2\psi_{eq}g_{2,\psi})S_2\right)\\
J_0 &= -e^{\sigma}\left(g_{1,\nu}S_1 + T_0 g_{2,\nu}S_2\right) - e^{\sigma}\epsilon^{ijk}\left[\frac{C}{3}A_i\nabla_j A_k + \frac{C}{3}A_0 A_i \nabla_j a_k\right]\\
J^i &= \hat{T}\left(2g_1(S_1\frac{(\zeta^{eq})^i}{\hat{T}^2\psi_{eq}} - \frac{V_3^i}{\hat{T}^2\psi_{eq}}) + T_0 g_2(S_2\frac{(\zeta^{eq})^i}{\hat{T}^2\psi_{eq}} - \frac{V_4^i}{\hat{T}^2\psi_{eq}}) + \hat{T}V_1^i g_{1,\hat{T}} - \frac{1}{T_0}V_2^i g_{1,\nu} - V_5^i g_{1,\psi_{eq}}\right)\\
&+ \frac{2}{\hat{T}}\zeta^i(S_1 g_{1,\psi_{eq}} + T_0 S_2 g_{2,\psi_{eq}})\\
&+ e^{-\sigma}\left[2\left(\frac{C}{3}A_0\right)\frac{1}{\hat{T}^2\psi_{eq}}((\zeta^{eq})^i S_1 - V_3^i) + \left(\frac{C}{6}A_0^2\right)\frac{1}{\hat{T}^2\psi_{eq}}((\zeta^{eq})^i S_2 - V_4^i) + \frac{C}{3}\epsilon^{ijk}A_k\nabla_j A_0\right]\\
T_0^i &= \hat{T}\Bigg(\frac{(T_0 g_2 - 2A_0 g_1)}{\hat{T}^2\psi_{eq}}(S_1(\zeta^{eq})^i - V_3^i) - \frac{T_0 A_0 g_2}{\hat{T}^2\psi_{eq}}(S_2(\zeta^{eq})^i - V_4^i) + T_0(\hat{T}V_1^i(g_{2,\hat{T}} - \hat{\nu}g_{1,\hat{T}})\\
&- \frac{1}{T_0}V_2^i(g_{2,\hat{\nu}} - \hat{\nu}g_{1,\nu}) - V_5^i(g_{2,\psi_{eq}} - \hat{\nu}g_{1,\psi_{eq}})) - \frac{2A_0}{\hat{T}}\zeta^i(S_1 g_{1,\psi_{eq}} + T_0 S_2 g_{2,\psi_{eq}})\Bigg)\\
&- \frac{1}{2}CA_0^2 e^{-\sigma}(\frac{1}{\hat{T}^2\psi_{eq}}(\zeta^{eq})^i S_1 - \frac{1}{\hat{T}^2\psi_{eq}}V_3^i) + (2C_1 - \frac{C}{6}A_0^3)e^{-\sigma}(\frac{1}{\hat{T}^2\psi_{eq}}(\zeta^{eq})^i S_2 - \frac{1}{\hat{T}^2\psi_{eq}}V_4^i)],
\end{aligned} \tag{6.27}$$

where

$$\psi_{eq} = \frac{\zeta_i^{eq}\zeta_j^{eq}g^{ij}}{\hat{T}^2} \tag{6.28}$$

$$\begin{aligned}
S_1 &= \epsilon^{ijk}\zeta_i^{eq}\partial_j\zeta_k^{eq},\ S_2 = \epsilon^{ijk}\zeta_i^{eq}\partial_j a_k\\
V_1^i &= \epsilon^{ijk}\zeta_j^{eq}\partial_k\sigma,\ V_2^i = \epsilon^{ijk}\zeta_j^{eq}\partial_k A_0,\ \ V_3^i = \epsilon^{ijk}\zeta_j^{eq}F_{kl}(\zeta^{eq})^l\\
V_4^i &= \epsilon^{ijk}\zeta_j^{eq}f_{kl}(\zeta^{eq})^l,\ V_5^i = \epsilon^{ijk}\zeta_j^{eq}\partial_k\psi_{eq}\\
V_6^i &= \epsilon^{ijk}F_{jk},\ V_7^i = \epsilon^{ijk}f_{jk}.
\end{aligned} \tag{6.29}$$

The symbols for V_6^i and V_7^i have been introduced for notational convenience only; these vectors are determined in terms of the other quantities above by

$$\begin{aligned}
V_6^i &= \frac{2}{\hat{T}^2\psi_{eq}}((\zeta^{eq})^i S_1 - V_3^i)\\
V_7^i &= \frac{2}{\hat{T}^2\psi_{eq}}((\zeta^{eq})^i S_2 - V_4^i).
\end{aligned} \tag{6.30}$$

As we have emphasized, the formulas above determine the consistent current. The covariant current is obtained from the consistent current by an additional shift (see section 2.4 of [47] for a review). We find that the one derivative contribution to the covariant current in equilibrium is given by

$$
\begin{split}
J_0 &= -e^{\sigma}\left(g_{1,\hat{\nu}} S_1 + T_0 g_{2,\hat{\nu}} S_2\right) \\
J^i &= \hat{T}\left(2g_1(S_1\frac{(\zeta^{eq})^i}{\hat{T}^2\psi_{eq}} - \frac{V_3^i}{\hat{T}^2\psi_{eq}}) + T_0 g_2(S_2\frac{(\zeta^{eq})^i}{\hat{T}^2\psi_{eq}} - \frac{V_4^i}{\hat{T}^2\psi_{eq}}) + \hat{T}V_1^i g_{1,\hat{T}} - \frac{1}{T_0}V_2^i g_{1,\hat{\nu}} - V_5^i g_{1,\psi_{eq}}\right) \\
&\quad + \frac{2}{\hat{T}}(\zeta^{eq})^i(S_1 g_{1,\psi_{eq}} + T_0 S_2 g_{2,\psi_{eq}}) \\
&\quad + e^{-\sigma}\left[C\frac{1}{\hat{T}^2\psi_{eq}}((\zeta^{eq})^i S_1 - V_3^i) + \left(\frac{C}{2}A_0^2\right)\frac{1}{\hat{T}^2\psi_{eq}}((\zeta^{eq})^i S_2 - V_4^i)\right]
\end{split}
\tag{6.31}
$$

6.4 Constraints on parity even corrections to constitutive relations at first order

In this subsection we will determine parity even first order corrections to the superfluid constitutive relations both from the method of entropy increase as well as from the partition function of the previous section, and demonstrate their equality.

Let us first consider the almost trivial case of parity even superfluids that also preserve time reversal invariance. As we have explained in the previous section, in this case $W_1 = 0$. It follows immediately from this result that all non dissipative superfluid transport coefficients must vanish. Exactly this conclusion was reached in [4] from the requirement of point wise positivity of the divergence of the entropy current in an arbitrary fluid flow.

The study of time reversal non invariant superfluids is more involved. In this case the constraints from the local second law have not previously been analyzed. In this section we first present this analysis. We then study the constraints obtained from the analysis of the partition function. As mentioned above, we will find that these two methods yield identical constraints.

6.4.1 Constraints from the local second law

In this subsubsection (but nowhere else in this section) we consider the non equilibrium flow of a superfluid on a (generically) non stationary spacetime. We continue to denote the background metric of our spacetime by $G_{\mu\nu}$. The background gauge field is denoted by $\mathcal{A}_\mu$. The variables of superfluid hydrodynamics are the temperature field $T(x^\mu)$, velocity field $u^\mu(x^\mu)$ and the gradient of the phase field $\xi_\mu = -\partial_\mu\phi + \mathcal{A}_\mu$. We often work in terms of the fluid dynamical field

$$(\zeta_f)_\mu = \xi_\mu + \mu u_\mu$$

Note that, in equilibrium and at lowest order in the derivative expansion $(\zeta_f)_0 = 0$ and

$$(\zeta_f)_i = \xi_i - \mathcal{A}_0 a_i = \zeta_i.$$

We specify some additional notation that we will use extensively below.

$$
\begin{aligned}
&P^{\mu\nu} = u^\mu u^\nu + G^{\mu\nu}, \quad \tilde{P}^{\mu\nu} = P^{\mu\nu} - \frac{(\zeta_f)^\mu(\zeta_f)^\nu}{(\zeta_f)^2}, \quad V^\mu = \frac{E^\mu}{T} - P^{\mu\nu}\partial_\nu\nu \\
&R = \frac{q}{\epsilon+P}, \quad K = \nabla_\mu\left(f\xi^\mu\right) = s(u.\partial)\left(\frac{q}{s}\right), \quad \Theta = (\nabla.u) = -\frac{u.\partial s}{s} \\
&\mathfrak{a}_\mu = (u.\nabla)u_\mu \\
&H_1 = T, \quad H_2 = \nu, \quad H_3 = (\zeta_f)^2
\end{aligned}
\tag{6.32}
$$

In words, $P^{\mu\nu}$ projects onto the three dimensional subspace orthogonal to the normal fluid, while $\tilde{P}^{\mu\nu}$ projects onto the two dimensional subspace orthogonal to both the normal and superfluid velocities. $\mathfrak{a}_\mu$ and Θ are the normal fluid acceleration and expansion respectively. V^μ is the 'Einstein combination' of the electric field and derivative of the chemical potential that vanishes in equilibrium. H_1, H_2 and H_3 are new names for the three scalar hydrodynamical fields; note that H_2 is $\nu = \frac{\mu}{T}$ rather than the chemical potential itself. Finally K is the term that is set to zero by the first order equation of motion of the Goldstone phase, while R is a combination of zero order thermodynamical fields that often appears in the formulas below.

In order to analyze the constraints that follow from the local form of the second law, we follow the procedure described in section 3 of [4]. Briefly, we first write down the most general onshell independent first order entropy current allowed by symmetry. We then compute the divergence of this current (this is mere algebra) and then use the equations of hydrodynamics, together with the corrected constitutive relations

$$
\begin{aligned}
T^{\mu\nu} &= (\epsilon+p)u^\mu u^\nu + pG^{\mu\nu} + f\xi^\mu\xi^\nu + \pi^{\mu\nu} \\
J^\mu &= qu^\mu - f\xi^\mu + j^\mu,
\end{aligned}
\tag{6.33}
$$

(here $\pi^{\mu\nu}$ and j^μ refer to as yet unspecified one and higher derivative corrections to the constitutive relations) to re express this divergence as the sum of a linear form in onshell independent two derivative data and a quadratic form in onshell independent one derivative data. Point wise positivity of the divergence requires the linear form to vanish (this imposes several constraints on the entropy current). Once these conditions are imposed, the divergence of the entropy current is purely a quadratic form in one derivative data. We require this quadratic form to be positive definite. This requirement further constrains the entropy current as well as the first order contributions to $\pi^{\mu\nu}$ and j^μ in a manner we now schematically describe.

As we will see below, the quadratic form so obtained has the property that it vanishes when projected onto a subset of one derivative terms. In other words, all independent one derivative terms can be divided into y type 'entropically dissipative' terms and x type entropically nondissipative terms, and the quadratic form takes the schematic form

$$
A_{ij}y^iy^j + B_{im}y^ix^m
$$

Note that the structure of this quadratic form is preserved under x redefinitions

$$
x_m \to x_m + C_{mi}y^i
$$

but not under analogous redefinitions of y^i. In other words there exists a well defined subspace of dissipative data but no definite subspace of nondissipative data.

Positivity of the quadratic form described above requires that A_{ij} is a positive matrix, and $B_{im} = 0$ for all i and m. The last set of constraints yield relations between otherwise apparently independent transport coefficients. [61]

In order to actually implement this process we need first to choose a basis for onshell independent data. As explained in [4] (see e.g. Table 3), at first order in the derivative expansion there exist 7 (4 dissipative and 3 non dissipative) onshell independent scalars , 7 (2 dissipative and 5 nondissipative) onshell independent vectors and 2 (1 dissipative and one nondissipative) independent tensors constructed out of thermodynamical fields and background fields. For the purposes of this section, we will find it useful to choose our onshell independent basis as follows.

Basis of independent scalars:

$$\frac{V.(\zeta_f)}{(\zeta_f)^2}, \quad (u.\partial H_a), \quad ((\zeta_f).\partial H_a), \quad a = \{1,2,3\}$$

The four of these scalars are dissipative (they vanish in equilibrium) while the remaining three are nondissipative (they are non vanishing in equilibrium, and do not cause entropy production).

Basis of independent vectors:

$$\tilde{P}^{\mu\alpha}V_\alpha, \quad \tilde{P}^{\mu\alpha}(\zeta_f)_\beta\sigma^\beta_\alpha, \quad \tilde{P}^\mu_\alpha(\zeta_f)_\nu f^{\nu\alpha}, \quad \tilde{P}_{\alpha\mu}(\zeta_f)_\nu F^{\nu\alpha}, \quad \tilde{P}^{\mu\alpha}\partial_\alpha H_a, \quad a = \{1,2,3\}$$

The first two vectors are dissipative (they vanish in equilibrium) and the remaining five vectors are nondissipative.

Basis of independent tensors

$$\begin{aligned}
\tilde{\sigma}_{\mu\nu} &= \tilde{P}^\alpha_\mu \tilde{P}^\beta_\nu \left[\frac{\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \tilde{P}^{\lambda\phi}(\nabla_\lambda u_\phi)\eta_{\alpha\beta}}{2}\right], \\
\sigma^{(\zeta_f)}_{\mu\nu} &= \tilde{P}^\alpha_\mu \tilde{P}^\beta_\nu \left[\frac{\nabla_\alpha(\zeta_f)_\beta + \nabla_\beta(\zeta_f)_\alpha - \tilde{P}^{\lambda\phi}(\nabla_\lambda(\zeta_f)_\phi)\eta_{\alpha\beta}}{2}\right]
\end{aligned}$$

The first is dissipative (it vanishes in equilibrium) while the second is nondissipative.

In this subsubsection we wish to constrain the equations of superfluid hydrodynamics presented in a 'fluid frame' (see [5] for an explanation of what this means). Throughout this

[61] Assuming that the matrix A is positive definite, entropy is always produced whenever any of the y^i are nonzero. It follows that all y^i must always vanish in equilibrium. This observation motivates the following definition, utilized in [47]. Expressions that vanish in (arbitrary stationary) equilibrium are referred to as dissipative data. It follows from that entropically dissipative data is necessarily dissipative. However the converse is not necessarily true; it is possible for data to vanish in arbitrary stationary equilibrium but yet be entropically nondissipative. We will see an example of this phenomenon later in this section.

section we will further restrict our attention to fluid frames with $\mu_{diss} = 0$ (again see [5] for definitions). This choice still permits the freedom of field redefinitions of the temperature and normal velocity fields (as well as field redefinitions of the superfluid phase, as we will exploit later in this section). Even though we work specifically frames in which $\mu_{diss} = 0$ our final results may easily be lifted to an arbitrary $\mu_{diss} \neq 0$ frame using the frame invariant formalism of [4].

The most general form of the entropy current, consistent with the absence of linear two derivative terms in its divergence was determined in [4] (see equation 3.19) and takes the form

$$\begin{aligned} J_S^\mu &= J_{can}^\mu + J_{new}^\mu \\ J_{can}^\mu &= s u^\mu - \nu j^\mu - \frac{u_\nu \pi^{\mu\nu}}{T} \\ J_{new}^\mu &= \sum_a c_a (\partial_\nu H_a) Q^{\mu\nu} + \nabla_\nu (c\, Q^{\mu\nu}) \\ &\text{where} \;\; Q^{\mu\nu} = f(u^\mu (\zeta_f)^\nu - u^\nu (\zeta_f)^\mu) \end{aligned} \tag{6.34}$$

The divergence of J_{can}^μ was worked out in [4, 5] (see for example, equation 3.9 [4], and recall we work with $\mu_{diss} = 0$).

$$\nabla_\mu J_{can}^\mu = -\pi^{\mu\nu} \nabla_\mu \left(\frac{u_\nu}{T}\right) + j^\mu V_\mu + (u_\mu j^\mu)(u.\partial\nu) \tag{6.35}$$

The RHS of (6.35) is given schematically by

(one derivative correction to constitutive relation) $\times$ (entropicallydissiaptive data),

[62] We will now rewrite the RHS of (6.35) as a quadratic form in the basis of independent dissipative one derivative data chosen above. In order to achieve this we need to rewrite all the y type terms in (6.35) in terms of the independent basis of dissipative scalars, vectors and tensors listed above. To achieve this we use the equations of motion

$$\begin{aligned} \frac{(\zeta_f).\partial T}{T} + \mathfrak{a}.(\zeta_f) &= RT(V.(\zeta_f)) - \frac{(\zeta_f)^2 K}{\epsilon + P} \\ \frac{(\zeta_f)_\mu (\zeta_f)_\nu \sigma^{\mu\nu}}{(\zeta_f)^2} + \frac{\Theta}{3} &= -T(1-\mu R)\frac{V.(\zeta_f)}{(\zeta_f)^2} - \frac{\mu K}{\epsilon + P} - \frac{(u.\partial)(\zeta_f)^2}{2(\zeta_f)^2} \end{aligned} \tag{6.36}$$

[62] Note that the one derivative expressions that appear here are always entropically dissipative, as contributions to changes in the proportional to these one derivative expressions yield quadratic terms in entropy production.

we find

$$
\begin{aligned}
\nabla_\mu J^\mu_{can} = &- \left(\frac{u_\mu u_\nu \pi^{\mu\nu}}{T^2}\right)(u.\partial T) + (j.u)(u.\partial\nu) + \frac{(j.(\zeta_f))(V.(\zeta_f))}{(\zeta_f)^2} \\
&+ \frac{u_\mu (\zeta_f)_\nu \pi^{\mu\nu}}{T}\left[RT\left(\frac{V.(\zeta_f)}{(\zeta_f)^2}\right) - \frac{K}{\epsilon+P}\right] - \frac{1}{2T}\left(\pi^{\mu\nu}\tilde{P}_{\mu\nu}\right)\Theta \\
&- \frac{1}{T}\left(\pi^{\mu\nu}P_{\mu\nu} - \frac{3}{2}\pi_{\mu\nu}\tilde{P}^{\mu\nu}\right)\left(\frac{(\zeta_f)_\mu(\zeta_f)_\nu\sigma^{\mu\nu}}{(\zeta_f)^2} + \frac{\Theta}{3}\right) \\
&- 2\left(\frac{(\zeta_f)_\alpha \pi^{\alpha\nu}\tilde{P}_{\nu\mu}\sigma^{\mu\beta}(\zeta_f)_\beta}{T(\zeta_f)^2}\right) + \left(j^\mu + Ru_\alpha\pi^{\alpha\mu}\right)\tilde{P}_{\mu\nu}V^\nu - \frac{1}{T}\tilde{\sigma}_{\mu\nu}\tilde{\pi}^{\mu\nu} \\
= &- \left(\frac{u_\mu u_\nu \pi^{\mu\nu}}{T^2}\right)(u.\partial T) + (j.u)(u.\partial\nu) - \left(\frac{\pi^{\mu\nu}\tilde{P}_{\mu\nu}}{2T}\right)\Theta \\
&+ \left(\pi^{\mu\nu}P_{\mu\nu} - \frac{3}{2}\pi^{\mu\nu}\tilde{P}_{\mu\nu}\right)\left(\frac{u.\partial(\zeta_f)^2}{2T(\zeta_f)^2}\right) \\
&+ \left(\frac{V.(\zeta_f)}{(\zeta_f)^2}\right)\left[(j.(\zeta_f)) + R(u_\mu(\zeta_f)_\nu\pi^{\mu\nu}) + (1-\mu R)\left(\pi^{\mu\nu}P_{\mu\nu} - \frac{3}{2}\pi^{\mu\nu}\tilde{P}_{\mu\nu}\right)\right] \\
&+ \left(\frac{K}{\epsilon+P}\right)\left[-\left(\frac{-u_\nu(\zeta_f)_\mu\pi^{\mu\nu}}{T}\right) + \nu\left(\pi^{\mu\nu}P_{\mu\nu} - \frac{3}{2}\pi^{\mu\nu}\tilde{P}_{\mu\nu}\right)\right] \\
&- 2\left(\frac{(\zeta_f)_\alpha \pi^{\alpha\nu}\tilde{P}_{\nu\mu}\sigma^{\mu\beta}(\zeta_f)_\beta}{T(\zeta_f)^2}\right) + \left(j^\mu + Ru_\alpha\pi^{\alpha\mu}\right)\tilde{P}_{\mu\nu}V^\nu - \frac{1}{T}\tilde{\sigma}_{\mu\nu}\tilde{\pi}^{\mu\nu} \\
= &\sum_{a=1}^{3}\mathfrak{S}_a(u.\partial)H_a + \mathfrak{S}_4\left(\frac{V.(\zeta_f)}{(\zeta_f)^2}\right) - 2\mathfrak{V}_2^\nu\left(\frac{\tilde{P}_{\nu\mu}\sigma^{\mu\beta}(\zeta_f)_\beta}{T(\zeta_f)^2}\right) + \mathfrak{V}_1^\mu\tilde{P}_{\mu\nu}V^\nu - \frac{1}{T}\tilde{\sigma}_{\mu\nu}\tilde{\pi}^{\mu\nu}
\end{aligned}
\tag{6.37}
$$

where

$$
\begin{aligned}
\mathfrak{S}_a = &\left[\left(\frac{s}{\epsilon+P}\right)\frac{\partial}{\partial H_a}\left(\frac{q}{s}\right)\right]\left[-\left(\frac{-u_\nu(\zeta_f)_\mu\pi^{\mu\nu}}{T}\right) + \nu\left(\pi^{\mu\nu}P_{\mu\nu} - \frac{3}{2}\pi^{\mu\nu}\tilde{P}_{\mu\nu}\right)\right] \\
&+ \left(\frac{1}{s}\frac{\partial s}{\partial H_a}\right)\left(\frac{\pi^{\mu\nu}\tilde{P}_{\mu\nu}}{2T}\right) - \left(\frac{u_\mu u_\nu\pi^{\mu\nu}}{T^2}\right)\delta_{a,1} + (j.u)\delta_{a,2} \\
&+ \left(\frac{1}{2T(\zeta_f)^2}\right)\left(\pi^{\mu\nu}P_{\mu\nu} - \frac{3}{2}\pi^{\mu\nu}\tilde{P}_{\mu\nu}\right)\delta_{a,3} \\
\mathfrak{S}_4 = &\left[(j.(\zeta_f)) + R(u_\mu(\zeta_f)_\nu\pi^{\mu\nu}) + (1-\mu R)\left(\pi^{\mu\nu}P_{\mu\nu} - \frac{3}{2}\pi^{\mu\nu}\tilde{P}_{\mu\nu}\right)\right] \\
\mathfrak{V}_2^\nu = &(\zeta_f)_\alpha\pi^{\alpha\nu} \\
\mathfrak{V}_1^\mu = &\left(j^\mu + Ru_\alpha\pi^{\alpha\mu}\right) \\
\tilde{\pi}^{\mu\nu} = &\tilde{P}^{\mu\alpha}\tilde{P}^{\nu\beta}\left[\pi_{\alpha\beta} - \frac{\eta_{\alpha\beta}}{2}\left(\tilde{P}_{\theta\phi}\pi^{\theta\phi}\right)\right] \\
H_1 = &T, \quad H_2 = \nu, \quad H_3 = (\zeta_f)^2
\end{aligned}
\tag{6.38}
$$

The last line of (6.37) is the final result of this manipulation. It expresses the divergence of the entropy current as a linear sum over the four dissipative onshell scalars and two dissipative onshell vectors and one dissipative tensor listed earlier in this subsection. These expressions appear multiplied by frame invariant linear combinations of $\pi^{\mu\nu}$ and j^μ.

The frame invariant quantities $\mathfrak{S}_a$ and $\mathfrak{V}_a$ will be used extensively below. For later use we will find it useful to regard these quantities as linear functions of $\pi^{\mu\nu}$ and j^μ, i.e.

$$\mathfrak{S}_a = \mathfrak{S}_a(\pi^{\mu\nu}, j^\mu), \quad \mathfrak{V}_a = \mathfrak{V}_a(\pi^{\mu\nu}, j^\mu) \tag{6.39}$$

The divergence of the 'new' part of the entropy current, J^μ_{new} (see (6.34)) is given by

$$\begin{aligned}
&\nabla_\mu J^\mu_{new} \\
&= \sum_{(a,b)} f\left(\frac{\partial c_a}{\partial H_b} - \frac{\partial c_b}{\partial H_a}\right)((\zeta_f).\partial H_a)(u.\partial H_b) + \sum_a (\partial_\nu H_a)\nabla_\mu Q^{\mu\nu} \\
&= \sum_{(a,b)} f\left(\frac{\partial c_a}{\partial H_b} - \frac{\partial c_b}{\partial H_a}\right)((\zeta_f).\partial H_a)(u.\partial H_b) + \sum_a (\partial_\mu H_a)\tilde{P}^\mu_\nu(\nabla_\alpha Q^{\alpha\nu}) \\
&\quad - \sum_a \left[(u.\partial H_a)\left(u_\nu \nabla_\mu Q^{\mu\nu}\right) + ((\zeta_f).\partial H_a)\left(\frac{(\zeta_f)_\nu \nabla_\mu Q^{\mu\nu}}{(\zeta_f)^2}\right)\right]
\end{aligned} \tag{6.40}$$

where $Q^{\mu\nu}$ was defined in (6.34).

Using equations of motion we can express $(u_\nu \nabla_\mu Q^{\mu\nu})$, $\left(\frac{(\zeta_f)_\nu \nabla_\mu Q^{\mu\nu}}{(\zeta_f)^2}\right)$ and $\tilde{P}^\mu_\nu(\nabla_\alpha Q^{\alpha\nu})$ in terms of the onshell independent basis scalars of this subsection (spanned by $(u.\partial H_a)$, $((\zeta_f).\partial H_a)$, $\left(\frac{V.(\zeta_f)}{(\zeta_f)^2}\right)$) and vectors (spanned by $\tilde{P}^{\mu\alpha}V_\alpha$, $\tilde{P}^{\mu\alpha}(\zeta_f)_\beta\sigma^\beta_\alpha$, $\tilde{P}^{\mu\alpha}\partial_\alpha H_a$).

$$\begin{aligned}
(u_\nu \nabla_\mu Q^{\mu\nu}) &= s(u.\partial)\left(\frac{f\mu}{s}\right) + \left(1 - \frac{f(\zeta_f)^2}{\epsilon + P}\right)K + f\left(\frac{(\zeta_f).\partial T}{T}\right) - f(V.(\zeta_f)) \\
\left(\frac{(\zeta_f)_\nu \nabla_\mu Q^{\mu\nu}}{(\zeta_f)^2}\right) &= s(u.\partial)\left(\frac{f}{s}\right) + fT\left[(1-\mu R)\left(\frac{V.(\zeta_f)}{(\zeta_f)^2}\right) + \frac{\nu K}{\epsilon + P} + \frac{u.\partial(\zeta_f)^2}{T(\zeta_f)^2}\right] \\
\tilde{P}^\mu_\nu(\nabla_\alpha Q^{\alpha\nu}) &= -P^\mu_\nu \sum_a f c_a\left[T(1-\mu R)V^\nu + 2(\zeta_f)_\alpha\sigma^{\nu\alpha}\right]
\end{aligned} \tag{6.41}$$

From equations (6.37), (6.38), (6.40) and (6.41) we conclude that, no matter what form the fluid constitutive relations take, the divergence of the entropy current cannot contain any expressions of the form $((\zeta_f).\partial H_a)^2$ or $(\tilde{P}^{\mu\nu}\partial_\mu H_a \partial_\nu H_b)$. In other words the scalars $(\zeta_f).\partial H_a$ and the vectors $(\tilde{P}^{\mu\nu}\partial_\mu H_a)$ are nondissipative. It follows that the positivity of $(\nabla_\mu J^\mu_S)$ requires that the divergence contain no term linear in $((\zeta_f).\partial H_a)$ or $(\tilde{P}^{\mu\nu}\partial_\mu H_a)$ (see e.g. [4] for repeated use of similar arguments.) To ensure this $\pi^{\mu\nu}$ and j^μ have to satisfy the following conditions.

$$
\begin{aligned}
\mathfrak{S}_a = & -\sum_{b=1}^{3}((\zeta_f).\partial H_b)\Big\{ f\left(\frac{\partial c_b}{\partial H_a}-\frac{\partial c_a}{\partial H_b}\right)-\frac{f c_a}{T}\delta_{b,1}+\frac{f c_b}{(\zeta_f)^2}\delta_{a,3} \\
& \qquad + c_b\left[s\frac{\partial}{\partial H_a}\left(\frac{f}{s}\right)+\left(\frac{s\nu}{\epsilon+P}\right)\frac{\partial}{\partial H_a}\left(\frac{q}{s}\right)\right]\Big\} \\
& + \text{dissipative terms} \\
= & -\sum_{b=1}^{3}((\zeta_f).\partial H_b)\Big\{\left[\frac{\partial}{\partial H_a}(f c_b)-\frac{f}{T}\frac{\partial}{\partial H_b}(T c_a)\right]+\frac{f c_b}{(\zeta_f)^2}\delta_{a,3} \\
& \qquad + c_b\left[-\frac{1}{s}\frac{\partial s}{\partial H_a}+\left(\frac{s\nu}{\epsilon+P}\right)\frac{\partial}{\partial H_a}\left(\frac{q}{s}\right)\right]\Big\} \\
& + \sum_{b=1}^{3} M_{ab}(u.\partial H_b)+M_{a4}\left(\frac{V.(\zeta_f)}{(\zeta_f)^2}\right) \qquad (a=\{1,2,3\}) \\
\mathfrak{S}_4 = & (j.(\zeta_f))+R(u_\mu(\zeta_f)_\nu\pi^{\mu\nu})+(1-\mu R)\left(\pi^{\mu\nu}P_{\mu\nu}-\frac{3}{2}\pi^{\mu\nu}\tilde{P}_{\mu\nu}\right)+\text{dissipative terms} \\
= & -\sum_{b}((\zeta_f).\partial H_b) f T(1-\mu R)c_b+\sum_{b=1}^{3}M_{4b}(u.\partial H_b)+M_{44}\left(\frac{V.(\zeta_f)}{(\zeta_f)^2}\right)
\end{aligned}
\tag{6.42}
$$

$$
\begin{aligned}
\mathfrak{V}_{1\mu} = & \left(j^\nu+R u_\alpha\pi^{\alpha\nu}\right)\tilde{P}_{\mu\nu}=T(1-\mu R)f\sum_b c_b(\tilde{P}^\nu_\mu\partial_\nu H_b)+\text{dissipative terms} \\
= & \, T(1-\mu R)f\sum_b c_b(\tilde{P}^\nu_\mu\partial_\nu H_b)+N_{11}\left(\tilde{P}_{\mu\nu}V^\nu\right)-N_{12}\left(\frac{\tilde{P}_{\mu\beta}(\zeta_f)_\alpha\sigma^{\alpha\beta}}{2T(\zeta_f)^2}\right) \\
\mathfrak{V}_{2\mu} = & (\zeta_f)_\alpha\pi^{\alpha\nu}\tilde{P}_{\nu\mu}=-T(\zeta_f)^2 f\sum_b c_b(\tilde{P}^\nu_\mu\partial_\nu H_b)+\text{dissipative terms} \\
= & -T(\zeta_f)^2 f\sum_b c_b(\tilde{P}^\nu_\mu\partial_\nu H_b)+N_{21}\left(\tilde{P}_{\mu\nu}V^\nu\right)-N_{22}\left(\frac{\tilde{P}_{\mu\beta}(\zeta_f)_\alpha\sigma^{\alpha\beta}}{2T(\zeta_f)^2}\right) \\
\tilde{\pi}^{\mu\nu} = & \tilde{P}^{\mu\alpha}\tilde{P}^{\nu\beta}\left[\pi_{\alpha\beta}-\frac{\eta_{\alpha\beta}}{2}\left(\tilde{P}_{\theta\phi}\pi^{\theta\phi}\right)\right]=\text{dissipative term}=-\eta\,\tilde{\sigma}^{\mu\nu}
\end{aligned}
\tag{6.43}
$$

where M is a 4×4 matrix of dissipative transport coefficients in the scalar sector and N is a 2×2 matrix of dissipative transport coefficients in the vector sector.

Equations (6.42),(6.43) are the main result of this subsection. It expresses the equality type constraints that follow from the local second law. Once (6.42),(6.43) are satisfied the final expressions for the divergence of the entropy current takes the following form.

$$
\begin{aligned}
&\nabla_\mu J^\mu_s \\
&= -\sum_{a,b} c_a \left\{ \mu s \frac{\partial \left(\frac{f}{s}\right)}{\partial H_b} + s\left(1 - \frac{f(\zeta_f)^2}{\epsilon + p}\right)\frac{\partial\left(\frac{q}{s}\right)}{\partial H_b} + f\left(T\delta_{b,2} + \nu\delta_{b,1}\right)\right\}(u.\partial H_a)(u.\partial H_b) \\
&\quad - f(V.(\zeta_f))\sum_a c_a (u.\partial H_a) \\
&\quad + \sum_{a,b=1}^{3} M_{ab}(u.\partial H_a)(u.\partial H_b) + \sum_{a=1}^{3}\left(M_{4a} + M_{a4}\right)(u.\partial H_a)\left(\frac{V.(\zeta_f)}{(\zeta_f)^2}\right) + M_{44}\left(\frac{V.(\zeta_f)}{(\zeta_f)^2}\right)^2 \\
&\quad + \tilde{P}_{\mu\nu}\Big[N_{11}\left(V^\mu V^\nu\right) + (N_{12} + N_{21})(V^\mu (\zeta_f)_\alpha \sigma^{\alpha\nu}) + N_{22}((\zeta_f)_\alpha (\zeta_f)_\beta \sigma^{\alpha\mu}\sigma^{\beta\nu})\Big] \\
&\quad + \frac{\eta}{T}\tilde{\sigma}_{\mu\nu}\tilde{\sigma}^{\mu\nu}
\end{aligned}
\tag{6.44}
$$

The positivity of this quadratic form imposes additional inequality type constraints on transport coefficients that we will not further explore here.

6.4.2 Constraints from the partition function

In this subsubsection we now reproduce the conditions (6.42),(6.43) using considerations independent of those of the previous subsection. The procedure we adopt is very similar to that described in [47], and we describe it only briefly, highlighting only those elements of the analysis that are unique to the superfluid.

The starting point of our analysis is the expressions (6.22) and (6.23) which represent the first order for the corrections to the stress tensor and charge current that follow by varying the local action for the Goldstone mode w.r.t. the metric and background gauge field. Once we substitute in the solution for the field $\xi^\mu(x)$, according to its equations of motion, (6.22) and (6.23) yield first order corrections $\delta T_{\mu\nu}$ and δJ^μ to the values of the stress tensor and charge current in thermal equilibrium.

From the hydrodynamical point of view, $\delta T_{\mu\nu}$ and δJ^μ are the first order contributions in (6.33) once we substitute

$$
T(x) = \hat{T}(x) + T_1(x), \qquad \mu(x) = \hat{\mu}(x) + \mu_1(x), \quad u^\mu(x) = \hat{u}^\mu + u_1^\mu(x) \tag{6.45}
$$

into those expressions. Here $T_1(x)$, $\mu_1(x)$ and $u_1^\mu(x)$ a are the first derivative corrections to the equilibrium configurations of temperature, chemical potential and velocity.

Upon substituting (6.45) into (6.33) we get first derivative contributions of two sorts. First we have the corrections to constitutive relations evaluated on the zero order equilibrium configurations $\Pi^{\mu\nu}(\hat{T}, \hat{\mu}, \hat{u}^\mu, (\zeta^{eq})^\mu)$ and $j^\mu(\hat{T}, \hat{\mu}, \hat{u}^\mu, (\zeta^{eq})^\mu)$. Second we have contributions from terms proportional to T_1, μ_1 and u_1^μ when (6.45) is plugged into the perfect fluid constitutive relations. Contributions of the second sort, however, *precisely* cancel out in the frame invariant linear combinations $\mathfrak{S}_a$ $(a = 1 \ldots 4)$ and $\mathfrak{V}_a$ $(a = 1 \ldots 2)$. In other words

$$\begin{aligned} \mathfrak{S}_a(\delta T_{\mu\nu}, \delta J_\mu) &= \mathfrak{S}_a(\pi_{\mu\nu}, j_\mu) \\ \mathfrak{V}_a(\delta T_{\mu\nu}, \delta J_\mu) &= \mathfrak{V}_a(\pi_{\mu\nu}, j_\mu) \end{aligned} \tag{6.46}$$

($\pi^{\mu\nu}$ and j^μ on the RHS of (6.46) are evaluated on the zero order equilibrium configurations). In the general formulation of hydrodynamics, however, it is precisely the frame invariants that appear on the RHS of (6.46) that are expanded in the most general symmetry allowed constitutive relations (see e.g. [4])

$$\begin{aligned} \mathfrak{S}_a(\pi^{\mu\nu}, j^\mu) &= \alpha_{am} S^m \quad (a = 1 \ldots 4), \quad (m = 1 \ldots 7) \\ \mathfrak{V}^\mu_a(\pi^{\mu\nu}, j^\mu) &= \gamma_{am} V^\mu_m \quad (a = 1 \ldots 2), \quad (m = 1 \ldots 5) \end{aligned} \tag{6.47}$$

where S^m and V^μ_m are the independent one derivative scalars and vectors and the coefficients α_{am} and γ_{am} are arbitrary functions of the scalars T, μ and $\xi^\mu \xi_\mu$.

α_{am} and γ_{am} are the constitutive coefficients we wish to constrain, and this is achieved as follows. In the LHS of (6.46) we substitute the expressions (6.22) and (6.23)) for $\delta T^{\mu\nu}$ and δJ^μ. This determines the LHS of (6.46) completely in terms of the functions f_1, f_2 and f_3 that appear in the partition function. In the RHS of (6.46) we substitute (6.47), and evaluate these expressions in equilibrium

$$T = \hat{T}, \quad \mu = \hat{\mu}, \quad \zeta = \zeta^{eq}.$$

Under the last substitution those of S^m and V^m that are dissipative vanish. The non dissipative one derivative scalars and vectors evaluate to geometric expressions. Equating the coefficients of these expressions we determine α_{am} and γ_{am} for those values of m that correspond to non dissipative terms. In other words this procedure completely determines all non dissipative transport coefficients. [63]

In the rest of this subsection we implement the procedure described above to explicitly determine all nondissipative transport coefficients in terms of the three free functions f_1, f_2 and f_3 that enter the local action for the Goldstone field. We demonstrate that our results agree exactly with (6.42),(6.43), obtained from the local form of the second law, once we identify the three unknown functions c_1, c_2 and c_3 in the entropy current of the previous subsection in terms of the functions in the partition function according to

$$c_1 = \frac{f_1}{fT} + \frac{1}{T}\frac{\partial f_3}{\partial T}, \quad c_2 = \frac{f_2}{fT} + \frac{1}{T}\frac{\partial f_3}{\partial \nu}, \quad c_3 = \frac{1}{T}\frac{\partial f_3}{\partial \zeta^2} \tag{6.48}$$

[63] There is an important subtlety here. All of the operations described above may only be performed in equilibrium, i.e. once we have solved for $(\zeta^{eq})^\mu$ as a function of background fields and substituted this back into the partition function. We implement our programme without explicitly solving, simply by treating $(\zeta^{eq})^\mu(x)$ as formally independent of the other background fields, except for those local combinations of $(\zeta^{eq})^\mu$ that appear in terms of its equation of motion and derivatives there off. The reason for this that the expressions for ξ^μ as a function of background fields is highly nonlocal. The only situation in which cancellations are possible between local expressions in $(\zeta^{eq})^\mu$ and local expressions in the background fields is when we get derivatives combining with $(\zeta^{eq})^\mu$ in the form of the ϕ equations of motion.

We will also demonstrate that the identification (6.48) may be argued for directly by comparing the thermodynamical entropy in equilibrium with the integral of the equilibrium entropy current over a spatial slice.

It will be useful in the computation below to note that $P^{\mu\nu}$ and $\tilde{P}^{\mu\nu}$ are given by

$$P_{ij} = g_{ij}, \quad \tilde{P}_{ij} = g_{ij} - \frac{\zeta^{eq}_i \zeta^{eq}_j}{(\zeta^{eq})^2}$$

We turn now to the explicit computation, starting with the vectors.

$$\begin{aligned}
\mathfrak{V}_{10}(\delta T_{\mu\nu}, \delta J_\mu) &= \mathfrak{V}_{20}(\delta T_{\mu\nu}, \delta J_\mu) = 0 \\
\mathfrak{V}_{1i}(\delta T_{\mu\nu}, \delta J_\mu) &= \tilde{P}_{ij}\left(\delta J^j + \hat{R}\hat{u}^0 \delta T^j_0\right) \\
&= \tilde{P}_{ij}\left(\delta J^j - Re^{-\sigma} A_0 \delta J^j\right) \\
&= (1-\hat{\mu}\hat{R})\tilde{P}_{ij}\delta J^j \\
&= \tilde{P}_{ij} g^{jk}(1-\hat{\mu}\hat{R})\left(\frac{f_1}{\hat{T}}\partial_k \hat{T} + \frac{f_2}{\hat{T}}\partial_k \hat{\nu} + \frac{f}{\hat{T}}\partial_k f_3\right) \\
\mathfrak{V}_{2i}(\delta T_{\mu\nu}, \delta J_\mu) &= \tilde{P}_{ij}\zeta_k \; \delta T^{kj} = -\zeta^2 \tilde{P}_{ij}\delta J^j \\
&= -\tilde{P}_{ij} g^{jk}\left(\frac{f_1}{\hat{T}}\partial_k \hat{T} + \frac{f_2}{\hat{T}}\partial_k \hat{\nu} + \frac{f}{\hat{T}}\partial_k f_3\right)
\end{aligned} \tag{6.49}$$

The last line of (6.49) exactly matches (6.42),(6.43)) upon using the identification of the parameters (6.48).

We turn next to the scalars; let us start with $\mathfrak{S}_4$.

$$\begin{aligned}
\mathfrak{S}_4(\delta T_{\mu\nu}, \delta J_\mu) &= (j.\zeta) + R(u_\mu \zeta_\nu \pi^{\mu\nu}) + (1-\mu R)\left(\pi^{\mu\nu}P_{\mu\nu} - \frac{3}{2}\pi^{\mu\nu}\tilde{P}_{\mu\nu}\right) \\
&= -(1-\hat{\mu}\hat{R})\left[f_1(\zeta^{eq}.\partial\hat{T}) + f_2(\zeta^{eq}.\partial\hat{\nu}) + f(\zeta^{eq}.\partial f_3)\right] \\
&= \hat{T}(1-\hat{\mu}\hat{R})\sum_b f c_b(\zeta^{eq}.\partial H_b)
\end{aligned} \tag{6.50}$$

In the last step we have used (6.48), and have obtained manifest agreement with (6.42),(6.43).

Next we shall calculate the remaining three scalars $\mathfrak{S}_a, \quad a = \{1,2,3\}$. The algebraic manipulations here are a little more involved than in previous cases, and we provide some details.

$$\begin{aligned}
\mathfrak{S}_a(\delta T_{\mu\nu}, \delta J_\mu) = &\left[\left(\frac{s}{\epsilon + P}\right)\frac{\partial}{\partial H_a}\left(\frac{q}{s}\right)\right]\left[-\left(\frac{-u_\nu \zeta_\mu \pi^{\mu\nu}}{T}\right) + \nu\left(\pi^{\mu\nu}P_{\mu\nu} - \frac{3}{2}\pi^{\mu\nu}\tilde{P}_{\mu\nu}\right)\right] \\
&+ \left(\frac{1}{s}\frac{\partial s}{\partial H_a}\right)\left(\frac{\pi^{\mu\nu}\tilde{P}_{\mu\nu}}{2T}\right) - \left(\frac{u_\mu u_\nu \pi^{\mu\nu}}{T^2}\right)\delta_{a,1} + (j.u)\delta_{a,2} \\
&+ \left(\frac{1}{2T\zeta^2}\right)\left(\pi^{\mu\nu}P_{\mu\nu} - \frac{3}{2}\pi^{\mu\nu}\tilde{P}_{\mu\nu}\right)\delta_{a,3}, \qquad (a = \{1,2,3\})
\end{aligned} \tag{6.51}$$

The first line in (6.51) can be evaluated as

$$\begin{aligned}
&\left[\left(\frac{s}{\epsilon+P}\right)\frac{\partial}{\partial H_a}\left(\frac{q}{s}\right)\right]\left[-\left(\frac{-u_\nu\zeta_\mu\pi^{\mu\nu}}{T}\right)+\nu\left(\pi^{\mu\nu}P_{\mu\nu}-\frac{3}{2}\pi^{\mu\nu}\tilde{P}_{\mu\nu}\right)\right]\\
=&\left[\left(\frac{s}{\epsilon+P}\right)\frac{\partial}{\partial H_a}\left(\frac{q}{s}\right)\right]\left[-\hat{\nu}f(\zeta^{eq}.\partial f_3)-\hat{\nu}f_2(\zeta^{eq}.\partial\hat{\nu})-\hat{\nu}f_1(\zeta^{eq}.\partial\hat{T})\right] \qquad (6.52)\\
=&-\left[\left(\frac{s\hat{\nu}}{\epsilon+P}\right)\frac{\partial}{\partial H_a}\left(\frac{q}{s}\right)\right]\sum_b c_b(\zeta^{eq}.\partial H_b)
\end{aligned}$$

In the last step we have used (6.48).

The second line of (6.51) may be evaluated as follows

$$\begin{aligned}
&\left(\frac{1}{s}\frac{\partial s}{\partial H_a}\right)\left(\frac{\pi^{\mu\nu}\tilde{P}_{\mu\nu}}{2T}\right)\\
=&\left(\frac{1}{s}\frac{\partial s}{\partial H_a}\right)\left[\frac{f_1}{\hat{T}}(\zeta^{eq}.\partial)\hat{T}+\frac{f_2}{\hat{T}}(\zeta^{eq}.\partial)\hat{\nu}+\frac{f}{\hat{T}}(\zeta^{eq}.\partial)f_3\right] \qquad (6.53)\\
=&\left(\frac{1}{s}\frac{\partial s}{\partial H_a}\right)\sum_b f c_b(\zeta^{eq}.\partial H_b)
\end{aligned}$$

In the last step we have used (6.48).

Finally we evaluate the last three terms of (6.51) together.

$$
\begin{aligned}
&-\left(\frac{u_\mu u_\nu \pi^{\mu\nu}}{T^2}\right)\delta_{a,1} + (j.u)\delta_{a,2} + \left(\frac{\pi^{\mu\nu}P_{\mu\nu} - \frac{3}{2}\pi^{\mu\nu}\tilde{P}_{\mu\nu}}{2T\zeta^2}\right)\delta_{a,3} \\
= \;& -\left[\frac{\partial}{\partial\hat{T}}\left(\frac{f_1}{\hat{T}}\right)(\zeta^{eq}.\partial\hat{T}) + \frac{\partial}{\partial\hat{T}}\left(\frac{f_2}{\hat{T}}\right)(\zeta^{eq}.\partial\nu) + \frac{\partial}{\partial\hat{T}}\left(\frac{f}{\hat{T}}\right)(\zeta^{eq}.\partial f_3) - \frac{f}{\hat{T}}\zeta^{eq}.\partial\left(\frac{f_1}{f}\right)\right]\delta_{a,1} \\
& -\left[\frac{\partial}{\partial\nu}\left(\frac{f_1}{\hat{T}}\right)(\zeta^{eq}.\partial\hat{T}) + \frac{\partial}{\partial\nu}\left(\frac{f_2}{\hat{T}}\right)(\zeta^{eq}.\partial\nu) + \frac{\partial}{\partial\nu}\left(\frac{f}{\hat{T}}\right)(\zeta^{eq}.\partial f_3) - \frac{f}{\hat{T}}\zeta^{eq}.\partial\left(\frac{f_2}{f}\right)\right]\delta_{a,2} \\
& -\frac{1}{\hat{T}}\left[\frac{\partial}{\partial(\zeta^{eq})^2}\left(\frac{f_1}{\hat{T}}\right)(\zeta^{eq}.\partial\hat{T}) + \frac{\partial}{\partial(\zeta^{eq})^2}\left(\frac{f_2}{\hat{T}}\right)(\zeta^{eq}.\partial\nu) + \frac{\partial}{\partial(\zeta^{eq})^2}\left(\frac{f}{\hat{T}}\right)(\zeta^{eq}.\partial f_3)\right]\delta_{a,3} \\
& -\left[\frac{f_1}{\hat{T}}(\zeta^{eq}.\partial\hat{T}) + \frac{f_2}{\hat{T}}(\zeta^{eq}.\partial\nu) + \frac{f}{\hat{T}}(\zeta^{eq}.\partial f_3)\right]\delta_{a,3} \\
\\
= \;& \sum_b(\zeta^{eq}.\partial H_b)\left[-\left(\frac{\partial f_3}{\partial H_b}\right)\frac{\partial}{\partial H_a}\left(\frac{f}{\hat{T}}\right) + \frac{f}{\hat{T}}\frac{\partial}{\partial H_b}\left(\frac{f_1}{f}\right)\delta_{a,1} + \frac{f}{\hat{T}}\frac{\partial}{\partial H_b}\left(\frac{f_2}{f}\right)\delta_{a,2}\right] \\
& -\frac{\partial}{\partial H_a}\left(\frac{f_1}{\hat{T}}\right)(\zeta^{eq}.\partial\hat{T}) - \frac{\partial}{\partial H_a}\left(\frac{f_2}{\hat{T}}\right)(\zeta^{eq}.\partial\nu) \\
& -\left[\frac{f_1}{\hat{T}}(\zeta^{eq}.\partial\hat{T}) + \frac{f_2}{\hat{T}}(\zeta^{eq}.\partial\nu) + \frac{f}{\hat{T}}(\zeta^{eq}.\partial f_3)\right]\delta_{a,3} \\
\\
= \;& -\sum_b(\zeta^{eq}.\partial H_b)\left[\frac{\partial}{\partial H_a}(fc_b) - \frac{f}{\hat{T}}\frac{\partial}{\partial H_b}(\hat{T}c_a) + \frac{fc_b}{(\zeta^{eq})^2}\;\delta_{a,3}\right]
\end{aligned}
\tag{6.54}
$$

In the last step we have used (6.48).

Combining (6.52), (6.53) and (6.54) it is straightforward to verify that the expressions for $\mathfrak{S}_a, \quad a = \{1,2,3,4\}$ as derived from partition function in this subsection, match exactly with (6.42),(6.43). Note that both methods leave dissipative contributions to constitutive relations completely unconstrained.

6.4.3 Entropy from the partition function

In this subsubsection we will explain how the nondissipative part of the entropy current of the superfluid may be read off in a rather direct way from the partition function. Our analysis is largely structural, and applies equally well to normal (non super) fluids. However our presentation applies only at first order in the derivative expansion.

For any system the entropy S_T in equilibrium may be evaluated from the logarithm of partition function $W = \ln Z$ via the thermodynamical relation

$$S_T = W + T_0\frac{\partial W}{\partial T_0} \tag{6.55}$$

We will now rewrite this expression in terms of the goldstone action that generates the partition function. Let this action take the form

$$S = \int \sqrt{g}\mathcal{L} d^3x$$

and also suppose

$$\mathcal{L}^{eq} = \mathcal{L}(\zeta_\mu = \zeta_\mu^{eq})$$

Now we can think of the partition function as

$$W = S(\hat{T}, \hat{\mu}, T_0 a_i, \zeta_\mu^{eq}) = \int \sqrt{g}\ \mathcal{L}^{eq}(\hat{T}, \hat{\mu}, T_0 a_i, \zeta_\mu^{eq}) d^3x$$

Using the simple rescaling of the time coordinate employed in subsection 2.3.1 of [47] one may show that

$$\begin{aligned}
T_0 \frac{\partial \hat{T}}{\partial T_0} &= \hat{T} \\
T_0 \frac{\partial \hat{\nu}}{\partial T_0} &= -\hat{\nu} \\
T_0 \frac{\partial a_i}{\partial T_0} &= 0 \\
T_0 \frac{\partial \zeta_i}{\partial T_0} &= 0
\end{aligned} \tag{6.56}$$

It follows that

$$\begin{aligned}
&\frac{\partial W}{\partial T_0} \\
&= \int d^3y \sqrt{g} \left[\left(\frac{\delta \mathcal{L}^{eq}}{\delta \hat{T}(y)} \right) \left(\frac{\partial \hat{T}(y)}{\partial T_0} \right) + \left(\frac{\delta \mathcal{L}^{eq}}{\delta \hat{\nu}(y)} \right) \left(\frac{\partial \hat{\nu}(y)}{\partial T_0} \right) + \left(\frac{\delta \mathcal{L}^{eq}}{\delta a_i(y)} \right) \left(\frac{a_i(y)}{T_0} \right) \right] \\
&= \int d^3y \sqrt{g} e^{-\sigma} \left[\frac{T_{00}}{T_0^2} + \frac{\hat{\nu} J_0}{T_0} + a_i \left(\frac{T_0^i + A_0 \delta J^i}{\hat{T}^2} \right) \right] \\
&= \int d^3y \sqrt{g} e^{\sigma} \left[\frac{1}{T_0^2} \left(T_{00} e^{-2\sigma} + a_i T_0^i \right) + \frac{\hat{\nu}}{T_0} \left(J_0 e^{-2\sigma} + a_i J^i \right) \right] \\
&= \int d^3y \sqrt{-G} \left[\frac{1}{T_0^2} \left(-\frac{T_{00}}{G_{00}} + \frac{G_{0i}}{G_{00}} T_0^i \right) + \frac{\hat{\nu}}{T_0} \left(-\frac{J_0}{G_{00}} + \frac{G_{0i}}{G_{00}} J^i \right) \right] \\
&= \int d^3y \frac{\sqrt{-G}}{T_0} \left[-\frac{T_0^0}{T_0} - \hat{\nu} J^0 \right] \\
&= \int d^3y \frac{\sqrt{-G}}{T_0} \left[-\frac{\hat{u}^\mu T_\mu^0}{\hat{T}} - \hat{\nu} J^0 \right]
\end{aligned} \tag{6.57}$$

so that

$$S_T = W + \frac{\partial W}{\partial T_0} = \int d^3y \frac{\sqrt{-G}}{T_0} \left[\hat{T} \mathcal{L}^{eq} - \frac{\hat{u}^\mu T_\mu^0}{\hat{T}} - \hat{\nu} J^0 \right] \tag{6.58}$$

This expression may be expanded to first order in derivatives employing

$$\begin{aligned}\mathcal{L}^{eq} &= \frac{\hat{P}}{\hat{T}} + \mathcal{L}_1^{eq} \\ T_0^0 &= (T_0^0)_{perf} + \delta T_0^0 \\ J^0 &= J^0_{perf} + \delta J^0\end{aligned} \tag{6.59}$$

where, from (6.14)

$$\begin{aligned}(T_0^0)_{perf} &= -\hat{\epsilon} - \hat{f} e^{-2\sigma} A_0^2 - \hat{f} A_0 a^i \zeta_i^{eq} \\ J^0_{perf} &= e^{-\sigma}\hat{q} - \hat{f}(e^{-2\sigma} A_0 + a^i \zeta_i^{eq}),\end{aligned} \tag{6.60}$$

δT_0^0 is defined in (6.23) and δJ^0 is defined in (6.22).

Using the Gibbs Duham relation

$$s = \frac{P + \epsilon - q\mu}{T}$$

we find that

$$\begin{aligned}S_T = &\int d^3y \sqrt{g}\hat{s} \\ &+ \int d^3y \frac{\sqrt{-G}}{T_0}\left[\hat{T}\mathcal{L}_1^{eq} - \frac{\hat{u}^\mu \delta T^0_\mu}{\hat{T}} - \hat{\nu}\delta J^0\right]\end{aligned} \tag{6.61}$$

(all proportional to $\hat{f}$ cancel out at zero order in the derivative expansion).

Now let us recall that

$$\delta T^\mu_\nu = (\pi^\mu_\nu)_0 + (T^\mu_\nu)^1_{perf}$$

where $(\pi^\mu_\nu)_0$ refers to π^μ_ν evaluated on the zero order equilibrium solution and $(T^\mu_\nu)^1_{perf}$ refers to the one derivative correction in T^μ_ν from the first order correction to the equilibrium solution. Similarly

$$\delta J^\mu = (j^\mu)_0 + (J^\mu_{perf})^1.$$

It follows that

$$\begin{aligned}S_T = &\int d^3y \sqrt{-G}\left[\hat{s}\hat{u}^0 - \frac{\hat{u}^\mu \delta (T^0_\mu)^1_{perf}}{\hat{T}} - \hat{\nu}\delta (J^0)^1_{perf}\right] \\ &+ \int d^3y \frac{\sqrt{-G}}{T_0}\left[\hat{T}\mathcal{L}_1^{eq} - \frac{\hat{u}^\mu \delta \pi^0_\mu}{\hat{T}} - \hat{\nu}\delta j^0\right]\end{aligned} \tag{6.62}$$

so that

$$\begin{aligned}S_T = &\int d^3y \sqrt{-G} s u^0 \\ &+ \int d^3y \frac{\sqrt{-G}}{T_0}\left[\hat{T}\mathcal{L}_1^{eq} - \frac{\hat{u}^\mu \delta \pi^0_\mu}{\hat{T}} - \hat{\nu}\delta j^0\right]\end{aligned} \tag{6.63}$$

where su^0 in (6.63) refers to the entropy evaluated on the first order corrected solution. In going from (6.62) to (6.63) we have used the fact that the frame invariance (see [4] for a definition and extensive discussion of frame invariance) of the canonical entropy current

$$J^\mu_{can} = su^\mu - \nu j^\mu - \frac{u_\nu \pi^{\mu\nu}}{T}$$

implies that

$$su^\mu - \hat{s}\hat{u}^\mu + \nu (J^\nu)^1_{perf} + \frac{u_\nu (T^{\mu\nu})^1_{perf}}{T} = 0.$$

It follows from (6.63) that

$$S_T = \int d^3y \sqrt{-G} \left[J^0_{can} + \frac{1}{T_0} \hat{T} \mathcal{L}^{eq}_1 \right] \tag{6.64}$$

Comparing with

$$J^\mu_S = J^\mu_{can} + J^\mu_{new} \tag{6.65}$$

we conclude that

$$\int d^3y \sqrt{-G} J^0_{new} = \int d^3y \sqrt{-G} \frac{\hat{T}}{T_0} \mathcal{L}^{eq}_1 \tag{6.66}$$

In other words the integral of J^0_{new} matches with the first order correction to the Goldstone action. (6.66) is the principal formal result of this subsection. It expresses a very simple relationship between the correction to the canonical entropy current of our system and the first order correction to the partition function.

To what extent does (6.66) determine J^μ_{new}? The most general first order correction to the entropy current takes the form

$$J^\mu_{new} = S_u u^\mu + S_\zeta \zeta^\mu + V^\mu_s \tag{6.67}$$

where S_u and S_ζ are first order scalars while V^μ_s is a first order vector. Notice that, to first order, $X^\mu = S_\zeta \zeta^\mu + V^\mu_s$ is orthogonal to $\hat{u}$. It follows immediately from this observation that

$$X^0 = -a_i X^i$$

Plugging this relation into (6.66) we conclude that the contribution from X^μ to the total entropy is not Kaluza Klein gauge invariant and so must vanish (see [47] for a discussion on related issues). It follows that S_ζ and V^μ_s vanish in equilibrium. Upto dissipative corrections, therefore, it follows that

$$J^\mu_{new} = S_u u^\mu \tag{6.68}$$

Now comparing with (6.66) it follows that

$$\int d^3y \sqrt{g} \left(S_u - \mathcal{L}^{eq}_1 \right) = 0$$

so that

$$S_u = \mathcal{L}_1^{eq} + \text{total} \;\; \text{derivatives} \tag{6.69}$$

Let us now turn to the case at hand. $\mathcal{L}_1^{eq}$ was listed in (6.17). It is easily verified that there exist no total derivative scalars at one derivative order. Consequently we conclude that

$$S_u = \frac{f_1}{\hat{T}}(\zeta.\partial)\hat{T} + \frac{f_2}{\hat{\nu}}(\zeta.\partial)\hat{\nu} - f_3 \nabla_i \left(\frac{f}{\hat{T}} \zeta^i \right) + \text{dissipative}$$

It is not difficult to verify that this expression, together with (6.68), agree exactly with (6.34) in equilibrium once we employ the identification of parameters (6.48).

In summary, the positive divergence entropy current - which we determined earlier in this section - is also uniquely determined by comparison with the partition function for parity even superfluids at first order in the derivative expansion.

6.4.4 Consistency with field redefinitions

We will now verify that the dependence of the constitutive relations and entropy current of the superfluid on f_3 is consistent with the transformation (6.21) of f_3 under the field redefinition (6.18).

Recall that the stress tensor and currents of our system take the form

$$\begin{aligned} T^{\mu\nu} &= (\epsilon + P) u^\mu u^\nu + P G^{\mu\nu} + f \xi^\mu \xi^\nu + \pi^{\mu\nu} \\ J^\mu &= q u^\mu - f \xi^\mu + j^\mu \\ J_s^\mu &= J_{can}^\mu + J_{new}^\mu = s u^\mu - \frac{u_\nu \pi^{\mu\nu}}{T} - \nu j^\mu + J_{new}^\mu . \end{aligned} \tag{6.70}$$

Substituting the field redefinition (6.18) into this equation and setting

$$\delta\phi(x^i) = h(x^i)$$

(recall h is a function only of space) we recover a new form of the stress tensor and currents

$$\begin{aligned} T^{\mu\nu} &= (\tilde{\epsilon} + \tilde{P}) u^\mu u^\nu + \tilde{P} G^{\mu\nu} + \tilde{f} \tilde{\xi}^\mu \tilde{\xi}^\nu + \tilde{\pi}^{\mu\nu} \\ J^\mu &= \tilde{q} u^\mu - \tilde{f} \tilde{\xi}^\mu + \tilde{j}^\mu \\ J_s^\mu &= \tilde{J}_{can}^\mu + \tilde{J}_{new}^\mu = \tilde{s} u^\mu - \frac{u_\nu \tilde{\pi}^{\mu\nu}}{T} - \nu \tilde{j}^\mu + \tilde{J}_{new}^\mu \end{aligned} \tag{6.71}$$

with

$$\begin{aligned} \tilde{\pi}^{\mu\nu} &= \pi^{\mu\nu} - \left[\frac{\partial(\epsilon + P)}{\partial\chi} u^\mu u^\nu + \frac{\partial P}{\partial\chi} G^{\mu\nu} + \frac{\partial f}{\partial\chi} \xi^\mu \xi^\nu \right] (-2\xi.\nabla^{(4)} h) \\ &\qquad - f(\xi^\mu G^{\nu\alpha} \nabla^{(4)}_\alpha h + \xi^\nu G^{\mu\alpha} \nabla^{(4)}_\alpha h) \\ \tilde{j}^\mu &= j^\mu - \left[\frac{\partial q}{\partial\chi} u^\mu - \frac{\partial f}{\partial\chi} \xi^\mu \right] (-2\xi.\nabla^{(4)} h) + f G^{\mu\alpha} \nabla^{(4)}_\alpha h \\ \tilde{J}_{new}^\mu &= J_{new}^\mu - (-2\xi.\nabla^{(4)} h) \left(\frac{\partial s}{\partial\chi} \right) u^\mu - \frac{u_\nu (\pi^{\mu\nu} - \tilde{\pi}^{\mu\nu})}{T} - \nu (j^\mu - \tilde{j}^\mu) \\ &= J_{new}^\mu + \frac{f}{T} \left(u^\mu \xi^\nu - u^\nu \xi^\mu \right) \nabla^{(4)}_\nu h = J_{new}^\mu + \frac{Q^{\mu\nu}}{T} \nabla^{(4)}_\nu h \end{aligned} \tag{6.72}$$

All Greek indices in (6.72) and (6.71) run from $1\dots4$ and are raised and lowered with the full four dimensional metric $G^{\mu\nu}$. χ derivatives in (6.72) are taken at fixed T and ν. In deriving last equality in (6.72) we have used the first law of thermodynamics.

$$d\epsilon = Tds + \nu dq - \frac{f}{2}d\chi$$

We will now independently verify that our final answers for J^μ_{new} and the constitutive relations have this symmetry. To start with recall that, from (6.21) and (6.48),

$$\tilde{c}_a - c_a = \frac{1}{T}\frac{\partial h}{\partial H_a} \tag{6.73}$$

It follows immediately from (6.73) that the expression for J^μ_{new}

$$J^\mu_{new} = \sum_a c_a(\partial_\nu H_a)Q^{\mu\nu}$$

(see (6.34)) transforms as predicted by the last of (6.72).

We now turn to the verification that our results for transport coefficients, (6.42),(6.43), transform as predicted by (6.72). The algebra involved in a direct verification is formidable, so we will content ourselves with an indirect check. We first recall that we have already verified (see (6.46)) that

$$\begin{aligned}\mathfrak{S}_a(\delta T_{\mu\nu}, \delta J_\mu) &= \mathfrak{S}_a(\pi_{\mu\nu}, j_\mu)\\ \mathfrak{V}_a(\delta T_{\mu\nu}, \delta J_\mu) &= \mathfrak{V}_a(\pi_{\mu\nu}, j_\mu)\end{aligned} \tag{6.74}$$

in fact this equation formed the basis of one of our two methods of determining constitutive relations. It follows that if we can show that $\delta T_{\mu\nu}$ and δJ_μ obey (6.72), then the same will be true of (6.42),(6.43). (Recall $\delta T_{\mu\nu}$ was the first order shift in the stress tensor arising from first order corrections to the Goldstone action; δJ^μ was similarly defined.) We will now check that this is indeed the case. In order to do this we first simplify the (6.72) specializing to the case of stationary equilibrium

$$\begin{aligned}(j_0 - \tilde{j}_0) &= -2e^\sigma\left[\frac{\partial}{\partial\zeta_f^2}(q+\mu f)\right](\zeta^{eq}.\partial)h\\ &= e^\sigma\left[\frac{\partial}{\partial\nu}\left(\frac{f}{T}\right)\right](\zeta^{eq}.\partial)h\\ (j^i - \tilde{j}^i) &= -f\nabla^i h - 2(\zeta^{eq}.\partial h)\frac{\partial f}{\partial\zeta_f^2}(\zeta^{eq})^i\end{aligned} \tag{6.75}$$

and

$$
\begin{aligned}
(\pi_{00} - \tilde{\pi}_{00}) &= -2e^{2\sigma} \left[\frac{\partial(\epsilon + \mu^2 f)}{\partial \zeta_f^2} \right] (\zeta^{eq}.\partial) h \\
&= 2e^{2\sigma} \left[\frac{\partial}{\partial \zeta_f^2} \left(T \frac{\partial P}{\partial T} - P \right) \right] (\zeta^{eq}.\partial) h \\
&= 2e^{2\sigma} \left(-\frac{T}{2} \frac{\partial f}{\partial T} + \frac{f}{2} \right) (\zeta^{eq}.\partial) h \\
&= -T^2 e^{2\sigma} \left[\frac{\partial}{\partial T} \left(\frac{f}{T} \right) \right] (\zeta^{eq}.\partial) h
\end{aligned} \tag{6.76}
$$

$$
(\pi_0^i - \tilde{\pi}_0^i) = -A_0 (j^i - \tilde{j}^i)
$$

$$
(\pi^{ij} - \tilde{\pi}^{ij}) = 2(\zeta^{eq}.\partial h) \left[-\frac{f}{2} g^{ij} + \left(\frac{\partial f}{\partial \zeta_f^2} \right) (\zeta^{eq})^i (\zeta^{eq})^j \right] + f \left[(\zeta^{eq})^i \nabla^j h + (\zeta^{eq})^j \nabla^i h \right]
$$

Where each of the scalar thermodynamic functions are evaluated on the zeroth order equilibrium solution

$$
T = \hat{T}, \quad \nu = \hat{\nu}, \quad (\zeta_f)_i = \zeta_i^{eq}
$$

In obtaining (6.75) and (6.76)

$$
dP = \left(\frac{\epsilon + P + \mu^2 f}{T} \right) dT + T(q + \nu f) d\nu - \frac{f}{2} d\zeta_f^2
$$

In those equations all spatial indices are raised and lowered by use of the spatial metric g_{ij} (all the free indices will run from 1 to 3).

We now turn to the explicit expressions for $\delta T_{\mu\nu}$ and δJ_μ listed in (6.22). Substituting

$$
\tilde{f}_3 = f_3 + h
$$

(see (6.21)) in those expressions we obtain immediate agreement with (6.75) and (6.76). This completes our verification.

6.5 Constraints on parity violating constitutive relations at first order

In this subsection we use the partition function to derive constraints on parity violating contributions to constitutive relations by comparison with the local goldstone action (6.26). As in the previous subsection, we find perfect agreement with the constraints obtained from the local form of the second law. It turns out in this case that the second law analysis has already been performed, in full detail, in [4]. We begin this section by reviewing the results of [4], before turning to a re derivation of those results by comparison with (6.26).

6.5.1 Review of constraints from the second law

Basis of Frame Invariants As we have seen above, the constitutive relations are an expansion of frame invariant combinations of $\pi^{\mu\nu}$ and j^μ in terms of independent one derivative scalars, vectors and tensors. Before even specifying the constitutive relations, we must first specify a basis of frame invariant expressions that we will expand in this manner. In the previous section we choose to work with the frame invariant scalars $\mathfrak{S}_a$ and frame invariant vectors $\mathfrak{V}_a$. A different choice for frame invariants was made in [4]; in order to ease comparison with the results of that paper, we will adapt that choice in this section. In this subsubsection we describe the basis of frame invariants used in [4].

Let

$$\begin{aligned}
\mathbf{s}_1 &= \pi^{\mu\nu}\tilde{P}_{\mu\nu} & \zeta_f^2\mathbf{s}_2 &= \zeta_f\cdot\pi\cdot\zeta_f \\
\mathbf{s}_3 &= u\cdot\pi\cdot u & \mathbf{s}_4 &= u\cdot\pi\cdot\zeta_f \\
\mathbf{s}_5 &= u\cdot j & \mathbf{s}_6 &= \zeta_f\cdot j \\
\mathbf{s}_7 &= -\mu_{diss} & & \\
\mathbf{v}_1^\nu &= u_\mu\pi^{\mu\alpha}\tilde{P}^\nu{}_\alpha & \mathbf{v}_2^\nu &= (\zeta_f)_\mu\pi^{\mu\alpha}\tilde{P}^\nu{}_\alpha \\
\mathbf{v}_3^\nu &= \tilde{P}^\nu{}_\alpha j^\alpha & & \\
\mathbf{t} &= \tilde{P}_\mu{}^\alpha\tilde{P}_\nu{}^\beta\pi_{\alpha\beta} - \frac{1}{2}\tilde{P}^{\mu\nu}\tilde{P}^{\alpha\beta}\pi_{\alpha\beta}\,, & &
\end{aligned} \tag{6.77}$$

Throughout this section $\mathbf{s}_7 = -\mu_{diss} = 0$ (μ_{diss} was defined in [5]). However we will retain s_7 in all our formulas, in order to permit easy adaptation of our final results to frames in which $\mu_{diss} \neq 0$.

$$P^{\mu\nu} = G^{\mu\nu} + u^\mu u^\nu \qquad \tilde{P}^{\mu\nu} = P^{\mu\nu} - \frac{(\zeta_f)^\mu(\zeta_f)^\nu}{(\zeta_f)^2}\,. \tag{6.78}$$

Following [4] we define the row vectors

$$\begin{aligned}
\mathbf{s} &= \begin{pmatrix}\mathbf{s}_1 & \mathbf{s}_2 & \mathbf{s}_3 & \mathbf{s}_4 & \mathbf{s}_5 & \mathbf{s}_6 & \mathbf{s}_7\end{pmatrix} \\
\mathbf{v} &= \begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{pmatrix}\,.
\end{aligned} \tag{6.79}$$

We also define the matrices

$$A^s = \begin{pmatrix}
\frac{Rs}{2qT\psi_f} & \frac{B_3}{3T} - \frac{A_3}{2T\psi_f} & \frac{B_2}{3T} - \frac{A_2}{2T\psi_f} & \frac{B_1}{3T} - \frac{A_1}{2T\psi_f} \\
-\frac{Rs}{q\psi_f T} & \frac{B_3}{3T} + \frac{A_3}{T\psi_f} & \frac{B_2}{3T} + \frac{A_2}{T\psi_f} & \frac{B_1}{3T} + \frac{A_1}{T\psi_f} \\
0 & \frac{1}{T^2} & 0 & 0 \\
-\frac{R}{T^2\psi_f} & \frac{K_3}{T} & \frac{K_2}{T} & \frac{K_1}{T} \\
0 & 0 & -1 & 0 \\
-\frac{1}{T^2\psi_f} & 0 & 0 & 0 \\
0 & \frac{(\rho+P)K_3}{T} & \frac{(\rho+P)K_2}{T} & \frac{(\rho+P)K_1}{T}
\end{pmatrix}\,, \quad A^v = \begin{pmatrix} -R & 0 \\ 0 & \frac{2}{T^3\psi_f} \\ -1 & 0 \end{pmatrix} \tag{6.80}$$

where

$$R = \frac{q}{\rho+P} \qquad V^\mu = \frac{E^\mu}{T} - P^{\mu\nu}\partial_\nu\nu$$

and the A_i's B_i's, C_i's and K_i's defined as follows.

$$
\begin{aligned}
&\nu = \frac{\mu}{T}, \quad \psi_f = \frac{\zeta_f^2}{T^2}, \quad K = \frac{\nabla_\theta[f\xi^\theta]}{\epsilon+P}, \quad R = \frac{q}{\epsilon+P} \\
&B_1 = -\frac{\partial}{\partial\psi_f}[\log(s)], \quad B_2 = -\frac{\partial}{\partial\nu}[\log(s)], \quad B_3 = -\frac{\partial}{\partial T}[\log(s)] \\
&K_1 = \frac{s}{\epsilon+P}\frac{\partial}{\partial\psi_f}\left[\frac{q}{s}\right], \quad K_2 = \frac{s}{\epsilon+P}\frac{\partial}{\partial\nu}\left[\frac{q}{s}\right], \quad K_3 = \frac{s}{\epsilon+P}\frac{\partial}{\partial T}\left[\frac{q}{s}\right] \\
&A_1 = -\frac{1}{2} - \nu\psi_f(1-\mu R)\left[\frac{\partial}{\partial\psi_f}\left(\frac{q}{s}\right)\right] + \frac{\psi_f}{3s}\frac{\partial s}{\partial\psi_f} \\
&A_2 = -\nu\psi_f(1-\mu R)\left[\frac{\partial}{\partial\nu}\left(\frac{q}{s}\right)\right] + \frac{\psi_f}{3s}\frac{\partial s}{\partial\nu} \\
&A_3 = -\nu\psi_f(1-\mu R)\left[\frac{\partial}{\partial T}\left(\frac{q}{s}\right)\right] + \frac{\psi_f}{3s}\left(\frac{\partial s}{\partial T} - \frac{3s}{T}\right) \\
&V_\mu = \frac{E_\mu}{T} - P_\mu^\sigma\nabla_\sigma\left[\frac{\mu}{T}\right] \\
&\Omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}u_\nu\nabla_\lambda(\zeta_f)_\sigma, \quad \omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}u_\nu\nabla_\lambda u_\sigma, \quad B^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}u_\nu F_{\lambda\sigma}\,.
\end{aligned}
\tag{6.81}
$$

In terms of (6.77)-(6.80), the frame invariant scalar, vector and tensor combinations of $\pi^{\mu\nu}$, j^μ and μ_{diss} are given by the row vectors

$$
\mathbf{s}A^s, \qquad \mathbf{v}_\mu A^v, \qquad \mathbf{t}_{\mu\nu}\,. \tag{6.82}
$$

By scalars, vectors and tensors we mean expressions which transform as spin 0, ±1 and ±2 representations of the $SO(2)$ symmetry that is left invariant by the two vectors u^μ and ξ^μ at each point in spacetime.

Constitutive Relations We have 4 frame invariant scalars, 2 frame invariant vectors and one frame invariant tensor. The most general symmetry allowed parity odd first derivative constitutive relations take the form

$$
\begin{aligned}
\mathbf{t}^{\mu\nu} &= -\tilde{\eta}\tilde{\mathcal{T}}_1^{\mu\nu} \\
\mathbf{v}_i^\mu A_{ij}^v &= -\sum_{i=1}^{2}\tilde{\mathcal{V}}_i\tilde{\kappa}_{ij} - \left(\sum_{i=3}^{7}\tilde{\mathcal{V}}_i\tilde{\kappa}_{ij}\right) \\
\mathbf{s}_i A_{ij}^s &= -\left(\sum_{i=1}^{2}\sum_{j=1}^{4}\tilde{\mathcal{S}}_i\tilde{\beta}_{ij}\right)
\end{aligned}
\tag{6.83}
$$

with $\mathcal{T}$, $\tilde{\mathcal{T}}$, $\mathcal{V}$, $\tilde{\mathcal{V}}$, $\mathcal{S}$ and $\tilde{\mathcal{S}}$ a basis of onshell independent $SO(2)$ invariant tensors, vectors and scalars given in table

6.5.2 Constraints on constitutive relations from the local second law

Notice that both pseudo tensors that appear in (6.83) are nondissipative. Further, the five pseudo vectors $\mathcal{V}_i$ $i = 3 \ldots 7$, are nondissipative. It may therefore come as no surprise to

vector	definition	dual parity odd vector $\left(\epsilon^{\mu\nu\alpha\beta}u_\nu\xi_\alpha\mathcal{V}_\beta\right)$ evaluated in equilibrium
$\mathcal{V}_1^\mu$	$\left(\frac{E^\mu}{T} - \nabla^\mu\left(\frac{\mu}{T}\right)\right)$	0
$\mathcal{V}_2^\mu$	$\tilde{P}^{\mu\beta}\left(\zeta_f^\alpha\sigma_{\alpha\beta}\right)$	0
$\mathcal{V}_3^\mu$	$\tilde{P}^{\mu\sigma}\nabla_\sigma T$	$-\hat{T}V_1^i$
$\mathcal{V}_4^\mu$	$\tilde{P}^{\mu\sigma}\nabla_\sigma\left(\frac{\mu}{T}\right)$	$\frac{1}{T_0}V_2^i$
$\mathcal{V}_5^\mu$	$\tilde{P}^{\mu\sigma}\nabla_\sigma\left(\frac{\zeta_f^2}{T^2}\right)$	$\epsilon^{ijk}V_5^i$
$\mathcal{V}_6^\mu$	$\frac{\mathcal{V}_2^{c\,\mu} - \tilde{P}^{\mu\alpha}\zeta_f^\nu\partial_\alpha u_\nu}{\zeta_f^2}$	$\frac{1}{2(\zeta^{eq})^2}e^\sigma V_4^i$
$\mathcal{V}_7^\mu$	$-\frac{P^{\mu\nu}F_{\nu\alpha}\zeta_f^\alpha}{\zeta_f^2}$	$-\frac{1}{(\zeta^{eq})^2}(\xi_0V_4^i + V_3^i)$

Table 19. Independent fluid vector data. Here V_m^i for m=1,2,3,4,5 are independent vectors in equilibrium defined in (6.28)

pseudo scalars	definition	In equilibrium
$\tilde{\mathcal{S}}_1$	$\omega.\xi$	$-\frac{1}{2}e^\sigma S_2$
$\tilde{\mathcal{S}}_2$	$B.\xi$	$S_1 + \xi_0 S_2$
pseudo tensors	definition	In equilibrium
$\tilde{\mathcal{T}}_1^{\mu\nu}$	$*\sigma^u_{\mu\nu}$	0
$\tilde{\mathcal{T}}_2^{\mu\nu}$	$*\sigma^\xi_{\mu\nu}$	won't need

Table 20. Independent fluid scalar and tensor data. Here S_m for m=1,2 are independent vectors in equilibrium defined in (6.28). σ^u and σ^ξ are the shear tensors for normal and superfluid velocity respectively and $*\sigma_{\mu\nu} = \epsilon^{\mu\rho\alpha\beta}u^\rho\xi^\alpha\sigma_\beta{}^\nu + (\mu \leftrightarrow \nu)$

the reader that [4] was able to use the principle of local entropy increase to determine $\tilde{\kappa}_{im}$ ($i = 3\ldots 7$ and $m = 1\ldots 2$), together with $\tilde{\beta}_{ij}$ ($i = 1\ldots 2$ and $j = 1\ldots 4$) in terms of two free functions that appeared in the parameterization of the entropy current. These two functions were called σ_8 and σ_{10} in [4]. The results of [4] were presented in terms of σ_8 and σ_{10} and four additional auxiliary fields which were determined in terms of σ_8 and σ_{10} by the relations

[64]

$$
\begin{aligned}
\sigma_3 &= -T\frac{\partial}{\partial T}(\sigma_{10}-\nu\sigma_8)\\
\sigma_4 &= \sigma_8 + C\nu + 2\tilde{h} - \frac{\partial}{\partial \nu}(\sigma_{10}-\nu\sigma_8)\\
\sigma_5 &= -\frac{\partial}{\partial \psi_f}(\sigma_{10}-\nu\sigma_8)\\
\sigma_9 &= 2\nu(\sigma_{10}-\nu\sigma_8) - \frac{2}{3}C\nu^3 - 2\tilde{h}\nu^2 + s_9
\end{aligned}
\tag{6.84}
$$

In terms of all these fields, it was demonstrated in [4] that point wise positivity of the the divergence of the entropy current determines

$$
\tilde{\eta} = 0, \quad \tilde{\kappa}_{m2} = 0 \tag{6.85}
$$

and

$$
\begin{aligned}
\tilde{\kappa}_{31} &= -RT\sigma_3 - T\partial_T\sigma_8\\
\tilde{\kappa}_{41} &= -RT^2\sigma_4 - T\partial_\nu\sigma_8\\
\tilde{\kappa}_{51} &= -RT^2\sigma_5 - T\partial_\psi\sigma_8\\
\tilde{\kappa}_{61} &= -2RT^3\sigma_9 + 2T^2\sigma_{10}\\
\tilde{\kappa}_{71} &= -RT^2\sigma_{10} + 2T\sigma_8 + CT\nu + 2\tilde{h}T
\end{aligned}
\tag{6.86}
$$

$$
-\tilde{\beta}_{ij} = \begin{pmatrix} \frac{2RT\sigma_9}{\psi_f} - \frac{2\sigma_{10}}{\psi_f} & -2\sigma_3 - 2T^2K_3\sigma_9 & -2T\sigma_4 - 2T^2K_2\sigma_9 & -2T\sigma_5 - 2K_1T^2\sigma_9 \\ -\frac{C\nu+2\tilde{h}}{T\psi_f} - \frac{2\sigma_8}{T\psi_f} + \frac{R\sigma_{10}}{\psi_f} & \partial_T\sigma_8 - K_3T\sigma_{10} & \partial_\nu\sigma_8 - K_2T\sigma_{10} & \partial_{\psi_f}\sigma_8 - K_1T\sigma_{10} \end{pmatrix}. \tag{6.87}
$$

6.5.3 Constraints on constitutive relations from the Goldstone action

As in the previous subsubsection, we use the Goldstone action to constrain transport coefficients as follows. All constraints follow from the analogue of (6.46)

$$
\begin{aligned}
\mathbf{t}^{\mu\nu}(\delta T_{\mu\nu}, \delta J_\mu) &= \mathbf{t}^{\mu\nu}(\pi_{\mu\nu}, j_\mu)\\
\mathbf{v}_i^\mu A_{ij}^v(\delta T_{\mu\nu}, \delta J_\mu) &= \mathbf{v}_i^\mu A_{ij}^v(\pi_{\mu\nu}, j_\mu)\\
\mathbf{s}_i A_{ij}^s(\delta T_{\mu\nu}, \delta J_\mu) &= \mathbf{s}_i A_{ij}^s(\pi_{\mu\nu}, j_\mu)
\end{aligned}
\tag{6.88}
$$

The LHS in this equation may be determined in terms of the functions g_1 and g_2 in the Goldstone action using (6.28). The RHS of the same equation is simplified using (6.83) under

[64]All terms in (6.84) proportional to the constant $\tilde{h}$ were omitted in [4]. The reason for this is that [4] assumed that the entropy current was gauge invariant. As explained in [47] this does not seem to be physically necessary as long as the divergence of the entropy current is gauge invariant. This allows the addition of the new term proportional to $\tilde{h}$ in (6.97), which allows for a slight modification of the results of [4], captured by the shifts described below. As we will see later, the requirement of CPT invariance forces $\tilde{h}$ to vanish.

the substitution $T \to \hat{T}$, $\mu \to \hat{\mu}$, $\zeta_f \to \zeta^{eq}$. Under the last substitution, the parity odd first derivative vectors and scalars evaluate to geometric expressions. Substituting these relations into the RHS of (6.88) and equation coefficients of independent vectors and tensors yields an expression for all non dissipative transport coefficients in terms of the functions g_1 and g_2. Using Eq.(6.27) one obtains

$$
\begin{aligned}
\mathbf{v}_1^i &= u_\mu \pi^{\mu\alpha} \tilde{P}^i{}_\alpha \\
&= \hat{T}^3(-\hat{\nu}\partial_{\hat{T}} g_1 + \partial_{\hat{T}} g_2) V_1^i + \frac{\hat{T}^2}{T_0}(\hat{\nu}\partial_{\hat{\nu}} g_1 - \partial_{\hat{\nu}} g_2) V_2^i - \frac{\hat{T}^2}{(\zeta^{eq})^2}(g_2 - 2g_1\hat{\nu}) V_3^i \\
&+ T_0 \hat{\nu} \frac{\hat{T}^2}{(\zeta^{eq})^2} V_4^i + \hat{T}^2(\hat{\nu}\partial_{\psi_{eq}} g_1 - \partial_{\psi_{eq}} g_2) V_5^i \\
\mathbf{v}_2^i &= (\zeta_f)_\mu \pi^{\mu\alpha} \tilde{P}^i{}_\alpha = 0 \\
\mathbf{v}_3^i &= \tilde{P}^\nu_{\alpha} j^\alpha \\
&= \hat{T}(\hat{T}\partial_{\hat{T}} g_1 \; V_1^i - \frac{1}{T_0}\partial_{\hat{\nu}} g_1 \; V_2^i - \frac{1}{(\zeta^{eq})^2}(2g_1 \; V_3^i + g_2 T_0 V_4^i + (\zeta^{eq})^2 \partial_{\psi_{eq}} g_1 \; V_5^i)) \\
\mathbf{s}_1 &= \pi^{\mu\nu}\tilde{P}_{\mu\nu} = 0 \\
\mathbf{s}_2 &= \frac{1}{(\zeta_f)^2}\zeta_f \cdot \pi \cdot \zeta_f = -\frac{2}{\hat{T}}(\zeta^{eq})^2\big(\partial_{\psi_{eq}} g_1 \; S_1 + T_0 \partial_{\psi_{eq}} g_2 \; S_2\big) \\
\mathbf{s}_3 &= u\cdot\pi\cdot u = \hat{T}(\hat{T}\partial_{\hat{T}} g_1 - 2\psi_{eq}\partial_{\psi_{eq}} g_1) S_1 + \hat{T} \; T_0(\hat{T}\partial_{\hat{T}} g_2 - 2\psi_{eq}\partial_{\psi_{eq}} g_2) S_2 \\
\mathbf{s}_4 &= u\cdot\pi\cdot\zeta_f = \big(\hat{T}^2(g_2 - 2g_1) - 2(\zeta^{eq})^2\hat{\nu}\partial_{\psi_{eq}} g_1\big) S_1 - S_2 T_0 \hat{\nu}(g_2 \hat{T}^2 + 2(\zeta^{eq})^2 \partial_{\psi_{eq}} g_2) \\
&+ 2C_1 e^{-\sigma} S_2 T_0^3 - C\frac{1}{6}A_0^2 e^{-\sigma}\left(A_0 S_2 + 3S_1\right) \\
\mathbf{s}_5 &= u\cdot j = -(\partial_{\hat{\nu}} g_1 S_1 + T_0 \partial_{\hat{\nu}} g_2 S_2) \\
\mathbf{s}_6 &= \zeta_f \cdot j = 2\hat{T}\big(g_1 + \frac{(\zeta^{eq})^2}{\hat{T}^2}\partial_{\psi_{eq}} g_1\big) S_1 + \hat{T} T_0 \big(g_2 + \frac{(\zeta^{eq})^2}{\hat{T}^2}\partial_{\psi_{eq}} g_2\big) S_2 + \frac{1}{2} C A_0 e^{-\sigma}\left(A_0 S_2 + 2S_1\right) \\
\mathbf{s}_7 &= -\mu_{diss} = 0
\end{aligned}
\tag{6.89}
$$

Now using Eq.(6.83) one can find out the transport coefficients $\tilde{\kappa}_{ij}$ in terms of partition function coefficients $g_1,\ g_2$ as follows

$$
\begin{aligned}
&\tilde{\eta} = 0, \quad \tilde{\kappa}_{i2} = 0 \quad for \quad i \in (3\ to\ 7) \\
\kappa_{31} &= -\frac{\hat{T}\left((-\hat{\nu}q\hat{T}+\epsilon+P)\partial_{\hat{T}}g_1 + q\hat{T}\partial_{\hat{T}}g_2\right)}{P+\epsilon} \\
\kappa_{41} &= -\frac{\hat{T}\left((-\hat{\nu}q\hat{T}+\epsilon+P)\partial_{\hat{\nu}}g_1 + q\hat{T}\partial_{\hat{\nu}}g_2\right)}{P+\epsilon} \\
\kappa_{51} &= -\frac{\hat{T}\left((-\hat{\nu}q\hat{T}+\epsilon+P)\partial_{\psi_{eq}}g_1 + q\hat{T}\partial_{\psi_{eq}}g_2\right)}{P+\epsilon} \\
\kappa_{61} &= \frac{2\hat{T}^2}{\epsilon+P}\left(-g_2(-2\hat{\nu}q\hat{T}+\epsilon+P) + 2g_1\hat{\nu}(-\hat{\nu}q\hat{T}+\epsilon+P)\right) \\
&\quad + \frac{C\hat{\nu}^2\hat{T}^2(3p-2\hat{\nu}q\hat{T}+3\epsilon)}{3(P+\epsilon)} - \frac{4C_1q\hat{T}^3}{P+\epsilon} \\
\kappa_{71} &= \frac{\hat{T}}{\epsilon+P}\left(g_2q\hat{T} + 2g_1(-\hat{\nu}q\hat{T}+\epsilon+P)\right) + \frac{C\hat{\nu}\hat{T}(2p-\hat{\nu}q\hat{T}+2\epsilon)}{2(P+\epsilon)}.
\end{aligned}
\tag{6.90}
$$

Similarly, the transport coefficients β_{ij} in terms of partition function coefficients $g_1,\ g_2$ as follows

$$
\begin{aligned}
-\beta_{11} &= \frac{4R\hat{T}\hat{\nu}(-g_2+g_1\hat{\nu})}{\psi_{eq}} - \frac{2(-g_2+2g_1\hat{\nu})}{\psi_{eq}} + C\frac{\hat{\nu}^2\hat{T}^2(-3P+2\hat{\nu}q\hat{T}-3\epsilon)}{3(\zeta^{eq})^2(P+\epsilon)} + C_1\frac{4q\hat{T}^3}{(\zeta^{eq})^2(P+\epsilon)} \\
-\beta_{12} &= -\frac{2g_1}{\hat{T}\psi_{eq}} + \frac{R(-g_2+2g_1\hat{\nu})}{\psi_{eq}} - C\frac{\hat{\nu}\hat{T}(2P-\hat{\nu}q\hat{T}+2\epsilon)}{2(\zeta^{eq})^2(P+\epsilon)} \\
-\beta_{21} &= -2\hat{T}(-\hat{\nu}\partial_{\hat{T}}g_1+\partial_{\hat{T}}g_2) - 4\hat{\nu}T^2K_3(-g_2+g_1\hat{\nu}) - \frac{2}{3}CK_3\hat{T}^2\hat{\nu}^3 - 4C_1K_3\hat{T}^2 \\
-\beta_{22} &= \partial_{\hat{T}}g_1 - K_3\hat{T}(-g_2+2g_1\nu) - \frac{1}{2}CK_3\hat{T}\hat{\nu}^2 \\
-\beta_{31} &= -2\hat{T}(-\hat{\nu}\partial_{\hat{\nu}}g_1+\partial_{\hat{\nu}}g_2) - 4\hat{\nu}\hat{T}^2K_2(-g_2+g_1\hat{\nu}) - \frac{2}{3}CK_2\hat{T}^2\hat{\nu}^3 - 4C_1K_2\hat{T}^2 \\
-\beta_{32} &= \partial_{\hat{\nu}}g_1 - K_2\hat{T}(-g_2+2g_1\hat{\nu}) - \frac{1}{2}CK_2\hat{T}\hat{\nu}^2 \\
-\beta_{41} &= -2T(-\hat{\nu}\partial_{\psi_{eq}}g_1+\partial_{\psi_{eq}}g_2) - 4K_1\hat{\nu}\hat{T}^2(-g_2+g_1\hat{\nu}) - \frac{2}{3}CK_1\hat{T}^2\hat{\nu}^3 - 4C_1K_1\hat{T}^2, \\
-\beta_{42} &= \partial_{\psi_{eq}}g_1 - K_1\hat{T}(-g_2+2g_1\hat{\nu}) - \frac{1}{2}CK_1\hat{T}\hat{\nu}^2.
\end{aligned}
\tag{6.91}
$$

In equations (6.89), (6.90) and (6.91) the functions g_1, g_2 and all the other thermodynamics functions (like ϵ, P, q etc) as arbitrary functions of $\hat{T}$, $\hat{\nu}$ and ψ_{eq}.

If we make the substitution

$$
g_1 = \sigma_8 + \tilde{h}, \quad g_2 = -\sigma_{10} + 2\hat{\nu}\sigma_8 + \frac{1}{2}C\hat{\nu}^2 + 2\tilde{h}\hat{\nu}. \tag{6.92}
$$

and introduce the auxiliary fields σ_3, σ_4, σ_5 and σ_9 which are written in terms of σ_8 and σ_{10} in (6.84) then our results for nondissipative transport coefficients agree precisely[65] with (6.85), (6.86), (6.87).

6.5.4 Entropy

As in the parity even case, we may determine the parity odd contribution to the entropy current by a simple direct comparison with the the partition function. The relevant equation here is

$$\begin{aligned} W_1^{odd} + W_{anom} = &= \int d^3y\sqrt{-G}\left[\hat{\nu}(\delta J^0_{consistent} - \delta J^0_{covariant}) + J^0_{S\,\mathrm{new}}\right] \\ &= \int d^3y\sqrt{-G}\left[\hat{\nu}\delta J^0_{shift} + J^0_{S\,\mathrm{new}}\right] \end{aligned} \tag{6.93}$$

The term in (6.93) proportional to δJ^0_{shift} has its origin in the fact that (6.58) is correct when J^0 is taken to be the consistent $U(1)$ current. On the other hand the canonical entropy current of hydrodynamics is defined in terms of the covariant $U(1)$ current. As explained in [47] these two currents differ by the shift

$$j^{\mu}_{shift} = \frac{C}{6}\epsilon^{\mu\nu\rho\sigma}\mathcal{A}_\nu\mathcal{F}_{\rho\sigma}. \tag{6.94}$$

The contribution of this shift to the RHS of (6.93) evaluates to

$$\begin{aligned} &\int d^3y\sqrt{-G}\hat{\nu}\delta J^0_{shift} \\ &= \frac{C}{6}\int d^3y\sqrt{-G}e^{-\sigma}\hat{\nu}\epsilon^{ijk}\mathcal{A}_i\mathcal{F}_{jk} \\ &= \frac{C}{3}\int d^3y\sqrt{g}\hat{\nu}\epsilon^{ijk}\Big(A_i\partial_jA_k + A_0A_i\partial_ja_k - A_ia_j\partial_kA_0 + A_0a_i\partial_jA_k + A_0^2a_i\partial_ja_k\Big) \\ &= \frac{C}{3}\int d^3y\sqrt{g}\hat{\nu}\epsilon^{ijk}\Big(A_i\partial_jA_k + \frac{1}{2}A_0A_i\partial_ja_k + \frac{3}{2}A_0a_i\partial_jA_k + A_0^2a_i\partial_ja_k\Big) \\ &= \frac{C}{3}\int d^3y\sqrt{g}\hat{\nu}\epsilon^{ijk}\Bigg(A_i\partial_jA_k + \frac{1}{2}A_0A_i\partial_ja_k + \frac{3}{2\hat{T}^2\psi_{eq}}A_0(a.(\zeta^{eq})S_1 - a.V_3) \\ &\qquad\qquad + \frac{A_0^2}{\hat{T}^2\psi_{eq}}(a.(\zeta^{eq})S_2 - a.V_4)\Bigg) \end{aligned} \tag{6.95}$$

65 We also need to make identification $s_9 = 2C_1$, as will be clear below.

Taking this contribution to the LHS of eq.(6.93) we find

$$\begin{aligned}
&\int d^3y\sqrt{-G}J^0_{S\,\text{new}}\\
&= W_1^{odd} + W_{anom} - \int d^3y\sqrt{-G}\delta J^0_{shift}\\
&= \int d^3y\sqrt{g}\Bigg(g_1 S_1 + T_0 g_2 S_2 + \frac{1}{\hat{T}^2\psi_{eq}}(C_1 T_0^2 - \frac{C}{3}\hat{\nu}A_0^2)(a.(\zeta^{eq})S_2 - a.V_4)\\
&\qquad\qquad - \frac{1}{2\hat{T}^2\psi_{eq}}C\hat{\nu}A_0(a.(\zeta^{eq})S_1 - a.V_3)\Bigg)
\end{aligned} \tag{6.96}$$

In rest of the subsubsection we will use (6.96) to constrain the new part of the entropy current. The most general form of the first order entropy current is given by

$$\begin{aligned}
J^\mu_{S\,\text{new}} = \epsilon^{\mu\nu\rho\sigma}\partial_\nu\left(\sigma_1 T u_\rho \zeta_\sigma\right) + \sigma_3\tilde{\mathcal{V}}_3^{c\,\mu} + T\sigma_4\tilde{\mathcal{V}}_4^{c\,\mu} + T\sigma_5\tilde{\mathcal{V}}_5^{c\,\mu}\\
+ \frac{\sigma_8}{2}\epsilon^{\mu\nu\rho\sigma}\xi_\nu F_{\rho\sigma} + T^2\sigma_9\omega^\mu + T\sigma_{10}B^\mu\\
+ \alpha_1\tilde{\mathcal{V}}_1^{c\,\mu} + \alpha_2\tilde{\mathcal{V}}_2^{c\,\mu} + \zeta_f^\mu\left[\alpha_3(\omega\cdot\zeta) + \alpha_4(B\cdot\zeta)\right] + \tilde{h}\epsilon^{\mu\nu\lambda\sigma}\mathcal{A}_\nu\partial_\lambda\mathcal{A}_\sigma\\
\text{where } \tilde{h} \text{ is a constant}
\end{aligned} \tag{6.97}$$

Since the first term proportional to σ_1 is a total derivative, it is not determined. The term proportional to α_1 and α_2 is also undetermined as $\tilde{\mathcal{V}}_1^{c\,\mu}$ and $\tilde{\mathcal{V}}_2^{c\,\mu}$ both are zero at equilibrium. We now evaluate (6.97) in equilibrium. Using Table 1 and Table 2 and

$$\begin{aligned}
&\tilde{\mathcal{V}}_0^{c\,I} = 0, \qquad \tilde{\mathcal{V}}^{c\,0,I} = -a_i\tilde{\mathcal{V}}^{c\,i,I} \quad \textit{where } I \in (1 \textit{ to } 7)\\
&\omega^0 = \frac{e^\sigma}{2(\zeta^{eq})^2}\big((a.(\zeta^{eq}))S_2 - (a.V_4)\big),\\
&B^0 = -\frac{1}{(\zeta^{eq})^2}\big((a.(\zeta^{eq}))S_1 - (a.V_3)\big) - \frac{A_0}{(\zeta^{eq})^2}\big((a.(\zeta^{eq}))S_2 - (a.V_4)\big)\\
&\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\xi_\nu F_{\rho\sigma} = e^\sigma(S_1 + A_0 S_2) + \frac{A_0 e^\sigma}{(\zeta^{eq})^2}\big((a.(\zeta^{eq}))S_1 - (a.V_3)\big) + \frac{A_0^2 e^\sigma}{(\zeta^{eq})^2}\big((a.(\zeta^{eq}))S_2 - (a.V_4)\big)\\
&\qquad\qquad + e^\sigma(a.V_2)\\
&\epsilon^{0\nu\lambda\sigma}\mathcal{A}_\nu\partial_\lambda\mathcal{A}_\sigma = e^{-\sigma}\epsilon^{ijk}\left[A_i\partial_j A_k + 2T_0\hat{\nu}a_i\partial_j A_k + T_0^2\hat{\nu}^2 a_i\partial_j a_k + \partial_i\left(T_0\hat{\nu}a_j A_k\right)\right]
\end{aligned} \tag{6.98}$$

where V_i and S_i are listed in Eq.6.28. Now using the fact that $(\zeta^{eq})_i = A_i + \partial_i \phi$

$$
\begin{aligned}
&\int \sqrt{-G}\epsilon^{0\nu\lambda\sigma}\mathcal{A}_\nu\partial_\lambda\mathcal{A}_\sigma \\
&= \int \epsilon^{ijk}\left[A_i\partial_j A_k + 2T_0\hat{\nu}a_i\partial_j A_k + T_0^2\hat{\nu}^2 a_i\partial_j a_k\right] \\
&= \int \sqrt{g}\epsilon^{ijk}\left[(\zeta^{eq})_i\partial_j(\zeta^{eq})_k + 2T_0\hat{\nu}a_i\partial_j(\zeta^{eq})_k + T_0^2\hat{\nu}^2 a_i\partial_j a_k\right] \\
&= \int \sqrt{g}\big(S_1 + \frac{1}{(\zeta^{eq})^2}2T_0\hat{\nu}((a.(\zeta^{eq}))S_1 - a.V_3) + \frac{1}{(\zeta^{eq})^2}T_0^2\hat{\nu}^2((a.(\zeta^{eq}))S_2 - a.V_4)\big)
\end{aligned}
\tag{6.99}
$$

we obtain

$$
\begin{aligned}
\int d^3x\sqrt{-G}J^0_{S\,\text{new}} = &\int d^3x\sqrt{g}\Big(e^\sigma\hat{T}\sigma_3(a.V_1) + (\sigma_8-\sigma_4)(a.V_2) - \hat{T}e^\sigma\sigma_5(a.V_5) \\
&+ \sigma_8(S_1 + A_0S_2) + \tilde{h}S_1 - (a.(\zeta^{eq}))\big(-\frac{1}{2}\alpha_3 e^\sigma S_2 + \alpha_4(S_1 + A_0S_2)\big) \\
&+ \frac{1}{(\zeta^{eq})^2}(-\hat{T}e^\sigma\sigma_{10} + \sigma_8A_0 + 2\tilde{h}T_0\hat{\nu})\big((a.(\zeta^{eq}))S_1 - (a.V_3)\big) \\
&+ \frac{1}{(\zeta^{eq})^2}(\frac{e^{2\sigma}}{2}\hat{T}^2\sigma_9 - \hat{T}e^\sigma\sigma_{10}A_0 + \sigma_8A_0^2 + \tilde{h}T_0^2\hat{\nu}^2)\big((a.(\zeta^{eq}))S_2 - (a.V_4)\big)\Big)
\end{aligned}
\tag{6.100}
$$

It is convenient to introduce the following redefinitions

$$
\sigma_3 = -\hat{T}\partial_{\hat{T}}X,\ \ \sigma_8 - \sigma_4 = \partial_{\hat{\nu}}X + Y,\ \ \sigma_5 = -\partial_{\psi_{eq}}X + Z. \tag{6.101}
$$

Now using

$$
\partial_k X = \partial_{\hat{T}}X\ \partial_k\hat{T} + \partial_{\hat{\nu}}X\ \partial_k\hat{\nu} + \partial_{\psi_{eq}}X\ \partial_k\psi_{eq}, \tag{6.102}
$$

the first line of the Eq.6.100 can be rewritten as

$$
\begin{aligned}
&\int d^3x\sqrt{g}\Big(e^\sigma\hat{T}\sigma_3(a.V_1) + (\sigma_8-\sigma_4)(a.V_2) - \hat{T}e^\sigma\sigma_5(a.V_5)\Big) \\
&= \int d^3x\sqrt{g}\Big(T_0\epsilon^{ijk}a_i(\zeta^{eq})_j\partial_k X + Y(a.V_2) - \hat{T}e^\sigma Z(a.V_5)\Big) \\
&= \int d^3x\sqrt{g}\Big(-T_0X\epsilon^{ijk}(\zeta^{eq})_i\partial_j a_k + T_0X\epsilon^{ijk}a_i\partial_j(\zeta^{eq})_k + Y(a.V_2) - \hat{T}e^\sigma Z(a.V_5)\Big) \\
&= \int d^3x\sqrt{g}\Big(-T_0XS_2 + T_0X\frac{1}{(\zeta^{eq})^2}\big((a.(\zeta^{eq}))S_1 - (a.V_3)\big) + Y(a.V_2) - \hat{T}e^\sigma Z(a.V_5)\Big).
\end{aligned}
\tag{6.103}
$$

So we obtain

$$\begin{aligned}\int d^3x\sqrt{-G}J^0_{S\,\text{new}} =& \int d^3x\sqrt{g}\Big(- T_0 X S_2 + \sigma_8(S_1 + A_0 S_2) + \tilde{h}S_1 \\ &+ (T_0 X - \hat{T}e^{\sigma}\sigma_{10} + \sigma_8 A_0 + 2\tilde{h}T_0\nu)\frac{1}{(\zeta^{eq})^2}\big((a.(\zeta^{eq}))S_1 - (a.V_3)\big) \\ &+ \frac{1}{(\zeta^{eq})^2}(\frac{e^{2\sigma}}{2}\hat{T}^2\sigma_9 - \hat{T}e^{\sigma}\sigma_{10}A_0 + \sigma_8 A_0^2 + \tilde{h}T_0^2\hat{\nu}^2)\big((a.(\zeta^{eq}))S_2 - (a.V_4)\big) \\ &- (a.(\zeta^{eq}))\big(- \frac{1}{2}\alpha_3 e^{\sigma}S_2 + \alpha_4(S_1 + A_0 S_2)\big) + Y(a.V_2) - \hat{T}e^{\sigma}Z(a.V_5)\Big).\end{aligned} \tag{6.104}$$

Now using (6.96) we obtain

$$\begin{aligned}&Y = Z = 0, \quad \alpha_3 = \alpha_4 = 0 \\ &X = \sigma_{10} - \hat{\nu}\sigma_8 - \frac{1}{2}C\hat{\nu}^2 - 2\tilde{h}\hat{\nu}, \\ &\sigma_3 = -\hat{T}\partial_{\hat{T}}(\sigma_{10} - \hat{\nu}\sigma_8),\ \sigma_4 = \sigma_8 - \partial_{\hat{\nu}}(\sigma_{10} - \hat{\nu}\sigma_8) + C\hat{\nu} + 2\tilde{h},\ \sigma_5 = -\partial_{\psi_{eq}}(\sigma_{10} - \hat{\nu}\sigma_8) \\ &\sigma_9 = 2\hat{\nu}(\sigma_{10} - \hat{\nu}\sigma_8) + 2(C_1 - \frac{C}{3}\hat{\nu}^3) - 2\tilde{h}\hat{\nu}^2 \\ &g_1 = \sigma_8 + \tilde{h}, \quad g_2 = -\sigma_{10} + 2\hat{\nu}\sigma_8 + \frac{1}{2}C\hat{\nu}^2 + 2\tilde{h}\hat{\nu}.\end{aligned} \tag{6.105}$$

[66] It may be verified that (6.105) is consistent with (6.84). In other words the entropy current determined by comparison with partition function agrees exactly with the non dissipative part of the entropy current determined from the requirement of positivity of divergence. [67]

6.6 CPT Invariance

In this subsection we explore the constraints imposed on the partition function (6.17) and (6.26) by the requirement of 4 dimensional CPT invariance. In Table 3 we list the action of CPT on various fields appearing in the partition function.

- **Parity even case:** Using this table one easily see that, demanding CPT invariance of the action (6.17), the functions $f_1,\ f_2,\ f_3$ are even under CPT. Instead had we demanded only time reversal invariance, then the we would conclude that $f_1 = f_2 = f_3 = 0$.
- **Parity odd case:** Now demanding CPT invariance of the action (6.26), we conclude that g_1 is odd function of A_0 and hence it can not contain any constant. This in particular implies $\tilde{h} = 0$, since $g_1 = \sigma_8 + \tilde{h}$. So the gauge non invariant piece in entropy current in (6.97) vanishes once we demand CPT invariance. The function g_2 appearing in (6.26) is even function in A_0. It is also easy to see that the requirement of CPT invariance of the partition function forces $C_1 = 0$.

[66] The expression $2C_1$ was referred to as s_9 in [4].

[67] Note however that the entropy positivity method, in addition, determines two dissipative terms in the entropy current, and so, in that sense, carries more information about the entropy current.

Field	C	P	T	CPT
σ	+	+	+	+
a_i	+	−	−	+
g_{ij}	+	+	+	+
A_0	−	+	+	−
A_i	−	−	−	−
ζ_i	−	−	−	−

Table 21. Action of CPT

6.7 Discussion

In this section we have studied the equality type constraints between transport coefficients for relativistic superfluids at first order in the derivative expansion. Our central result is that the constraints obtained from a local form of the second law of thermodynamics agree exactly with those obtained from a study of the equilibrium partition function.

As the constraints obtained from both methods are numerous and rather involved in structure, the perfect agreement found in this section strengthens our earlier conjecture [47] that the constraints obtained from the partition function agree with those obtained from the local version of the second law of thermodynamics under all circumstances. It would be interesting to find either a proof for or a counterexample against this conjecture.

In the special case that the superfluid is nondissipative, [31] has presented a framework for describing superfluid dynamics from an action formalism. It would be interesting to understand the connection of the formalism of [31] to that described in this paper.

As we have explained above, a central object in our analysis was a local Euclidean action for the superconducting Goldstone field. In the neighborhood of a second order phase transition familiar Landau-Ginzburg action for the order parameter is the natural analogue of the Goldstone boson action utilized in this paper. It seems likely that the methods of the current paper generalize to the study of hydrodynamics in the neighborhood of second order phase transitions (see [56] for a review). It would be interesting to perform this generalization.

Finally, in this section we have discussed only the equality type constraints on nondissipative transport coefficients that follow from the local second law. We have neither discussed Onsager type equality constraints on dissipative coefficients nor the inequalities on dissipative coefficients. It is possible that these constraints follow the imposition of reasonable conditions (like stability) to time fluctuations about equilibrium. We leave the study of time dependence for future work.

7 Conclusion

In the work reported in this thesis we have developed a more field theoretic approach based on the existence of a partition function for equilibrium systems in hydrodynamic regime to constrain the non-dissipative transport coefficients. This approach apart from capturing independent non-dissipative transport coefficients beautifully captures the effect of anomalies in global symmetries as certain Chern-Simons terms in the equilibrium partition function. In a wide variety of examples we have shown that the constraints thus obtained match precisely with those obtained using a local second law of thermodynamics. This lead us to conjecture that the equality holds at all derivative orders for all fluid systems.

We hope that the work done in this thesis would lead to further studies of anomalous transport ranging from non abelian internal symmetries to weyl and diffeomorphism anomalies and provide a platform for better understanding of inequality relations leading to a fuller understanding of 2nd law of thermodynamics (at least in hydrodynamical context) and Wald entropy via fluid-gravity map.

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