Generalized unitarity limits on the mass of thermal dark matter in (non-) standard cosmologies

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(Work done in collaboration with Satyanarayan Mukhopadhyay)
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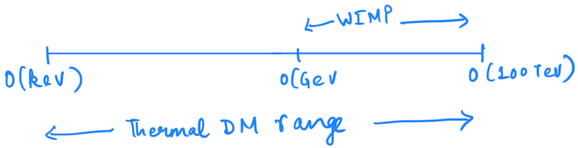

- WIMPs are most popular thermal relics, with mass range between $\mathcal{O}$(GeV)–$\mathcal{O}$(100 TeV).

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- These bounds are derived assuming DM freeze-out in the standard scenario.

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- Note that these limits on mass can certainly be different if some of the assumptions of the standard freeze-out are lifted.

Assumptions (keep/lift)

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- Presence of tiny coupling within DM-SM sector to establish kinetic equilibrium but small enough to not have any significant contribution in the annihilation/creation of DM.
- What are the implications of S-matrix unitarity on these kinds of models?

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- Conservation of probability or the unitarity condition of S-matrix gives the generalized optical theorem, which for n-body scattering can be written as:

$$\begin{aligned} 2\mathrm{Im}\mathcal{M}_{\rm el}(\alpha_n \to \alpha_n) &= \int d\Pi_n (2\pi)^4 \delta^4(p_{\alpha_n} - p_{\beta_n}) |\mathcal{M}_{\rm el}(\alpha_n \to \beta_n)|^2 \\ &+ \sum_{n'} \int d\Pi_2 (2\pi)^4 \delta^4(p_{\alpha_n} - p_{\beta(n')}) |\mathcal{M}_{\rm in}(\alpha_n \to \beta_{n'}|^2 \end{aligned}$$

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- Model independent bounds are determined by expanding the matrix amplitude in terms of orthogonal functions of angular variables i.e. by doing partial wave expansion.

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- We use the equality of rates of forward-backward processes, and estimate the annihilation cross-section in terms of the creation cross-section i.e.

$$n_1^{\rm eq} \dots n_n^{\rm eq} \langle \sigma_{n \to 2} v^{n-1} \rangle = n_{n+1}^{\rm eq} n_{n+2}^{\rm eq} \langle \sigma_{2 \to n} v \rangle$$

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- We now solve for optical theorem for 2 particle state scattering i.e.

$$\begin{aligned} 2{\rm Im}\mathcal{M}_{\rm el}(\alpha_2 \to \alpha_2) &= \int d\Pi_n (2\pi)^4 \delta^4 (p_{\alpha_2} - p_{\beta_2}) |\mathcal{M}_{\rm el}(\alpha_2 \to \beta_2)|^2 \\ &+ \sum_{n'} \int d\Pi_2 (2\pi)^4 \delta^4 (p_{\alpha_2} - p_{\beta(n')}) |\mathcal{M}_{\rm in}(\alpha_2 \to \beta_{n'}|^2 \end{aligned}$$

S-matrix unitarity continued ...

- The total inelastic scattering cross-section can be written in terms of $2 \to 2$ elastic scattering

$$\sigma_{\text{in,total}} = \frac{2}{F_2}\text{Im}\mathcal{M}_{\text{el}}(\theta=0) - \frac{|q|}{4F_2 S_2 E_{\text{tot}}}\int \frac{d\Omega}{(2\pi)^2}|\mathcal{M}_{\text{el}}(\theta,\phi)|^2$$
$$= \frac{2}{F_2}\text{Im}\mathcal{M}_{\text{el}}(\theta=0) - \sigma_{\text{el}}$$

- Expanding the amplitude in partial basis:

$$\mathcal{M}_{\text{el}} = \langle \hat{q}|T_{\rm el}|\hat{z}\rangle = \sum_{\ell,\ell',m,m'} \langle \hat{q}|\ell',m'\rangle\langle \ell',m'|T^{\ell}_{\rm el}|\ell,m\rangle\langle \ell,m|\hat{z}\rangle$$

- Unitarity of S-matrix in angular basis:

$$\langle \ell,m|S^\dagger S|\ell,m\rangle = \langle \ell,m|S^\dagger_{\rm el}S_{\rm el} + S^\dagger_{\rm in}S_{\rm in}|\ell,m\rangle = 1$$

with $\langle \ell,m|S^\dagger_{\rm el}S_{\rm el}|\ell,m\rangle \to 0$ and $\langle \ell,m|S^\dagger_{\rm in}S_{\rm in}|\ell,m\rangle \to 1$.

S-matrix unitarity continued ...

for maximizing inelastic scattering.

$$\sigma_{\text{in,total}} \quad = \quad \sum_{\ell} \frac{\pi}{|\vec{p_1}|^2} S_2(2\ell+1) \left(1 - \langle \ell, m | S_{\mathrm{el}}^{\dagger} S_{\mathrm{el}} | \ell, m \rangle \right)$$

Thermal averaged cross-sections assuming domination of $2 \to n$ scattering:

$$\langle \sigma_{2\to n} v_{\mathrm{rel}} \rangle_{\mathrm{max}} = \sum_{\ell} (2\ell+1) \frac{4\sqrt{\pi}}{m_{\chi}^2} \sqrt{x} e^{-(n-2)x}$$

Using equality of rates, the annihilation cs for $n \to 2$ can be determined from $2 \to 2$.

$$\langle \sigma_{n\to 2} v^{n-1} \rangle_{\mathrm{max}} = \sum_{\ell} (2\ell+1) \frac{2^{\frac{3n-2}{2}} (\pi x)^{\frac{3n-5}{2}}}{g_{\chi}^{k-2} m_{\chi}^{3n-4}}.$$

General Boltzmann equation for $n \to 2$ processes

$$\frac{1}{a^3}\frac{d}{dt}(n_\chi a^3) = -\sum_i \Delta n_{\chi,i} \left[n_1 n_2 \ldots n_m \langle \sigma_{m\to 2} v^{m-1} \rangle_i - n_{m+1} n_{m+2} \langle \sigma_{2\to m} v \rangle_i \right] .$$

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$$\langle \sigma_{m\to 2} v^{m-1} \rangle = \frac{n^{\rm eq}_{m+1} n^{\rm eq}_{m+2}}{n^{\rm eq}_1 n^{\rm eq}_2 \ldots n^{\rm eq}_m} \langle \sigma_{2\to m} v \rangle$$

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Results for radiation domination

Symmetry	Annihilation channels	$\ell = 0$	$\ell = 1$	$\ell = 0 + 1$
Z_2	$\chi + \chi^* \to \mathrm{SM} + \mathrm{SM}$	127.7 TeV	220 TeV	253.5 TeV
Z_3	$3\chi^{(*)} \to 2\chi^{(*)}$	1.15 GeV	1.72 GeV	1.91 GeV
Z_2	$4\chi \to 2\chi$	6.9 MeV	9.4 MeV	10.1 MeV
Z_5	$5\chi^{(*)} \to 2\chi^{(*)}$	112.5 keV	138 keV	145.5 keV

Table: Unitarity upper limits on thermal DM mass in a radiation dominated Universe

For ex., $3\chi \to 2\chi$, assuming CP-conservation, the Boltzmann equation is

$$\frac{1}{a^3}\frac{d}{dt}(n_\chi a^3) = -\frac{1}{n_{\chi,\mathrm{eq}}}\left(\langle\sigma_{\chi\chi^*\to\chi\chi\chi}v\rangle + \langle\sigma_{\chi^*\chi^*\to\chi\chi\chi^*}v\rangle\right) \times \left[n_\chi^3 - n_\chi^2 n_{\chi,\mathrm{eq}}\right] .$$

Why mass bounds decreases with increasing n in $n \to 2$

Freeze-out condition approx can be determined by $\Gamma \approx H$

$$\begin{aligned}
\Gamma_{2\to2} &= n_{\rm eq}\langle\sigma v\rangle_{2\to2} \\
\Gamma_{3\to2} &= n_{\rm eq}^2\langle\sigma v^2\rangle = n_{\rm eq}\langle\sigma v\rangle_{2\to2}e^{-x_F} \\
\Gamma_{4\to2} &= n_{\rm eq}^3\langle\sigma v^3\rangle = n_{\rm eq}\langle\sigma v\rangle_{2\to2}e^{-2x_F}
\end{aligned}$$

where, $\langle\sigma v\rangle_{2\to2} = \dfrac{4\sqrt{\pi}\sqrt{x_F}}{m_\chi^2}$

The rate decreases with increase in n, which must be accomplished by decrease in m_χ to get correct abundance.

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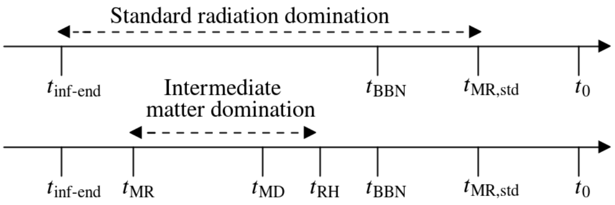
- We study the implications of these long lived fields on the freeze-out of dark matter.

Specifying the time-lines



t_{MR}: Epoch of matter-radiation equality

t_{MD}: Epoch where matter decays starts to become important

t_{RH}: Epoch where radiation domination is restored.

Regions	$a(t)$	$a(T)$	$H(T)$
$t_{\text{inf,end}} < t < t_{\text{MR}}$	$a(t) \propto t^{1/2}$	$a(T) \propto T^{-1}$	$\sqrt{\frac{\pi^2 g_*(T)}{90}} \frac{T^2}{M_{\text{pl}}}$
$t_{\text{MR}} < t < t_{\text{MD}}$	$a(t) \propto t^{2/3}$	$a(T) \propto T^{-1}$	$\sqrt{\frac{\pi^2 g_*(T) T_{\text{MR}}}{90}} \frac{T^{3/2}}{M_{\text{pl}}}$
$t_{\text{MD}} < t < t_{\text{RH}}$	$a(t) \propto t^{2/3}$	$a(T) \propto T^{-8/3}$	$\sqrt{\frac{\pi^2 g_*^2(T)}{90 g_*(T_{\text{RH}})}} \frac{T^4}{M_{\text{pl}} T_{\text{RH}}^2}$

Model independent effect of IMD on cosmology

This problem can be solved by defining the $T_{\rm MR}$ and Γ (decay rate) which can be determined in terms of $T_{\rm RH}$.

$$\begin{aligned} \frac{d\rho_R(t)}{dt} + 4H(t)\rho_R(t) &= \Gamma_\Phi \rho_\Phi(t) \\ \frac{d\rho_\Phi(t)}{dt} + 3H(t)\rho_\Phi(t) &= -\Gamma_\Phi \rho_\Phi(t), \end{aligned}$$

and the Friedmann equation determining the Hubble parameter

$$H^2(t) = \frac{8\pi G}{3}\left(\rho_R(t) + \rho_\Phi(t)\right),$$

- Ideally one should solve for Boltzmann equation of DM along with these.
- Assuming that the matter field decays only to radiation, we restrict ourselves to thermal production of dark matter.
- This helps us in decoupling the dark matter dynamics from the rest and also helps in setting maximum mass limits by unitarity.

The solution to energy density of radiation can be calculated analytically:

$$\rho_R(t) \simeq \frac{4M_P^2\Gamma_\Phi}{3}\sum_{n=0}^{\infty}\frac{(-\Gamma_\Phi)^n\, t^{n-1}}{n!\,\left(n+\frac{5}{3}\right)} + \frac{4M_P^2}{3}\frac{t_{\rm MR}^{2/3}}{t^{8/3}},$$

The effect of intermediate period of matter domination has two fold effect on DM:

- The expansion rate of the universe changes, changing the temperature at which freeze-out occurs (in other words x_F).

$$\begin{aligned}\Gamma &\approx H \quad \text{(Freeze-out condition)}\\ n(T_F) &\approx \frac{H(T_F)}{\langle\sigma v\rangle}\end{aligned}$$

- The restoration to RD phase is accompanied by release in entropy which dilutes the relic density of the dark matter.

  To balance out the correct abundance, one must hence first overproduce dark matter $\Rightarrow$ larger masses can be allowed.

Entropy dilution

Whether in absence/presence of matter decay, DM number density after freeze-out in comoving volume remains conserved

$$n(t_0)a(t_0)^3 = n(t_{F.O.})a(t_{F.O.})^3$$

This equation can be rearranged as:

$$\begin{aligned} n(t_0) &= n(t_{F.O.})\frac{a(t_{F.O.})^3}{a(t_0)^3} \times \frac{s(t_{F.O.})}{s(t_{F.O.})} \times \frac{s(t_0)}{s(t_0)} \\ \frac{n(t_0)}{s(t_0)} &= \frac{n(t_{F.O.})}{s(t_{F.O.})}\frac{S(t_{F.O.})}{S(t_0)} \end{aligned}$$

Entropy conservation will imply $S(t_{F.O.}) = S(t_0)$
and violation imply n/s no longer conserved and dilutes with increase in entropy.

All we have to do is evaluate entropy increase.

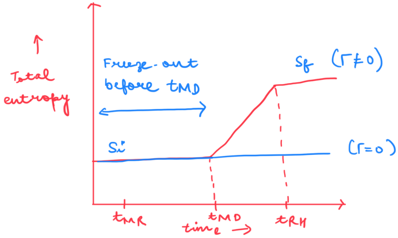
Entropy dilution in different epochs

A. DM freeze-out in constant entropy phase ($t_{\rm F.O.} < t_{\rm M.D.}$)
(either radiation domination or stable matter domination)

Assumption: No entropy increase after reheat.

$$\frac{S(t_0)}{S(t_{F.O.})} = \frac{S(t_{\rm RH})}{S(t_{\rm MD})}$$

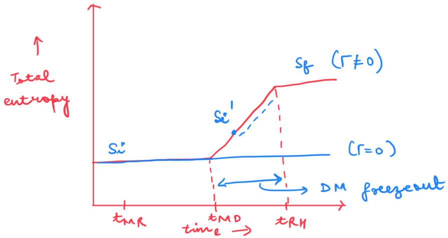
This ratio can be calculated in the instantaneous decay approximation.



$$\frac{S(t_{\rm RH})}{S(t_{\rm MD})} \approx \frac{s(t_{\rm RH})}{s^{\Gamma=0}(t_{\rm RH})} \approx \frac{\rho(t_{\rm RH})^{3/4}}{\rho^{\Gamma=0}(t_{\rm RH})^{3/4}} \approx \frac{T_{\rm MR}}{T_{\rm RH}}$$

Entropy dilution in different epochs

B. DM freeze-out in decaying matter phase/increasing entropy case ($t_{\mathrm{F.O.}} > t_{\mathrm{M.D.}}$)



$$\frac{S(t_0)}{S(t_{F.O.})} = \frac{S(t_{\mathrm{RH}})}{S(t_{\mathrm{F.O.}})}$$

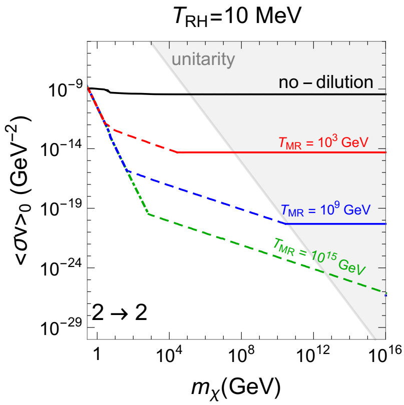
Instantaneous decay approximation doesn't work.

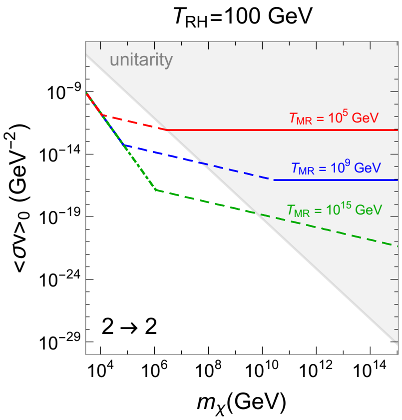
$$\frac{S(t_{\mathrm{RH}})}{S(t_{\mathrm{F.O.}})} = \frac{s(t_{\mathrm{RH}})}{s(t_{\mathrm{F.O.}})} \times \frac{a(t_{\mathrm{RH}})^3}{a(t_{\mathrm{F.O.}})^3} \approx \frac{T_{\mathrm{FO}}^5}{T_{\mathrm{RH}}^5}$$

Three regions of freeze-out:

1. $T_{\rm FO} > T_{MR}$, which translates to $m_\chi > x_F T_{\rm MR}$
2. $T_{\rm MD} < T_{\rm FO} < T_{\rm MR}$, which translates to $x_F T_{\rm MD} < m_\chi < x_F T_{MR}$
3. $T_{\rm RH} < T_{\rm FO} < T_{\rm MD}$, which translates to $x_F T_{\rm RH} < m_\chi < x_F T_{\rm MD}$

Implications of unitarity on 2 → 2



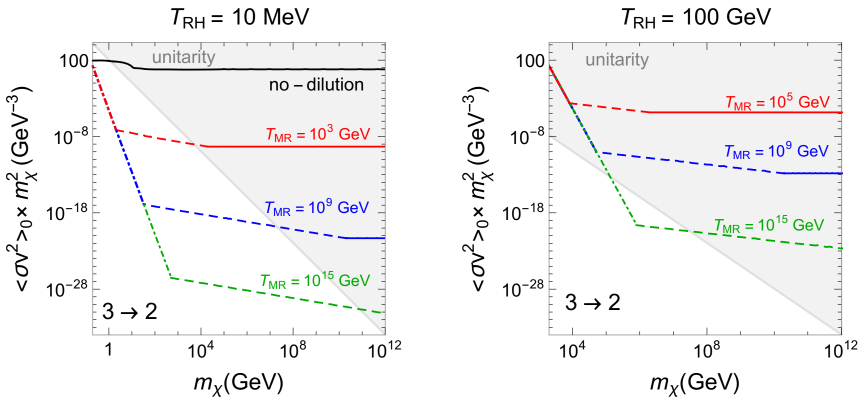


solid line : freeze-out in radiation dom
dashed line: freeze-out in stable matter dom
dot-dashed line: freeze-out in decaying matter phase

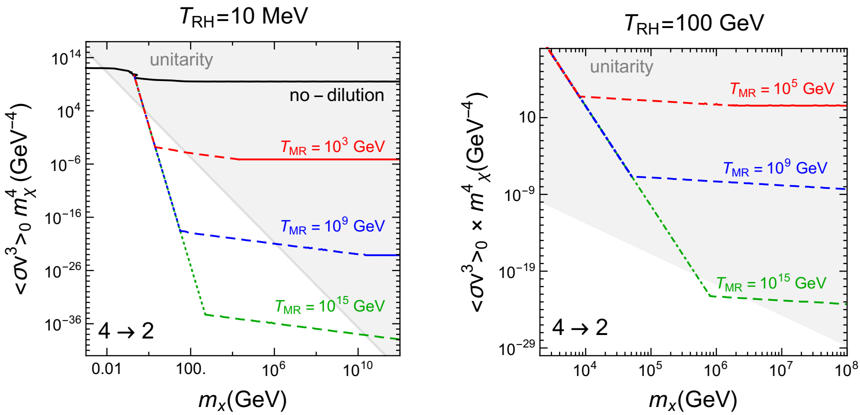
Main highlights of the results

- The regions below the shaded regions are allowed by unitarity.
- The contours satisfy relic density constraint.
- Freeze-out in radiation domination, $\Omega \propto 1/\langle \sigma v \rangle$, and doesn't depend explicitly of mass $\Rightarrow$ one value of c.s. satisfying relic.
- Freeze-out in stable matter domination, $\Omega \propto 1/\sqrt{m_\chi}\langle \sigma v \rangle \Rightarrow$ for low masses, higher values of annihilation c.s required for satisfying relic.
- Freeze-out in decaying matter phase, $\Omega \propto 1/m_\chi^3 \langle \sigma v \rangle \Rightarrow$ for low masses, higher values of annihilation c.s required for satisfying relic.
- For increasing dilution $T_{\mathrm{MR}}/T_{\mathrm{RH}}$, the mass bound of DM allowed by unitarity increases.
- For increasing reheat temp, and fixing matter-radiation equality temperature, dilution decreases, hence corresponding mass bound also decreases.
- The values of T_{MR} is chosen in accordance with coincidence with famous NP scales — TeV scale, see-saw, GUT scale.

Implications of unitarity on $3 \rightarrow 2$



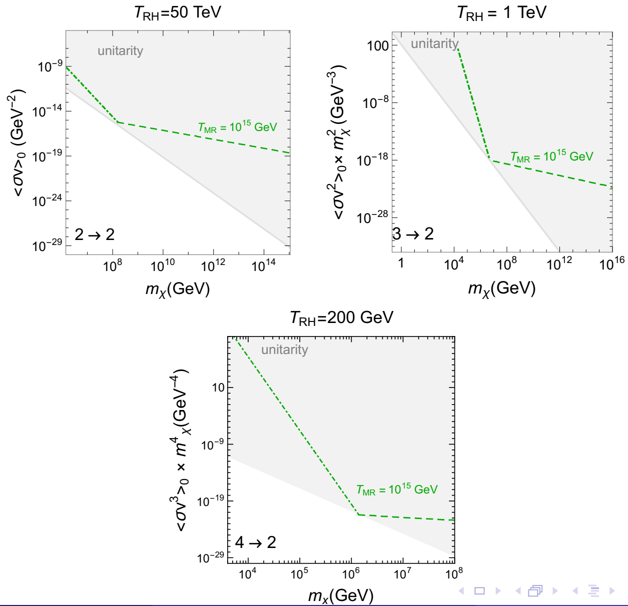
Implications of unitarity on $4 \rightarrow 2$



Combined results:

- Qualitative results same for $n \to 2$.
- With increasing n, the maximum mass allowed by unitarity for a fixed value of dilution decreases.
- We do not take T_{MR} to be more than GUT scale.
- Unitarity rules out more region for smaller mass as $\sigma v \propto 1/m_\chi^2$, therefore the allowed regions for $4 \to 2$ is smaller than $3 \to 2$ which is smaller than $2 \to 2$.
- Using this, we fix T_{MR} to its max value, and determine upto what values of T_{RH} is allowed by combined constraints from unitarity and relic density.

Combined results:



Can we constrain matter-radiation equality temperature?

- For DM freeze-out purely within the dark sector

$$
\begin{aligned}
\Gamma_{\chi\chi\to SMSM} &< \Gamma_{\chi\chi\chi\to\chi\chi} \\
n_{\rm eq}\frac{\alpha^2}{m_\chi^2} &\leq n_{\rm eq}^2\langle\sigma v^2\rangle \\
\alpha^2 &\leq 4\sqrt{\pi x_F}e^{-x_F}
\end{aligned}
$$

For $x_F \approx 20$, $\alpha \leq 10^{-4}$

- Also the temperatures, when the two sectors — DM and SM came in thermal contact is given as:

$$\Gamma_{\chi SM\to\chi SM} \geq H \Rightarrow T_{\rm eq} \leq \alpha^2 M_{\rm pl}$$

Substituting α_{max}, we get $T_{\rm eq} \leq 10^{10}$.

T_{MR} and m_χ should be less than $T_{\rm eq}$ $\Rightarrow$ one way perhaps to constrain the matter-radiation equality temp, theoretically.

Sumarizing

- We solved for the Boltzmann equations for generalized $n \to 2$ dark matter annihilations.
- The $n \to 2$ annihilations evaluated in terms of knowledge of $2 \to 2$ elastic scattering.
- The exercise is repeated for matter dominated universe for epochs greater than Big bang nucleosynthesis.
- Future direction to see if somehow we can constrain the extent of entropy dilution.

Why unitarity gives bound on the maximum mass

The c.s. fixed by unitarity is for strong coupling for a given value of mass.

$$\sigma^{\text{max}}_{\text{in,total}} = \frac{\pi}{m_\chi^2 v^2} \tag{1}$$

The c.s. in terms of couplings can be parameterized as:

$$\sigma^{\text{max}}_{\text{in,total}} = \frac{\alpha^2}{m_\chi^2} \tag{2}$$

Identifying $\alpha \rightarrow \frac{\pi}{v^2}$
For larger masses, the c.s. decreases leading in overabundance.
For smaller masses, one can decrease the coupling which leads to the correct σ for establishing dark matter abundances.

Unitarity derivation

The total energy-momentum is conserved, and the non-trivial part of $\mathcal{T}$ matrix doesn't act on the four momentum states, hence $\mathcal{T} = \mathbb{I} \otimes \mathrm{T}$ and

$$\begin{aligned} \langle P'_\mu, \hat{q}|\mathcal{T}_{\rm el}|P_\mu, \hat{z}\rangle &= \mathrm{i}(2\pi)^4\delta(P'_\mu - P_\mu)\langle\hat{q}|\mathcal{T}_{\rm el}|\hat{z}\rangle = \mathrm{i}(2\pi)^4\delta(P'_\mu - P_\mu)\mathcal{M}_{\rm el}(\theta,\phi) \\ \langle P'_\mu, \alpha'|T_{\rm in}|P_\mu, \alpha\rangle &= \mathrm{i}(2\pi)^4\delta(P'_\mu - P_\mu)\langle\alpha'|T_{\rm in}|\alpha\rangle = \mathrm{i}(2\pi)^4\delta(P'_\mu - P_\mu)\mathcal{M}_{\rm in}(\alpha,\alpha') \end{aligned}$$

Substituting these, we get

$$2\mathrm{Im}\mathcal{M}_{\mathrm{el}}(\theta=0) = \frac{|q|}{4E_{\mathrm{tot}}S_2}\int\frac{d\Omega}{(2\pi)^2}|\mathcal{M}_{\mathrm{el}}(\theta,\phi)|^2 + \sum_x \frac{c}{S_x}(2\pi)^4\int d\alpha|\mathcal{M}_{\mathrm{in}}(\alpha)|^2$$

$$\begin{aligned} \sigma_{\mathrm{in,total}} &= \frac{2}{F_2}\mathrm{Im}\mathcal{M}_{\mathrm{el}}(\theta=0) - \frac{|q|}{4F_2S_2E_{\mathrm{tot}}}\int\frac{d\Omega}{(2\pi)^2}|\mathcal{M}_{\mathrm{el}}(\theta,\phi)|^2 \\ &= \frac{2}{F_2}\mathrm{Im}\mathcal{M}_{\mathrm{el}}(\theta=0) - \sigma_{\mathrm{el}} \end{aligned}$$

$$\mathcal{M}_{\mathrm{el}} = \langle\hat{q}|T_{\rm el}|\hat{z}\rangle = \sum_{\ell,\ell',m,m'}\langle\hat{q}|\ell',m'\rangle\langle\ell',m'|T^{\ell}_{\rm el}|\ell,m\rangle\langle\ell,m|\hat{z}\rangle$$

Unitarity continued ...

$$\langle \theta, \phi | \ell, m \rangle = Y_\ell^m(\theta, \phi) \quad (3)$$

$$\langle \ell, m | \hat{z} \rangle = \sqrt{\frac{(2\ell + 1)}{4\pi}} \delta_{m0} \quad (4)$$

Eqn. (3) can be written as

$$\langle \hat{q} | T_{\rm el} | \hat{z} \rangle = 16\pi \sum_\ell (2\ell + 1)\, a_\ell P_\ell(\cos\theta) \quad (5)$$

Boltzmann's equation for the number density of χ

In equilibrium, $\chi(p_1)\chi(p_2)\chi(p_3) \leftrightarrows \chi(p_4)\chi(p_5)$.
Let us study the evolution of particle 1, whose number density is given by n_1.

$$
\begin{aligned}
\frac{1}{a^3}\frac{d}{dt}(n_1 a^3) &= -\sum_{\text{spins}}\int \left[f(p_1)f(p_2)f(p_3)|M_{3\to 2}|^2 - f(p_4)f(p_5)|M_{2\to 3}|^2 \right] \\
&\times (2\pi)^4\delta^4(p_1+p_2+p_3-p_4-p_5) \\
&\times \frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2}\frac{d^3p_3}{(2\pi)^3 2E_3}\frac{d^3p_4}{(2\pi)^3 2E_4}\frac{d^3p_5}{(2\pi)^3 2E_5} \\
&= -\int \frac{g_1 f(p_1)d^3p_1}{(2\pi)^3 2E_1}\frac{g_2 f(p_2)d^3p_2}{(2\pi)^3 2E_2}\frac{g_3 f(p_3)d^3p_3}{(2\pi)^3 2E_3}\mathrm{F}_{3\to 2}\sigma_{3\to 2} \\
&\quad + \int \frac{g_4 f(p_4)d^3p_4}{(2\pi)^3 2E_4}\frac{g_5 f(p_5)d^3p_5}{(2\pi)^3 2E_5}\mathrm{F}_{2\to 3}\sigma_{2\to 3} \\
&= -n_1 n_2 n_3\langle\sigma_{3\to 2}v^2\rangle + n_4 n_5\langle\sigma_{2\to 3}v\rangle
\end{aligned}
$$

Where the thermal averaged cross-section are given as:

$$\langle\sigma_{3\to2}v^2\rangle = \frac{\int\left(\frac{F_{3\to2}\sigma_{3\to2}}{8E_1E_2E_3}\right)dn_1^{\mathrm{eq}}dn_2^{\mathrm{eq}}dn_3^{\mathrm{eq}}}{\int dn_1^{\mathrm{eq}}dn_2^{\mathrm{eq}}dn_3^{\mathrm{eq}}}$$

$$\langle\sigma_{2\to3}v\rangle = \frac{\int\left(\frac{F_{2\to3}\sigma_{2\to3}}{4E_1E_2}\right)dn_1^{\mathrm{eq}}dn_2^{\mathrm{eq}}}{\int dn_1^{\mathrm{eq}}dn_2^{\mathrm{eq}}}$$

In equilibrium, the number density of χ in comoving volume remains constant i.e. $\frac{d(n_1a^3)}{dt}=0$, which implies

$$n_1^{\mathrm{eq}}n_2^{\mathrm{eq}}n_3^{\mathrm{eq}}\langle\sigma_{3\to2}v^2\rangle = n_4^{\mathrm{eq}}n_5^{\mathrm{eq}}\langle\sigma_{2\to3}v\rangle$$

Using this we can write the Boltzmann equation as:

$$\frac{1}{a^3}\frac{d}{dt}(na^3) = -\langle\sigma_{3\to2}v^2\rangle(n^3-n^2n_{\mathrm{eq}}) \quad \text{or,}$$

$$\frac{1}{a^3}\frac{d}{dt}(na^3) = -\frac{\langle\sigma_{2\to3}v\rangle}{n_{\mathrm{eq}}}(n^3-n^2n_{\mathrm{eq}})$$